MONASH University

ETC2420

Statistical methods in Insurance

Week 2.
Introduction to Decision Theory

29 July 2016

Decision Theory

- Decision Theory is concerned with the mathematical analysis of decision making when the **state of the world is uncertain** but information can be obtained about it by means of observation or experimentation.
- Some **action must be chosen** from a well defined set of alternatives, but the exact circumstances in which the action must be taken are unknown.
- Different actions implies **different consequences** and therefore have different merit according to the **decision maker's preference**.

Decision Theory

- Assuming that **some numerical value** can be assigned to the different combinations of circumstances and actions provides a basis for assessing **how reasonable a particular action is in different situations**.
- It may be possible to obtain **data** that will yield information about the prevailing circumstances, or prior information concerning the **frequency with which different circumstances arise**.
- The aim of **decision theory** is to provide a means of **exploiting such information** to determine a **reasonable (optimal?) course of action**.

Example 1

	SUN	RAIN	SNOW
a_1	\$49	\$25	\$25
a_2	\$36	\$100	\$0
a_3	\$81	\$0	\$0

a_1? a_2? a_3?

Example 1

	SUN	RAIN	SNOW
a_1	\$49	\$25	\$25
a_2	\$36	\$100	\$0
a_3	\$81	\$0	\$0

a_1? a_2? a_3?

	SUN (1/2)	RAIN (1/4)	SNOW (1/4)
a_1	\$49	\$25	\$25
a_2	\$36	\$100	\$0
a_3	\$81	\$0	\$0

a_1? a_2? a_3?

Decision under ignorance/risk

- Decision under **ignorance**: probability of the possible outcomes **unknown or do not exist**
- Decision under **risk**: probability of the possible outcomes **known**
- Decision under **uncertainty**: synonym for ignorance, or as a broader term referring to both risk and ignorance

Example 2

Zero-sum two-person games:

- **Player A** has strategies labelled I, II, III, ...
- **Player B** has strategies labeled 1, 2, 3, ...
- The **payoff** is the amount of 'money' each player receive after choosing their respective strategies
- Whatever one **player loses**, the **other player wins**.
- Each player must choose his own strategy **without knowing** what his opponent is going to do.
- The objective is to determine **optimal strategies**.

		Player A	
		I	*II*
Player B	1	7, -7	-4, 4
	2	8, -8	10, -10

		Player A	
		I	*II*
Player B	1	7	-4
	2	8	10

A taxonomy of games

- **Zero-sum** versus nonzero-sum games
- Non-cooperative versus cooperative games
- Simultaneous-move versus sequential-move games
- Games with perfect information versus games with imperfect information
- Non-symmetric versus symmetric games
- **Two-person** versus *n*-person games
- Non-iterated versus iterated games

Zero-sum two-person games: dominance

Consider the following **payoff matrix** (losses to A, gains to B):

		Player A	
		I	*II*
Player B	1	7	-4
	2	8	10

Player B: For Player B, Strategy 1 is *never* better than Strategy 2 *regardless* of what Player A does.

$\implies$ Strategy 1 is **dominated** by Strategy 2: discard Strategy 1

Player A: A's optimum strategy is now obviously *I*, since a loss of 8 is preferable to a loss of 10.

$\implies$ the value of the game is 8.

Zero-sum two-person games: dominance

- Discarding dominated strategies can help, but doesn’t generally lead to a complete solution.
- Also, dominant strategies may not even exist.
- Another example:

	Good chef	Bad chef
Monkfish	good monkfish	terrible monkfish
Hamburger	edible hamburger	edible hamburger
No main course	hungry	hungry

- so must consider other approaches...

Zero-sum two-person games: minimax

Consider the following payoff matrix (**losses** to A, gains to B):

		Player A		
		I	*II*	*III*
	1	−1	6	−2
Player B	2	2	4	6
	3	−2	−6	12

Player A:

- The worst that can happen if A chooses Strategy *I* is a loss of 2.
- Worst outcome for Strategy *II* is a loss of 6 and for Strategy *III* a loss of 12.
- Thus Player A could *minimise his maximum loss* by choosing Strategy *I*.

Zero-sum two-person games: minimax

Now consider the same table from B's POV (**losses** to A, gains to B):

		Player A		
		I	*II*	*III*
	1	-1	6	-2
Player B	2	2	4	6
	3	-2	-6	12

Player B:

- *minimizing maximum loss* $\equiv$ *maximising minimum gain*
- B's maximum loss is minimised for Strategy 2 $\implies$ *Value* of the game is 2.
- Even if A knew B would choose Strategy 2, Strategy I would still be A's optimal choice (and vice-versa). This is called a *saddle point* or *equilibrium*.

Zero-sum two-person games: minimax

This is not always the case, as is shown by the following payoff table:

		Player A	
		I	*II*
Player B	1	8	-5
	2	2	6

- Player A's minimax strategy is II and Player B's is Strategy 2.
- But if Player A knew that **Player B was going to choose Strategy 2**, Player A could switch to Strategy I and reduce the value of the game from 6 to 2.
- And if Player B knew that **Player A would act this way** Player B could in turn switch to Strategy 1 and increase the value of the game from 6 to 8.

In other words, knowledge of the other player's choice of strategy is advantageous in this case

- Consistently choosing the *same* strategy cannot be optimal
- This suggests that each player should mix up (*randomize*) their behaviour patterns
- ie, introduce a stochastic element into their choice of strategy.

		Player A	
		I	*II*
Player B	1	8	−5
	2	2	6

Suppose Player A employs a random device which leads to the selection of Strategy I with a probability p and Strategy II with a probability $1 - p$ (a Bernoulli trial).

If **Player B chooses Strategy 1**, Player A's expected loss is

$$L_1(p) = 8p - 5(1 - p)\,,$$

and if **Player B chooses Strategy 2** Player A can expect to lose

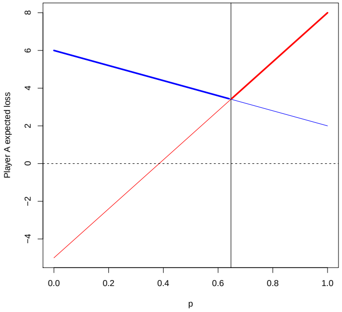
$$L_2(p) = 2p + 6(1 - p)\,.$$

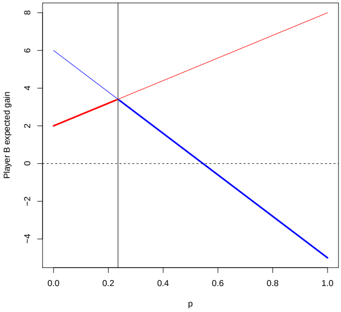
B chooses Strategy 1 or 2:

$$L_1(p) = 8p - 5(1-p)$$
$$L_2(p) = 2p + 6(1-p)$$

A chooses Strategy I or II:

$$L_I(p) = 8p + 2(1-p)$$
$$L_{II}(p) = -5p + 6(1-p)$$





Three cases: $p < 11/17$, $p = 11/17$ and $p > 11/17$.

Zero-sum two-person games: randomized strategies

- So if Player A applies the minimax criterion they should **mix** their two strategies randomly in the proportions **11:6**
- this will hold A's expected loss down to $58/17$.
- Similarly, for Player B, the expected minimum gain is maximised by choosing Strategies 1 and 2 randomly in the proportions **4:13**, giving the expected gain of $58/17$.
- Such strategies are called **mixed** or **randomized**
- The original strategies are referred to as **pure** strategies.

Decision theory model

1. There is a well defined set of possible **actions**, a, that constitutes an ***action space*** A.
2. The **state of the world**, or state of nature, is represented by a parameter θ. The set of possible states of nature, the **state space** (or *parameter space*) Θ, is known.
3. There is a **loss function** $\ell(a, \theta)$ defined on the space of consequences $A \times \Theta$ which assigns a value to the loss incurred if action a is taken when the prevailing state of nature is θ.
4. **Data** x from a random experiment with ***sample space*** Ω is available that provides information on the possible state of nature that prevails.

Examples of decision problems: hypothesis testing, parameter estimation, games, etc.

The Decision Theory Model as a 2-person game

Analogy with a zero sum two person game?

- The **decision maker** (the scientist or statistician, say) and "**nature**" replace the **two players**,
- The **payoff** is replaced by the corresponding **loss** (the loss function is assumed to be given).
- The **data** may be thought of as a form of "**spying**".
- The aim is to select **the best action** with respect to the loss function having regard to the extent and basis of any information that is available concerning the prevailing state of nature.
- *Statistical inference* can be thought of as a game between **the statistician**, who needs to make a decision about the population, and "**nature**", meaning the relevant features of the population of interest.

The Decision Theory Model: Losses vs. Regrets

An alternative basis for assessing actions is **regret** (rather than loss).

- The regret function is defined as

$$r(a,\theta) = \ell(a,\theta) - \min_{a \epsilon A} \ell(a,\theta) \qquad a \in A\,.$$

- $\min_{a \epsilon A} \ell(a,\theta)$ is the smallest loss for that θ, so if we *knew* θ (and took the correct action!) this would be the loss we'd face
- So $r(a,\theta)$ represents the loss that *could* have been avoided had the state of nature been known with certainty.
- Using regrets to assess the merits of different consequences rather than losses **can** lead to a different "optimal" strategy.

Losses to Regrets: Example

Suppose we have two "states of nature" $\Theta = \{\theta_1, \theta_2\}$ and three possible actions $A = \{a_1, a_2, a_3\}$. The losses for all combinations $A \times \Theta$ are:

		Actions a_1	a_2	a_3
States of	θ_1	4	5	2
Nature	θ_2	4	0	5

(Losses).

- The optimal actions are a_3 if θ_1 prevails (with a loss of 2) and a_2 if θ_2 prevails (with a loss of 0).
- Subtracting these minima from the losses for each state yields:

		Actions a_1	a_2	a_3
States of	θ_1	2	3	0
Nature	θ_2	4	0	5

(Regrets).

The No-Data/Data situations

- The No-Data situation
  - There is no data available containing auxiliary information regarding the true state of nature
  - If θ is **known**: minimize $l(a, \theta)$ over a
  - If θ is **unknown**: **minimax** or **Bayes actions**
- Using Data in making decisions (not covered today)
  - We are able to observe the value of a random variable X which we believe depends on θ, and we have $f(x|\theta)$
  - Use $f(x|\theta)$ to compute **frequentist risk**, then apply either the **minimax** or the **Bayes** principle to select an optimal action
  - Use $f(x|\theta)$ to compute **posterior risk** to refine an assumed prior distribution for θ, then compute **Bayes actions**

The Minimax Principle

Consider the following table of losses $\ell(a, \theta)$ plus the maximum (worst-case) loss for each action:

		Actions		
		a_1	a_2	a_3
States of Nature	θ_1	4	5	2
	θ_2	4	0	5
	$\max \ell(a, \theta)$	4	5	5

(Losses).

- If action a_1 is selected the maximum loss is 4, incurred for either state of nature.
- This maximum is smaller than the maximum of 5 encountered for a_2 or a_3, so a_1 is the minimax action.

$$a_M = \arg \min_{a \epsilon A} \max_{\theta \epsilon \Theta} \ell(a, \theta) = \arg \max_{a \epsilon A} \min_{\theta \epsilon \Theta} g(a, \theta)$$

The Minimax Principle: Regrets

		Actions		
		a_1	a_2	a_3
States of	θ_1	2	3	0
Nature	θ_2	4	0	5
	$\max r(a,\theta)$	4	3	5

(Regrets).

$$a_R = \arg\min_{a \epsilon A} \max_{\theta \epsilon \Theta} r(a,\theta)$$

- The minimum of the maximum regrets is 3, achieved for action a_2.
- So a_1 provides the minimax solution to the table of losses, but the minimax *regret* action is a_2.
- $\Longrightarrow$ the minimax principle can lead to different actions depending on whether it is applied to losses or regrets!

The minimax *regret* action is not necessarily the same as the minimax *loss* action, i.e. $a_R \neq a_M$

The Minimax principle: mixed strategies?

- As in the analysis of zero-sum games, the optimal minimax strategy may be a **mixed** strategy.
- We would now have

$$\begin{aligned} L(\boldsymbol{p},\theta_1) &= 4p_1 + 5p_2 + 2p_3 \\ L(\boldsymbol{p},\theta_2) &= 4p_1 + 0p_2 + 5p_3 \end{aligned}$$

  where $\boldsymbol{p} = [p_1, p_2, p_3]'$ and $p_1 + p_2 + p_3 = 1$.
- $$\boldsymbol{p}_M = \arg\min_{\boldsymbol{p}} \max_{\theta \epsilon \Theta} L(\boldsymbol{p},\theta)$$
- Will find that a_1, a_2 and a_3 mixed in the proportions 0:3:5 yields the minimax expected loss strategy.
- Again the optimal mixed strategy would have been different if regrets had been used rather than losses.

The Minimax principle: critique

- Basing a decision-making principle on game-theory ideas implicitly means we are supposing that nature attempts to maximise its own gain (and so the decision makers loss)
- The minimax approach means we are focused on trying to avoid the **worst case**
- But what if the worst state has only a very remote chance of being the actual state of affairs that will be realised?
- In this case basing a strategy around the worst possible state of affairs may not be optimal.

$\Longrightarrow$ our second decision-making principle: minimizing **expected** loss with respect to an assumed distribution on θ

Bayes Actions

- This distribution might be based on past experiences, or reflect personal degrees of belief in the possibility of different values of θ occurring.
- Given this distribution function, $\pi(\theta)$ say, it seems natural to average the prospective losses for each action with respect to π :

$$B(a) = \sum_i \ell(a, \theta_i)\pi(\theta_i) \qquad (\Theta \text{ is discrete})$$

- this "expected" loss $B(a)$ is called the **Bayes loss** for action a
- the action that minimizes the Bayes loss is called the Bayes action, a_B :

$$a_B = \arg\min_{a \in A} B(a)$$

Bayes Actions – Remark

- The distribution $\pi(\theta)$ is often called the **prior** or **a priori** distribution because it represents the decision makers beliefs before data is taken into consideration.
- But none of this uses Bayes Theorem (yet!)
- It is only after data-based information is incorporated that the connection between Bayes' actions and Bayes' probability theorem becomes apparent!

Bayes Actions: Example

Consider again the decision problem of the proceeding example:

		Actions		
		a_1	a_2	a_3
States of	θ_1	4	5	2
Nature	θ_2	4	0	5

(Losses).

- Suppose that the prior distribution for the states is $\pi(\theta_1) = 0.2$, $\pi(\theta_2) = 0.8$.
- The Bayes losses would be

$$\begin{aligned} B(a_1) &= 4(0.2) + 4(0.8) = 4 \\ B(a_2) &= 5(0.2) + 0(0.8) = 1 \\ B(a_3) &= 2(0.2) + 5(0.8) = 4.4 \end{aligned}$$

- So a_2 has the smallest Bayes loss: $a_B = a_2$.

Bayes Actions: unknown prior

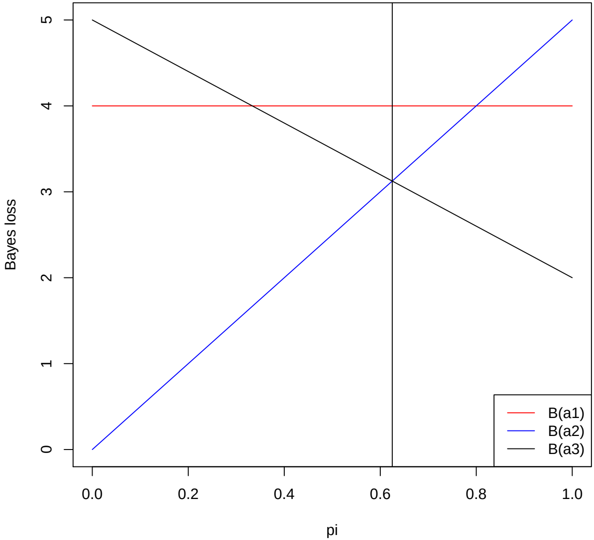
Now suppose we don't actually know $\pi(\theta_1)$. Instead write only $\pi(\theta_1) = \pi$, $\pi(\theta_2) = (1 - \pi)$, $0 \leq \pi \leq 1$.

- The Bayes losses are now expressed in terms of π :

$$\begin{aligned} B(a_1) &= 4\pi + 4(1-\pi) = 4 \\ B(a_2) &= 5\pi + 0(1-\pi) = 5\pi \\ B(a_3) &= 2\pi + 5(1-\pi) = 5 - 3\pi \end{aligned}$$

- If we plot these losses as functions of π we see that for any $\pi < \frac{5}{8}$ action a_2 has smallest Bayes loss, for $\pi > \frac{5}{8}$ a_3 minimises $B(a)$, whilst if $\pi = \frac{5}{8}$ $B(a_2) = B(a_3) < B(a_1)$ and either a_2 or a_3 provides the Bayes action.

Bayes Actions: unknown prior



Bayes vs minimax

- The minimum *Bayes* loss for the **least favourable prior distribution** is precisely the *minimax* loss for a mixed strategy.
- This is because the prior distribution can be thought of as specifying a **mixed strategy for nature**
- $\implies$ the prior distribution that *maximises the minimum Bayes loss* is equivalent to “nature” playing a mixed strategy that **maximises her minimum gain**.
- A result in game-theory states that both players act optimally if they choose their **minimax (maximum) mixed strategies**, the loss of one player equalling the gain of the other.
- Hence the worst-case prior distribution could be regarded as a “malevolent nature” prior.