Problem Set 6

This problem set is due **at 10:00pm** on **Wednesday, April 15, 2020**.

Please make note of the following instructions:

- This assignment, like later assignments, consists of *exercises* and *problems*. **Hand in solutions to the problems only.** However, we strongly advise that you work out the exercises for yourself, since they will help you learn the course material. You are responsible for the material they cover.
- Remember that the problem set must be submitted on Gradescope. If you haven't done so already, please signup for 6.046 Spring 2020 on Gradescope, with the entry code MNEBKP, to submit this assignment.
- We require that the solution to the problems is submitted as a PDF file, **typeset on LaTeX**, using the template available in the course materials. Each submitted solution should start with your name, the course number, the problem number, your recitation section, the date, and the names of any students with whom you collaborated.
- You will often be called upon to "give an algorithm" to solve a certain problem. Your write-up should take the form of a short essay. A topic paragraph should summarize the problem you are solving and what your results are. The body of your essay should provide the following:
  1. A description of the algorithm in English and, if helpful, pseudocode.
  2. A proof (or indication) of the correctness of the algorithm.
  3. An analysis of the asymptotic running time behavior of the algorithm.
  4. Optionally, you may find it useful to include a worked example or diagram to show more precisely how your algorithm works.

EXERCISES (NOT TO BE TURNED IN)

Max Flow

- Do Exercises 26.1-6 in CLRS (pg. 714)
- Do Exercises 26.2-10 in CLRS (pg. 731)

Linear Programming

- Do Exercise 29.1-9 in CLRS (pg. 858)
- Do Exercise 29.2-6 in CLRS (pg. 864)
- Do Exercise 29.5-9 in CLRS (pg. 893)

Problem 6-1. Intercepting the Smugglers [40 points]

You are commanding a small border patrol outpost and you are tasked with stopping the smuggling of contraband from the infamous Sorkis system to the Tarconis system. To do that, you need to secure some of the intermediate systems (unfortunately, both Sorkis and Tarconis are off limits). Once a system is secured, any smuggler traveling through that system will be intercepted.

The number of security details at your disposal is quite limited, though. Can you find a deployment that is guaranteed to intercept any smuggler, no matter which route he/she chooses, while using the minimal possible number of security details (one per system)? Your algorithm should identify at which systems security details should be placed.

Assume you are given a hyperspace map represented as a graph G with each vertex corresponding to a system and each directed edge representing a hyperspace lane. (Assume also that there is no hyperspace lane that directly connects Sorkis to Tarconis.)

(a) [20 points] Devise an efficient (i.e., polynomial-time) algorithm that finds a minimum-size subset of systems such that once each one of these systems is secured, any smuggler trying to get from Sorkis to Tarconis will be intercepted. State the running time of your algorithm and briefly justify its correctness.

(b) [20 points] It turns out that smugglers' ships have a one-time, hyperboosting capability, which allows a smuggler to evade interception by a security detail *once*. So, as long as a smuggling route passes through *at most one* secured system, smugglers still will succeed. Adjust your algorithm from the previous part to accommodate this modification and still intercept all smugglers. No justification is required. (Assume that there is no pair of hyperspace lanes that connect Sorkis to Tarconis while passing through just one intermediate system.)

Problem 6-2. A little LP [10 points]

Show that

$$\max 3x + 4y + 5z$$

$$\begin{aligned} x + y &\le 2 \\ y + z &\le 3 \\ x + z &\le 4 \\ x, y, z &\ge 0 \end{aligned}$$

has a value of no more than 20.

Problem 6-3. Routing Tours [40 points]

Tourists spawn in Kendall Square, represented as a node s, at a fixed rate of p. We would like to send them along MIT's network of hallways, which we model as a connected directed graph $G = (V, E)$ with no duplicate edges, to the student center, represented as a node t. Each hallway $(i, j) \in E$ has a maximum rate of tourists it can sustain, u_{ij}, as well as a disturbance ratio, c_{ij}, such that the disturbance caused by sending tourists at a rate of k down this hallway is $k \cdot c_{ij}$. Our goal is to distribute the tourists–that is, to pick a rate of tourists for each hallway–such as to minimize the total (additive) disturbance across all hallways. That is, if we let k_{ij} denote the rate of tourists on hallway (i, j), we want to minimize $\sum_{(i,j)\in E} k_{ij} \cdot c_{ij}$.

(a) [10 points] Formulate this problem as an LP (not necessarily in standard form).

(b) [10 points] Consider a Linear Program P with an equality constraint $\mathbf{a}^T \cdot \mathbf{x} = b$. Describe a how you would deal with this equality constraint when writing the dual to P. In particular, show that such an equality constraint maps to a variable in the dual that does not have to be non-negative.

(c) [10 points] Using the recipe from part **(b)**, write the dual of the program you gave in part **(a)**.

(d) [10 points] Restrict your attention to the case where $p = 1$ and $u_{ij} = \infty$ for each edge (i, j). To what common problem X does the tourist routing problem reduce to? Explain why your LP formulation from either part a) or part b) corresponds to X.

Problem 6-4. Feedback Form [10 points] Please fill out a feedback form about this problem set at

`https://forms.gle/fXDMDjt5Ka1FCCGV7`.

It should not take more than a few minutes and will greatly help us improve teaching and material for future semesters!