CARTAN SUBALGEBRAS SEMINAR

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1. Introduction

In the Winter Semester of 2022 we will run a seminar on Cartan subalgebras, and study why twisted groupoids and inverse semigroup actions model them in a natural way. The seminar will be based on several papers and surveys by multiple authors. (It will not be based on a unique canonical text, because, to the best of our knowledge, there is no such text.)

2. Meeting info

We may hold a **preliminary meeting** about the seminar in mid-August, but for now, we hope that this document is informative enough. Please email us (`becky.armstrong@uni-muenster.de` and `diego.martinez@uni-muenster.de`) to sign up for a specific talk or to ask us questions. We will maintain an up-to-date schedule of the talks at `beckyarmstrong.com.au/seminars`, and you can use this to check which talks have already been assigned.

The **seminar** itself will take place at:

Time: TBD (let us know if you have a preference/suggestion).
Location: TBD.

3. Prerequisites

No prior knowledge of groupoids or Cartan inclusions will be needed to participate in this seminar. The seminar is aimed at people who have studied an introductory course on operator algebras, particularly C*-algebras. The seminar will include introductory talks on groupoids, twists, their associated C*-algebras, Cartan inclusions, Fell bundles over groups, and inverse semigroup actions. Talks 1–4 in particular should be quite introductory.

4. Seminar abstract

Cartan inclusions of C*-algebras (i.e. inclusions of the form $D \subseteq A$, where D is a maximal abelian subalgebra of A, and A itself is "not too big" with respect to D), are a class of inclusions of C*-algebras that appear naturally when one considers dynamical systems. For instance, the following are classic examples of Cartan inclusions.

- $D_n \subseteq M_n$, where D_n is the set of diagonal matrices of size $n \times n$, and M_n is the set of matrices of the same size.
- Given any free minimal action of a discrete group G on a Hausdorff space X, the inclusion $C_0(X) \subseteq C_0(X) \rtimes_r G$ is, again, a Cartan inclusion. In this case it is important that the C*-norm in the crossed product is the so-called *reduced* one.
- Given a twist Σ over a topologically principal second-countable Hausdorff étale groupoid G, the inclusion $C_0(G^{(0)}) \subseteq C_r^*(G; \Sigma)$ is a Cartan inclusion (by [10, Theorem 5.2]). As before, it is crucial that we consider the *reduced* C*-algebra of the groupoid.

Morally speaking, if $D \subseteq A$ is a Cartan inclusion, then every element of A is almost a linear combination of *normalisers*; i.e. elements of A that behave like matrix units with respect to the open sets of X (when $D \cong C_0(X)$). These normalisers "send" open subsets of X to other open subsets of X, and hence fix D. In the case of crossed products, for instance, normalisers are given by the starting action of G on X. If X is totally disconnected, then the matrix unit behaviour is even clearer, as those normalisers may be taken to have clopen (rather than just open) support, as is the case for the matrix units of M_n.

It is by now well known that any Cartan inclusion $D \subseteq A$ can be modelled by a twisted groupoid inclusion of the form $C_0(G^{(0)}) \subseteq C_r^*(G;\Sigma)$, where Σ is a twist over a topologically principal second-countable Hausdorff étale groupoid G. In other words, the third class of examples above exhausts the class of Cartan inclusions. This result is due to Renault [10, Theorem 5.9], and was based on previous ideas of Feldman and Moore [5, 6] relating to Cartan inclusions of von Neumann algebras.

In this seminar we shall study in detail the correspondence between Cartan inclusions and twisted C*-algebras of Hausdorff étale groupoids. As motivation, we will first study the von Neumann algebra setting, and we will then introduce Cartan inclusions of C*-algebras and twisted groupoids. Afterwards, we will construct both the full and reduced C*-algebras associated to an étale groupoid and study some basic properties of these C*-algebras that can be deduced from properties of the groupoid. We will then see some concrete groupoid models, including groupoid models for the Jiang–Su algebra $\mathcal{Z}$. In the last few talks we will diverge from the original setting, and explore two related areas of research.

- **The study of uniform Roe algebras and coarse geometry.** Nowhere in the discussion above did we mention that X must be metrisable, and indeed, this is not necessary in general. In particular, if $X = \beta G$ is the *Stone–Čech compactification* of G, then $C(\beta G) \rtimes_r G$ is the *uniform Roe algebra* of G, and we can study it using similar methods to those mentioned above.
- **Generalisations of Cartan inclusions, and Fell bundles over groups.** As mentioned earlier, if $D \subseteq A$ is a Cartan inclusion, then the elements of A can be approximated by linear combinations of normalisers (i.e. elements of A that fix D). Using this idea, we can drop the requirement that D be abelian, which brings us into the realm of *Fell bundles*, and again, these may be studied using similar methods to the ones mentioned above.

The Hausdorff question: We have not yet decided whether we will work with non-Hausdorff groupoids. These do not appear as Cartan inclusions, but they do appear in more general contexts, and we believe it would be useful and interesting to explore how to handle them. Please let us know if you have any opinions on this either way.

5. Talks schedule

Talk 1. *Feldman–Moore theory, and diagonals in von Neumann algebras.*
The notion of a *Cartan pair*, or *Cartan inclusion*, in the setting of measure spaces predates the one in the topological setting. The aim of this talk is to motivate the rest of the seminar from a historical point of view. Interesting topics would include:

- Countable group actions and equivalence relations, and [5, Theorem 1].
- [6, Definition 3.1 and Theorem 1].

Date: Week 10–14, October, 2022
Speaker: TBA

Talk 2. *Groupoids, twists, and (non-)Hausdorffness.*
This introductory talk will cover the basics of groupoids and twists, which will be used throughout the rest of the seminar. There are numerous references for this, but we will follow [10, Section 3] (see also [13, Chapters 8 and 9]). Several examples of groupoids will be introduced, such as those associated to group actions and partial group actions, groupoids of germs, and graph groupoids. Twists coming from continuous 2-cocycles on groupoids will also be studied.
Date: Week 17–21, October, 2022
Speaker: TBA

Talk 3. *Full and reduced twisted and untwisted groupoid C*-algebras.*
In this talk we will construct the full and reduced C*-algebras associated to a (not necessarily Hausdorff) étale groupoid. Again, there are several references for this, but we will follow [13, Chapter 9]. We will also construct full and reduced twisted groupoid C*-algebras.

Date: Week 24–28, October, 2022
Speaker: TBA

Talk 4. *C*-algebraic Cartan inclusions.*
This talk will introduce the central topic of the seminar; namely, what it means for $D \subseteq A$ to be a *Cartan inclusion* (see [10, Definition 5.1]). It would be desirable to present numerous examples of such inclusions; for instance, those covered in [10, Section 6]:

- Crossed products by discrete groups (Section 6.1).
- AF-algebras, and the diagonalisation method of Strătilă and Voiculescu (Section 6.2).
- Cuntz–Krieger algebras and graph algebras (Section 6.3).
- Cartan subalgebras in continuous trace C*-algebras (Section 6.4).

Date: Week 24–28, October, 2022
Speaker: TBA

Talk 5. *From Cartan inclusions to groupoids: the Weyl groupoid and Weyl twist.*
The aim of this talk is to present the most important idea of the seminar: how to construct a groupoid model for a Cartan inclusion. In this talk we will investigate why the class of Cartan inclusions coming from twists over effective Hausdorff étale groupoids exhausts the class of arbitrary Cartan inclusions. We will follow [10, Section 5] in detail, and, given a Cartan inclusion $D \subseteq A$, we will see how to construct its *Weyl groupoid* and *Weyl twist*. For this we will need the notion of a *groupoid of germs*, which we will see are *always* Hausdorff in this setting. Of particular interest to us are the proofs of Theorem 5.9 and Corollary 5.10 in [10].

Date: Week 7–11, November, 2022
Speaker: TBA

Talk 6. *Groupoid properties vs Cartan inclusions.*
Given that every Cartan inclusion comes from a twisted groupoid, an interesting area of study is the relationships between properties of the inclusion and properties of the underlying groupoid and twist. We will follow [13, Chapter 10], and interesting topics include:

- The relationship between amenability of G and nuclearity of $C_r^*(G)$ (see Theorem 10.1.5).
- The relationship between amenability of G and $C_r^*(G)$ satisfying the UCT (see Theorem 10.1.7). (This is based on a very deep result of Tu [14], and we do not intend to prove it.)
- Effective groupoids and ideals (see Theorems 10.2.7 and 10.3.3).
- A characterisation of simplicity of $C_r^*(G)$ in terms of properties of the groupoid G (see Theorem 10.3.6).

Date: Week 14–18, November, 2022
Speaker: TBA

Talk 7. *Examples of C*-algebras with groupoid models.*
This will be a talk about examples, with a different point of view to those of Talks 2 and 3. Here, given a Cartan inclusion $D \subseteq A$, we will construct an associated groupoid and twist, and we will see that this groupoid model may not be unique. Of particular interest to us are:

- Irrational rotation algebras, which have both twisted and untwisted models.
- Graph C*-algebras and their groupoid models.
- AF algebras, and their groupoid models coming from odometers and partial actions of $\mathbb{Z}$ [2].

Date: Week 7–11, November, 2022
Speaker: TBA

Talk 8. *Groupoid models for the Jiang–Su algebra.*
A particularly important C*-algebra is the Jiang–Su algebra. In this seminar we will *not* look at the role the Jiang–Su algebra plays in the classification programme, but we will look at several groupoid models for it. Among many others, the following are interesting models:

- Dimension drop algebras and inductive limits (see [11, Theorem 3.4]).
- Minimal homeomorphisms on point-like spaces (see [1, Section 2]).
- Menger's curve and its equivalence relations (see [8, Section 8]).

Date: Week 14–18, November, 2022
Speaker: TBA

Talk 9. *Coarse geometry.*
In this talk we will venture into the realm of coarse geometry, and follow the canonical book [9]. We will introduce the notion of bounded geometry and coarse equivalence between metric spaces (see Section 1.4), and prove that discrete groups naturally give rise to (uniquely determined) coarse metric spaces. Likewise, we will construct the uniform Roe algebra of a metric space of bounded geometry, and give a canonical *coarse* groupoid model for it (see [12, Section 3]). In this setting, the groupoids are usually not second-countable, meaning the associated C*-algebras are not separable.
Date: Week 21–25, November, 2022
Speaker: TBA

Talk 10. *Fell bundles over groups.*
Fell bundles over groups were introduced in [7], and later studied by Exel in [3]. This talk will involve an introduction to Fell bundles, along with gradings over discrete groups. As an application we will investigate the sense in which Fell bundles generalise twisted group actions. Moreover, we will follow [3, Section 2] to construct the reduced cross-sectional algebra of a Fell bundle over a discrete group G. This is similar to the construction of the reduced twisted groupoid C*-algebras covered in Talk 3. (Note that there is also a notion of a Fell bundle over a Hausdorff étale groupoid, but this is outside the scope of the seminar.)
Date: Week 5–9, December, 2022
Speaker: TBA

Talk 11. *The approximation property for Fell bundles over groups.*
In [3] Exel introduced the *approximation property* of a Fell bundle. In this talk we will introduce this property (Definition 4.4), and prove that Fell bundles with the approximation property also have the *weak containment property*; i.e. that the canonical left-regular representation $\Lambda\colon C^*(\mathcal{B}) \to C^*_r(\mathcal{B})$ is injective (Theorem 4.6). This will enable us to prove that $C^*_r(\mathcal{B})$ is nuclear in this setting.
Date: Week 12–17, December, 2022
Speaker: TBA

Talk 12. *Free group gradings of Cuntz–Krieger algebras.*
In this talk we will follow [3, Section 5], and give a particular example of a Fell bundle over the free group $\mathbb{F}_n$ for the Cuntz–Krieger algebra $\mathcal{O}_A$, associated to a $\{0,1\}$-matrix A. The main idea behind this is due to Quigg and Raeburn, but we will follow [3], as its approach is more suitable from our perspective. We are particularly interested in Theorems 5.2 and 5.5.
Date: Week 9–13, January, 2023
Speaker: TBA

Talk 13. *Noncommutative Cartan subalgebras.*
The intention of the final talk of the seminar is to further generalise Renault's theorem [10, Theorem 5.9] (discussed in Talk 5). We will follow Exel's discussion in [4]. In particular, given an inclusion $D \subseteq A$, where D is *not* assumed to be abelian, we will give conditions that characterise

when the inclusion $D \subseteq A$ can be obtained from an inclusion of the reduced cross-sectional C*-algebras coming from a Fell bundle (see Theorem 14.8). This is a far-reaching result that greatly generalises Renault's, and we will only be able to briefly sketch some of the main ideas behind the proof.

Date: Week 9–13, January, 2023
Speaker: TBA

References

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