Assignment #2

Diego Oniarti - 257835

1 Missing Progress Cases

Write the proof for the progress theorem for the following cases

- $t \equiv \ inl \ t_1$
- $t \equiv \text{case}t_0 \text{ of } \left| \begin{array}{l} inl \ x_1 \mapsto t_1 \\ inr \ x_2 \mapsto t_2 \end{array} \right.$

  Theorem:

$$\text{if } \emptyset \vdash t : \tau \text{ then either } \vdash t.VAL \text{ or } \exists t'.t \leadsto t'$$

Prof by induction on the typing derivation of t.
Base cases *t-var* and *t-nat* seen in class

1.1 $t \equiv \ inl \ t_1$

$$\text{if } \emptyset \vdash \ inl \ t : \tau_1 \uplus \tau_2 \text{ then either } \vdash \ inl \ t.\text{VAL or } \exists t'. \ inl \ t \leadsto t'$$

$$\begin{array}{rll} & \dfrac{\emptyset \vdash t : \tau_1}{\emptyset \vdash \ inl \ t : \tau_1 \uplus \tau_2}\text{T-inl} & \\ \text{t.s. either} & \vdash \ inl \ t.\text{VAL} & \\ \text{or} & \exists t'. \ inl \ t \leadsto t' & \\ \text{by I.H. either} & \vdash t.\text{VAL} & \text{I1} \\ \text{or} & \exists t".t \leadsto t" & \text{I2} \end{array}$$

Assuming I1 $\quad$ *inl* t.VAL by I1 and definition of *inl* . $\square$

Assuming I2

$$\begin{array}{rll} \text{by inversion on I2 w.h.} & t \equiv \mathbb{E}[t_0] & \text{HE1} \\ & t" \equiv \mathbb{E}[t_0"] & \text{HE1'} \\ & t_0 \leadsto^p t_0" & \text{HPR} \\ \text{by HE1, HE1' t.s.} & \ inl \ \mathbb{E}[t_0] \leadsto \ inl \ \mathbb{E}[t_0"] & \\ \text{by ctx with HPR and} & \mathbb{E}' = \ inl \ \mathbb{E}_\square & \end{array}$$

1.2 $t \equiv \textbf{case} t_0 \textbf{ of } \left| \begin{array}{l} inl\ x_1 \mapsto t_1 \\ inr\ x_2 \mapsto t_2 \end{array} \right.$

$$\text{if } \underbrace{\emptyset \vdash \text{case } t_0 \text{ of } \left| \begin{array}{l} inl\ x_1 \mapsto t_1 \\ inr\ x_2 \mapsto t_2 \end{array} \right. : \tau}_{\text{HY}} \text{ then either}$$

$$\cancel{\vdash \text{case} t_0 \text{ of } \left| \begin{array}{l} inl\ x_1 \mapsto t_1 \\ inr\ x_2 \mapsto t_2 \end{array} \right. .\text{VAL}}$$

$$\text{or } \exists t'.\text{case} t_0 \text{ of } \left| \begin{array}{l} inl\ x_1 \mapsto t_1 \\ inr\ x_2 \mapsto t_2 \end{array} \right. \rightsquigarrow t'$$

$$\begin{array}{rll} byI.H.w.h. \text{ either} & \vdash t_0.\text{VAL} & \text{I1} \\ & \text{or} \exists t_0'.t_0 \rightsquigarrow t_0' & \text{I2} \end{array}$$

Assuming I1

$$\text{by typing definition of } case \text{ and HY } \begin{array}{rl} \text{either} & t_0 \equiv\ inl\ t_0^l \\ \text{or} & t_0 \equiv\ inr\ t_0^r \end{array}$$

- $t_0 \equiv\ inl\ t_0^l$

  by COS rule for *case*: case $inl\ t_0^l$ of $\left| \begin{array}{l} inl\ x_1 \mapsto t_1 \\ inr\ x_2 \mapsto t_2 \end{array} \right. \rightsquigarrow^p t_1[t_0^l/x_1]\square$

- $t_0 \equiv\ inr\ t_0^r$

  by COS rule for case: case $inr\ t_0^r$ of $\left| \begin{array}{l} inl\ x_1 \mapsto t_1 \\ inr\ x_2 \mapsto t_2 \end{array} \right. \rightsquigarrow^p t_2[t_0^r/x_2]\square$

Assuming I2

$$\begin{array}{rll} \text{by inversion on I2 w.h.} & t_0 \equiv \mathbb{E}[t_0^z] & \text{HE1} \\ & t_0' \equiv \mathbb{E}[t_0'^z] & \text{HE1'} \\ & t_0^z \rightsquigarrow^p t_0'^z & \text{HPR} \\ \text{by HE1, HE1' t.s.} & \text{case } \mathbb{E}[t_0^z] \text{ of } \left| \begin{array}{l} inl\ x_1 \mapsto t_1 \\ inr\ x_2 \mapsto t_2 \end{array} \right. \rightsquigarrow \text{case } \mathbb{E}[t_0'^z] \text{ of } \left| \begin{array}{l} inl\ x_1 \mapsto t_1 \\ inr\ x_2 \mapsto t_2 \end{array} \right. & \\ \text{by ctx with HPR and} & \mathbb{E}' = \text{case } \mathbb{E} \text{ of } \left| \begin{array}{l} inl\ x_1 \mapsto t_1 \\ inr\ x_2 \mapsto t_2 \end{array} \right. \square & \end{array}$$

2 Missing Compatibility Lemmas

Write the proof for these compatibility lemmas:

- Pairs
  Assuming:
  - $\Gamma \vDash t_1 : \tau_1$ (HP1)
  - $\Gamma \vDash t_2 : \tau_2$ (HP2)

  Prove:
  - $\Gamma \vDash \langle t_1, t_2 \rangle : \tau_1 \times \tau_2$

$$\begin{aligned}
\text{by def. ST take } & \gamma \in G[\tau] \\
\text{t.s. } & \langle t_1, t_2 \rangle \gamma \in \mathcal{E}[\tau_1, \tau_2] \\
\text{by def. } \mathcal{E} \text{ t.s. } & \exists v. \langle t_1, t_2 \rangle \gamma \rightsquigarrow^* v \text{ and } v \in \mathcal{V}[\tau_1 \times \tau_2] \\
\text{by HP1 w.h. } & t_1 \in \mathcal{E}[\tau_1] && (A1) \\
\text{by A1 and def. } \mathcal{E} \text{ w.h. } & \exists v_1 . t_1 \rightsquigarrow^* v_1 \text{ and } v_1 \in \mathcal{V}[\tau_1] && (A2) \\
\text{by HP2 w.h. } & t_2 \in \mathcal{E}[\tau_2] && (B1) \\
\text{by B1 and def. } \mathcal{E} \text{ w.h. } & \exists v_2 . t_2 \rightsquigarrow^* v_2 \text{ and } v_2 \in \mathcal{V}[\tau_2] && (B2) \\
\text{by A2,B2 w.h. } & \langle t_1, t_2 \rangle \gamma \rightsquigarrow^* \langle v_1, v_2 \rangle && (C1) \\
\text{by C1 and def. } \mathcal{V}[\tau_1 \times \tau_2] \text{ w.h. } & v \equiv \langle v_1, v_2 \rangle \in \mathcal{V}[\tau_1 \times \tau_2]_\square
\end{aligned}$$

- Projection
  Assuming:
  - $\Gamma \vDash t : \tau_1 \times \tau_2$ (HP)

  Prove:
  - $\Gamma \vDash t.1 : \tau_1$

$$\begin{aligned}
\text{by def. ST take } & \gamma \in G[\tau] \\
\text{t.s. } & t.1\gamma \in \mathcal{E}[\tau_1] \\
\text{by def } \mathcal{E} \text{ t.s. } & \exists v_1 .\ t.1 \rightsquigarrow^* v_1 \text{ and } v_1 \in \mathcal{V}[\tau_1] && (B1) \\
\text{by HP and def. ST w.h. } & \exists v .\ t \rightsquigarrow^* v \text{ and } v \in \mathcal{V}[\tau_1 \times \tau_2] && (A1) \\
\text{by A1 and def. } \mathcal{V}_\times \text{ w.h. } & v \equiv \langle v_1', v_2' \rangle \text{ and } v_1' \in \mathcal{V}[\tau_1] && (A2) \\
\text{by A1,A2 with } & v_1' = v_1 \ B1 \ holds_\square
\end{aligned}$$

- Unpack
  Assuming:
  - $\Delta \vdash \tau'$ (HP1)
  - $\Delta, \Gamma \vDash t : \exists \alpha . \tau$ (HP2)
  - $\Delta; \alpha, \Gamma; x : \tau \vDash t' : \tau'$ (HP3)

Prove:

– $\Delta, \Gamma \vDash \text{unpack } t \text{ as } \langle \alpha, x \rangle \text{ in } t' \tau'$

$$\begin{aligned} &\text{by def. ST take } \delta \in \mathcal{D}[\Delta], \gamma \in G[\Gamma]^{\delta} \\ &\qquad \text{t.s. } (\text{unpack } t \text{ as } \langle \alpha, x \rangle \text{ in } t')^{\delta}\gamma \in \mathcal{E}[\tau']^{\delta} \qquad (TY) \\ &\text{by def. } \mathcal{E} \text{ t.s. } \exists v.(\text{unpack } t \text{ as } \langle \alpha, x \rangle \text{ in } t')^{\delta}\gamma \in \mathcal{E}[\tau']^{\delta} \rightsquigarrow^{*} v \\ &\qquad \text{and } v \in \mathcal{V}[\tau']^{\delta} \end{aligned}$$

From this point on the proof is trivial and left to the reader

3 Adding Cycles

Add `for` and `while` constructs to STLC. Add their syntax, their typing and their operational semantics in COS.

3.1 while

$$\begin{aligned} t &= \cdots | \text{while } t \text{ do } t \text{ end} \\ v &= \cdots | \Xi \\ \tau &= \cdots | \xi \\ \mathbb{E} &= \cdots | \text{while } \mathbb{E} \text{ do } t \text{ end} \end{aligned}$$

$$\frac{\Gamma \vdash t_0 : \mathit{bool} \quad \Gamma \vdash t_1 : \tau}{\Gamma \vdash \text{while } t_0 \text{ do } t_1 \text{ end} : \xi}\text{t-while}$$

$$\frac{}{\text{while } \mathit{true} \text{ do } t \text{ end} \rightsquigarrow^{p} t; \text{while } \mathit{true} \text{ do } t \text{ end}}\text{c-whileT}$$

$$\frac{}{\text{while } \mathit{false} \text{ do } t \text{ end} \rightsquigarrow^{p} \Xi}\text{c-whileF}$$

3.2 for

$$\begin{aligned} t &= \cdots | \text{for } i = t \text{ to } t \text{ do } t \text{ end} \\ v &= \cdots | \Xi \\ \tau &= \cdots | \xi \\ \mathbb{E} &= \cdots | \text{for } i = \mathbb{E} \text{ to } t \text{ do } t \text{ end} \\ &\quad\;\; | \text{for } i = n \text{ to } \mathbb{E} \text{ do } t \text{ end} \end{aligned}$$

$$\frac{\Gamma \vdash t_0 : \mathbb{N} \quad \Gamma \vdash t_1 : \mathbb{N} \quad \Gamma, i : \mathbb{N} \vdash t_2 : \tau}{\Gamma \vdash \text{for } i = t_0 \text{ to } t_1 \text{ do } t_2 \text{ end} : \xi}\text{t-for}$$

$$\frac{n_1 < n_2 \quad n_1' = n_1 + 1}{\text{for } i = n_1 \text{ to } n_2 \text{ do } t \text{ end} \leadsto^p t[n_1/i]; \text{for } i = n_1' \text{ to } n_2 \text{ do } t \text{ end}}\text{c-for1}$$

$$\frac{n_1 \geq n_2}{\text{for } i = n_1 \text{ to } n_2 \text{ do } t \text{ end} \leadsto^p \Xi}\text{c-for2}$$