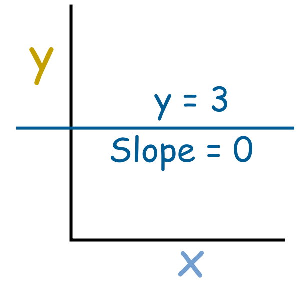
Derivative Rules

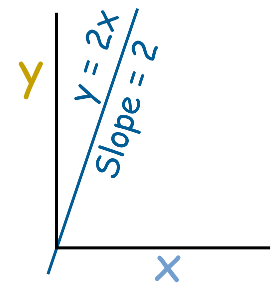
The Derivative tells us the slope of a function at any point.



There are **rules** we can follow to find many derivatives.

For example:

- The slope of a **constant** value (like 3) is always 0
- The slope of a **line** like 2x is 2, or 3x is 3 etc
- and so on.



Here are useful rules to help you work out the derivatives of many functions (with examples below). Note: the little mark ’ means "Derivative of", and f and g are functions.

Common Functions	Function	Derivative
Constant	c	0
Line	x	1
	ax	a
Square	x^2	$2x$
Square Root	$\sqrt{x}$	$(½)x^{-½}$
Exponential	e^x	e^x
	a^x	$\ln(a)\ a^x$
Logarithms	$\ln(x)$	$1/x$
	$\log_a(x)$	$1 / (x \ln(a))$
Trigonometry (x is in radians)	$\sin(x)$	$\cos(x)$
	$\cos(x)$	$-\sin(x)$
	$\tan(x)$	$\sec^2(x)$
Inverse Trigonometry	$\sin^{-1}(x)$	$1/\sqrt{(1-x^2)}$
	$\cos^{-1}(x)$	$-1/\sqrt{(1-x^2)}$

$\tan^{-1}(x)$	$1/(1+x^2)$

Rules	Function	Derivative
Multiplication by constant	cf	cf’
Power Rule	x^n	nx^{n-1}
Sum Rule	f + g	f’ + g’
Difference Rule	f - g	f’ − g’
Product Rule	fg	f g’ + f’ g
Quotient Rule	f/g	$(f' g - g' f)/g^2$
Reciprocal Rule	1/f	$-f'/f^2$
Chain Rule (as "Composition of Functions")	f º g	(f’ º g) × g’
Chain Rule (using ’)	f(g(x))	f’(g(x))g’(x)
Chain Rule (using $\frac{d}{dx}$)	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$	

"The derivative of" is also written $\frac{d}{dx}$

So $\frac{d}{dx}\sin(x)$ and $\sin(x)'$ both mean "The derivative of sin(x)"

Examples

Example: what is the derivative of sin(x) ?

From the table above it is listed as being **cos(x)**

It can be written as:

$$\frac{d}{dx}\sin(x) = \cos(x)$$

Or:

$$\sin(x)' = \cos(x)$$

Power Rule

Example: What is $\frac{d}{dx}x^3$?

The question is asking "what is the derivative of x^3 ?"

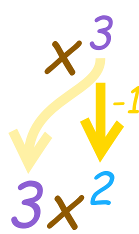
We can use the Power Rule, where n=3:

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}x^3 = 3x^{3-1} = \mathbf{3x^2}$$

(In other words the derivative of x^3 is $3x^2$)

So it is simply this:



"multiply by power
then reduce power by 1"

It can also be used in cases like this:

Example: What is $\frac{d}{dx}(1/x)$?

1/x is also $\mathbf{x^{-1}}$

We can use the Power Rule, where n = −1:

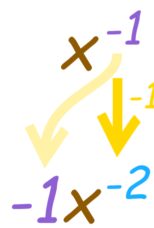
$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}x^{-1} = -1x^{-1-1}$$

$$= -x^{-2}$$

$$= \frac{-1}{x^2}$$

So we just did this:



which simplifies to $\mathbf{-1/x^2}$

Multiplication by constant

Example: What is $\frac{d}{dx}5x^3$?

the derivative of cf = cf’

the derivative of 5f = 5f’

We know (from the Power Rule):

$$\frac{d}{dx}x^3 = 3x^{3-1} = 3x^2$$

So:

$$\frac{d}{dx}5x^3 = 5\frac{d}{dx}x^3 = 5 \times 3x^2 = \mathbf{15x^2}$$

Sum Rule

Example: What is the derivative of x^2+x^3 ?

The Sum Rule says:

the derivative of f + g = f’ + g’

So we can work out each derivative separately and then add them.

Using the Power Rule:

-

- $\frac{d}{dx}x^2 = 2x$
- $\frac{d}{dx}x^3 = 3x^2$

And so:

the derivative of $x^2 + x^3 = \mathbf{2x + 3x^2}$

Difference Rule

It doesn't have to be **x**, we can differentiate with respect to, for example, **v**:

Example: What is $\frac{d}{dv}(v^3 - v^4)$?

The Difference Rule says

the derivative of $f - g = f' - g'$

So we can work out each derivative separately and then subtract them.

Using the Power Rule:

- $\frac{d}{dv}v^3 = 3v^2$
- $\frac{d}{dv}v^4 = 4v^3$

And so:

the derivative of $v^3 - v^4 = \mathbf{3v^2 - 4v^3}$

Sum, Difference, Constant Multiplication And Power Rules

Example: What is $\frac{d}{dz}(5z^2 + z^3 - 7z^4)$?

Using the Power Rule:

- $\frac{d}{dz}z^2 = 2z$
- $\frac{d}{dz}z^3 = 3z^2$
-

$$\frac{d}{dz}z^4 = 4z^3$$

And so:

$$\frac{d}{dz}(5z^2 + z^3 - 7z^4) = 5 \times 2z + 3z^2 - 7 \times 4z^3 = \mathbf{10z + 3z^2 - 28z^3}$$

Product Rule

Example: What is the derivative of cos(x)sin(x) ?

The Product Rule says:

the derivative of fg = f g′ + f′ g

In our case:

- f = cos
- g = sin

We know (from the table above):

- $\frac{d}{dx}\cos(x) = -\sin(x)$
- $\frac{d}{dx}\sin(x) = \cos(x)$

So:

the derivative of cos(x)sin(x) = cos(x)cos(x) − sin(x)sin(x)

$$= \mathbf{\cos^2(x) - \sin^2(x)}$$

Quotient Rule

To help you remember:

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

The derivative of "High over Low" is:

"Low dHigh minus High dLow, over the line and square the Low"

Example: What is the derivative of cos(x)/x ?

In our case:

- f = cos
- g = x

We know (from the table above):

- f' = −sin(x)
- g' = 1

So:

$$\text{the derivative of } \frac{\cos(x)}{x} = \frac{\text{Low dHigh minus High dLow}}{\text{over the line and square the Low}}$$

$$= \frac{x(-\sin(x)) - \cos(x)(1)}{x^2}$$

$$= -\frac{x\sin(x) + \cos(x)}{x^2}$$

Reciprocal Rule

Example: What is $\frac{d}{dx}$(1/x) ?

The Reciprocal Rule says:

$$\text{the derivative of } \frac{1}{f} = \frac{-f'}{f^2}$$

With f(x)= x, we know that f'(x) = 1

So:

$$\text{the derivative of } \frac{1}{x} = \frac{-1}{x^2}$$

Which is the same result we got above using the Power Rule.

Chain Rule

Example: What is $\frac{d}{dx}\sin(x^2)$?

$\sin(x^2)$ is made up of **sin()** and **x^2**:

- $f(g) = \sin(g)$
- $g(x) = x^2$

The Chain Rule says:

$$\text{the derivative of } f(g(x)) = f'(g(x))g'(x)$$

The individual derivatives are:

- $f'(g) = \cos(g)$
- $g'(x) = 2x$

So:

$$\frac{d}{dx}\sin(x^2) = \cos(g(x))\ (2x)$$

$$= 2x \cos(x^2)$$

Another way of writing the Chain Rule is: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

Let's do the previous example again using that formula:

Example: What is $\frac{d}{dx}\sin(x^2)$?

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Have $u = x^2$, so $y = \sin(u)$:

$$\frac{d}{dx}\sin(x^2) = \frac{d}{du}\sin(u)\frac{d}{dx}x^2$$

Differentiate each:

$$\frac{d}{dx}\sin(x^2) = \cos(u)\ (2x)$$

Substitue back $u = x^2$ and simplify:

$$\frac{d}{dx}\sin(x^2) = 2x\cos(x^2)$$

Same result as before (thank goodness!)

Another couple of examples of the Chain Rule:

Example: What is $\frac{d}{dx}(1/\cos(x))$?

1/cos(x) is made up of **1/g** and **cos()**:

- $f(g) = 1/g$
- $g(x) = \cos(x)$

The Chain Rule says:

$$\text{the derivative of } f(g(x)) = f'(g(x))g'(x)$$

The individual derivatives are:

- $f'(g) = -1/(g^2)$
- $g'(x) = -\sin(x)$

So:

$$\mathbf{(1/\cos(x))' = -1/(g(x))^2 \times -\sin(x)}$$

$$\mathbf{= \sin(x)/\cos^2(x)}$$

Note: $\mathbf{\sin(x)/\cos^2(x)}$ is also $\mathbf{\tan(x)/\cos(x)}$, or many other forms.

Example: What is $\frac{d}{dx}(5x-2)^3$?

The Chain Rule says:

the derivative of $f(g(x)) = f'(g(x))g'(x)$

$(5x\text{-}2)^3$ is made up of **g^3** and **5x-2**:

- $f(g) = g^3$
- $g(x) = 5x-2$

The individual derivatives are:

- $f'(g) = 3g^2$ (by the Power Rule)
- $g'(x) = 5$

So:

$$\frac{d}{dx}(5x-2)^3 = 3g(x)^2 \times 5 = 15(5x-2)^2$$

Question 1 Question 2 Question 3 Question 4 Question 5
Question 6 Question 7 Question 8 Question 9 Question 10
Question 11 Question 12 Question 13