Noether's Theorem

Didi Park

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1 Preliminaries

Noether's theorem, in a sentence, states that for every differentiable (infinitesimal) symmetry, there exists a corresponding conservation law. In order to fully understand this, we must understand what symmetry means. We will define symmetry in terms of extremals and invariance.

2 Functionals

2.0.1 Formal statement

A functional is a mapping from a set of functions to the real numbers, given by

$$J = \int_a^b L(t, q_i, \dot{q}_i)dt.$$

The domain of the mapping is the set of twice differentiable functions $q(t)$ on the closed interval $[a, b]$.

The integrand of the functional, $L(t, q_i, \dot{q}_i)$, is called the *Lagrangian* of the functional.

Each q_i represents a generalized coordinate; $i = 1, 2, \ldots, N$.

3 Extremals

3.1 Euler-Lagrange Equation

Given functional

$$J = \int_a^b L(t, q_i, \dot{q}_i)dt,$$

The $\{q_i(t)\}$ that make J an extremal are the N solutions of the Euler-Lagrange equations

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}.$$

3.2 Canonically Conjugate Momentum

The momentum canonically conjugate to q_i is defined by

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}.$$

This allows us to rewrite the Euler-Lagrange equation as

$$\frac{\partial L}{\partial q_i} = \dot{p}_i,$$

which leads to a conservation law! p_i is conserved when $\frac{\partial L}{\partial q_i} = 0$. Using the chain rule, we take the derivative of the Lagrangian with respect to time:

$$\begin{aligned} \frac{dL}{dt} &= \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \\ &= \frac{\partial L}{\partial t} + \dot{p}_i\dot{q}_i + p_i\ddot{q}_i \\ &= \frac{d}{dt}[L - p_i\dot{q}_i]. \end{aligned}$$

This leads to…

3.3 The Hamiltonian

$$H \equiv H(t, q_i, p_i) \equiv p_i\dot{q}_i - L$$

The Hamiltonian is different from the Lagrangian in that it depends on the canonical momenta.
It also has its own symmetry:

$$H \text{ is constant iff } \frac{\partial L}{\partial t} = 0.$$

It gives us two ways of writing the Euler-Lagrange equation:

$$\frac{\partial L}{\partial t} = -\dot{H}$$

$$\frac{\partial L}{\partial q_i} = \dot{p}_i.$$

4 Invariance

The functional

$$J = \int_a^b L(t, q^\mu, \dot{q}^\mu) dt$$

is said to be invariant under the infinitesimal transformation

$$t' = t + \epsilon\tau + \dots,$$

$$q_i' = q_i + \epsilon\zeta_i + \dots$$

iff

$$J' - J \sim \epsilon^s, \text{ where } s > 1.$$

4.1 Rund-Trautman Identity

If the functional

$$J = \int_a^b L(t, q_i, \dot{q}_i) dt,$$

is invariant under the infinitesimal transformation

$$t' = t + \epsilon\tau + \dots,$$

$$q_i' = q_i + \epsilon\zeta_i + \dots,$$

then

$$\frac{\partial L}{\partial q_i}\zeta_i + p_i\dot{\zeta}_i + \frac{\partial L}{\partial t}\tau - H\dot{\tau} = 0.$$

Another equivalent form:

$$-(\zeta_i - \dot{q}_i\tau)[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}] = \frac{d}{dt}[p_i\zeta_i - H\tau].$$

5 Statement of the theorem

If the functional

$$J = \int_a^b L(t, q_i, \dot{q}_i) dt$$

is an extremal, and if under the infinitesimal transformation

$$t' = t + \epsilon\tau + \dots, q_i' = q_i + \epsilon\zeta_i + \dots$$

the functional is invariant according to the definition

$$L'\frac{dt'}{dt} - L = \epsilon\frac{dF}{dt} + O(\epsilon^s), \text{ where } s > 1,$$

then the following conservation law holds:

$$p + \mu\zeta^\mu - H\tau - F = \text{const.}$$

5.1 Cute proof

If J is an extremal, then the Euler-Lagrange equations hold:

$$\frac{\partial L}{\partial t} = -\dot{H}$$

$$\frac{\partial L}{\partial x_i} = \dot{p_i}.$$

If J is also invariant, then the Euler-Lagrange equations, when substituted into the Rund-Trautman identity, make

$$\frac{d}{dt}[p_i \zeta_i - H\tau - F] = 0.$$

Therefore,

$$p_i \zeta_i - H\tau - F = \text{const}.$$

5.2 Simple proof

Suppose that the functional

$$J = \int_{t_1}^{t_2} L(t, q_i, \dot{q}_i) dt.$$

is perturbed by a symmetric, infinitesimal translation $q_i \to q_i + \epsilon f_i(q)$.

$$\delta A = \int_{t_1}^{t_2} \frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i = 0$$

Integrating by parts, we get

$$= \int_{t_1}^{t_2} [\frac{\partial L}{\partial q_i} \delta q_i - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \delta q_i] + \frac{\partial L}{\partial \dot{q}_i} \Big|_{t_1}^{t_2}$$

By the Euler-Lagrange equation, the integrand equals 0.

$$= \frac{\partial L}{\partial \dot{q}_i} \Big|_{t_1}^{t_2} = 0.$$

This can be rewritten as

$$= \frac{\partial L}{\partial q_i} \epsilon f_i(q) = p_i f_i(q) = 0.$$

This proof, being simpler, gives a different conserved quantity from the first proof, but equivalently shows that there exists a corresponding conservation law. We call the function $f_i(q)$ a Noether charge.

5.3 Conservation of Energy

We translate a particle through time using the translation

$$q(t) \to q(t-\epsilon)$$

which implies

$$\delta q(t) = -\frac{dq}{dt}\epsilon$$
$$\delta q = -\dot{q}\epsilon$$

We assume that it is a symmetry, so

$$\delta L = 0$$
$$= \int_{t_1}^{t_2} dq \frac{\partial L}{\partial q}(-\dot{q}\epsilon) + \frac{\partial L}{\partial q}\delta\dot{q} + A - B$$

Integrating by parts, we obtain

$$= \int_{t_1}^{t_2} [\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}]\delta q + \delta q \frac{\partial L}{\partial \dot{q}}\Big|_{t_1}^{t_2} + A - B$$

By the Euler-Lagrange equation,

$$= \delta q \frac{\partial L}{\partial \dot{q}}\Big|_{t_1}^{t_2} + A - B$$

$$= -\epsilon\dot{q}\frac{\partial L}{\partial \dot{q}}\Big|_{t_1}^{t_2} + A - B$$

We know $A = L(t_2)\epsilon$ and $B = L(t_1)\epsilon$, so

$$= [L(t) - \epsilon\dot{q}\frac{\partial L}{\partial \dot{q}}]\Big|_{t_1}^{t_2}.$$

This is the conserved quantity, so we can simply negate it:

$$= [\epsilon\dot{x}\frac{\partial L}{\partial \dot{q}} - L(t)]\Big|_{t_1}^{t_2}$$

which is called the Hamiltonian H. It turns out that this equals energy. Setting $L(t) = K - U$,

$$= [\epsilon\dot{x}\dot{p} - \frac{1}{2}m\dot{x}^2 + U(x)]\Big|_{t_1}^{t_2}$$

$$= [m\dot{x}^2 - \frac{1}{2}m\dot{x}^2 + U(x)]\Big|_{t_1}^{t_2}$$

$$= [\frac{1}{2}m\dot{x}^2 + U(x)]\Big|_{t_1}^{t_2} = 0.$$

Yay, energy is conserved!

5.4 Conservation of Linear Momentum

We translate a particle using the infinitesimal translation

$$\delta x = -\epsilon$$

$$\delta y = 0$$

Assuming that this translation is a symmetry, we calculate the Noether charge:

$$\epsilon f_x = -\epsilon \Rightarrow f_x = -1$$

$$\epsilon f_y = 0 \Rightarrow f_y = 0$$

So the Noether charge (conserved quantity) is:

$$p_x \text{ (linear momentum).}$$

5.5 Conservation of Angular Momentum

We rotate a particle about the origin using the infinitesimal translation

$$\delta x = -\epsilon y$$

$$\delta y = \epsilon x.$$

Assuming that this translation is a symmetry, we calculate the Noether charge:

$$\epsilon f_x = -\epsilon y \Rightarrow f_x = -y$$

$$\epsilon f_y = \epsilon x \Rightarrow f_y = x$$

So the Noether charge (conserved quantity) is:

$$-p_x y + p_y x = L \text{ (angular momentum).}$$

6 Problems

1. Derive Snell's law
$$n_1 sin(\theta_1) = n_2 sin(\theta_2)$$
using the functional $T = \frac{1}{c}[n_1 s_1 + n_2 s_2]$.
Hint: Fermat's principle states that the extremal path T must be a minimum.

2. Consider the damped oscillator with Lagrangian
$$L = [\frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2] e^{\frac{bt}{m}}.$$
Apply the transformation
$$t' = t + \epsilon\tau, x' = x + \epsilon\zeta, \text{ where } \tau = 1 \text{ and } \zeta = \frac{-bx}{2m}.$$
What is the corresponding conservation law?

3. Consider the functional with Lagrangian $L = L(t, \dot{x}) = t\dot{x}^2$.
   (a) Find the canonical momentum and the Hamiltonian.
   (b) Using $\tau = At$ and $\zeta = x_0$, find the corresponding conservation law.

7 Further reading/Works Cited

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