Assignment #1

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1 Big step - call by name

Write the operational semantics rules for a big-step, call-by-name reduction for ULC. Write the semantically correct ones only, but write them all

$$\frac{}{v \Downarrow v} \quad \text{val}$$

$$\frac{t \Downarrow n' \quad t' \Downarrow n" \quad n' \oplus n" = n}{t \oplus t' \Downarrow n} \quad \text{bs-bop}$$

$$\frac{t \Downarrow \lambda x.t" \quad t"[t'/x] \Downarrow v}{t\ t' \Downarrow v} \quad \text{bs-app}$$

$$\frac{t_1 \Downarrow v_1 \quad t_2 \Downarrow v_2}{\langle t_1, t_2 \rangle \Downarrow \langle v_1, v_2 \rangle} \quad \text{pair}$$

$$\frac{t \Downarrow \langle v, v' \rangle}{t.1 \Downarrow v} \quad \text{first-projection}$$

$$\frac{t \Downarrow \langle v', v \rangle}{t.2 \Downarrow v} \quad \text{first-projection}$$

$$\frac{t \Downarrow v}{inL\ t \Downarrow inL\ v} \quad \text{inLeft}$$

$$\frac{t \Downarrow v}{inR\ t \Downarrow inR\ v} \quad \text{inRight}$$

$$\frac{t \Downarrow\ inL\ v' \quad t_1[v'/x_1] \Downarrow v}{\text{case } t \text{ of } \left| \begin{array}{l} inL\ x_1 \mapsto t_1 \\ inR\ x_2 \mapsto t_2 \end{array} \right. \Downarrow v} \quad \text{pattern matching L}$$

$$\frac{t \Downarrow\ inR\ v' \quad t_2[v'/x_2] \Downarrow v}{\text{case } t \text{ of } \left| \begin{array}{l} inL\ x_1 \mapsto t_1 \\ inR\ x_2 \mapsto t_2 \end{array} \right. \Downarrow v} \quad \text{pattern matching R}$$

2 Equivalence of SOS and COS

2.1 if $t \to t'$ then $t \rightsquigarrow t'$

Proof by induction on $\to$.
Inductive hypothesis: $\forall t_h, t'_h . if\ t_h \to t'_h\ then\ t_h \rightsquigarrow t'_h$

2.1.1 App-2

$t = (\lambda x.t_1)\ t_2$
$t' = (\lambda x.t_1)\ t'_2$

$$\begin{array}{rll}
HP1 & (\lambda x.t_1)\ t_2 \to (\lambda x.t_1)\ t'_2 & \\
HP2 & t_2 \to t'_2 & \\
TH & (\lambda x.t_1)\ t_2 \rightsquigarrow (\lambda x.t_1)\ t'_2 & \\
by\ IH\ with\ HP2\ w.h & t_2 \rightsquigarrow t'_2 & HT1 \\
by\ inversion\ on\ HT1\ w.h & t_2 \equiv E[t_0] & HEI \\
 & t'_2 \equiv E[t'_0] & HEI' \\
 & t_0 \rightsquigarrow^p t'_0 & HPR \\
by\ HEI,\ HEI'\ t.s & (\lambda x.t_1)\ E[t_0] \rightsquigarrow (\lambda x.t_1)\ E[t'_0] & \\
by\ ctx\ with\ E' = (\lambda x.t_1)\ E & & \\
and\ HPR & (\lambda x.t_1)\ E[t_0] \equiv E'[t_0] \rightsquigarrow E'[t'_0] &
\end{array}$$

2.1.2 Op-1

$t = t_1 \oplus t_2$
$t' = t'_1 \oplus t_2$

$$\begin{array}{rll}
HP1 & t_1 \oplus t_2 \to t'_1 \oplus t_2 & \\
HP2 & t_1 \to t'_1 & \\
TH & t_1 \oplus t_2 \rightsquigarrow t'_1 \oplus t_2 & \\
by\ IH\ with\ HP2\ w.h & t_1 \rightsquigarrow t'_1 & HT1 \\
by\ inversion\ on\ HT1\ w.h & t_1 \equiv E[t_0] & HEI \\
 & t'_1 \equiv E[t'_0] & HEI' \\
 & t_0 \rightsquigarrow^p t'_0 & HPR \\
by\ HEI,\ HEI'\ t.s & E[t_0] \oplus t_2 \rightsquigarrow E[t'_0] \oplus t_2 & \\
by\ ctx\ with\ E' = E \oplus t_2 & & \\
and\ HPR & E[t_0] \oplus t_2 \equiv E'[t_0] \rightsquigarrow E'[t'_0] &
\end{array}$$

2.1.3 Op-2

$t = n \oplus t_1$
$t' = n \oplus t'_1$

$$\begin{array}{rll}
HP1 & n \oplus t_1 \to n \oplus t'_1 & \\
HP2 & t_1 \to t'_1 & \\
TH & n \oplus t_1 \rightsquigarrow n \oplus t'_1 & \\
by\ IH\ with\ HP2\ w.h & t_1 \rightsquigarrow t'_1 & HT1 \\
by\ inversion\ on\ HT1\ w.h & t_1 \equiv E[t_0] & HEI \\
 & t'_1 \equiv E[t'_0] & HEI' \\
 & t_0 \rightsquigarrow^p t'_0 & HPR \\
by\ HEI,\ HEI'\ t.s & n \oplus E[t_0] \rightsquigarrow n \oplus E[t'_0] & \\
by\ ctx\ with\ E' = n \oplus E & & \\
and\ HPR & n \oplus E[t_0] \equiv E'[t_0] \rightsquigarrow E'[t'_0] &
\end{array}$$

2.2 if $t \rightsquigarrow t'$ then $t \to t'$

Helper lemma $\forall t, t', E.\ if\ t \to t'\ then\ E[t] \to E[t']$
Proof by induction on E.
Inductive hypothesis: $if\ t \to t'\ then\ E'[t] \to E'[t']$

2.2.1 $E = (\lambda x.t")\ E'$

$$\begin{array}{rll}
by\ IH\ with\ HP, E'\ w.h & E'[t] \to E'[t'] & HE \\
t.s & ((\lambda x.t")\ E')[t] \to ((\lambda x.t")\ E')[t'] & \\
by\ [\cdot]\ t.s & (\lambda x.t")E'[t] \to (\lambda x.t")E'[t'] & \\
by\ APP-1\ with\ HE & \square &
\end{array}$$

2.2.2 $E = E' \oplus t"$

$$\begin{array}{rll}
by\ IH\ with\ HP, E'\ w.h & E'[t] \to E'[t'] & HE \\
t.s & (E \oplus t")[t] \to (E \oplus t")[t'] & \\
by\ [\cdot]\ t.s & E'[t] \oplus t" \to E'[t'] \oplus t" & \\
by\ BOP-1\ with\ HE & \square &
\end{array}$$

2.2.3 $E = n \oplus E'$

$$\begin{array}{rll}
by\ IH\ with\ HP, E'\ w.h & E'[t] \to E'[t'] & HE \\
t.s & (n \oplus E')[t] \to (n \oplus E')[t'] & \\
by\ [\cdot]\ t.s & n \oplus E'[t] \to n \oplus E'[t'] & \\
by\ BOP-1\ with\ HE & \square &
\end{array}$$

3 Distinguish terms

$$t \overset{def}{=} \lambda d : (\mathbb{N} \to \mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N} \to \mathbb{N}) \to \mathbb{N} \\ .d\ (\lambda a : \mathbb{N}.\lambda b : \mathbb{N}.b)\ (\lambda i : \mathbb{N}.\lambda j : \mathbb{N}.i)$$

Reduction 1

$$\begin{array}{ll}
(\lambda d.d\ (\lambda a.\lambda b.b)\ (\lambda i.\lambda j.i))\ (\lambda x.\lambda y.x\ 0\ (y\ 0\ 0)) & \rightsquigarrow \\
(\lambda x.\lambda y.x\ 0\ (y\ 0\ 0))\ (\lambda a.\lambda b.b)\ (\lambda i.\lambda j.i) & \rightsquigarrow \\
(\lambda y.(\lambda a.\lambda b.b)\ 0\ (y\ 0\ 0))\ (\lambda i.\lambda j.i) & \rightsquigarrow \\
(\lambda a.\lambda b.b)\ 0\ ((\lambda i.\lambda j.i)\ 0\ 0) & \rightsquigarrow \\
(\lambda b.b)\ ((\lambda i.\lambda j.i)\ 0\ 0) & \rightsquigarrow \\
(\lambda b.b)\ ((\lambda j.0)\ 0) & \rightsquigarrow \\
(\lambda b.b)\ 0 & \rightsquigarrow \\
0 &
\end{array}$$

Reduction 2

$$\begin{array}{ll}
(\lambda d.d\ (\lambda a.\lambda b.b)\ (\lambda i.\lambda j.i))\ (\lambda x.\lambda y.x\ 0\ (y\ 1\ 0)) & \rightsquigarrow \\
(\lambda x.\lambda y.x\ 0\ (y\ 1\ 0))\ (\lambda a.\lambda b.b)\ (\lambda i.\lambda j.i) & \rightsquigarrow \\
(\lambda y.(\lambda a.\lambda b.b)\ 0\ (y\ 1\ 0))\ (\lambda i.\lambda j.i) & \rightsquigarrow \\
(\lambda a.\lambda b.b)\ 0\ ((\lambda i.\lambda j.i)\ 1\ 0) & \rightsquigarrow \\
(\lambda b.b)\ ((\lambda i.\lambda j.i)\ 1\ 0) & \rightsquigarrow \\
(\lambda b.b)\ ((\lambda j.1)\ 0) & \rightsquigarrow \\
(\lambda b.b)\ 1 & \rightsquigarrow \\
1 &
\end{array}$$

If you want the two reductions to result in generic n_1 and n_2 values you can apply t_d to the result of the above mentioned reductions.

$$t_d \overset{def}{=} \lambda i.(i * n_2) + ((1 - i) * n_1)$$

4 Safe untypable term

$$(\lambda x.35 * 12)\ (\lambda y.1\ y)$$

typing derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x : \mathbb{N} \vdash 35 : \mathbb{N}}\text{nat} \quad \dfrac{}{x : \mathbb{N} \vdash 12 : \mathbb{N}}\text{nat}}{x : \tau \vdash 35 * 12 : \mathbb{N}}\text{bop}}{\emptyset \vdash \lambda x.35 * 12 : \tau \to \mathbb{N}}\text{lam} \quad \dfrac{\dfrac{}{y : \tau' \vdash 1\ y : \tau}\text{fail}}{\emptyset \vdash \lambda y.1\ y : \tau' \to \tau}\text{lam}}{\emptyset \vdash (\lambda x.35 * 12)\ (\lambda y.1\ y) :}\text{app}$$

COS-SM-CBV

$$
\begin{aligned}
&(\lambda x.35 * 12)\ (\lambda x.1\ x) \rightsquigarrow \\
&35 * 12 \rightsquigarrow \\
&420
\end{aligned}
$$

5 Typing derivation

$$
\dfrac{\dfrac{f:\mathbb{N}\to\mathbb{N}\in\Gamma}{\Gamma\vdash f:\mathbb{N}\to\mathbb{N}}\,\text{var} \qquad \dfrac{\dfrac{\dfrac{\dfrac{x:\mathbb{N}\in\Gamma'}{\Gamma'\vdash x:\mathbb{N}}\,\text{var} \qquad \dfrac{}{\Gamma'\vdash 2:\mathbb{N}}\,\text{nat}}{\Gamma'\begin{cases} f:\mathbb{N}\to\mathbb{N}, \\ x:\mathbb{N}\end{cases}\vdash x+2:\mathbb{N}}\,\text{op}}{\Gamma\vdash \lambda x.x+2:\mathbb{N}\to\mathbb{N}}\,\text{lam} \qquad \dfrac{}{\Gamma\vdash 4:\mathbb{N}}\,\text{nat}}{\Gamma\vdash(\lambda x.x+2)\ 4:\mathbb{N}}\,\text{app}}{\Gamma\left\{f:\mathbb{N}\to\mathbb{N}\vdash f\ ((\lambda x.x+2)\ 4):\mathbb{N}\right.}\,\text{app}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x:\mathbb{N}\to\mathbb{N}\in\Gamma'}{\Gamma'\vdash x:\mathbb{N}\to\mathbb{N}}\,\text{var} \qquad \dfrac{y:\mathbb{N}\in\Gamma'}{\Gamma'\vdash y:\mathbb{N}}\,\text{var}}{\Gamma'\begin{cases}\Gamma, \\ x:\mathbb{N}\to\mathbb{N}, \\ y:\mathbb{N}\end{cases}\vdash x\ y:\mathbb{N}}\,\text{app}}{\substack{\Gamma, \\ x:\mathbb{N}\to\mathbb{N}}\vdash \lambda y.x\ y:\mathbb{N}\to\mathbb{N}}\,\text{lam}}{\Gamma\vdash\lambda x.\lambda y.x\ y:(\mathbb{N}\to\mathbb{N})\to\mathbb{N}\to\mathbb{N}}\,\text{lam} \qquad \dfrac{f:\mathbb{N}\to\mathbb{N}\in\Gamma}{\Gamma\vdash f:\mathbb{N}\to\mathbb{N}}\,\text{var}}{\Gamma\vdash(\lambda x.\lambda y.x\ y)\ f:\mathbb{N}\to\mathbb{N}}\,\text{app} \qquad \dfrac{}{\Gamma\vdash 3:\mathbb{N}}\,\text{nat}}{\Gamma\left\{f:\mathbb{N}\to\mathbb{N}\vdash((\lambda x.\lambda y.x\ y)\ f)\ 3:\mathbb{N}\right.}\,\text{app}
$$

6 Encoding

6.1 Sequencing

$$
t ::= \cdots | t; t'
$$
$$
t; t' \equiv (\lambda a:\tau.\lambda b:\tau'.b)\ t\ t'
$$

reductions

$$
\begin{array}{rll}
2+4; 8+1 \equiv & (\lambda a.\lambda b.b)\ (2+4)\ (8+1) & \rightsquigarrow \\
(t \text{ gets evaluated first}) & (\lambda a.\lambda b.b)\ 6\ (8+1) & \rightsquigarrow \\
 & (\lambda b.b)\ (8+1) & \rightsquigarrow \\
(t' \text{ gets evaluated second}) & (\lambda b.b)\ 9 & \rightsquigarrow 9
\end{array}
$$

Let-in

$$
t ::= \cdots | let\ x = t\ in\ t'
$$
$$
let\ x = t\ in\ t' \equiv (\lambda x:\tau.t')\ t
$$

reductions

$$\begin{aligned} let\ x = (\lambda a.\lambda b.a * b)\ in\ (x\ 11\ 13) \equiv\ & (\lambda x.x\ 11\ 13)\ (\lambda a.\lambda b.a * b) \rightsquigarrow \\ & (\lambda a.\lambda b.a * b)\ 11\ 13 \qquad \rightsquigarrow 143 \end{aligned}$$

Arrays of Length 4

$$t ::= \cdots | [t, t, t, t]$$
$$v ::= \cdots | [v, v, v, v]$$

$$\begin{aligned} [t_1, t_2, t_3, t_4] \equiv\ & (\lambda a : \tau_1.\lambda b : \tau_2.\lambda c : \tau_3. \\ & \lambda d : \tau_4.\lambda x : \tau_1 \to \tau_2 \to \tau_3 \to \tau_4 \to \tau.x\ a\ b\ c\ d)\ t_1\ t_2\ t_3\ t_4 \\ [v_1, v_2, v_3, v_4] \equiv\ & \lambda x : \tau_1 \to \tau_2 \to \tau_3 \to \tau_4 \to \tau.x\ v_1\ v_2\ v_3\ v_4 \end{aligned}$$

reductions $[85, \lambda x.\lambda y.x + y, 29 + 44, 3]$

$$\begin{aligned} & (\lambda a.\lambda b.\lambda c.\lambda d.\lambda x.x\ a\ b\ c\ d)\ (85)\ (\lambda x.\lambda y.x + y)\ (29 + 44)\ 3 \rightsquigarrow \\ & (\lambda b.\lambda c.\lambda d.\lambda x.x\ 85\ b\ c\ d)\ (\lambda x.\lambda y.x + y)\ (29 + 44)\ 3 \rightsquigarrow \\ & (\lambda c.\lambda d.\lambda x.x\ 85\ (\lambda x.\lambda y.x + y)\ c\ d)\ (29 + 44)\ 3 \rightsquigarrow \\ & (\lambda c.\lambda d.\lambda x.x\ 85\ (\lambda x.\lambda y.x + y)\ c\ d)\ 73\ 3 \rightsquigarrow \\ & (\lambda d.\lambda x.x\ 85\ (\lambda x.\lambda y.x + y)\ 73\ d)\ 3 \rightsquigarrow \\ & \lambda x.x\ 85\ (\lambda x.\lambda y.x + y)\ 73\ 3 \end{aligned}$$

Array field access

$$t ::= \cdots | t.i\ (i \in 0..3)$$
$$t.0 \equiv t\ (\lambda a : \tau_1.\lambda b : \tau_2.\lambda c : \tau_3.\lambda d : \tau_4.a)$$
$$t.1 \equiv t\ (\lambda a : \tau_1.\lambda b : \tau_2.\lambda c : \tau_3.\lambda d : \tau_4.b)$$
$$t.2 \equiv t\ (\lambda a : \tau_1.\lambda b : \tau_2.\lambda c : \tau_3.\lambda d : \tau_4.c)$$
$$t.3 \equiv t\ (\lambda a : \tau_1.\lambda b : \tau_2.\lambda c : \tau_3.\lambda d : \tau_4.d)$$

reductions $[1, 1, 2, 3].0$

$$\begin{aligned} & ((\lambda a.\lambda b.\lambda c.\lambda d.\lambda x.x\ a\ b\ c\ d)\ 1\ 1\ 2\ 3)\ (\lambda a.\lambda b.\lambda c.\lambda d.a) \rightsquigarrow \\ & ((\lambda b.\lambda c.\lambda d.\lambda x.x\ 1\ b\ c\ d)\ 1\ 2\ 3)\ (\lambda a.\lambda b.\lambda c.\lambda d.a) \rightsquigarrow \\ & ((\lambda c.\lambda d.\lambda x.x\ 1\ 1\ c\ d)\ 2\ 3)\ (\lambda a.\lambda b.\lambda c.\lambda d.a) \rightsquigarrow \\ & ((\lambda d.\lambda x.x\ 1\ 1\ 2\ d)\ 3)\ (\lambda a.\lambda b.\lambda c.\lambda d.a) \rightsquigarrow \\ & (\lambda x.x\ 1\ 1\ 2\ 3)\ (\lambda a.\lambda b.\lambda c.\lambda d.a) \rightsquigarrow \\ & (\lambda a.\lambda b.\lambda c.\lambda d.a)\ 1\ 1\ 2\ 3 \rightsquigarrow \\ & (\lambda b.\lambda c.\lambda d.1)\ 1\ 2\ 3 \rightsquigarrow \\ & (\lambda c.\lambda d.1)\ 2\ 3 \rightsquigarrow \\ & (\lambda d.1)\ 3 \rightsquigarrow \\ & 1 \end{aligned}$$

Array update

$$t ::= \cdots | t.i = t \ (i \in 0..3)$$

$$t.0 = t' \equiv t \ ((\lambda i : \tau'.\lambda a : \tau_1.\lambda b : \tau_2.\lambda c : \tau_3.\lambda d : \tau_4.\lambda z : \tau' \to \tau_2 \to \tau_3 \to \tau_4 \to \tau.z \ i \ b \ c \ d) \ t')$$
$$t.1 = t' \equiv t \ ((\lambda i : \tau'.\lambda a : \tau_1.\lambda b : \tau_2.\lambda c : \tau_3.\lambda d : \tau_4.\lambda z : \tau_1 \to \tau' \to \tau_3 \to \tau_4 \to \tau.z \ a \ i \ c \ d) \ t')$$
$$t.2 = t' \equiv t \ ((\lambda i : \tau'.\lambda a : \tau_1.\lambda b : \tau_2.\lambda c : \tau_3.\lambda d : \tau_4.\lambda z : \tau_1 \to \tau_2 \to \tau' \to \tau_4 \to \tau.z \ a \ b \ i \ d) \ t')$$
$$t.3 = t' \equiv t \ ((\lambda i : \tau'.\lambda a : \tau_1.\lambda b : \tau_2.\lambda c : \tau_3.\lambda d : \tau_4.\lambda z : \tau_1 \to \tau_2 \to \tau_3 \to \tau' \to \tau.z \ a \ b \ c \ i) \ t')$$

reductions $[3, 3, 3, 2].3 = 3$

$$\begin{array}{ll}
((\lambda a.\lambda b.\lambda c.\lambda d.\lambda x.x \ a \ b \ c \ d) \ 3 \ 3 \ 3 \ 2) \ ((\lambda i.\lambda a.\lambda b.\lambda c.\lambda d.\lambda z.z \ a \ b \ c \ i) \ 3) & \rightsquigarrow \\
((\lambda b.\lambda c.\lambda d.\lambda x.x \ 3 \ b \ c \ d) \ 3 \ 3 \ 2) \ ((\lambda i.\lambda a.\lambda b.\lambda c.\lambda d.\lambda z.z \ a \ b \ c \ i) \ 3) & \rightsquigarrow \\
((\lambda c.\lambda d.\lambda x.x \ 3 \ 3 \ c \ d) \ 3 \ 2) \ ((\lambda i.\lambda a.\lambda b.\lambda c.\lambda d.\lambda z.z \ a \ b \ c \ i) \ 3) & \rightsquigarrow \\
((\lambda d.\lambda x.x \ 3 \ 3 \ 3 \ d) \ 2) \ ((\lambda i.\lambda a.\lambda b.\lambda c.\lambda d.\lambda z.z \ a \ b \ c \ i) \ 3) & \rightsquigarrow \\
(\lambda x.x \ 3 \ 3 \ 3 \ 2) \ ((\lambda i.\lambda a.\lambda b.\lambda c.\lambda d.\lambda z.z \ a \ b \ c \ i) \ 3) & \rightsquigarrow \\
(\lambda x.x \ 3 \ 3 \ 3 \ 2) \ (\lambda a.\lambda b.\lambda c.\lambda d.\lambda z.z \ a \ b \ c \ 3) & \rightsquigarrow \\
(\lambda a.\lambda b.\lambda c.\lambda d.\lambda z.z \ a \ b \ c \ 3) \ 3 \ 3 \ 3 \ 2 & \rightsquigarrow \\
(\lambda b.\lambda c.\lambda d.\lambda z.z \ 3 \ b \ c \ 3) \ 3 \ 3 \ 2 & \rightsquigarrow \\
(\lambda c.\lambda d.\lambda z.z \ 3 \ 3 \ c \ 3) \ 3 \ 2 & \rightsquigarrow \\
(\lambda d.\lambda z.z \ 3 \ 3 \ 3 \ 3) \ 2 & \rightsquigarrow \\
\lambda z.z \ 3 \ 3 \ 3 \ 3 &
\end{array}$$