Econometrics: Endogeneity and Instrumental Variables

Diego López Tamayo * Based on MOOC by Erasmus University Rotterdam

Contents

"There are two things you are better off not watching in the making: sausages and econometric estimates." -Edward Leamer

*El Colegio de México, diego.lopez@colmex.mx

Endogeneity

What is endogeneity?

```
dataset4 <- read_csv(
  "https://raw.githubusercontent.com/diego-eco/diego-eco.github.io/master/downloads/dataset4.csv")
```

Dataset 4

Simulated data on performance of 1000 participants of an Engineering MOOC. Performance is measured by Grade Point Average. Background variables are gender, whether participant followed a preparatory mathematics MOOC, and whether the participant received an email invitation for this preparatory MOOC. Variables: - GPA: Grade point average scale 0 to 10, 10 being the best - PARTICIPATION: 0/1 variable indicating participation (1) in preparatory MOOC or not (0) - GENDER: 0/1 variable indicating gender: male (1), female (0) - EMAIL: 0/1 variable indicating whether participant received email invitation for preparatory course (0: no invitation, 1: invitation)

Ordinary least squares (OLS) is a great tool to uncover relationships in economics and business. But we must be aware that this tool does not always work. There are circumstances where OLS breaks down. These circumstances relate to the **difference between correlation and causality**. Luckily, econometrics also has the solution. But we before we discuss this, let's consider a motivating example.

Suppose we want to explain:
- the monthly number of departing flights at an airport (y) - using the number of travel insurances sold in the month before. (x)

What kind of relationship would you expect if you regress flights as the variable y on a constant, and insurances as the variable x? Most likely we will obtain a positive relationship. Suppose OLS yields:

$$y = 10,000 + .25x + e$$

How should we interpret the obtained coefficients? What does the estimate .25 really mean? Suppose we have 4,000 travel insurances sold in the month before:

- **Correct:** 4, 000 insurances sold $\rightarrow$ expected number of flights $= 10,000 + .25Ö4,000 = 11,000$. Because High x tends to go together with high y. The identified correlation yields adequate predictions. Note that this statement merely relies on a correlation.
- **Incorrect:** Selling 4,000 additional insurances causes $.25Ö4,000 = 1,000$ additional flights. The regression does not identify a causal impact! A third variable (travel demand) affects y (flights) and x (insurances).

This example shows that we cannot always interpret least squares estimation results, as causal effects. However, identifying causal effects is one of the main goals of econometrics.

Ordinary least squares requires some assumptions for it to correctly estimate causal effects. One important assumption is that **explanatory variables are exogenous**. The violation of this assumption is called endogeneity.

In the following sections you will:

- Understand/recognize endogeneity
- Know the consequences of endogeneity
- Estimate parameters under endogeneity
- Know the intuition of the new estimator
- Test assumptions underlying this new estimator

Sources of endogeneity

Let us start by studying the source of endogeneity.

The formal assumption that we violate is the assumption that explanatory variables X in the linear model are non-stochastic. (Assumption A2) Explanatory variables are non-stochastic.

Literally speaking, non-stochastic means that if you would obtain new data only the y values would be different and the values for X would stay the same. This is like a *controlled experiment* where the researcher determines the experimental conditions coded in X. This assumption is crucial for the OLS estimator to be consistent. Consistent means that the estimator b converges to the true coefficient beta when the data set grows larger and larger. $b \to \beta$ for $n \to \infty$.

In economics however, controlled experiments are rare. X variables are often the consequence of an economic process, or of individual decision making. In our example, the travelers together determine the number of insurances sold. From the researcher's point of view, the X variables should therefore be seen as stochastic.

Once we allow X to be stochastic, we acknowledge that we would get different X values in a new data set. And if variables are stochastic, they can also be correlated with other variables, even with variables that are not included in the model!

In the context of our example, the number of insurances will be correlated with the travel demand. Although travel demand is difficult to observe and not included in the model, it does influence the number of flights. In the model, travel demand is therefore part of the error term epsilon. As a consequence, the X variable, insurances sold, is correlated with the error term ϵ.

- If X is endogenous $\rightarrow$ there is another variable(s) that affect y and X.
- OLS does not properly estimates β (inconsistent)

Usually, this correlation is due to an omitted factor.

Now let's consider three possible sources of endogeneity in more detail.

1. Endogeneity is often due to an **omitted variable**. In our example, the omitted variable was travel demand. Let's consider this situation formally.

Suppose that the true model for a variable y contains two blocks of explanatory variables, X1 and X2. And that in this true model, all assumptions are satisfied $y = X_1\beta_1 + X_2\beta_2 + \eta$ but we do not observe X_2 and perform OLS on $y = X_1\beta_1 + \epsilon$. The error term in the second model is:

$$\epsilon = X_2\beta_2 + \eta$$

From this relationship we can see that in the second model X_1 will be correlated with epsilon if X_1 and X_2 are correlated and β_2 does not equal 0: $Cov(X_1, X_2)\beta_2 \neq 0$ notice that $Cov(X_1, \eta) = 0$ due to orthogonality.

$$Cov(X_1, \epsilon) = Cov(X_1, X_2\beta_2 + \eta) = Cov(X_1, X_2)\beta_2 + Cov(X_1, \eta)$$

When thinking about whether certain variables in a model are endogenous, it is good to think about potential omitted variables. If you can think of an omitted variable that is related to the included variables, and the dependent variable, you will have endogeneity.

Suppose we run a regression to explain a student's grade using only the number of attended lectures. What omitted variable leads to endogeneity here? There are many possible ommited factors:

- Difficulty of exam? Probably NOT correlated with attendance.
- Motivation of the students? Probably correlates with attendance and affects grade.
- Compulsory attendance yes/no? Does not directly impact the grade

The omission of the motivation of students does lead to endogeneity. Highly motivated students are likely to attend many lectures and obtain high grades. So a regression of grades on attendance will not show the true impact of attendance. It will partly capture the unobserved motivation as well.

2. A second cause of endogeneity is **strategic behavior**.

Consider a model in which you explain the demand for products using only its price. If the salesperson strategically sets high prices when a high demand is expected, high demand will often go together with high prices! A simple regression may then yield a positive price coefficient. This is of course not the true impact of price. **Price is endogenous in this regression** as it correlates with the market information, which in turn, determines demand.

3.A third reason for endogeneity, is **measurement error**.

Suppose that we have a variable y, say, salary, That depends on a factor that is difficult to measure. For example, intelligence. Let's denote the intelligence by x^*. We can obtain a noisy measurement of intelligence, for example through an IQ test. The test score is called x and is equal to the true intelligence plus the measurement error.

$$x = x^* + measurementerror$$

To summarize, endogeneity is a common and serious challenge in econometrics as OLS is not useful under endogeneity.

Formalizing endogeneity

We will show that such measurement error leads to endogeneity in a model that explains why using the IQ test score x in the salary example:

We want to explain the income yi of an individual $i = 1, ..., n$ using the individual's intelligence x_i^*. Suppose that the true relationship between these two variables is:

$$y_i = \alpha + \beta x_i^* + u_i$$

where β gives the impact of intelligence on income. Furthermore, suppose that this model satisfies all the standard assumptions of the linear model. However, the intelligence x_i^* cannot be observed directly. We can only observe a test score that equals the true intelligence plus a measurement error, that is $(1)x = x_i^* + w_i$ The measurement error process w_i satisfies the following conditions:
- Mean zero $E(w_i) = 0$ - Constant variance $Var(w_i) = \sigma_w^2$ - Zero correlation across individuals: $Cov(w_i, w_j) = 0$ $\forall i \neq j$ - Uncorrelated with unexplained income and true intelligene: $Cov(w_i, u_i) = 0$ and $Cov(w_i, x_i^*) = 0$

We have data on (y_i, x_i) for $i = 1, ..., n$ Suppose we ignore measurement error and simply apply OLS to:

$$y_i = \alpha + \beta x_i + \epsilon_i$$

By definition $\epsilon_i = -\beta w_i + u_i$ we can show this just by equalizing the true relation and the estimated model:

$$\alpha + \beta x_i^* + u_i = \alpha + \beta x_i + \epsilon_i$$

Which can be rewritten as:

$$\epsilon_i = \beta(x_i^* - x_i) + u_i = -\beta w_i + u_i$$

Using this $(1)x = x_i^* + w_i$ equation and $(2)\epsilon_i = -\beta w_i + u_i$ we can show that the covariance between x_i and ϵ_i is $Cov(x_i, \epsilon) = -\beta\sigma_w^2$

$$Cov(x_i, \epsilon) = Cov(x_i^* + w_i, -\beta w_i + u_i) = \beta Cov(x_i^*, w_i) + Cov(x_i^*, u_i) - \beta Cov(w_i, w_i) + Cov(w_i, u_i)$$

Where we know by definition that $Cov(x_i^*, w_i) = 0$, $Cov(x_i^*, u_i) = 0$ and $Cov(w_i, u_i) = 0$

$$Cov(x_i, \epsilon) = -\beta\sigma_w^2$$

So x_i is endogenous if the $Cov(x_i, \epsilon) \neq 0$ in this case using the last resutl, it means that the variance of the measurement error $\sigma_w^2 \neq 0$ and that the true impact of inteligence on income $\beta \neq 0$.

Consequences

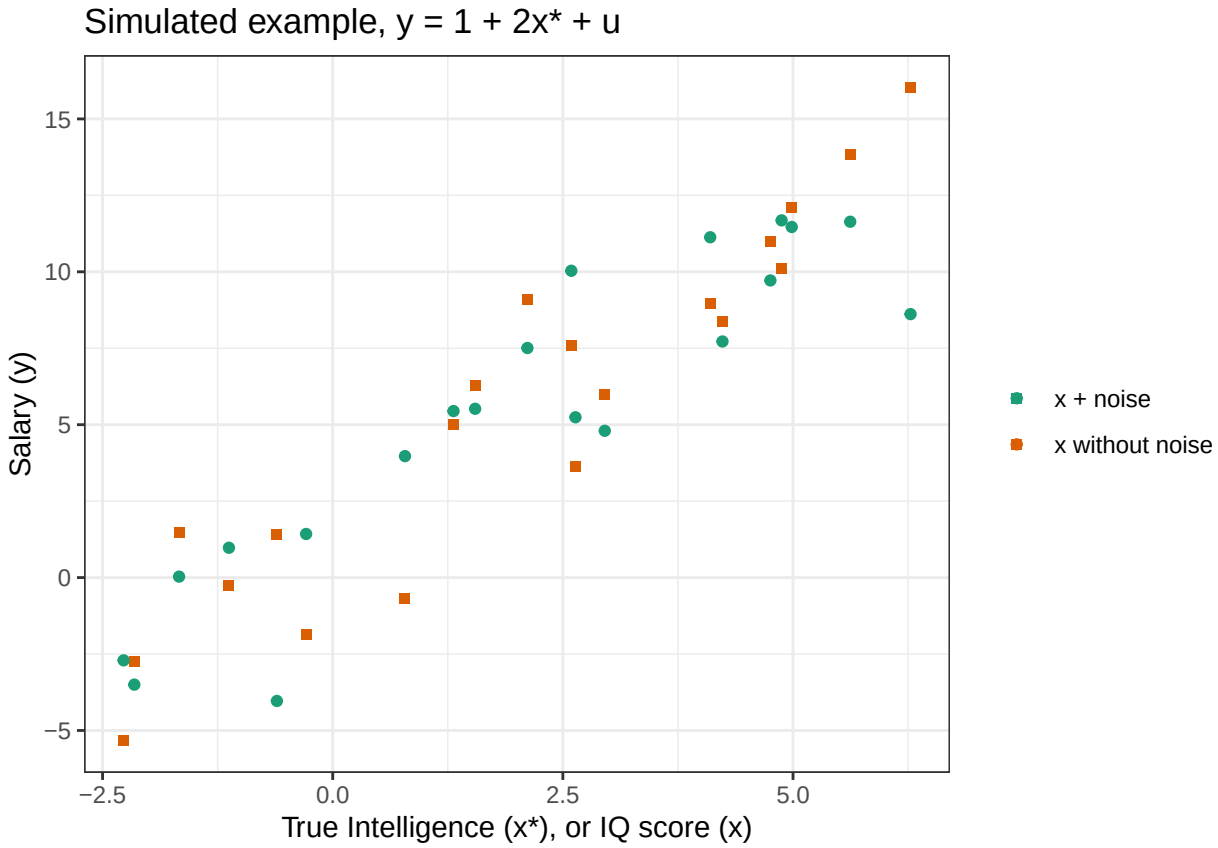
We have discussed three main causes of endogeneity: omitted variables, strategic behavior by people in a market, and the presence of measurement error in an explanatory variable. All three lead to a correlation between explanatory variables X, and the unexplained part in the econometric model, epsilon. This violates the standard assumptions underlying OLS estimation. But, .. how bad is this?

Let's reconsider the measurement error example where salary, denoted by y, depends on intelligence, denoted by x-star. However, in practice we cannot observe intelligence and can only get a noisy measurement, say an IQ score. The noisy measurement is denoted by x. As an illustration, we will use hypothetical data. where we randomly generate intelligence, x^*, and generate $y = 1 + 2x^*$. The IQ score x is generated as $x = x^* + noise$.

Here you see a scatter between y and intelligence x^*. In this new graph, I add a scatter of y versus the IQ score x using orange squares.

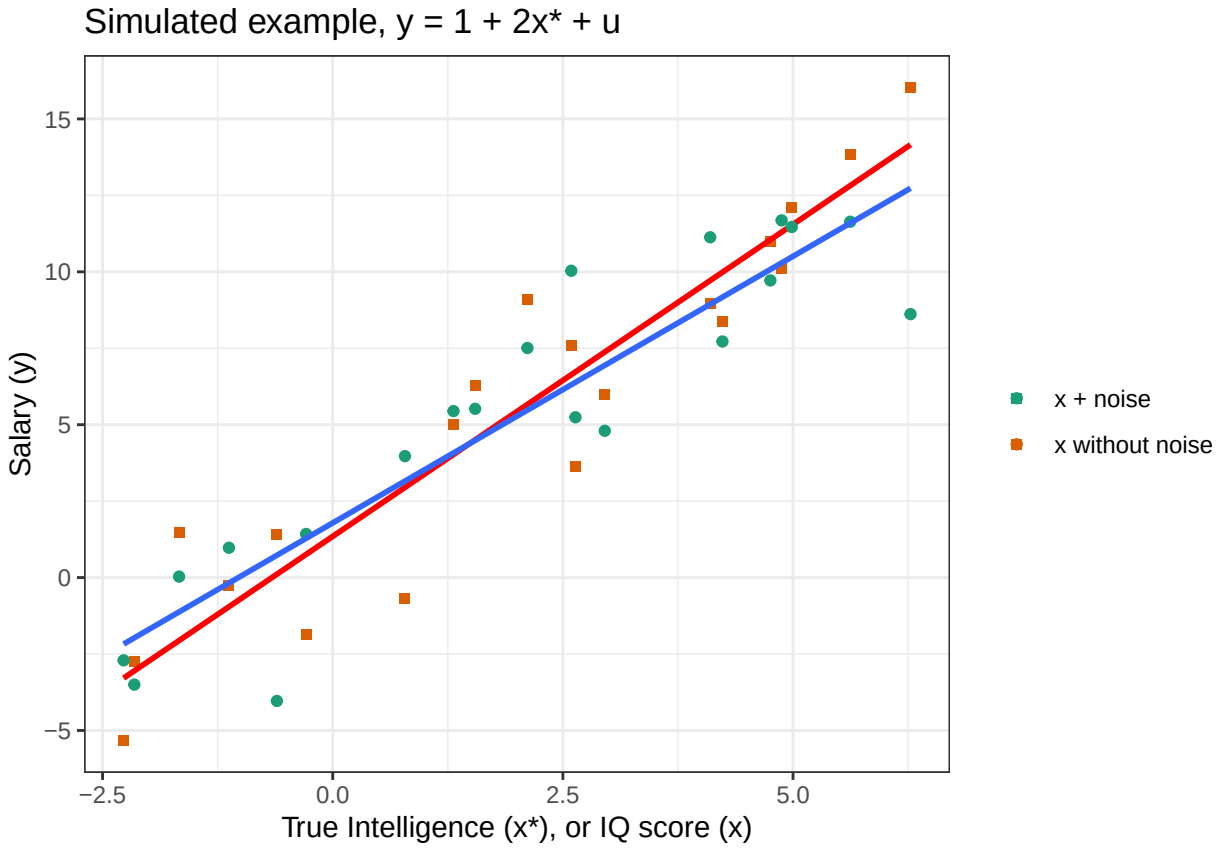
We create the random variables for our example:

```
set.seed(124)
n <-20
x_1 <- rnorm(n, mean = 2, sd = 3)
x_star <- rnorm(n, mean = 2, sd = 3) + rnorm(n, mean = 0, sd = 4)
y_1 <- c(1+2%*%x_1 + rnorm(n, mean = 0, sd = 2))
y_star <- c(1+2%*%x_1 + rnorm(n, mean = 0, sd = 2))
sample <- tibble(x_1,x_star,y_1,y_star)
#str(sample)
plot1 <- ggplot(data=sample, aes(x=x_1)) +
  geom_point(aes(y=y_1, col = "x without noise"),shape=15) +
  geom_point(aes(y=y_star, col = "x + noise")) +
  labs(x = "True Intelligence (x*), or IQ score (x)", y = "Salary (y)",
  title = "Simulated example, y = 1 + 2x* + u") +
  theme_bw() +
  scale_color_brewer(name= NULL, palette = "Dark2")
plot1
```



In practice, we would only have these orange squares as data. The OLS regression, through this cloud of points, is given by this blue line. However, this is not the line we want to have. The true effect of intelligence on salary is stronger (steeper)! This can clearly be seen by the regression line through the green dots. This red line shows the true effect we would like to estimate!

```
plot2 <- ggplot(data=sample, aes(x_1,y_1)) +
  geom_point(aes(y=y_1, col = "x without noise"),shape=15) +
  geom_smooth(method = lm, se = FALSE, colour="red") +
  geom_point(aes(y=y_star, col = "x + noise")) +
  geom_smooth(formula = y_star ~ x, method = lm, se = FALSE) +
  labs(x = "True Intelligence (x*), or IQ score (x)", y = "Salary (y)",
  title = "Simulated example, y = 1 + 2x* + u") +
  theme_bw() +
  scale_color_brewer(name= NULL, palette = "Dark2")
plot2
```



If we use noisy x variables, we obtain the wrong coefficients. This also hold on the endogeneity in general. Can we say anything about the sign of the difference between the true and the estimated effect in case of measurement error?
Under measurement error, OLS is biased towards zero. The estimated line (blue) is not steep enough (as red). As a result, points on the left of the scatter are likely due to negative measurement errors. While points on the right are likely due to positive errors. In other words, measurement error stretches the cloud of points horizontally. This results in a flatter regression line.

> This example illustrates that OLS is biased under endogeneity. However, we only looked at one particular data set with a small number of observations. Would it help to have more data points or different data sets?

Let's consider what happens if we repeat the same experiment many times and for differently sized data sets. For each repetition, we generate a new data set and of course get different estimates. Even for a very large data set, we do not get close to the correct value of the slope parameter.

Things are very different if we would have the noise-free explanatory variable available. In the context of our example, we do as if we can perfectly measure intelligence. It is clear that for all sizes of the data set, OLS on average gives the correct value. And for large data sets, it almost exactly gives the correct value.

If the assumptions underlying OLS are satisfied, OLS is unbiased and consistent.

> If X is endogenous, when n grows the OLS estimator converges to the wrong value. OLS is inconsistent.

We can also show this inconsistency mathematically. Let's consider the standard linear model $y = X\beta + \epsilon$ in combination with the OLS estimator $b = (X'X)^{-1}X'y$. For the y in the formula, we insert the model definition, and next, work out the matrix multiplications.

$$b = (X'X)^{-1}X'(X\beta + \epsilon) = b = (X'X)^{-1}X'X\beta + (X'X)^{-1}X'\epsilon = \beta + (X'X)^{-1}X'\epsilon$$

In the resulting equation, you can see that the first term reduces to beta. So, we can split the OLS estimator into beta, plus a random term that depends on X and epsilon. We use this formulation to see what happens to the estimator when the sample size gets very large.

The first part, beta, is constant, so we only need to study what happens to the second part. Both the term X'X and the term X'epsilon have sums over the observations as elements:

$$X'X = \begin{pmatrix} \sum_{i=1}^{n} x_{1i}^2 & \sum_{i=1}^{n} x_{1i}x_{2i} & \dots & \sum_{i=1}^{n} x_{1i}x_{ki} \\ \sum_{i=1}^{n} x_{2i}x_{1i} & \sum_{i=1}^{n} x_{2i}^2 & \dots & \sum_{i=1}^{n} x_{2i}x_{ki} \\ \dots & \dots & \dots & \dots \\ \sum_{i=1}^{n} x_{ki}x_{1i} & \sum_{i=1}^{n} x_{ki}x_{2i} & \dots & \sum_{i=1}^{n} x_{ki}^2 \end{pmatrix}$$

$$X'\epsilon = \begin{pmatrix} \sum_{i=1}^{n} x_{1i}\epsilon_i \\ \sum_{i=1}^{n} x_{2i}\epsilon_i \\ \dots \\ \sum_{i=1}^{n} x_{ki}\epsilon_i \end{pmatrix}$$

If the number of observations increases, these terms will therefore diverge as $n \to \infty$ However, we can rewrite the estimator such that we are left with terms that do converge. In this equation, we have inserted two $\frac{1}{n}$ terms that cancel against each other.

$$b = \beta + (\frac{1}{n}X'X)^{-1}(\frac{1}{n}X'\epsilon)$$

Notice that $(\frac{1}{n}X'X)^{-1}$ is an average, in general converges to, say Q. The population mean.

The result is that the two matrices now have elements that are averages over the observations. Under mild condition the term $(\frac{1}{n}X'X)^{-1}$ now converges to it's population mean, Q. The second term $(\frac{1}{n}X'\epsilon)$ is also an average over observations and converges in general. The OLS estimator, b, will now **converge to the true parameter beta if three conditions are true**:

$b \to \beta$ as $n \to \infty$ if: 1. $(\frac{1}{n}X'X)^{-1} \to Q$ 2. Q^{-1} exists. 3. $(\frac{1}{n}X'\epsilon) \to 0$

OLS is consistent under these conditions. This third condition is equal to X being exogenous, that is, no correlation between X and the error term.

We have seen what happens when n grows large. However, we have not discussed the **bias**, that is, what happens in small samples. To study the bias, we will need the expected value of $E(\frac{1}{n}X'\epsilon)$. Here, we need to take into account that X is stochastic, and perhaps, correlated with ϵ. Without further assumptions, we just cannot simplify this expectation. However, under endogeneity, this expectation tends to be unequal to zero.

To summarize, if X is endogenous, some variable in X is correlated with the error term epsilon. And OLS is not consistent. This means that even with an infinite amount of data, OLS will not give useful estimates. We will study an alternative estimation method that solves this in the next lecture.

Example on endogeneity

```
dataset <- read_csv(
  "https://raw.githubusercontent.com/diego-eco/diego-eco.github.io/master/downloads/trainexer42.csv")
```

TrainExer42

Simulated data on 250 observations of sales and prices of ice cream under various scenarios of the Data Generating Process [DGP]. The DGP equals:

$$Sales = 100 - 1 \cdot Price + \alpha \cdot Event + \epsilon_1$$

and

$$Price = 5 \cdot \beta Event + \epsilon_2$$

where Event is a 0/1 dummy variable indicating whether a certain event took place. This variable is not included in the data set.

Variables: - PRICE_i: Price variable under β_i for $i = 0, 1, 5, 10$ - SALESj_i: Sales under α_j and β_i

The dataset contains sales and price data for different values of α and β. For each scenario the same simulated values for ϵ_1 and ϵ_2 were used. Specifically, the data contains 4 price series and 16 sales series.

Price variables "Price_i" give the price assuming that $\beta = B_i$ for $B_i = 0, 1, 5, 10$ Sales variables "Sales_ B" give the sales for $\alpha = A_i$ and $\beta = B_i$, for $A_i = 0, 1, 5, 10$.

1. First consider the case where the event only directly affects price ($\alpha = 0$). Estimate and report the price coefficients under all 4 scenarios for β and calculate the R^2 for all these regressions. Do the estimated price coefficients signal any endogeneity problem for these values of α and β? Can you also explain the pattern you find for the R^2?

```
lm1 <- lm(SALES0_0 ~ PRICE0 , data = dataset)
lm2 <- lm(SALES0_1 ~ PRICE1 , data = dataset)
lm3 <- lm(SALES0_5 ~ PRICE5 , data = dataset)
lm4 <- lm(SALES0_10 ~ PRICE10 , data = dataset)
```

Regression Results

Dependent variable:

SALES0_0

SALES0_1

SALES0_5

SALES0_10

lm1

lm1

lm3

lm4

PRICE0

-0.976***

(0.032)

PRICE1

-0.966***

(0.030)

PRICE5

-0.973***

(0.017)

PRICE10

-0.985***

(0.010)

Constant

99.862***

99.808***

99.833***

99.890***

(0.161)

(0.156)

(0.100)

(0.068)

Observations

250

250

250

250

R2

0.794

0.808

0.930

0.977

Adjusted R2

0.794

0.807

0.930

0.977

Residual Std. Error (df = 248)

0.525

0.524

0.523

0.523

F Statistic (df = 1; 248)

958.478***

1,044.203***

3,314.297***

10,491.330***

Note:

p<0.1; ***p<0.05;*** p<0.01

As you can see, the coefficients are all close enough to the true value of -1 so there's no problem here, price is NOT endogenous, as the event does not influence Sales directly. The R^2 increases for higher values of β this is due to the fact that for higher β, more of the variations in sales can be explained.

In other words: for higher $\beta \rightarrow$ Variation in Sales increases $\rightarrow$ Perfectly explained by the Price.

2. Repeat the exercise above, but now consider the case where the event only directly affects sales, that is, set ($\beta = 0$) and check the results for the four different values of α.

```
lm5 <- lm(SALES0_0 ~ PRICE0 , data = dataset)
lm6 <- lm(SALES1_0 ~ PRICE0 , data = dataset)
lm7 <- lm(SALES5_0 ~ PRICE0 , data = dataset)
lm8 <- lm(SALES10_0 ~ PRICE0 , data = dataset)
```

Regression Results

Dependent variable:

SALES0_0

SALES1_0

SALES5_0

SALES10_0

lm5

lm6

lm7

lm8

PRICE0

-0.976***

-0.969***

-0.942***

-0.909***

(0.032)

(0.039)

(0.106)

(0.201)

Constant

99.862***

99.948***

100.294***

100.727***

(0.161)

(0.197)

(0.539)

(1.027)

Observations

250

250

250

250

R2

0.794

0.718

0.243

0.076

Adjusted R2

0.794

0.717

0.240

0.072

Residual Std. Error (df = 248)

0.525

0.642

1.757

3.349

F Statistic (df = 1; 248)

958.478***

631.998***

79.714***

20.395***

Note:

p<0.1; ***p<0.05;*** p<0.01

We can see that all the coefficients again are relatively close to the true value -1, again, Price is not endogenous, as the Event only affects Sales, not Price. So the omission of Event does not lead to a correlation between the Error and Price.

We can see that the R^2 drops significantly for higher values of α. At a high value of alpha, a lot of variation in Sales is due to the Event. However, this variation is not captured in the regression, that's why at higher levels of α the model explains less of the variation.

3. Finally consider the parameter estimates for the cases where the event affects price and sales, that is, look at $\alpha = \beta = 0, 1, 5, 10$. Can you see the impact of endogeneity in this case?

```
lm9 <- lm(SALES0_0 ~ PRICE0 , data = dataset)
lm10 <- lm(SALES1_1 ~ PRICE1 , data = dataset)
lm11 <- lm(SALES5_5 ~ PRICE5 , data = dataset)
lm12 <- lm(SALES10_10 ~ PRICE10 , data = dataset)
```

Regression Results

Dependent variable:

SALES0_0

SALES1_1

SALES5_5

SALES10_10

lm9

lm10

lm11

lm12

PRICE0

-0.976***

(0.032)

PRICE1

-0.874***

(0.036)

PRICE5

-0.273***

(0.033)

PRICE10

-0.085***

(0.021)

Constant

99.862***

99.458***

96.515***

95.515***

(0.161)

(0.187)

(0.197)

(0.146)

Observations

250

250

250

250

R2

0.794

0.706

0.214

0.064

Adjusted R2

0.794

0.705

0.211

0.061

Residual Std. Error (df = 248)

0.525

0.627

1.026

1.119

F Statistic (df = 1; 248)

958.478***

596.815***

67.648***

17.053***

Note:

p<0.1; ***p<0.05;*** p<0.01

We now can see consequences of endogeneity, if $\alpha = \beta \neq 0$, the omition of the Event dummy will lead to correltaion between the Error term $Corr(Price, \epsilon) \neq 0$. As a consequence of the correlation, the estimate can be completely off, as $\alpha = \beta = 10$ shows an estimate close to 0.

The following tables summarize these results.

	beta_0	beta_1	beta_5	beta_10
alpha_0	-0.976	-0.966	-0.973	-0.985
alpha_1	-0.969	-0.874	-0.976	NA
alpha_5	-0.942	NA	-0.273	NA
alpha_10	-0.909	NA	NA	-0.085

Note:
Regression coefficients

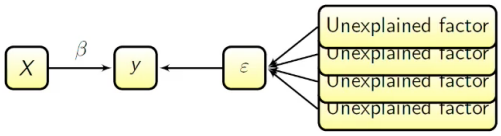
	beta_0	beta_1	beta_5	beta_10
alpha_0	0.794	0.808	0.930	0.977
alpha_1	0.718	0.706	NA	NA
alpha_5	0.243	NA	0.214	NA
alpha_10	0.076	NA	NA	0.064

Note:
Regressions R^2

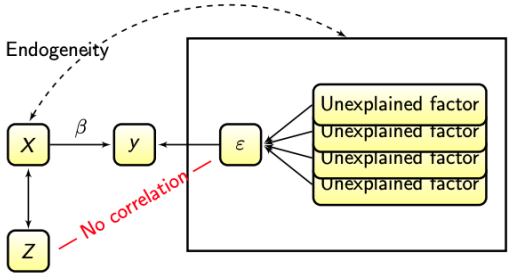
Correcting endogeneity: Instruments

When applying econometrics in practice, there are often important factors that cannot be included in the model due to a lack of data. This often leads to endogeneity and in turn inconsistency of OLS. To gain some intuition, we first represent endogeneity in a graphical way.

Here, you see the standard setup, with the dependent variable y, explanatory variables X, and an error term epsilon. Hidden in the epsilon term are different, unexplained factors. These are factors that affect y but are not included in the model, usually because we have no data on them.



Endogeneity appears if at least one of these unexplained factors is correlated with an X variable. The key to consistently estimating the impact of X on y is to find a set of additional variables. Such variables are called instruments and are usually denoted by Z. The instruments need to satisfy two important properties. First of all they should be correlated with X. Secondly they should not be correlated with the unexplained factors.



To correct for endogeneity we need instruments Z such that, Z and X are correlated but Z does not correlate with epsilon. Under these two conditions any correlation that we find between instruments and y will be due to X. This information can be used to form a new estimator for beta.

Z variables are instruments if:

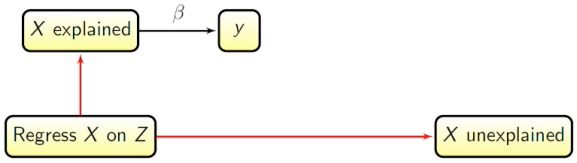
- Z and X are correlated.
- Z does NOT correlate with ϵ

Correlation between instruments and y is only due to X. With $Cov(Z\epsilon) = 0$

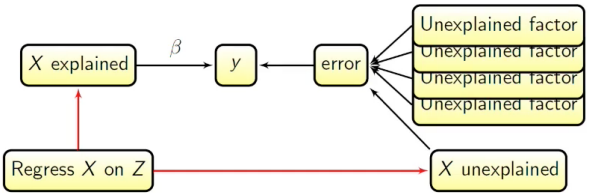
$$Cov(Z, y) = Cov(Z, X\beta + \epsilon) = Cov(Z, X\beta) + Cov(Z\epsilon) = Cov(Z, X)\beta$$

There are two steps to the new estimation procedure:

1. First, we use Z to decompose X in two parts, a part that can be explained by Z and a part that cannot be explained.



2. In the second step, we regress only the explained part of X on y.



The theoretical impact of "X explained" on y equals the true effect size, beta, as the unexplained part of X is simply added to the error term. This solves endogeneity as the unexplained part of X is by construction uncorrelated with the explained part. **So X explained is now exogenous!**

2SLS

This procedure is known as **two-stage least squares, or 2SLS**. Given the linear model and a matrix of instruments Z, we literally need to perform the two steps.

Given model:

$$y = X\beta + \epsilon, Var(\epsilon) = \sigma^2 I$$

And instruments matrix Z

1. Regress X on Z to get explained part : $X = Z\gamma + \eta$

The standard OLS formula applies here. Only the role of X has changed. It is now the dependent variable.

1. Obtain OLS estimator : $\hat{X} = (Z'Z)^{-1}Z'X = H_Z X$

Next, we calculate the explained part of X. Let's denote this part as X hat. X hat can be written as a projection matrix, H of Z times the original matrix of regressors X. (To recall projection matrices look General coefficient estimation). Recall the properties of $H_Z = H'_Z = H'_Z H_Z$, H_Z is symmetric and idempotent.

In the next step, we regress y on X hat using OLS.

2. Regress y on $\hat{X}$

The estimator in this step is the **2SLS estimator, also known as the IV or instrumental variable estimator**. In the first line it is very clear that this estimator is obtained using a standard regression with X hat as explanatory variable.

2. Obtain the 2SLS estimator : $b_{2SLS} = (\hat{X}'\hat{X})^{-1}\hat{X}'y = (X'H_Z H_Z X)^{-1}X'H'_Z y$

Using the properties of $H_Z = H'_Z = H'_Z H_Z$

$$b_{2SLS} = (X'H_ZX)^{-1}X'H_Zy$$

Properties of 2SLS

The **variance** of the 2SLS estimator is calculated as follows, first rewrite the 2SLS estimator:

$$b_{2SLS} = (X'H_ZX)^{-1}X'H_Zy = (X'H_ZX)^{-1}X'H_Z(X\beta + \epsilon) = \beta + (X'H_ZX)^{-1}X'H_Z\epsilon$$
$$Var(b_{2SLS}) = Var((X'H_ZX)^{-1}X'H_Z\epsilon) = (X'H_ZX)^{-1}X'H_ZVar(\epsilon)((X'H_ZX)^{-1}X'H_Z)'$$
$$Var(b_{2SLS}) = \sigma^2(X'H_ZX)^{-1}X'H_ZH_Z'X(X'H_ZX)^{-1} = \sigma^2(X'H_ZX)^{-1}I$$

$$Var(b_{2SLS}) = \sigma^2(X'H_ZX)^{-1}$$

The **standard errors** can easily be obtained from the variance matrix. To estimate sigma squared, it is important to use the correct residuals. The residuals should be in terms of the real X variables, **not** the variables used in the second stage regression.

$$\hat{\sigma}^2 = \frac{1}{n-k}(y - Xb_{2SLS})'(y - Xb_{2SLS})$$

2SLS is consistent if some large sample conditions are satisfied. $2SLS \to \beta$ when $n \to \infty$:

1. Z and ϵ are not correlated: $\frac{1}{n}Z'\epsilon \to 0$.
2. Z itself is not multicollinear: $\frac{1}{n}Z'Z \to Q_{ZZ}$ and Q_{ZZ} invertible.
3. X and Z are sufficiently correlated: $\frac{1}{n}X'Z \to Q_{XZ}$ and Q_{XZ} rank k

The third condition also implies that we must at least have as many instruments as explanatory variables.

Given these conditions the following derivation argues that the 2SLS estimator converges to beta as n grows large. You should take some time to look at the steps:

Use the fact that $H_Z = Z(Z'Z)^{-1}Z'$

$$b_{2SLS} = \beta + (X'H_ZX)^{-1}X'H_Z\epsilon = \beta + (X'Z(Z'Z)^{-1}Z'X)^{-1}X'Z(Z'Z)^{-1}Z'\epsilon$$

$$b_{2SLS} = \beta + (\frac{1}{n}X'Z(\frac{1}{n}Z'Z)^{-1}\frac{1}{n}Z'X)^{-1}\frac{1}{n}X'Z(\frac{1}{n}Z'Z)^{-1}\frac{1}{n}Z'\epsilon$$
$$b_{2SLS} = \beta + (Q_{XZ}Q_{ZZ}^{-1}Q_{XZ}')^{-1}Q_{XZ}Q_{ZZ}^{-1}(0) = \beta + 0 = \beta$$

How to select instruments?

So, if we have instruments Z we can consistently estimate beta. But **how can we obtain instruments?**

First of all, all exogenous explanatory variables in X qualify as instruments.

If there are endogenous variables, additional instruments are needed. To find these, we often need expert knowledge on the topic of the model. For every endogenous variable, we will need to obtain at least one additional instrument. In general, the stronger the correlation between Z and X, the better. However, we need to make sure that there is no correlation between Z and epsilon.

Let us reconsider an earlier example. Suppose we want to explain the grades on a course using the attendance at lectures. We argued before that attendance is endogenous due to omitted variables, such as the student's motivation. Which variables would be good instruments in this case?

They need to be related to attendance but should not affect the grade itself. Two variables that are likely to be good instruments are travel time to university, or if data over multiple years are available, a variable that indicates an introduction of obligatory attendance. Both variables are not likely to impact grades, but are likely to affect attendance. Students living far away may be less likely to attend all classes. And the policy change will likely increase attendance.

Another example: Recall the case where we wanted to explain demand using price, and where a salesperson strategically sets prices. Suppose that the product is ice cream. What variables can you think of as instruments for price?

- Prices of raw materials (valid)
- Competitor prices (direct influence on sales, so part of ϵ)
- Outside temperature (direct influence on sales, so part of ϵ)

In this case, the price of raw materials is likely to be an instrument. An increase in price of raw materials will increase the consumer price. However, the raw materials' price will not likely affect demand directly. In the end, the consumers only care about the price that they need to pay. Variables like competitor price or outside temperature are not valid instruments. These variables are likely to affect demand themselves.

Statistical properties of 2SLS

Consider the linear mode $y = X\beta + \epsilon$ where some variable in the n × k matrix X may be correlated with ϵ. As a result X may be endogenous. Denote by Z an (n × m) matrix of instruments. In general the 2SLS estimator is given by

$$b_{2SLS} = (X'H_ZX)^{-1}X'H_Zy, with H_Z = H_Z = Z(Z'Z)^{-1}Z'$$

We can show that if $m = k$ we can rewrite the 2SLS estimator to $b_{2SLS} = (Z'X)^{-1}Z'y$

Notice first that the dimensions of $X'Z$,$Z'Z$ and $Z'X$ are all dimension (kxk) and by assumption they have an inverse assumming n is large enough. Therefore we can use the rule $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

$$b_{2SLS} = (X'Z(Z'Z)^{-1}Z'X)^{-1}X'Z(Z'Z)^{-1}Z'y = (Z'X)^{-1}(Z'Z)^{-1}(X'Z)^{-1}X'Z(Z'Z)^{-1}Z'y$$

Symplifying:

$$b_{2SLS} = (Z'X)^{-1}IZ'y = (Z'X)^{-1}Z'y$$

Now suppose that there is only a single explanatory variable, that is, the model equals $y = X\beta + \epsilon$ and that there is only a single instrument z, so $m = k = 1$. Furthermore suppose that the means of x, y and z over the sample are equal to 0. We show that we can write the 2SLS estimator of β as $b_{2SLS} = \frac{Cov(y,z)}{Cov(z,x)}$.

Where $Cov(u, v)$ denotes the (sample) covariance between u and v, which is defined as $Cov(u, v) = \frac{1}{n-2}\sum_{i=i}^{n}(u_i - \bar{u})(v_i - \bar{v})$ where $\bar{u}$ and $\bar{v}$ denote the sample variance of u,v respectively.

We can use the definition obtained $b_{2SLS} = (Z'X)^{-1}Z'y$, furthermore, $Z = \begin{pmatrix} Z_1 \\ ... \\ Z_n \end{pmatrix}$ and $X = \begin{pmatrix} X_1 \\ ... \\ X_n \end{pmatrix}$ therefore $X'Z = \sum_{i=i}^{n} z_i x_i$ and $Z'y = \sum_{i=i}^{n} z_i y_i$. The 2SLS estimator can be written as:

$$b_{2SLS} = \frac{\sum_{i=i}^{n} z_i y_i}{\sum_{i=i}^{n} z_i x_i}$$

And as the sample means are equal to 0 this can be rewritten as:

$$b_{2SLS} = \frac{\sum_{i=i}^{n} z_i y_i}{\sum_{i=i}^{n} z_i x_i} = \frac{\sum_{i=i}^{n} (z_i - \bar{z})(y_i - \bar{y})}{\sum_{i=i}^{n} (z_i - \bar{z})(x_i - \bar{x})} = \frac{Cov(y,z)}{Cov(z,x)}$$

Notice that factors $\frac{1}{n-2}$ cancel eachother.

Now we use this last formula in to explain what happens to the 2SLS estimator when the correlation between instruments and the endogenous variable is very small:

From $b_{2SLS} = \frac{Cov(y,z)}{Cov(z,x)}$ we can see that when the correlation between Z,X is zero, the 2SLS estimator is not defined, furthermore correlation between y,Z will also be zero as any correlation between these two variables shoud be due to X. The 2SLS estimator will be 0/0. Although in practice is hard to find 0 correlation.

As consequence you may obtain any number as an estimate when correlation is very low.

Conclusion

2SLS solves endogeneity. However, there is a price that we need to pay. We should only use 2SLS if the explanatory variables are really endogenous. If X is in fact exogenous, OLS and 2SLS are both consistent. However, the Gauss-Markov theorem says that the variance of OLS will never be larger than that of 2SLS.