Math 317: Theory Of Linear Algebra, Spring 2024
Homework Assignment 2

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1. Use row operations to describe the set of solutions of

$$\begin{aligned} x + 3y - 5z &= 4 \\ x + 4y - 8z &= 7 \\ -3x - 7y + 9z &= -6 \end{aligned}$$

Sol.

First, we can write the system of equations as an augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 3 & -5 & 4 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & -6 \end{array}\right]$$

Then, we perform the following operations

$$R_2 - R_1 = R_2 \left[\begin{array}{ccc|c} 1 & 3 & -5 & 4 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & -6 \end{array}\right]$$

$$R_3 + 3R_1 = R_3 \left[\begin{array}{ccc|c} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ -3 & -7 & 9 & -6 \end{array}\right]$$

$$R_3 - 2R_2 = R_3 \left[\begin{array}{ccc|c} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 2 & -6 & 6 \end{array}\right]$$

$$R_3 - 2R_2 = R_3 \left[\begin{array}{ccc|c} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R1 - 3R2 = R1 \left[\begin{array}{ccc|c} 1 & 0 & 4 & -5 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Then, we can write the system of equations as

$$\begin{aligned} x + 4z &= -5 \\ y - 3z &= 3 \end{aligned}$$

Or,

$$\begin{aligned} x &= -5 - 4z \\ y &= 3 + 3z \end{aligned}$$

Thus, the set of solutions is given by

$$\mathbf{x} = \begin{bmatrix} -5 - 4z \\ 3 + 3z \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} + z \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$$

$\square$

2. Let $x_1, \ldots, x_{n+1} \in \mathbb{R}$ be distinct, and let $y_1, \ldots, y_{n+1} \in \mathbb{R}$. Prove that there exists a unique degree n polynomial p such that $p(x_k) = y_k$ for all k.

Sol.

Another way we can describe the polynomial $p(x_k)$ is

$$p(x_k) = a_0 + a_1 x_k + a_2 x_k^2 + \ldots + a_n x_k^n = y_k$$

$$p(x_k) = \sum_{i=0}^{n} a_i x_k^i = y_k$$

Consider the Lagrange interpolating polynomial

$$p(x) = \sum_{k=1}^{n+1} y_k \ell_k(x)$$

Where

$$\ell_k(x) = \prod_{i=1, i \neq k}^{n+1} \frac{x - x_i}{x_k - x_i}$$

Note that there are two main scenarios to consider,

If $i = k$ then

$$\frac{x - x_i}{x_k - x_i} = \frac{x - x_k}{x_k - x_k} = 1$$

[] If $i \neq k$ then

$$\frac{x - x_i}{x_k - x_i} = \frac{x - x_i}{x_k - x_i}$$

will tend towards zero and these terms will vanish.

Therefore,

$$p(x_k) = \sum_{k=1}^{n+1} y_k \ell_k(x_k) = y_k$$

As far as uniqueness goes, consider the following.

Suppose that there are two polynomials $p(x)$ and $q(x)$ of degree at most n such that

$$p(x_k) = y_k = q(x_k)$$

Then

$$p(x_k) - q(x_k) = 0$$

has $n + 1$ distinct roots (since it is zero at n + 1 distinct points).

Therefore,

$$p(x) - q(x) = 0$$

$$p(x) = q(x)$$

and the polynomial is unique. □

3. Let V be a vector space. Use induction to show that

$$nv = \sum_{k=1}^{n} v = \underbrace{v + \ldots + v}_{n \text{ times}}$$

holds for all $n \in \{1, 2, 3, \ldots\}$ and $v \in V$.

Sol.

Note, for $n = 1$

$$1v = v$$
$$\sum_{k=1}^{1} v = v\checkmark$$

holds.

Suppose that for $n = 1, 2, 3 \ldots n$

$$nv = \sum_{k=1}^{n} v$$

holds.

Then for $n + 1$

$$(n+1)v = nv + v$$
$$(n+1)v = \sum_{k=1}^{n} v + v$$
$$(n+1)v = \sum_{k=1}^{n+1} v\checkmark$$

Since the statement holds for $n = 1, 2, 3 \ldots n$ and $n + 1$, then it holds for all $n \in \mathbb{N}$. $\square$

4. (a) Prove that $\{[x_1, x_2, x_3] \in \mathbb{R}^3 \colon x_1 + x_2 + x_3 = 0\}$ is a subspace of $\mathbb{R}^3$

Sol.

We want to show that the set is closed under the zero vector property, addition and scalar multiplication.

Firstly,

For any vector $[x_1, x_2, x_3] \in M$ we know that $x_1 + x_2 + x_3 = 0$

Consider the vector $[-x_1, -x_2, -x_3]$

$$\begin{aligned} -x_1 - x_2 - x_3 &= (-1)(x_1 + x_2 + x_3) \\ -x_1 - x_2 - x_3 &= (-1)(0) \\ -x_1 - x_2 - x_3 &= 0 \end{aligned}$$

Thus, $[-x_1, -x_2, -x_3]$, its additive inverse, $\in M$.

Therefore, M is closed under the zero vector property. ✓

Secondly,

For any two vectors, $[x_1, x_2, x_3], [y_1, y_2, y_3] \in M$ we know that

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ y_1 + y_2 + y_3 &= 0 \end{aligned}$$

Then,

$$\begin{aligned} (x_1 + y_1) + (x_2 + y_2) + (x_3 + y_3) &= (x_1 + x_2 + x_3) + (y_1 + y_2 + y_3) \\ (x_1 + y_1) + (x_2 + y_2) + (x_3 + y_3) &= 0 + 0 \\ (x_1 + y_1) + (x_2 + y_2) + (x_3 + y_3) &= 0 \end{aligned}$$

Thus, for any vectors $[x_1, x_2, x_3], [y_1, y_2, y_3] \in M$ we have that

$$[x_1, x_2, x_3] + [y_1, y_2, y_3] \in M$$

Therefore, M is closed under addition. ✓

Lastly,

For any vector $[x_1, x_2, x_3] \in M$ and any scalar $\alpha \in \mathbb{R}$ we know that

$$x_1 + x_2 + x_3 = 0$$

Then for $\alpha \in \mathbb{R}$

$$\alpha[x_1, x_2, x_3] = [\alpha x_1, \alpha x_2, \alpha x_3]$$

Additionally,

$$\begin{aligned} \alpha x_1 + \alpha x_2 + \alpha x_3 &= \alpha(x_1 + x_2 + x_3) \\ \alpha x_1 + \alpha x_2 + \alpha x_3 &= 0 \end{aligned}$$

Therefore, M is closed under scalar multiplication. ✓

Since M is closed under the zero vector property, addition and scalar multiplication, then M is a subspace of $\mathbb{R}^3$. $\square$

(b) Prove that $\{[x_1, x_2, x_3] \in \mathbb{R}^3 \colon x_1 x_2 x_3 = 0\}$ is not a subspace of $\mathbb{R}^3$

Sol.

We want to show that the set is not closed under the zero vector property, addition or scalar multiplication.

Consider the vectors $[1, 0, 0], [0, 1, 0] \in M$

$$\begin{aligned} 1 \cdot 0 \cdot 0 = 0 \\ 0 \cdot 1 \cdot 0 = 0 \end{aligned}$$

Then,

$$[1, 0, 0] + [0, 1, 0] = [1, 1, 0] \notin M$$

Therefore, M is not closed under addition and cannot be a subspace of $\mathbb{R}^{\nVDash}$. $\square$