

INTRODUCCIÓN A LA VARIABLE COMPLEJA

INTEGRACIÓN EN EL CAMPO COMPLEJO. SERIES DE TAYLOR Y DE LAURENT. TEOREMA DEL RESIDUO. RESOLUCIÓN DE INTEGRALES REALES.

Manuel Carlevaro

Departamento de Ingeniería Mecánica

Grupo de Materiales Granulares - UTN FRLP

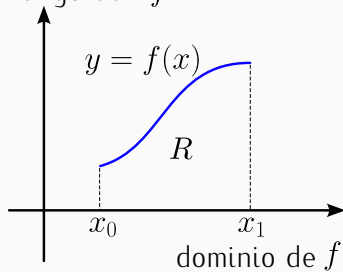
manuel.carlevaro@gmail.com

Revisión:

$$\int_{x_0}^{x_1} f(x) dx = \lim_{\substack{\max \\ \Delta x \rightarrow 0}} \sum_{k=1}^n f(c_k^*) \Delta x_k$$

$$= F(x_1) - F(x_0), F' = f$$

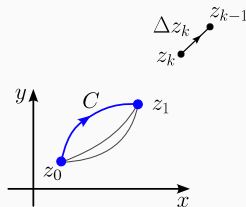
rango de f



$$\int_{z_0}^{z_1} f(z) dz \stackrel{?}{=} \lim_{\substack{\max \\ \Delta z \rightarrow 0}} \sum_{k=1}^n f(c_k^*) \Delta z_k \stackrel{?}{}$$

$$C : \begin{cases} x = x(t) \\ y = y(t) \end{cases} = \begin{cases} \vec{R} = x(t)\hat{i} + y(t)\hat{j} \\ z = x(t) + iy(t) \end{cases}$$

$$t_0 \leq t \leq t_1$$



$$\int_{C: z_0}^{z_1} f(z) dz = \lim_{\substack{\max \\ \Delta z \rightarrow 0}} \sum_{k=1}^n f(c_k(t_k)) \frac{\Delta z_k}{\Delta t_k} \Delta t_k$$

$$\therefore \int_{C: z_0}^{z_1} f(z) dz = \int_{t_0}^{t_1} f(z(t)) z'(t) dt$$

En términos de u y v : $f(z) = u + iv$,

$$\Delta z = \Delta x + i\Delta y:$$

$$\begin{aligned} \int_C^{z_1} f(z) dz &= \int_C^{(x_1, y_1)} (u + iv)(dx + idy) = \\ &= \int_C^{(x_1, y_1)} (u dx - v dy) + i \int_C^{(x_1, y_1)} (v dx + u dy) \\ C: &\begin{cases} x = x(t) \\ y = y(t) \end{cases} \end{aligned}$$

Si $u + iv$ es **analítica**: $u_x = v_y$, $u_y = -v_x$.

$$\therefore \begin{cases} u dx - v dy \\ v dx + u dy \end{cases} \text{ es diferencial exacta.}$$

\therefore Si $f = u + iv$ en analítica:

$$\int_{z_0}^{z_1} f(z) dz$$

es **independiente** de C , y

$$\oint_C f(z) dz = 0, \quad \forall C$$

f analítica \rightarrow

$$\int_{z_0}^{z_1} f(z) dz = F(z_1) - F(z_0), \quad F' = f$$

Nota:

$$\oint_C f(z) dz$$

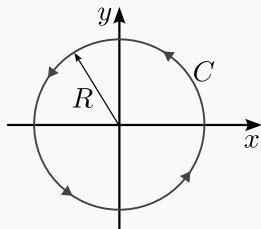
no necesariamente es 0 si f no es analítica.

EJEMPLO

Calcular:

$$\oint_C \frac{dz}{z}$$

donde



El integrando es analítico en \mathbb{C} excepto en $z = 0$.

Método #1:

$$\begin{aligned}\oint_C \frac{dz}{z} &= \oint_C \frac{dx + idy}{x + iy} \\ &= \oint_C \frac{(x - iy)(dx + idy)}{x^2 + y^2} = \\ &= \oint_C \frac{x dx + y dy}{x^2 + y^2} + i \oint_C \frac{-y dx + x dy}{x^2 + y^2}\end{aligned}$$

$$\begin{aligned}\text{En } C: x &= R \cos \theta, y = R \sin \theta, \\ dx &= -R \sin \theta d\theta, \\ dy &= R \cos \theta d\theta, \\ x^2 + y^2 &= R^2, 0 \leq \theta \leq 2\pi.\end{aligned}$$

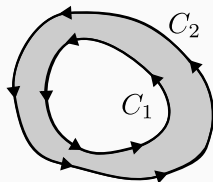
$$\begin{aligned}\therefore \oint_C \frac{dz}{z} &= 0 \\ &+ i \int_0^{2\pi} \frac{R^2(\sin^2 \theta + \cos^2 \theta) d\theta}{R^2} \\ &= \boxed{2\pi i}\end{aligned}$$

Método #2:

$$C : z = Re^{i\theta}, 0 \leq \theta \leq 2\pi.$$

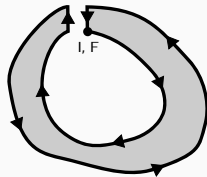
$$\begin{aligned}\frac{dz}{d\theta} &= iRe^{i\theta} \\ \oint_C \frac{dz}{z} &= \int_0^{2\pi} \frac{1}{z(\theta)} \frac{dz}{d\theta} d\theta \\ &= \int_0^{2\pi} \frac{iRe^{i\theta}}{Re^{i\theta}} d\theta \\ &= \boxed{2\pi i}\end{aligned}$$

Geometría elástica (topología):



Si f es analítica en C_1 y C_2 , y en la región entre ellas, entonces:

$$\oint_{C_1} f(z) dz = \oint_{C_2} f(z) dz$$

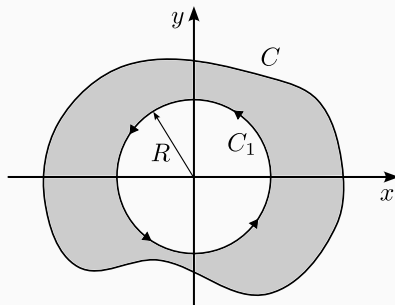


$$\oint_{\emptyset} f(z) dz = \oint_{C_2} f(z) dz - \oint_{C_1} f(z) dz = 0$$

Ejemplo: calcular

$$\oint_C \frac{dz}{z}$$

donde



$$\oint_C \frac{dz}{z} = \oint_{C_1} \frac{dz}{z} = 2\pi i$$

- ▶ E. Kreyszig, H. Kreyszig y E.J. Norminton. *Advanced Engineering Mathematics*. Hoboken, USA: John Wiley & Sons, Inc, 2011. Capítulo 13.
- ▶ M.R. Spiegel et al. *Variable compleja*. Mexico: McGraw-Hill, 1991. Capítulo 1.