# Multivariate Regression

Statistics 407, ISU

# Definition

Multivariate regression refers to modeling multiple responses, in addition to possibly multiple predictors.

$$\mathbf{Y}_{n\times q} = \mathbf{X}_{n\times p}\boldsymbol{\beta}_{p\times q} + \boldsymbol{\varepsilon}_{n\times q}$$

Examples might include (1) a collection of psychological variables and a collection of test scores, (2) a collection of health variables and a collection of eating habits variables, or (3) measurements on the physical characteristics of plants and variables describing the growing conditions.

# Example

Relationships between properties of pulp fiber and paper, Lee (1992) unpublished

$Y_1 =$ breaking length (BL)
$Y_2 =$ elastic modulus (EM)
$Y_3 =$ stress at failure (SF)
$Y_4 =$ burst strength (BS)

$X_1 =$ arithmetic fiber length (AFL)
$X_2 =$ long fiber fraction (LFF)
$X_3 =$ fine fiber fraction (FFF)
$X_4 =$ zero span tensile (ZST)

q=4, p=4

# Example

| | BL | EM | SF | BS | AFL | LFF | FFF | ZST |
|---|---|---|---|---|---|---|---|---|
| Mean | 21.72 | 7.27 | 5.64 | 1.02 | -0.02 | 39.03 | 26.68 | 1.07 |
| Std dev | 2.88 | 0.72 | 1.46 | 0.69 | 0.25 | 14.87 | 17.56 | 0.03 |

## Corr(Y)

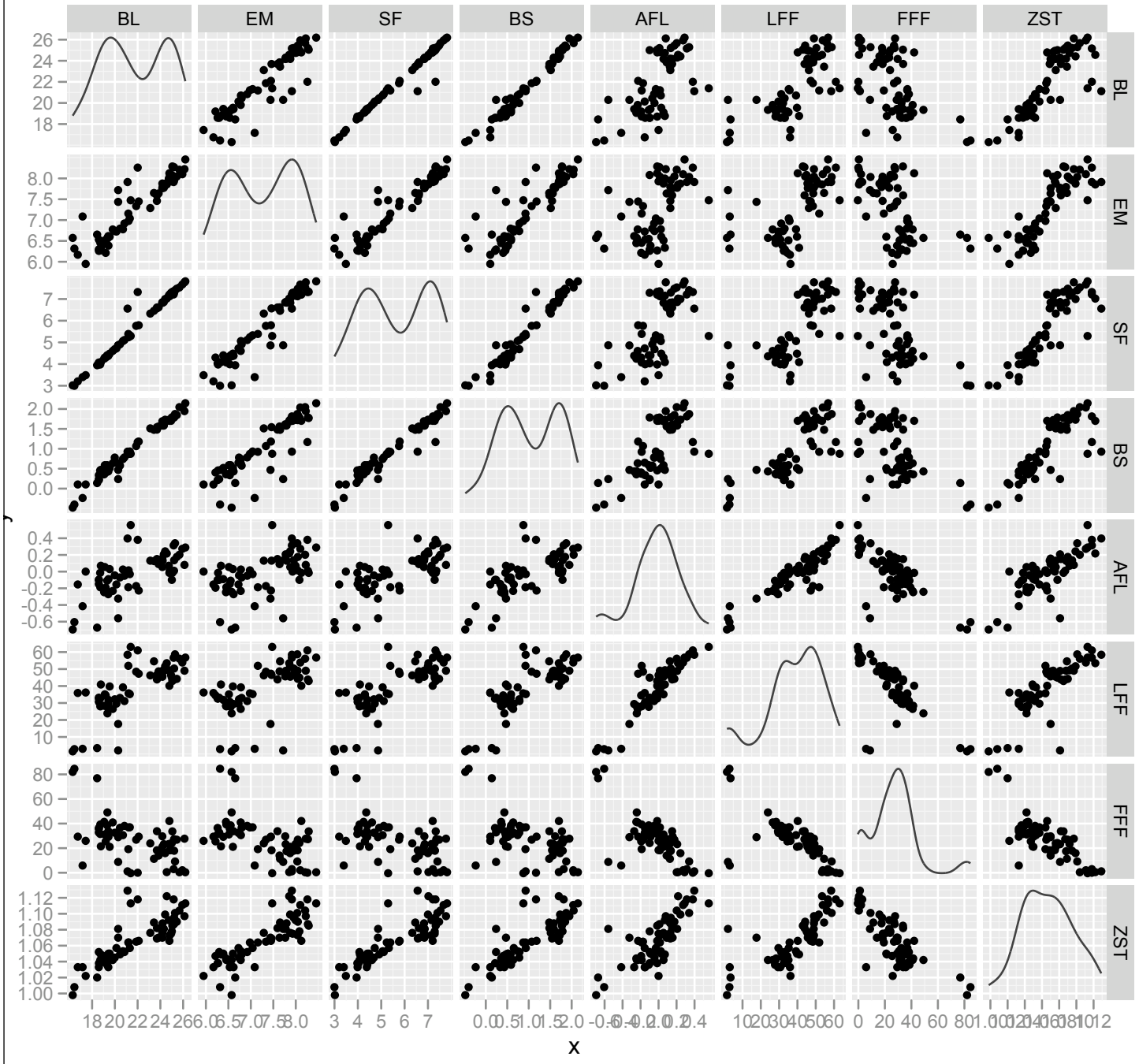
| | BL | EM | SF | BS |
|---|---|---|---|---|
| BL | 1.00 | 0.91 | 0.98 | 0.99 |
| EM | 0.91 | 1.00 | 0.94 | 0.87 |
| SF | 0.98 | 0.94 | 1.00 | 0.97 |
| BS | 0.99 | 0.87 | 0.97 | 1.00 |

## Corr(X)

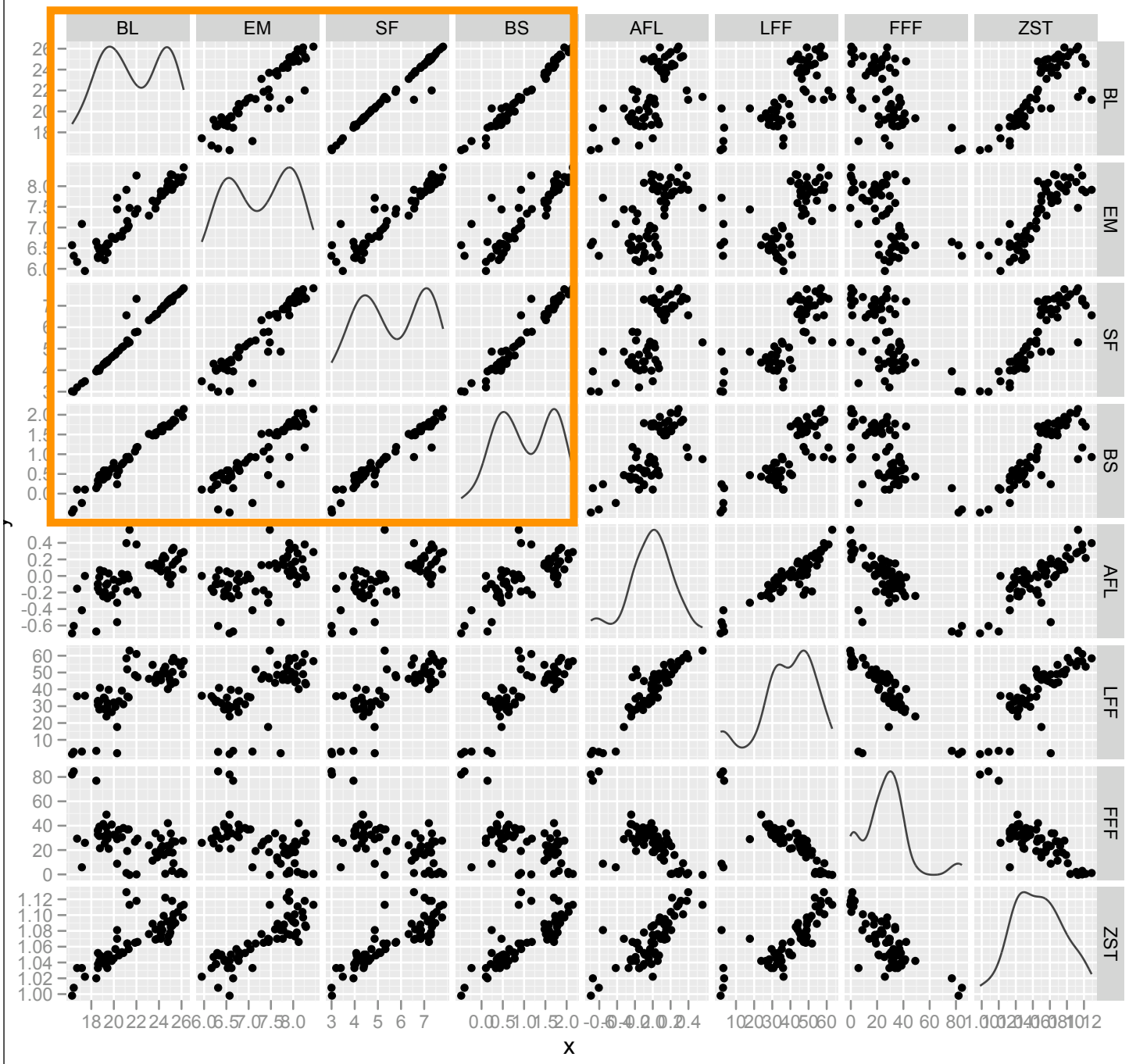
| | AFL | LFF | FFF | ZST |
|---|---|---|---|---|
| AFL | 1.00 | 0.91 | -0.73 | 0.78 |
| LFF | 0.91 | 1.00 | -0.71 | 0.79 |
| FFF | -0.73 | -0.71 | 1.00 | -0.78 |
| ZST | 0.78 | 0.79 | -0.78 | 1.00 |

## Corr(Y,X)

| | AFL | LFF | FFF | ZST |
|---|---|---|---|---|
| BL | 0.65 | 0.74 | -0.54 | 0.82 |
| EM | 0.54 | 0.61 | -0.56 | 0.85 |
| SF | 0.68 | 0.76 | -0.57 | 0.87 |
| BS | 0.71 | 0.80 | -0.56 | 0.81 |




Response variables strongly positively correlated, outliers.
Predictors somewhat correlated both negative and positive, outliers.
Response vs predictors some correlation, clusters and outliers.

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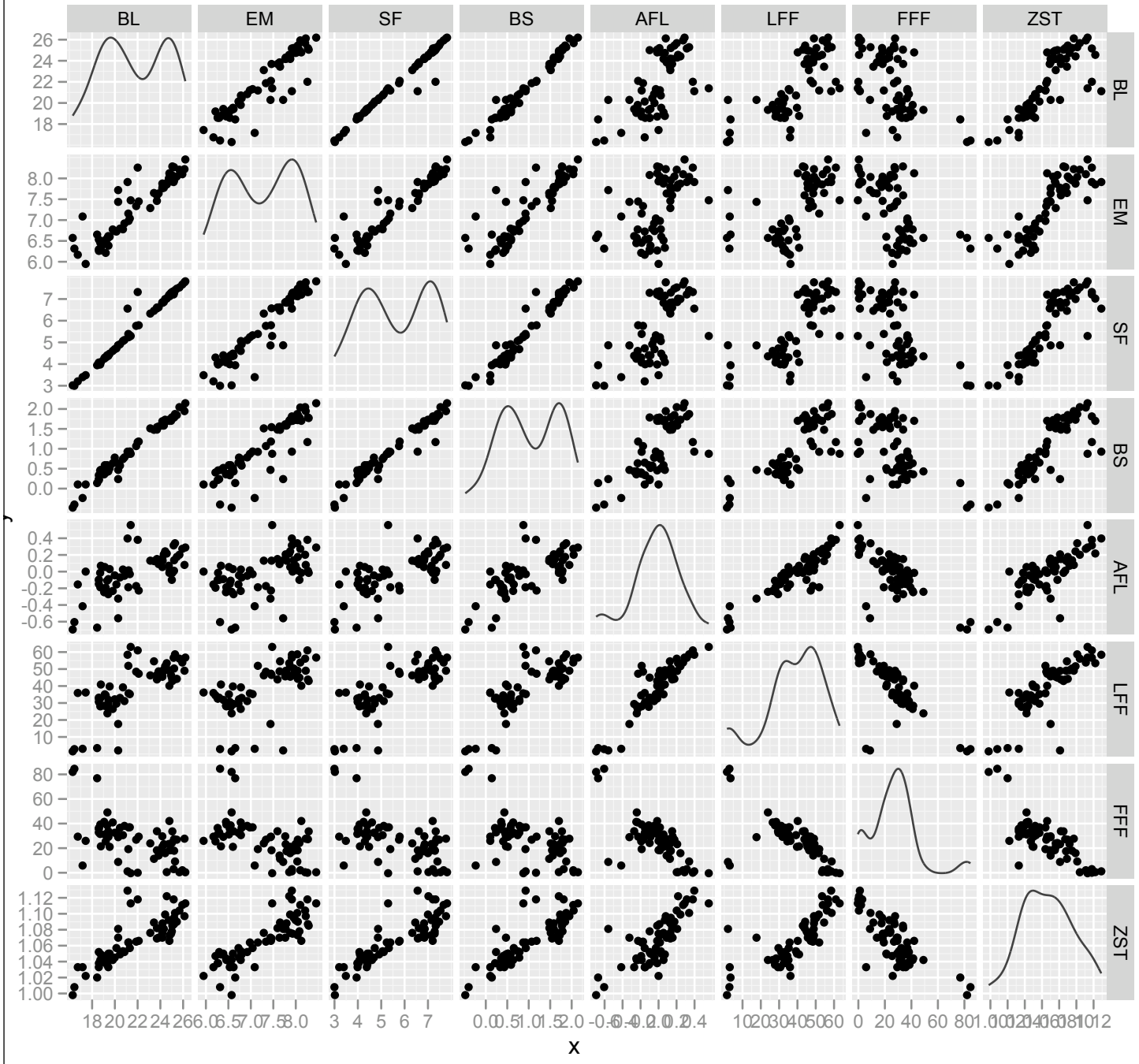


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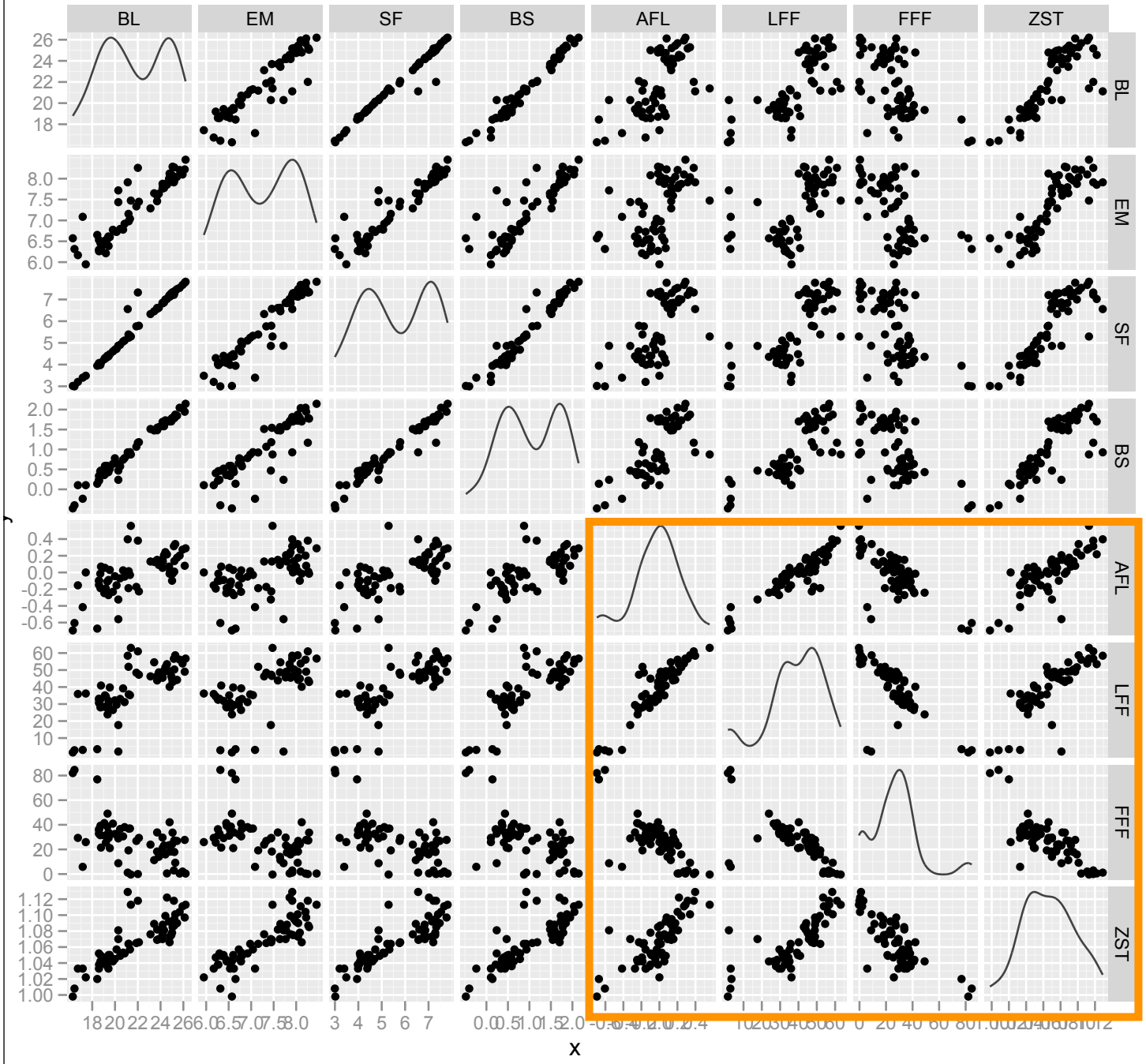


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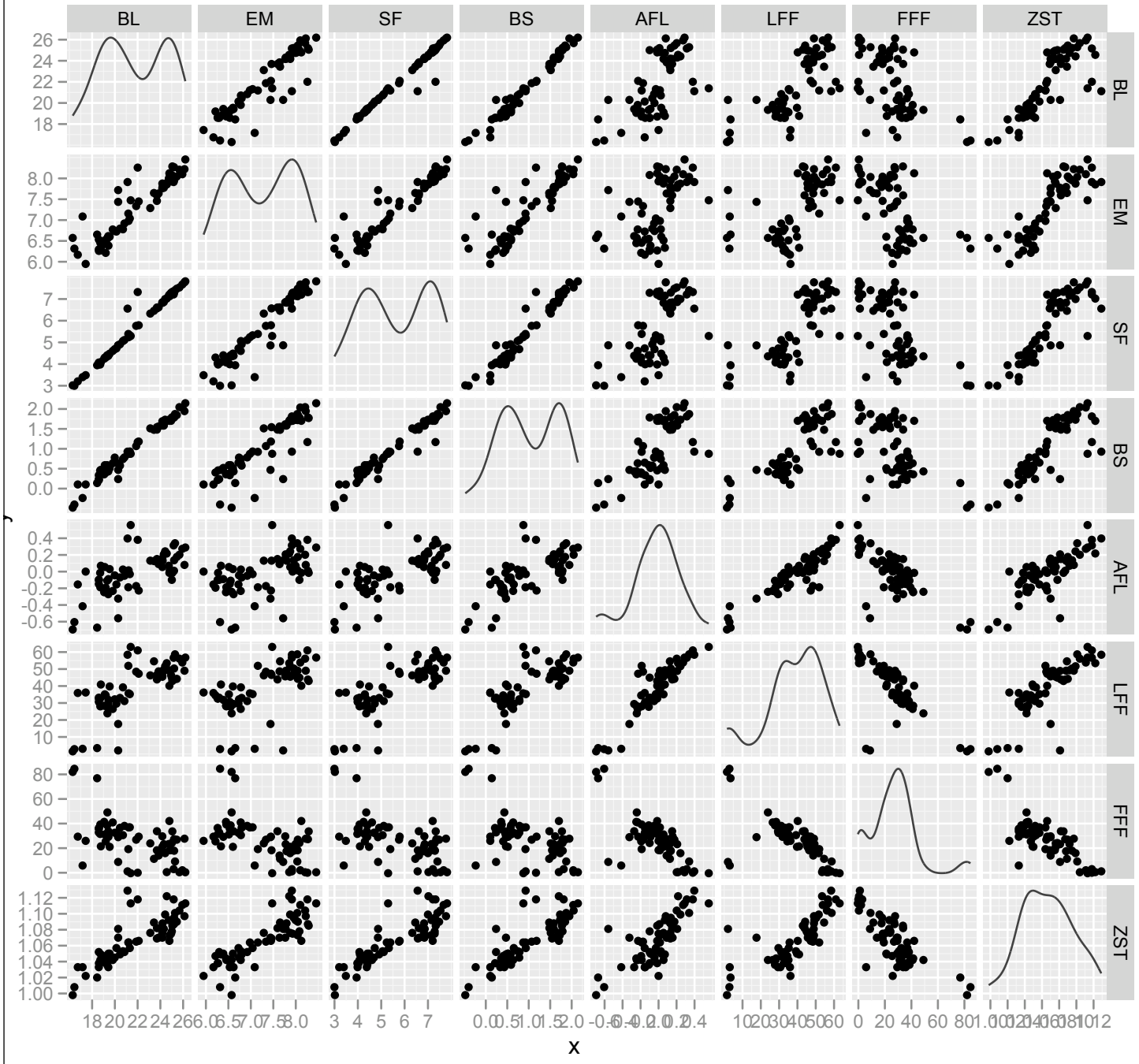


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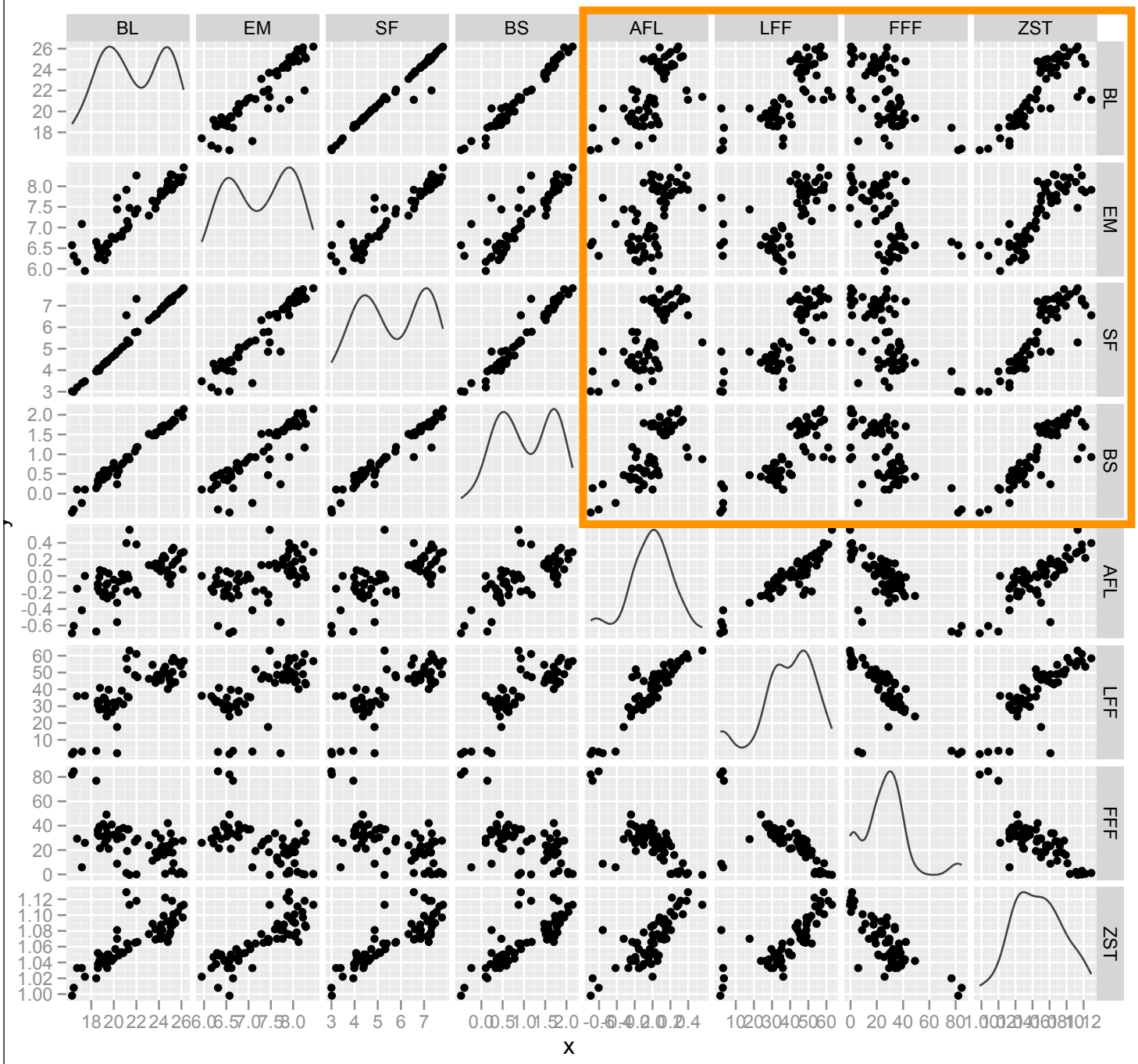


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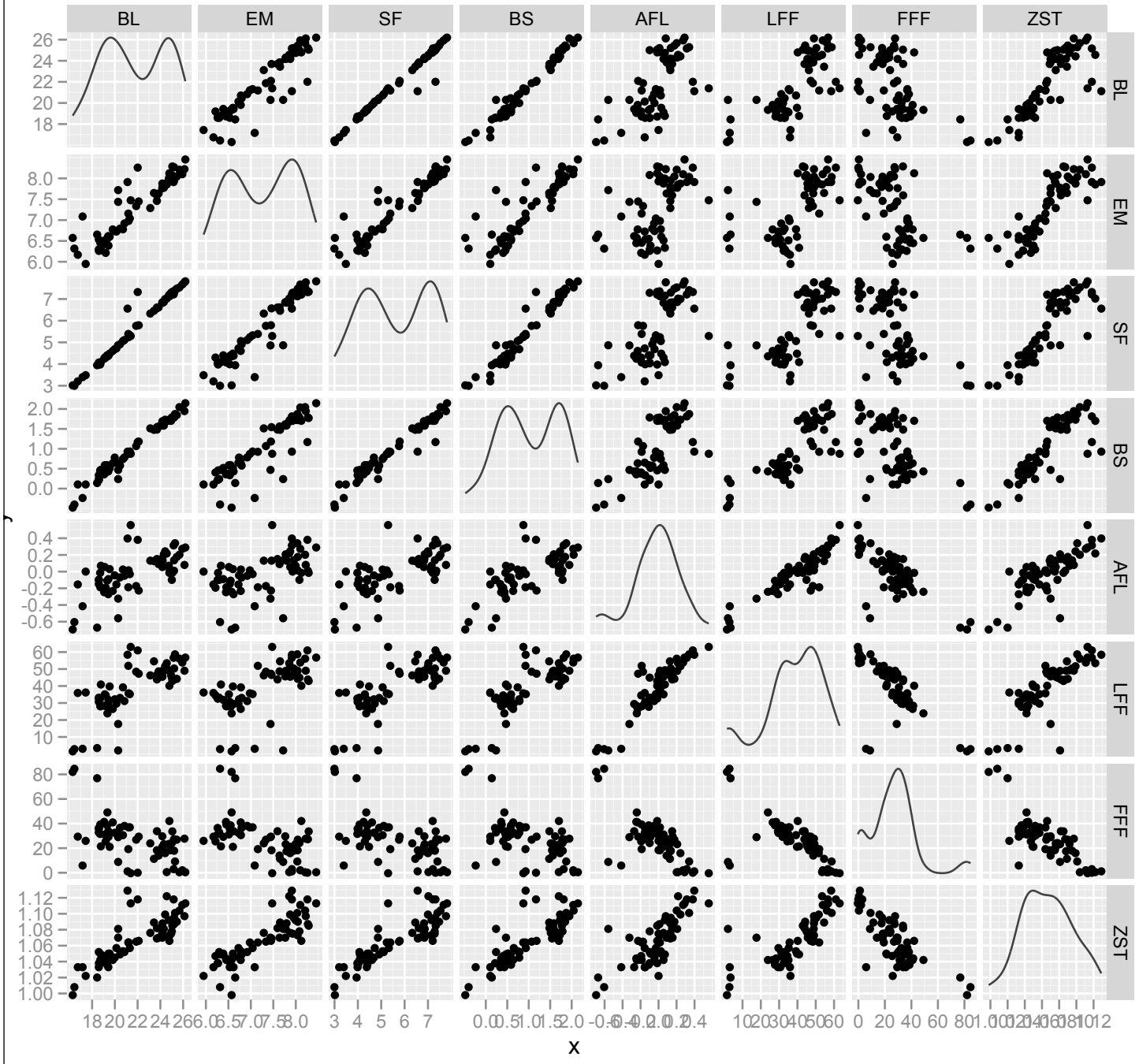


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Response vs predictors some correlation, clusters and outliers.

# Fit separate models

| | (Intercept) | AFL | LFF | FFF | ZST |
|---|---|---|---|---|---|
| BL | -74.23 | -3.12 | 0.10** | 0.05** | 85.08*** |
| EM | -24.01 | -1.18** | 0.01 | 0.01 | 28.75*** |
| SF | -45.76 | -1.49 | 0.05** | 0.03** | 45.80*** |
| BS | -17.73 | -0.55 | 0.03*** | 0.01* | 16.22*** |

- Coefficients are similar for each response
- Significance varies, though
- ZST is most important, AFL least important.

# Multivariate regression

| | (Intercept) | AFL | LFF | FFF | ZST |
|---|---|---|---|---|---|
| BL | -74.23 | -3.12 | 0.10 | 0.05 | 85.08 |
| EM | -24.01 | -1.18 | 0.01 | 0.01 | 28.75 |
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WAIT!!! These are the same as fitting separate models.

# Multivariate tests of coefficients

$$H_o : \boldsymbol{\beta} = \mathbf{0}$$

$$H_A : \text{at least one } \beta \text{ is not } 0$$

The matrix of coefficients are jointly tested for significance. This is the difference from fitting separate models.

# Multivariate tests of coefficients

$$\mathbf{E} = (\mathbf{Y} - \hat{\mathbf{Y}})(\mathbf{Y} - \hat{\mathbf{Y}})' \qquad \mathbf{H} = \hat{Y}\hat{Y}'$$

$$\text{Wilks } \Lambda = \frac{|\mathbf{E}|}{|\mathbf{H} + \mathbf{E}|}$$

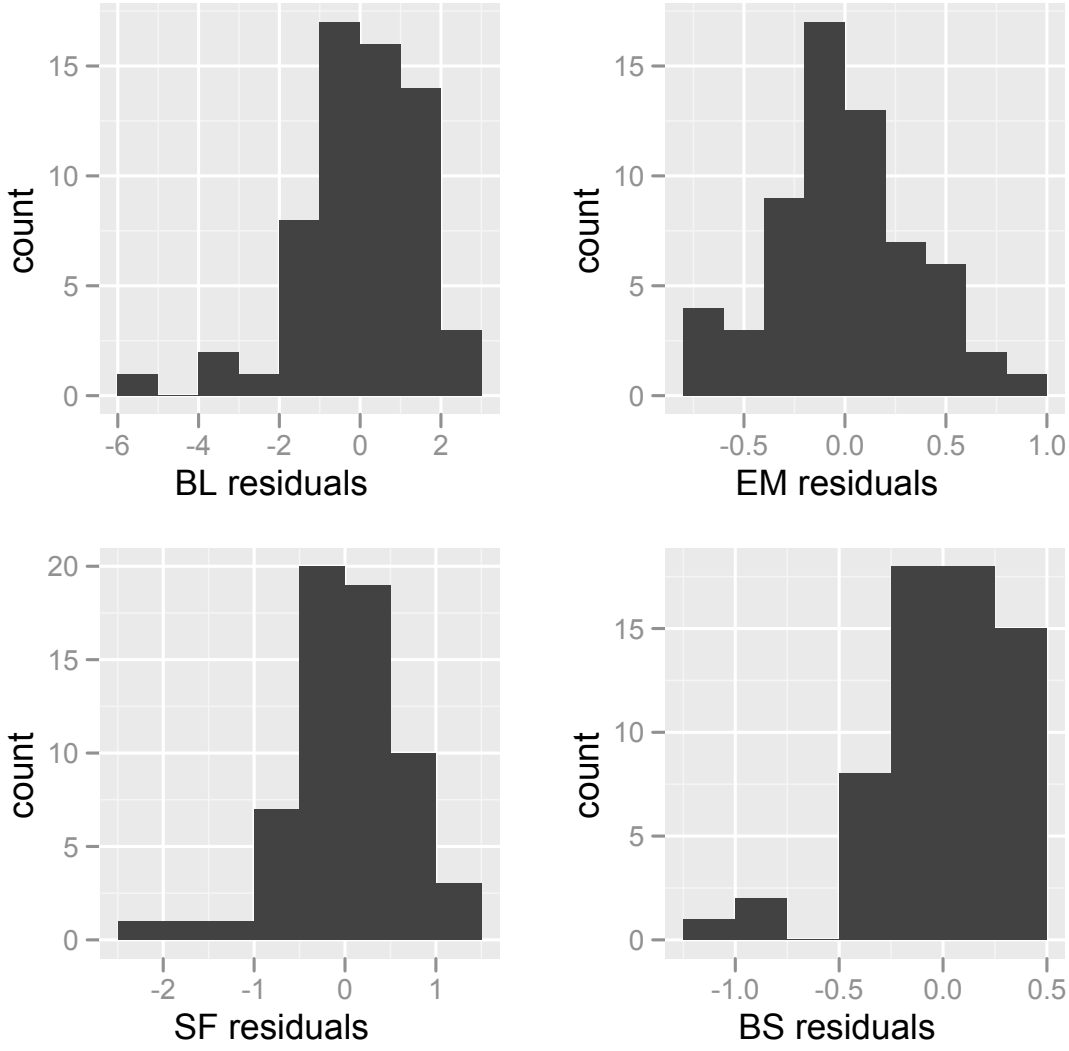
Yes, its the same Wilks. Here the comparison is between the variation due to the model (model sum of squares), and the variation due to error (residual sum of squares).

# Example

| | Df | Wilks | approx F | num Df | den Df | Pr(>F) |
|---|---|---|---|---|---|---|
| (Intercept) | 1 | 0.00 | 54983.25 | 4 | 54 | 0.0000 |
| AFL | 1 | 0.21 | 51.51 | 4 | 54 | 0.0000 |
| LFF | 1 | 0.54 | 11.49 | 4 | 54 | 0.0000 |
| FFF | 1 | 0.69 | 6.10 | 4 | 54 | 0.0004 |
| ZST | 1 | 0.30 | 30.80 | 4 | 54 | 0.0000 |
| Residuals | 57 | | | | | |

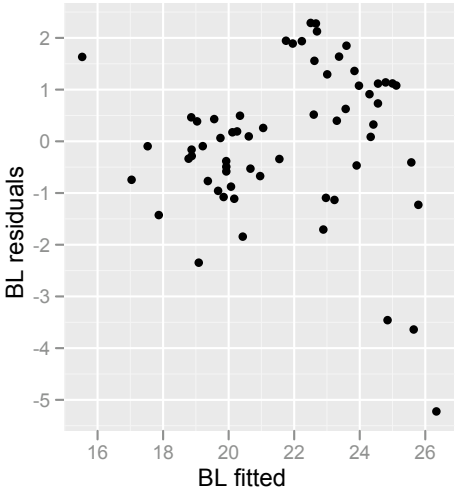
- All four predictors are very important for modeling the paper quality.
- Interestingly AFL is most important, followed by ZST, and FFF is least important

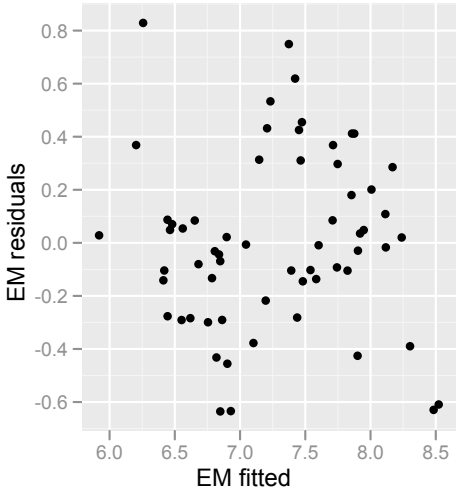
# Diagnostics

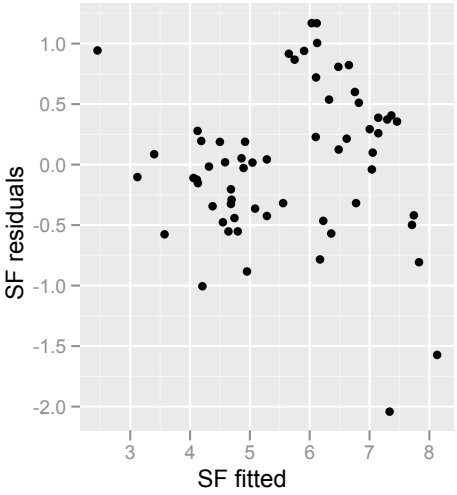


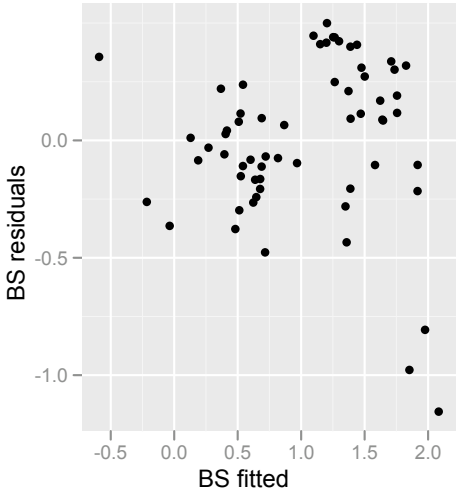

Residuals on all variables should be samples from a normal distribution. Some problems with outliers here.

# Diagnostics




Residuals vs fitted.
Indicates some problems with a few outliers.

# Alternatives

- PCA on the response variables, use a linear combination of these variables as the response. (And similarly PCA on predictors.)
- Canonical correlation analysis - which combination of responses is most correlated with which combination of predictors.
- Useful explanation: http://www.ats.ucla.edu/stat/Stata/dae/mvreg.htm