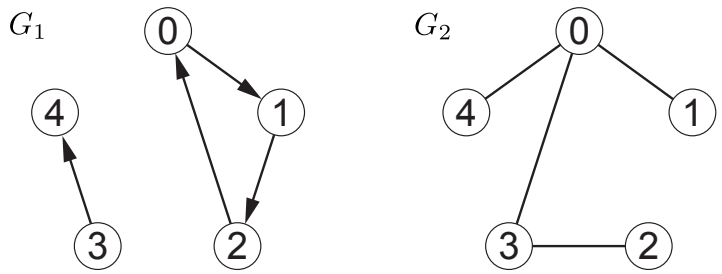
Recitation 9

Graphs

A graph $G = (V, E)$ is a mathematical object comprising a set of **vertices** V (also called nodes) and a set of **edges** E, where each edge in E is a two-element subset of vertices from V. A vertex and edge are **incident** or **adjacent** if the edge contains the vertex. Let u and v be vertices. An edge is **directed** if its subset pair is **ordered**, e.g., (u, v), and **undirected** if its subset pair is **unordered**, e.g., $\{u, v\}$. A directed edge $e = (u, v)$ extends from vertex u (e's **tail**) to vertex v (e's **head**), with e an **incoming** edge of v and an **outgoing** edge of u. In an undirected graph, every edge is incoming and outgoing. The **degree** of a vertex v is the number of edges from the graph incident to v, while **in-degree** and **out-degree** denote the number of incoming and outgoing edges respectively.

As their name suggest, graphs are often depicted **graph**ically, with vertices drawn as points, and edges drawn as lines connecting the points. If an edge is directed, its corresponding line typically includes an indication of the direction of the edge, for example via an arrowhead near the edge's head. Below are examples of a directed graph G_1 and an undirected graph G_2.

$$G_1 = (V_1, E_1) \qquad V_1 = \{0, 1, 2, 3, 4\} \qquad E_1 = \{(0, 1), (1, 2), (2, 0), (3, 4)\}$$
$$G_2 = (V_2, E_2) \qquad V_2 = \{0, 1, 2, 3, 4\} \qquad E_2 = \{\{0, 1\}, \{0, 3\}, \{0, 4\}, \{2, 3\}\}$$



Graph G_2 is **connected**, i.e., for every pair of vertices $u, v \in V$, there exists a sequence of vertices $(u, \ldots, v)$ such that every adjacent pair of vertices are contained in an edge of the graph. By contrast, graph G_1 is not connected, e.g., there is no connection between vertices v_0 and v_4. A **connected component** of a graph is a maximal connected sub-graph. Graph G_1 contains two connected components, while G_2 contains only one. A **path** in a graph is a sequence of vertices $(v_0, \ldots, v_k)$ such that for every ordered pair of vertices (v_i, v_{i+1}), there exists an outgoing edge in the graph from v_i to v_{i+1}. The **length** of a path is the number of edges in the path, or one less than the number of vertices. A graph is called **strongly connected** if there is a path from every node to every other node in the graph. Note that every connected undirected graph is also strongly connected because every undirected edge incident to a vertex is also outgoing. Of the two connected components of directed graph G_1, only one of them is strongly connected.

Graph Representations

There are many ways to represent a graph in code. A common representation is to store with each vertex v, an array $v.Adj$ containing v's outgoing (or incoming) edges. Such an array is called an **adjacency list**. Because every edge from vertex v's adjacency list is incident to v, it is common to store only a list of vertices which form edges with v, and not the edges themselves. When vertices are uniquely labeled from 0 to $|V| - 1$, it is common to store adjacency lists within a direct access array of length $|V|$, where array slot i points to the adjacency list of the vertex labeled i. For example, the following are adjacency list representations of G_1 and G_2.

```
A1 = [[1],                  A2 = [[1, 3, 4],            # 0
      [2],                       [0],                   # 1
      [0],                       [3],                   # 2
      [4],                       [0, 2],                # 3
      []]                        [0]]                   # 4
```

Adjacency lists are a good data structure if you need to loop over the edges incident to a vertex. Each edge appears in any adjacency list at most twice, so the size of an adjacency list representation is $\Theta(|V| + |E|)$. A drawback of this representation is that determining whether your graph contains a given edge (u, v) might require $\Omega(|V|)$ time to step through an adjacency list of u or v. An **adjacency matrix** representation supports checking for an edge in constant time by storing edge information in a direct access array of direct access arrays. Below are adjacency matrix representations of G_1 and G_2. Rows represent indices of the first vertex in an edge pair, while columns represent indices of the second; a '1' in row u and column v represents an edge from u to v.

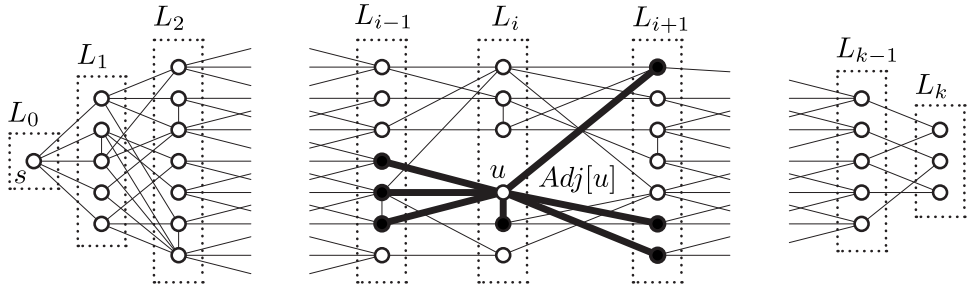
```
M1 = [[0, 1, 0, 0, 0],     M2 = [[0, 1, 0, 1, 1],       # 0
      [0, 0, 1, 0, 0],           [1, 0, 0, 0, 0],       # 1
      [1, 0, 0, 0, 0],           [0, 0, 0, 1, 0],       # 2
      [0, 0, 0, 0, 1],           [1, 0, 1, 0, 0],       # 3
      [0, 0, 0, 0, 0]]           [1, 0, 0, 0, 0]]       # 4
```

Note that the adjacency matrix of an undirected graph is always symmetric. While edge checking runs asymptotically faster with an adjacency matrix, it requires $\Theta(|V|^2)$ space; so when $|E| = o(|V|^2)$, an adjacency matrix will require asymptotically more space than an adjacency list representation. Also, if a vertex v has out-degree $deg(v)$, an adjacency matrix will take $\Theta(|V|)$ time to loop through edges incident to a vertex, while the same operation takes $O(deg(v))$ time using an adjacency list. **Adjacency sets** try to achieve both bounds by using hash tables instead of lists! An adjacency set representation supports edge checking in expected $O(1)$ time, looping over vertex v's edges in worst-case $O(deg(v))$, and can be stored using only $\Theta(|V| + |E|)$ space. Below are adjacency set representations of G_1 and G_2 using Python dictionaries and sets.

```
S1 = {0: {1},               S2 = {0: {1, 3, 4},          # 0
      1: {2},                     1: {0},                # 1
      2: {0},                     2: {3},                # 2
      3: {4}}                     3: {0, 2},             # 3
                                  4: {0}}                # 4
```

Breadth-First Search

Given a graph, a common query is to find the vertices reachable by a path from a queried vertex s. A **breadth-first search** (BFS) from s discovers the **level sets** of s: level L_i is the set of vertices reachable from s via a **shortest** path of length i (not reachable via a path of shorter length). Breadth-first search discovers levels in increasing order starting with $i = 0$, where $L_0 = \{s\}$ since the only vertex reachable from s via a path of length $i = 0$ is s itself. Then any vertex reachable from s via a shortest path of length $i + 1$ must have an incoming edge from a vertex whose shortest path from s has length i, so it is contained in level L_i. So to compute level L_{i+1}, include every vertex with an incoming edge from a vertex in L_i, that has not already been assigned a level. By computing each level from the preceding level, a growing frontier of vertices will be explored according to their shortest path length from s.



Below is Python code implementing breadth-first search for a graph represented using index-labeled adjacency lists, returning a parent label for each vertex in the direction of a shortest path back to s. Parent labels (**pointers**) together determine a **BFS tree** from vertex s, containing some shortest path from s to every other vertex in the graph.

```
1  def bfs(Adj, s):                          # Adj: adjacency list, s: starting vertex
2      parent = [None for v in Adj]          # O(V) (use hash if unlabeled)
3      parent[s] = s                         # O(1) root
4      level = [[s]]                         # O(1) initialize levels
5      while 0 < len(level[-1]):             # O(?) last level contains vertices
6          level.append([])                  # O(1) amortized, make new level
7          for u in level[-2]:               # O(?) loop over last full level
8              for v in Adj[u]:              # O(Adj[u]) loop over neighbors
9                  if parent[v] is None:     # O(1) parent not yet assigned
10                     parent[v] = u         # O(1) assign parent from level[-2]
11                     level[-1].append(v)   # O(1) amortized, add to border
12     return parent
```

How fast is breadth-first search? In particular, how many times can the inner loop on lines 9–11 be executed? A vertex is added to any level at most once in line 11, so the loop in line 7 processes each vertex v at most once. The loop in line 8 cycles through all $deg(v)$ outgoing edges from vertex v. Thus the inner loop is repeated at most $O(\sum_{v \in V} deg(v)) = O(|E|)$ times. Because the parent array returned has length $|V|$, breadth-first search runs in $O(|V| + |E|)$ time.

For graphs G_1 and G_2, conducting a breadth-first search from vertex v_0 yields the parent labels and level sets below.

```
P1 = [0,      L1 = [[0],     P2 = [0,     L2 = [[0],        # 0
      0,            [1],           0,           [1,3,4],    # 1
      1,            [2],           3,           [2],        # 2
      None,         []]            0,           []]         # 3
      None]                        0]                       # 4
```

We can use parent labels returned by a breadth-first search to construct a shortest path from a vertex s to vertex t, following parent pointers from t backward through the graph to s. Below is Python code to compute the shortest path from s to t which also runs in worst-case $O(|V| + |E|)$ time.

```
def unweighted_shortest_path(Adj, s, t):
    parent = bfs(Adj, s)                    # O(V + E) BFS tree from s
    if parent[t] is None:                   # O(1) t reachable from s?
        return None                         # O(1) no path
    i = t                                   # O(1) label of current vertex
    path = [t]                              # O(1) initialize path
    while i != s:                           # O(V) walk back to s
        i = parent[i]                       # O(1) move to parent
        path.append(i)                      # O(1) amortized add to path
    return path[::-1]                       # O(V) return reversed path
```

Exercise: Given an unweighted graph $G = (V, E)$, find a shortest path from s to t having an **odd** number of edges.

Solution: Construct a new graph $G' = (V', E')$. For every vertex u in V, construct two vertices u_E and u_O in V': these represent reaching the vertex u through an even and odd number of edges, respectively. For every edge (u, v) in E, construct the edges (u_E, v_O) and (u_O, v_E) in E'. Run breadth-first search on G' from s_E to find the shortest path from s_E to t_O. Because G' is bipartite between even and odd vertices, even paths from s_E will always end at even vertices, and odd paths will end at odd vertices, so finding a shortest path from s_E to t_O will represent a path of odd length in the original graph. Because G' has $2|V|$ vertices and $2|E|$ edges, constructing G' and running breadth-first search from s_E each take $O(|V| + |E|)$ time.