*5. Write a program that implements collisions by looking ahead, that is, extrapolating the free-fall solutions (3.17) forward in time to determine where and when the next collision will take place. Then treat the collision as described above. Compare your results with the method used in this section.

3.7 Behavior in the Frequency Domain: Chaos and Noise

Our intuitive ideas concerning what it means to be chaotic usually include some connection with terms such as *random*, *unpredictable*, and *noisy*. We have already explored the first two notions; in this section we consider how chaotic behavior is connected with noise. For this we require several tools for dealing with time-dependent signals. These tools are discussed in Appendix 2 and rely on the Fourier transform.[24]

Our goal in this section is to Fourier analyze the time-dependent signals obtained in our simulations of the damped, nonlinear pendulum in Section 3.2. We will consider only the signals associated with the angular position of the pendulum, $\theta(t)$, although the same sort of analysis could be used with the angular velocity, $\omega(t)$. Such a signal can, in general, be a complicated function of time. Nevertheless, we show in Appendix 2 how it can always be decomposed into component waveforms that are simple sines and cosines.

A program to calculate this Fourier decomposition is given in Appendix 4 and employs the fast Fourier transform (FFT) algorithm. Here we will be concerned only with the frequency spectrum associated with $\theta(t)$, which can be derived using the power-spectrum program given in the appendix. Here the term *power* is used in the following sense. Typical signals of interest include the amplitudes of pressure waves, electrical voltages, and light waves. In such cases the square of the amplitude of a particular frequency component of the signal is proportional to the *intensity* of the signal at that frequency, which is in turn proportional to the power carried by the signal. An examination of the intensity as a function of frequency leads to the *power spectrum* discussed in Appendix 2. This term is commonly used, even in cases (such as the present one) where the connection with power and energy is not direct.[25]

The power spectra of several pendulum waveforms are shown in Figure 3.24, where we show the power spectrum of $\theta(t)$ as a function of frequency. The area under each peak in the spectrum is proportional to the effective power, that is, the sum of the squares of the corresponding Fourier components of the signal. At low drive, $F_D = 0.5$, the pendulum is in a period-1 state, in which the $\theta(t)$ waveform is very close to a simple sine wave. The FFT result shows a single peak at the frequency of this sine wave, that is, at the drive frequency. This is completely analogous to what we found for the FFT of a sine wave in Appendix 2. At somewhat higher drive $F_D = 0.95$, the behavior is again period-1, and we see that the power spectrum is again dominated by a single peak at the drive frequency. We also notice a very small peak at approximately three times the drive frequency (≈ 0.3 Hz). This peak is not a plotting error! It is produced by the nonlinearity of the waveform at this drive and is an example of the phenomenon of nonlinear mixing we mentioned earlier. The most interesting result is found in the chaotic regime, $F_D = 1.2$. The spectrum is now very complicated as the power is broadly

[24] Readers who are not familiar with the concept of Fourier analysis should review Appendix 2 before tackling this section.

[25] Although with the pendulum, the power spectrum of $\theta(t)$ will be closely connected with the kinetic energy of the system.

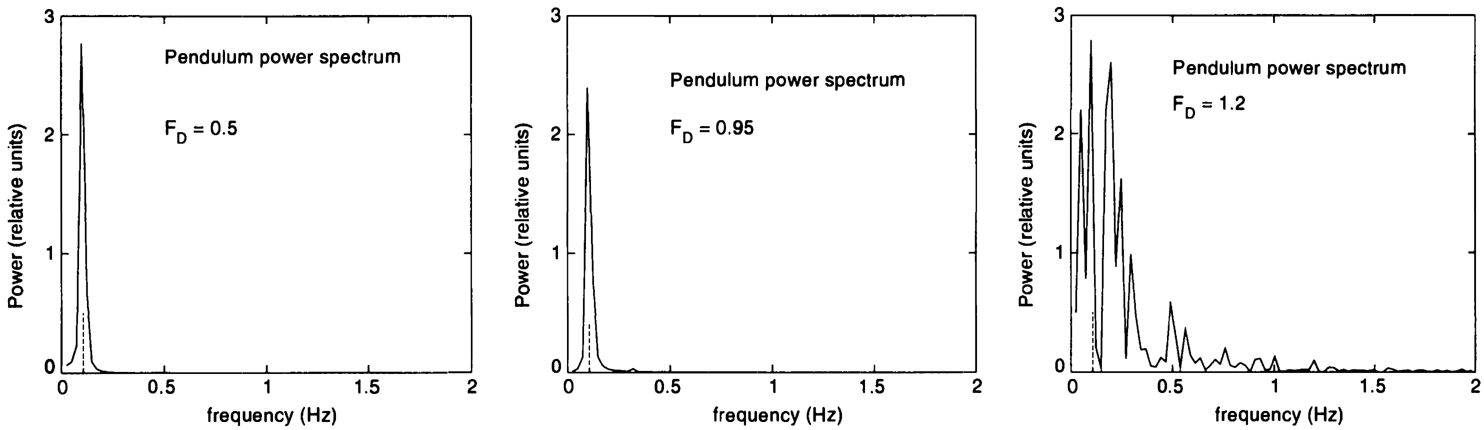

Figure 3.24: Fourier analysis of the results for $\theta(t)$ for the nonlinear pendulum at different values of the driving force. At $F_D = 0.5$ and $F_D = 0.95$ the pendulum is in the period-1 regime, while for $F_D = 1.2$ it is chaotic. The dashed lines indicate the drive frequency.

distributed over a wide range of frequencies. This is just the noise that we intuitively expect to find in a chaotic system.

It is also interesting to examine the power spectra when the pendulum undergoes period-doubling on its way to the chaotic regime. Results from the period-2 regime are shown in Figure 3.25. There is a large peak at the drive frequency $\Omega_D/2\pi \approx 0.1$ Hz, with additional peaks at integer multiples of this frequency, as we again have the nonlinear mixing observed earlier. However, there is now a strong component at half this frequency, ≈ 0.05 Hz. This corresponds to a component with twice the period

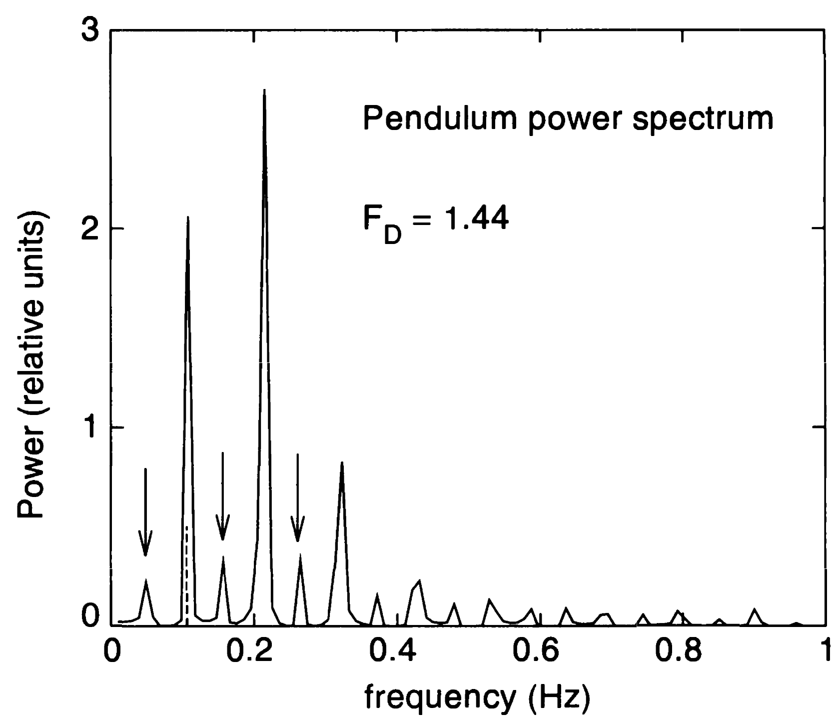

Figure 3.25: Fourier analysis of the results for $\theta(t)$ for the nonlinear pendulum in the period-2 regime. The dashed line indicates the drive frequency. The arrows indicate the subharmonic (period-doubled) component, which appears at half the drive frequency, and some of its harmonics.

and is, therefore, the result of *period-doubling*. A similar analysis can be used to examine the behavior in the period-4, period-8, etc., regimes. If the behavior is period-n, there will be a spectral component at a frequency $1/n$ times the drive frequency. Hence, spectral analysis reveals the period-doubling route to chaos in an extremely clear manner.

The key result of this section is that much can be learned about the behavior of a system by examination of its frequency spectrum. Here we have used this approach to study the pendulum in and near its chaotic states. In later chapters we will use the same method in connection with several other problems.

Exercises

1. In Figure 3.24 we saw that at a relatively high drive, $F_D = 0.95$, there was a small, but noticeable response of the pendulum at three times the frequency of the driving force. Calculate the size of this component as a function of the drive force in the range $F_D = 0.95 - 1.00$. Try also to observe a component at five times the drive frequency. The process in which these signals at multiples of the drive frequency are produced is an example of mixing.

2. Analyze the behavior of the nonlinear pendulum in the period-4 regime and show that the spectral component with the lowest frequency has a frequency of one-fourth the drive frequency.

*3. We saw in connection with Figures 3.8 and 3.9 that every time a period-doubling threshold is crossed a new subharmonic component is added to the $\theta(t)$ waveform. The size of this component can be readily extracted using the Fourier transform. Calculate $\theta(t)$ for values of the drive amplitude near the period-2 transition in Figure 3.9. Then use the FFT to obtain the amplitude of the period-2 component as a function of F_D. Try to determine the functional form that describes the way in which this amplitude vanishes at the transition.

4. Analyze the power spectrum of $\omega(t)$ of the nonlinear pendulum for different values of the driving force.

5. Calculate the frequency spectra for the waveforms $z(t)$ for the Lorenz model. Compare the behavior in the chaotic, nonchaotic, intermittent, and period-doubled regimes.

References

CROMER, A. 1981. "Stable Solutions using the Euler Approximation." *Am. J. Phys.* **49**, 455. Discusses the Euler-Cromer method, and shows analytically that it conserves energy for oscillatory problems.

BAKER, G. L. and J. P. GOLLUB. 1990. *Chaotic Dynamics: an Introduction*. Cambridge: Cambridge University Press. A very readable introduction to chaotic behavior in simple systems. Discusses the logistic map extensively and also has a nice description of the damped nonlinear pendulum.

HILBORN, R. C. 1994. *Chaos and Nonlinear Dynamics: An Introduction for Scientists and Engineers*. Oxford: Oxford University Press. A thorough introduction to chaos. Goes into much more depth than we have had space to discuss in this chapter.

LORENZ, E. N. 1963. "Deterministic Nonperiodic Flow." *J. Atmos. Sci.* **20**, 130 The classic treatment of the Lorenz equations.

MAY, R. M. 1976. "Simple Mathematical Models with very Complicated Dynamics." *Nature* **261**, 459. A nice introduction to interated maps, such as the logistic map (3.11).