

# Assignment #2

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## 1 Missing Progress Cases

Write the proof for the progress theorem for the following cases

- $t \equiv \text{inl } t_1$
- $t \equiv \text{caset}_0$  of  $\left\{ \begin{array}{l} \text{inl } x_1 \mapsto t_1 \\ \text{inr } x_2 \mapsto t_2 \end{array} \right.$

Theorem:

if  $\emptyset \vdash t : \tau$  then either  $\vdash t.\text{VAL}$  or  $\exists t'. t \rightsquigarrow t'$

Prof by induction on the typing derivation of  $t$ .  
Base cases  $t\text{-var}$  and  $t\text{-nat}$  seen in class

### 1.1 $t \equiv \text{inl } t_1$

if  $\emptyset \vdash \text{inl } t : \tau_1 \uplus \tau_2$  then either  $\vdash \text{inl } t.\text{VAL}$  or  $\exists t'. \text{inl } t \rightsquigarrow t'$

	$\frac{\emptyset \vdash t : \tau_1}{\emptyset \vdash \text{inl } t : \tau_1 \uplus \tau_2}$	T-inl
t.s. either	$\vdash \text{inl } t.\text{VAL}$	
	or $\exists t'. \text{inl } t \rightsquigarrow t'$	
by I.H. either	$\vdash t.\text{VAL}$	I1
	or $\exists t''. t \rightsquigarrow t''$	I2

**Assuming I1**  $\text{inl } t.\text{VAL}$  by I1 and definition of  $\text{inl}$ .  $\square$

**Assuming I2**

by inversion on I2 w.h.	$t \equiv \mathbb{E}[t_0]$	HE1
	$t'' \equiv \mathbb{E}[t_0'']$	HE1'
	$t_0 \rightsquigarrow^p t_0''$	HPR
by HE1, HE1' t.s.	$\text{inl } \mathbb{E}[t_0] \rightsquigarrow \text{inl } \mathbb{E}[t_0'']$	
by ctx with HPR and	$\mathbb{E}' = \text{inl } \mathbb{E}_\square$	

$$1.2 \quad t \equiv \text{caset}_0 \text{ of } \left| \begin{array}{l} \text{inl } x_1 \mapsto t_1 \\ \text{inr } x_2 \mapsto t_2 \end{array} \right.$$

$$\underbrace{\text{if } \emptyset \vdash \text{case } t_0 \text{ of } \left| \begin{array}{l} \text{inl } x_1 \mapsto t_1 \\ \text{inr } x_2 \mapsto t_2 \end{array} \right. : \tau \text{ then either}}_{\text{HY}}$$

$$\vdash \text{caset}_0 \text{ of } \left| \begin{array}{l} \text{inl } x_1 \mapsto t_1 \\ \text{inr } x_2 \mapsto t_2 \end{array} \right. . \text{VAL}$$

$$\text{or } \exists t'. \text{caset}_0 \text{ of } \left| \begin{array}{l} \text{inl } x_1 \mapsto t_1 \\ \text{inr } x_2 \mapsto t_2 \end{array} \right. \rightsquigarrow t'$$

$$\text{by } I.H.w.h. \text{ either } \vdash t_0 . \text{VAL} \quad \text{I1}$$

$$\text{or } \exists t'_0. t_0 \rightsquigarrow t'_0 \quad \text{I2}$$

**Assuming I1**

$$\text{by typing definition of } case \text{ and HY } \begin{array}{l} \text{either } t_0 \equiv \text{inl } t_0^l \\ \text{or } t_0 \equiv \text{inr } t_0^r \end{array}$$

$$\bullet t_0 \equiv \text{inl } t_0^l$$

$$\text{by COS rule for } case: \text{case } \text{inl } t_0^l \text{ of } \left| \begin{array}{l} \text{inl } x_1 \mapsto t_1 \\ \text{inr } x_2 \mapsto t_2 \end{array} \right. \rightsquigarrow^p t_1[t_0^l/x_1] \square$$

$$\bullet t_0 \equiv \text{inr } t_0^r$$

$$\text{by COS rule for case: case } \text{inr } t_0^r \text{ of } \left| \begin{array}{l} \text{inl } x_1 \mapsto t_1 \\ \text{inr } x_2 \mapsto t_2 \end{array} \right. \rightsquigarrow^p t_2[t_0^r/x_2] \square$$

**Assuming I2**

$$\begin{array}{l} \text{by inversion on I2 w.h. } t_0 \equiv \mathbb{E}[t_0^z] \quad \text{HE1} \\ t'_0 \equiv \mathbb{E}[t_0'^z] \quad \text{HE1'} \\ t_0^z \rightsquigarrow^p t_0'^z \quad \text{HPR} \end{array}$$

$$\text{by HE1, HE1' t.s. case } \mathbb{E}[t_0^z] \text{ of } \left| \begin{array}{l} \text{inl } x_1 \mapsto t_1 \\ \text{inr } x_2 \mapsto t_2 \end{array} \right. \rightsquigarrow \text{case } \mathbb{E}[t_0'^z] \text{ of } \left| \begin{array}{l} \text{inl } x_1 \mapsto t_1 \\ \text{inr } x_2 \mapsto t_2 \end{array} \right.$$

$$\text{by ctx with HPR and } \mathbb{E}' = \text{case } \mathbb{E} \text{ of } \left| \begin{array}{l} \text{inl } x_1 \mapsto t_1 \\ \text{inr } x_2 \mapsto t_2 \end{array} \right. \square$$

## 2 Missing Compatibility Lemmas

Write the proof for these compatibility lemmas:

- Pairs

Assuming:

- $\Gamma \models t_1 : \tau_1$  (HP1)
- $\Gamma \models t_2 : \tau_2$  (HP2)

Prove:

- $\Gamma \models \langle t_1, t_2 \rangle : \tau_1 \times \tau_2$

by def. ST take  $\gamma \in G[\tau]$

t.s.  $\langle t_1, t_2 \rangle \gamma \in \mathcal{E}[\tau_1, \tau_2]$

by def.  $\mathcal{E}$  t.s.  $\exists v. \langle t_1, t_2 \rangle \gamma \rightsquigarrow^* v$  and  $v \in \mathcal{V}[\tau_1 \times \tau_2]$

by HP1 w.h.  $t_1 \in \mathcal{E}[\tau_1]$  (A1)

by A1 and def.  $\mathcal{E}$  w.h.  $\exists v_1. t_1 \rightsquigarrow^* v_1$  and  $v_1 \in \mathcal{V}[\tau_1]$  (A2)

by HP2 w.h.  $t_2 \in \mathcal{E}[\tau_2]$  (B1)

by B1 and def.  $\mathcal{E}$  w.h.  $\exists v_2. t_2 \rightsquigarrow^* v_2$  and  $v_2 \in \mathcal{V}[\tau_2]$  (B2)

by A2, B2 w.h.  $\langle t_1, t_2 \rangle \gamma \rightsquigarrow^* \langle v_1, v_2 \rangle$  (C1)

by C1 and def.  $\mathcal{V}[\tau_1 \times \tau_2]$  w.h.  $v \equiv \langle v_1, v_2 \rangle \in \mathcal{V}[\tau_1 \times \tau_2] \square$

- Projection

Assuming:

- $\Gamma \models t : \tau_1 \times \tau_2$  (HP)

Prove:

- $\Gamma \models t.1 : \tau_1$

by def. ST take  $\gamma \in G[\tau]$

t.s.  $t.1 \gamma \in \mathcal{E}[\tau_1]$

by def  $\mathcal{E}$  t.s.  $\exists v_1. t.1 \rightsquigarrow^* v_1$  and  $v_1 \in \mathcal{V}[\tau_1]$  (B1)

by HP and def. ST w.h.  $\exists v. t \rightsquigarrow^* v$  and  $v \in \mathcal{V}[\tau_1 \times \tau_2]$  (A1)

by A1 and def.  $\mathcal{V}_\times$  w.h.  $v \equiv \langle v'_1, v'_2 \rangle$  and  $v'_1 \in \mathcal{V}[\tau_1]$  (A2)

by A1, A2 with  $v'_1 = v_1$  B1 holds  $\square$

- Unpack

Assuming:

- $\Delta \vdash \tau'$  (HP1)
- $\Delta, \Gamma \models t : \exists \alpha. \tau$  (HP2)
- $\Delta; \alpha, \Gamma; x : \tau \models t' : \tau'$  (HP3)

Prove:

–  $\Delta, \Gamma \models \text{unpack } t \text{ as } \langle \alpha, x \rangle \text{ in } t' \tau'$

by def. ST take  $\delta \in \mathcal{D}[\Delta], \gamma \in G[\Gamma]^\delta$

t.s.  $(\text{unpack } t \text{ as } \langle \alpha, x \rangle \text{ in } t')^\delta \gamma \in \mathcal{E}[\tau']^\delta$  (TY)

by def.  $\mathcal{E}$  t.s.  $\exists v. (\text{unpack } t \text{ as } \langle \alpha, x \rangle \text{ in } t')^\delta \gamma \in \mathcal{E}[\tau']^\delta \rightsquigarrow^* v$

and  $v \in \mathcal{V}[\tau']^\delta$

From this point on the proof is trivial and left to the reader

### 3 Adding Cycles

Add **for** and **while** constructs to STLC. Add their syntax, their typing and their operational semantics in COS.

#### 3.1 while

$t = \dots | \text{while } t \text{ do } t \text{ end}$

$v = \dots | \Xi$

$\tau = \dots | \xi$

$\mathbb{E} = \dots | \text{while } \mathbb{E} \text{ do } t \text{ end}$

$$\frac{\Gamma \vdash t_0 : \text{bool} \quad \Gamma \vdash t_1 : \tau}{\Gamma \vdash \text{while } t_0 \text{ do } t_1 \text{ end} : \xi} \text{t-while}$$

$$\overline{\text{while } \text{true} \text{ do } t \text{ end} \rightsquigarrow^p t; \text{while } \text{true} \text{ do } t \text{ end}}^{\text{c-whileT}}$$

$$\overline{\text{while } \text{false} \text{ do } t \text{ end} \rightsquigarrow^p \Xi}^{\text{c-whileF}}$$

#### 3.2 for

$t = \dots | \text{for } i = t \text{ to } t \text{ do } t \text{ end}$

$v = \dots | \Xi$

$\tau = \dots | \xi$

$\mathbb{E} = \dots | \text{for } i = \mathbb{E} \text{ to } t \text{ do } t \text{ end}$

$| \text{for } i = n \text{ to } \mathbb{E} \text{ do } t \text{ end}$

$$\frac{\Gamma \vdash t_0 : \mathbb{N} \quad \Gamma \vdash t_1 : \mathbb{N} \quad \Gamma, i : \mathbb{N} \vdash t_2 : \tau}{\Gamma \vdash \text{for } i = t_0 \text{ to } t_1 \text{ do } t_2 \text{ end} : \xi} \text{t-for}$$

$$\frac{n_1 < n_2 \quad n'_1 = n_1 + 1}{\text{for } i = n_1 \text{ to } n_2 \text{ do } t \text{ end} \rightsquigarrow^p t[n_1/i]; \text{for } i = n'_1 \text{ to } n_2 \text{ do } t \text{ end}} \text{c-for1}$$

$$\frac{n_1 \geq n_2}{\text{for } i = n_1 \text{ to } n_2 \text{ do } t \text{ end} \rightsquigarrow^p \Xi} \text{c-for2}$$