MONASH University

ETC2420

Statistical methods in Insurance

Week 10.
Bayesian Reasoning: Introduction

22 September 2016

Outline

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2	Hypothesis Testing II & Decision Theory	Souhaib
3	Statistical Distributions	Di
4	Model fitting & Linear regression	Di
5	Linear models	Di
6	Bootstrap, Permutation and Linear models	Di
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8	Compiling data for problem solving	Di
9	Bayesian Reasoning I & II	Souhaib
10	Bayesian Reasoning III & Time series models I	Souhaib
10	Time series models II & III	Souhaib
11	Project presentation	Souhaib

References

- Berger, J. O. 2013. **Statistical Decision Theory and Bayesian Analysis**. Springer Series in Statistics. Springer New York.
- Blitzstein, Joseph K., and Jessica Hwang. 2014. **Introduction to Probability** (Chapman & Hall/CRC Texts in Statistical Science). Chapman and Hall/CRC.
- Wasserman, Larry. 2004. **All of Statistics: A Concise Course in Statistical Inference** (Springer Texts in Statistics). 1st Corrected ed. 20 edition. Springer.

Frequentist philosopy

- ”Probability refers to **limiting relative frequencies**. Probabilities are objective properties of the real world.”
- ”Parameters are **fixed, unknown constants**. Because they are not fluctuating, no useful probability statements can be made about parameters.”
- ”Statistical procedures should be designed to have **well-defined long run frequency properties**. For example, a 95 percent confidence interval should trap the value of the parameter with limiting frequency at least 95 percent.”

Bayesian philosopy

- Probability describes **degree of belief**, not limiting frequency. As such, we can make probability statements about anything that is subject to random variation. For example, *the probability that it will rain on Sunday is* .30. This does not refer to any limiting frequency. It reflects my strength of belief that the proposition is true.
- We can make **probability statements about parameters**, even though they are fixed constants.
- We make inferences about a parameter by producing **a probability distribution for the parameter**. Inferences, such as point estimates and interval estimates, may then be extracted from this distribution.

Conditional probability

If A and B are events with $P(B) > 0$, then the conditional probability of A given B is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

For any events A and B with positive probabilities

$$P(A \cap B) = P(B)P(A|B) = P(A)P(B|A)$$

Bayes' rule

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

(or Bayes' theorem)

Example

A patient name Bob is tested for a disease that **afflicts 1% of the population**. The test result is **positive**, i.e., the test claims that the Bob has the disease. Let D be the event that Bob has the disease and T be the event that he tests positive. Suppose that the test is 95% accurate (we suppose $P(T|D) = 0.95$ and $P(T|D^c) = 0.95$). What is $P(D|T)$?

$$P(D) = 0.01 \rightarrow P(D|T) = ?$$

1. P(D|T) = 0.02
2. P(D|T) = 0.16
3. P(D|T) = 0.99

Example

$$P(D|T) = \frac{P(T|D)P(D)}{P(T)}$$
$$= \frac{0.95 \cdot 0.01}{P(T)}$$

(Law of total probability) Let $A_1, \ldots, A_n$ be a partition of the sample space S (i.e. the A_i are disjoint events and their union is S), with $P(A_i) > 0$ for all i. Then

$$P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i)$$

Example

$$P(D|T) = \frac{P(T|D)P(D)}{P(T)} = \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D^c)P(D^c)}$$
$$= \frac{0.95 \cdot 0.01}{0.95 \cdot 0.01 + 0.05 \cdot 0.99}$$
$$\approx 0.16$$

Why such a small chance to have the disease given that he tested positive, and the test is reliable ($P(T|D) = 0.95$)?

Example

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Why such a small chance to have the disease given that he tested positive, and the test is reliable ($P(T|D) = 0.95$)?

- Two factors: the **evidence** from the test and the **prior information** about the disease.
- Although the test provides evidence in favour of disease, the disease is rare.
- The conditional probability reflects a balance between these two factors

The Bayesian method

We are interested in the unknown parameter θ.

- We choose a probability density $\pi(\theta)$, called the **prior distribution**, that expresses our beliefs about the parameter θ before we see any data.
- We choose a probability distribution $f(x|\theta)$ that reflects our beliefs about the random variable X given θ.
- After observing data point x, we update our beliefs and calculate the **posterior** distribution $\pi(\theta|x)$:

$$\pi(\theta|x) = \frac{f(x|\theta)\pi(\theta)}{f(x)} \propto f(x|\theta)\pi(\theta)$$

where

$$f(x) = \int_{\Theta} f(x|\theta)\pi(\theta)d\theta$$

Comparison with MLE

Suppose we use an "uninformative" prior ($\pi(\theta) = 1$), then

$$\begin{aligned}
\pi(\theta|x) &= \frac{f(x|\theta)\pi(\theta)}{f(x)} \\
&= f(x|\theta)k(x) \\
&\propto f(x|\theta) = f(x;\theta) \\
\mathcal{L}(\theta|x) &= f(x;\theta)
\end{aligned}$$

$$\implies (MLE)\,\text{maximize}_\theta\,\mathcal{L}(\theta|x) \equiv (MAP)\,\text{maximize}_\theta\,\pi(\theta|x)$$

Note:

- Any prior includes information, including priors that state that no information is known or that do not favour some values over others.
- $\hat{\theta}_{\text{MLE}} \neq E[\theta|x]$

Example

Consider the random variable $X \in \{H, T\}$ with $X \sim$ Bernouilli(p) and p is the probability a coin will turn up heads. Compare the Bayes estimate and the MLE for the following steps.

1. Before tossing the coin, what is p?
   - Bayes estimate: if $\pi(p) = 1$, then $\hat{p} = E[p] = \frac{1}{2}$
   - MLE: $0/0$
2. We toss the coin one time, and we see heads. What is p?
   - Bayes estimate: if $\pi(p) = 1$, then $\hat{p} = E[p|x] = \frac{2}{3}$
   - MLE: $\hat{p} = 1/1$

The Bayesian method

If we have n i.i.d. observations $x_1, \ldots, x_n$, we can replace $f(x|\theta)$ with

$$f(x_1, \ldots, x_n|\theta) = \Pi_{i=1}^{n} f(x_i|\theta) = \mathcal{L}_n(\theta)$$

and compute

$$\pi(\theta|x_1, \ldots, x_n) = \frac{\mathcal{L}_n(\theta)\pi(\theta)}{f(x_1, \ldots, x_n)}$$

where

$$f(x_1, \ldots, x_n) = \int_\Theta \mathcal{L}_n(\theta)\pi(\theta)d\theta = c_n$$

$$\pi(\theta|x_1, \ldots, x_n) = \frac{\mathcal{L}_n(\theta)\pi(\theta)}{c_n} \propto \mathcal{L}_n(\theta)\pi(\theta)$$

Since c_n does not depend on θ, Posterior is proportional to Likelihood times Prior.

Comparison with MLE

Let $X_1, \dots, X_n$ be i.i.d. with PDF $f(x;\theta)$. The likelihood function is defined by

$$\mathcal{L}_n(\theta) = \Pi_{i=1}^{n} f(X_i;\theta).$$

The log-likelihood is defined by $l_n(\theta) = log\mathcal{L}_n(\theta)$.

The likelihood is the joint density of the data, *treated as a function of the parameter* θ. Thus, $\mathcal{L}_n : \Theta \to [0, \infty]$. $\mathcal{L}_n$ is not a density function: in general, $\int_\Theta \mathcal{L}_n(\theta) \neq 1$.

The maximum likelihood estimator (MLE) is defined by

$$\hat{\theta}_n = \operatorname{argmax}_\theta \mathcal{L}_n(\theta).$$

Example I with the Bayesian method

Suppose that $X_1, \ldots, X_n \sim$ Bernouilli(p). We want to estimate p.

- $\pi(p) = 1; 0 \leq p \leq 1$ (uniform prior distribution)
- We observe $x_1, \ldots, x_n$. Compute $\pi(p|x_1, \ldots, x_n)$.

If $s = \sum_{i=1}^{n} x_i$ is the number of successes, then

$$\pi(p|x_1, \ldots, x_n) = \frac{\mathcal{L}_n(p)\pi(p)}{c_n} = \frac{p^s(1-p)^{n-s} \cdot 1}{c_n},$$

where

$$c_n = \int_0^1 p^s(1-p)^{n-s} dp = \frac{\Gamma(s+1)\Gamma(n-s+1)}{\Gamma(n+2)}$$

and $\Gamma(n) = (n-1)!$

Example I with the Bayesian method

The pdf of the **beta distribution**, for $0 \leq x \leq 1$, is given by

$$f(x;\alpha;\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

with shape parameters $\alpha, \beta > 0$.

- We can see that $p|x_1, \ldots, x_n \sim \text{Beta}(s+\alpha, n-s+\beta)$ with $\alpha = \beta = 1$.
- It is also possible to consider different values for α and β which will lead to different prior assumptions.
- From the posterior, we can compute multiple posterior quantities (analytically or via simulation): mean, standard deviation, credible intervals, etc.

Example I with MLE

Suppose that $X_1, \ldots, X_n \sim$ Bernouilli(p). We want to compute the Maximum Likelihood Estimate (MLE) of p. The likelihood function is defined by

$$\mathcal{L}_n(p) = \Pi_{i=1}^{n} p^{X_i}(1-p)^{1-X_i} = p^S(1-p)^{n-S}$$

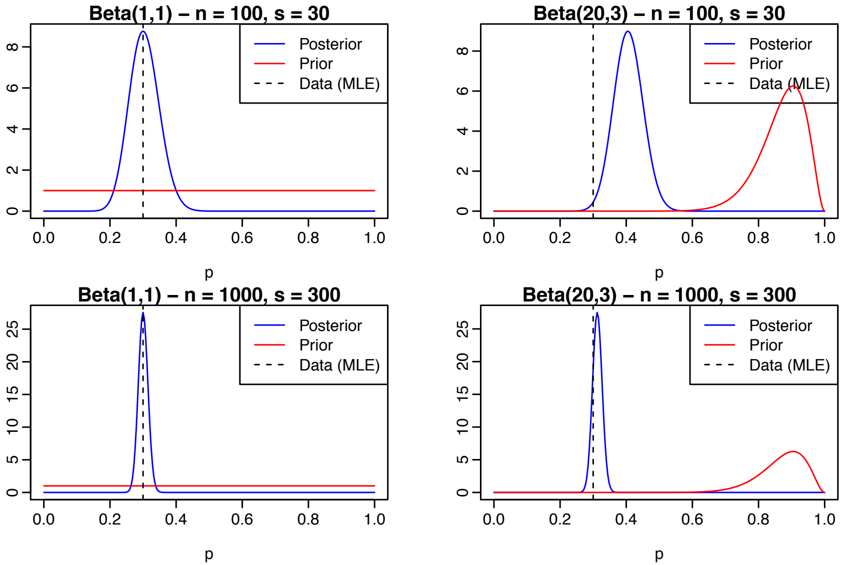
where $S = \sum_i X_i$.
The log-likelihood is given by

$$l_n(p) = S log(p) + (n-S) log(1-p)$$

The MLE estimate is then given by

$$\hat{p}_n = \text{argmax}_p \, l_n(p) = \frac{S}{n}$$

Example I: Bayesian method vs MLE



Example II

Suppose that $X_1, \ldots, X_n \sim N(\theta, \sigma_0^2)$ where σ_0^2 is known; and $\pi(\theta) \sim N(\mu, \tau^2)$. Determine $\pi(\theta \mid x_1, \ldots, x_n)$.

Since the data is *i.i.d.* the likelihood is

$$
\begin{aligned}
& f(x_1, \ldots, x_n \mid \theta) \\
& \propto \prod_{i=1}^{n} \exp\left\{-\frac{1}{2}\left(\frac{x_i - \theta}{\sigma_0}\right)^2\right\} \\
& \propto \exp\left\{-\frac{1}{2\sigma_0^2}\left(\sum_{i=1}^{n}(x_i - \bar{x})^2 + n(\bar{x} - \theta)^2\right)\right\} \\
& \propto \exp\left\{-\frac{n}{2\sigma_0^2}(\bar{x} - \theta)^2\right\} \exp\left\{-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n}(x_i - \bar{x})^2\right\} \\
& \propto \exp\left\{-\frac{n}{2}\left(\frac{\bar{x} - \theta}{\sigma_0}\right)^2\right\}
\end{aligned}
$$

Example II

Prior $\pi(\theta) \sim N(\mu, \tau^2) \implies \pi(\theta) \propto \exp\left\{-\frac{1}{2}\left(\frac{\theta-\mu}{\tau}\right)^2\right\}$

Expanding the square and omitting terms independent of θ, we obtain

$$\pi(\theta \mid x_1, \ldots, x_n) \propto \exp\left\{-\frac{\theta^2}{2}\left(\frac{n}{\sigma_0^2} + \frac{1}{\tau^2}\right) + \theta\left(\frac{n\bar{x}}{\sigma_0^2} + \frac{\mu}{\tau^2}\right)\right\}$$

which can be rewritten as

$$\pi(\theta \mid x_1, \ldots, x_n) \propto \exp\left\{-\frac{1}{2}\left(\frac{n}{\sigma_0^2} + \frac{1}{\tau^2}\right)^{-1}\left(\theta - \frac{n\bar{x}/\sigma_0^2 + \mu/\tau^2}{n/\sigma_0^2 + 1/\tau^2}\right)^2\right\}$$

Example II

We recognize the formula for a normal distribution (without the scaling constant):

$$\pi\left(\theta \mid x_1, \ldots, x_n\right) \propto \exp\left\{-\frac{1}{2\bar{\sigma}^2}\left(\theta - \bar{\mu}\right)^2\right\}$$

where $\bar{\mu} = \frac{n\bar{x}/\sigma_0^2 + \mu/\tau^2}{n/\sigma_0^2 + 1/\tau^2}$ and $\bar{\sigma}^2 = \frac{1}{n/\sigma_0^2 + 1/\tau^2}$.
Thus we conclude that

$$\pi(\theta \mid x_1 \ldots x_n) = \phi(\bar{\mu}, \bar{\sigma}^2)$$

where $\bar{\mu}$ and $\bar{\sigma}^2$ are the **posterior mean** and **variance** respectively.

Example II

Reconsider the posterior mean and variance as n (the sample size) and τ (the prior standard deviation) become *large*. Rewriting $\bar{\mu}$ and $\bar{\sigma}^2$ as

$$\bar{\mu} = \frac{\bar{x}\tau^2 + \mu\sigma_0^2/n}{\tau^2 + \sigma_0^2/n} = \frac{\bar{x} + \mu\frac{\sigma_0^2/n}{\tau^2}}{1 + \frac{\sigma_0^2/n}{\tau^2}} \qquad \text{and} \qquad \bar{\sigma}^2 = \frac{\sigma_0^2/\,n}{1 + \frac{\sigma_0^2/n}{\tau^2}}$$

we observe:

1. As $n \to \infty$ (fixing τ and σ_0^2): $\bar{\mu} \to \bar{x}$ and $\bar{\sigma}^2 \to \sigma_0^2/n \to 0$
2. As $\tau \to \infty$ (fixing n and σ_0^2): $\bar{\mu} \to \bar{x}$ and $\bar{\sigma}^2 \to \sigma_0^2/n \to 0$

Summary

In the previous examples , we had

$$X_1, \ldots, X_n \overset{i.i.d}{\sim} \text{Bernouilli}(p)$$

$$\hat{p}_{MLE} = \frac{s}{n}$$

$$p|x_1, \ldots, x_n \sim \text{Beta}(s + \alpha, n - s + \beta) = \frac{\mathcal{L}_n(p) \times \text{Beta}(\alpha, \beta)}{c_n}$$

and

$$X_1, \ldots, X_n \overset{i.i.d}{\sim} N(\theta, \sigma_0^2)$$

$$\hat{\theta}_{MLE} = \bar{x}$$

$$\theta \mid x_1 \ldots x_n \sim N(\bar{\mu}, \bar{\sigma}^2) = \frac{\mathcal{L}_n(\theta) \times N(\mu, \tau^2)}{c_n}$$

Important: we are making inference on the **parameter** of a distribution

Conjugate priors

Did you see anything special about the posterior distributions?

- When the posterior distributions are in the same distribution family as the prior distribution, they are called **conjugate distributions**, and the prior is called a **conjugate prior** for the likelihood function
- Conjugate priors are convenient since we have a closed-form expression for the posterior (we avoid numerical integration).
- In practice, we have conjugacy only for very simple models. In most cases, the posterior distribution has to be found **numerically**.