

# INTRODUCCIÓN A LA VARIABLE COMPLEJA

NÚMEROS COMPLEJOS (REPASO). FUNCIONES DE VARIABLE COMPLEJA. LÍMITE Y CONTINUIDAD.  
DIFERENCIABILIDAD Y FUNCIONES ANALÍTICAS.

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# LOS NÚMEROS COMPLEJOS

Sistema de enteros:

$$2x = 3$$

$$x = ?$$

Números "reales":  $\{x : x^2 \geq 0\}$

$$x^2 = -1$$

$$x = ?$$

Motivación:  $x^2 + 1 = 0$  ¿tiene solución?

Ejemplo: usar  $y = e^{rx}$  para resolver:

$$y'' + y = 0$$

$$r^2 e^{rx} + e^{rx} = 0$$

$$\therefore r^2 + 1 = 0 \therefore r = \pm\sqrt{-1} = \pm i$$

$$\therefore y = e^{ix} \text{ o } y = e^{-ix}$$

$$\boxed{y = \cos x \text{ o } y = \sin x}$$

De "alguna manera"  $i$  debe existir y  $e^{ix}$  debe estar relacionado a  $\sin x$  y  $\cos x$ .

El sistema de **números complejos**:

$$\mathbb{C} = \{x + iy : x \text{ y } y \text{ son reales.}\}$$

con una estructura.

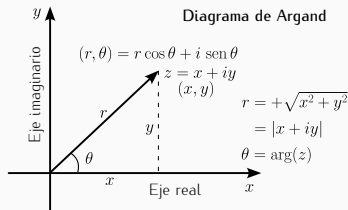
$$(1) x_1 + iy_1 = x_2 + iy_2 \Leftrightarrow$$

$$x_1 = x_2 \text{ y } y_1 = y_2$$

$$(2) (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

$$(3) r(x + iy) = rx + iry$$

$r$  real.



Estructura adicional de  $\mathbb{C}$ :

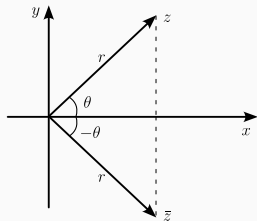
$$(4) (a + ib)(c + id) = \\ = (ac - bd) + i(bc + ad)$$

Caso especial:

$$(a + ib)(a - ib) = a^2 + b^2 \geq 0 \\ = |a + ib|^2$$

**Definición:** el complejo  
conjugado de  $z = x + yi$  es

$$\bar{z} = x - yi$$



$$\frac{c + di}{a + bi} = \frac{c + di}{a + bi} \frac{a - bi}{a - bi} \\ = \frac{(ac + bd) + (ad - bc)i}{a^2 + b^2}$$

$$\frac{3 + 2i}{4 + i} = \frac{(3 + 2i)(4 - i)}{(4 + i)(4 - i)} \\ = \frac{14 + 5i}{17} \\ = \frac{14}{17} + \frac{5}{17}i$$

$$\therefore \frac{\text{complejo}}{\text{complejo}} = \text{complejo}$$

(excepto para división por  
cero).

Producto en coordenadas polares:

$$(r_1, \theta_1)(r_2, \theta_2) = (r_1 \cos \theta_1 + ir_1 \sin \theta_1) \\ (r_2 \cos \theta_2 + ir_2 \sin \theta_2) = \\ r_1 r_2 (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + \\ ir_1 r_2 (\sin \theta_1 \cos \theta_2 + \sin \theta_2 \cos \theta_1) = \\ r_1 r_2 \cos(\theta_1 + \theta_2) + ir_1 r_2 \sin(\theta_1 + \theta_2) = \\ \boxed{(r_1 r_2, \theta_1 + \theta_2)}$$

Por inducción:

$$(r_1, \theta_1) \cdots (r_n, \theta_n) = \\ (r_1 \cdots r_n, \theta_1 + \cdots + \theta_n)$$

Caso especial:

$$(r, \theta)^n = (r^n, n\theta) \\ \therefore r = 1 \rightarrow (1, \theta)^n = (1, n\theta)$$

Teorema de De Moivre:

$$(\cos \theta + i \operatorname{sen} \theta)^n = \cos n\theta + i \operatorname{sen} n\theta$$

Ejemplo:

$$\begin{aligned} (\cos \theta + i \operatorname{sen} \theta)^2 &= \cos 2\theta + i \operatorname{sen} 2\theta \\ (\cos^2 \theta - \operatorname{sen}^2 \theta) + i 2 \operatorname{sen} \theta \cos \theta \end{aligned}$$

$$\begin{aligned} \therefore \operatorname{sen} 2\theta &= 2 \operatorname{sen} \theta \cos \theta \\ \cos 2\theta &= \cos^2 \theta - \operatorname{sen}^2 \theta \end{aligned}$$

Raíces: encontrar

$$\sqrt[6]{i} = x + iy \rightarrow i = (x + iy)^6 = 0 + 1i$$

$$\begin{aligned} \therefore x^6 + 15x^4(iy)^2 + 15x^2(iy)^4 + (iy)^6 \\ + 6x^5(iy) + 20x^3(iy)^3 + 6x(iy)^5 \end{aligned}$$

Sistema complicado a resolver:

$$\begin{cases} x^6 + 15x^4(iy)^2 + 15x^2(iy)^4 + (iy)^6 = 0 \\ 6x^5(iy) + 20x^3(iy)^3 + 6x(iy)^5 = 1 \end{cases}$$

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En coordenadas polares:

$$i = (1, \pi/2) \therefore \sqrt[6]{i} = (r, \theta)$$

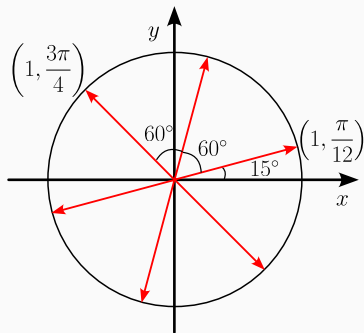
$$\rightarrow i = (r, \theta)^6 = (r^6, 6\theta)$$

$$\therefore r = 1, \quad 6\theta = \frac{\pi}{2} + 2\pi k = \frac{1 + 4k}{2}\pi$$

$$r = 1,$$

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{9\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{21\pi}{12}, \frac{25\pi}{12}, \dots$$

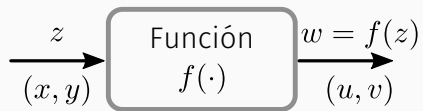
$$\begin{aligned} \left(1, \frac{3\pi}{4}\right) &= \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \\ &= \frac{1}{\sqrt{2}}(-1 + i) \end{aligned}$$



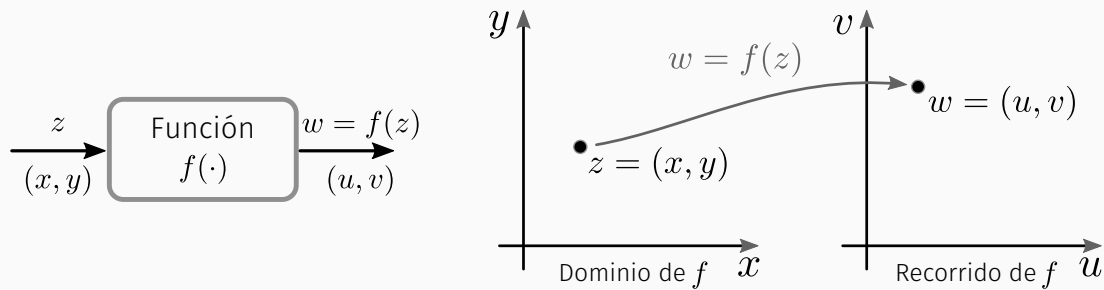
### Sistema de números complejos:

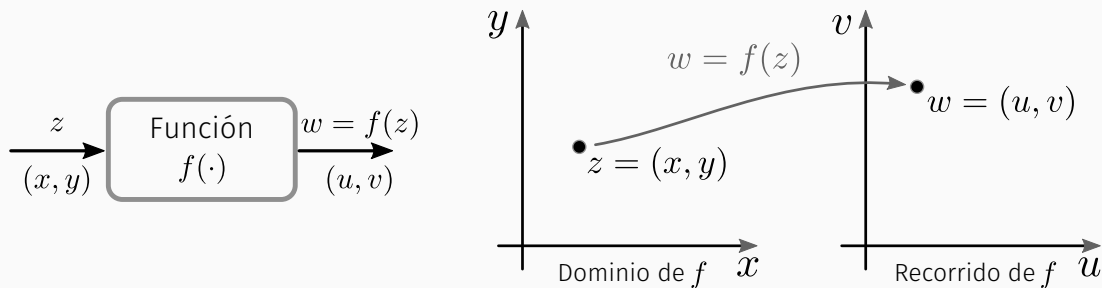
Los números complejos son **cerrados** respecto de la radicación.

PAUSA PARA RESOLVER PROBLEMAS: 1 – 8.



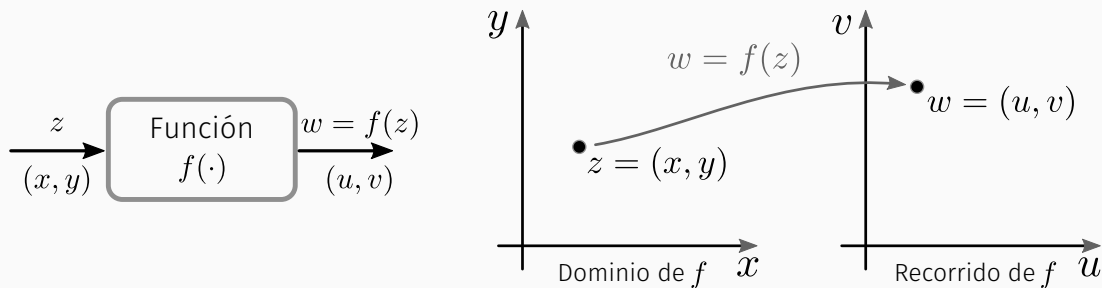






## Ejemplo:

$$\begin{aligned}
 f(z) &= z^2 = (x + iy)^2 \\
 &= x^2 + 2xiy + i^2 y^2 = x^2 - y^2 + 2ixy \\
 \therefore f(x, y) &= (x^2 - y^2, 2xy)
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 \end{aligned}$$

$\therefore f(z) = z^2$  es equivalente al sistema real:

$$\begin{cases} u = x^2 + y^2 \\ v = 2xy \end{cases}$$

$\mathbb{C}$ : números complejos

$f : \mathbb{C} \mapsto \mathbb{C}, a \in \mathbb{C}$

## Definición:

$$\lim_{z \rightarrow a} f(z) = L$$

dado  $\epsilon > 0$  existe  $\delta > 0$  tal que

$$0 < |z - a| < \delta \mapsto |f(z) - L| < \epsilon$$

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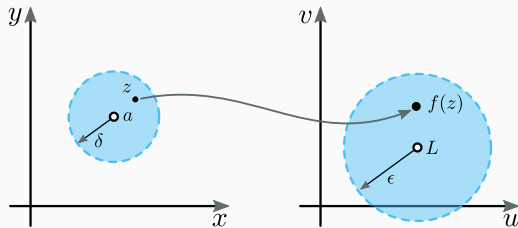
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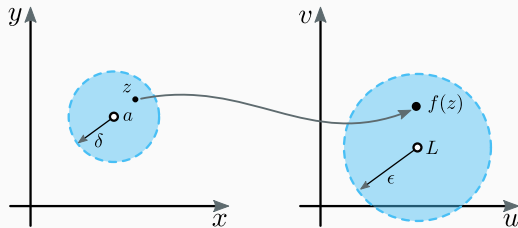
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Los teoremas usuales sobre límites **son válidos**. En particular:

Si

Entonces:

$$f(z) = u(x, y) + iv(x, y)$$

$$L = L_1 + iL_2$$

$$a = a_1 + ia_2$$

$$\lim_{z \rightarrow a} f(z) = L \iff \begin{cases} \lim_{(x,y) \rightarrow (a_1, a_2)} u(x, y) = L_1 \\ \lim_{(x,y) \rightarrow (a_1, a_2)} v(x, y) = L_2 \end{cases}$$

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Si  $w = f(z) = u(x, y) + iv(x, y)$ :

$$\begin{aligned} f'(z_0) &= \frac{dw}{dz} = \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z} \\ &= \lim_{\Delta z \rightarrow 0} \left[ \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y} \right] \end{aligned}$$

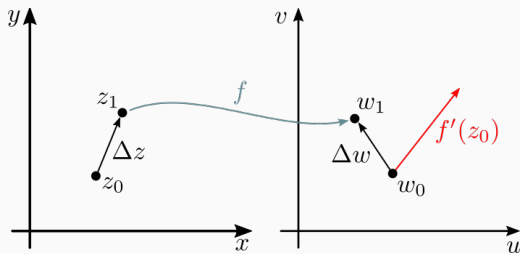


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**Caso 1:**  $\Delta y \equiv 0$ .

$$\begin{aligned}\therefore f'(z_0) &= \lim_{\Delta x \rightarrow 0} \left[ \frac{\Delta u}{\Delta x} + i \frac{\Delta v}{\Delta x} \right] \\ &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \Big|_{z_0=(x_0, y_0)}\end{aligned}$$

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**Caso 2:**  $\Delta x \equiv 0$ .

$$\begin{aligned}\therefore f'(z_0) &= \lim_{\Delta y \rightarrow 0} \left[ \frac{\Delta u}{i \Delta y} + \frac{\Delta v}{\Delta y} \right] = \frac{\partial v}{\partial y} + \frac{1}{i} \frac{\partial u}{\partial y} \Big|_{z_0=(x_0, y_0)} \\ &= \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \Big|_{z_0=(x_0, y_0)}\end{aligned}$$

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### Ecuaciones de Cauchy-Riemann

Si  $f = u + iv$  es diferenciable (**analítica**), entonces:

$$u_x = v_y$$

$$u_y = -v_x$$

Ecuaciones de Cauchy-Riemann:

$$\begin{aligned} f(z) &= z^2 = (x + iy)^2 \\ &= (x^2 - y^2) + i(2xy) \end{aligned}$$

$$\left. \begin{aligned} u_x &= 2x, & v_x &= 2y \\ u_y &= -2y, & v_y &= 2x \end{aligned} \right\} \Rightarrow \begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}$$

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Derivada por definición:

$$\begin{aligned}\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} &= \frac{(z_0 + \Delta z)^2 - z_0^2}{\Delta z} \\&= \frac{2z_0\Delta z + \Delta z^2}{\Delta z} \quad (\Delta z \neq 0) \\&= 2z_0 + \Delta z\end{aligned}$$

$$\therefore \boxed{f'(z_0) = 2z_0}$$

$$f(z) = \bar{z} = x - iy$$
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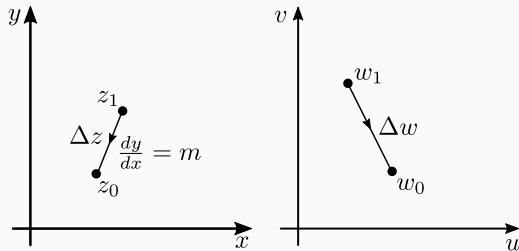
$$\begin{aligned}\frac{\Delta w}{\Delta z} &= \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} \\ &= \frac{1 - i\frac{\Delta y}{\Delta x}}{1 + i\frac{\Delta y}{\Delta x}} \rightarrow \frac{1 - \frac{dy}{dx}}{1 + i\frac{dy}{dx}}\end{aligned}$$



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$$\frac{\Delta w}{\Delta z} = \frac{1 - im}{1 + im} = g(m)$$

$u(x, y)$  satisface la ecuación de Laplace si:

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$$u_x = v_y \quad \therefore \quad u_{xx} = v_{yx}$$

$$u_y = -v_x \quad \therefore \quad u_{yy} = -v_{xy}$$

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$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

**Ejemplo:**

$$f(z) = z^2 \longrightarrow \begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}$$

$$\therefore \begin{cases} u_{xx} + u_{yy} = 0 \\ v_{xx} + v_{yy} = 0 \end{cases}$$

- ▶ E. Kreyszig, H. Kreyszig y E.J. Norminton. *Advanced Engineering Mathematics*. Hoboken, USA: John Wiley & Sons, Inc, 2011. Capítulo 13.
- ▶ M.R. Spiegel et al. *Variable compleja*. Mexico: McGraw-Hill, 1991. Capítulo 1.