Today: Linear programming (LP)
- → Examples
- → Standard form
- → Geometric view
- → Algorithms
- → LP duality

Example: Optimal campaigning

How to campaign to win an election?
→ Staff estimates votes obtained per $ spent advertising in support of a particular issue

E.g.

	Policy/issue	Affected demographic: Urban	Suburban	Rural
y_1	Building roads	-2	5	3
y_2	Gun control	8	2	-5
y_3	Farm subsidies	0	0	10
y_4	Gasoline tax	10	0	-2

→ Each entry: How many votes will we win/lose per $ spent on advertising on a given issue

→ Goal: Want to win majority in each demographic

	Urban	Suburban	Rural
Population:	100,000	200,000	50,000
Majority:	50,000	100,000	25,000

while spending as little dollars as needed

Let's phrase it in algebraic language!

Algebraic setup:

→ Let y_1, y_2, y_3, y_4 denote \$ spent on advertising on given issue

→ Goal: minimize $y_1 + y_2 + y_3 + y_4$ ← Total \$ spent

subject to

$$-2y_1 + 8y_2 + 0 \cdot y_3 + 10y_4 \geq 50{,}000$$
$$5y_1 + 2y_2 + 0 \cdot y_3 + 0 \cdot y_4 \geq 100{,}000$$
$$3y_1 - 5y_2 + 10y_3 - 2y_4 \geq 25{,}000$$
$$y_1, y_2, y_3, y_4 \geq 0$$

Ensure that we get all majorities

Can't "unadvertise"

→ Optimal solution:

$$\left.\begin{aligned} y_1^* &= \frac{2{,}050{,}000}{111} \approx 18{,}468 \\ y_2^* &= \frac{425{,}000}{111} \approx 3{,}829 \\ y_3^* &= 0 \\ y_4^* &= \frac{625{,}000}{111} \approx 5{,}631 \end{aligned}\right\} \Rightarrow$$

Total \$: $y_1^* + y_2^* + y_3^* + y_4^* =$

$= \frac{3{,}100{,}000}{111}$

$\approx 27{,}928$

→ Ok, but: Where is this coming from?

Linear programming (LP):

Minimize (or maximize) linear objective function

Subject to linear constraints (= inequalities & equalities)

→ Variables: $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \Longleftrightarrow \vec{x} \in \mathbb{R}^n$

→ objective function: $c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \Longleftrightarrow \vec{c} \cdot \vec{x}$

→ constraints: $\sum_i A_{ij} x_i \leq b_j, \; \forall j \in \{1, \ldots, m\}$

$\Longleftrightarrow \quad A \cdot \vec{x} \leq \vec{b} \in \mathbb{R}^m$

↳ $m \times n$ constraint matrix with entries $A_{ij} \in \mathbb{R}$

Standard LP form:

$$\max \; \vec{c} \cdot \vec{x}$$
$$A \cdot \vec{x} \leq \vec{b}$$
$$\vec{x} \geq 0$$

What if the LP I wrote is not in this form?

No problem! Any LP can be transformed into this form!
(i.e., this formulation is universal)

To change:
1. min → max : $\vec{c} \to -\vec{c}$
2. $\geq \; \to \; \leq$: multiply both sides by (-1)
3. $= \; \to \; \leq$: use both $\leq$ & $\geq$ + ②
4. $x_i \in \mathbb{R} \to x_i \geq 0$: introduce new variables $x_i^+ \geq 0$ & $x_i^- \geq 0$ and substitute occurrences of x_i with $(x_i^+ - x_i^-)$

Example:

$$\begin{array}{ll} \min & -x_1+x_2+x_3 \\ \text{s.t.} & x_1-x_3 \le 4 \\ & x_2-x_3=2 \\ & x_1+x_3 \ge 5 \\ & x_2,x_3 \ge 0 \end{array} \quad \longrightarrow \quad \begin{array}{ll} \max & (x_1^+-x_1^-)-x_2-x_3 \\ \text{s.t.} & (x_1^+-x_1^-)-x_3 \le 4 \\ & x_2-x_3 \le 2 \\ & -x_2+x_3 \le -2 \\ & -(x_1^+-x_1^-)-x_3 \le -5 \\ & x_1^+,x_1^-,x_2,x_3 \ge 0 \end{array}$$

Geometric view of LPs:

→ $\vec{x}$ is a point in $\mathbb{R}^n$

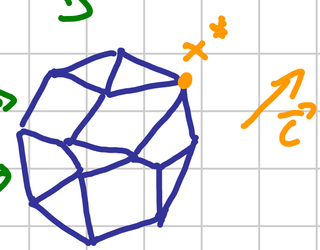
→ $\vec{c}$ is a direction vector (Length irrelevant for determining the optimal solution - gives only scaling of the objective)

⇒ $\max \vec{c}\cdot\vec{x}$ = want most extreme $\vec{x}$ in direction of $\vec{c}$ s.t. constraints

→ constraint $\vec{a}\cdot\vec{x} \le b$ = halfspace bounded by the (hyper) plane

→ together constraints form a polytope with $\le m$ polygonal facets

# of constraints / # of rows of A



• **Important:** the polytope could be unbounded / empty

POSSIBLY NO best solution, i.e., optimal obj. value = +∞ ⇐ unbounded

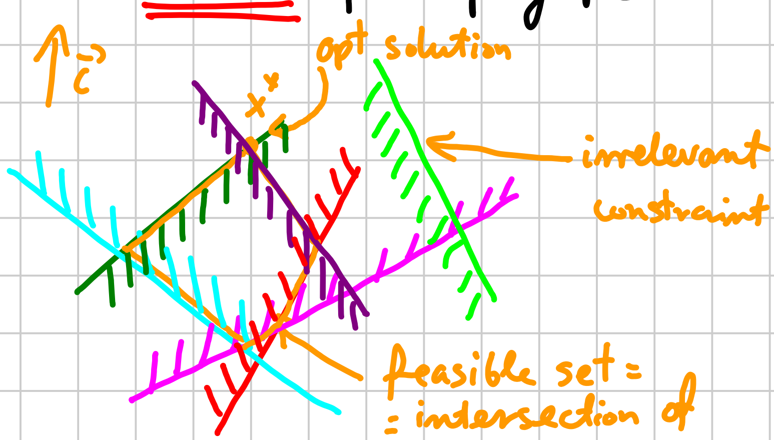
empty ⇒ no solution (= LP is infeasible)

(Note: Polytope might be unbounded but still have a finite opt. obj. value)

→ Can show: if finite opt. obj. value then exists optimal solution x^* that is a vertex of the polytope

LP in 2D:

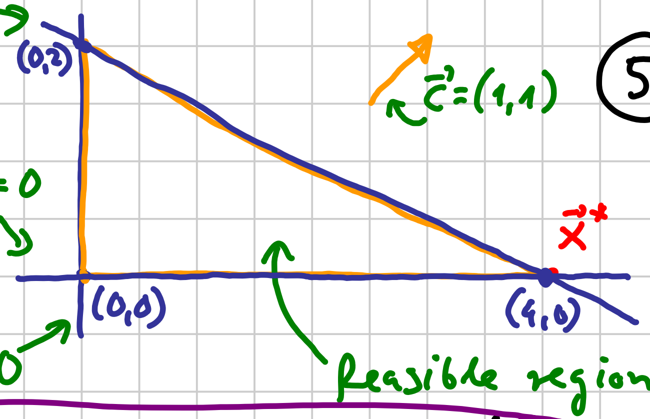
polytope → polygon
halfspace → halfplane
(hyper) plane → line



Example: $\vec{c}=(1,1)$ $\quad x_1+2x_2=4$

$$\max \; x_1+x_2$$
$$\text{s.t.} \; x_1+2x_2 \le 4$$
$$x_1, x_2 \geq 0$$

$x_2=0$ $\quad x_1=0$



$\Rightarrow$ Polygon bounded $\Rightarrow$ need to just check all vertices

$\Rightarrow$ $\vec{x}=(0,0) \to \vec{c}\vec{x}=0$, $\vec{x}=(0,2) \to \vec{c}\vec{x}=2$, $\underbrace{\vec{x}=(4,0) \to \vec{c}\vec{x}=4}_{\text{best obj. value!}}$

$\Rightarrow$ $\boxed{\text{opt. solution } \vec{x}^*=(4,0) \;\&\; \text{opt. objective value } \vec{c}\vec{x}^*=4}$

Exercise: Why is this a case?

How to solve an LP in general?

LP solving algorithms:

① **Simplex:** [Dantzig 1947]

→ current feasible solution $\vec{x}$ walks from vertex to vertex of the feasible polytope, in direction of $\vec{c}$

→ **Runtime:** Practical but exponential in the worst-case

② **Ellipsoid method:** [Khachiyan 1979]

→ Maintains ellipsoid that is guaranteed to contain $\vec{x}^*$

→ In each step, shrinks the ellipsoid

→ **Runtime:** Polynomial in the worst-case, useful in theory, but impractical

# of bits to describe $A, \vec{b}, \vec{c}$

③ **Interior-point method:** [Karmarkar 1984]

$O(m^{3.5} L)$

# of constrains

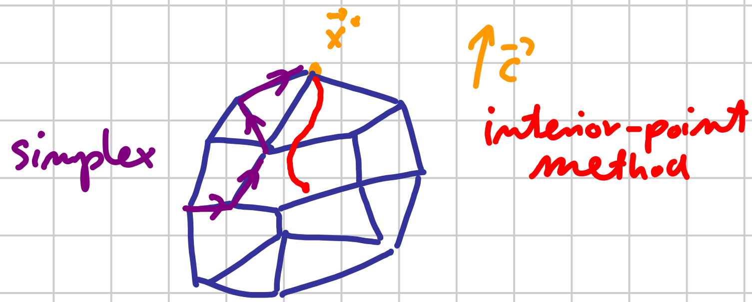
→ Current feasible solution $\vec{x}$ moves inside the feasible polytope, vaguely in direction of $\vec{c}$

→ **Runtime:** Polynomial in the worst case & quite practical

Simplex vs. Interior-point method

Simplex
- → moves on the edges of the polytope
- → highly attuned to the combinatorial structure of the constraints

Interior-point method
- → moves inside the polytope
- → ignores much of the combinatorial structure of the constraints



CAUTION: A "close" relative to linear programming, called integer linear programming (ILP) is NP-complete ← means: (probably) REALLY HARD ! (in the worst case)

ILP: $\max\ \vec{c}\cdot\vec{x}$
s.t. $A\vec{x} \leq \vec{b}$
$\vec{x} \geq 0$
& $x_1, x_2, \dots, x_n$ integral ← very difficult to deal with algorithmically

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LP Duality:

Recall the example LP we considered earlier:

$$\min\ y_1 + y_2 + y_3 + y_4$$

$$\text{s.t.}\quad -2y_1 + 8y_2 + 0\cdot y_3 + 10y_4 \geq 50{,}000 \quad (1)$$

$$5y_1 + 2y_2 + 0\cdot y_3 + 0\cdot y_4 \geq 100{,}000 \quad (2)$$

$$3y_1 - 5y_2 + 10y_3 - 2y_4 \geq 25{,}000 \quad (3)$$

$$y_1, y_2, y_3, y_4 \geq 0$$

Earlier: we exhibited a feasible solution with obj. value $\frac{3{,}100{,}000}{111}$ and claimed this is best possible.

Can we somehow substantiate this claim?

Yes! **Note:** Any feasible solution to that LP has to satisfy constraints (inequalities) (1), (2), (3)

$\Rightarrow$ Consider taking the following linear combination of them:

$$\left.\begin{array}{l} \frac{25}{222}\cdot(1) \\ \frac{46}{222}\cdot(2) \\ \frac{14}{222}\cdot(3) \end{array}\right\} + \Rightarrow \underbrace{\overbrace{y_1 + y_2 + \frac{140}{222} y_3 + y_4}^{(B)} \geq \frac{3{,}100{,}000}{111}}_{(A)}$$

$\Rightarrow$ (A) has to be satisfied by any feasible solution

$\Rightarrow$ But: $\vec{c}\cdot\vec{y} = y_1 + y_2 + y_3 + y_4 \geq (B) \underset{(A)}{\geq} \frac{3{,}100{,}000}{111}$

$\Rightarrow$ Every feasible solution has to have obj. value $\geq \frac{3{,}100{,}000}{111}$

$\Rightarrow$ This proves that $\frac{3{,}100{,}000}{111}$ is indeed best possible here!

This is great, but where did the "magic" coefficients $\left(\frac{25}{222}, \frac{46}{222}, \frac{14}{222}\right)$ come from ???

Key insight: Finding such (in fact, "best" such coeff.) "magic" coefficients can be phrased by itself as an LP!

Specifically: Given an LP (in standard form)

"Primal" LP: (P)

$$\begin{aligned} \max\ & \sum_i c_i x_i \\ \text{s.t.}\ & \sum_i A_{ij} x_i \leq b_j \quad \forall j \\ & x_i \geq 0 \quad \forall i \end{aligned}$$

Compactly:

$$\begin{aligned} \max\ & \vec{c}\cdot\vec{x} \\ \text{s.t.}\ & A\vec{x} \leq \vec{b} \\ & \vec{x} \geq 0 \end{aligned}$$

We consider a "dual" LP

"Dual" LP: (D)

$$\begin{aligned} \min\ & \sum_j b_j y_j \\ \text{s.t.}\ & \sum_j A_{ij} y_j \geq c_i \quad \forall i \\ & y_j \geq 0 \quad \forall j \end{aligned}$$

$$\begin{aligned} \min\ & \vec{b}\cdot\vec{y} \\ \text{s.t.}\ & A^T\vec{y} \geq \vec{c} \\ & \vec{y} \geq 0 \end{aligned}$$

Intuition:
- $\vec{x} \in \mathbb{R}^{m} \rightarrow x_i$ corresponds to i-th constraint in (D)
- $\vec{y} \in \mathbb{R}^{m} \rightarrow y_j$ — " — j-th constraint in (P)

Weak LP Duality: If $\vec{x}$ is a feasible solution to (P) & $\vec{y}$ is a feasible solution to (D)

Then: $\boxed{\vec{c}\cdot\vec{x} \leq \vec{b}\cdot y}$

In other words: Obj. value of any feasible $\vec{y}$ provides an upper bound on obj. value of any feasible $\vec{x}$ & vice versa

$\Rightarrow$ If $\vec{x}^*$ is opt. solution to (P) & $\vec{y}^*$ an opt. solution to (D) then

$$\boxed{\max_{\substack{A\vec{x}\leq\vec{b} \\ \vec{x}\geq 0}} \vec{c}\cdot\vec{x} = \vec{c}\cdot\vec{x}^* \leq \vec{b}\cdot\vec{y}^* = \min_{\substack{A^T\vec{y}\geq\vec{c} \\ \vec{y}\geq 0}} \vec{b}\cdot y}$$

Proof of Weak LP Duality:

(Mirrors our reasoning from the example. Now, $\vec{y}$ gives the "magic" coefficients for (P) & vice versa)

$\rightarrow$ Taking a linear combination of constraints of (P) with $\vec{y}$ as coeff.:

All $y_j \geq 0$, so no $\leq$ flips!

$$\Rightarrow \quad \sum_j y_j \left(\sum_i A_{ij} x_i\right) \leq \sum_j y_j b_j$$

$$\Rightarrow \quad \vec{y} A \vec{x} = \sum_{i,j} A_{ij}\, y_j x_i \leq \sum_j y_j b_j = \vec{b}\,\vec{y} \qquad (*)$$

$\rightarrow$ Taking a linear combination of constraints of (D) with $\vec{x}$ as coeff.

Again: $\vec{x} \geq 0$

$$\Rightarrow \quad \sum_i x_i \sum_j A_{ij} y_j \geq \sum_i x_i c_i = \vec{c}\,\vec{x}$$

$$\Rightarrow \quad \vec{x} A^T \vec{y} = \sum_{i,j} A_{ij} x_i y_j \geq \vec{c}\cdot\vec{x} \qquad (**)$$

identity $vMu = uM^Tv$

$$\Rightarrow \quad \vec{c}\,\vec{x} \overset{(**)}{\leq} \vec{x} A^T \vec{y} = \vec{y} A \vec{x} \overset{(*)}{\leq} \vec{b}\,\vec{y}$$

$\blacksquare$

So, we know that if OPT(P) is the best obj. value for (P) & OPT(D) is the best obj. value for (D) then

$$\boxed{OPT(P) \leq OPT(D)}$$

(Note: We assume here that both OPT(P) and OPT(D) are defined & finite, i.e., that neither (P) nor (D) are infeasible or unbounded)

Question: Is this inequality ever strict? That is, are the "magic coefficients arguments" not exhaustive?

Suprisingly (?): NO!

Strong LP duality: If $\vec{x}^*$ & $\vec{y}^*$ are optimal feasible solutions to (P) & (D), respectively, then:

[We skip the proof of this thm]

$$\boxed{\vec{c} \cdot \vec{x}^* = \vec{b}\, \vec{y}^*}$$

Moreover: Only one of the following four possibilities exists:

→ Both (P) & (D) have optimal solutions (as above)

→ (P) is unbounded & (D) is infeasible

→ (D) is unbounded & (P) is infeasible

symmetric

→ Both (P) & (D) are infeasible (not too interesting but possible!)

Note: → y^* delivers best "magic" coefficients to prove optimality of x^* & vice versa

→ Roles of (P) & (D) are completely symmetric

⇒ $\boxed{\text{"Dual of (D)"} = (P)}$

→ Max Flow Min Cut Thm is a special case of the above strong duality → See Recitation notes