BERNOULLI POLYNOMIALS

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1. Bernoulli Numbers

1.1. **Series definition.** The Bernoulli numbers are generated by the following series:

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}; |x| < 2\pi.$$

To find the value of the nth Bernoulli number B_n, we can take the nth derivative with respect to x at $x = 0$:

$$\left[\frac{d^n}{dx^n} (\frac{x}{e^x - 1}) \right]_{x=0} = B_n.$$

For example, we can find the values for B_0 and B_1:

$$B_0 = \frac{x}{e^x - 1}\bigg|_{x=0} = 1.$$

$$\begin{aligned} B_1 = \frac{d}{dx}(\frac{x}{e^x-1})\bigg|_{x=0} &= \frac{1}{e^x-1} - \frac{xe^x}{(e^x-1)^2}\bigg|_{x=0} \\ &= \frac{1}{x + \frac{x^2}{2} + \dots} - \frac{xe^x}{(x + \frac{x^2}{2} + \dots)^2}\bigg|_{x=0} \\ &= \frac{1}{x}(1 - \frac{x}{2} + \dots) - \frac{1}{x}(1 + x + \dots)(1 - \frac{x}{2} + \dots)^2\bigg|_{x=0} \\ &= -\frac{1}{2}. \end{aligned}$$

1.2. **Recursion relation.**