ETC3250: Classification

Semester 1, 2019

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Econometrics and Business Statistics
Monash University

Week 3 (a)

Outline

- Classification
  - Categorical response

In classification, the output Y is a categorical variable. For example,

- Loan approval: $Y \in \{successful, unsuccessful\}$
- Type of business culture: $Y \in \{clan, adhocracy, market, hierarchical\}$
- Historical document author: $Y \in \{Austen, Dickens, Imitator\}$
- Email: $Y \in \{spam, ham\}$

Map the categories to a numeric variable, or possibly a binary matrix.

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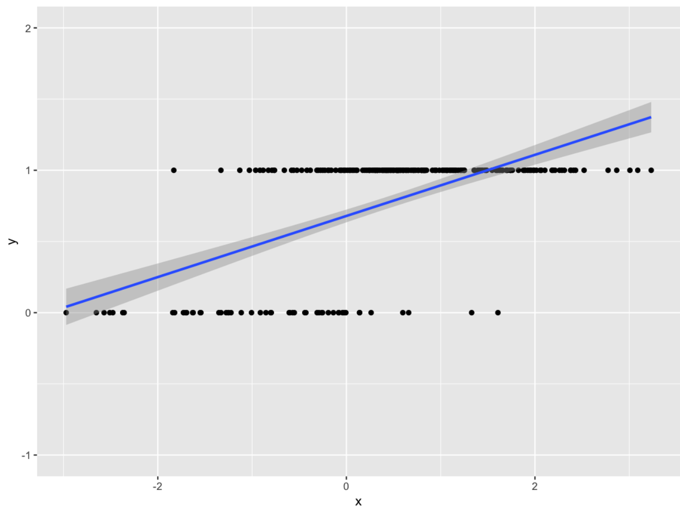
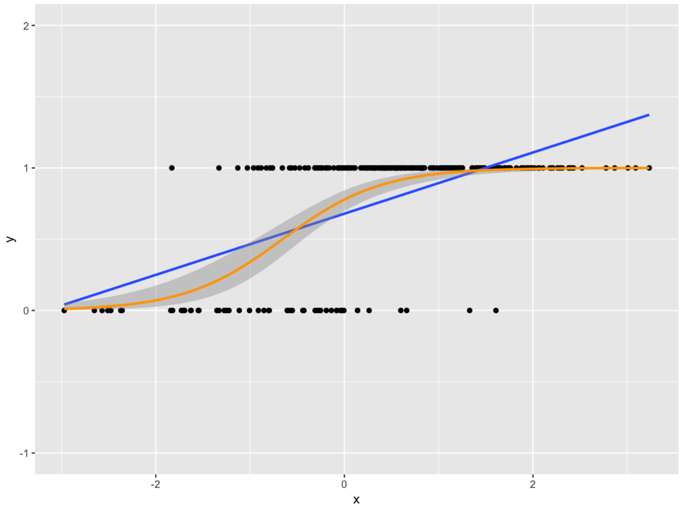
Outline

- Classification
  - Categorical response
  - Why not linear reg?



Outline

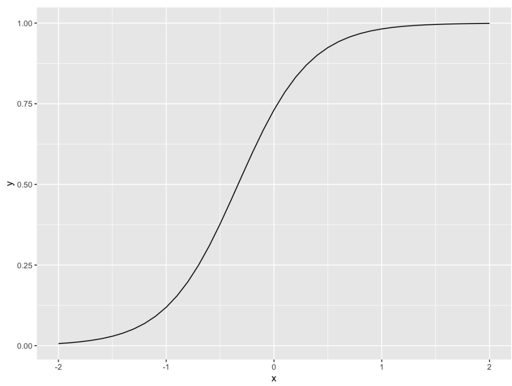
- Classification
- Logistic regression



Outline

- Classification
- Logistic regression
  - Logistic function

$$f(x) = \frac{e^{\beta_0+\beta_1 x}}{1+e^{\beta_0+\beta_1 x}}$$



Transform the function:

$$f(x) = \frac{e^{\beta_0+\beta_1 x}}{1+e^{\beta_0+\beta_1 x}}$$

$\rightarrow f(x) = \frac{1}{1/e^{\beta_0+\beta_1 x}+1}$

$\rightarrow 1/f(x) = 1/e^{\beta_0+\beta_1 x} + 1$

$\rightarrow 1/f(x) - 1 = 1/e^{\beta_0+\beta_1 x}$

$\rightarrow \frac{1}{1/f(x)-1} = e^{\beta_0+\beta_1 x}$

$\rightarrow \ldots$

$\rightarrow \ldots$

$\rightarrow \frac{f(x)}{1-f(x)} = e^{\beta_0+\beta_1 x}$

$\rightarrow \log_e \frac{f(x)}{1-f(x)} = \beta_0 + \beta_1 x$

The left-hand side $\log_e \frac{f(x)}{1-f(x)}$ is called the log-odds ratio or logit.

The fitted model is then written as:

$\log_e \frac{P(Y=1|X)}{1-P(Y=1|X)} = \beta_0 + \beta_1 X$

and then

$$P(Y=0|X) = 1 - P(Y=1|X)$$

Multiple categories: This formula can be extended to more than binary response variables. Writing the equation is not simple, but follows from the above, extending it to provide probabilities for each level/category. The sum of all probabilities is 1.

Outline

- Classification
- Logistic regression
  - Logistic function
  - Logistic model - GLM
  - Interpretation

- Linear regression
  - β_1 gives the average change in Y associated with a one-unit increase in X
- Logistic regression
  - Increasing X by one unit changes the log odds by β_1, or equivalently it multiplies the odds by e^{β_1}
  - However, because the model is not linear in X, β_1 does not correspond to the change in response associated with a one-unit increase in X

Outline

- Classification
- Logistic regression
  - Logistic function
  - Logistic model - GLM
  - Interpretation
  - Estimation

Maximum Likelihood Estimation

Given the logistic $p(x_i) = \frac{1}{e^{-(\beta_0+\beta_1 x)}+1}$

We choose parameters β_0, β_1 to maximize the likelihood of the data given the model. The likelihood function is

$$l_n(\beta_0, \beta_1) = \prod_{y_i=1,i}^{n} p(x_i) \prod_{y_i=0,i}^{n} (1 - p(x_i)).$$

It is more convenient to maximize the *log-likelihood*:

$$\max_{\beta_0,\beta_1} l_n(\beta_0, \beta_1) = -\sum_{i=1}^{n} \log(1 + e^{-(\beta_0+\beta_1 x)})$$

Outline

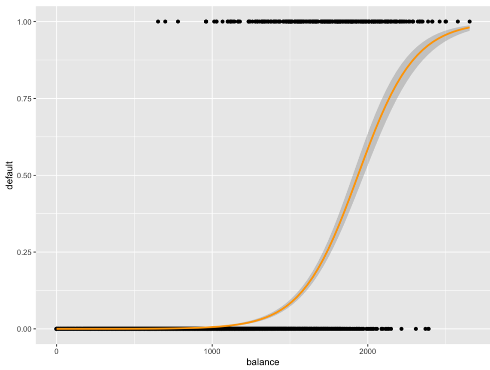
With estimates from the model fit, $\hat{\beta}_0, \hat{\beta}_1$, predict the response using:

$$\hat{p}(x) = \frac{e^{\hat{\beta}_0+\hat{\beta}_1 x}}{1+e^{\hat{\beta}_0+\hat{\beta}_1 x}}$$

It is a proportion, but can be rounded to 0 or 1 for class prediction.

Outline

Simulated data to predict which customers will default on their credit card debt.



Outline


```
library(broom)
fit <- glm(default~balance, data=simcredit, family="binomial")
tidy(fit)
```

```
## # A tibble: 2 x 5
##   term        estimate std.error statistic   p.value
##   <chr>          <dbl>     <dbl>     <dbl>     <dbl>
## 1 (Intercept) -10.7      0.361       -29.5 3.62e-191
## 2 balance       0.00550  0.000220     25.0 1.98e-137
```

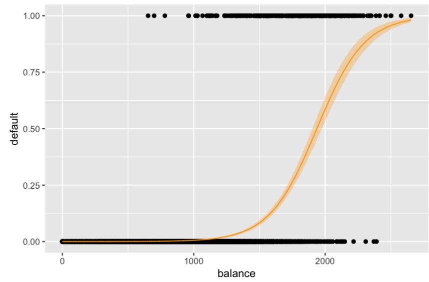

Outline


```
glance(fit)
```

```
## # A tibble: 1 x 7
##   null.deviance df.null logLik   AIC   BIC deviance df.residual
##           <dbl>   <int>  <dbl> <dbl> <dbl>    <dbl>       <int>
## 1         2921.    9999  -798. 1600. 1615.    1596.        9998
```

Outline


```
simcredit_fit <- augment(fit, simcredit, type.predict="response")
ggplot(simcredit_fit, aes(x=balance, y=default_bin)) +
  geom_point() +
  geom_ribbon(aes(ymin=.fitted-2*.se.fit, ymax=.fitted+2*.se.fit),
              fill = "orange", alpha=0.3) +
  geom_line(aes(y=.fitted), colour = "orange") +
  ylab("default")
```



Outline

Logistic regression involves directly modeling $Pr(Y = k|X = x)$ using the logistic function. Rounding the probabilities produces class predictions, in two class problems; selecting the class with the highest probability produces class predictions in multi-class problems.

Another approach for building a classification model is linear discriminant analysis. This involves directly estimating the distribution of the predictors, separately for each class.

Outline

- Classification
- Logistic regression
- Linear discriminant analysis
  - Bayes theorem

Let $f_k(x)$ be the density function for predictor x for class k. If f is small, the probability that x belongs to class k is small, and conversely if f is large.

Bayes theorem (for K classes) states:

$$Pr(Y = k | X = x) = p_k(x) = \frac{\pi_k f_k(x)}{\sum_{i=1}^{K} \pi_k f_k(x)}$$

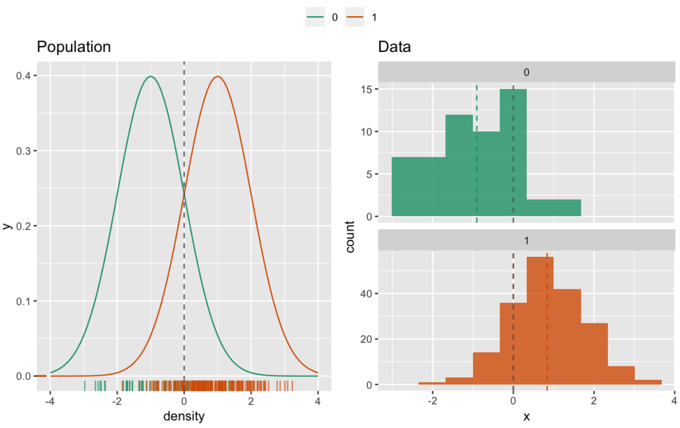
where

$$\pi_k = \Pr(Y = k)$$

is the prior probability that the observation comes from class, *k*.

Outline

- Classification
- Logistic regression
- Linear discriminant analysis
  - Bayes theorem
  - When $p = 1$



Outline

- Classification
- Logistic regression
- Linear discriminant analysis
  - Bayes theorem
  - When $p = 1$

We assume $f_k(x)$ is Normal or Gaussian:

$$f_k(x) = \frac{1}{\sqrt{2\pi}\sigma_k} \exp\left(-\frac{1}{2\sigma_k^2}(x-\mu_k)^2\right)$$

where μ_k and σ_k^2 are the mean and variance parameters for the kth class. Further assume that $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_K^2$; then the conditional probabilities are

$$p_k(x) = \frac{\pi_k \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(x-\mu_k)^2\right)}{\sum_{l=1}^{K} \pi_l \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(x-\mu_l)^2\right)}$$

Outline

- Classification
- Logistic regression
- Linear discriminant analysis
  - Bayes theorem
  - When $p = 1$

The Bayes classifier is assign new observation $X = x_0$ to the class with the highest $p_k(x_0)$. A simplification of $p_k(x_0)$ yields the discriminant functions:

$$\delta_k(x_0) = x_0 \frac{\mu_k}{\sigma^2} - \frac{\mu_k^2}{2\sigma^2} + log(\pi_k)$$

and the rule Bayes classifier will assign x_0 to the class with the largest value.

Outline

- Classification
- Logistic regression
- Linear discriminant analysis
  - Bayes theorem
  - When $p = 1$

When $K = 2$ and $\pi_1 = \pi_2$, then the Bayes classifier is :

Assign x_0 to class 1, if

$$\delta_1(x_0) > \delta_2(x_0)$$

$$x_0 \frac{\mu_1}{\sigma^2} - \frac{\mu_1^2}{2\sigma^2} + log(\pi) > x_0 \frac{\mu_2}{\sigma^2} - \frac{\mu_2^2}{2\sigma^2} + log(\pi)$$

which simplifies to

$$x_0 > \frac{\mu_1 + \mu_2}{2}$$

This is estimated on the data with $x_0 > \frac{\bar{x}_1 + \bar{x}_2}{2}$.

Outline

- Classification
- Logistic regression
- Linear discriminant analysis
  - Bayes theorem
  - When $p = 1$
  - Multivariate

To indicate that a p-dimensional random variable X has a multivariate Gaussian distribution with $E[X] = \mu$ and $\mathrm{Cov}(X) = \Sigma$, we write $X \sim N(\mu, \Sigma)$.

The multivariate normal density function is:

$$f(x) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp\{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\}$$

with x, μ are p-dimensional vectors, Σ is a $p \times p$ variance-covariance matrix.

Outline

The discriminant functions are:

$$\delta_k(x) = x'\Sigma^{-1}\mu_k - \frac{1}{2}\mu_k'\Sigma^{-1}\mu_k + \pi_k$$

and Bayes classifier is assign a new observation x_0 to the class with the highest $\delta_k(x_0)$.

When $K = 2$ and $\pi_1 = \pi_2$ this reduces to

Assign observation x_0 to class 1 if

$$x_0'\Sigma^{-1}(\mu_1 - \mu_2) > \frac{1}{2}(\mu_1 + \mu_2)'\Sigma^{-1}(\mu_1 - \mu_2)$$

Outline

Discriminant space: a benefit of LDA is that it provides a low-dimensional projection of the p-dimensional space, where the groups are the most separated. For $K = 2$, this is

$$\Sigma^{-1}(\mu_1 - \mu_2)$$

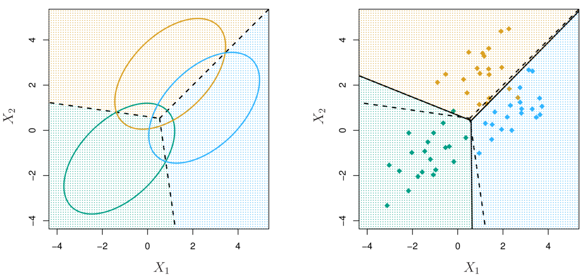
For $K > 2$, the discriminant space is found be taking an eigen-decomposition of $\Sigma^{-1}\Sigma_B$, where

$$\Sigma_B = \frac{1}{K}\sum_{i=1}^{K}(\mu_i - \mu)(\mu_i - \mu)'$$

Outline

- Classification
- Logistic regression
- Linear discriminant analysis
  - Bayes theorem
  - When $p=1$
  - Multivariate

The dashed lines are the Bayes decision boundaries. Ellipses that contain 95% of the probability for each of the three classes are shown. Solid line corresponds to the class boundaries from the LDA model fit to the sample.



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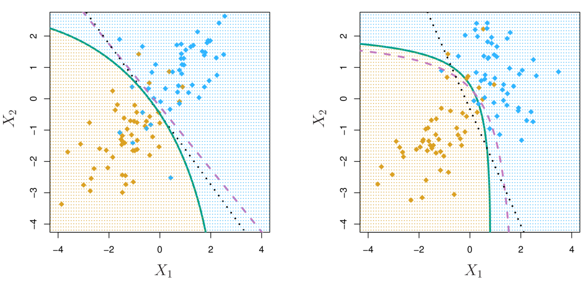
Outline

- Classification
- Logistic regression
- Linear discriminant analysis
  - Bayes theorem
  - When $p=1$
  - Multivariate
  - Quadratic

A quadratic boundary is obtained by relaxing the assumption of equal variance-covariance, and assume that $\Sigma_k \neq \Sigma_l,\ \ k \neq l, k, l = 1, \ldots, K$



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Slides at https://monba.dicook.org.

Code and data at https://github.com/dicook/Business_Analytics.

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