Assignment #3

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1 Polymorphic behaviour

Prove that for any closed term f of type $\forall\alpha.\forall\beta.\alpha \to (\alpha \uplus \beta)$ and for any closed types τ_1, τ_2 value $v : \tau_1$, we have $f\ \tau_1\ \tau_2\ v \rightsquigarrow^*\ \mathit{inl}\ v$

Assuming:

$$
\begin{array}{ll}
\Delta, \Gamma \vDash f : \forall\alpha.\forall\beta.\alpha \to (\alpha \uplus \beta) & HP1 \\
\Delta \vdash \tau_1 & HP2 \\
\Delta \vdash \tau_2 & HP3 \\
\vdash v : \tau_1 & HP4
\end{array}
$$

Prove:

$$
f\ \tau_1\ \tau_2\ v \rightsquigarrow^*\ \mathit{inl}\ v \qquad THS
$$

$$
\begin{array}{rl}
\text{by HP1 w.h.} & \forall\delta \in D[\Delta], \forall\gamma \in G[\Gamma]^{\delta}, f\gamma\delta \in \mathcal{E}[\forall\alpha.\forall\beta.\alpha \to (\alpha \uplus \beta)]^{\delta} \\
\text{by def. of } \mathcal{E} \text{ w.h.} & \exists v_1. f\gamma\delta \rightsquigarrow^* v_1 \text{ and } v_1 \in \mathcal{V}[\forall\alpha.\forall\beta.\alpha \to (\alpha \uplus \beta)]^{\delta} \\
\text{by def. of } \mathcal{V} \text{ w.h.} & v_1 \equiv \Lambda\alpha.t \text{ and } \forall\tau'.\forall S \in \text{semty}(\tau').t[\tau'/\alpha] \in \mathcal{E}[\forall\beta.\alpha \to (\alpha \uplus \beta)]^{\delta, \alpha\mapsto\tau',S} \\
\text{by HP2 pick} & \tau' = \tau_1, S = \{v\} \\
\text{by def. of } \mathcal{E} \text{ w.h.} & \exists v_2.t[\tau'/\alpha] \rightsquigarrow^* v_2 \text{ and } v_2 \in \mathcal{V}[\forall\beta.\alpha \to (\alpha \uplus \beta)]^{\delta, \alpha\mapsto\tau_1,\{v\}} \\
\text{by def. of } \mathcal{V} \text{ w.h.} & v_2 \equiv \Lambda\beta.t' \text{ and } \forall\tau".\forall S' \in \text{semty}(\tau").t'[\tau"/\beta] \in \mathcal{E}[\alpha \to (\alpha \uplus \beta)]^{\overbrace{\delta, \alpha\mapsto\tau_1,\{v\}, \beta\mapsto\tau",S'}^{\delta'}} \\
\text{pick} & \tau" = \tau_2, S' = \emptyset \\
\text{by def. of } \mathcal{E} \text{ w.h.} & \exists v_3.t'[\tau"/\beta] \rightsquigarrow^* v_3 \text{ and } v_3 \in \mathcal{V}[\alpha \to (\alpha \uplus \beta)]^{\delta'} \\
\text{by def. of } \mathcal{V} \text{ w.h.} & v_3 \equiv \lambda x : \alpha.t" \text{ and } \forall v_4. \text{ if } v_4 \in \mathcal{V}[\alpha]^{\delta'} \text{ then } t"[v_4/x] \in \mathcal{E}[\alpha \uplus \beta]^{\delta'} \\
\text{by def. of } \mathcal{V} \text{ w.h.} & v_3 \equiv \lambda x : \alpha.t" \text{ and } \forall v_4. \text{ if } v_4 \in \{v\} \text{ then } t"[v_4/x] \in \mathcal{E}[\alpha \uplus \beta]^{\delta'} \\
\text{by } v_4 = v \text{ w.h.} & v_3 \equiv \lambda x : \alpha.t" \text{ and } t"[v/x] \in \mathcal{E}[\alpha \uplus \beta]^{\delta'} \\
\text{by def. of } \mathcal{E} \text{ w.h.} & \exists v_5.t"[v/x] \rightsquigarrow^* v_5 \text{ and } v_5 \in \mathcal{V}[\alpha \uplus \beta]^{\delta'} \\
\text{by def. of } \mathcal{V} \text{ w.h.} & \begin{array}{rl} \text{either} & v_5 \equiv \mathit{inl}\ v_6 \text{ and } v_6 \in \mathcal{V}[\alpha]^{\delta'} \\ \text{or} & v_5 \equiv \mathit{inr}\ v_6 \text{ and } v_6 \in \mathcal{V}[\beta]^{\delta'} \end{array} \\
\text{by def. of } \mathcal{V} \text{ w.h.} & \begin{array}{rl} \text{either} & v_5 \equiv \mathit{inl}\ v_6 \text{ and } v_6 \in \{v\} \\ \text{or} & \cancel{v_5 \equiv \mathit{inr}\ v_6 \text{ and } v_6 \in \emptyset} \end{array} \\
\text{by no val in } \emptyset \text{ w.h.} & v_5 \equiv \mathit{inl}\ v \square
\end{array}
$$

2 Free Theorems

$$\nexists t . \emptyset ; \emptyset \vdash t : \forall \alpha . \forall \beta . \beta \to \alpha (THM)$$

Proof by contradiction. We assume that $\emptyset ; \emptyset \vdash t : \forall \alpha . \forall \beta . \beta \to \alpha$.

$$\begin{array}{rl}
\text{By semantic soundness w.h.} & \emptyset ; \emptyset \vDash t : \forall \alpha . \forall \beta . \beta \to \alpha \\
\text{By semantic typing w.h.} & t \in \mathcal{E}[\forall \alpha . \forall \beta . \beta \to \alpha] \\
\text{By definition of } \mathcal{E} \text{ w.h.} & \exists v . t \rightsquigarrow^{*} v \text{ and } v \in \mathcal{V}[\forall \alpha . \forall \beta . \beta \to \alpha] \\
\text{By definition of } \mathcal{V}_{\forall} \text{ w.h.} & v \equiv \Lambda \alpha . t_1 \text{ and } \forall \tau', \forall S \in \mathrm{SemTy}(\tau') . t_1[\tau'/\alpha] \in \mathcal{E}[\forall \beta . \beta \to \alpha]^{\alpha \mapsto \tau', S} \\
\text{pick} & S = \emptyset \\
\text{By definition of } \mathcal{E} \text{ w.h.} & \exists v_1 . t_1[\tau'/\alpha] \rightsquigarrow^{*} v_1 \text{ and } v_1 \in \mathcal{V}[\forall \beta . \beta \to \alpha]^{\alpha \mapsto \tau', \emptyset} \\
\text{By definition of } \mathcal{V}_{\forall} \text{ w.h.} & v_1 \equiv \Lambda \beta . t_2 \text{ and } \forall \tau'', \forall S' \in \mathrm{SemTy}(\tau'') . t_2[\tau''/\beta] \in \mathcal{E}[\beta \to \alpha]^{\left. \begin{smallmatrix} \alpha \mapsto \tau', \emptyset \\ \beta \mapsto \tau'', S' \end{smallmatrix} \right\} \delta} \\
\text{By definition of } \mathcal{E} \text{ w.h.} & \exists v_2 . t_2[\tau''/\beta] \rightsquigarrow^{*} v_2 \text{ and } v_2 \in \mathcal{V}[\beta \to \alpha]^{\delta} \\
\text{by definition of } \mathcal{V}_{\to} \text{ w.h.} & v_2 \equiv \lambda x : \beta . t_3 \text{ and } \forall v_3 . \text{ if } v_3 \in \mathcal{V}[\beta]^{\delta} \text{ then } t_3[v_3/x] \in \mathcal{E}[\alpha]^{\delta} \\
\text{by definition of } \mathcal{E} \text{ w.h.} & \exists v_4 . t_3[v_3/x] \rightsquigarrow^{*} v_4 \text{ and } v_4 \in \mathcal{V}[\alpha] \\
\text{by definition of } \mathcal{V}_{\alpha} \text{ w.h.} & v_4 \in \emptyset \text{ . Contradiction} \square
\end{array}$$

3 A Register Machine Language

$$
\begin{aligned}
t ::= & r := n \\
& |sum\ r\ r \\
& |sub\ r\ r \\
& |cmp\ r\ r \\
& |jmp\ r \\
& |jiz\ r \\
& |jeq\ r \\
r ::= & ar|br|cr|dr|er|fr|gr|hr \\
& |ir|jr|kr|lr|mr|nr|or \\
C ::= & \emptyset|C, \mathbb{N} \mapsto t \\
F ::= & \emptyset|F, \mathbb{N} \mapsto b \\
R ::= & \emptyset|R, r \mapsto \mathbb{N}
\end{aligned}
$$

Judgement:

$$n; C; R; F \rightrightarrows n; C; R; F$$

Codebase, registers, and flags:

$$\frac{C = C', n \mapsto t}{C(n) = t} \qquad \frac{R = R', r \mapsto n}{R(r) = n} \qquad \frac{F = F', n \mapsto B}{F(n) = B}$$

$$\frac{C = C', n' \mapsto \_ \quad C'(n) = t}{C(n) = t} \qquad \frac{R = R', r' \mapsto \_ \quad R'(r) = n}{R(r) = n} \qquad \frac{F = F', n' \mapsto \_ \quad F'(n) = B}{F(n) = B}$$

Rules:

$$\frac{C(n) = r_1 := n_1 \quad R' = R, r_1 \mapsto n_1 \quad n' = n + 1}{n; C; R; F \rightrightarrows n'; C; R'; F}\text{load}$$

$$\frac{C(n) = sum\ r_1\ r_2\ R(r_1) = n_1\ R(r_2) = n_2\ R' = R, r_1 \mapsto n_1 + n_2\ n' = n + 1}{n; C; R; F \rightrightarrows n'; C; R'; F}\text{sum}$$

$$\frac{C(n) = sub\ r_1\ r_2\ R(r_1) = n_1\ R(r_2) = n_2\ R' = R, r_1 \mapsto n_1 - n_2\ n' = n + 1\ F' = F, 0 \mapsto n_2 > n_1}{n; C; R; F \rightrightarrows n'; C; R'; F'}\text{sub}$$

$$\frac{C(n) = cmp\ r_1\ r_2\ R(r_1) = n_1\ R(r_2) = n_2\ F' = F, 1 \mapsto n_1 == n_2\ n' = n + 1}{n; C; R; F \rightrightarrows n'; C; R; F'}\text{cmp}$$

$$\frac{C(n) = jmp\ r_1 \quad R(r_1) = n_1 \quad n' = n_1}{n; C; R; F \rightrightarrows n'; C; R; F}\text{jmp}$$

$$\frac{C(n) = jiz\ r_1 \quad R(r_1) = n_1 \quad F(1) = b \quad n' = \text{ if } b \text{ then } n_1 \text{ else } n + 1}{n; C; R; F \rightrightarrows n'; C; R; F}\text{jiz}$$

$$\frac{C(n) = jeq\ r_1 \quad R(r_1) = n_1 \quad F(0) = b \quad n' = \text{ if } b \text{ then } n_1 \text{ else } n + 1}{n; C; R; F \rightrightarrows n'; C; R; F}\text{jeq}$$

4 From the Register Machine to an Assembly Language

$$
\begin{aligned}
t ::=& r := n \\
&|sum\ r\ r \\
&|sub\ r\ r \\
&|cmp\ r\ r \\
&|jmp\ r \\
&|jiz\ r \\
&|jeq\ r \\
&|read\ r\ r \\
&|write\ r\ r \\
r ::=& ar|br|cr|dr|er|fr|gr|hr \\
&|ir|jr|kr|lr|mr|nr|or \\
C ::=& \emptyset|C, \mathbb{N} \mapsto t \\
F ::=& \emptyset|F, \mathbb{N} \mapsto b \\
R ::=& \emptyset|R, r \mapsto \mathbb{N} \\
M ::=& \emptyset|M, \mathbb{N} \mapsto \mathbb{N}
\end{aligned}
$$

Judgement:

$$n; C; R; F; M \rightrightarrows n; C; R; F; M$$

Codebase, registers, flags, and memory:

$$\frac{C = C', n \mapsto t}{C(n) = t} \qquad \frac{R = R', r \mapsto n}{R(r) = n} \qquad \frac{F = F', n \mapsto B}{F(n) = B}$$

$$\frac{C = C', n' \mapsto \_ \quad C'(n) = t}{C(n) = t} \qquad \frac{R = R', r' \mapsto \_ \quad R'(r) = n}{R(r) = n} \qquad \frac{F = F', n' \mapsto \_ \quad F'(n) = B}{F(n) = B}$$

$$\frac{M = M', n \mapsto n'}{M(n) = n'} \qquad \frac{M = M', n" \mapsto \_ \quad M'(n) = n'}{M(n) = n'}$$

Rules:

$$\frac{C(n) = r_1 := n_1 \quad R' = R, r_1 \mapsto n_1 \quad n' = n+1}{n; C; R; F; M \rightrightarrows n'; C; R'; F; M}\text{load}$$

$$\frac{C(n) = sum\ r_1\ r_2\ R(r_1) = n_1\ R(r_2) = n_2\ R' = R, r_1 \mapsto n_1 + n_2\ n' = n+1}{n; C; R; F; M \rightrightarrows n'; C; R'; F; M}\text{sum}$$

$$\frac{C(n) = sub\ r_1\ r_2\ R(r_1) = n_1\ R(r_2) = n_2\ R' = R, r_1 \mapsto n_1 - n_2\ n' = n+1\ F' = F, 0 \mapsto n_2 > n_1}{n; C; R; F; M \rightrightarrows n'; C; R'; F'; M}\text{sub}$$

$$\frac{C(n) = cmp\ r_1\ r_2\ R(r_1) = n_1\ R(r_2) = n_2\ F' = F, 1 \mapsto n_1 == n_2\ n' = n+1}{n; C; R; F; M \rightrightarrows n'; C; R; F'; M}\text{cmp}$$

$$\frac{C(n) = jmp\ r_1 \quad R(r_1) = n_1 \quad n' = n_1}{n; C; R; F; M \rightrightarrows n'; C; R; F; M}\text{jmp}$$

$$\frac{C(n) = jiz\ r_1 \quad R(r_1) = n_1 \quad F(1) = b \quad n' = \text{ if } b \text{ then } n_1 \text{ else } n+1}{n; C; R; F; M \rightrightarrows n'; C; R; F; M}\text{jiz}$$

$$\frac{C(n) = jeq\ r_1 \quad R(r_1) = n_1 \quad F(0) = b \quad n' = \text{ if } b \text{ then } n_1 \text{ else } n+1}{n; C; R; F; M \rightrightarrows n'; C; R; F; M}\text{jeq}$$

$$\frac{C(n) = read\ r_1\ r_2 \quad R(r_2) = n_2 \quad M(n_2) = n\text{"} \quad R' = R, r_1 \mapsto n\text{"} \quad n' = n+1}{n; C; R; F; M \rightrightarrows n'; C; R'; F; M}\text{read}$$

$$\frac{C(n) = write\ r_1\ r_2 \quad R(r_1) = n_1 \quad R(r_2) = n_2 \quad M' = M, n_1 \mapsto n_2 \quad n' = n+1}{n; C; R; F; M \rightrightarrows n'; C; R; F; M'}\text{write}$$