

# SERIE Y TRANSFORMADA DE FOURIER

## FUNCIONES ORTOGONALES. SERIES DE FOURIER. EJEMPLOS

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 · X<sub>Y</sub>LaTeX · 

# FUNCIONES ORTOGONALES

Secuencia infinita  $\{\phi_n(x)\}$   
integrable en  $[a, b]$  y

$$\int_a^b \phi_i(x)\phi_j(x) dx = 0, \quad i \neq j$$

Supongamos que existe

$$\int_a^b f(x) dx$$

y

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

$$c_3 = ?$$

$$f(x)\phi_3(x) = \sum_{i=1}^{\infty} c_n \phi_n(x)\phi_3(x)$$

$$\begin{aligned} \int_a^b f(x)\phi_3(x) dx &= \\ \int_a^b \sum_{i=1}^{\infty} c_n \phi_n(x)\phi_3(x) dx &= \\ \sum_{i=1}^{\infty} \int_a^b c_n \phi_n(x)\phi_3(x) dx &= \\ c_3 \int_a^b \phi_3^2(x) dx \end{aligned}$$

$$c_k = \frac{\int_a^b f(x)\phi_k(x) dx}{\int_a^b \phi_k^2(x) dx} \quad (1)$$

## Definición

Producto interno:

$$\langle \phi_i, \phi_j \rangle = \int_a^b \phi_i(x)\phi_j(x) dx$$

Con  $c_k$  definida por (1):

$$f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x) = F(x)$$

$F(x)$  es la **representación de Fourier** de  $f(x)$  con respecto de  $\{\phi_n(x)\}$ .

Se puede demostrar:

$$\begin{aligned} \int_a^b \left[ f(x) - \sum_{k=1}^n c_k \phi_k(x) \right]^2 dx &\leq \\ \int_a^b \left[ f(x) - \sum_{k=1}^n d_k \phi_k(x) \right]^2 dx \end{aligned}$$

### Caso especial:

$$\{\cos x, \cos 2x, \cos 3x, \dots, \\ \sin x, \sin 2x, \sin 3x, \dots\}$$

ortogonales en  $[-\pi, \pi]$ .

Por ejemplo:

$$\begin{aligned} \int_{-\pi}^{\pi} \cos mx \cos nx \, dx &= \\ \frac{1}{2} \int_{-\pi}^{\pi} [\cos(m+n)x + \cos(m-n)x] \, dx &= \\ \frac{1}{2} \left[ \frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n} \right]_{x=-\pi}^{\pi} &= 0 \end{aligned}$$

### Resultado clave:

Supongamos que  $f(x)$  sea suave a tramos en  $[-\pi, \pi]$ , y que

$$F(x) = \sum_{n=0}^{\infty} a_n \cos nx + \sum_{n=0}^{\infty} b_n \sin nx$$

es la representación de Fourier de  $f(x)$ . Entonces, para  $x \in [-\pi, \pi]$ :

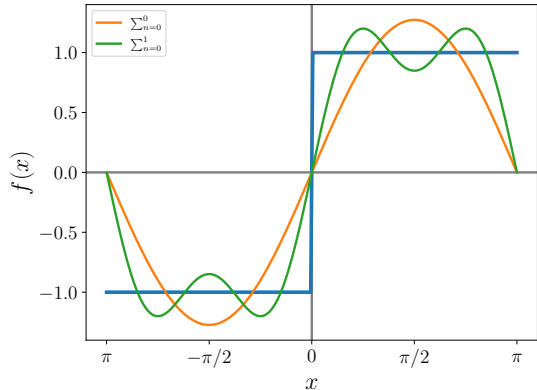
$$F(x) = \frac{f(x^+) + f(x^-)}{2}$$

### Ejemplo:

$$f(x) = \begin{cases} -1, & -\pi \leq x < 0 \\ 1, & 0 \leq x < \pi \end{cases}$$

$$\begin{aligned}
 F(x) &= \frac{4}{\pi} \sum_{n \text{ par}}^{\infty} \frac{\text{sen } nx}{n} = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\text{sen}(2n+1)x}{2n+1} \\
 &= \frac{4}{\pi} \left( \text{sen } x + \frac{\text{sen } 3x}{3} + \frac{\text{sen } 5x}{5} + \dots \right)
 \end{aligned}$$

Primeros dos términos:

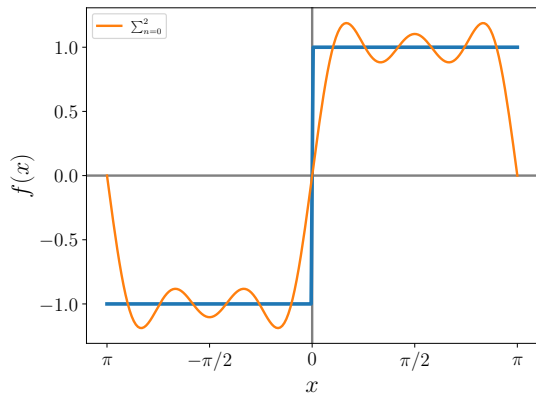
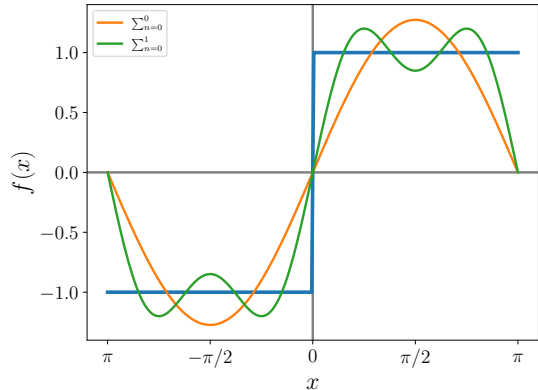


Agregando más términos:

$$F(x) = \frac{4}{\pi} \sum_{n \text{ par}} \frac{\text{sen } nx}{n} = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\text{sen}(2n+1)x}{2n+1}$$

$$= \frac{4}{\pi} \left( \text{sen } x + \frac{\text{sen } 3x}{3} + \frac{\text{sen } 5x}{5} + \dots \right)$$

Primeros dos términos:



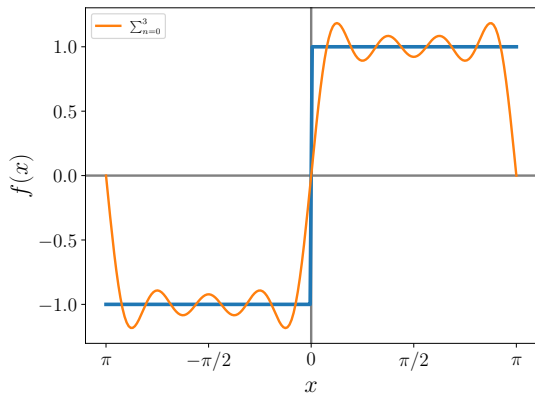
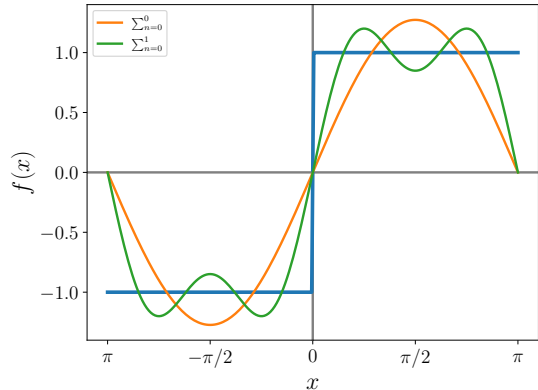
$$F_2(x) = \frac{4}{\pi} \left( \text{sen } x + \frac{\text{sen } 3x}{3} + \frac{\text{sen } 5x}{5} \right)$$

Agregando más términos:

$$F(x) = \frac{4}{\pi} \sum_{n \text{ par}} \frac{\text{sen } nx}{n} = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\text{sen}(2n+1)x}{2n+1}$$

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Primeros dos términos:



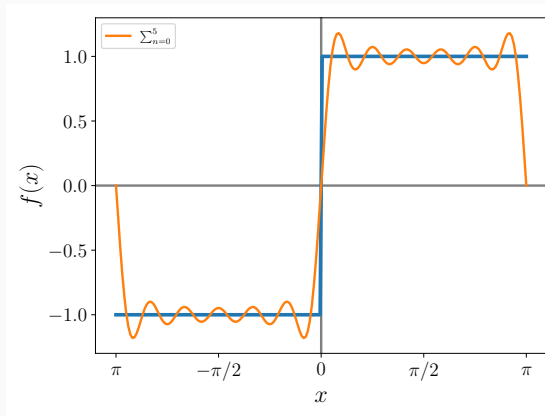
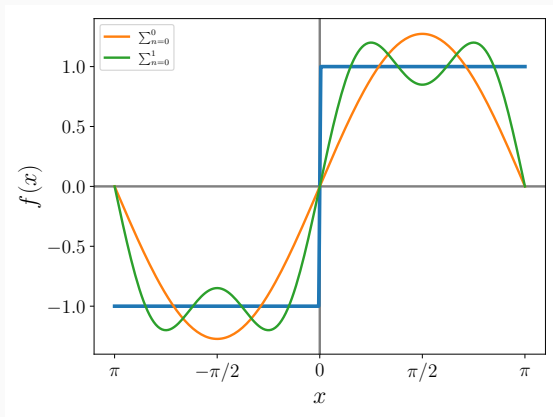
$$F_3(x) = \frac{4}{\pi} \left( \text{sen } x + \frac{\text{sen } 3x}{3} + \frac{\text{sen } 5x}{5} + \frac{\text{sen } 7x}{7} \right)$$

Agregando más términos:

$$F(x) = \frac{4}{\pi} \sum_{n \text{ par}} \frac{\text{sen } nx}{n} = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\text{sen}(2n+1)x}{2n+1}$$

$$= \frac{4}{\pi} \left( \text{sen } x + \frac{\text{sen } 3x}{3} + \frac{\text{sen } 5x}{5} + \dots \right)$$

Primeros dos términos:

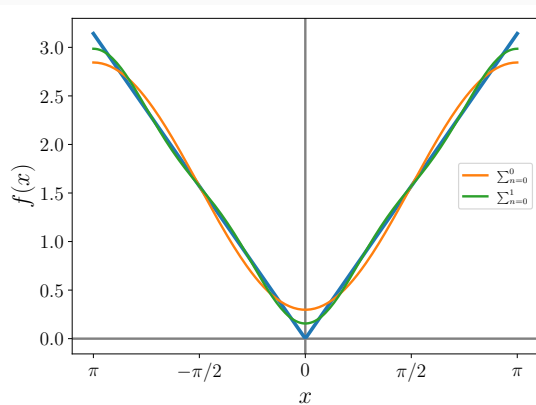


$$F_5(x) = \frac{4}{\pi} \left( \dots + \frac{\text{sen } 9x}{9} + \frac{\text{sen } 11x}{11} + \frac{\text{sen } 13x}{13} \right)$$

### Otro ejemplo:

$$f(x) = |x|, \quad -\pi < x < \pi$$

$$F(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ impar}}^{\infty} \frac{\cos nx}{n^2}$$

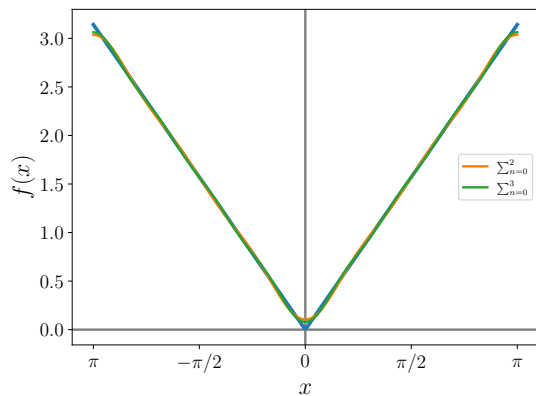




Otro ejemplo:

$$f(x) = |x|, \quad -\pi < x < \pi$$

$$F(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ impar}}^{\infty} \frac{\cos nx}{n^2}$$



Comparación con serie de potencias:

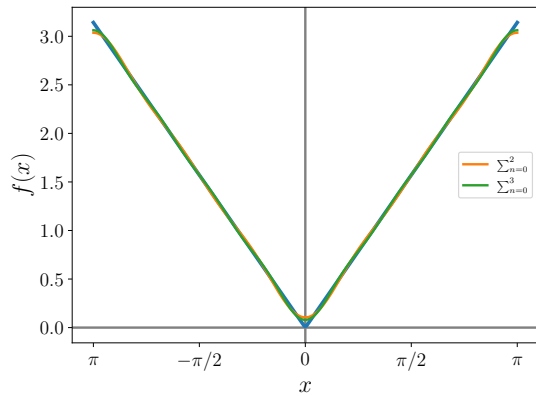
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \dots$$

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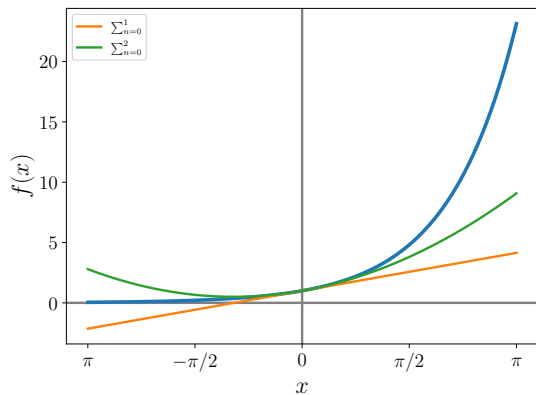
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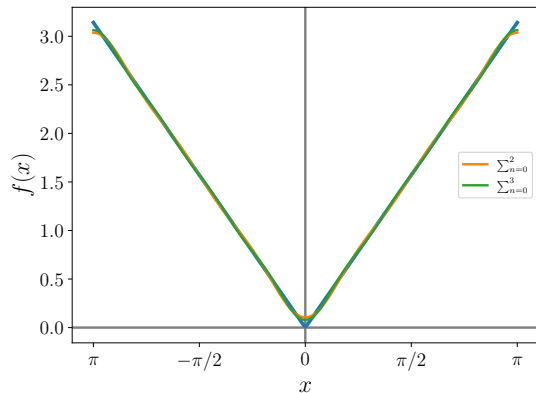
$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \dots$$



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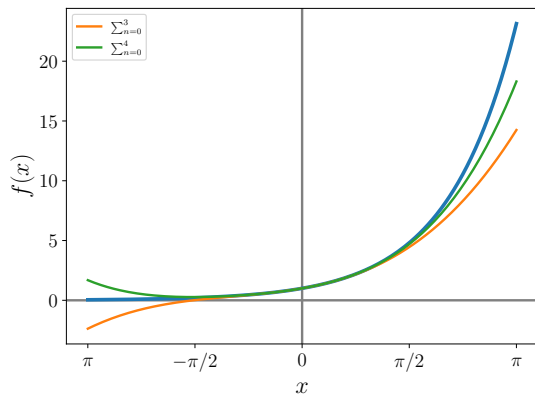
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- ▶ Peter V O'Neil. *Matemáticas avanzadas para ingeniería*. 7.<sup>a</sup> ed. México, DF: CENGAGE Learning Custom Publishing, 2012. Capítulo 2.
- ▶ K A Stroud y Dexter J Booth. *Advanced Engineering Mathematics*. 6.<sup>a</sup> ed. London, England: Bloomsbury Academic, 2020. Capítulos 7 y 8.