Finding two largest

Code ———

Bound comparisons including constants

$1 + 2(n-2) = 2n-3$ = WORST CASE

Example

~~2 1 4 3 5 0~~

(2,1) (4,2) (4,3)

+1 +2 +2

3 1 4 2 5 0 $n = 6$

(3,1) (4,3) (4,3) (5,4) (5,4)

+1 +2 +1 +2 +1 = 7 comparisons

worst case $2 \cdot 6 - 3 = 9$ comparisons.

Turns out: Many real life instances require

~~O(n + logn)~~

$n + O(\log n)$ comparisons.

~~How to~~

Why?

What's the worst case input

A. $\langle 1, 2, 3, \ldots, n \rangle$ → new elt max

B. $\langle 1, 3, 2, 4, 6, 5, 7 \ldots n \rangle$ — new elt second max

unlikely inputs

A : there is only 1 in $n!$

B : there is one 1 in $n!$

How to reason about ~~the~~ "frequency of things"

- Probability is the common sense reduced to calculation

Laplace (1749–1827)

Let S = permute (S) } Don't have to permute in reality.

Run max2 on S

~~Input is of~~

We're modeling the world as throwing at us randomly permuted sequences

2

Finding the kth Smallest

Find the k^{th} smallest elt in a sequence

Example $\quad S = \langle 1, 2, 3, 3, 4, 5 \rangle$

3^{rd} smallest is 3 $\qquad$ 4th smallest $= 3$

$S = \langle 3, 3, 2, 1, 4, 5, 3 \rangle$

3^{rd} small is 3

4th ___ 3.

How to find the k^{th} smallest?

→ Algo 1: Sort & Pull $\qquad O(n \lg n)$

$O(\lg^2 n)$

→ Consider contraction.

Insert Algo 9.16.

$$\le \frac{2}{n^2} \cdot \frac{3n^2}{4} \cdot \frac{1}{2} \le \frac{3}{4}.$$

In other words we expect the largest fraction to decrease by a factor of 3/4 — Input decreases by at least a factor of 3/4.

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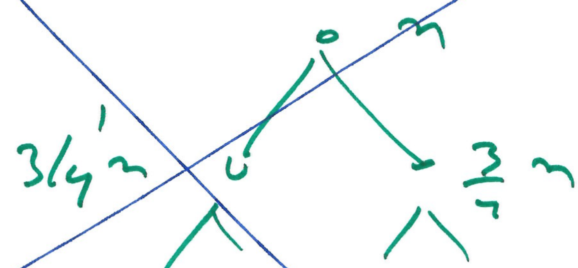
size at step i

Y_i The expected size of the i^{th} recursive call Y_i

$$E[Y_i] = \left(\frac{3}{4}\right)^i n$$



$$E[W(n)] = \sum_{i=0}^{n} \left(\frac{3}{4}\right)^i n = O(n).$$

$$X$$
$$\downarrow$$
$$3/4\,X$$
$$\downarrow$$
$$3/4 \cdot 3/4\,X$$
$$\downarrow$$
$$\vdots$$

Are we done? <u>No</u> The problem is that this fraction is only in expectation!

<u>what if ~~the~~ get unlucky</u>?

→ The idea is that the probabil we get unlu is small

Consider step $m = 10 \log_2 n$.

$$E[Y_m] = n \cdot \left(\frac{3}{4}\right)^{10 \log_2 n}.$$

$$= n \cdot \left(\cancel{4}/3\right)^{-10 \log_2 n} = n \cdot n^{\underbrace{-10 \cdot \log_2 4/3}_{-\cancel{4}.15}}$$

$$= n^{-3.15}$$

<u>Now</u> use Markov's inequality.

$$\Pr[X \geq \alpha] \leq \frac{E[X]}{\alpha}$$

$$\Pr[Y_m \geq 1] \leq E[Y_m]/1 = n^{-3.15}$$

$\Rightarrow$ The probability that the problem is not solved is $\leq \dfrac{1}{n^3}$

<u>Thus</u> <u>W.H.P.</u> # steps is $\Theta(\log n)$

Each step has $O(\log n)$ span

8.D

The $O(\log^2 n)$ span <u>W.H.P.</u>

$\Rightarrow$ equivalent Bound in Expectation

Review Kth Smallest

X: ratio of max(|L|, |R|) / |S|

show $E[X] = \frac{3}{4}$

$E[Y_i] = \left(\frac{3}{4}\right)^i$ because each step is independent (pivot ~~is chosen~~ ~~independently~~ different)

$$E[W(n)] = \sum_{i=1}^{n} \left(\frac{3}{4}\right)^i n = \Theta(n)$$

upper bound known.

SPAN ~~E~~ using markov to convert expectation into high probability

$$\Pr[X \geq \alpha] \leq \frac{E[X]}{\alpha}$$

Let Y_i = size of step i.

$$\Pr[Y_m \geq 1] \leq E[Y_m]/1$$

Let Y_m = size of input at m

$M = 10 \log_2 n \Rightarrow E[Y_m] = n^{-3.15}$

– QUICKSORT –

Insert 9.19 page 173

~~Call Pivot the representative~~

How to pick the pivot?

- First
- Median of three elt, first, middle, last
- Random elt.

Intuition: if we choose a random pivot we should get good balance

Why?

We'll choose the pivot as one of elt UNIFORMLY RANDOMLY.

PIVOT TREE.

Quicksort analysis

1 – One comparison instead of 3

2 – assign priorities.

uniformly random [0, 1].

pick the elt with the highest prio

to be the pivot

SOME OBSERVATIONS

1. A comparison involves a pivot & another key

2. A pivot is never sent to a recursive call.

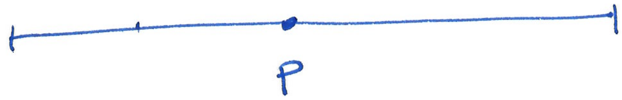
Once a pivot is considered it is not

involved in further comparison.

3. Two keys are never compared more than <u>once</u>.

why?

Consider one call



$p = T_i$ or $T_j \Rightarrow A_{ij} = 1$

~~p~~

$T_i < p < T_j \Rightarrow A_{ij} = 0$

$p < T_i \Rightarrow A_{ij}$ -- determined by recursive call

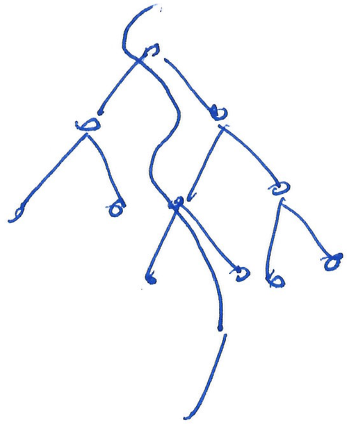
$p > T_j \Rightarrow$ —— " ——.

<u>Claim</u> For $i < j$ T_i & T_j are compared iff p_i & p_j has the highest among $\{p_i, \ldots, p_j\}$

$$\boxed{\Pr[A_{ij} = 1] = \frac{2}{j-i+1}}$$

Quicksort Span

Depth of the pivot tree.



Take any path = path kthsmallest (quickselect) takes.

Recall we proved that

$$Pr[Y_m \geq 1] \leq \frac{1}{n^3} \quad \text{for } m = 10 \log n$$

$\Rightarrow$ probability any path in quicksort tree depth $\geq m$ is $1/n^3$.

there are n paths in the tree

$$Pr[\underline{\text{any}} \text{ path} \geq m] \leq \frac{1}{n^3} \cdot n \leq \frac{1}{n^2}$$

Union bound