

SERIE Y TRANSFORMADA DE FOURIER

FUNCIONES ORTOGONALES. SERIES DE FOURIER. EJEMPLOS

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FUNCIONES ORTOGONALES

Secuencia infinita $\{\phi_n(x)\}$
integrable en $[a, b]$ y

$$\int_a^b \phi_i(x)\phi_j(x) dx = 0, \quad i \neq j$$

Supongamos que existe

$$\int_a^b f(x) dx$$

y

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

$$c_3 = ?$$

$$f(x)\phi_3(x) = \sum_{i=1}^{\infty} c_i \phi_i(x)\phi_3(x)$$

$$\begin{aligned} \int_a^b f(x)\phi_3(x) dx &= \\ \int_a^b \sum_{i=1}^{\infty} c_i \phi_i(x)\phi_3(x) dx &= \\ \sum_{i=1}^{\infty} \int_a^b c_i \phi_i(x)\phi_3(x) dx &= \\ c_3 \int_a^b \phi_3^2(x) dx \end{aligned}$$

$$c_k = \frac{\int_a^b f(x)\phi_k(x) dx}{\int_a^b \phi_k^2(x) dx} \quad (1)$$

Definición

Producto interno:

$$\langle \phi_i, \phi_j \rangle = \int_a^b \phi_i(x)\phi_j(x) dx$$

Con c_k definida por (??):

$$f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x) = F(x)$$

$F(x)$ es la **representación de Fourier** de $f(x)$ con respecto de $\{\phi_n(x)\}$.

Se puede demostrar:

$$\begin{aligned} \int_a^b \left[f(x) - \sum_{k=1}^n c_k \phi_k(x) \right]^2 dx &\leq \\ \int_a^b \left[f(x) - \sum_{k=1}^n d_k \phi_k(x) \right]^2 dx \end{aligned}$$

Caso especial:

$$\{\cos x, \cos 2x, \cos 3x, \dots, \\ \sin x, \sin 2x, \sin 3x, \dots\}$$

ortogonales en $[-\pi, \pi]$.

Por ejemplo:

$$\begin{aligned} \int_{-\pi}^{\pi} \cos mx \cos nx \, dx &= \\ \frac{1}{2} \int_{-\pi}^{\pi} [\cos(m+n)x + \cos(m-n)x] \, dx &= \\ \frac{1}{2} \left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n} \right]_{x=-\pi}^{\pi} &= 0 \end{aligned}$$

Resultado clave:

Supongamos que $f(x)$ sea suave a tramos en $[-\pi, \pi]$, y que

$$F(x) = \sum_{n=0}^{\infty} a_n \cos nx + \sum_{n=0}^{\infty} b_n \sin nx$$

es la representación de Fourier de $f(x)$. Entonces, para $x \in [-\pi, \pi]$:

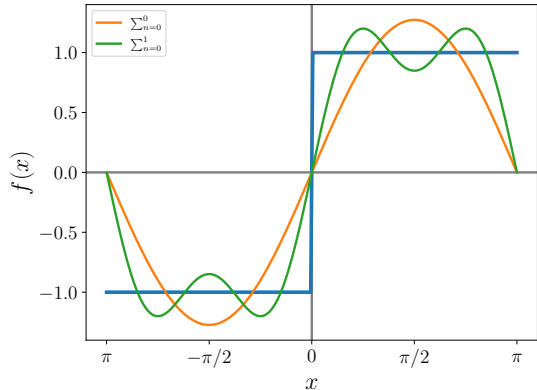
$$F(x) = \frac{f(x^+) + f(x^-)}{2}$$

Ejemplo:

$$f(x) = \begin{cases} -1, & -\pi \leq x < 0 \\ 1, & 0 \leq x < \pi \end{cases}$$

$$\begin{aligned}
 F(x) &= \frac{4}{\pi} \sum_{n \text{ par}}^{\infty} \frac{\text{sen } nx}{n} = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\text{sen}(2n+1)x}{2n+1} \\
 &= \frac{4}{\pi} \left(\text{sen } x + \frac{\text{sen } 3x}{3} + \frac{\text{sen } 5x}{5} + \dots \right)
 \end{aligned}$$

Primeros dos términos:

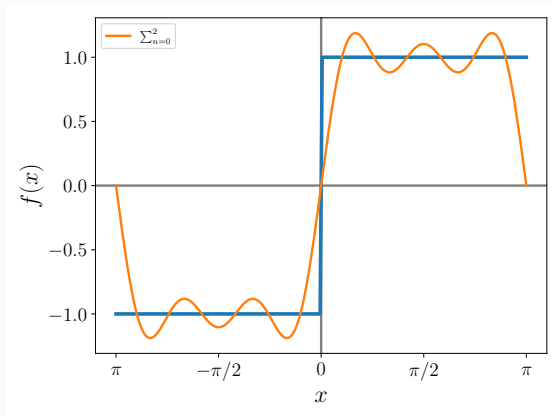
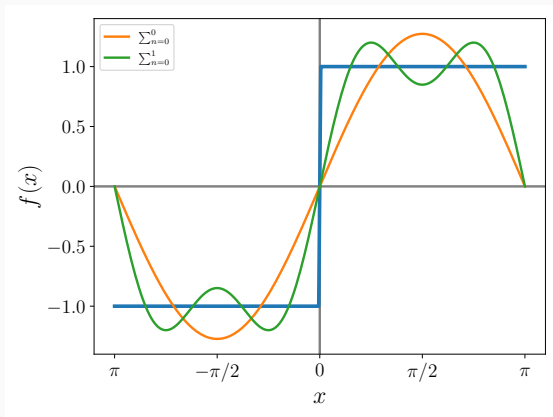


Agregando más términos:

$$F(x) = \frac{4}{\pi} \sum_{n \text{ par}} \frac{\text{sen } nx}{n} = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\text{sen}(2n+1)x}{2n+1}$$

$$= \frac{4}{\pi} \left(\text{sen } x + \frac{\text{sen } 3x}{3} + \frac{\text{sen } 5x}{5} + \dots \right)$$

Primeros dos términos:



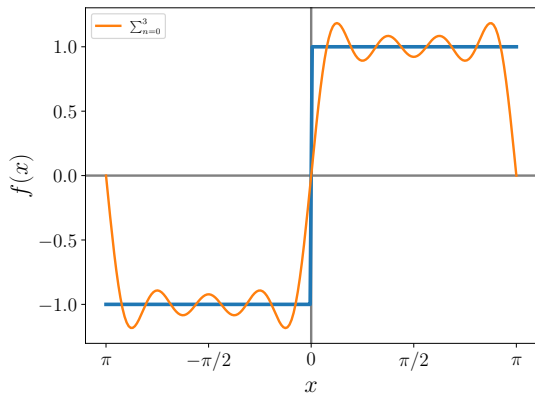
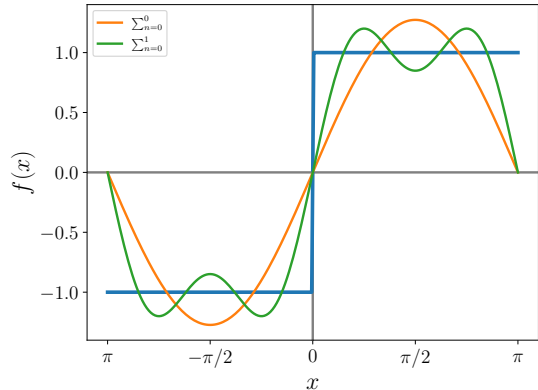
$$F_2(x) = \frac{4}{\pi} \left(\text{sen } x + \frac{\text{sen } 3x}{3} + \frac{\text{sen } 5x}{5} \right)$$

Agregando más términos:

$$F(x) = \frac{4}{\pi} \sum_{n \text{ par}} \frac{\text{sen } nx}{n} = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\text{sen}(2n+1)x}{2n+1}$$

$$= \frac{4}{\pi} \left(\text{sen } x + \frac{\text{sen } 3x}{3} + \frac{\text{sen } 5x}{5} + \dots \right)$$

Primeros dos términos:



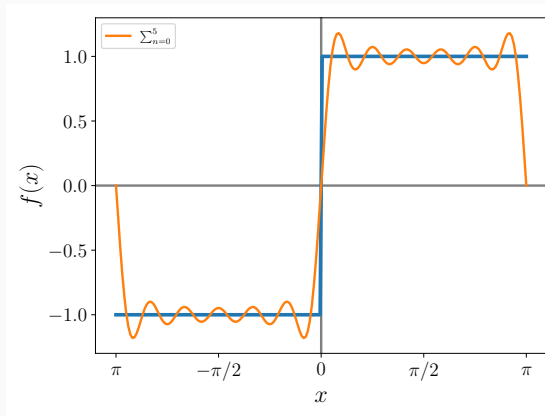
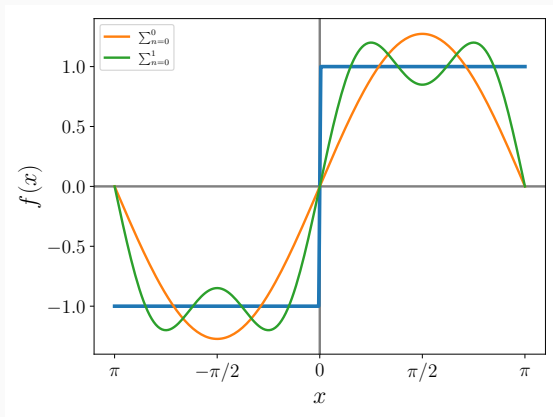
$$F_3(x) = \frac{4}{\pi} \left(\text{sen } x + \frac{\text{sen } 3x}{3} + \frac{\text{sen } 5x}{5} + \frac{\text{sen } 7x}{7} \right)$$

Agregando más términos:

$$F(x) = \frac{4}{\pi} \sum_{n \text{ par}} \frac{\text{sen } nx}{n} = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\text{sen}(2n+1)x}{2n+1}$$

$$= \frac{4}{\pi} \left(\text{sen } x + \frac{\text{sen } 3x}{3} + \frac{\text{sen } 5x}{5} + \dots \right)$$

Primeros dos términos:

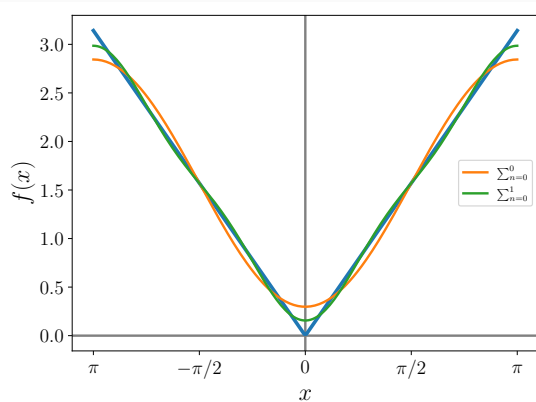


$$F_5(x) = \frac{4}{\pi} \left(\dots + \frac{\text{sen } 9x}{9} + \frac{\text{sen } 11x}{11} + \frac{\text{sen } 13x}{13} \right)$$

Otro ejemplo:

$$f(x) = |x|, \quad -\pi < x < \pi$$

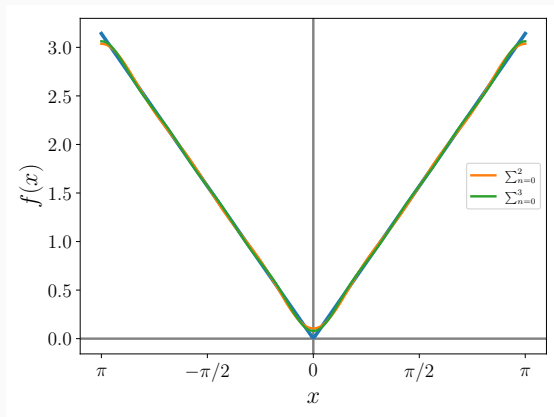
$$F(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ impar}}^{\infty} \frac{\cos nx}{n^2}$$



Otro ejemplo:

$$f(x) = |x|, \quad -\pi < x < \pi$$

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Comparación con serie de potencias:

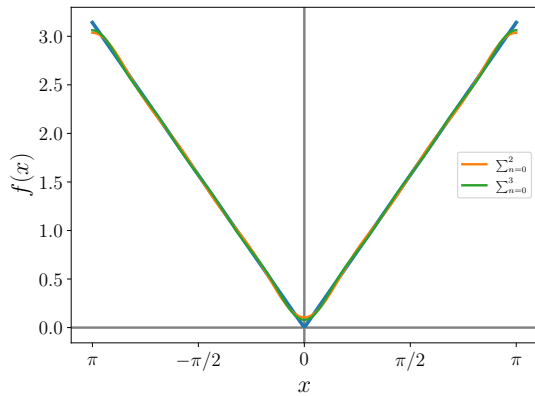
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \dots$$

Otro ejemplo:

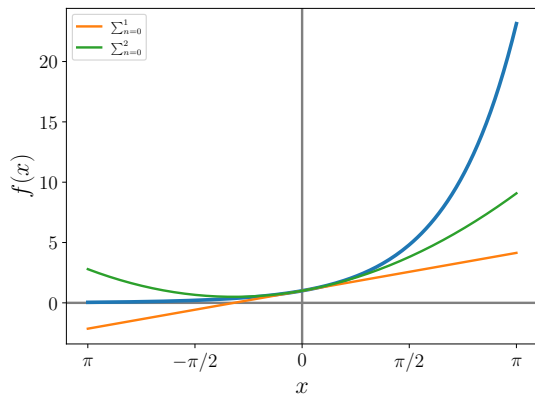
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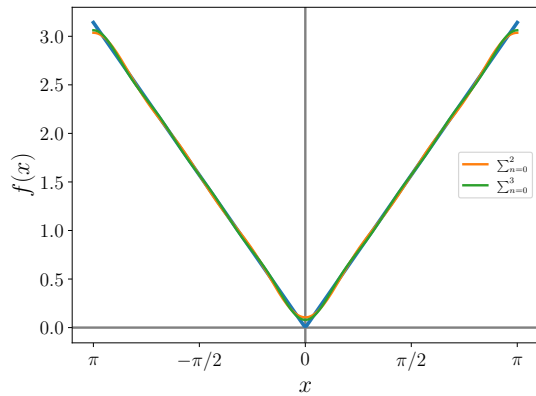
$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \dots \end{aligned}$$



Otro ejemplo:

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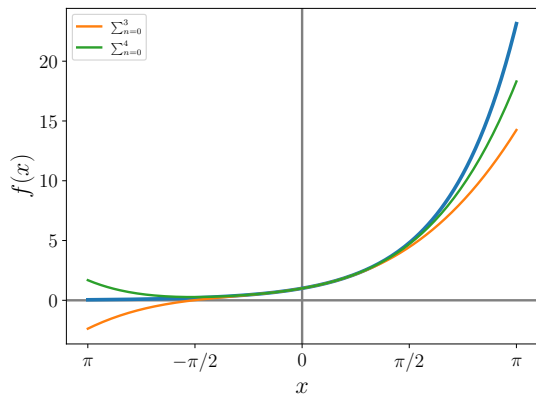
$$F(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ impar}}^{\infty} \frac{\cos nx}{n^2}$$



Comparación con serie de potencias:

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- ▶ Peter V O'Neil. *Advanced Engineering Mathematics*. 7.^a ed. Mason (OH), USA: CENGAGE Learning Custom Publishing, 2011. Capítulo 13.
- ▶ K A Stroud y Dexter J Booth. *Advanced Engineering Mathematics*. 6.^a ed. London, England: Bloomsbury Academic, 2020. Capítulos 7 y 8.