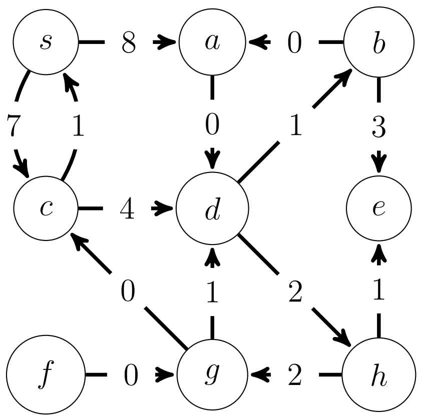
Introduction to Algorithms: 6.006
Massachusetts Institute of Technology — October 25, 2019
Instructors: Jason Ku, Julian Shun, and Virginia Williams — Problem Set 7

Problem Set 7

All parts are due on Novemeber 1, 2019 at 6PM. Please write your solutions in the LaTeX and Python templates provided. Aim for concise solutions; convoluted and obtuse descriptions might receive low marks, even when they are correct. Solutions should be submitted on the course website, and any code should be submitted for automated checking on `alg.mit.edu`.

Problem 7-1. [10 points] **Dijkstra Practice**

(a) [8 points] Run Dijkstra on the following graph from vertex s to every vertex in $V = \{a, b, c, d, e, f, g, h, s\}$. Write down (1) the minimum-weight path weight $\delta(s, v)$ for each vertex $v \in V$, and (2) the order that vertices are removed from Dijkstra's queue.



Solution:

Vertex v	a	b	c	d	e	f	g	h	s
$\delta(s, v)$	8	9	7	8	11	∞	12	10	0
Removal order	3	5	2	4	7	9	8	6	1

Nodes are processed in increasing order of $\delta(s, v)$, which for this graph is unique. The processing order is not always unique; if two vertices have the same shortest path distance, it is possible that Dijkstra could process them in either order, depending on how the implementation of Dijkstra breaks ties in the priority queue. While a and d have the same shortest path distance from s, d is only reachable through a, so a must be processed first.

Rubric:

- 4 points for minimum-weight path weights
- -1 point per incorrect minimum-weight path weight, minimum zero points
- 4 points for vertex order
- -1 point per vertex order inversion, minimum zero points

(b) [2 points] Change the weight of edge (g, c) to -6. Identify a vertex v for which running Dijkstra from s as in part (a) would change the shortest path estimate to v at a time when v is not in Dijkstra's queue.

Solution: If the weight of edge (g, c) were changed to -6, running Dijkstra's would remove vertices from the queue in the same order as in part (a). But then when g is removed from the q, c would have already been popped from the queue and would change the shortest path estimate to c from 7 to 6, so c has the requested property.

Rubric:

- 2 points for correctly identifying vertex c

Problem 7-2. [10 points] **Faster SSSP with Negative Weights**

Given a weighted **undirected dense**[1] graph $G = (V, E)$ (where edge weights are integers that may be either positive or negative), describe an $O(|V|^2)$-time algorithm to solve single-source shortest paths from a given source vertex $s \in V$.

Solution: Note that in an undirected graph, every vertex v is reachable from every other vertex in the same connected component as v. Further, if a connected component contains an edge with negative weight, traversing that edge back and forth constitutes a negative-weight cycle. Thus if the connected component C containing s contains a negative weight edge, then $\delta(s, v) = -\infty$ for every v in C, and $\delta(s, v) = \infty$ for every v not in C. Alternatively, if there is no edge with negative weight in C, then all edge weights in C are non-negative, so we can use Dijkstra to find shortest paths from s to vertices reachable from s. So, find C using breadth-first search or depth-first search in $O(|V| + |E|) = O(|V|^2)$ time, and then check whether any edge in C has negative weight in $O(|V|^2)$ time. If so, then set $\delta(s, v)$ to $-\infty$ for v in C and to ∞ to all other vertices in $O(|V|)$ time. Otherwise, initialize all $\delta(s, v)$ to ∞ and run Dijkstra's algorithm from s in $O(|V| \log |V| + |E|) = O(|V|^2)$ time.

Rubric:

- 2 points for a description of a correct algorithm
- 1 points for a correct argument of correctness
- 1 points for a correct argument of running time
- 4 points if correct algorithm is efficient, i.e., $O(|V|^2)$
- Partial credit may be awarded

Problem 7-3. [10 points] **Weighted Graph Radius**

In a weighted directed graph $G = (V, E)$, the **weighted eccentricity** $\epsilon(u)$ of a vertex $u \in V$ is the shortest weighted distance to its farthest vertex v, i.e., $\epsilon(u) = \max\{\delta(u, v) \mid v \in V\}$. The

[1]Recall, a graph is dense if the number of edges is at least asymptotically quadratic in the number of vertices, i.e., $|E| = \Omega(|V|^2)$.

weighted radius $R(G)$ of a weighted directed graph $G = (V, E)$ is the smallest eccentricity of any vertex, i.e., $R(G) = \min\{\epsilon(v) \mid u \in V\}$. Given a weighted directed graph G, where edge weights may be positive or negative but G contains no negative-weight cycle, describe an $O(|V|^3)$-time algorithm to determine the graph's weighted radius (compare with Problem 2 from PS5).

Solution: Use Johnson's algorithm to compute all-pairs shortest paths distances $\delta(a, b)$ for every ordered distinct pair of vertices (a, b) in $O(|V||E| + |V|^2 \log |V|) = O(|V|^3)$ time (none of which are $-\infty$ since the graph contains no negative-weight cycles). Then, simply compute $\epsilon(u)$ directly for each $u \in V$, by computing a maximum in $O(|V|)$ time per vertex and $O(|V|^2)$ time in total, and then find the smallest $\epsilon(u)$ for $u \in V$ in $O(|V|)$ time. This algorithm is correct because we are performing the requested computation directly and Johnson's computes every $\delta(a, b)$ correctly for any graph not containing negative-weight cycles. The running time for this algorithm is bounded by Johnson's, so runs in $O(|V|^3)$ in total.

The Floyd-Warshall algorithm could also be used in combination with Bellman-Ford, but since we have not yet talked about this algorithm, students would need to fully describe and analyze Floyd-Warshall in order to use it.

Rubric:

- 2 points for a description of a correct algorithm
- 1 points for a correct argument of correctness
- 1 points for a correct argument of running time
- 4 points if correct algorithm is efficient, i.e., $O(|V|^3)$
- Partial credit may be awarded

Problem 7-4. [15 points] **Under Games**

Atniss Keverdeen is a rebel spy who has been assigned to go on a reconnaissance mission to the mansion of the tyrannical dictator, President Rain. To limit exposure, she has decided to travel via an underground sewer network. She has a map of the sewer, composed of n bidirectional pipes which connect to each other at junctions. Each junction connects to at most four pipes, and every junction is reachable from every other junction via the sewer network. Every pipe is marked with its positive integer length, while some junctions are marked as containing identical motion sensors, any of which will be able to sense Atniss if her distance to that sensor (as measured along pipes in the sewer network) is too close. Unfortunately, Atniss does not know the sensitivity of the sensors. Describe an $O(n \log n)$-time algorithm to find a path along pipes from a given entrance junction to the junction below President Rain's mansion that stays as far from motion sensors as possible.

Solution: Construct a graph G with a vertex for every junction in the sewer network and an undirected edge between junction a and junction b with weight w if a and b are connected by a pipe with length w. Since each junction connects to at most a constant number of pipes, the number of edges and vertices in this graph are both upper bounded by $O(n)$. Let s be the vertex associated with the entrance junction, and let t be the junction below President Rain's mansion. Define G_k to be the subgraph of G on only the vertices whose distance to any sensor is strictly larger than k.

Our goal will be to find k^* such that s and t are in the same connected component in G_{k^*}, but s and t are not in the same connected component in G_{k^*+1}, i.e., k^* is the maximum sensor sensativity for which it is still possible for Atniss to reach t from s undetected. If we could find k^*, then we could just return any path from s to t in G_{k^*}.

To find k^*, first label each junction with its shortest distance to any sensor. To do this, construct a new graph G' from G by adding an auxilliary vertex x and an undirected edge of weight zero from x to every vertex in G, and then run Dijkstra from x in $O(n \log n)$ time. Then $\delta(x, v)$ corresponds to the shortest distance labels from v to any sensor as desired. Then we can construct G_k for any k in $O(n)$ time by looping through vertices, removing any vertex v for which $\delta(v, k) \leq k$ (also removing any edge adjacent v). Further, we can check whether t is reachable from s in G_k in $O(n)$ time via breath-first search or depth-first search. We could find k^* in $O(nk^*)$ time by simply constructing G_k for every $k \in \{1, \ldots, k^* + 1\}$, but we can find k^* faster via binary search. Sort the vertices by their distance $\delta(x, v)$ from any sensor in $O(n \log n)$ time using any optimal comparison sort algorithm (e.g., merge sort), and let d_i be the i^{th} largest distance in this sorted order for $i \in \{1, \ldots, n\}$, where d_1 is the smallest and d_n is the largest. Then construct G_{d_i} for $i = \lceil n/2 \rceil$ and check whether t is reachable from s in G_{d_i} in $O(n)$ time. If it is, recurse for $i > \lceil n/2 \rceil$, and if not, recurse for $i < \lceil n/2 \rceil$. The binary search proceeds in $O(\log n)$ iterations that each take at most $O(n)$ time until finding the smallest $d_i = d^*$ such that t is not reachable from s. Then t is reachable from s for $k < d^*$ but not for $k \geq d^*$, so $k^* = d^* - 1$. Finally, we can use either breadth-first search or depth-first search while storing parent pointers to reconstruct some path from s to t in G_{k^*} in $O(n)$ time. If $k^* = 0$, every path from s to t goes through a junction containing a sensor, so any path from s to t may be returned.

Rubric:

- 6 points for description of a correct algorithm
- 2 points for a correct argument of correctness
- 2 points for a correct analysis of running time
- 5 points if correct algorithm is $O(n \log n)$
- Partial credit may be awarded

Problem 7-5. [15 points] **Critter Collection**

Ashley Getem (from PS3) is trying to walk from Trundle Town to Blue Bluff, which are both clearings in the Tanko region. She has a map of all clearings and two-way trails in Tanko. Each of the n clearings connects to at most five trails, while each trail t directly connects two clearings and is marked with its length ℓ_t and capacity c_t of critters living on it, both positive integers. Ashley is a compulsive collector and will collect every critter she comes across by throwing an empty Pocket Sphere at it (which will fill the Pocket Sphere so it can no longer be used). If she encounters a critter without an empty Pocket Sphere, she will be sad. Whenever Ashley reaches a clearing, all critters on all trails will respawn to max capacity. Some clearings contain stores where Ashley can buy empty Pocket Spheres, and deposit full ones. Ashley has more money than she knows what to do with, but her backpack can only hold k Pocket Spheres at a time. Given Ashley's map, describe an $O(nk\log(nk))$-time algorithm to return the shortest route for Ashley to walk from Trundle Town (with a backpack full of empty Pocket Spheres) to Blue Bluff without ever being sad, or return that sadness is unavoidable.

Solution: Construct a graph $G = (V, E)$ with $k + 1$ vertices for each clearing, where vertex $v_{c,i}$ corresponds to being at clearing c while having i empty pocket spheres. Then, for each directed pair of clearings (a, b) for which a and b are connected by a trial t, having length ℓ_t and critters, add the following directed edges:

- If clearing a does not contain a store, add an edge of weight ℓ_t from $v_{a,i}$ to $v_{b,i-c_t}$ for every $i \in \{c_t, \dots, k\}$, since Ashley will use up c_i pocket spheres by traversing the trail.
- Otherwise, if clearing a contains a store, add an edge of weight ℓ_t from $v_{a,i}$ to $v_{b,k-c_t}$ for every $i \in \{k - c_t, \dots, k\}$, since it is never bad for Ashley to completely fill her backpack with empty pocket spheres whenever she leaves a store.

G has $(k + 1)n = O(kn)$ vertices and at most k edges per trail. Since there are at most five trails per clearing, there are at most $5kn = O(kn)$ edges in G, so G has size $O(nk)$. Let s be the vertex associated with Trundle Town and t be the vertex associated with Blue Bluff. Then any path from $v_{s,k}$ to any vertex $v_{t,i}$ for $i \in \{0, \dots, k\}$ in G avoids sadness, and every path that avoids sadness corresponds to a path in G. Since the weights in G are non-negative, we can run Dijkstra, storing parent pointers to reconstruct a shortest route from s to t that avoids sadness in $O(nk\log(nk))$ time. If the shortest path weight to every $v_{t,i}$ in G is infinite, then return that sadness is unavoidable.

Rubric:

- 6 points for description of a correct algorithm
- 2 points for a correct argument of correctness
- 2 points for a correct analysis of running time
- 5 points if correct algorithm is $O(nk\log(nk))$
- Partial credit may be awarded

Problem 7-6. [40 points] **Shipping Servers**

The video streaming service UsTube has decided to relocate across country, and needs to ship their servers by truck from San Francisco, CA to Cambridge, MA. They will pay third-party trucking companies to transport servers from city to city. An intern at UsTube has compiled a list R of all n available trucking routes; each available route $r_i \in R$ is a tuple (s_i, t_i, w_i, c_i) where s_i and t_i are respectively the names[2] of the starting and ending cities of the trucking route, w_i is the positive integer weight capacity of the truck, and c_i is the positive integer cost of shipping along that route (cost is the same for shipping any weight from 0 to w_i). Note that the existence of a shipping route from s_i to t_i does not imply a shipping route from t_i to s_i. Some of UsTube's servers are too heavy

[2] Assume names are ASCII strings that can each be read in constant time.

to fit on any truck, so they will need to transfer them to smaller servers for the move. Assume that it is possible to ship some finite weight from San Francisco to Cambridge via some routes in R. Help UsTube evaluate their shipping options.

(a) [5 points] (Useful Digression) Given a weighted path π, its **bottleneck** is the minimum weight of any edge along the path. Given a directed graph containing vertices s and t, let $b(s,t)$ denote the maximum bottleneck of any path from s to t, and let $I(t)$ denote the set of incoming neighbors of t. Argue that $b(s,t) \geq \min(b(s,v), w(v,t))$ for every $v \in I(t)$, and that $b(s,t) = \min(b(s,v^*), w(v^*,t))$ for at least one $v^* \in I(t)$.

Solution: First, we argue that $b(s,t) \geq \min(b(s,v), w(v,t))$ for every $v \in I(t)$. Suppose for contradiction that there exist two vertices s and t such that $b(s,t) < \min(b(s,v), w(v,t))$. Then there exists some path from s to v for which the minimum weight of any edge along the path is strictly greater than $b(s,t)$. So extending that path along the edge from v to t would result in a path from s to t for which every edge along the path has weight greater than $b(s,t)$, i.e., a larger bottleneck, contradicting that $b(s,t)$ is the maximum bottleneck over all paths.

Next, we argue that $b(s,t) = \min(b(s,v^*), w(v^*,t))$ for at least one $v^* \in I(t)$. Suppose for contradiction that $b(s,t) > \min(b(s,v), w(v,t))$ for every $v \in I(t)$. Then there exists a path π from s to t whose bottleneck equals $b(s,t)$. Let v be the vertex appearing before t along the path. The bottleneck of π cannot be the weight of the edge (v,t) or else $b(s,t) = w(v,t)$; but the bottleneck can also not occur for some other edge in the path from s to v in π, or else $b(s,t) = b(s,v)$, a contradiction.

Rubric:

- 5 points for a correct argument of the claim
- Partial credit may be awarded

(b) [10 points] Assuming that the number of cities appearing in any of the n trucking routes is less than $3\sqrt{n}$, describe an $O(n)$-time algorithm to return both: (1) the weight w^* of the largest single server that can be shipped from San Francisco to Cambridge via a sequence of trucking routes, and (2) the minimum cost to ship such a server with weight w^*.

Solution: Suppose we knew w^*. Then we could construct a graph $G_c = (V_c, E_c)$ with a vertex for every city and a directed edge from vertex s_i to t_i weighted by c_i for every trucking route (s_i, t_i, w_i, c_i) for which $w_i \geq w^*$, and then run Dijkstra from the vertex corresponding to San Francisco to find the minimum cost of shipping weight w^* to Cambridge. This graph has n edges and at most $O(\sqrt{n})$ vertices (by the problem statement), so Dijkstra runs in $O(n)$ time, even when a direct access array or hash table is used for Dijkstra's priority queue.

To find w^*, we construct a graph $G_w = (V_w, E_w)$, similar to G_c, with a vertex for every city and a directed edge from vertex s_i to t_i weighted by w_i for every trucking route (s_i, t_i, w_i, c_i), and let s and t be the vertices associated with San Francisco and Cambridge respectively. The problem is then to determine the maximum bottleneck of any path from s to t. We can modify Dijkstra to compute the bottleneck by replacing shortest path distance estimates with bottleneck estimates $b_s(v)$. Specifically, initialize all bottleneck estimates to zero (since not shipping anything acheives this), except for $b_s(s)$ which we set equal to positive infinity, since we have no limits at the start; and then add them all to a maximum priority queue. Then, repeatedly remove the vertex with largest bottleneck estimate from the priority queue and relax all outgoing edges from it. To relax an edge (u, v) with weight w_i, set $b_s(v) = \max(\min(b_s(u), w_i), b_s(v))$ (either a path through u to v improves on the estimate for $b_s(v)$ or it does not).

To prove that this modification to Dijkstra is correct, we can follow an identical proof structure as the one provided in the Lecture 13 notes, proving correctness of normal Dijkstra. Specifically, we can prove the analogue of Claim 2: for every $v \in V_w$, when v is removed from Dijkstra's queue, $b_s(v) = b(s, v)$. We omit the full proof here. For this problem, students can receive full points for a description of a correct description, without a full proof of correctness.

After the modified Dijkstra has completed, $w^* = b_s(t) = b(s, t)$ by definition of w^*. Since we only changed the relaxation step from one constant time operation to another, and follow the same structure as Dijkstra on a graph with $O(n)$ edges and $O(\sqrt{n})$ vertices, this modified Dijkstra also runs in $O(n)$ time (using a direct access array or hash table for Dijkstra's priority queue).

Rubric:

- 4 points for description of a correct algorithm
- 1 point for a correct argument of correctness (bottleneck proof may be omitted)
- 1 point for a correct analysis of running time
- 4 points if correct algorithm is $O(n)$
- Partial credit may be awarded

(c) [25 points] Write a Python function `ship_server_stats(R, s, t)` that implements your algorithm from (b). You can download a code template containing some test cases from the website. Submit your code online at `alg.mit.edu`.

Solution:

```python
def dijkstra(Adj, w, s):                # modified from R13
    d = [float('inf') for _ in Adj]
    d[s] = 0
    Q = [i for i in range(len(Adj))]    # direct access as priority queue
    while len(Q) > 0:
        u = Q[0]                        # find min distance in Q
        for v in Q:
            if d[v] < d[u]:
                u = v
        Q.remove(u)                     # remove min
        for v in Adj[u]:                # relax outgoing edges
            d[v] = min(d[v], d[u] + w(u, v))
    return d

def dijkstra_bottleneck(Adj, w, s):     # modified from R13
    d = [0 for _ in Adj]                # initially vertices not reached
    d[s] = float('inf')                 # source has arbitrary capacity
    Q = [i for i in range(len(Adj))]
    while len(Q) > 0:
        u = Q[0]                        # find max capacity in Q
        for v in Q:
            if d[v] > d[u]:
                u = v
        Q.remove(u)
        for v in Adj[u]:                # relax outgoing edges
            d[v] = max(d[v], min(d[u], w(u, v)))
    return d

def ship_server_stats(R, s, t):
    n = 0
    city_idx = {}                       # label cities with integers
    for (_s, _t, _w, _c) in R:
        for city in (_s, _t):
            if city not in city_idx:
                city_idx[city] = n
                n += 1
    Adj = [[] for i in range(n)]        # construct graph
    w_w, w_c = {}, {}
    for (_s, _t, _w, _c) in R:
        si, ti = city_idx[_s], city_idx[_t]
        Adj[si].append(ti)
        w_w[(si, ti)], w_c[(si, ti)] = _w, _c
    si, ti = city_idx[s], city_idx[t]   # find max bottleneck weight
    w = dijkstra_bottleneck(Adj, lambda u,v: w_w[(u,v)], si)[ti]
    for i in range(n):                  # remove edges above bottleneck
        for j in Adj[i]:
            if w_w[(i, j)] < w:
                w_c[(i, j)] = float('inf')
    c = dijkstra(Adj, lambda u,v: w_c[(u,v)], si)[ti] # compute min length
    return w, c
```