Introducción a la variable compleja

Integración en el campo complejo. Series de Taylor y de Laurent. Teorema del residuo. Resolución de integrales reales.

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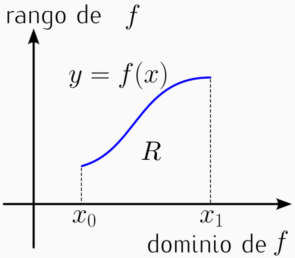
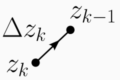
Revisión:

$$\int_{x_0}^{x_1} f(x)\,dx = \lim_{\substack{\text{máx} \\ \Delta x \to 0}} \sum_{k=1}^{n} f(c_k^*)\Delta x_k$$

$$= F(x_1) - F(x_0), \; F' = f$$

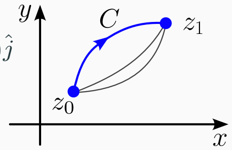


$$\int_{z_0}^{z_1} f(z)\,dz \overset{?}{=} \lim_{\substack{\text{máx} \\ \Delta z \to 0}} \sum_{k=1}^{n} f(c_k^*)\overset{?}{\Delta} z_k$$



$$C : \begin{cases} x = x(t) \\ y = y(y) \end{cases} = \begin{cases} \vec{R} = x(t)\hat{\imath} + y(t)\hat{\jmath} \\ z = x(t) + iy(t) \end{cases}$$

$$t_0 \leq t \leq t_1$$



$$\int_{C:z_0}^{z_1} f(z)\,dz = \lim_{\substack{\text{máx} \\ \Delta z \to 0}} \sum_{k=1}^{n} f(c_k(t_k))\frac{\Delta z_k}{\Delta t_k}\Delta t_k$$

$$\therefore \int_{C:z_0}^{z_1} f(z)\,dz = \int_{t_0}^{t_1} f(z(t))z'(t)\,dt$$

En términos de u y v: $f(z) = u + iv$,
$\Delta z = \Delta x + i\Delta y$:

$$\int_{\substack{C\\z_0}}^{z_1} f(z)\,dz = \int_{\substack{C\\(x_0,y_0)}}^{(x_1,y_1)} (u+iv)(dx+idy) =$$

$$\int_{\substack{C\\(x_0,y_0)}}^{(x_1,y_1)} (u\,dx - v\,dy) + i\int_{\substack{C\\(x_0,y_0)}}^{(x_1,y_1)} (v\,dx + u\,dy)$$

$$C : \begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

Si $u + iv$ es analítica: $u_x = v_y,\ u_y = -v_x$.

$$\therefore \left.\begin{array}{r} u\,dx - v\,dy \\ v\,dx + u\,dy \end{array}\right\} \text{ es diferencial exacta.}$$

$\therefore$ Si $f = u + iv$ en analítica:

$$\int_{z_0}^{z_1} f(z)\,dz$$

es **independiente** de C, y

$$\boxed{\oint_C f(z)\,dz = 0, \quad \forall C}$$

f analítica $\rightarrow$

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0), \quad F' = f$$

Nota:

$$\oint_C f(z)\,dz$$

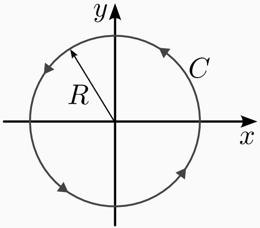
no necesariamente es 0 si f no es analítica.

Ejemplo

Calcular:

$$\oint_C \frac{dz}{z}$$

donde



El integrando es analítico en $\mathbb{C}$ excepto en $z = 0$.

Método #1:

$$\oint_C \frac{dz}{z} = \oint_C \frac{dx + i\,dy}{x + iy}$$

$$= \oint_C \frac{(x - iy)(dx + i\,dy)}{x^2 + y^2} =$$

$$\oint_C \frac{x\,dx + y\,dy}{x^2 + y^2} + i \oint_C \frac{-y\,dx + x\,dy}{x^2 + y^2}$$

En C: $x = R\cos\theta,\ y = R\,\mathrm{sen}\,\theta,$
$dx = -R\,\mathrm{sen}\,\theta\,d\theta,$
$dy = R\cos\theta\,d\theta,$
$x^2 + y^2 = R^2,\ 0 \le \theta \le 2\pi.$

$$\therefore \oint_C \frac{dz}{z} = 0$$

$$+\,i \int_0^{2\pi} \frac{R^2(\mathrm{sen}^2\,\theta + \cos^2\theta)\,d\theta}{R^2}$$

$$= \boxed{2\pi i}$$

Método #2:

$C : z = Re^{i\theta},\ 0 \le \theta \le 2\pi.$

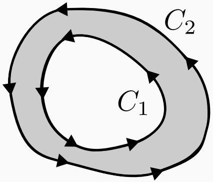
$$\frac{dz}{d\theta} = iRe^{i\theta}$$

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} \frac{1}{z(\theta)} \frac{dz}{d\theta} d\theta$$

$$= \int_0^{2\pi} \frac{iRe^{i\theta}}{Re^{i\theta}} d\theta$$

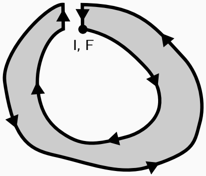
$$= \boxed{2\pi i}$$

Geometría elástica (topología):



Si f es analítica en C_1 y C_2, y en la región entre ellas, entonces:

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

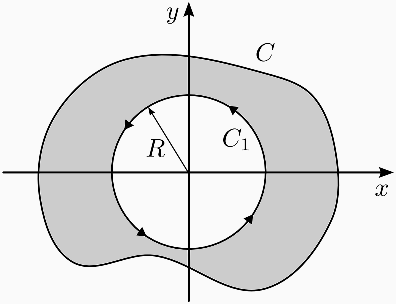


$$\oint_{\cancel{C}} f(z)\,dz = \oint_{C_2} f(z)\,dz - \oint_{C_1} f(z)\,dz = 0$$

Ejemplo: calcular

$$\oint_C \frac{dz}{dz}$$

donde



$$\oint_C \frac{dz}{z} = \oint_{C_1} \frac{dz}{z} = 2\pi i$$

- E. Kreyszig, H. Kreyszig y E.J. Norminton. *Advanced Engineering Mathematics*. Hoboken, USA: John Wiley & Sons, Inc, 2011. Capítulo 13.
- M.R. Spiegel et al. *Variable compleja*. Mexico: McGraw-Hill, 1991. Capítulo 1.