

# Assignment #1

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## 1 Big step - call by name

Write the operational semantics rules for a big-step, call-by-name reduction for ULC. Write the semantically correct ones only, but write them all

$\frac{}{v \Downarrow v}$	val
$\frac{t \Downarrow n' \quad t' \Downarrow n'' \quad n' \oplus n'' = n}{t \oplus t' \Downarrow n}$	bs-bop
$\frac{t \Downarrow \lambda x. t'' \quad t''[t'/x] \Downarrow v}{t \ t' \Downarrow v}$	bs-app
$\frac{t_1 \Downarrow v_1 \quad t_2 \Downarrow v_2}{\langle t_1, t_2 \rangle \Downarrow \langle v_1, v_2 \rangle}$	pair
$\frac{t \Downarrow \langle v, v' \rangle}{t.1 \Downarrow v}$	first-projection
$\frac{t \Downarrow \langle v', v \rangle}{t.2 \Downarrow v}$	first-projection
$\frac{t \Downarrow v}{inL \ t \Downarrow inL \ v}$	inLeft
$\frac{t \Downarrow v}{inR \ t \Downarrow inR \ v}$	inRight
$\frac{t \Downarrow inL \ v' \quad t_1[v'/x_1] \Downarrow v}{\text{case } t \text{ of } \left  \begin{array}{l} inL \ x_1 \mapsto t_1 \\ inR \ x_2 \mapsto t_2 \end{array} \right. \Downarrow v}$	pattern matching L
$\frac{t \Downarrow inR \ v' \quad t_2[v'/x_2] \Downarrow v}{\text{case } t \text{ of } \left  \begin{array}{l} inL \ x_1 \mapsto t_1 \\ inR \ x_2 \mapsto t_2 \end{array} \right. \Downarrow v}$	pattern matching R

## 2 Equivalence of SOS and COS

### 2.1 if $t \rightarrow t'$ then $t \rightsquigarrow t'$

Proof by induction on  $\rightarrow$ .

Inductive hypothesis:  $\forall t_h, t'_h. \text{if } t_h \rightarrow t'_h \text{ then } t_h \rightsquigarrow t'_h$

#### 2.1.1 App-2

$$t = (\lambda x.t_1) t_2$$

$$t' = (\lambda x.t_1) t'_2$$

$$\begin{array}{llll}
& HP1 & (\lambda x.t_1) t_2 \rightarrow (\lambda x.t_1) t'_2 & \\
& HP2 & t_2 \rightarrow t'_2 & \\
& TH & (\lambda x.t_1) t_2 \rightsquigarrow (\lambda x.t_1) t'_2 & \\
\text{by IH with HP2 w.h} & & t_2 \rightsquigarrow t'_2 & HT1 \\
\text{by inversion on HT1 w.h} & & t_2 \equiv E[t_0] & HEI \\
& & t'_2 \equiv E[t'_0] & HEI' \\
& & t_0 \rightsquigarrow^p t'_0 & HPR \\
\text{by HEI, HEI' t.s} & & (\lambda x.t_1) E[t_0] \rightsquigarrow (\lambda x.t_1) E[t'_0] & \\
\text{by ctx with } E' = (\lambda x.t_1) E & & & \\
\text{and HPR} & & (\lambda x.t_1) E[t_0] \equiv E'[t_0] \rightsquigarrow E'[t'_0] & 
\end{array}$$

#### 2.1.2 Op-1

$$t = t_1 \oplus t_2$$

$$t' = t'_1 \oplus t_2$$

$$\begin{array}{llll}
& HP1 & t_1 \oplus t_2 \rightarrow t'_1 \oplus t_2 & \\
& HP2 & t_1 \rightarrow t'_1 & \\
& TH & t_1 \oplus t_2 \rightsquigarrow t'_1 \oplus t_2 & \\
\text{by IH with HP2 w.h} & & t_1 \rightsquigarrow t'_1 & HT1 \\
\text{by inversion on HT1 w.h} & & t_1 \equiv E[t_0] & HEI \\
& & t'_1 \equiv E[t'_0] & HEI' \\
& & t_0 \rightsquigarrow^p t'_0 & HPR \\
\text{by HEI, HEI' t.s} & & E[t_0] \oplus t_2 \rightsquigarrow E[t'_0] \oplus t_2 & \\
\text{by ctx with } E' = E \oplus t_2 & & & \\
\text{and HPR} & & E[t_0] \oplus t_2 \equiv E'[t_0] \rightsquigarrow E'[t'_0] & 
\end{array}$$

### 2.1.3 Op-2

$$t = n \oplus t_1$$

$$t' = n \oplus t'_1$$

$$\begin{array}{llll}
HP1 & n \oplus t_1 \rightarrow n \oplus t'_1 & & \\
HP2 & t_1 \rightarrow t'_1 & & \\
TH & n \oplus t_1 \rightsquigarrow n \oplus t'_1 & & \\
\text{by IH with HP2 w.h} & t_1 \rightsquigarrow t'_1 & HT1 & \\
\text{by inversion on HT1 w.h} & t_1 \equiv E[t_0] & HEI & \\
& t'_1 \equiv E[t'_0] & HEI' & \\
& t_0 \rightsquigarrow^P t'_0 & HPR & \\
\text{by HEI, HEI' t.s} & n \oplus E[t_0] \rightsquigarrow n \oplus E[t'_0] & & \\
\text{by ctx with } E' = n \oplus E & & & \\
\text{and HPR} & n \oplus E[t_0] \equiv E'[t_0] \rightsquigarrow E'[t'_0] & & 
\end{array}$$

### 2.2 if $t \rightsquigarrow t'$ then $t \rightarrow t'$

**Helper lemma**  $\forall t, t', E.$  if  $t \rightarrow t'$  then  $E[t] \rightarrow E[t']$

Proof by induction on  $E$ .

Inductive hypothesis: if  $t \rightarrow t'$  then  $E'[t] \rightarrow E'[t']$

#### 2.2.1 $E = (\lambda x.t'') E'$

$$\begin{array}{llll}
\text{by IH with HP, } E' \text{ w.h} & E'[t] \rightarrow E'[t'] & HE & \\
& t.s & ((\lambda x.t'') E')[t] \rightarrow ((\lambda x.t'') E')[t'] & \\
& \text{by } [\cdot] \text{ t.s} & (\lambda x.t'')E'[t] \rightarrow (\lambda x.t'')E'[t'] & \\
\text{by APP - 1 with HE} & \square & & 
\end{array}$$

#### 2.2.2 $E = E' \oplus t''$

$$\begin{array}{llll}
\text{by IH with HP, } E' \text{ w.h} & E'[t] \rightarrow E'[t'] & HE & \\
& t.s & (E \oplus t'')[t] \rightarrow (E \oplus t'')[t'] & \\
& \text{by } [\cdot] \text{ t.s} & E'[t] \oplus t'' \rightarrow E'[t'] \oplus t'' & \\
\text{by BOP - 1 with HE} & \square & & 
\end{array}$$

#### 2.2.3 $E = n \oplus E'$

$$\begin{array}{llll}
\text{by IH with HP, } E' \text{ w.h} & E'[t] \rightarrow E'[t'] & HE & \\
& t.s & (n \oplus E')[t] \rightarrow (n \oplus E')[t'] & \\
& \text{by } [\cdot] \text{ t.s} & n \oplus E'[t] \rightarrow n \oplus E'[t'] & \\
\text{by BOP - 1 with HE} & \square & & 
\end{array}$$

### 3 Distinguish terms

$$t \stackrel{def}{=} \lambda d : (\mathbb{N} \rightarrow \mathbb{N} \rightarrow \mathbb{N}) \rightarrow (\mathbb{N} \rightarrow \mathbb{N} \rightarrow \mathbb{N}) \rightarrow \mathbb{N} \\ .d (\lambda a : \mathbb{N}. \lambda b : \mathbb{N}. b) (\lambda i : \mathbb{N}. \lambda j : \mathbb{N}. i)$$

#### Reduction 1

$$\begin{aligned} & (\lambda d. d (\lambda a. \lambda b. b) (\lambda i. \lambda j. i)) (\lambda x. \lambda y. x \ 0 \ (y \ 0 \ 0)) \rightsquigarrow \\ & (\lambda x. \lambda y. x \ 0 \ (y \ 0 \ 0)) (\lambda a. \lambda b. b) (\lambda i. \lambda j. i) \rightsquigarrow \\ & (\lambda y. (\lambda a. \lambda b. b) \ 0 \ (y \ 0 \ 0)) (\lambda i. \lambda j. i) \rightsquigarrow \\ & (\lambda a. \lambda b. b) \ 0 \ ((\lambda i. \lambda j. i) \ 0 \ 0) \rightsquigarrow \\ & (\lambda b. b) ((\lambda i. \lambda j. i) \ 0 \ 0) \rightsquigarrow \\ & (\lambda b. b) ((\lambda j. 0) \ 0) \rightsquigarrow \\ & (\lambda b. b) \ 0 \rightsquigarrow \\ & 0 \end{aligned}$$

#### Reduction 2

$$\begin{aligned} & (\lambda d. d (\lambda a. \lambda b. b) (\lambda i. \lambda j. i)) (\lambda x. \lambda y. x \ 0 \ (y \ 1 \ 0)) \rightsquigarrow \\ & (\lambda x. \lambda y. x \ 0 \ (y \ 1 \ 0)) (\lambda a. \lambda b. b) (\lambda i. \lambda j. i) \rightsquigarrow \\ & (\lambda y. (\lambda a. \lambda b. b) \ 0 \ (y \ 1 \ 0)) (\lambda i. \lambda j. i) \rightsquigarrow \\ & (\lambda a. \lambda b. b) \ 0 \ ((\lambda i. \lambda j. i) \ 1 \ 0) \rightsquigarrow \\ & (\lambda b. b) ((\lambda i. \lambda j. i) \ 1 \ 0) \rightsquigarrow \\ & (\lambda b. b) ((\lambda j. 1) \ 0) \rightsquigarrow \\ & (\lambda b. b) \ 1 \rightsquigarrow \\ & 1 \end{aligned}$$

If you want the two reductions to result in generic  $n_1$  and  $n_2$  values you can apply  $t_d$  to the result of the above mentioned reductions.

$$t_d \stackrel{def}{=} \lambda i. (i * n_2) + ((1 - i) * n_1)$$

### 4 Safe untypable term

$$(\lambda x. 35 * 12) (\lambda y. 1 \ y)$$

#### typing derivation

$$\frac{\frac{\frac{}{x : \mathbb{N} \vdash 35 : \mathbb{N}}{\text{nat}} \quad \frac{\frac{}{x : \mathbb{N} \vdash 12 : \mathbb{N}}{\text{nat}}}{\text{bop}}}{\frac{}{x : \tau \vdash 35 * 12 : \mathbb{N}}{\text{lam}}} \quad \frac{\frac{}{y : \tau' \vdash 1 \ y : \tau}}{\text{lam}}}{\frac{}{\emptyset \vdash \lambda x. 35 * 12 : \tau \rightarrow \mathbb{N}}{\text{app}}} \quad \frac{}{\emptyset \vdash \lambda y. 1 \ y : \tau' \rightarrow \tau}$$

$$\emptyset \vdash (\lambda x. 35 * 12) (\lambda y. 1 \ y) :$$

## COS-SM-CBV

$$\begin{aligned}
& (\lambda x.35 * 12) (\lambda x.1 \ x) \rightsquigarrow \\
& 35 * 12 \rightsquigarrow \\
& 420
\end{aligned}$$

## 5 Typing derivation

$$\frac{\frac{f : \mathbb{N} \rightarrow \mathbb{N} \in \Gamma}{\Gamma \vdash f : \mathbb{N} \rightarrow \mathbb{N}} \text{var} \quad \frac{\frac{\frac{x : \mathbb{N} \in \Gamma'}{\Gamma' \vdash x : \mathbb{N}} \text{var} \quad \frac{}{\Gamma' \vdash 2 : \mathbb{N}} \text{nat}}{\Gamma' \{ \frac{f : \mathbb{N} \rightarrow \mathbb{N}}{x : \mathbb{N}}, \vdash x + 2 : \mathbb{N} \}} \text{op}}{\Gamma \vdash \lambda x.x + 2 : \mathbb{N} \rightarrow \mathbb{N}} \text{lam} \quad \frac{}{\Gamma \vdash 4 : \mathbb{N}} \text{nat}}{\Gamma \{ f : \mathbb{N} \rightarrow \mathbb{N} \vdash f ((\lambda x.x + 2) \ 4) : \mathbb{N} \}} \text{app}$$

$$\frac{\frac{\frac{x : \mathbb{N} \rightarrow \mathbb{N} \in \Gamma'}{\Gamma' \vdash x : \mathbb{N} \rightarrow \mathbb{N}} \text{var} \quad \frac{y : \mathbb{N} \in \Gamma'}{\Gamma' \vdash y : \mathbb{N}} \text{var}}{\Gamma' \left\{ \begin{array}{l} \Gamma, \\ x : \mathbb{N} \rightarrow \mathbb{N}, \vdash x \ y : \mathbb{N} \\ y : \mathbb{N} \end{array} \right\}} \text{lam}}{\frac{\frac{\Gamma, x : \mathbb{N} \rightarrow \mathbb{N} \vdash \lambda y.x \ y : \mathbb{N} \rightarrow \mathbb{N}}{\Gamma \vdash \lambda x.\lambda y.x \ y : (\mathbb{N} \rightarrow \mathbb{N}) \rightarrow \mathbb{N} \rightarrow \mathbb{N}} \text{lam} \quad \frac{f : \mathbb{N} \rightarrow \mathbb{N} \in \Gamma}{\Gamma \vdash f : \mathbb{N} \rightarrow \mathbb{N}} \text{var}}{\Gamma \vdash (\lambda x.\lambda y.x \ y) \ f : \mathbb{N} \rightarrow \mathbb{N}} \text{app} \quad \frac{}{\Gamma \vdash 3 : \mathbb{N}} \text{nat}}{\Gamma \{ f : \mathbb{N} \rightarrow \mathbb{N} \vdash ((\lambda x.\lambda y.x \ y) \ f) \ 3 : \mathbb{N} \}} \text{app}$$

## 6 Encoding

### 6.1 Sequencing

$$\begin{aligned}
& t ::= \dots | t; t' \\
& t; t' \equiv (\lambda a : \tau. \lambda b : \tau'. b) \ t \ t'
\end{aligned}$$

#### reductions

$$\begin{aligned}
& 2 + 4; 8 + 1 \equiv (\lambda a. \lambda b. b) \ (2 + 4) \ (8 + 1) \rightsquigarrow \\
& \text{($t$ gets evaluated first)} \quad (\lambda a. \lambda b. b) \ 6 \ (8 + 1) \rightsquigarrow \\
& \quad (\lambda b. b) \ (8 + 1) \rightsquigarrow \\
& \text{($t'$ gets evaluated second)} \quad (\lambda b. b) \ 9 \rightsquigarrow 9
\end{aligned}$$

#### Let-in

$$\begin{aligned}
& t ::= \dots | \text{let } x = t \text{ in } t' \\
& \text{let } x = t \text{ in } t' \equiv (\lambda x : \tau. t') \ t
\end{aligned}$$

### reductions

$$\text{let } x = (\lambda a. \lambda b. a * b) \text{ in } (x \ 11 \ 13) \equiv (\lambda x. x \ 11 \ 13) (\lambda a. \lambda b. a * b) \rightsquigarrow \\ (\lambda a. \lambda b. a * b) \ 11 \ 13 \rightsquigarrow 143$$

### Arrays of Length 4

$$t ::= \dots \mid [t, t, t, t] \\ v ::= \dots \mid [v, v, v, v]$$

$$[t_1, t_2, t_3, t_4] \equiv (\lambda a : \tau_1. \lambda b : \tau_2. \lambda c : \tau_3. \\ \lambda d : \tau_4. \lambda x : \tau_1 \rightarrow \tau_2 \rightarrow \tau_3 \rightarrow \tau_4 \rightarrow \tau. x \ a \ b \ c \ d) \ t_1 \ t_2 \ t_3 \ t_4 \\ [v_1, v_2, v_3, v_4] \equiv \lambda x : \tau_1 \rightarrow \tau_2 \rightarrow \tau_3 \rightarrow \tau_4 \rightarrow \tau. x \ v_1 \ v_2 \ v_3 \ v_4$$

### reductions [85, $\lambda x. \lambda y. x + y$ , 29 + 44, 3]

$$(\lambda a. \lambda b. \lambda c. \lambda d. \lambda x. x \ a \ b \ c \ d) \ (85) \ (\lambda x. \lambda y. x + y) \ (29 + 44) \ 3 \rightsquigarrow \\ (\lambda b. \lambda c. \lambda d. \lambda x. x \ 85 \ b \ c \ d) \ (\lambda x. \lambda y. x + y) \ (29 + 44) \ 3 \rightsquigarrow \\ (\lambda c. \lambda d. \lambda x. x \ 85 \ (\lambda x. \lambda y. x + y) \ c \ d) \ (29 + 44) \ 3 \rightsquigarrow \\ (\lambda c. \lambda d. \lambda x. x \ 85 \ (\lambda x. \lambda y. x + y) \ c \ d) \ 73 \ 3 \rightsquigarrow \\ (\lambda d. \lambda x. x \ 85 \ (\lambda x. \lambda y. x + y) \ 73 \ d) \ 3 \rightsquigarrow \\ \lambda x. x \ 85 \ (\lambda x. \lambda y. x + y) \ 73 \ 3$$

### Array field access

$$t ::= \dots \mid t.i \ (i \in 0..3) \\ t.0 \equiv t \ (\lambda a : \tau_1. \lambda b : \tau_2. \lambda c : \tau_3. \lambda d : \tau_4. a) \\ t.1 \equiv t \ (\lambda a : \tau_1. \lambda b : \tau_2. \lambda c : \tau_3. \lambda d : \tau_4. b) \\ t.2 \equiv t \ (\lambda a : \tau_1. \lambda b : \tau_2. \lambda c : \tau_3. \lambda d : \tau_4. c) \\ t.3 \equiv t \ (\lambda a : \tau_1. \lambda b : \tau_2. \lambda c : \tau_3. \lambda d : \tau_4. d)$$

### reductions [1, 1, 2, 3].0

$$((\lambda a. \lambda b. \lambda c. \lambda d. \lambda x. x \ a \ b \ c \ d) \ 1 \ 1 \ 2 \ 3) \ (\lambda a. \lambda b. \lambda c. \lambda d. a) \rightsquigarrow \\ ((\lambda b. \lambda c. \lambda d. \lambda x. x \ 1 \ b \ c \ d) \ 1 \ 2 \ 3) \ (\lambda a. \lambda b. \lambda c. \lambda d. a) \rightsquigarrow \\ ((\lambda c. \lambda d. \lambda x. x \ 1 \ 1 \ c \ d) \ 2 \ 3) \ (\lambda a. \lambda b. \lambda c. \lambda d. a) \rightsquigarrow \\ ((\lambda d. \lambda x. x \ 1 \ 1 \ 2 \ d) \ 3) \ (\lambda a. \lambda b. \lambda c. \lambda d. a) \rightsquigarrow \\ (\lambda x. x \ 1 \ 1 \ 2 \ 3) \ (\lambda a. \lambda b. \lambda c. \lambda d. a) \rightsquigarrow \\ (\lambda a. \lambda b. \lambda c. \lambda d. a) \ 1 \ 1 \ 2 \ 3 \rightsquigarrow \\ (\lambda b. \lambda c. \lambda d. 1) \ 1 \ 2 \ 3 \rightsquigarrow \\ (\lambda c. \lambda d. 1) \ 2 \ 3 \rightsquigarrow \\ (\lambda d. 1) \ 3 \rightsquigarrow$$

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## Array update

$$t ::= \dots | t.i = t \ (i \in 0..3)$$

$$t.0 = t' \equiv t \ ((\lambda i : \tau'. \lambda a : \tau_1. \lambda b : \tau_2. \lambda c : \tau_3. \lambda d : \tau_4. \lambda z : \tau' \rightarrow \tau_2 \rightarrow \tau_3 \rightarrow \tau_4 \rightarrow \tau. z \ i \ b \ c \ d) \ t')$$

$$t.1 = t' \equiv t \ ((\lambda i : \tau'. \lambda a : \tau_1. \lambda b : \tau_2. \lambda c : \tau_3. \lambda d : \tau_4. \lambda z : \tau_1 \rightarrow \tau' \rightarrow \tau_3 \rightarrow \tau_4 \rightarrow \tau. z \ a \ i \ c \ d) \ t')$$

$$t.2 = t' \equiv t \ ((\lambda i : \tau'. \lambda a : \tau_1. \lambda b : \tau_2. \lambda c : \tau_3. \lambda d : \tau_4. \lambda z : \tau_1 \rightarrow \tau_2 \rightarrow \tau' \rightarrow \tau_4 \rightarrow \tau. z \ a \ b \ i \ d) \ t')$$

$$t.3 = t' \equiv t \ ((\lambda i : \tau'. \lambda a : \tau_1. \lambda b : \tau_2. \lambda c : \tau_3. \lambda d : \tau_4. \lambda z : \tau_1 \rightarrow \tau_2 \rightarrow \tau_3 \rightarrow \tau' \rightarrow \tau. z \ a \ b \ c \ i) \ t')$$

**reductions**  $[3, 3, 3, 2].3 = 3$

$$\begin{aligned} & ((\lambda a. \lambda b. \lambda c. \lambda d. \lambda x. x \ a \ b \ c \ d) \ 3 \ 3 \ 3 \ 2) \ ((\lambda i. \lambda a. \lambda b. \lambda c. \lambda d. \lambda z. z \ a \ b \ c \ i) \ 3) \rightsquigarrow \\ & ((\lambda b. \lambda c. \lambda d. \lambda x. x \ 3 \ b \ c \ d) \ 3 \ 3 \ 2) \ ((\lambda i. \lambda a. \lambda b. \lambda c. \lambda d. \lambda z. z \ a \ b \ c \ i) \ 3) \rightsquigarrow \\ & ((\lambda c. \lambda d. \lambda x. x \ 3 \ 3 \ c \ d) \ 3 \ 2) \ ((\lambda i. \lambda a. \lambda b. \lambda c. \lambda d. \lambda z. z \ a \ b \ c \ i) \ 3) \rightsquigarrow \\ & ((\lambda d. \lambda x. x \ 3 \ 3 \ 3 \ d) \ 2) \ ((\lambda i. \lambda a. \lambda b. \lambda c. \lambda d. \lambda z. z \ a \ b \ c \ i) \ 3) \rightsquigarrow \\ & (\lambda x. x \ 3 \ 3 \ 3 \ 2) \ ((\lambda i. \lambda a. \lambda b. \lambda c. \lambda d. \lambda z. z \ a \ b \ c \ i) \ 3) \rightsquigarrow \\ & (\lambda x. x \ 3 \ 3 \ 3 \ 2) \ (\lambda a. \lambda b. \lambda c. \lambda d. \lambda z. z \ a \ b \ c \ 3) \rightsquigarrow \\ & (\lambda a. \lambda b. \lambda c. \lambda d. \lambda z. z \ a \ b \ c \ 3) \ 3 \ 3 \ 3 \ 2 \rightsquigarrow \\ & (\lambda b. \lambda c. \lambda d. \lambda z. z \ 3 \ b \ c \ 3) \ 3 \ 3 \ 2 \rightsquigarrow \\ & (\lambda c. \lambda d. \lambda z. z \ 3 \ 3 \ c \ 3) \ 3 \ 2 \rightsquigarrow \\ & (\lambda d. \lambda z. z \ 3 \ 3 \ 3 \ 3) \ 2 \rightsquigarrow \\ & \lambda z. z \ 3 \ 3 \ 3 \ 3 \end{aligned}$$