The Discrete Fourier Transform

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1 The Fourier Transform

Let us define the Fourier transform $\hat{f}$, $f : \mathbb{R} \to \mathbb{C}$ to be

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} \, \mathrm{d}x, \text{ for real } \xi.$$

$f(x)$ must be square integrable, that is,

$$\int_{-\infty}^{\infty} \|f(x)\|^2 \, \mathrm{d}x < \infty.$$

The Fourier transform also has an easy inverse:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} \, \mathrm{d}\xi.$$

Speaking in linear algebra terms, the Fourier transform is a *linear transformation*. This can easily be seen from the linearity and scalar multiplicity of the integral. Both the domain and codomain, f and $\hat{f}$, effectively form infinite-dimensional vector spaces.

1.1 Applications

The Fourier transform relates a function's time or spatial domain to a frequency domain. It can be used to solve a wide variety of problems, including the following:

- Partial (and ordinary) differential equations: the Fourier transform of a function's derivative can be used to change a differential equation to an algebraic equation
- Quantum mechanics: Fourier transforms on the solutions to Schrödinger's equations form momentum space wave functions
- Signal processing: Fourier transforms are employed to filter out noise
- Data compression: JPEG's and mp3's, among other file formats, use Fourier transforms to compress data.

2 The Discrete Fourier Transform

As you can see, the Fourier transform is continuous, making it difficult for computers to perform the transform to high accuracy. It also requires that the function being transformed be square integrable, thus limiting the possible inputs significantly.
The discrete fourier transform (DFT) is discrete, making it much more conducive to computation. Unlike the Fourier transform, it maps a countable set of numbers to a new countable set of numbers. If you're only interested in transforming a small set of data, use the DFT.

2.1 Definition

We simply discretize the Fourier transform by changing the integral to a sum, appropriately:

$$X_k = \sum_{n=0}^{N-1} x_n e^{-2\pi i n \frac{k}{N}}.$$

Note that the $\frac{k}{N}$ serves the same purpose as ξ in the continuous case.
The inverse is given by:

$$x_k = \frac{1}{N} \sum_{n=0}^{N-1} X_k e^{2\pi i n \frac{k}{N}}.$$

2.2 Matrix representation

The DFT, unsurprisingly, can be written as a matrix transformation.

$$F_n = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & w & w^2 & \dots & w^{N-1} \\ 1 & w^2 & w^4 & \dots & w^{2(N-1)} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & w^{N-1} & w^{2(N-1)} & \dots & w^{(N-1)^2} \end{pmatrix}$$

where $w = e^{i\frac{2\pi}{N}}$.
The inverse transform is given by:

$$\frac{1}{N} F^*,$$

where * denotes the complex conjugate.
Notice that this matrix is symmetrical. In fact, its columns are also orthogonal (keep in mind that the complex dot product requires one to take the conjugate transpose).
These w's can best be thought of as complex roots of unity.

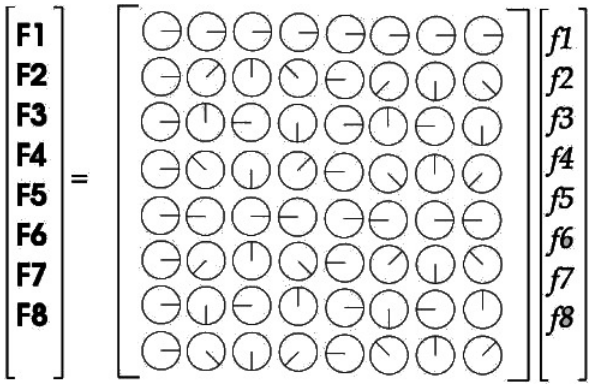

Figure 1: The zeroth row transforms by the 0th roots, first row 1st roots, second row square roots, etc.[6]

3 The Fast Fourier Transform

The DFT has efficiency $O(n^2)$. The fast Fourier transform (FFT) takes advantage of the symmetry and orthogonality of the DFT matrix, and recursively factors it until we reach an efficiency of $O(nlog(n))$. Let us take the example of the 8×8 DFT matrix, F_8.

$$F_8 = \begin{pmatrix} 1 & D \\ 1 & -D \end{pmatrix} \begin{pmatrix} F_4 & 0 \\ 0 & F_4 \end{pmatrix} P_8$$

Where

$$P_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & w & 0 & 0 \\ 0 & 1 & w^2 & 0 \\ 0 & 0 & 0 & w^3 \end{pmatrix}.$$

The permutation matrix splits the original F_8 into its even and odd columns.
Now we can recursively factorize by factorizing the F_4 matrices:

$$\begin{pmatrix} 1 & D & & \\ 1 & -D & & \\ & & 1 & D \\ & & 1 & -D \end{pmatrix} \begin{pmatrix} 1 & D & & 0 \\ 1 & -D & & \\ & 0 & 1 & D \\ & & 1 & -D \end{pmatrix} \begin{pmatrix} F_{16} & 0 & & \\ 0 & F_{16} & & \\ & & F_{16} & 0 \\ & & 0 & F_{16} \end{pmatrix} \begin{pmatrix} F_{16} & 0 & & 0 \\ 0 & F_{16} & & \\ & 0 & F_{16} & 0 \\ & & 0 & F_{16} \end{pmatrix} \begin{pmatrix} P_4 & 0 \\ 0 & P_4 \end{pmatrix} P_{64}$$

...and we find that the efficiency of this process is $O(nlog_2(n))$.

4 Further reading/Works Cited

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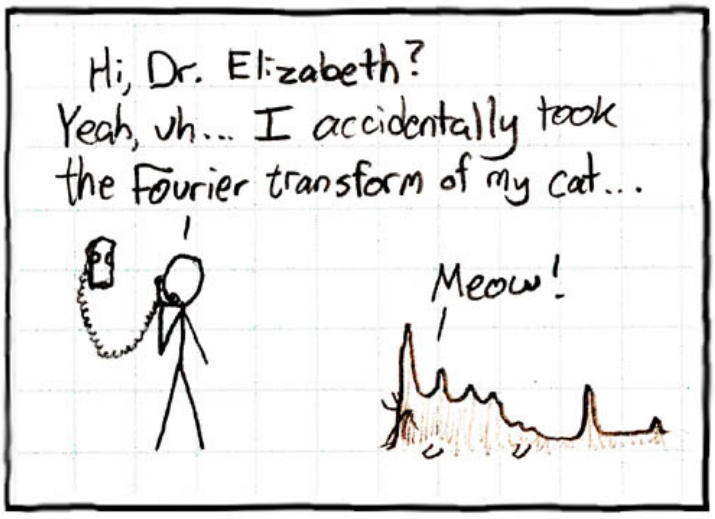

http://xkcd.com/26/