The Gamma Function

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[I'm starting a series of posts on the gamma function and its representations]

First off, what is the gamma function?

Well, it's like the factorial function, only extended to the complex numbers.

Recall that $n! = n(n-1)!^{\dagger}$

Anyway, back to the gamma function.

We want a function which can compute factorials for all real and complex numbers, not just non-negative integers.

Here's a clever way of writing $x!$:

$$x! = \frac{(n+x)!}{(n+x)(n-1+x)\dots(1+x)}$$

$$= \frac{(n+x)(n+x-1)(n+x-2)\dots(n+1)n!}{(n+x)(n-1+x)\dots(1+x)}$$

In the numerator, we're subtracting from x until x goes away, and in the denominator we're subtracting from n until n goes away.

Now, if we let $n >> x$, then the above expression equals

$$\lim_{n\to\infty} \frac{n^x n!}{(n+x)(n-1+x)\dots(1+x)}$$

And at this point, we've gotten an expression which we can evaluate at non-integer x, so we will go ahead and define the gamma function to be

$$\Gamma(z) = (z-1)!^{\ddagger} = \lim_{n\to\infty} \frac{n^z n!}{(n+z)(n-1+z)\dots(1+z)z}$$

There we go!

Comments: † You probably thought of it as $n(n-1)(n-2)\dots$, but this recursive definition is a little more useful because it allows us to figure out what $0!$ is; by the recursive definition, $1! = 1 \times 0!$, which implies that $0! = 1$ ‡ Why shifted to $(z-1)!$ rather than simply $z!$? It's going to make things nicer later on; just trust me.