Serie y Transformada de Fourier

Funciones ortogonales. Series de Fourier. Ejemplos

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Funciones ortogonales

Secuencia infinita $\{\phi_n(x)\}$
integrable en $[a, b]$ y

$$\int_a^b \phi_i(x)\phi_j(x)\,dx = 0, \quad i \neq j$$

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$$c_3 \int_a^b \phi_3^2(x)\, dx$$

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Definición : .

Producto interno:

$$\langle f, g\rangle = \int_a^b f(x)\,g(x)\,dx$$

Con c_k definida por (1):

$$f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x) = F(x)$$

$F(x)$ es la **representación de Fourier** de $f(x)$ con respecto de $\{\phi_n(x)\}$.

Se puede demostrar:

$$\int_a^b \left[f(x) - \sum_{k=1}^{n} c_k \phi_k(x)\right]^2 dx \leq$$

$$\int_a^b \left[f(x) - \sum_{k=1}^{n} d_k \phi_k(x)\right]^2 dx$$

Caso especial:

$$\{\cos x, \cos 2x, \cos 3x, \cdots ,$$
$$\operatorname{sen} x, \operatorname{sen} 2x, \operatorname{sen} 3x, \cdots\}$$

ortogonales en $[-\pi, \pi]$.
Por ejemplo:

$$\int_{-\pi}^{\pi} \cos mx \, \cos nx \, dx =$$
$$\frac{1}{2} \int_{-\pi}^{\pi} [\cos(m+n)x + \cos(m-n)x] \, dx =$$
$$\frac{1}{2} \left[\frac{\operatorname{sen}(m+n)x}{m+n} + \frac{\operatorname{sen}(m-n)x}{m-n} \right]_{x=-\pi}^{\pi}$$
$$= 0$$

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Resultado clave:
Supongamos que $f(x)$ sea suave a tramos en $[-\pi, \pi]$, y que

$$F(x) = \sum_{n=0}^{\infty} a_n \cos nx + \sum_{n=0}^{\infty} b_n \operatorname{sen} nx$$

es la representación de Fourier de $f(x)$. Entonces, para $x \in [-\pi, \pi]$:

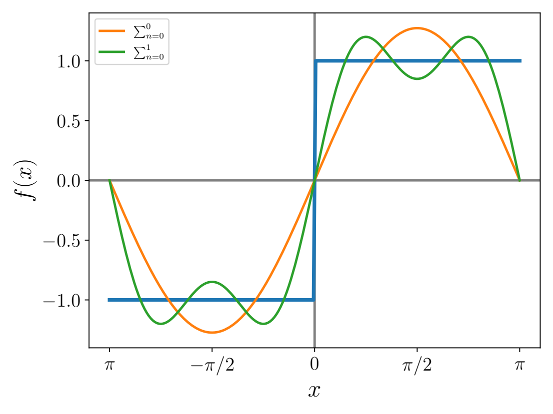
$$F(x) = \frac{f(x^+) + f(x^-)}{2}$$

Ejemplo:

$$f(x) = \begin{cases} -1, & -\pi \leq x < 0 \\ 1, & 0 \leq x < \pi \end{cases}$$

$$F(x) = \frac{4}{\pi}\sum_{n \text{ par}}^{\infty} \frac{\operatorname{sen} nx}{n} = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\operatorname{sen}(2n+1)x}{2n+1}$$
$$= \frac{4}{\pi}\left(\operatorname{sen} x + \frac{\operatorname{sen} 3x}{3} + \frac{\operatorname{sen} 5x}{5} + \cdots\right)$$

Primeros dos términos:



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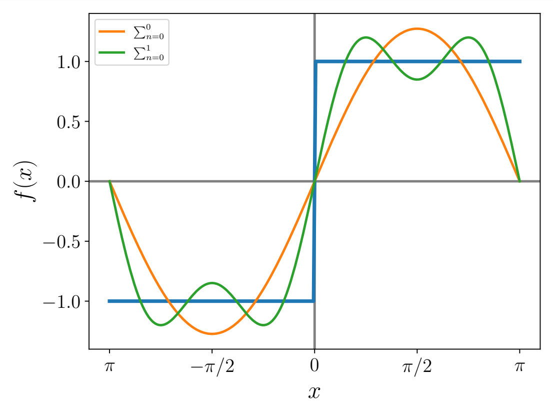
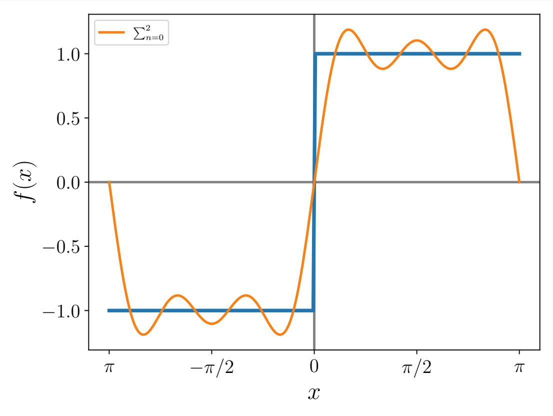


Agregando más términos:



$$F_2(x) = \frac{4}{\pi} \left(\operatorname{sen} x + \frac{\operatorname{sen} 3x}{3} + \frac{\operatorname{sen} 5x}{5} \right)$$

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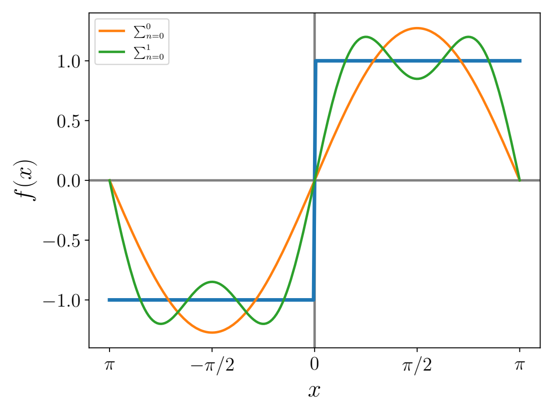

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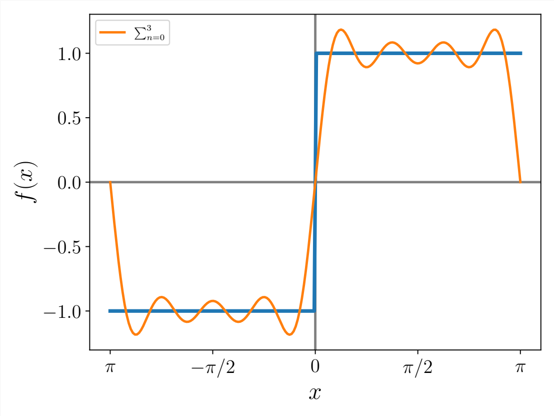

$$F_3(x) = \frac{4}{\pi}\left(\operatorname{sen} x + \frac{\operatorname{sen} 3x}{3} + \frac{\operatorname{sen} 5x}{5} + \frac{\operatorname{sen} 7x}{7}\right)$$

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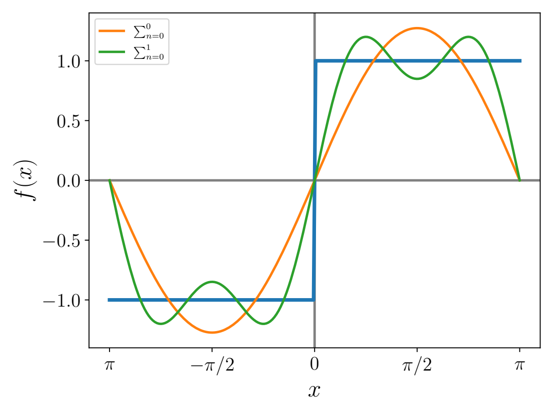

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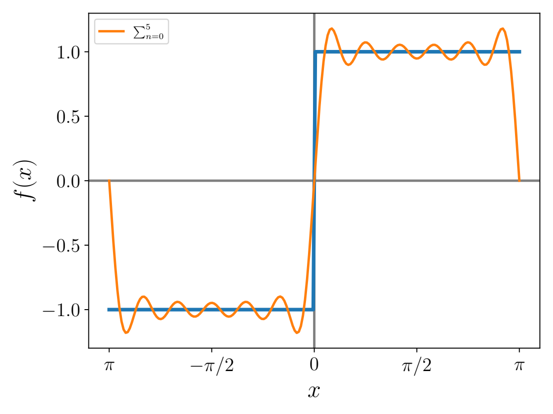
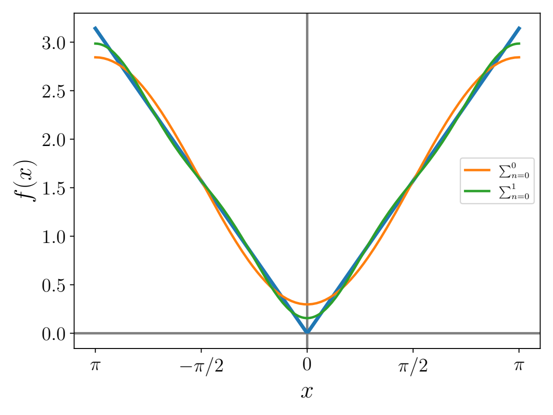


$$F_5(x) = \frac{4}{\pi} \left(\cdots + \frac{\operatorname{sen} 9x}{9} + \frac{\operatorname{sen} 11x}{11} + \frac{\operatorname{sen} 13x}{13} \right)$$

Otro ejemplo:

$$f(x) = |x|, \quad -\pi < x < \pi$$

$$F(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ impar}}^{\infty} \frac{\cos nx}{n^2}$$



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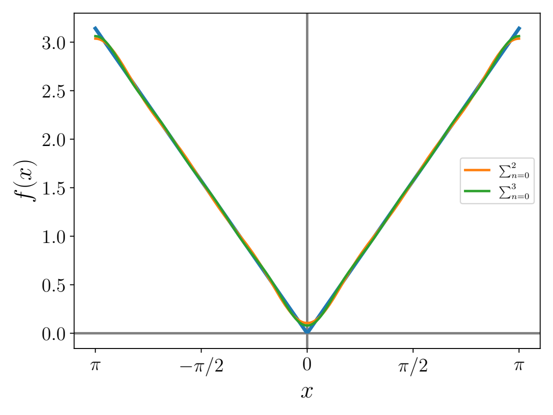

Comparación con serie de potencias:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots$$

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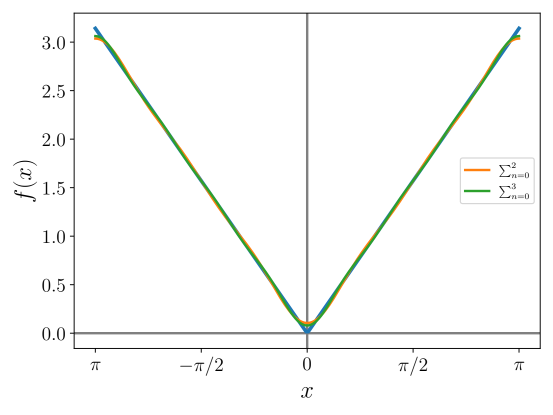

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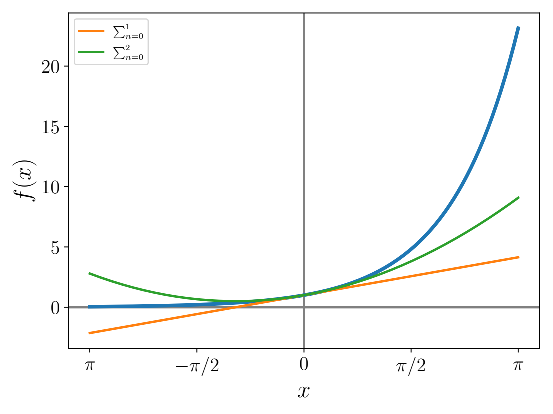

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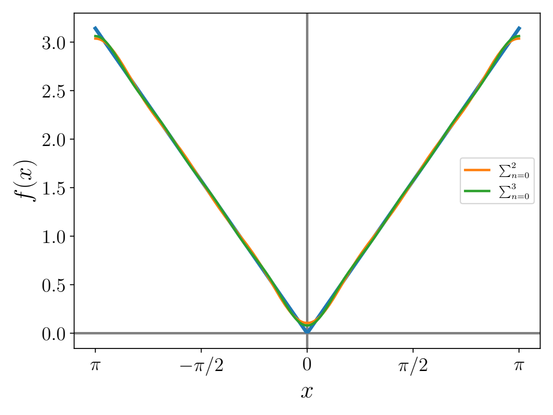

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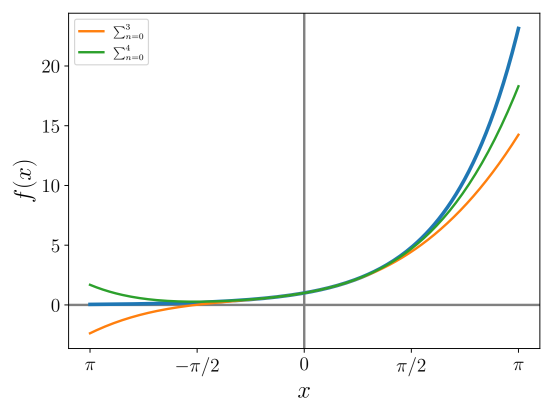

Lecturas recomendadas I

- E. Kreyszig, H. Kreyszig y E.J. Norminton. *Advanced Engineering Mathematics*. Hoboken, USA: John Wiley & Sons, Inc, 2011. Capítulo 11 (11.1 – 11.6).
- Peter V O'Neil. *Matemáticas avanzadas para ingeniería*. 7.ª ed. México, DF: CENGAGE Learning Custom Publishing, 2012. Capítulo 2.
- K A Stroud y Dexter J Booth. *Advanced Engineering Mathematics*. 6.ª ed. London, England: Bloomsbury Academic, 2020. Capítulos 7 y 8.