MATH 317, Spring 2024 **Problem Set #3**

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1. Let V be a vector space, and let $v_1, v_2, v_3, v_4 \in V$ be such that $\text{span}(v_1, v_2, v_3, v_4) = V$. Prove that $V = \text{span}(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$.
Sol.

We know that $V = \text{span}(v_1, v_2, v_3, v_4)$.

Then, for any $v \in V$, we can write

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 = v$$
$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 = \mathbf{0}v \to \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$$

Now, suppose we have scalars $\alpha, \beta, \gamma, \delta$ such that

$$\alpha(v_1 - v_2) + \beta(v_2 - v_3) + \gamma(v_3 - v_4) + \delta v_4 = v$$

for some $v \in V$

To show linear independence, we need to show that $\alpha = \beta = \gamma = \delta = 0$

We can rewrite the above equation as

$$\alpha v_1 + (\beta - \alpha)v_2 + (\gamma - \beta)v_3 + (\delta - \gamma)v_4 = v$$

Then setting

$$\alpha_1 = \alpha, \alpha_2 = \beta - \alpha, \alpha_3 = \gamma - \beta, \alpha_4 = \delta - \gamma$$

Yields

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 = v$$

Which we know is linearly independent.

Since we have a four linearly independent vectors, we have a basis for V. □

2. Find a number t such that

$$\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ 9 \\ t \end{bmatrix}$$

is not linearly independent.

If these are not linearly independent, we must have

$$\alpha \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} + \gamma \begin{bmatrix} 5 \\ 9 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

where one of α, β, γ is non-zero.

This can be rewritten as a augmented matrix

$$\begin{bmatrix} 3 & 2 & 5 & | & 0 \\ 1 & -3 & 9 & | & 0 \\ 4 & 5 & t & | & 0 \end{bmatrix}$$

$R_1 - 3R_2 \to R_1$

$$\begin{bmatrix} 0 & 11 & -22 & | & 0 \\ 1 & -3 & 9 & | & 0 \\ 4 & 5 & t & | & 0 \end{bmatrix}$$

$R_3 - 4R_2 \to R_3$

$$\begin{bmatrix} 0 & 11 & -22 & | & 0 \\ 1 & -3 & 9 & | & 0 \\ 0 & 17 & t-36 & | & 0 \end{bmatrix}$$

$\frac{1}{11}R_1 \to R_1$

$$\begin{bmatrix} 0 & 1 & -2 & | & 0 \\ 1 & -3 & 9 & | & 0 \\ 0 & 17 & t-36 & | & 0 \end{bmatrix}$$

$\frac{1}{3}R_2 \to R_2$

$$\begin{bmatrix} 0 & 1 & -2 & | & 0 \\ 1 & -1 & 3 & | & 0 \\ 0 & 17 & t-36 & | & 0 \end{bmatrix}$$

With this we can write the following system of equations

$$\beta - 2\gamma = 0$$
$$\frac{1}{3}\alpha - \beta + 3\gamma = 0$$
$$17\beta + (t-36)\gamma = 0$$

When evaluting it, we get

$$
\begin{aligned}
&\beta = 2\gamma \\
&\frac{1}{3}\alpha - 2\gamma + 3\gamma = 0 \rightarrow \alpha = -3\gamma \\
&17\beta + (t-36)\gamma = 0 \\
&34\gamma + (t-36)\gamma = 0 \\
&\gamma(34 + t - 36) = 0
\end{aligned}
$$

Since we know that $\gamma \neq 0$

$$
\begin{aligned}
&34 + t - 36 = 0 \\
&t - 2 = 0 \\
&t = 2
\end{aligned}
$$

Therefore, the value of t that makes the vectors linearly dependent is $t = 2$. $\square$

3. Suppose that v_1, v_2, v_3, v_4 is a basis for a vector space V. Prove that

$$v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$$

is also a basis.
Sol.

We know that v_1, v_2, v_3, v_4 is a basis for a vector space V.
Therefore, we can write any vector $v \in V$ as a linear combination of v_1, v_2, v_3, v_4 as

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 = v$$

Now, suppose we have scalars $\alpha, \beta, \gamma, \delta$ such that

$$\alpha(v_1 + v_2) + \beta(v_2 + v_3) + \gamma(v_3 + v_4) + \delta v_4 = \mathbf{0}$$

To show linear independence, we need to show that $\alpha = \beta = \gamma = \delta = 0$
Expanding the above equation, we get

$$\begin{aligned} &\alpha v_1 + \alpha v_2 + \beta v_2 + \beta v_3 + \gamma v_3 + \gamma v_4 + \delta v_4 = \mathbf{0} \\ &\alpha v_1 + (\alpha + \beta) v_2 + (\beta + \gamma) v_3 + (\gamma + \delta) v_4 = \mathbf{0} \end{aligned}$$

For this equation to hold, we must have

$$\begin{aligned} &\alpha = 0 \\ &\alpha + \beta = 0 \\ &\beta + \gamma = 0 \\ &\gamma + \delta = 0 \end{aligned}$$

It is trivial to see that the only solution to this system of equations is $\alpha = \beta = \gamma = \delta = 0$.
Therefore, the set $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ is linearly independent.
Since we have a set of four linearly independent vectors, we have a basis for V. $\square$

4. Let $\mathbb{R}_m[x]$ be the vector space of polynomials of degree at most m, and let $p_0, \ldots, p_m \in \mathbb{R}_m[x]$ be such that each p_k has degree k. Prove that $p_0, \ldots, p_m$ is a basis of $\mathbb{R}_m[x]$.
Sol.

> **Note:** p_0 is a polynomial of degree 0.
> Thus,
>
> $$p_0 = \alpha_{0,0}$$
>
> where the scalar for the zeroth degree polynomial is attached to the 0th degree variable.
> As we continue
>
> $$\begin{aligned} &p_1 = \alpha_{1,0} + \alpha_{1,1}t \\ &p_2 = \alpha_{2,0} + \alpha_{2,1}t + \alpha_{2,2}t^2 \\ &\ldots \\ &p_m = \alpha_{m,0} + \alpha_{m,1}t + \alpha_{m,2}t^2 + \cdots + \alpha_{m,m}t^m \end{aligned}$$

Now, suppose we have $\beta_0, \beta_1, \ldots, \beta_m$ such that

$$\beta_0 p_0 + \beta_1 p_1 + \cdots + \beta_m p_m = 0$$

This is equivalent to

$$(\beta_0 + \alpha_{0,0} + \alpha_{1,0} \cdots + \alpha_{m,0}) + \cdots + (\beta_m + \alpha m, 0 + \cdots + \alpha_{m,m})t^m = 0$$

We know that $\alpha_{m,m} \neq 0$ because p_m is a polynomial of degree m.
Therefore

$$\begin{aligned} 0 &= \beta_m \alpha_{m,m} \\ \beta_m &= 0 \end{aligned}$$

Continuing for all $\beta_k | 0 \leq k \leq m$, we get

$$\beta_0 = \beta_1 = \cdots = \beta_m = 0$$

Thus the set $p_0, p_1, \ldots, p_m$ is linearly independent.
Since we have a set of $m+1$ linearly independent vectors, we have a basis for $\mathbb{R}_m[x]$. $\square$