Problem Set 7

This problem set is due **at 10:00pm** on **Wednesday, April 22, 2020**.

Please make note of the following instructions:

- This assignment, like later assignments, consists of *exercises* and *problems*. **Hand in solutions to the problems only.** However, we strongly advise that you work out the exercises for yourself, since they will help you learn the course material. You are responsible for the material they cover.
- Remember that the problem set must be submitted on Gradescope. If you haven't done so already, please signup for 6.046 Spring 2020 on Gradescope, with the entry code MNEBKP, to submit this assignment.
- We require that the solution to the problems is submitted as a PDF file, **typeset on LaTeX**, using the template available in the course materials. Each submitted solution should start with your name, the course number, the problem number, your recitation section, the date, and the names of any students with whom you collaborated.
- You will often be called upon to "give an algorithm" to solve a certain problem. Your write-up should take the form of a short essay. A topic paragraph should summarize the problem you are solving and what your results are. The body of your essay should provide the following:
  1. A description of the algorithm in English and, if helpful, pseudocode.
  2. A proof (or indication) of the correctness of the algorithm.
  3. An analysis of the asymptotic running time behavior of the algorithm.
  4. Optionally, you may find it useful to include a worked example or diagram to show more precisely how your algorithm works.

EXERCISES (NOT TO BE TURNED IN)

There are no exercises from the textbook this week.

Problem 7-1. Link Sharing [40 points]

Alice and Bob are renting an apartment together. Although they get along pretty well, the issue of sharing an internet connection has become quite contentious. Specifically, we model their sharing as a game in which the action of Alice (resp. Bob) corresponds to specifying a fraction $0 \leq x_A \leq 1$ (resp. $0 \leq x_B \leq 1$) of the connection they want to claim. Then, the utility u_A of Alice and the utility u_B of Bob become:

$$u_A = \begin{cases} 0 & \text{if } x_A + x_B > 1, \\ x_A(1 - x_A - x_B) & \text{otherwise;} \end{cases} \quad \text{and} \quad u_B = \begin{cases} 0 & \text{if } x_A + x_B > 1, \\ x_B(1 - x_A - x_B) & \text{otherwise.} \end{cases}$$

In the above, the top pair of cases ($x_A + x_B > 1$) corresponds to the internet link being overloaded, and the bottom pair reflects the need for having enough bandwidth left to maintain network speed.

(a) [10 points] Find a pair of actions x_A and x_B that give rise to maximal *total* utility $u_A + u_B$ for Alice and Bob.

(b) [15 points] Show that the pair of actions $x_A = \frac{1}{3}$ and $x_B = \frac{1}{3}$ constitutes a Nash equilibrium of this game. What is the *total* utility $u_A + u_B$ it achieves?

(c) [15 points] Alice and Bob managed to provide bandwidth via alternative means. As a result, their utilities are the following:

$$u_A = \begin{cases} 0 & \text{if } x_A + x_B > 1, \\ x_A & \text{otherwise;} \end{cases} \quad \text{and} \quad u_B = \begin{cases} 0 & \text{if } x_A + x_B > 1, \\ x_B & \text{otherwise.} \end{cases}$$

List *all* the *deterministic* Nash equilibria of this game.

Problem 7-2. The Interview [15 points]

After stints at eLake and Pewlett-Hackard (PH), Wheg Mitman is now the CEO of Whobi, and she is determined to make Whobi the next big thing. She hears of your experience with algorithms and invites you to interview for a position on her board. Answer the following interview questions correctly to get the job.

(a) [5 points] **True or False:** In every instance of the Stable Matching Problem, there is a stable matching containing a pair (a, b) such that a is ranked first on the preference list of b and b is ranked first on the preference list of a. Justify your response with a short explanation or a counterexample.

(b) [10 points] Suppose we have two television networks, called A and B. There are n prime-time programming slots, and each network has n TV shows. Each network needs to devise a *schedule*—an assignment of each show to a distinct slot, to attract as much market share as possible.

Each show has a fixed *rating*, and we'll assume that no two shows have exactly the same rating. The winner of each time slot is the network whose show has a larger rating. Each network wants to win as many slots as possible.

This season, network A reveals a schedule S, and network B reveals a schedule T. We will define a pair of schedules (S, T) to be *stable* if neither network can unilaterally change its own schedule and win more time slots. That is, there is no schedule S' such that A wins more time slots with (S', T) than it did with (S, T), and symmetrically, there is no schedule T' such that B wins more time slots with (S, T') than it did with (S, T).

For every set of TV shows and ratings, is there always a stable pair of schedules? Justify with a short explanation or a counterexample.

Problem 7-3. Who Goes Where? [35 points]

Wheg Mitman is pleased with your performance and offers you a spot on her board! She decides that the first step to improving the productivity at Whobi is to promote good connections between employees and the divisions of Whobi to which they belong.

In order to do so, Wheg organizes the hiring and matching process as follows. Whobi has m divisions, and each division d_i has c_i openings, where we define $c = \sum_{i=1}^{m} c_i$. Wheg has selected $n > c$ potential hires to come on-site and interview with each of the m divisions of Whobi. Each potential hire, h_j, ranks all m divisions of Whobi, and each division of Whobi, d_i, ranks all n potential hires. Finally, they all submit their rankings to Wheg, who matches the potential hires to the divisions, ensuring that each division's positions are filled.

Wheg recruits you to help her with this problem. She explains to you that after researching the stable matching problem, she has determined that she would like her matching to satisfy some notion of stability. In particular, she would like to prevent certain types of instabilities that indicate inefficient matchings. We say that an assignment of potential hires to divisions that fills all openings is *stable* if no such type of instability is present in the assignment.

One type of instability that she has borrowed from the stable matching problem with two groups of equal size can be stated as follows:

Instability Type #1: There are potential hires h_α and h_β and divisions d_i and d_j such that

- h_α is assigned to d_i
- h_β is assigned to d_j
- d_i prefers h_β to h_α
- h_β prefers d_i to d_j

However, Wheg understands that her problem is slightly different, since her divisions would like to hire multiple employees, and she has an excess of potential hires. Therefore, she asks you to consider if there might be some other type of instability that can exist in an assignment.

(a) [5 points] Help Wheg precisely formulate another type of instability that can exist in an assignment of potential hires to divisions. (Hint: consider those potential hires who are not assigned to any divisions by a given assignment.) Formally state the conditions for this type of instability.

(b) [30 points] As discussed above, an assignment of potential hires to departments that exhibits neither the instability type proposed by Wheg nor the one that you proposed in part (a) is called a stable assignment. Propose an efficient algorithm to make a stable assignment of potential hires to divisions, and prove that this algorithm always terminates with the guarantee of a stable assignment. In addition, analyze the number of offers made by this algorithm.

Problem 7-4. Feedback Form [10 points] Please fill out a feedback form about this problem set at

`https://forms.gle/85beLrcf1ZA4cnMB8`.

It should not take more than a few minutes and will greatly help us improve teaching and material for future semesters!