The nearest point to a variety problem, near the variety

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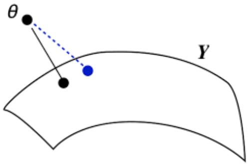
SIAM Conference on Applied Algebraic Geometry - 2017

Nearest point problems

Given a variety $Y \subset \mathbb{R}^n$, and a point $\theta \in \mathbb{R}^n$,

$$\begin{aligned} \min_y \quad & \|y - \theta\|^2 \\ \text{s.t.} \quad & y \in Y \end{aligned}$$



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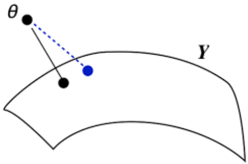

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- *SDP relaxations* have been successful in several applications.

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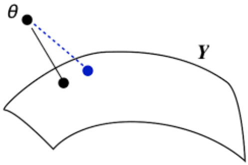

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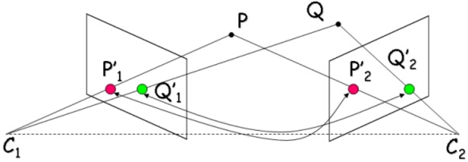
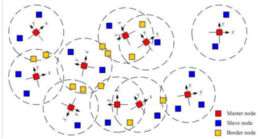
Goal

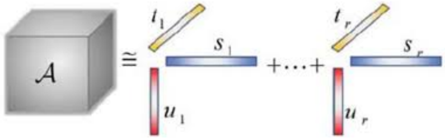
Study the behavior of SDP relaxations in the *low noise* regime: when x is sufficiently close to X.

Nearest point problems

Many different applications







Nearest point to the twisted cubic

$$\min_{y \in Y} \quad \|y - \theta\|^2, \qquad \text{where} \qquad Y := \{(y_1, y_2, y_3) : y_2 = y_1^2,\, y_3 = y_1 y_2\}$$

The twisted cubic Y can be parametrized as $t \mapsto (t : t^2 : t^3)$.

Its Lagrangian dual is the following SDP:

$$\max_{\gamma,\lambda_1,\lambda_2 \in \mathbb{R}} \quad \gamma, \qquad \text{s.t.} \qquad \begin{pmatrix} \gamma+\|\theta\|^2 & -\theta_1 & \lambda_1-\theta_2 & \lambda_2-\theta_3 \\ -\theta_1 & 1-2\lambda_1 & -\lambda_2 & 0 \\ \lambda_1-\theta_2 & -\lambda_2 & 1 & 0 \\ \lambda_2-\theta_3 & 0 & 0 & 1 \end{pmatrix} \succeq 0.$$

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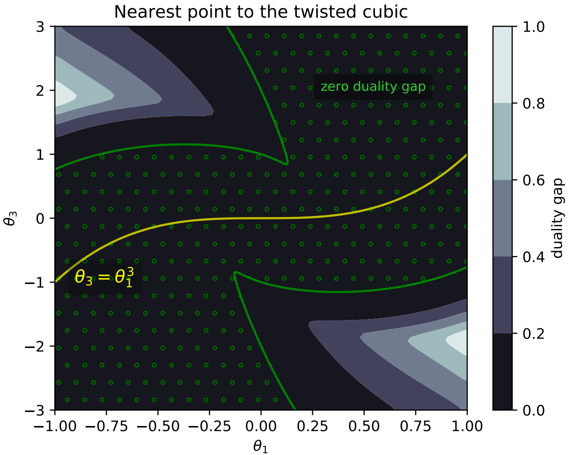

Nearest point problem to a quadratic variety

Theorem

Given quadratic equations f_i, consider

$$\min_{y \in Y} \quad \|y - \theta\|^2, \qquad \textit{where} \qquad Y := \{y \in \mathbb{R}^n : f_1(y) = \cdots = f_m(y) = 0\}$$

Let $\bar{\theta} \in Y$ be such that $\operatorname{rank}(\nabla f(\bar{\theta})) = \operatorname{codim}_{\bar{\theta}} Y$. Then there is zero-duality-gap for any $\theta \in \mathbb{R}^n$ that is sufficiently close to $\bar{\theta}$.

Applications:

- Triangulation problem [Aholt-Agarwal-Thomas]
- Nearest (symmetric) rank one tensor

Parametrized QCQPs

Consider a family of *quadratically constrained programs* (QCQPs):

$$\begin{aligned} \min_{x\in\mathbb{R}^N} \quad & g_\theta(x) \\ & h_\theta^i(x) = 0 \quad \text{for } i = 1,\ldots,m \end{aligned} \tag{P_θ}$$

where g_θ, h_θ^i are *quadratic*, and the dependence on θ is *continuous*.
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Example: For a nearest point problem

$$g_\theta(x) := \|x-\theta\|^2, \qquad h^i(x) \text{ independent of } \theta$$

The problem is trivial for any $\bar\theta \in X$.

SDP relaxation of a (homogeneous) QCQP

Primal problem

$$\min_{x\in\mathbb{R}^N} \quad x^T G_\theta x \qquad\qquad (P_\theta)$$

$$x^T H_\theta^i x = b_i \quad i = 1, \dots, m$$

Dual problem

$$\max_{\lambda\in\mathbb{R}^m} \quad d(\lambda) := -\textstyle\sum_i \lambda_i b_i \qquad\qquad (D_\theta)$$

$$\mathcal{Q}_\theta(\lambda) \succeq 0$$

where $\mathcal{Q}_\theta(\lambda)$ is the Hessian of the Lagrangian

$$\mathcal{Q}_\theta(\lambda) := G_\theta + \sum_i \lambda_i H_\theta^i \in \mathbb{S}^N.$$

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Problem statement

Assume that $\mathrm{val}(P_{\bar\theta}) = \mathrm{val}(D_{\bar\theta})$, i.e., $\bar\theta$ is a *zero-duality-gap* parameter.
Find conditions under which $\mathrm{val}(P_\theta) = \mathrm{val}(D_\theta)$ when θ is close to $\bar\theta$.

Characterization of zero-duality-gap

Given x_θ primal feasible, its *Lagrange multipliers* are:

$$\lambda \in \Lambda_\theta(x_\theta) \iff \lambda^T \nabla h_\theta(x_\theta) = -\nabla g_\theta(x_\theta) \iff \mathcal{Q}_\theta(\lambda) x_\theta = 0.$$

Lemma

Let $x_\theta \in \mathbb{R}^N, \lambda \in \mathbb{R}^m$. Then x_θ is optimal to (P_θ) and λ is optimal to (D_θ) with $\mathrm{val}(P_\theta) = \mathrm{val}(D_\theta)$ iff:

1. $h_\theta(x_\theta) = 0$ *(primal feasibility).*
2. $\mathcal{Q}_\theta(\lambda) \succeq 0$ *(dual feasibility).*
3. $\lambda \in \Lambda_\theta(x_\theta)$ *(complementarity).*

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Proof.

If $\mathcal{Q}_\theta(\lambda) x_\theta = 0$ and $h_\theta(x_\theta) = 0$, then

$$0 = x_\theta^T \mathcal{Q}_\theta(\lambda) x_\theta = x_\theta^T G_\theta x_\theta + \sum_i \lambda_i x_\theta^T H_i x_\theta = g_\theta(x_\theta) - d(\lambda).$$

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Let $\bar{\theta}$ be a zero-duality-gap parameter with $(\bar{x}, \bar{\lambda})$ primal/dual optimal. Assume that

1. $\mathcal{Q}_{\bar{\theta}}(\bar{\lambda})$ *has corank-one (strict-complementarity)*
2. $\exists x_\theta$ *feasible for* $(P_\theta), \lambda_\theta \in \Lambda_\theta(x_\theta)$ *s.t.* $(x_\theta, \lambda_\theta) \xrightarrow{\theta \to \bar{\theta}} (\bar{x}, \bar{\lambda})$.

Then there is zero-duality-gap when θ is close to $\bar{\theta}$.

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- $\mathcal{Q}_\theta(\lambda_\theta) \succeq 0$, so there is zero-duality-gap.

Nearest point to a quadratic variety

$$\min_{y \in Y} \quad \|y - \theta\|^2, \qquad \text{where} \qquad Y := \{y \in \mathbb{R}^n : f_1(y) = \cdots = f_m(y) = 0\}$$

Regularity: *ACQ* holds at $y \in Y$ if $\operatorname{rank}(\nabla f(\bar{\theta})) = \operatorname{codim}_{\bar{\theta}} Y$.

Theorem

Let $\bar{\theta} \in Y$ and assume that ACQ holds at $\bar{\theta}$. Then there is zero-duality-gap for θ close to $\bar{\theta}$.

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- Need to find $\lambda_\theta \in \Lambda_\theta(y_\theta)$ s.t. $\lambda_\theta \xrightarrow{\theta \to \bar{\theta}} 0$.
- ACQ implies $\|\lambda_\theta\| \leq \frac{2}{\sigma(\nabla f)}\|y_\theta - \theta\| \xrightarrow{\theta \to \bar{\theta}} 0$. $\square$

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$\square$

Remark: The theorem generalizes to the case of *strictly convex* objective.

Guaranteed region of zero-duality-gap

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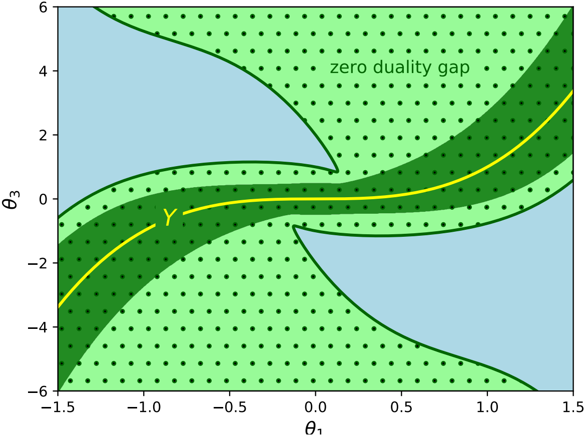


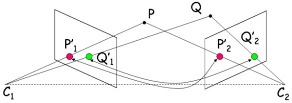
Application: Triangulation [Aholt-Agarwal-Thomas]

Problem

Given noisy images $\hat{u}_j \in \mathbb{R}^2$ of an unknown point,

$$\min_{u \in U} \quad \sum_j \|u_j - \hat{u}_j\|^2$$

where U is the *multiview variety* of the cameras.



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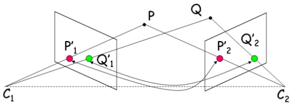

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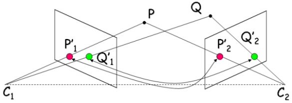

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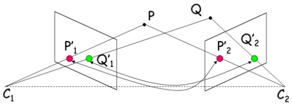

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- The regularity condition (ACQ) is easy to check.
- Under *low noise* the SDP relaxation is tight.

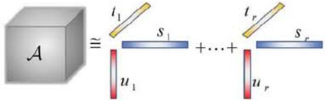
Application: Rank one approximation

Problem

Given a *tensor* $\hat{y} \in \mathbb{R}^{n_1 \times \cdots \times n_\ell}$, consider

$$\min_{y \in Y} \quad \|y - \hat{y}\|^2$$

where Y is the *Segre* variety.



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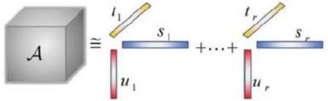

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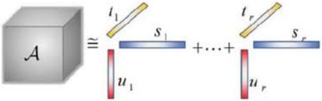
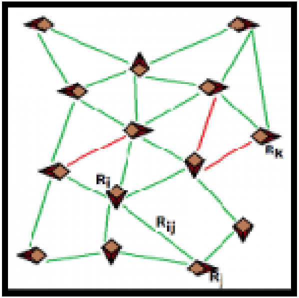


- The Segre variety is defined by quadratics (2×2 minors).
- Thus, the SDP relaxation is tight under low noise.

Application: Rotation synchronization



Problem

Given a graph $G = (V, E)$ and matrices $\hat{R}_{ij} \in \mathbb{R}^{d\times d}$ for $ij \in E$,

$$\min_{R_1,\ldots,R_n \in SO(d)} \sum_{ij\in E} \|R_j - \hat{R}_{ij} R_i\|_F^2$$

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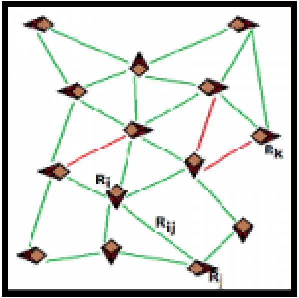

- The objective function is strictly convex.

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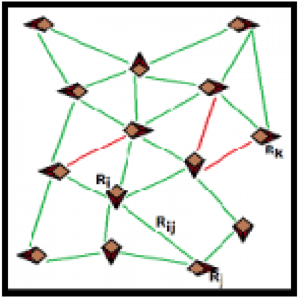

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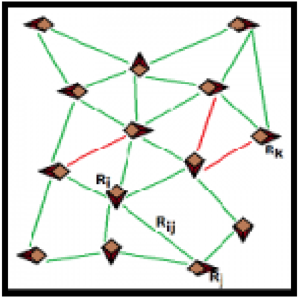


- The objective function is strictly convex.
- Thus, the SDP relaxation is tight under low noise.
- Similar tightness results have been shown [Fredriksson-Olsson], [Rosen-Carlone-Bandeira-Leonard], [Wang-Singer].

Application: Stability of unconstrained SOS

Consider a family of *polynomial optimization* problems

$$\min_{z \in \mathbb{R}^n} \; p_\theta(z), \qquad \text{where } p_\theta \in \mathbb{R}[z]_{2d}$$

and its *sum-of-squares* (SOS) relaxation.

Theorem

Let $\bar{\theta}$ be such that the relaxation is tight, and there is a unique minimizer $\bar{z}$. Consider the face of cone $\Sigma_{n,2d}$:

$$K_{\bar{z}} := \{f \in \Sigma_{n,2d} : f(\bar{z}) = 0\}.$$

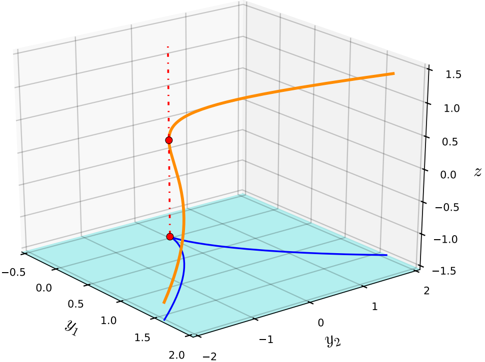
If $p_{\bar{\theta}} - \gamma_{\bar{\theta}} \in \operatorname{int} K_{\bar{z}}$, then the SOS relaxation is tight when θ is close to $\bar{\theta}$.

Nearest point to non-quadratic varieties

Any variety can be described by quadratics by using *auxiliary* variables.

Example: The nearest point problem to the cuspidal curve $y_2^2 = y_1^3$ can be phrased as

$$\min_{y\in\mathbb{R}^2, z\in\mathbb{R}} \quad \|y-\theta\|^2, \qquad \text{s.t.} \qquad y_2 = y_1 z, \quad y_1 = z^2, \quad y_2 z = y_1^2.$$



Beyond quadratic varieties

The nearest point problem to a variety can be phrased as

$$\min_{y\in\mathbb{R}^n, z\in\mathbb{R}^k} \|y-\theta\|^2, \quad \text{s.t.} \quad f_i(y,z)=0, \quad 1\leq i\leq m$$

with f_i quadratic.

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Applications (ongoing): Triangulation problem ($n=3$), camera resectioning, approximate GCD.

Summary

- We analyzed the local stability of SDP relaxations.
- Found sufficient conditions for zero-duality-gap nearby $\bar{\theta}$.
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If you want to know more:

- D. Cifuentes, S. Agarwal, P. Parrilo, R. Thomas, *On the local stability of semidefinite relaxations*, `arXiv:1708.?????`.

Thanks for your attention!