Math 317: Theory Of Linear Algebra, Spring 2024
Homework Assignment 1

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Problem 1. Let $x, y \in \mathbb{R}$. Prove that $(-x)(-y) = xy$ *Sol.*

We want:

$$(-x)(-y) = xy$$

Note:

$$\begin{aligned}
&0 + x = x \forall x \in \mathbb{R} \\
&0x = 0 \forall x \in \mathbb{R}
\end{aligned}$$

Therefore it follows that:

$$\begin{aligned}
(-x)(-y) &= (-x)(-y) + 0 \\
(-x)(-y) &= (-x)(-y) + 0y \\
(-x)(-y) &= (-x)(-y) + (-x + x)y \\
(-x)(-y) &= (-x)(-y) + (-x)y + xy \\
(-x)(-y) &= ((-x)(-y) + (-x)y) + xy \\
(-x)(-y) &= (-x)((-y) + y) + xy \\
(-x)(-y) &= (-x)(0) + xy \\
(-x)(-y) &= 0 + xy \\
(-x)(-y) &= xy. \square
\end{aligned}$$

Problem 2. Use induction to prove that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Sol.

Base case, $n = 1$

$$\begin{aligned}\sum_{k=1}^{1} k &= \frac{1(1+1)}{2}\\ 1 &= \frac{2}{2}\\ 1 &= 1\checkmark\end{aligned}$$

Assume true for $n = 1, 2, 3 \ldots n$

$$\begin{aligned}\sum_{k=1}^{n} k &= \frac{n(n+1)}{2}\\ \sum_{k=1}^{n} k + (n+1) &= \frac{n(n+1)}{2} + (n+1)\\ \frac{n(n+1)}{2} + \frac{(2n)+2}{2} &= \frac{n(n+1)}{2} + \frac{2(n+1)}{2}\\ \frac{n^2+3n+2}{2} &= \frac{n^2+3n+2}{2}\checkmark\end{aligned}$$

Since the statement holds for $n = 1, 2, 3 \ldots n$ and $n + 1$, then it holds for all $n \in \mathbb{N}$. $\square$

Problem 3. Prove that if $\alpha \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$ are such that $\alpha\mathbf{x} = 0$, then $\alpha = 0$ or $\mathbf{x} = \mathbf{0}$ *Sol.*

We have two cases, $\alpha = 0$ or $\alpha \neq 0$ in which case $\mathbf{x} = \mathbf{0}$.

Case 1: $\alpha = 0$

In this case, the proof is trivial since $\alpha = 0$. We know from class that multipying any vector by 0 will result in $\mathbf{0}$.

Case 2: $\alpha \neq 0$

Since,

$$\alpha \neq 0$$

We know that

$$\frac{1}{\alpha} \in \mathbb{R}$$

Thus,

$$\begin{aligned}
&\frac{1}{\alpha}\alpha\mathbf{x}_k = \frac{1}{\alpha}0 \\
&(\frac{1}{\alpha}\alpha)\mathbf{x}_k = 0 \\
&1\mathbf{x}_k = 0 \\
&\mathbf{x}_k = 0\checkmark
\end{aligned}$$

Since any element $\mathbf{x}_k = 0$, then we know that $\mathbf{x} = \mathbf{0}$. $\square$

Problem 4. Find a square roof i; i.e find a complex number z such that

$$z^2 = i$$

Sol.

Let $z \in \mathbb{C}$ satisfy the proposition

Therefore,

$$\begin{aligned} &z^2 = i \\ &(a+bi)^2 = i \end{aligned}$$

Where $a, b \in \mathbb{R}$ and $a = \Re z, b = \Im z$

Thus,

$$\begin{aligned} &(a+bi)^2 = i \\ &a^2 + 2abi - b^2 = i \\ &(a^2 - b^2) + 2abi = i \end{aligned}$$

We need $a^2 - b^2 = 0$ and $2ab = 1$

This gives:

$$a = \pm b$$

If $a = -b$, then

$$\begin{aligned} &2ab = 1 \\ &2b(-b) = 1 \\ &-2a^2 = 1 \\ &a^2 = -\frac{1}{2} \\ &\text{This is not possible since } a \in \mathbb{R} \end{aligned}$$

If $a = b$, then

$$
\begin{aligned}
2ab &= 1 \\
2b^2 &= 1 \\
b^2 &= \frac{1}{2} \\
b &= \pm\frac{1}{\sqrt{2}}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
z &= a + bi \\
z &= \pm(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i). \square
\end{aligned}
$$

Problem 5. A symmetric matrix is an $n \times n$ matrix such that $A^\top = A$. A skew-symmetric matrix is an $n \times n$ matrix such that $A^\top = -A$

(a) Prove that

$$\frac{1}{2}(A + A^\top) \text{ and } \frac{1}{2}(A - A^\top)$$

are symmetric and skew-symmetric, respectively.

Sol.

Consider

$$(\frac{1}{2}(A + A^\top))^\top$$
$$(\frac{1}{2}A + \frac{1}{2}A^\top))^\top$$
$$(\frac{1}{2}A^\top + \frac{1}{2}A)$$

Thus,

$$(\frac{1}{2}(A + A^\top))^\top = \frac{1}{2}(A + A^\top) \text{ is symmetric.}$$

Now, consider

$$(\frac{1}{2}(A - A^\top))^\top$$
$$(\frac{1}{2}A - \frac{1}{2}A^\top))^\top$$
$$(\frac{1}{2}A^\top - \frac{1}{2}A)$$

Thus,

$$(\frac{1}{2}(A - A^\top))^\top = (-1)(\frac{1}{2}(A - A^\top))$$

(b) Prove that for every $n \times n$ matrix A there exists unique matrices S and K such that S is symmetric, K is skew-symmetric, and $A = S + K$. *Sol.*

We know from (a) that

$$\begin{aligned}&\frac{1}{2}(A + A^\top) \text{ is symmetric. and}\\&\frac{1}{2}(A - A^\top) \text{ is skew-symmetric.}\end{aligned}$$

Thus, let

$$\begin{aligned}S &= \frac{1}{2}(A + A^\top) \text{ and}\\K &= \frac{1}{2}(A - A^\top)\end{aligned}$$

Evaluate,

$$\begin{aligned}S + K &= \frac{1}{2}(A + A^\top) + \frac{1}{2}(A - A\top)\\S + K &= \frac{1}{2}(A + A + A^\top - A^\top)\\S + K &= \frac{1}{2}(2A)\\S + K &= A\checkmark\end{aligned}$$

Therefore, S, K are a symmetric and skew-symmetric tuple of matrices such that $A = S + K$.

Assume that there exists another pair of matrices S', K' such that $A = S' + K'$ and S' is symmetric and K' is skew-symmetric.

Therefore,

$$\begin{aligned}(S + K) - (S' + K') &= A - A\\(S + K) - (S' + K') &= 0\\(S - S') + (K - K') &= 0\end{aligned}$$

Observe that $(S - S')$ is symmetric and $(K - K')$ is skew-symmetric:

$$\begin{aligned}&(S - S')^\top = S^\top - S'^\top\\&(S - S')^\top = S - S'\\&\text{Same applies for } (K - K')\end{aligned}$$

Thus, the matrix

$$(S - S') + (K - K') = 0$$

results in a symmetric and skew-symmetric matrix, which only happens in the zero matrix.

Therefore,

$$(S - S') = 0 \text{ and } (K - K') = 0$$

Thus,

$$S = S' \text{ and } K = K'$$

Finally, we have shown that there exists a unique pair of matrices S, K such that $A = S + K$ and S is symmetric and K is skew-symmetric. $\square$

Problem 6. An idempotent matrix is a matrix A such that $A^2 = A$. Prove that if A is idempotent, then $I - A$ is also idempotent.
Sol.

Let $A \in \mathbb{R}$ be an idempotent matrix.

Evaluate the following:

$$\begin{aligned}
(I-A)^2 &= (I-A)(I-A) \\
(I-A)^2 &= I^2 - IA - AI + A^2 \\
(I-A)^2 &= I - 2A + A^2 \\
(I-A)^2 &= I - 2A + A \\
(I-A)^2 &= I - A
\end{aligned}$$

Thus, $(I - A)$ is idempotent. $\square$