Canonical Correlation Analysis

Statistics 407, ISU

Definition

Find the linear combination of X, $\mathbf{a}'\mathbf{X}$ that is most correlated with a linear combination of Y, $\mathbf{b}'\mathbf{Y}$, ie maximize $r = cor(\mathbf{a}'\mathbf{X}, \mathbf{b}'\mathbf{Y})$.

Then find the next most correlated pairs of linear combinations. ...

Determine how many linear combinations are needed to describe most of the association.

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n x q

Then find the next most correlated pairs of linear combinations. ...

Determine how many linear combinations are needed to describe most of the association.

Solution

Find the eigenvectors of

$$S_{XX}^{-1} S_{XY} S_{YY}^{-1} S_{YX}$$

to get $\mathbf{a}$ and the eigenvectors of

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give $\mathbf{b}$. The eigenvalues are the correlations.

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Covariance between X and Y

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Var-Cov of Y

Example

- n=50, p=2, q=3

	pop15	pop75	sr	dpi	ddpi
Australia	29.35	2.87	11.43	2329.68	2.87
Austria	23.32	4.41	12.07	1507.99	3.93
Belgium	23.80	4.43	13.17	2108.47	3.82
Bolivia	41.89	1.67	5.75	189.13	0.22
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Pct of pop under 15 older than 75 respectively

Savings ratio, per capita disposable income, % growth in dpi

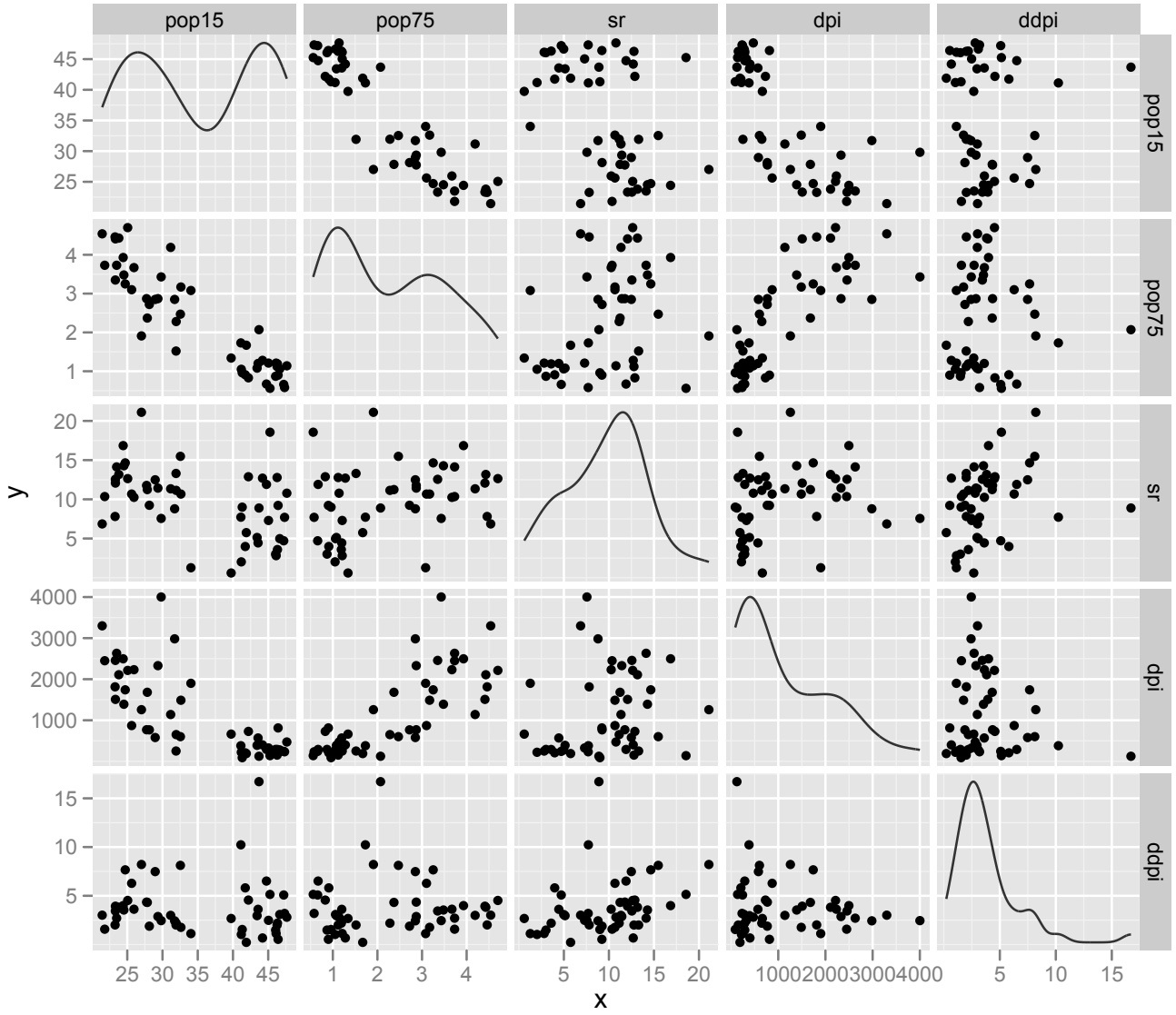
Summary Statistics

$$\bar{\mathbf{X}} = \begin{bmatrix} 35.1 \\ 2.29 \end{bmatrix} \quad \bar{\mathbf{Y}} = \begin{bmatrix} 9.67 \\ 1106.8 \\ 3.76 \end{bmatrix}$$

$$\mathbf{S}_{XX} = \begin{bmatrix} 1.00 & -0.91 \\ -0.91 & 1.00 \end{bmatrix} \quad \mathbf{S}_{YY} = \begin{bmatrix} 1.00 & 0.22 & 0.30 \\ 0.22 & 1.00 & -0.13 \\ 0.30 & -0.13 & 1.00 \end{bmatrix}$$

$$\mathbf{S}_{XY} = \left[\begin{array}{c|ccc} & sr & dpi & ddpi \\ \hline pop15 & -0.456 & -0.756 & -0.048 \\ pop75 & 0.317 & 0.787 & 0.025 \end{array}\right]$$

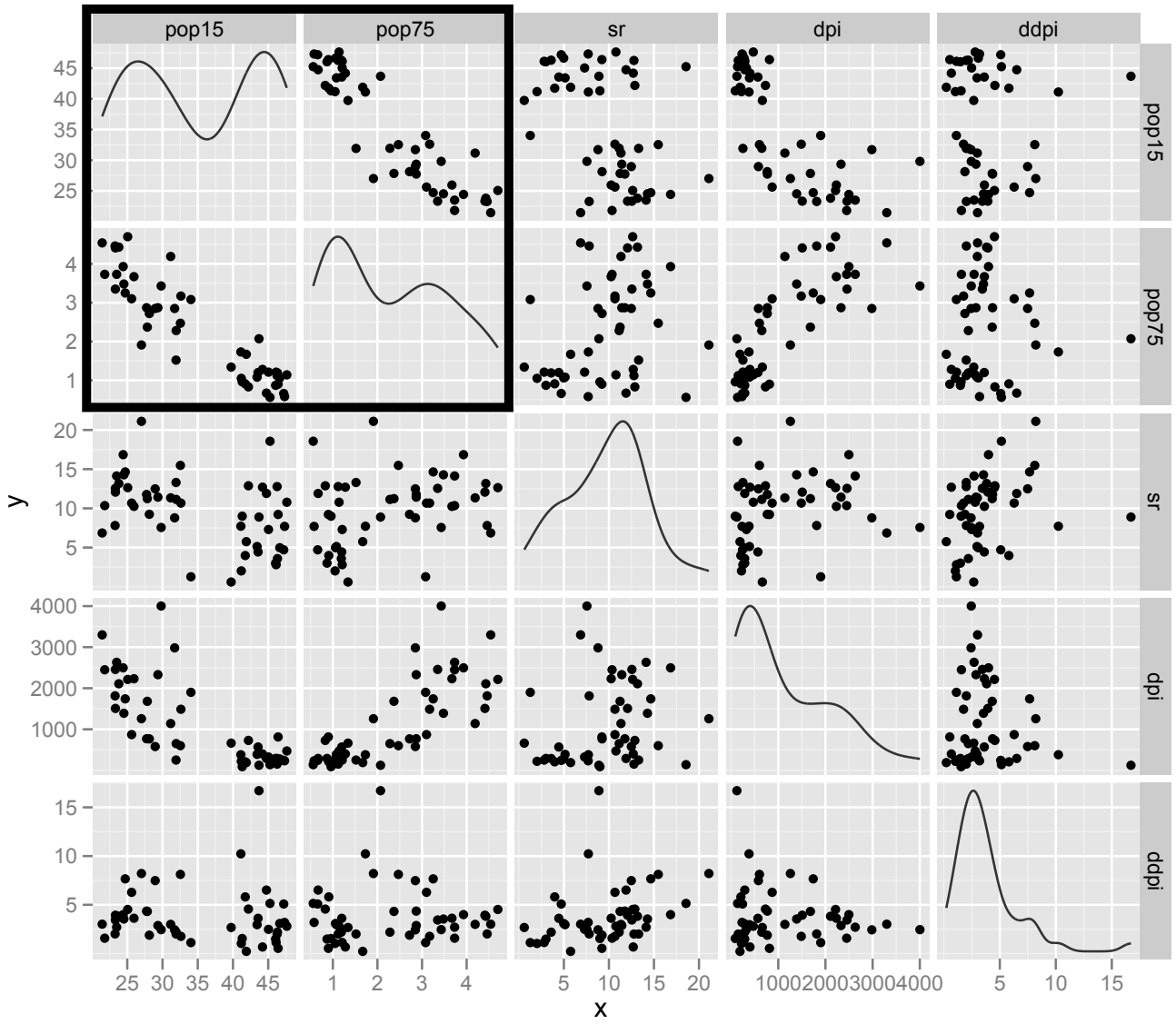
Example



Raw data: two groups.

Association between the groups

Example



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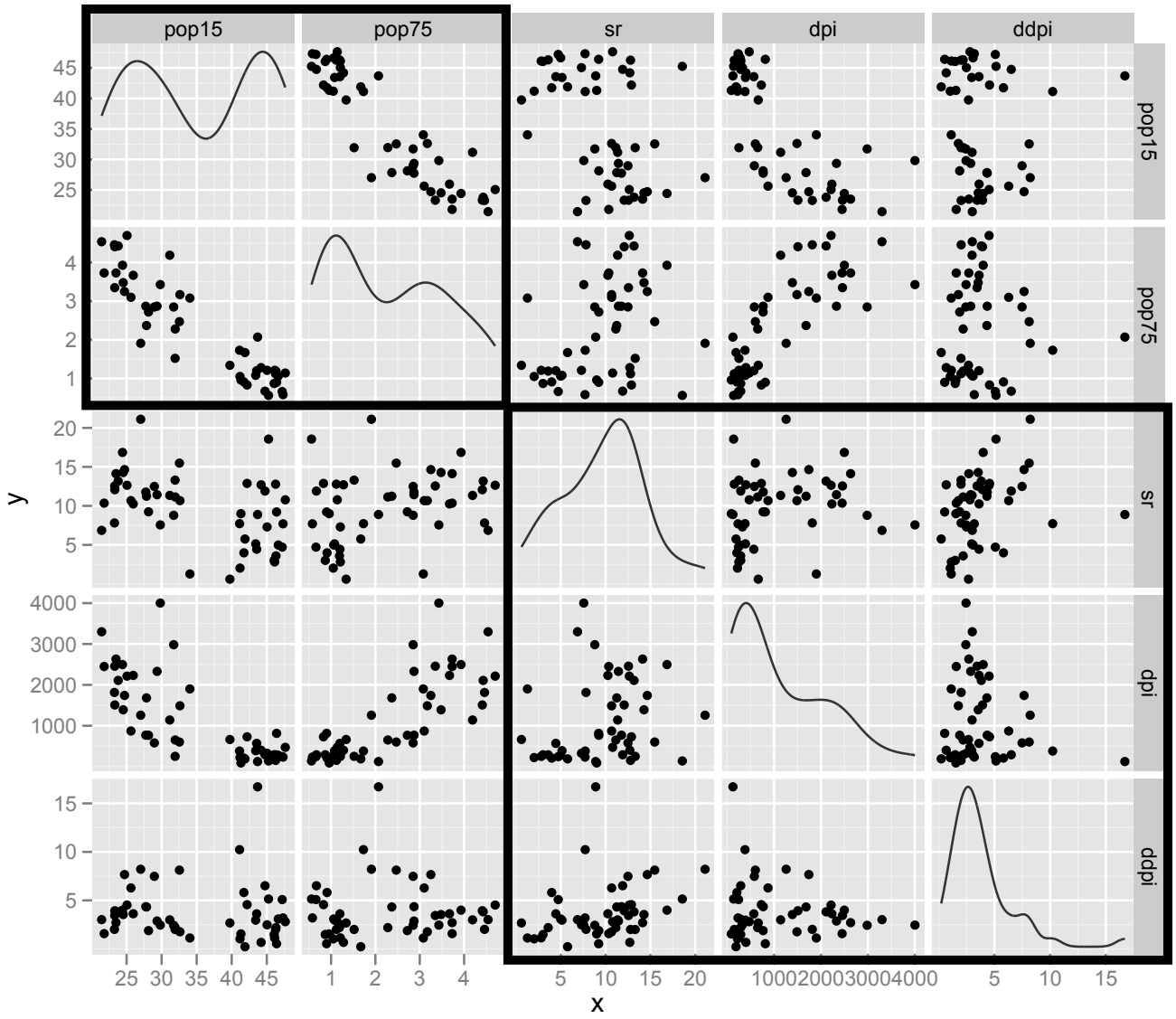

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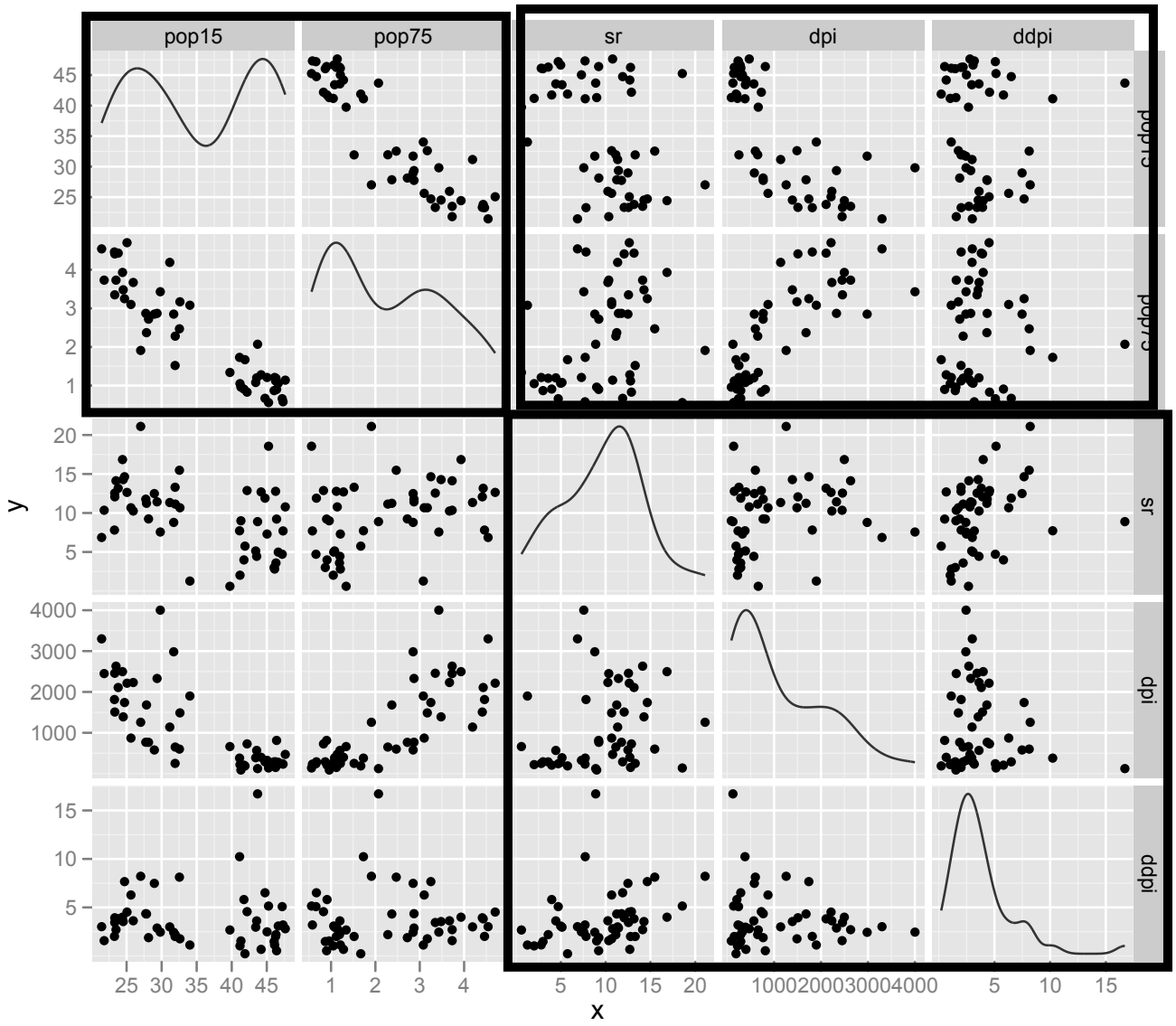

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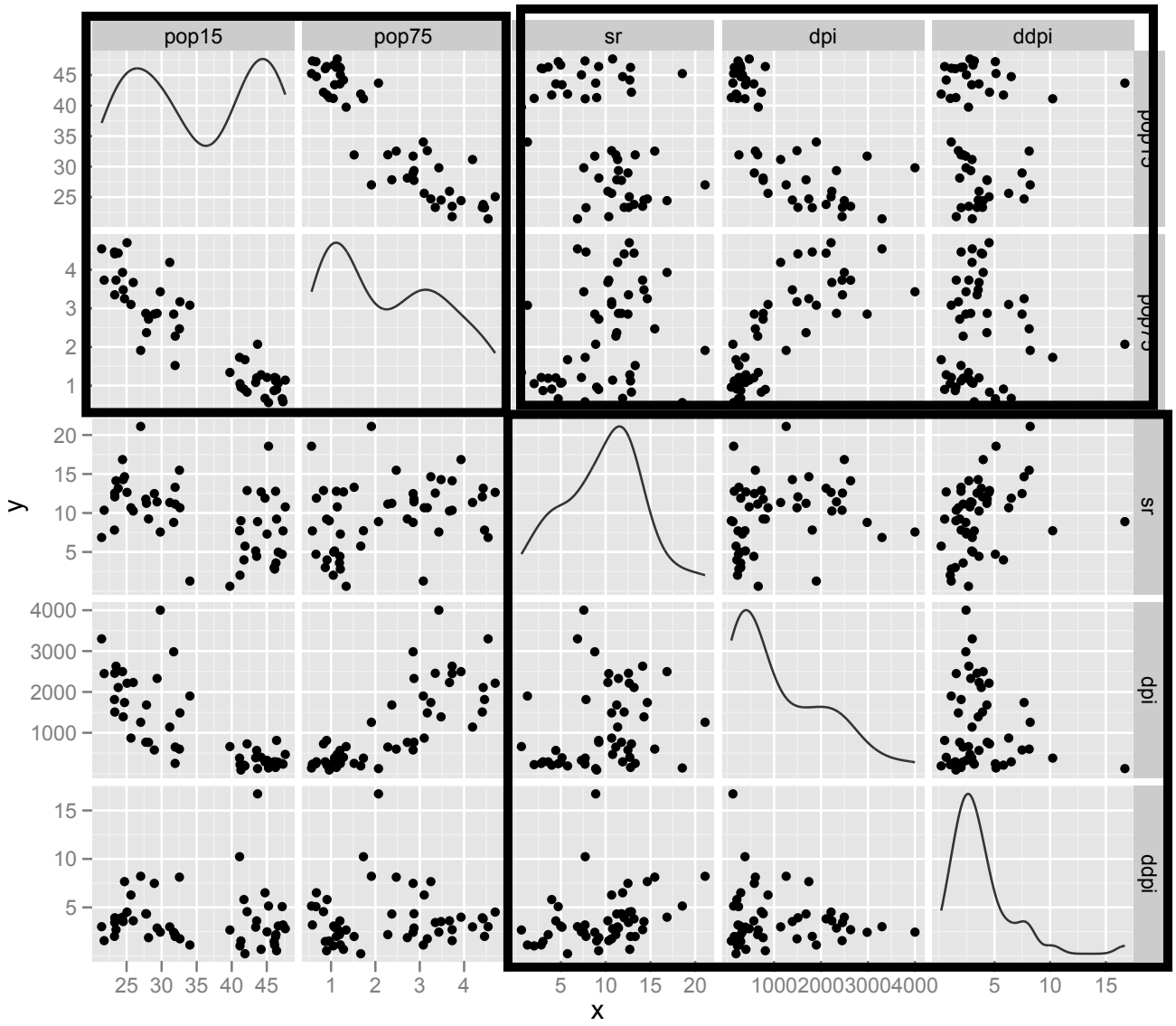
Example

Var-Cov of X



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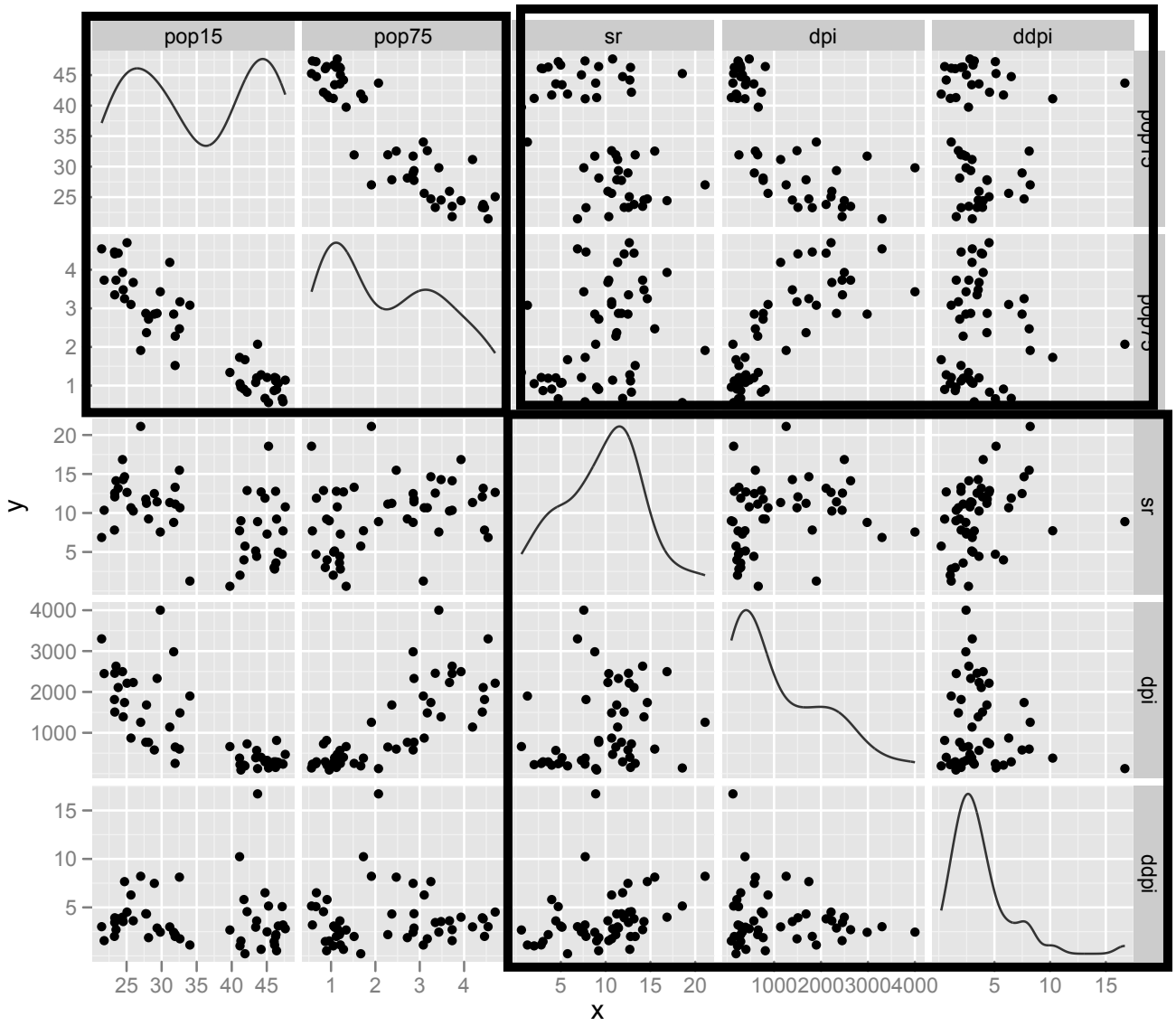

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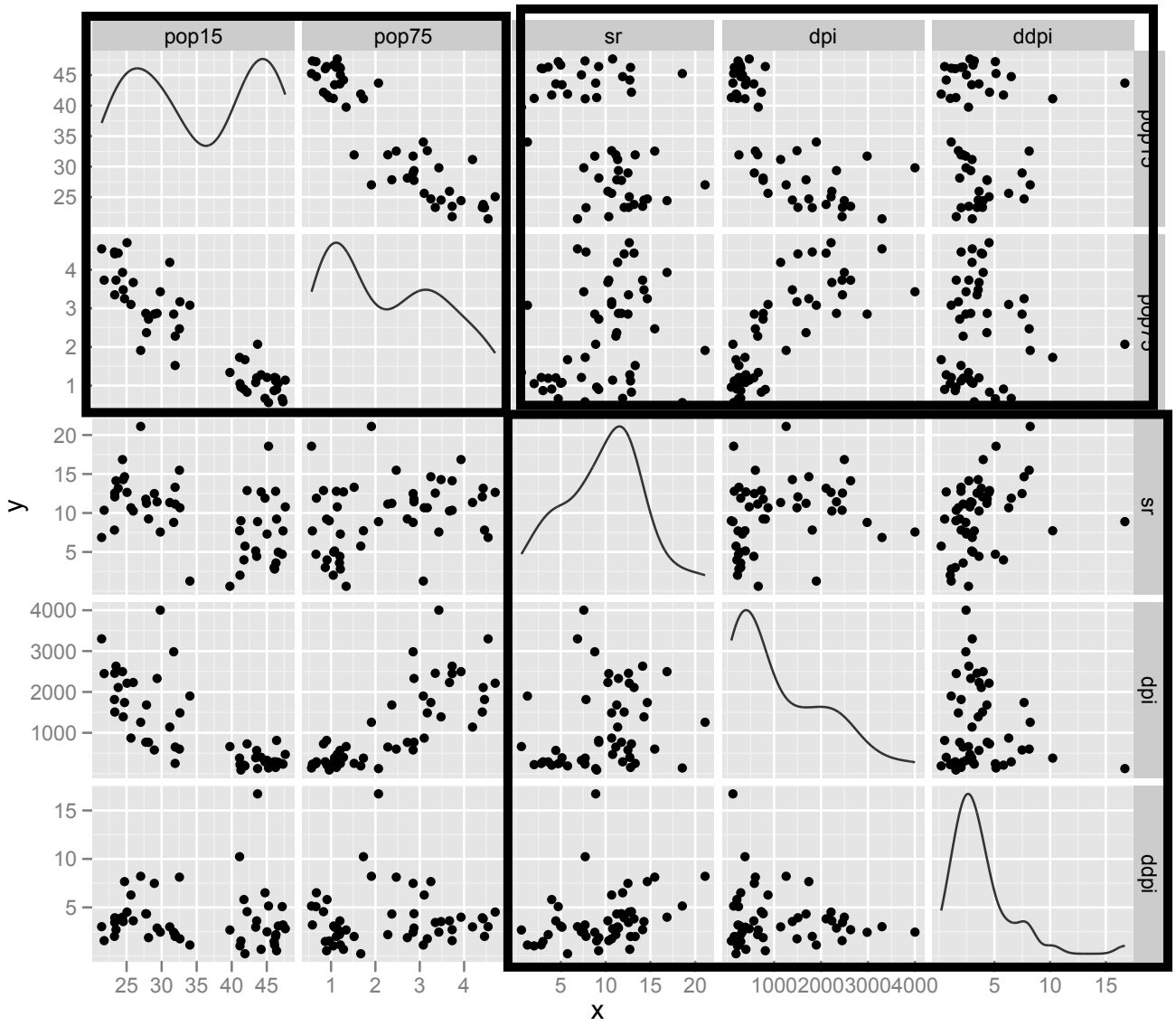
Example

Var-Cov of X

Covariance between X, Y



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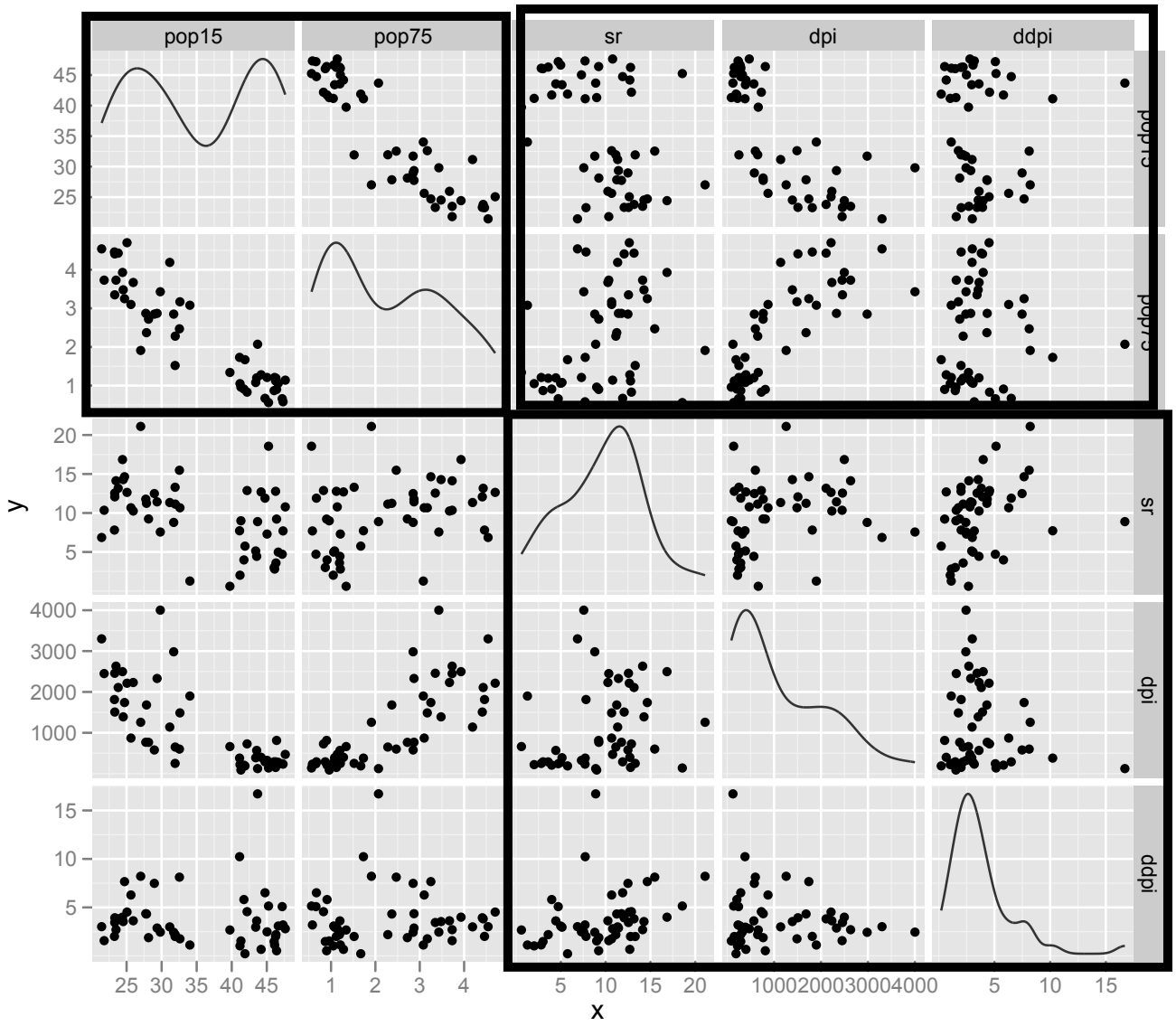

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Association between the groups

Data has some problems, outliers, nonlinear dependence.

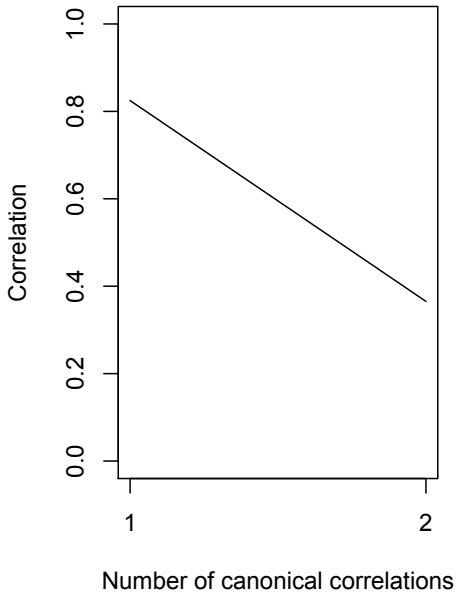
Var-Cov of Y

Canonical correlation

$$r_1 = 0.825, r_2 = 0.365$$

$$\mathbf{A} = \begin{bmatrix} -0.00911 & -0.0362 \\ 0.0486 & -0.260 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0.00847 & 0.0333 & -0.00516 \\ 0.000131 & -0.0000759 & 0.00000454 \\ 0.00417 & -0.00123 & 0.00519 \end{bmatrix}$$



Can't interpret coefficients because the variables had different scales

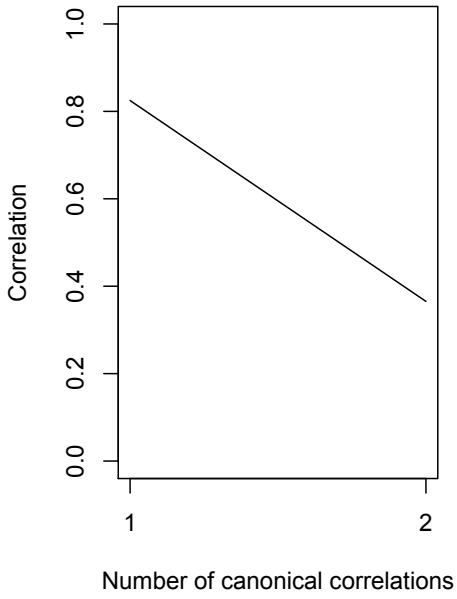
Canonical correlation

Results for standardized variables

$$r_1 = 0.825, r_2 = 0.365$$

$$\mathbf{A} = \begin{bmatrix} -0.083 & -0.331 \\ 0.062 & -0.336 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0.038 & 0.150 & -0.023 \\ 0.130 & -0.075 & 0.0045 \\ 0.012 & -0.035 & 0.149 \end{bmatrix}$$



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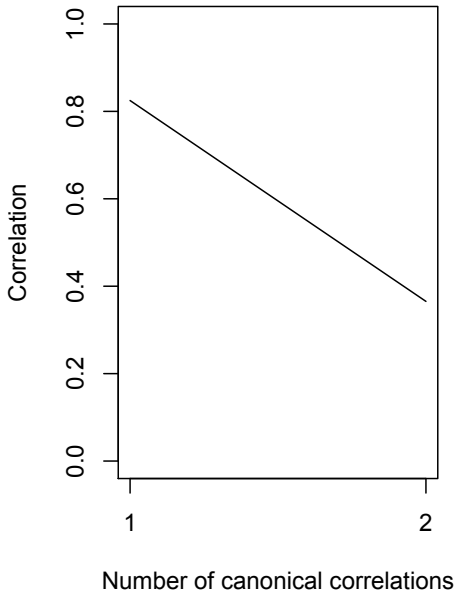

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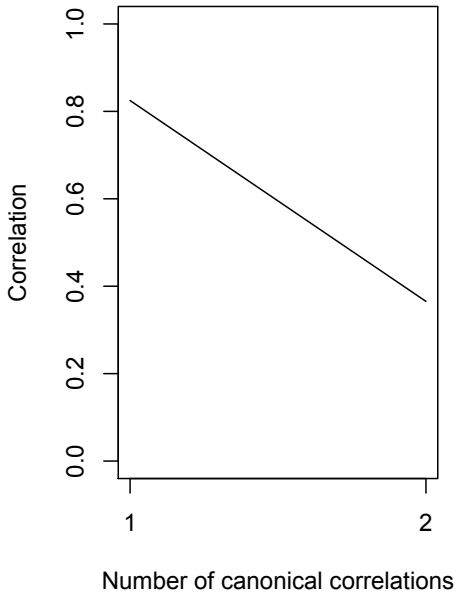
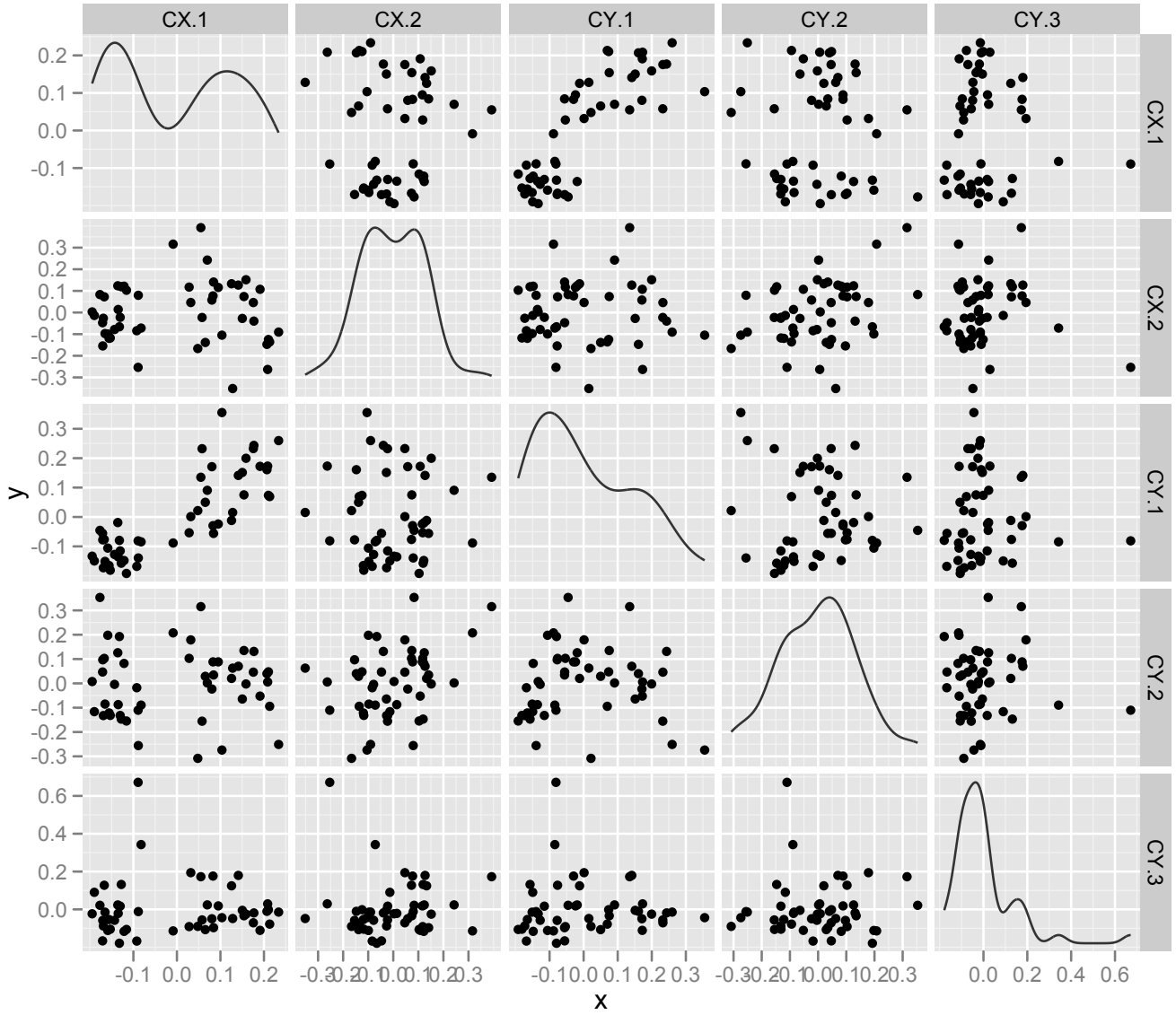


dpi most important variable in explaining contrast between age proportions

Example



Canonical correlations.

First correlation stronger than individual vars, just.

No association between canonical variables.

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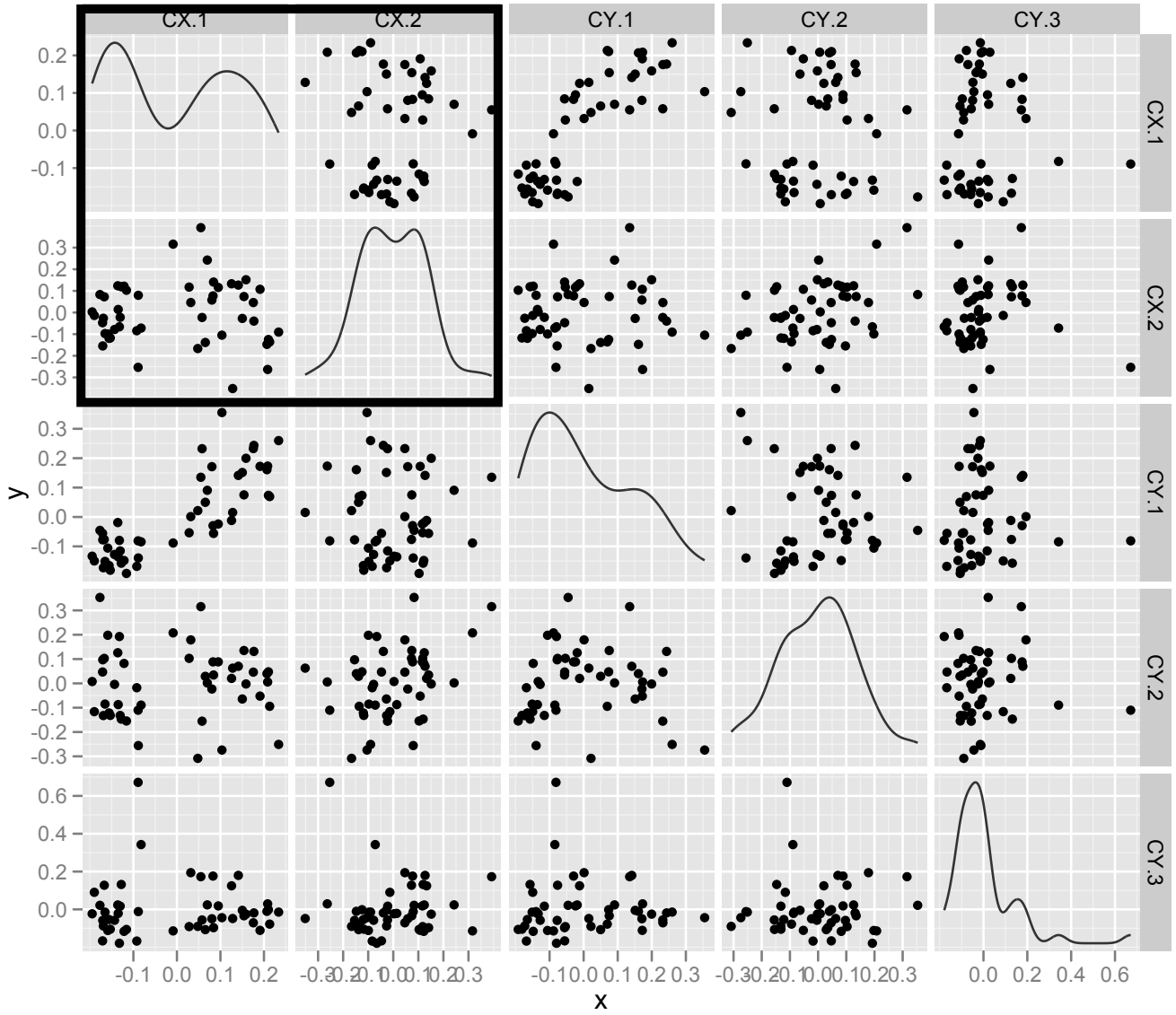

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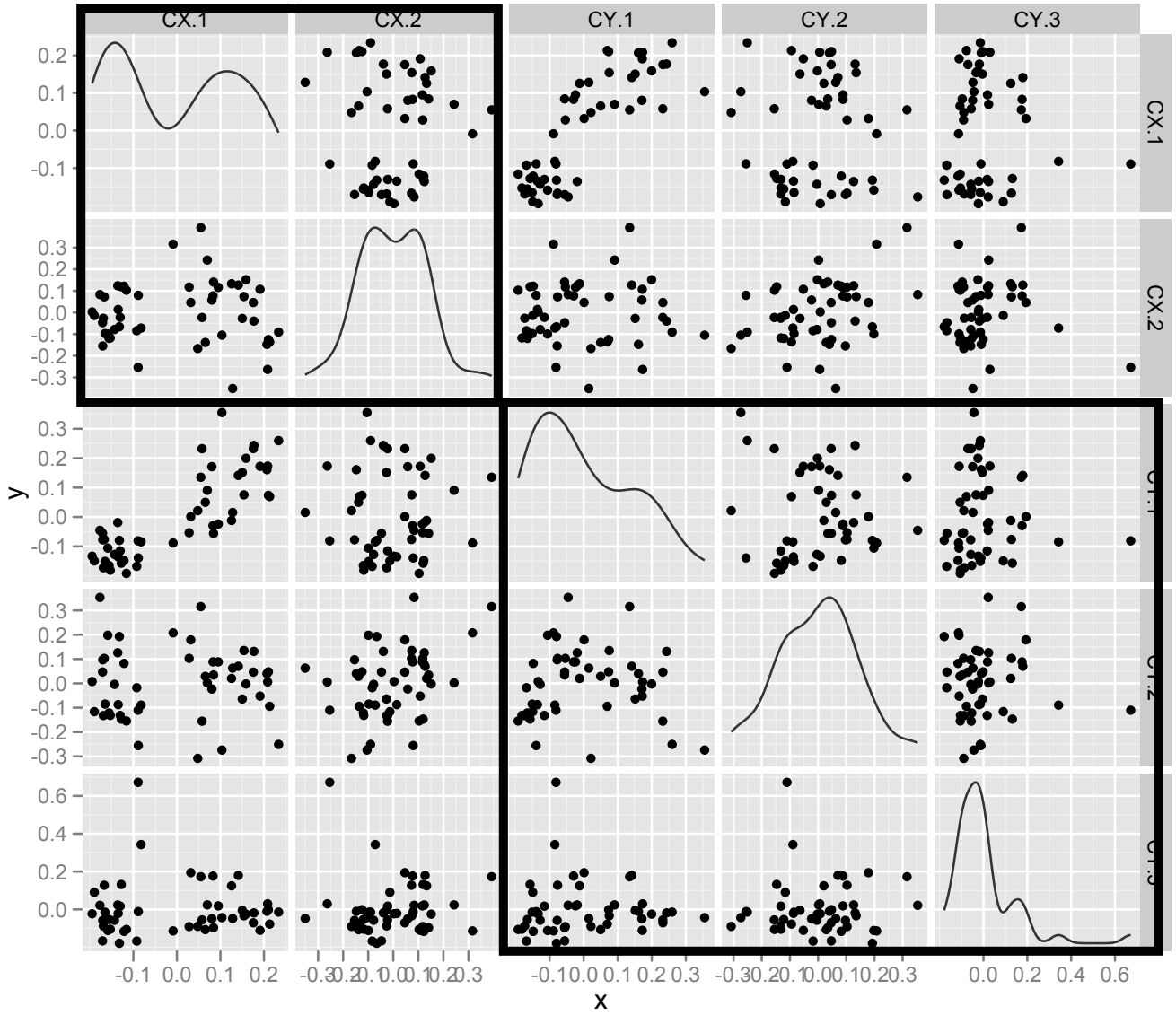

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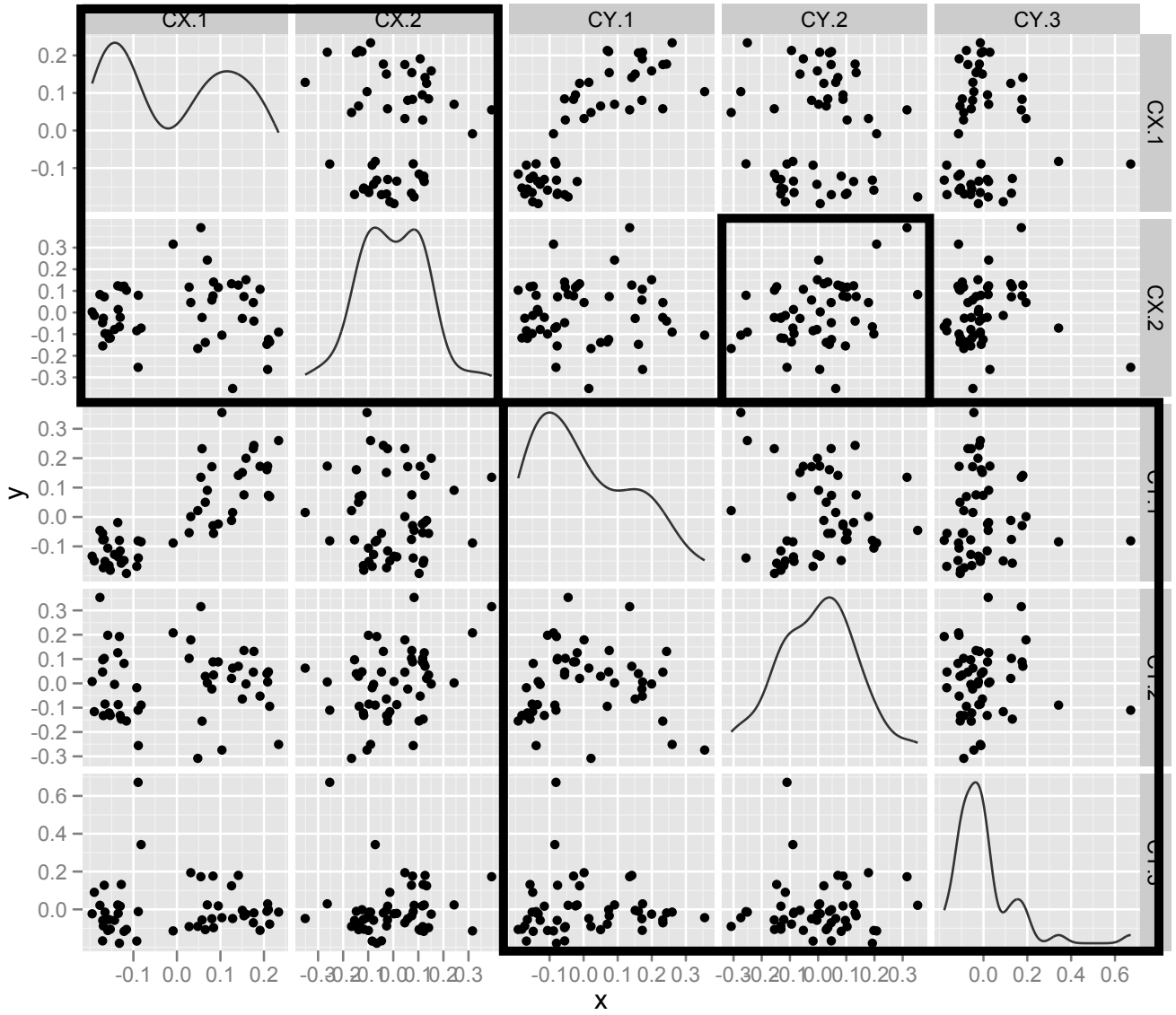

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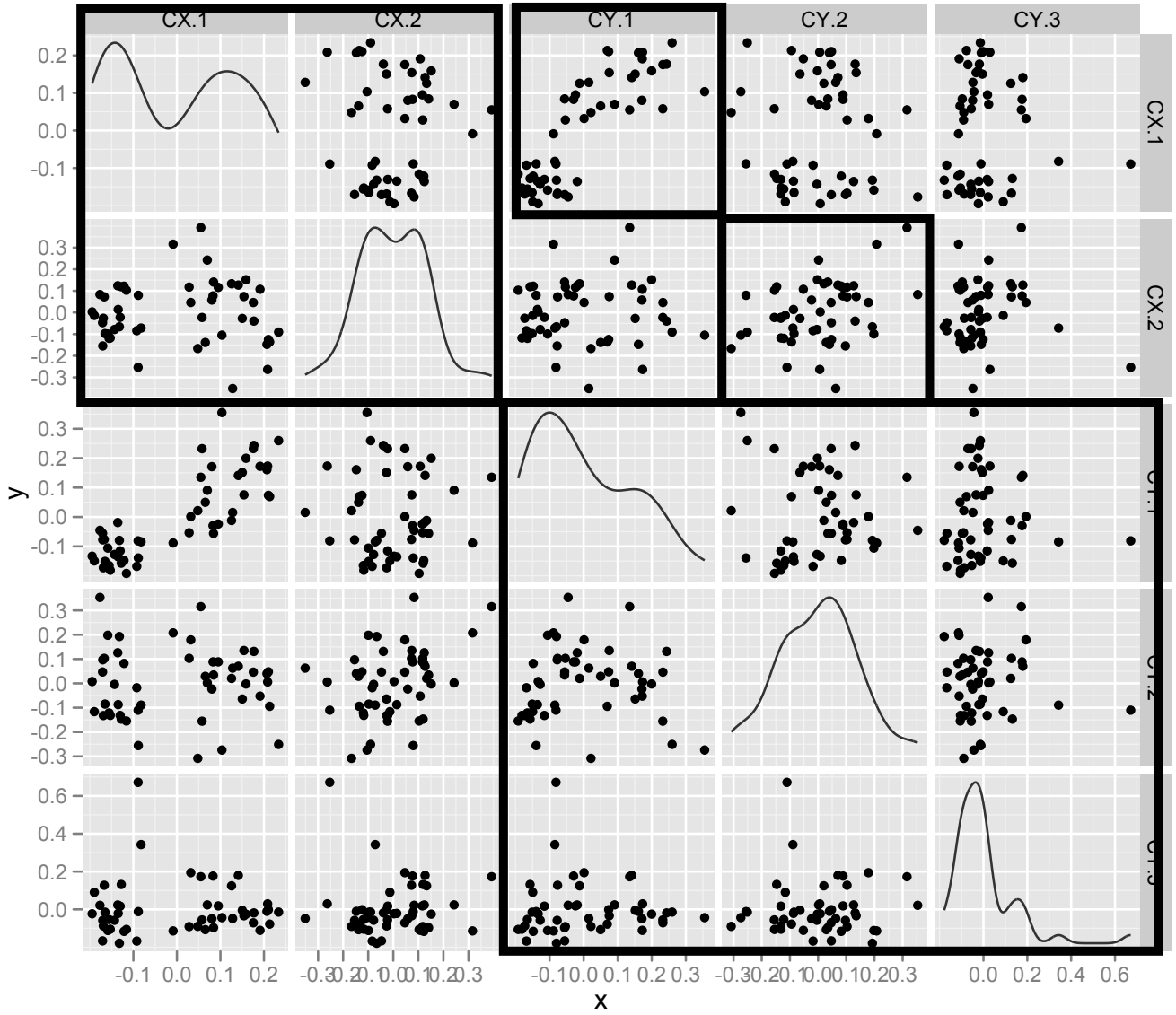

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Interpretation

- Highest correlation between dpi (with a little sr) and a contrast of age proportions.
- Low value on dpi (and sr) corresponds to high proportion of population under 15, and a low number over 75.