Problem Set 0

All parts are due Sunday, September 8 at 6PM.

Name: Diego Escobedo

Problem 0-1.
$A = \{1, 2, 4, 8, 16\}$ and $B = \{-1, 1, 3, 5\}$

(a) The intersection of A and B is $\{1\}$

(b) The cardinality of the union of A and B is 8

(c) The cardinality of A minus B is 4

Problem 0-2.

(a) $E[X] = 0 * \frac{2}{5} * \frac{1}{4} + 1 * \frac{2}{5} * \frac{3}{4} + 1 * \frac{3}{5} * \frac{1}{2} + 2 * \frac{3}{5} * \frac{1}{2}$ = 1.2

(b) $E[Y] = 1$

(c) $E[X + Y] = 2.2$

Problem 0-3.

(a) True, both evenly divisible by 2

(b) True, both evenly divisible by 3

(c) False, 606 is not evenly divisible by 4

Problem 0-4.
Theorem: When $a \neq 1,$ for any integer $n \geq 0,$

$$\sum_{i=1}^{n} a^i = \frac{(1 - a^{n+1})}{1 - a}$$

Base Case $n = 2$: If $n = 2$, the left side is $1 + a + a^2$ and the right side is $\frac{1-a^3}{1-a}$.If we move the $1 - a$ in the denominator to the other side, both sides become $1 - a^3$. So, the theorem holds when $n = 2$.

Inductive Step: Let $n = k + 1$. Then our left side is

$$\begin{aligned}
\sum_{i=1}^{k+1} a^i &= a^{k+1} + \sum_{i=1}^{k} a^i & (1)\\
&= \left(a^{k+1}\right) + \frac{(1 - a^{k+1})}{1 - a} \text{ , by our inductive hypothesis} & (2)\\
&= \frac{(a^{k+1})(1 - a)}{(1 - a)} + \frac{(1 - a^{k+1})}{1 - a} & (3)\\
&= \frac{(a^{k+1} - a^{k+2}) + (1 - a^{k+1})}{1 - a} & (4)\\
&= \frac{1 - a^{k+2}}{1 - a} & (5)
\end{aligned}$$

which is our right side. So, the theorem holds for $n = k + 1$. By the principle of mathematical induction, the theorem holds for all $n \geq 0$.

Problem 0-5.
Our inductive hypothesis is that the vertices of a tree (a connected acyclic undirected graph) can each be colored either red or blue such that no edge connects two vertices of the same color.
Consider the trivial base case when there is a tree with only one vertex. In that case, we can color it red or blue and there is no issue.
For our inductive step, consider the case where there are n+1 vertices, we know that there exists a vertex with degree one. If we remove it from the tree, then we know that we have a tree with n nodes, which by our inductive hypothesis can be colored appropriately. We can now add that vertex back to the node it was originally connected to, and simply color it differently. This proves the n+1 case, which completes the induction.