PART I - THEORY Q.

MLE

1- Let's say,

$$f(x_i) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}} \quad , \text{for each } i$$

$0 < i < n$.

We can rewrite it as

$$f(x_i, x_{i+1}, \dots x_n | \sigma, \mu) = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n e^{-\sum_{i=1}^{n} \frac{(x_i-\mu)^2}{2\sigma^2}}$$

since all ~~#~~ point are independent.

In order to find MLE, it is enough to take the partial derivative of the function above. However, we can simplify the process by ~~take~~ considering the log likelihood of the function which is:

$$\ln(f(x_1, \dots x_n | \sigma, \mu)) = \overbrace{-n\ln(\sqrt{2\pi}) - n\ln(\sigma)}^{\text{constant in terms of } \frac{\partial f}{\partial \mu}} \underbrace{- \sum_{i=1}^{n} \frac{(x_i-\mu)^2}{2\sigma^2}}_{\text{the part that needs to be considered in the process of partial derivative}}$$

To find the MLE for "μ":

$$\frac{\partial f(x_1, x_2 \dots x_n | \sigma, \mu)}{\partial \mu} = \sum_{i=1}^{n} \frac{(x_i-\mu)}{\sigma^2} = 0$$

MLE

$$\sum_{i=1}^{n} (x_i - \mu) = 0 \Rightarrow \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \mu = n.\mu$$

$$\Rightarrow \hat{\mu} = \frac{\sum_{i=1}^{n} x_i}{n} = \overline{X}$$

2-

3- As in the first example, we need to multiply the probabilities of each sample to find the likelihood, which is in this case:

$$L(\theta) = [P(x=0)]^2 \cdot [P(x=1)]^3 \cdot [P(x=2)]^3 \cdot [P(x=3)]^2$$

$$= \left(\frac{2\theta}{3}\right)^2 \left(\frac{\theta}{3}\right)^3 \cdot \left(\frac{2(1-\theta)}{3}\right)^3 \left(\frac{1-\theta}{3}\right)^2$$

To compute the partial derivative easily, we can use log likelihood as in the first example:

$$\Rightarrow 2\log\frac{2}{3} + 2\log\theta + 3\log\frac{1}{3} + 3\log\theta + 3\log\frac{2}{3} +$$
$$3\log(1-\theta) + 2\log\frac{1}{3} + 2\log(1-\theta)$$

Since we take the derivative with respect to θ, we can ignore constant values.

$$\Rightarrow 5\log(\theta) + 5\log(1-\theta)$$

$$\frac{d\,\ell(\theta)}{d\theta} = 5 \cdot \frac{1}{\theta} + 5 \cdot \frac{1}{1-\theta} = \emptyset$$

$$\Rightarrow \boxed{\hat{\theta} = 0.5}$$

NÄIVE BAYES

1 - Even though there is no example of Red, Domestic, SUV in our dataset, we still can find the probability of that car's being stolen thanks to Näive Bayes.

$P(\text{Is Stolen} \mid \text{Red, Domestic, SUV}) \Rightarrow$

(first case, it is stolen)

$$\frac{P(\text{Red} \mid \text{yes})\, P(\text{Domestic} \mid \text{yes})\, P(\text{SUV} \mid \text{yes})\, P(\text{Stolen}=\text{yes})}{P(\text{Red}) .\, P(\text{Domestic})\, P(\text{SUV})}$$

$$\Rightarrow \left(\frac{3}{5} \cdot \frac{3}{5} \cdot \frac{2}{5} \cdot \frac{5}{10}\right) \Big/ \left(\frac{1}{2} \cdot \frac{4}{6} \cdot \frac{1}{2}\right) = 0{,}43$$

For, stolen = no

$$\frac{P(\text{Red} \mid \text{no})\, P(\text{Domestic} \mid \text{no})\, P(\text{SUV} \mid \text{no})\, P(\text{stolen}=\text{no})}{P(\text{Red})\, P(\text{Domestic})\, P(\text{SUV})}$$

$$= \left(\frac{2}{5} \cdot \frac{2}{5} \cdot \frac{2}{5} \cdot \frac{5}{10}\right) \Big/ \frac{5}{10} \cdot \frac{4}{6} \cdot \frac{5}{10} = 0{,}19$$

Since $0{,}43 > 0{,}19$, the car is <u>STOLEN.</u>

NAÏVE BAYES

2- $P(\text{content} \mid \text{rich}=0, \text{marriage}=1, \text{health}=1) = ?$

This probability is equivalent to,

$$\frac{P(\text{rich}=0 \mid \text{content}) \cdot P(\text{marriage}=1 \mid \text{content}) \cdot P(\text{health}=1 \mid \text{content}) \cdot P(\text{content})}{P(\text{rich}=0) \cdot P(\text{marriage}=1) \, P(\text{health}=1)}$$

$$= \frac{\frac{1}{4} \cdot \frac{2}{4} \cdot \frac{3}{4} \cdot \frac{4}{9}}{\frac{5}{9} \cdot \frac{3}{9} \cdot \frac{4}{9}} = \underline{0,52}$$

Now, calculate the probability

$P(\text{not content} \mid \text{rich}=0, \text{marri.}=1, \text{health}=1)$

$$\frac{P(\text{rich}=0 \mid \text{not content}) \cdot P(\text{marr}=1 \mid \neg\text{content}) \cdot P(\text{health}=1 \mid \neg\text{content}) \, P(\neg\text{content})}{P(\text{rich}=0) \cdot P(\text{mar}=1) \cdot P(\text{health}=1)}$$

$$= \frac{\frac{4}{5} \cdot \frac{1}{5} \cdot \frac{1}{5} \cdot \frac{5}{9}}{\frac{5}{9} \cdot \frac{3}{9} \cdot \frac{4}{9}} = 0,22$$

So, the probability $\Rightarrow \dfrac{0,52}{0,52 + 0,22} = \underline{\underline{\%70}}$

NAÏVE BAYES

- $P(\text{content} \mid \text{rich}=0, \text{married}=1) = ?$

$$= \frac{P(\text{rich}=0 \mid \text{content}) \cdot P(\text{married}=1 \mid \text{content})\, P(\text{content})}{P(\text{rich}=0) \cdot P(\text{married}=1)}$$

$$= \frac{\frac{1}{4} \cdot \frac{2}{4} \cdot \frac{4}{9}}{\frac{5}{9} \cdot \frac{3}{9}} = \boxed{0{,}3}$$