Ecuaciones diferenciales parciales de segundo orden

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Ecuación diferencial parcial (EDP) lineal de segundo orden:

$$A\frac{\partial^2\phi}{\partial x^2} + B\frac{\partial^2\phi}{\partial x\partial y} + C\frac{\partial^2\phi}{\partial y^2} + D\frac{\partial\phi}{\partial x} + E\frac{\partial\phi}{\partial y} + F\phi = S$$

donde A, B, C, D, E, F y S son funciones de x y y en $D \in \mathbf{R}^2$.

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Tipos:

- Parabólica: $B^2 - 4AC = 0$
- Elíptica: $B^2 - 4AC < 0$
- Hiperbólica: $B^2 - 4AC > 0$

para todo $(x, y) \in D$.

Casos:

- Conducción de calor en sólidos, flujo de fluidos
- Ejemplos:
  - Conducción de calor:

$$\rho c\frac{\partial T}{\partial t} = k\frac{\partial^2 T(x,t)}{\partial x^2} + Q(x)$$

  - Transporte convectivo:

$$\frac{\partial\phi}{\partial t} = -\frac{\partial}{\partial x}u(x)\phi + D\frac{\partial^2}{\partial x^2}$$

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Casos:

- Problemas estacionarios de 2 y 3 dimensiones
- Conducción de calor en sólidos, vibración de membranas
- Ejemplos:
  - Ecuación de Poisson:

$$-\nabla^2\phi(x, y) = S(x, y)$$

  - Ecuación de Laplace:

$$-\nabla^2\phi(x, y) = 0$$

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para todo $(x, y) \in D$.

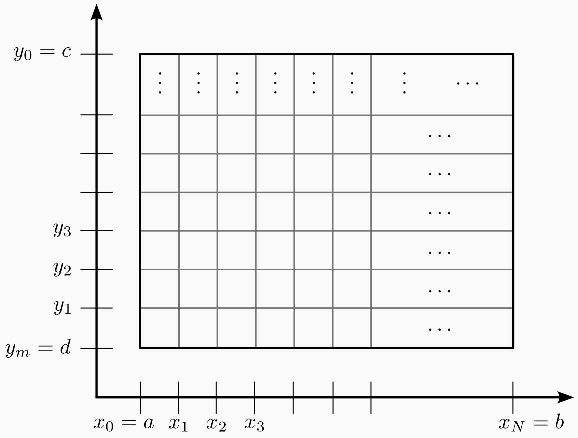
Casos:

- Problemas oscilatorios, propagación de ondas, fluidos
- Ejemplos:
  - Ecuación de onda:
    $$\frac{\partial^2 u(x,y,z,t)}{\partial t^2} = c^2\nabla^2 u(x,y,z,t)$$
  - Navier-Stokes (incompresible):
    $$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nu\nabla^2\boldsymbol{u} = -\nabla\left(\frac{p}{p_0}\right) + \boldsymbol{g}$$

Ecuación de Poisson (elíptica):

$$\nabla^2 u(x,y) = \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = f(x,y)$$

en $R = \{(x,y) | a < x < b, c < y < d\}$, con $u(x,y) = g(x,y)$ para $(x,y) \in S$, siendo S la frontera de R.



Malla:

- División $[a,b]$ y $[c,d]$ en n y m partes iguales
- $h = (b-a)/n$, $k = (d-c)/m$
- $x_i = a + ih, \ i = 0, 1, \ldots, n$
- $y_j = c + jk, \ j = 0, 1, \ldots, m$

Aproximación en diferencias finitas (serie de Taylor):

$$\frac{\partial^2 u}{\partial x^2} = \frac{u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j)}{h^2} + \mathcal{O}(h^2)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u(x_i, y_{j+1}) - 2u(x_i, y_j) + u(x_i, y_{j-1})}{k^2} + \mathcal{O}(k^2)$$

Con $u(x_i, y_j) \mapsto u_{i,j}$:

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = f_{i,j} + \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i, y_j) + \frac{k^2}{12}\frac{\partial^4 u}{\partial y^4}(x_i, \eta_j)$$

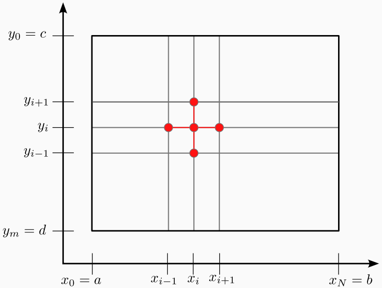
para $i = 1, 2, \ldots, n-1$, $j = 1, 2, \ldots, m-1$ y condiciones de contorno:

$$u_{0,j} = g_{0,j} \quad \text{y} \quad u_{n,j} = g_{n,j}, \quad j = 0, 1, \ldots m;$$

$$u_{i,0} = g_{i,0} \quad \text{y} \quad u_{i,m} = g_{i,m}, \quad i = 0, 1, \ldots n$$

Resulta:

$$2\left[\left(\frac{h}{k}\right)^2 + 1\right] u_{i,j} - (u_{i+1,j} + u_{i-1,j}) - \left(\frac{h}{k}\right)^2 (u_{i,j+1} + u_{i,j-1}) = -h^2 f_{i,j},\ i \in [1, n-1],\ j \in [1, m-1]$$



Con $u(x_i, y_j) \mapsto u_{i,j}$:

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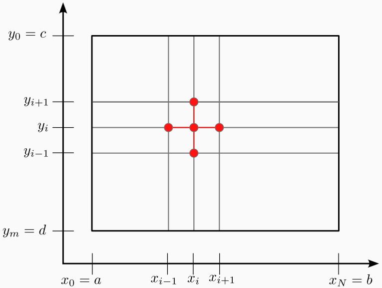
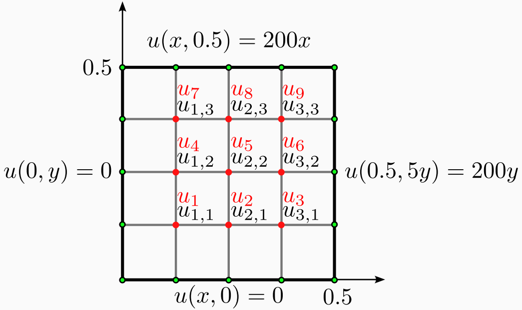


Ejemplo: Determinar la distribución estacionaria de temperaturas en una placa de 0.5×0.5 m usando $n = m = 4$. Dos bordes adyacentes se mantienen a 0 ℃ y la temperatura se incrementa linealmente en los otros bordes hasta llegar a 100 ℃ en la esquina de unión.

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0;\ (x,y) \in [0,0.5]^2$$

$$h = k = 1/8$$

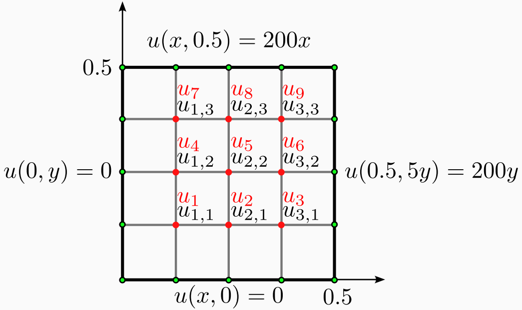


$$u_{i,j} \mapsto u_l,\ l = i + m(j-1)$$

$$u_{1,1} = u_1,\ u_{2,1} = u_2,\ u_{3,1} = u_3$$

$$u_{1,2} = u_4,\ u_{2,2} = u_5,\ u_{3,2} = u_6$$

$$u_{1,3} = u_7,\ u_{2,3} = u_8,\ u_{3,3} = u_9$$



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Ecuaciones:

$$4u_{i,j} - u_{i+1,j} - u_{i-1,j} - u_{i,j-1} - u_{i,j+1} = 0$$

para $i = 1,2,3;\ j = 1,2,3.$

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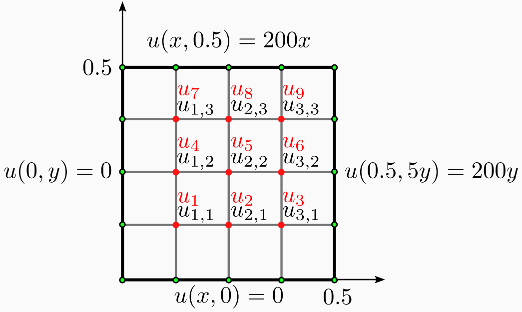

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$$\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = 0; \ (x, y) \in [0, 0.5]^2$$

$$h = k = 1/8$$

Ecuaciones:

$$4u_{i,j} - u_{i+1,j} - u_{i-1,j} - u_{i,j-1} - u_{i,j+1} = 0$$

para $i = 1, 2, 3; \ j = 1, 2, 3$.

Condiciones de borde:

$$u_{0,0} = u_{0,1} = u_{0,2} = u_{0,3} = u_{0,4} = 0$$

$$u_{1,0} = u_{2,0} = u_{3,0} = u_{4,0} = 0$$

$$u_{1,4} = u_{4,1} = 25; u_{2,4} = u_{4,2} = 50$$

$$u_{3,4} = u_{4,3} = 75; u_{4,4} = 100$$

$$
\begin{bmatrix}
4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4
\end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{bmatrix}
=
\begin{bmatrix} u_{0,1} + u_{1,0} \\ u_{0,2} \\ u_{0,3} + u_{4,1} \\ u_{0,2} \\ 0 \\ u_{4,2} \\ u_{0,3} + u_{1,4} \\ u_{2,4} \\ u_{3,4} + u_{3,4} \end{bmatrix}
\rightarrow
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{bmatrix}
=
\begin{bmatrix} 6.25 \\ 12.5 \\ 18.75 \\ 12.5 \\ 25 \\ 37.5 \\ 18.75 \\ 37.5 \\ 56.25 \end{bmatrix}
$$

$$\begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{bmatrix} = \begin{bmatrix} u_{0,1} + u_{1,0} \\ u_{0,2} \\ u_{0,3} + u_{4,1} \\ u_{0,2} \\ 0 \\ u_{4,2} \\ u_{0,3} + u_{1,4} \\ u_{2,4} \\ u_{3,4} + u_{3,4} \end{bmatrix} \rightarrow \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{bmatrix} = \begin{bmatrix} 6.25 \\ 12.5 \\ 18.75 \\ 12.5 \\ 25 \\ 37.5 \\ 18.75 \\ 37.5 \\ 56.25 \end{bmatrix}$$

Solución exacta: $u(x,y) = 400xy$ por lo que la aproximación en diferencias finitas no tiene error:

$$\frac{\partial^4 u}{\partial x^4} = \frac{\partial^4 u}{\partial y^4} = 0$$

Ecuación de calor (parabólica):

$$\frac{\partial u}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

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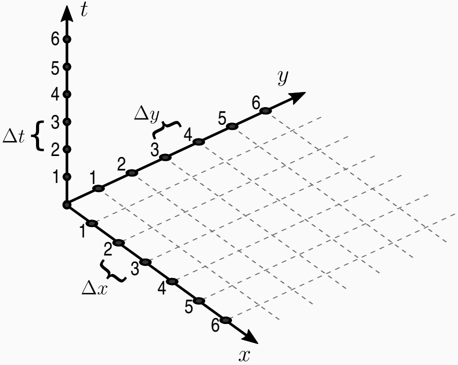

Grilla:

$$x_i = i\Delta x,\ y_j = j\Delta y,\ t_k = k\Delta t,\ u(x,y,t) = u_{i,j}^k$$

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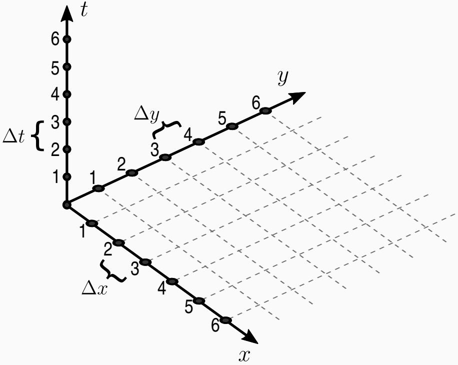

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$$x_i = i\Delta x,\; y_j = j\Delta y,\; t_k = k\Delta t,\; u(x,y,t) = u_{i,j}^k$$

Diferencias finitas (hacia adelante - explícito):

$$\frac{u_{i,j}^{k+1} - u_{i,j}^k}{\Delta t} = \alpha\left(\frac{u_{i+1,j}^k - 2u_{i,j}^k + u_{i-1,j}^k}{\Delta x^2}\right) + \left(\frac{u_{i,j+1}^k - 2u_{i,j}^k + u_{i,j-1}^k}{\Delta y^2}\right)$$

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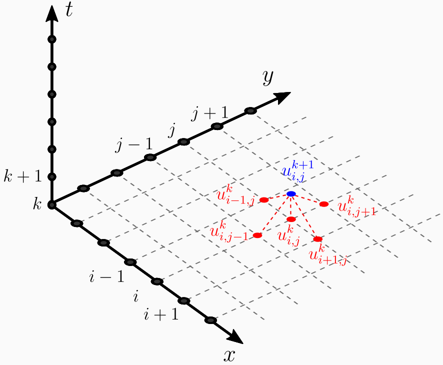

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$$\frac{u_{i,j}^{k+1} - u_{i,j}^k}{\Delta t} = \alpha \left(\frac{u_{i+1,j}^k - 2u_{i,j}^k + u_{i-1,j}^k}{\Delta x^2} \right) + \left(\frac{u_{i,j+1}^k - 2u_{i,j}^k + u_{i,j-1}^k}{\Delta y^2} \right)$$

Hacemos $\Delta x = \Delta y$, $\gamma = \alpha \dfrac{\Delta t}{\Delta x^2}$:

$$u_{i,j}^{k+1} = \gamma \left(u_{i+1,j}^k + u_{i-1,j}^k + u_{i,j+1}^k + u_{i,j-1}^k - 4u_{i,j}^k \right) + u_{i,j}^k$$

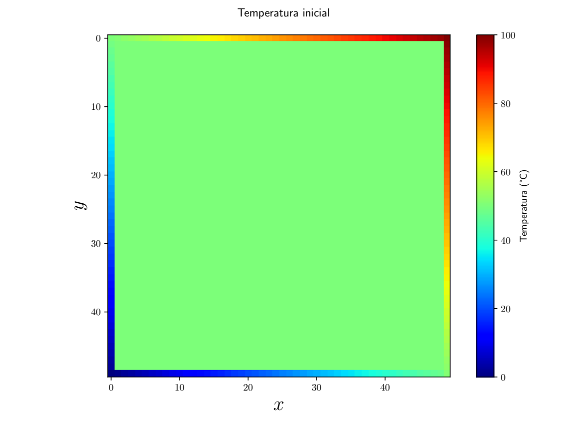
Método explícito: $\Delta t \leq \dfrac{\Delta x^2}{4\alpha} \longleftarrow$ estabilidad numérica.

Stencil:

$$u_{i,j}^{k+1} = \gamma \left(u_{i+1,j}^{k} + u_{i-1,j}^{k} + u_{i,j+1}^{k} + u_{i,j-1}^{k} - 4u_{i,j}^{k} \right) + u_{i,j}^{k}$$

$$\frac{\partial u}{\partial t} - \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0$$

$$0 \leq x \leq L_x,\ 0 \leq y \leq L_y$$



Condiciones de borde:

$$u(x,0) = 50 + \frac{x(100-50)}{L_x}$$

$$u(0,y) = 50 - y\frac{50}{L_y}$$

$$u(x,L_y) = 50\frac{x}{L_x}$$

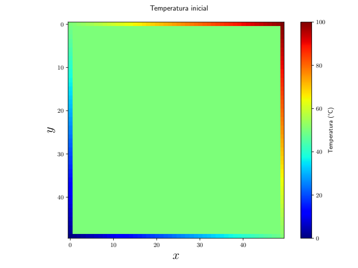
$$u(L_x,y) = 100 - y\frac{50}{L_y}$$

Condición inicial:

$$u(x,y)_{t=0} = 50$$

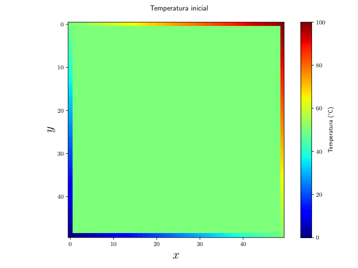
```python
#!/usr/bin/env python3

import numpy as np

Lx, Ly = 50, 50
max_iter_tiempo = 750
alpha = 5
delta_x = 1
delta_t = (delta_x ** 2)/(4 * alpha)
gamma = (alpha * delta_t) / (delta_x ** 2)

# Condiciones de borde
u_SO = 0.0
u_NO, u_SE = 50.0, 50.0
u_NE = 100.0

# Condición inicial interior
u_inicial = 50
```

```python
def inicializar_u(max_iter_tiempo, ni=Lx, nj=Ly):
    # Inicializar la solución: u(k, i, j)
    u = np.full((max_iter_tiempo, ni, nj), u_inicial)
    # Establecer condiciones de borde
    u[:, 0, :] = u_NO + np.arange(ni) * delta_x
                * (u_NE - u_NO) / Lx
    u[:, :, 0] = u_NO - np.arange(nj) * delta_x
                * u_NO / Ly
    u[:, -1, :] = np.arange(ni) * delta_x * u_SE / Lx
    u[:, :, -1] = u_NE - np.arange(nj) * delta_x
                * u_SE / Ly
    return u

u = inicializar_u(max_iter_tiempo)
```

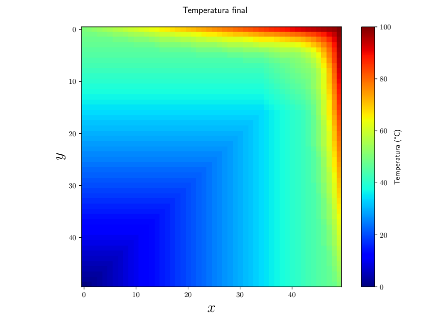


```python
import matplotlib
import matplotlib.pyplot as plt
matplotlib.rcParams.update({"text.usetex": True})
fig, ax = plt.subplots(figsize=(8,6))
mappable = ax.imshow(u[0], interpolation=None,
                     cmap=plt.cm.jet)
fig.colorbar(mappable, label="Temperatura (°C)", ax=ax)
ax.set_xlabel(r"$x$", fontsize=20)
ax.set_ylabel(r"$y$", fontsize=20)
fig.suptitle("Temperatura inicial")
fig.tight_layout()
fig.savefig("temp-inicial.pdf")
plt.close()
```

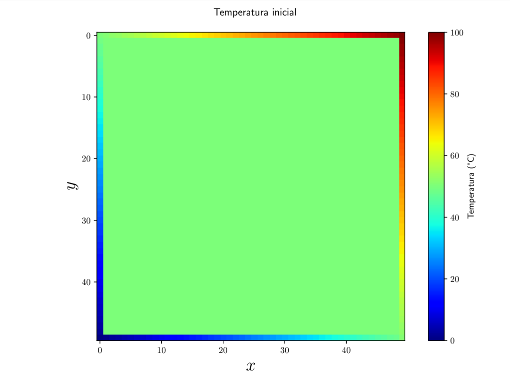
```python
# Código que aplica el stencil en la grilla (i, j) y en cada t
def calcular(u):
    nk, ni, nj = u.shape
    for k in range(0, nk-1):
        for i in range(1, ni-1):
            for j in range(1, nj-1):
                u[k + 1, i, j] = gamma * (u[k][i+1][j] + u[k][
                                          + u[k][i][j+1] + u[k
                                          - 4*u[k][i][j]) + u[
    return u

u = calcular(u)
```



```python
# Código que aplica el stencil en la grilla (i, j) y en cada tiempo k
def calcular(u):
    nk, ni, nj = u.shape
    for k in range(0, nk-1):
        for i in range(1, ni-1):
            for j in range(1, nj-1):
                u[k + 1, i, j] = gamma * (u[k][i+1][j] + u[k][i-1][j]
                                         + u[k][i][j+1] + u[k][i][j-1]
                                         - 4*u[k][i][j]) + u[k][i][j]
    return u

u = calcular(u)
fig, ax = plt.subplots(figsize=(8,6))
mappable = ax.imshow(u[-1], interpolation=None,
                     cmap=plt.cm.jet)
fig.colorbar(mappable, label="Temperatura (°C)", ax=ax)
ax.set_xlabel(r"$x$", fontsize=20)
ax.set_ylabel(r"$y$", fontsize=20)
fig.suptitle("Temperatura final")
fig.tight_layout()
fig.savefig("temp-final.pdf")
plt.close()
```



```python
def plotheatmap(u_k, k):
    # Limpiamos la figura
    plt.clf()

    plt.title(f"Temperatura en t = {k*delta_t:.2f} u.t.")
    plt.xlabel(r"$x$", fontsize=20)
    plt.ylabel(r"$y$", fontsize=20)

    # Ploteamos u_k (u_{i,j} en `paso de tiempo k)
    plt.imshow(u_k, cmap=plt.cm.jet,
               interpolation="bicubic", vmin=0, vmax=100)
    plt.colorbar()

    return plt

import matplotlib.animation as animation
from matplotlib.animation import FuncAnimation

def animate(k):
    plotheatmap(u[k], k)

anim = animation.FuncAnimation(plt.figure(),
    animate, interval=50, frames=max_iter_tiempo,
    repeat=False)
anim.save("solucion_ecuacion_calor_t.mp4")
```



```python
def plotheatmap(u_k, k):
    # Limpiamos la figura
    plt.clf()

    plt.title(f"Temperatura en t = {k*delta_t:.2f} u.t.")
    plt.xlabel(r"$x$", fontsize=20)
    plt.ylabel(r"$y$", fontsize=20)

    # Ploteamos u_k (u_{i,j} en `paso de tiempo k)
    plt.imshow(u_k, cmap=plt.cm.jet,
                    interpolation="bicubic", vmin=0, vmax=100)
    plt.colorbar()

    return plt

import matplotlib.animation as animation
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def animate(k):
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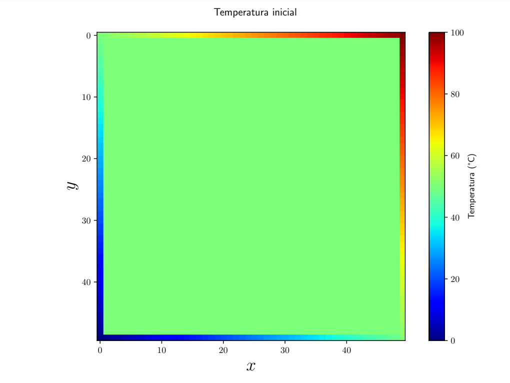

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