Quiz 2

- Do not open this quiz booklet until directed to do so. Read all the instructions on this page.
- When the quiz begins, write your name on the top of every page of this quiz booklet.
- You have 90 minutes to earn a maximum of 90 points. Do not spend too much time on any one problem. Skim them all first, and attack them in the order that allows you to make the most progress.
- **You are allowed two double-sided letter-sized sheet with your own notes**. No calculators, cell phones, or other programmable or communication devices are permitted.
- Write your solutions in the space provided. Pages will be scanned and separated for grading. If you need more space, write "Continued on S1" (or S2, S3, S4) and continue your solution on the referenced scratch page at the end of the exam.
- Do not waste time and paper rederiving facts that we have studied in lecture, recitation, or problem sets. Simply cite them.
- When writing an algorithm, a **clear** description in English will suffice. Pseudo-code is not required. Be sure to argue that your **algorithm is correct**, and analyze the **asymptotic running time of your algorithm**. Even if your algorithm does not meet a requested bound, you **may** receive partial credit for inefficient solutions that are correct.
- **Pay close attention to the instructions for each problem**. Depending on the problem, partial credit may be awarded for incomplete answers.

Problem	Parts	Points
0: Information	2	2
1: Through Cycle	1	10
2: Consistent Colors	1	10
3: Useless Vertices	1	15
4: Restaurant Relocation	1	15
5: Disjoint Dimensions	1	18
6: Military Movements	1	20
Total		90

Name: ______________________________

School Email: ______________________________

Problem 0. [2 points] **Information** (2 parts)

(a) [1 point] Write your name and email address on the cover page.

(b) [1 point] Write your name at the top of each page.

Problem 1. [10 points] **Through Cycle**

Given a directed graph $G = (V, E)$ and a vertex $v \in V$, describe an $O(|E|)$-time algorithm to determine whether G contains a cycle through v.

Problem 2. [10 points] **Consistent Colors**

A **consistent coloring** of an undirected graph is an assignment of vertices to colors such that two vertices have the same color if they share an edge. Describe a linear-time algorithm to determine the maximum number of distinct colors in any consistent coloring of a given undirected graph.

Problem 3. [15 points] **Useless Vertices**

A directed weighted graph is **special** if: for every ordered pair of vertices there exists a **unique** minimum weight path between them (having finite total weight). A vertex in a special graph is **useless** if it is not on the unique minimum weight path between any other pair of vertices. Given a special directed weighted graph $G = (V, E)$, possibly containing positive and negative weights, describe an $O(|V|^3)$-time algorithm to determine for every vertex $v \in V$ whether v is useless.

Problem 4. [15 points] **Restaurant Relocation**

Katé Srill, the owner of a food truck in Cambridge, wants to open a restaurant in San Francisco. She decides to drive there and sell food in cities along the way. She gets out her map of all US roads, and marks:

- each of the r roads directly connecting two cities on the map with a positive integer estimating the **cost of tolls** to travel between the two cities along that road, and
- each of the c cities on the map with a positive integer estimating the **profit** she thinks she can make by purchasing ingredients and selling food in that city for one day (she won't spend more than one day **consecutively** in any city).

Given her marked map, describe an efficient algorithm to determine whether Katé can start at Cambridge with some amount of money, and reach San Francisco with at least $1000 **more** than when she started (for her restaurant down payment).

Problem 5. [18 points] **Disjoint Dimensions**

An **interval** (a, b, w) is a triple of integers with left endpoint a, right endpoint $b > a$, and weight w.

- Two intervals (a_1, b_1, w_1) and (a_2, b_2, w_2) are **disjoint** if $b_1 \leq a_2$ or $b_2 \leq a_1$, and a set of intervals are **pairwise disjoint** if each pair of intervals in the set is disjoint.
- The **dimension** of a set of intervals is the sum total of their weights.

Given a set of n intervals, describe an efficient algorithm to determine the maximum dimension of any subset of the intervals that is pairwise disjoint.

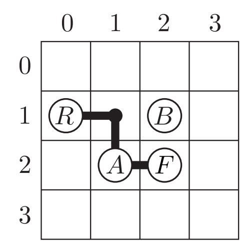
Problem 6. [20 points] **Military Movements**

Uncivilized is a turn-based game where armies move on the tiles of a game board consisting of an $n \times n$ grid of square **tiles**.

- Each **turn** an army may make at most $k = \Theta(\sqrt{n})$ sequential moves between edge-adjacent tiles on the board.
- Exactly n of the tiles contain **cities**; an army must end each turn on a tile containing a city so the army may rest (before the next turn can be taken). Each city will charge an army a positive integer **lodging fee** for resting there.

Given the name, location, and lodging fee of each city on the board, describe an $O(n^2)$-time algorithm to determine the minimum total fee that must be paid by an army that starts rested in **R**eemeen city, and moves to the city of **F**interwell to rest, using as many turns as needed.

For example, if $k = 2$ and the board contains $n = 4$ cities: R at $(0, 1)$ with fee 4, A at $(1, 2)$ with fee 2, B at $(2, 1)$ with fee 3, and F at $(2, 2)$ with fee 1, the minimum total fee for an army at R to move to F would be 3: by making two moves to rest at A on the first turn, and making one move to rest at F on the second turn (note that F is not reachable from R on the first turn).



SCRATCH PAPER 1. DO NOT REMOVE FROM THE EXAM.

You can use this paper to write a longer solution if you run out of space, but be sure to write "Continued on S1" on the problem statement's page.

SCRATCH PAPER 2. DO NOT REMOVE FROM THE EXAM.

You can use this paper to write a longer solution if you run out of space, but be sure to write "Continued on S2" on the problem statement's page.

SCRATCH PAPER 3. DO NOT REMOVE FROM THE EXAM.

You can use this paper to write a longer solution if you run out of space, but be sure to write "Continued on S3" on the problem statement's page.

SCRATCH PAPER 4. DO NOT REMOVE FROM THE EXAM.

You can use this paper to write a longer solution if you run out of space, but be sure to write "Continued on S4" on the problem statement's page.