Chordal networks of polynomial ideals

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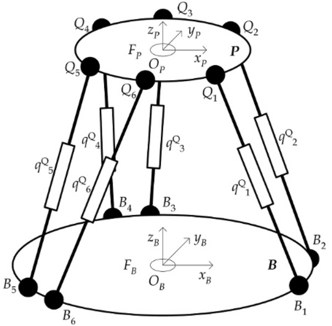
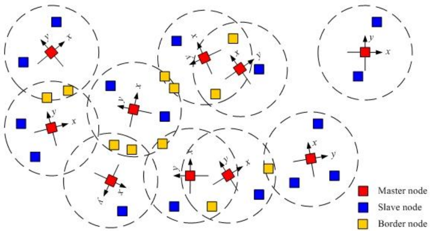
Polynomial systems

Systems of polynomial equations have been used to model problems in areas such as: robotics, cryptography, statistics, optimization, computer vision, power networks, graph theory.





$$S + E \underset{k_{-1}}{\overset{k_{+1}}{\rightleftharpoons}} ES \underset{k_{-2}}{\overset{k_{+2}}{\rightleftharpoons}} P + E$$

$$\frac{d[S]}{dt} = -k_1[E][S] + k_{-1}[ES]$$

$$\frac{d[E]}{dt} = -k_1[E][S] + (k_{-1} + k_2)[ES] - k_{-2}[E][P]$$

$$\frac{d[ES]}{dt} = k_1[E][S] - (k_{-1} + k_2)[ES] + k_{-2}[E][P]$$

Polynomial systems

Systems of polynomial equations have been used to model problems in areas such as: robotics, cryptography, statistics, optimization, computer vision, power networks, graph theory.

Given polynomial equations $F = \{f_1, \ldots, f_m\}$, let

$$\mathcal{V}(F) := \{x \in \mathbb{R}^n : f_1(x) = \cdots = f_m(x) = 0\}$$

denote the associated variety.

Depending on the application we might be interesting in:

Feasibility Is there any solution, i.e., $\mathcal{V}(F) \neq \emptyset$?

Counting How many solutions?

Dimension What is the dimension of $\mathcal{V}(F)$?

Components Decompose $\mathcal{V}(F)$ into irreducible components.

Polynomial systems and graphs

Systems coming from applications often have simple *sparsity structure*. We can represent this structure using graphs.

Given m equations in n variables, construct a graph as:

- Nodes are the variables $\{x_0, \ldots, x_{n-1}\}$.
- For each equation, add a clique connecting the variables appearing in that equation

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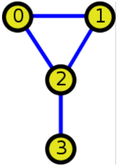
Example:

$$F = \{x_0^2 x_1 x_2 + 2x_1 + 1,\ \ x_1^2 + x_2,\ \ x_1 + x_2,\ \ x_2 x_3\}$$



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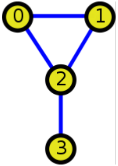

Question: Can the graph structure help *solve* polynomial systems?

Graphical modelling

Pervasive idea in many areas, in particular: numerical linear algebra, graphical models, constraint satisfaction, database theory, . . .

Key notions: chordality and treewidth.

Many names: Arnborg, Beeri/Fagin/Maier/Yannakakis, Blair/Peyton, Bodlaender, Courcelle, Dechter, Lauritzen/Spiegelhalter, Pearl, Robertson/Seymour, . . .

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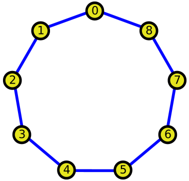
We hope to change this... ;)

Example 1: Coloring a cycle

Let $C_n = (V, E)$ be the cycle graph and consider the ideal I given by the equations



$$x_i^3 - 1 = 0, \qquad i \in V$$
$$x_i^2 + x_i x_j + x_j^2 = 0, \qquad ij \in E$$

These equations encode the proper 3-colorings of the graph. Note that coloring the cycle graph is very easy!

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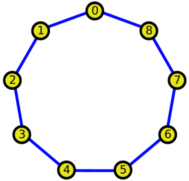

$$\begin{aligned} x_i^3 - 1 &= 0, & i &\in V \\ x_i^2 + x_i x_j + x_j^2 &= 0, & ij &\in E \end{aligned}$$

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However, a Gröbner basis is not so simple: one of its 13 elements is

$$\begin{gathered} x_0x_2x_4x_6 + x_0x_2x_4x_7 + x_0x_2x_4x_8 + x_0x_2x_5x_6 + x_0x_2x_5x_7 + x_0x_2x_5x_8 + x_0x_2x_6x_8 + x_0x_2x_7x_8 + x_0x_2x_8^2 + x_0x_3x_4x_6 + x_0x_3x_4x_7 \\ +x_0x_3x_4x_8 + x_0x_3x_5x_6 + x_0x_3x_5x_7 + x_0x_3x_5x_8 + x_0x_3x_6x_8 + x_0x_3x_7x_8 + x_0x_3x_8^2 + x_0x_4x_6x_8 + x_0x_4x_7x_8 + x_0x_4x_8^2 + x_0x_5x_6x_8 \\ +x_0x_5x_7x_8 + x_0x_5x_8^2 + x_0x_6x_8^2 + x_0x_7x_8^2 + x_0 + x_1x_2x_4x_6 + x_1x_2x_4x_7 + x_1x_2x_4x_8 + x_1x_2x_5x_6 + x_1x_2x_5x_7 + x_1x_2x_5x_8 \\ +x_1x_2x_6x_8 + x_1x_2x_7x_8 + x_1x_2x_8^2 + x_1x_3x_4x_6 + x_1x_3x_4x_7 + x_1x_3x_4x_8 + x_1x_3x_5x_6 + x_1x_3x_5x_7 + x_1x_3x_5x^8 + x_1x_3x_6x_8 + x_1x_3x_7x_8 \\ +x_1x_3x_8^2 + x_1x_4x_6x_8 + x_1x_4x_7x_8 + x_1x_4x_8^2 + x_1x_5x_6x_8 + x_1x_5x_7x_8 + x_1x_5x_8^2 + x_1x_6x_8^2 + x_1x_7x_8^2 + x_1 + x_2x_4x_6x_8 + x_2x_4x_7x_8 \\ +x_2x_4x_8^2 + x_2x_5x_6x_8 + x_2x_5x_7x_8 + x_2x_5x_8^2 + x_2x_6x_8^2 + x_2x_7x_8^2 + x_2 + x_3x_4x_6x_8 + x_3x_4x_7x_8 + x_3x_4x_8^2 + x_3x_5x_6x_8 + x_3x_5x_7x_8 \\ +x_3x_5x_8^2 + x_3x_6x_8^2 + x_3x_7x_8^2 + x_3 + x_4x_6x_8^2 + x_4x_7x_8^2 + x_4 + x_5x_6x_8^2 + x_5x_7x_8^2 + x_5 + x_6 + x_7 + x_8 \end{gathered}$$

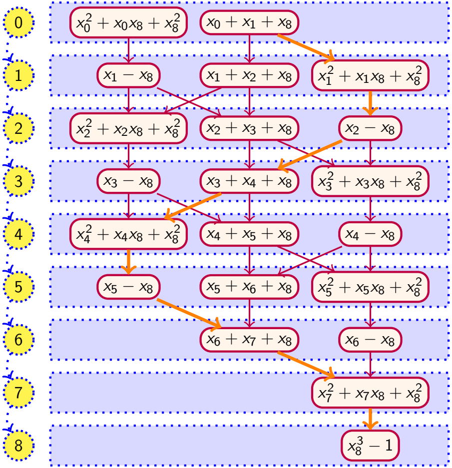
Example 1: Coloring a cycle

There is an alternative representation of the ideal, that respects its graphical structure.

The variety can be decomposed into *triangular* sets:

$$\mathcal{V}(I) = \bigcup_{T} \mathcal{V}(T)$$

where the union is overall all *maximal directed paths* (or *chains*).
The number of triangular sets is 21, which is the 8-th Fibonacci number.



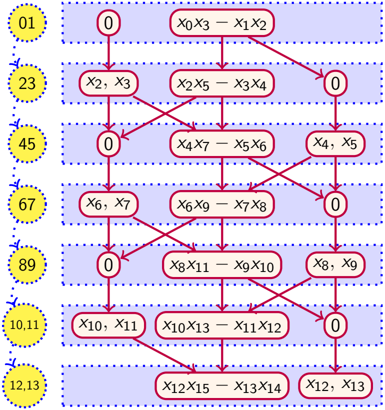
Example 2: Ideal of adjacent minors

$$I = \{x_{2i}x_{2i+3} - x_{2i+1}x_{2i+2} : 0 \leq i < n\}$$

This is the ideal of adjacent minors of the matrix

$$\begin{pmatrix} x_0 & x_2 & x_4 & \cdots & x_{2n-2} \\ x_1 & x_3 & x_5 & \cdots & x_{2n-1} \end{pmatrix}$$

The total number of irreducible components is the n-th Fibonacci number.



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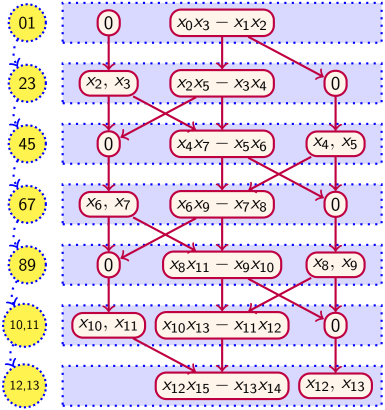


More generally, the ideal of adjacent minors of a $k \times n$ matrix also has a simple chordal network representation.

Our results

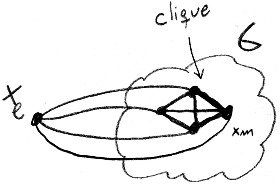
- We introduce the notion of chordal networks, a new representation of structured polynomial ideals.
- An algorithm to compute chordal network representations.
- We show that several families of polynomial systems admit a chordal network representation of size $O(n)$, even though the number of components is exponentially large.
- We show how to effectively use chordal networks to solve: feasibility, counting, dimension, elimination, radical membership and sometimes components.
- Implementation and experimental results.

Chordal graphs

For a graph G, an ordering of its vertices $x_0 > x_1 > \cdots > x_{n-1}$ is a *perfect elimination ordering* if for each x_ℓ

$$X_\ell := \{x_m : x_m \text{ is adjacent to } x_\ell,\ \ x_\ell > x_m\}$$

is a clique.



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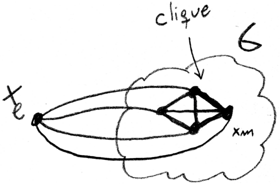
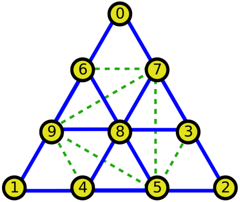


is a clique.
A graph is chordal if it has a perfect elimination ordering.

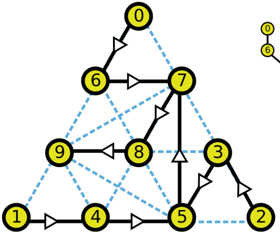
A *chordal completion* of G is supergraph that is chordal.

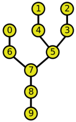


Elimination tree of a chordal graph

The elimination tree of a graph G is the following *directed spanning tree*:

For each ℓ there is an arc towards its smallest neighbor p, with $p > \ell$.

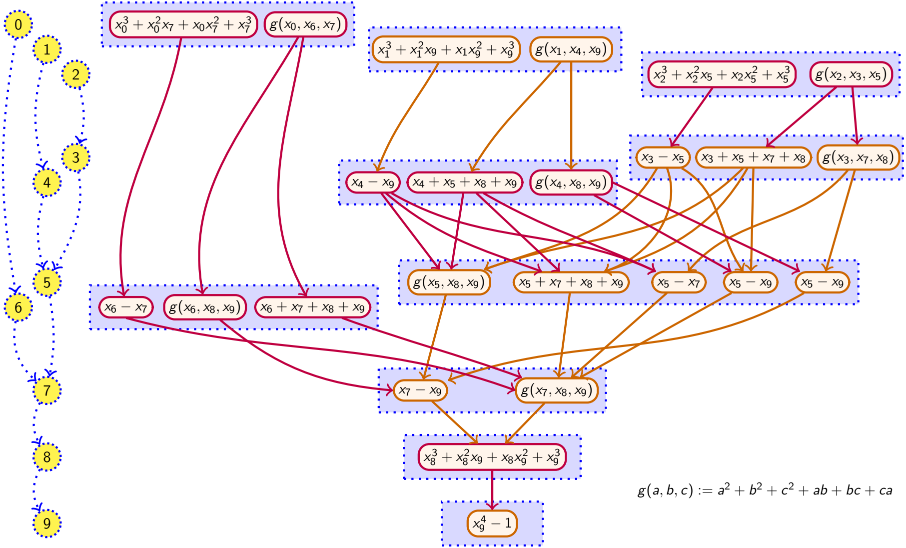




Chordal networks

A *G-chordal network* is a directed graph $\mathcal{N}$, whose nodes are polynomial sets, satisfying the following conditions

- arcs follow elimination tree: if (F_ℓ, F_p) is an arc, then (ℓ, p) is an arc of the elimination tree, where $\ell = \operatorname{rank}(F_\ell), p = \operatorname{rank}(F_p)$.
- nodes supported on cliques: each node F of $\mathcal{N}$ is given a rank $\ell := \operatorname{rank}(F)$, such that F only involves variables in the clique X_ℓ.



$$g(a,b,c) := a^2 + b^2 + c^2 + ab + bc + ca$$

Computing chordal networks: Triangular sets

Defn: A zero dimensional triangular set is $T = \{t_0, \ldots, t_{n-1}\}$ such that

$$\begin{aligned} t_0 &= x_0^{d_0} + g_0(x_0, x_1, \ldots, x_{n-1}), && (\deg_{x_0}(g_0) < d_0) \\ &\vdots \\ t_{n-2} &= x_{n-2}^{d_{n-2}} + g_{n-2}(x_{n-2}, x_{n-1}), && (\deg_{x_{n-2}}(g_1) < d_{n-2}) \\ t_{n-1} &= g_{n-1}(x_{n-1}) \end{aligned}$$

Remk: A triangular set is a Gröbner basis w.r.t. lexicographic order.

Defn: Let $I \subset \mathbb{K}[X]$ be a zero dimensional ideal. A triangular decomposition of I is a collection $\mathcal{T}$ of triangular sets, such that

$$\mathcal{V}(I) = \bigsqcup_{T \in \mathcal{T}} \mathcal{V}(T)$$

Computing chordal networks (Example)

The ideal

$$I = \langle x_0x_2 - x_2, x_0^3 - x_0, x_1 - x_2, x_2^2 - x_2, x_2 - x_3 \rangle$$

can be decomposed into three triangular sets

$$T_1 = (x_0^3 - x_0,\ \ x_1 - x_2,\ \ x_2,\ \ x_3),$$
$$T_2 = (x_0 - 1,\ \ x_1 - x_2,\ \ x_2 - 1,\ \ x_3),$$
$$T_3 = (x_0 - 1,\ \ x_1 - x_2,\ \ x_2 - 1,\ \ x_3 - 1).$$

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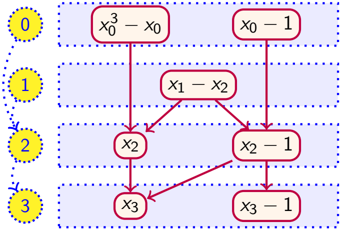
These triangular sets correspond to chains of a chordal network



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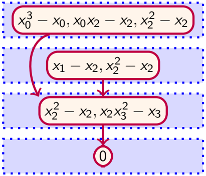

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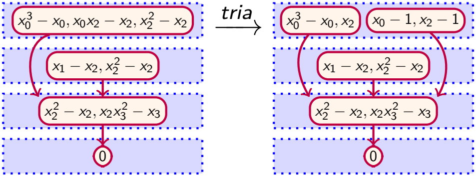

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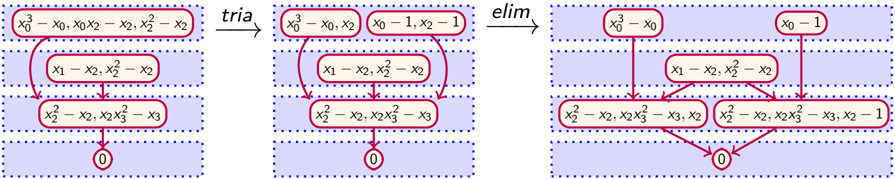

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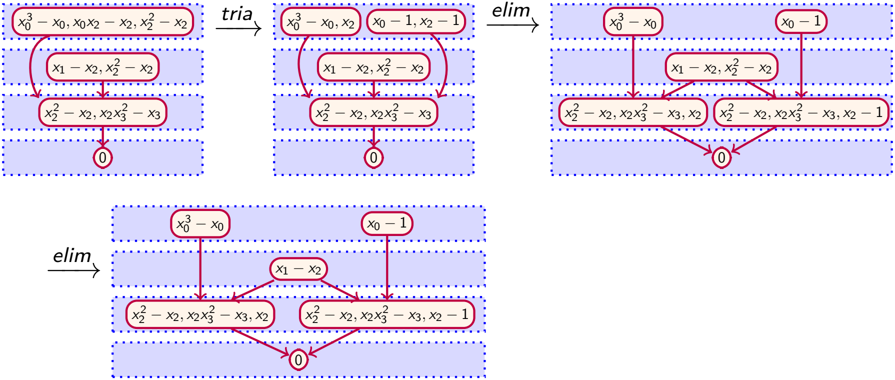

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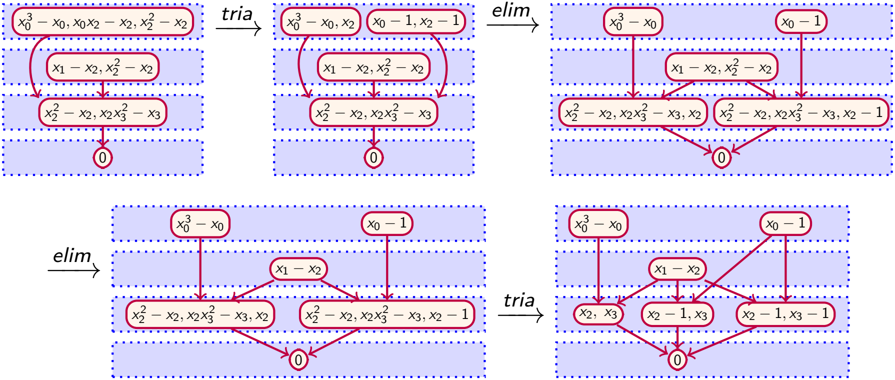

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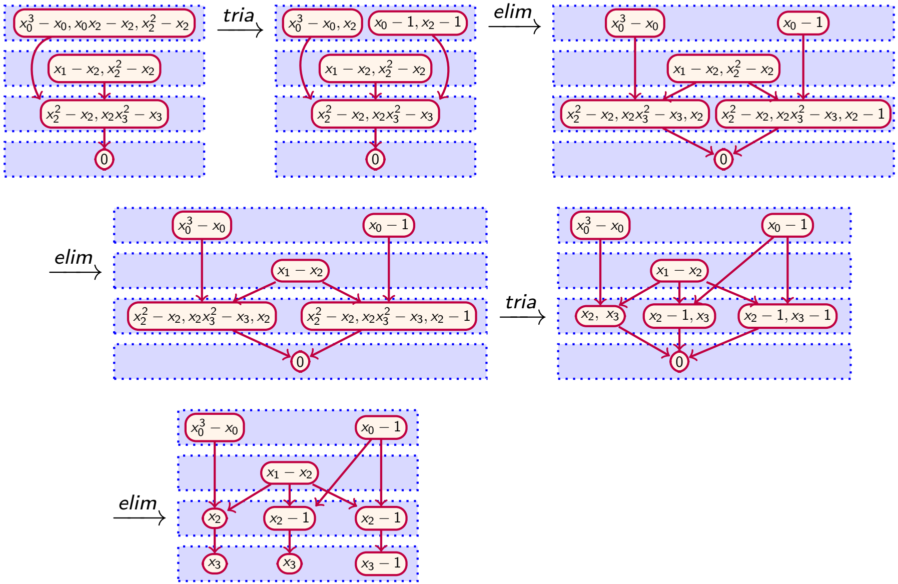

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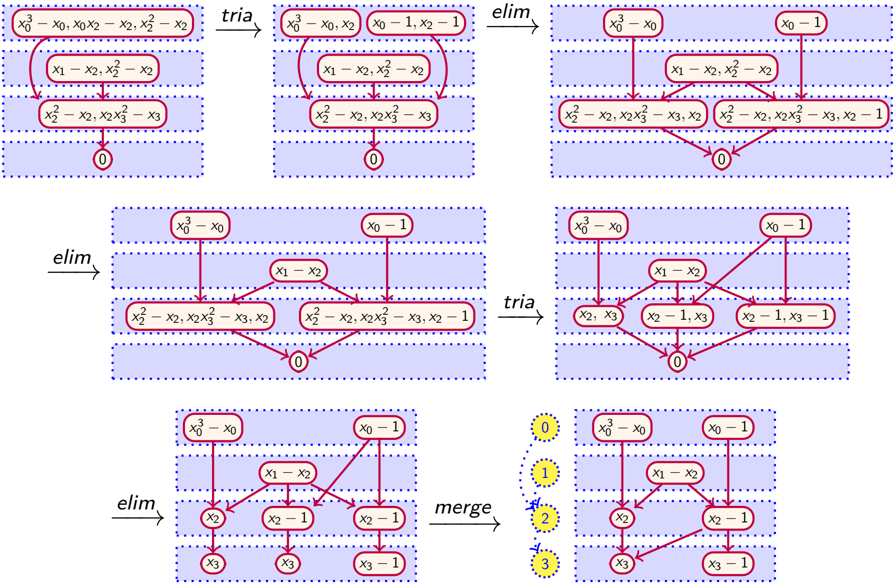

Main results

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For "nice" cases the chordal network obtained has linear size.

Thm 2: Let $\mathcal{F}$ be a family of structured polynomial systems such that $|\mathcal{V}(F \cap \mathbb{K}[X_l])|$ is bounded for any $F \in \mathcal{F}$ and for any maximal clique X_l. Then any $F \in \mathcal{F}$ admits a chordal network representation of size $O(n)$.

Chordal networks in computational algebra

Given a triangular chordal network $\mathcal{N}$ of an ideal I, we can compute in linear time:

- the cardinality of $\mathcal{V}(I)$.
- the dimension of $\mathcal{V}(I)$
- the top dimensional part of $\mathcal{V}(I)$.

We also show efficient algorithms for:

- radical ideal membership.
- computing equidimensional (sometimes irreducible) components.

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The main difficulty is that there might be exponentially many chains. It can be overcomed by cleverly using dynamic programming (or message-passing).

Radical ideal membership

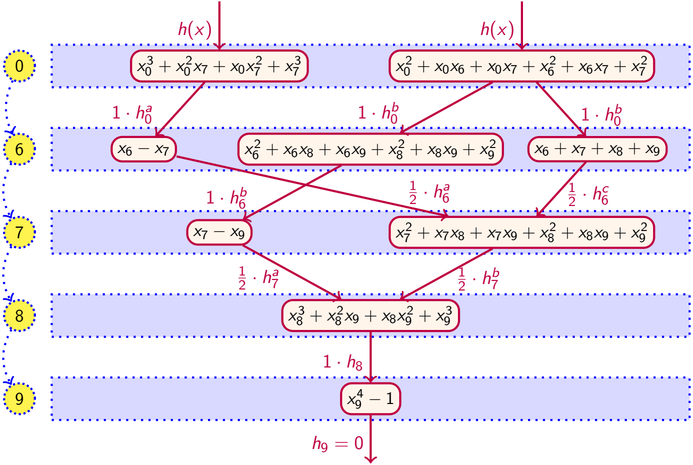
Problem: Determine if a polynomial h vanishes on a variety.

Given a chordal network, we need to determine whether

$$h \bmod C \equiv 0 \qquad\qquad \text{for all chains } C$$

Note that there might be exponentially many chains.

Radical ideal membership (Sketch)



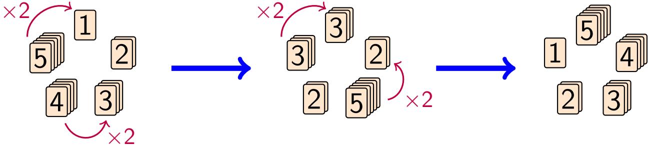
Implementation and examples

Implemented in Sage.

- Graph colorings (counting q-colorings)
- Cryptography ("baby" AES, Cid *et al.*)
- Sensor Network localization
- Discretization of differential equations
- Algebraic statistics
- Reachability in vector addition systems

Application: Vector addition systems

Given a set of vectors $\mathcal{B} \subset \mathbb{Z}^n$, construct a graph with vertex set $\mathbb{N}^n$ in which $u, v \in \mathbb{N}^n$ are adjacent if $u - v \in \pm\mathcal{B}$. The reachability problem is to describe the connected components of the graph.



Problem: There are n card decks on a circle. Given any four consecutive decks we can:

- take one card from the inner decks and place them in the outer decks.
- take one card from the outer decks and place them on the inner decks.

Initially the number of cards in the decks are $1, 2, \ldots, n$. Is it possible to reverse the number of cards in the decks?

Application: Vector addition systems

The problem is equivalent to determining whether $f_n \in I_n$, where

$$f_n := x_0 x_1^2 x_2^3 \cdots x_{n-1}^n - x_0^n x_1^{n-1} \cdots x_{n-1},$$
$$I_n := \{x_i x_{i+3} - x_{i+1} x_{i+2} : 0 \leq i < n\},$$

and where the indices are taken modulo n.

We compare our radical membership test with Singular (Gröbner bases) and Epsilon (triangular decompositions).

n	5	10	15	20	25	30	35	40	45	50	55
ChordalNet	0.7	3.0	8.5	14.3	21.8	29.8	37.7	48.2	62.3	70.6	84.8
Singular	0.0	0.0	0.2	17.9	1036.2	-	-	-	-	-	-
Epsilon	0.1	0.2	0.4	2.0	54.4	160.1	5141.9	17510.1	-	-	-
Test result	true	false	false	false	true	false	true	false	false	false	true

Application: Algebraic statistics (Evans et al.)

Consider the binomial ideal I^{n,n_2} that models a 2D dimensional generalization of the birth-death Markov process. We fix $n_2 = 1$.

We can compute faster all irreducible components of the ideal than the Macaulay2 package "Binomials".

n		1	2	3	4	5	6	7
#components		3	11	40	139	466	1528	4953
time	ChordTriaD	0:00:00	0:00:01	0:00:04	0:00:13	0:02:01	0:37:35	12:22:19
	Binomials	0:00:00	0:00:00	0:00:01	0:00:12	0:03:00	4:15:36	-

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We can compute faster all irreducible components of the ideal than the Macaulay2 package "Binomials".

n		1	2	3	4	5	6	7
#components		3	11	40	139	466	1528	4953
time	ChordTriaD	0:00:00	0:00:01	0:00:04	0:00:13	0:02:01	0:37:35	12:22:19
	Binomials	0:00:00	0:00:00	0:00:01	0:00:12	0:03:00	4:15:36	-

Our methods are particularly efficient to compute the highest dimensional components.

	Highest 5 dimensions					Highest 7 dimensions			
n	20	40	60	80	100	10	20	30	40
#comps	404	684	964	1244	1524	2442	5372	8702	12432
time	0:01:07	0:04:54	0:15:12	0:41:52	1:34:05	0:05:02	0:41:41	3:03:29	9:53:09

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- Chordal structure can notably help in computational algebraic geometry.
- Under assumptions (chordality + algebraic structure), tractable!
- Yields practical, implementable algorithms. A Macaulay2 package "Chordal" is in preparation.

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If you want to know more:

- D. Cifuentes, P.A. Parrilo (2017), Chordal networks of polynomial ideals. *SIAM J. of Applied Algebra and Geometry*, 1(1):73-170. `arXiv:1604.02618`.
- D. Cifuentes, P.A. Parrilo (2016), Exploiting chordal structure in polynomial ideals: a Gröbner basis approach. *SIAM J. Discrete Math.*, 30(3):1534-1570. `arXiv:1411.1745`.
- D. Cifuentes, P.A. Parrilo (2016), An efficient tree decomposition method for permanents and mixed discriminants. *Linear Alg. and its Appl.*, 493:45-81. `arXiv:1507.03046`.

Thanks for your attention!