Introducción a la variable compleja

Secuencias y series. Integración en el campo complejo.

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$$e^z = ?, \qquad \operatorname{sen} z = ?$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = ?$$

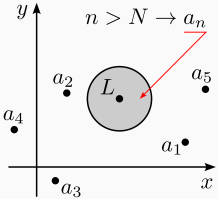
Definición

$$\lim_{n\to\infty} a_n = L \text{ significa que}$$

dado $\epsilon > 0$, $(\epsilon \in \mathbb{R})$, existe N, tal que

$$n > N \to |a_n - L| < \epsilon$$

Gráficamente:



En forma similar podemos definir:

$$\sum_{n=1}^{\infty} c_n = \lim_{n\to\infty} (c_1 + \cdots + c_n)$$

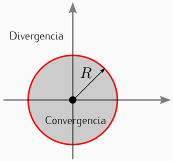
Por su estructura, **son válidos** todos los teoremas usuales.

En particular, si $S = \{z : \sum a_n z^n \text{ converge }\}$, entonces pueden darse los siguientes casos:

I) $S = \{0\}$.

II) $S = \mathbb{C}$ (todos los números complejos).

III) Existe un $R > 0$ tal que $S = \{z : |z| < R\}$, y la convergencia es absoluta e uniforme para $|z| \leq r < R$.



Entonces podemos **definir**:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\operatorname{sen} z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

$$\operatorname{sen} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Se puede entonces probar que:

$$e^{iz} = \cos z + i \operatorname{sen} z$$

o

$$e^{ix} = \cos x + i \operatorname{sen} x$$

$$\therefore (r, \theta) = r\cos\theta + ir\operatorname{sen}\theta$$
$$= re^{i\theta}$$

Tres observaciones:

1. $z = re^{i\theta} = re^{i(\theta+2\pi k)}$

$$\therefore \log z = \log r + \log e^{i(\theta+2\pi k)}$$
$$= \ln r + i(\theta + 2\pi k)$$

$\therefore \log z$ es multivaluada, el **valor principal** es $-\pi < \theta \leq \pi$.

2.

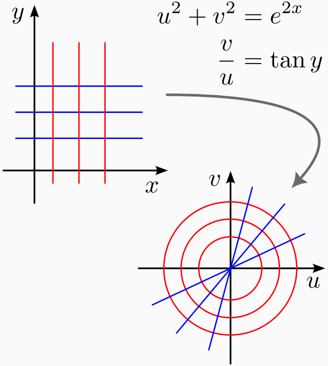
$$\cosh ix = \frac{e^{ix} + e^{-ix}}{2} =$$
$$\frac{(\cos x + i\operatorname{sen} x)}{2} +$$
$$\frac{\cos(-x) + i\operatorname{sen}(-x)}{2} = \cos x$$

3. $e^z = e^{x+iy} = e^x e^{iy} =$

$$e^x(\cos y + i\operatorname{sen} y) =$$
$$e^x\cos y + ie^x\operatorname{sen} y$$
$$u(x,y) + iv(x,y)$$

u y v representan un **mapeo conforme** real.

$$u^2 + v^2 = e^{2x}$$
$$\frac{v}{u} = \tan y$$



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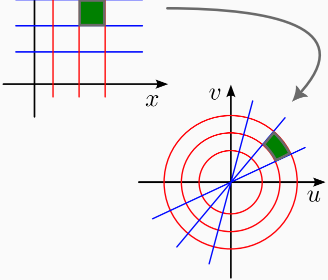
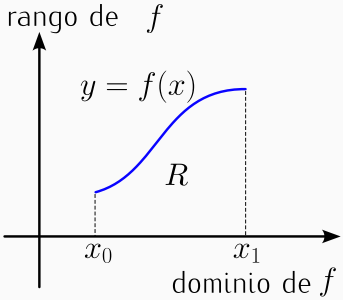


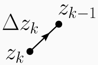
Revisión:

$$\int_{x_0}^{x_1} f(x)\,dx = \lim_{\max \Delta x \to 0} \sum_{k=1}^{n} f(c_k^*)\Delta x_k$$

$$= F(x_1) - F(x_0),\ F' = f$$

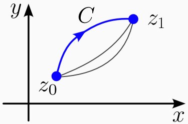


$$\int_{z_0}^{z_1} f(z)\,dz \overset{?}{=} \lim_{\max \Delta z \to 0} \sum_{k=1}^{n} f(c_k^*)\overset{?}{\Delta} z_k$$



$$C : \begin{cases} x = x(t) \\ y = y(y) \end{cases} = \begin{cases} \vec{R} = x(t)\hat{\imath} + y(t)\hat{\jmath} \\ z = x(t) + iy(t) \end{cases}$$

$$t_0 \leq t \leq t_1$$



$$\int_{C:z_0}^{z_1} f(z)\,dz = \lim_{\max \Delta z \to 0} \sum_{k=1}^{n} f(c_k(t_k))\frac{\Delta z_k}{\Delta t_k}\Delta t_k$$

$$\therefore \int_{C:z_0}^{z_1} f(z)\,dz = \int_{t_0}^{t_1} f(z(t))z'(t)\,dt$$

En términos de u y v: $f(z) = u + iv$,
$\Delta z = \Delta x + i\Delta y$:

$$\int_{\substack{C\\z_0}}^{z_1} f(z)\,dz = \int_{\substack{C\\(x_0,y_0)}}^{(x_1,y_1)} (u+iv)(dx+idy) =$$

$$\int_{\substack{C\\(x_0,y_0)}}^{(x_1,y_1)} (u\,dx - v\,dy) + i\int_{\substack{C\\(x_0,y_0)}}^{(x_1,y_1)} (v\,dx + u\,dy)$$

$$C : \begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

Si $u + iv$ es analítica: $u_x = v_y,\ u_y = -v_x$.

$$\therefore \left.\begin{matrix} u\,dx - v\,dy \\ v\,dx + u\,dy \end{matrix}\right\} \text{ es diferencial exacta.}$$

$\therefore$ Si $f = u + iv$ en analítica:

$$\int_{z_0}^{z_1} f(z)\,dz$$

es **independiente** de C, y

$$\boxed{\oint_C f(z)\,dz = 0, \quad \forall C}$$

f analítica $\rightarrow$

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0), \quad F' = f$$

Nota:

$$\oint_C f(z)\,dz$$

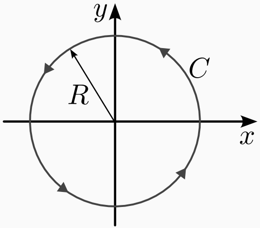
no necesariamente es 0 si f no es analítica.

Ejemplo

Calcular:

$$\oint_C \frac{dz}{z}$$

donde



El integrando es analítico en $\mathbb{C}$ excepto en $z = 0$.

Método #1:

$$\oint_C \frac{dz}{z} = \oint_C \frac{dx + idy}{x + iy}$$

$$= \oint_C \frac{(x - iy)(dx + idy)}{x^2 + y^2} =$$

$$\oint_C \frac{x\,dx + y\,dy}{x^2 + y^2} + i \oint_C \frac{-y\,dx + x\,dy}{x^2 + y^2}$$

En C: $x = R\cos\theta,\ y = R\operatorname{sen}\theta$,

$dx = -R\operatorname{sen}\theta\, d\theta$,

$dy = R\cos\theta\, d\theta$,

$x^2 + y^2 = R^2,\ 0 \le \theta \le 2\pi$.

$$\therefore \oint_C \frac{dz}{z} = 0$$

$$+ i \int_0^{2\pi} \frac{R^2(\operatorname{sen}^2\theta + \cos^2\theta)\, d\theta}{R^2}$$

$$= \boxed{2\pi i}$$

Método #2:

$C : z = Re^{i\theta},\ 0 \le \theta \le 2\pi$.

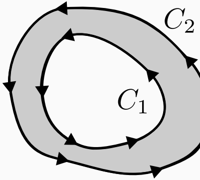
$$\frac{dz}{d\theta} = iRe^{i\theta}$$

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} \frac{1}{z(\theta)} \frac{dz}{d\theta} d\theta$$

$$= \int_0^{2\pi} \frac{iRe^{i\theta}}{Re^{i\theta}} d\theta$$

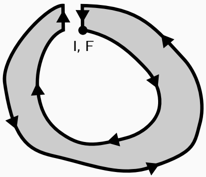
$$= \boxed{2\pi i}$$

Geometría elástica (topología):



Si f es analítica en C_1 y C_2, y en la región entre ellas, entonces:

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

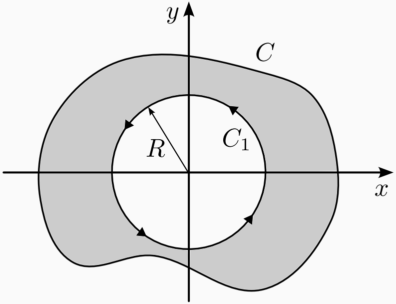


$$\oint_{\not C} f(z)\,dz = \oint_{C_2} f(z)\,dz - \oint_{C_1} f(z)\,dz = 0$$

Ejemplo: calcular

$$\oint_C \frac{dz}{dz}$$

donde



$$\oint_C \frac{dz}{z} = \oint_{C_1} \frac{dz}{z} = 2\pi i$$

Pausa para resolver problemas de la práctica Nro. 2.

Ejercicios 1 – 3.

- E. Kreyszig, H. Kreyszig y E.J. Norminton. *Advanced Engineering Mathematics*. Hoboken, USA: John Wiley & Sons, Inc, 2011. Capítulo 13.
- M.R. Spiegel et al. *Variable compleja*. Mexico: McGraw-Hill, 1991. Capítulo 1.