Recitation 1: Exercise Solutions

1 Randomized Algorithms

Exercise 1: Given a Monte Carlo algorithm $\mathcal{A}$ and a polynomial-time algorithm $\mathcal{V}$ which can verify whether $\mathcal{A}$ returned a correct answer, devise a Las Vegas algorithm that performs the same task using the repeated trials technique above.

Solution: Our Las Vegas algorithm $\mathcal{B}$ is as follows:
Repeat:

1. Run $\mathcal{A}$ and denote a as the output of $\mathcal{A}$.
2. Verify whether a is the correct answer with $\mathcal{V}$. If yes, terminate and output a. Otherwise, continue.

$\mathcal{B}$ always returns a correct answer, since we only terminate when we have a correct answer. We will write the expected runtime in terms of p, where p is the probability that $\mathcal{A}$ returns the correct answer. The expected runtime is

$$E\left[T\left(\mathcal{B}\right)\right] = \left(T\left(\mathcal{A}\right) + T\left(\mathcal{V}\right)\right)\frac{1}{p}$$

since $\left(T\left(\mathcal{A}\right) + T\left(\mathcal{V}\right)\right)$ is the amount of time each trial takes, and $\frac{1}{p}$ is the expected number of trials until the first success (in general, for a weighted coin with heads probability p, the expected number of coin flips until the first head is $1/p$). Since p must be at least some constant probability (i.e. $\frac{1}{2}$), we have $p \geq \frac{1}{2} \geq \frac{1}{n}$ for large enough n, so $\frac{1}{p} \leq n$. Thus, the expected runtime $E\left[T\left(\mathcal{B}\right)\right]$ is polynomial, so $\mathcal{B}$ is a Las Vegas algorithm.

Exercise 2: Given a Las Vegas algorithm $\mathcal{B}$, devise a Monte Carlo algorithm that performs the same task, and prove that it returns the correct output with high probability using a probability bounding technique above.

Solution: Let e be the expected runtime of $\mathcal{B}$. Our Monte Carlo algorithm $\mathcal{A}$ is as follows:

1. Run $\mathcal{B}$ until time $2e$, and terminate.

2. If $\mathcal{B}$ has output something, then output the same thing. Otherwise, output nothing.

Since $\mathcal{B}$ is always correct, $\mathcal{A}$ is correct exactly when $\mathcal{B}$ terminates before twice its expected runtime. By Markov's inequality, since the runtime $T(\mathcal{B})$ is a nonnegative random variable, the probability that this does not occur is

$$\Pr\left[T(\mathcal{B}) \leq 2E\left[T(\mathcal{B})\right]\right] \leq \frac{1}{2},$$

so the probability $\mathcal{A}$ returns the correct answer is at least $\frac{1}{2}$.