

Lecture 3: Sorting

Sorting

- Given a sorted array, we can leverage binary search to make an efficient set data structure.
- But how can we sort an array?
- **Input:** (static) array A of n numbers
- **Output:** (static) array B which is a sorted permutation of A
 - **Permutation:** array with same elements in a different order
 - **Sorted:** $B[i - 1] \leq B[i]$ for all $i \in \{1, \dots, n\}$
- Example: $[8, 2, 4, 9, 3] \rightarrow [2, 3, 4, 8, 9]$
- A sort is **destructive** if $B = A$
- A sort is **in place** if it uses $O(1)$ extra space (implies destructive)

Permutation Sort

- There are $n!$ permutations of A , at least one of which is sorted
- For each permutation, check whether sorted in $\Theta(n)$
- Example: $[2, 3, 1] \rightarrow \{[1, 2, 3], [1, 3, 2], [2, 1, 3], [2, 3, 1], [3, 1, 2], [3, 2, 1]\}$

```
1 def permutation_sort(A):
2     '''Sort A'''
3     for B in permutations(A):          # O(n!)
4         if is_sorted(B):              # O(n)
5             return B                  # O(1)
```

- permutation_sort analysis:
 - Correct by case analysis: try all possibilities (Brute Force)
 - Running time: $O(n! \cdot n)$ which is **exponential** :(

Solving Recurrences

- **Substitution:** Guess a solution, replace with representative function, recurrence holds true
- **Recurrence Tree:** Draw a tree representing the recursive calls and sum computation at nodes
- **Master Theorem:** A formula to solve many recurrences (R02)

Selection Sort

- Find a largest number in prefix $A[:i + 1]$ and swap it to $A[i]$
- Recursively sort prefix $A[:i]$
- Example: $[8, 2, 4, 9, 3]$, $[8, 2, 4, 3, 9]$, $[3, 2, 4, 8, 9]$, $[3, 2, 4, 8, 9]$, $[2, 3, 4, 8, 9]$

```

1 def selection_sort(A, i = None):                # T(i)
2     '''Sort A[:i + 1]'''
3     if i is None: i = len(A) - 1                # O(1)
4     if i > 0:                                    # O(1)
5         j = prefix_max(A, i)                    # S(i)
6         A[i], A[j] = A[j], A[i]                  # O(1)
7         selection_sort(A, i - 1)                 # T(i - 1)
8
9 def prefix_max(A, i):                            # S(i)
10    '''Return index of maximum in A[:i + 1]'''
11    if i > 0:                                     # O(1)
12        j = prefix_max(A, i - 1)                 # S(i - 1)
13        if A[i] < A[j]:                           # O(1)
14            return j                              # O(1)
15    return i                                       # O(1)

```

- `prefix_max` analysis:
 - Base case: for $i = 0$, array has one element, so index of max is i
 - Induction: assume correct for i , maximum is either the maximum of $A[:i]$ or $A[i]$, returns correct index in either case. \square
 - $S(1) = \Theta(1), S(n) = S(n - 1) + \Theta(1)$
 - * Substitution: $S(n) = \Theta(n), \quad cn = \Theta(1) + c(n - 1) \implies 1 = \Theta(1)$
 - * Recurrence tree: chain of n nodes with $\Theta(1)$ work per node, $\sum_{i=0}^{n-1} 1 = \Theta(n)$
- `selection_sort` analysis:
 - Base case: for $i = 0$, array has one element so is sorted
 - Induction: assume correct for i , last number of a sorted output is a largest number of the array, and the algorithm puts one there; then $A[:i]$ is sorted by induction \square
 - $T(1) = \Theta(1), T(n) = T(n - 1) + \Theta(n)$
 - * Substitution: $T(n) = \Theta(n^2), \quad cn^2 = \Theta(n) + c(n - 1)^2 \implies c(2n - 1) = \Theta(n)$
 - * Recurrence tree: chain of n nodes with $\Theta(i)$ work per node, $\sum_{i=0}^{n-1} i = \Theta(n^2)$

Insertion Sort

- Recursively sort prefix $A[:i]$
- Sort prefix $A[:i + 1]$ assuming that prefix $A[:i]$ is sorted by repeated swaps
- Example: $[8, 2, 4, 9, 3]$, $[2, 8, 4, 9, 3]$, $[2, 4, 8, 9, 3]$, $[2, 4, 8, 9, 3]$, $[2, 3, 4, 8, 9]$

```

1 def insertion_sort(A, i = None):                # T(i)
2     '''Sort A[:i + 1]'''
3     if i is None: i = len(A) - 1                # O(1)
4     if i > 0:                                    # O(1)
5         insertion_sort(A, i - 1)                # T(i - 1)
6         insert_last(A, i)                        # S(i)
7
8 def insert_last(A, i):                           # S(i)
9     '''Sort A[:i + 1] assuming sorted A[:i]'''
10    if i > 0 and A[i] < A[i - 1]:                 # O(1)
11        A[i], A[i - 1] = A[i - 1], A[i]          # O(1)
12        insert_last(A, i - 1)                    # S(i - 1)

```

- `insert_last` analysis:
 - Base case: for $i = 0$, array has one element so is sorted
 - Induction: assume correct for i , if $A[i] \geq A[i - 1]$, array is sorted; otherwise, swapping last two elements allows us to sort $A[:i]$ by induction \square
 - $S(1) = \Theta(1), S(n) = S(n - 1) + \Theta(1) \implies S(n) = \Theta(n)$
- `insertion_sort` analysis:
 - Base case: for $i = 0$, array has one element so is sorted
 - Induction: assume correct for i , algorithm sorts $A[:i]$ by induction, and then `insert_last` correctly sorts the rest as proved above \square
 - $T(1) = \Theta(1), T(n) = T(n - 1) + \Theta(n) \implies T(n) = \Theta(n^2)$

Merge Sort

- Recursively sort first half and second half (may assume power of two)
- Merge sorted halves into one sorted list (two finger algorithm)
- Example: [7, 1, 5, 6, 2, 4, 9, 3], [1, 7, 5, 6, 2, 4, 3, 9], [1, 5, 6, 7, 2, 3, 4, 9], [1, 2, 3, 4, 5, 6, 7, 9]

```

1  def merge_sort(A, a = 0, b = None):                                # T(b - a = n)
2      '''Sort A[a:b]'''
3      if b is None: b = len(A)                                     # O(1)
4      if 1 < b - a:                                                # O(1)
5          c = (a + b + 1) // 2                                     # O(1)
6          merge_sort(A, a, c)                                     # T(n / 2)
7          merge_sort(A, c, b)                                     # T(n / 2)
8          L, R = A[a:c], A[c:b]                                   # O(n)
9          merge(L, R, A, len(L), len(R), a, b)                   # S(n)
10
11 def merge(L, R, A, i, j, a, b):                                  # S(b - a = n)
12     '''Merge sorted L[:i] and R[:j] into A[a:b]'''
13     if a < b:                                                     # O(1)
14         if (j <= 0) or (i > 0 and L[i - 1] > R[j - 1]):        # O(1)
15             A[b - 1] = L[i - 1]                                  # O(1)
16             i = i - 1                                           # O(1)
17         else:                                                     # O(1)
18             A[b - 1] = R[j - 1]                                  # O(1)
19             j = j - 1                                           # O(1)
20     merge(L, R, A, i, j, a, b - 1)                               # S(n - 1)

```

- merge analysis:
 - Base case: for $n = 0$, arrays are empty, so vacuously correct
 - Induction: assume correct for n , item in $A[r]$ must be a largest number from remaining prefixes of L and R , and since they are sorted, taking largest of last items suffices; remainder is merged by induction \square
 - $S(0) = \Theta(1), S(n) = S(n - 1) + \Theta(1) \implies S(n) = \Theta(n)$
- merge_sort analysis:
 - Base case: for $n = 1$, array has one element so is sorted
 - Induction: assume correct for $k < n$, algorithm sorts smaller halves by induction, and then merge merges into a sorted array as proved above. \square
 - $T(1) = \Theta(1), T(n) = 2T(n/2) + \Theta(n)$
 - * Substitution: Guess $T(n) = \Theta(n \log n)$
 $cn \log n = \Theta(n) + 2c(n/2) \log(n/2) \implies cn \log(2) = \Theta(n)$
 - * Recurrence Tree: complete binary tree with depth $\log_2 n$ and n leaves, level i has 2^i nodes with $O(n/2^i)$ work each, total: $\sum_{i=0}^{\log_2 n} (2^i)(n/2^i) = \sum_{i=0}^{\log_2 n} n = \Theta(n \log n)$