Assignment #1

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1 Big step - call by name

Write the operational semantics rules for a big-step, call-by-name reduction for ULC. Write the semantically correct ones only, but write them all

$$\frac{}{v \Downarrow v} \quad \text{val}$$

$$\frac{t \Downarrow n' \quad t' \Downarrow n" \quad n' \oplus n" = n}{t \oplus t' \Downarrow n} \quad \text{bs-bop}$$

$$\frac{t \Downarrow \lambda x.t" \quad t"[t'/x] \Downarrow v}{t\ t' \Downarrow v} \quad \text{bs-app}$$

$$\frac{t_1 \Downarrow v_1 \quad t_2 \Downarrow v_2}{\langle t_1, t_2 \rangle \Downarrow \langle v_1, v_2 \rangle} \quad \text{pair}$$

$$\frac{t \Downarrow \langle v, v' \rangle}{t.1 \Downarrow v} \quad \text{first-projection}$$

$$\frac{t \Downarrow \langle v', v \rangle}{t.2 \Downarrow v} \quad \text{first-projection}$$

$$\frac{t \Downarrow v}{inL\ t \Downarrow inL\ v} \quad \text{inLeft}$$

$$\frac{t \Downarrow v}{inR\ t \Downarrow inR\ v} \quad \text{inRight}$$

$$\frac{t \Downarrow\ inL\ v' \quad t_1[v'/x_1] \Downarrow v}{\text{case } t \text{ of } \left| \begin{array}{l} inL\ x_1 \mapsto t_1 \\ inR\ x_2 \mapsto t_2 \end{array} \right. \Downarrow v} \quad \text{pattern matching L}$$

$$\frac{t \Downarrow\ inR\ v' \quad t_2[v'/x_2] \Downarrow v}{\text{case } t \text{ of } \left| \begin{array}{l} inL\ x_1 \mapsto t_1 \\ inR\ x_2 \mapsto t_2 \end{array} \right. \Downarrow v} \quad \text{pattern matching R}$$

2 Equivalence of SOS and COS

2.1 if $t \to t'$ then $t \rightsquigarrow t'$

Proof by induction on $\to$.
Inductive hypothesis: $\forall t_h, t'_h . if\ t_h \to t'_h\ then\ t_h \rightsquigarrow t'_h$

2.1.1 App-2

$t = (\lambda x.t_1)\ t_2$
$t' = (\lambda x.t_1)\ t'_2$

$$\begin{array}{rll}
HP1 & (\lambda x.t_1)\ t_2 \to (\lambda x.t_1)\ t'_2 & \\
HP2 & t_2 \to t'_2 & \\
TH & (\lambda x.t_1)\ t_2 \rightsquigarrow (\lambda x.t_1)\ t'_2 & \\
by\ IH\ with\ HP2\ w.h & t_2 \rightsquigarrow t'_2 & HT1 \\
by\ inversion\ on\ HT1\ w.h & t_2 \equiv E[t_0] & HEI \\
 & t'_2 \equiv E[t'_0] & HEI' \\
 & t_0 \rightsquigarrow^p t'_0 & HPR \\
by\ HEI,\ HEI'\ t.s & (\lambda x.t_1)\ E[t_0] \rightsquigarrow (\lambda x.t_1)\ E[t'_0] & \\
by\ ctx\ with\ E' = (\lambda x.t_1)\ E & & \\
and\ HPR & (\lambda x.t_1)\ E[t_0] \equiv E'[t_0] \rightsquigarrow E'[t'_0] &
\end{array}$$

2.1.2 Op-1

$t = t_1 \oplus t_2$
$t' = t'_1 \oplus t_2$

$$\begin{array}{rll}
HP1 & t_1 \oplus t_2 \to t'_1 \oplus t_2 & \\
HP2 & t_1 \to t'_1 & \\
TH & t_1 \oplus t_2 \rightsquigarrow t'_1 \oplus t_2 & \\
by\ IH\ with\ HP2\ w.h & t_1 \rightsquigarrow t'_1 & HT1 \\
by\ inversion\ on\ HT1\ w.h & t_1 \equiv E[t_0] & HEI \\
 & t'_1 \equiv E[t'_0] & HEI' \\
 & t_0 \rightsquigarrow^p t'_0 & HPR \\
by\ HEI,\ HEI'\ t.s & E[t_0] \oplus t_2 \rightsquigarrow E[t'_0] \oplus t_2 & \\
by\ ctx\ with\ E' = E \oplus t_2 & & \\
and\ HPR & E[t_0] \oplus t_2 \equiv E'[t_0] \rightsquigarrow E'[t'_0] &
\end{array}$$

2.1.3 Op-2

$t = n \oplus t_1$
$t' = n \oplus t'_1$

$$\begin{array}{rll}
HP1 & n \oplus t_1 \rightarrow n \oplus t'_1 & \\
HP2 & t_1 \rightarrow t'_1 & \\
TH & n \oplus t_1 \rightsquigarrow n \oplus t'_1 & \\
by\ IH\ with\ HP2\ w.h & t_1 \rightsquigarrow t'_1 & HT1 \\
by\ inversion\ on\ HT1\ w.h & t_1 \equiv E[t_0] & HEI \\
 & t'_1 \equiv E[t'_0] & HEI' \\
 & t_0 \rightsquigarrow^p t'_0 & HPR \\
by\ HEI,\ HEI'\ t.s & n \oplus E[t_0] \rightsquigarrow n \oplus E[t'_0] & \\
by\ ctx\ with\ E' = n \oplus E & & \\
and\ HPR & n \oplus E[t_0] \equiv E'[t_0] \rightsquigarrow E'[t'_0] &
\end{array}$$

2.2 if $t \rightsquigarrow t'$ then $t \rightarrow t'$

Helper lemma $\forall t, t', E.\ if\ t \rightarrow t'\ then\ E[t] \rightarrow E[t']$
Proof by induction on E.
Inductive hypothesis: $if\ t \rightarrow t'\ then\ E'[t] \rightarrow E'[t']$

2.2.1 $E = (\lambda x.t")\ E'$

$$\begin{array}{rll}
by\ IH\ with\ HP, E'\ w.h & E'[t] \rightarrow E'[t'] & HE \\
t.s & ((\lambda x.t")\ E')[t] \rightarrow ((\lambda x.t")\ E')[t'] & \\
by\ [\cdot]\ t.s & (\lambda x.t")E'[t] \rightarrow (\lambda x.t")E'[t'] & \\
by\ APP-1\ with\ HE & \Box &
\end{array}$$

2.2.2 $E = E' \oplus t"$

$$\begin{array}{rll}
by\ IH\ with\ HP, E'\ w.h & E'[t] \rightarrow E'[t'] & HE \\
t.s & (E \oplus t")[t] \rightarrow (E \oplus t")[t'] & \\
by\ [\cdot]\ t.s & E'[t] \oplus t" \rightarrow E'[t'] \oplus t" & \\
by\ BOP-1\ with\ HE & \Box &
\end{array}$$

2.2.3 $E = n \oplus E'$

$$\begin{array}{rll}
by\ IH\ with\ HP, E'\ w.h & E'[t] \rightarrow E'[t'] & HE \\
t.s & (n \oplus E')[t] \rightarrow (n \oplus E')[t'] & \\
by\ [\cdot]\ t.s & n \oplus E'[t] \rightarrow n \oplus E'[t'] & \\
by\ BOP-1\ with\ HE & \Box &
\end{array}$$

3 Distinguish terms

$$t \overset{def}{=} \lambda d : (\mathbb{N} \to \mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N} \to \mathbb{N}) \to \mathbb{N}$$
$$.d\ (\lambda a : \mathbb{N}.\lambda b : \mathbb{N}.b)\ (\lambda i : \mathbb{N}.\lambda j : \mathbb{N}.i)$$

Reduction 1

$$\begin{array}{ll}
(\lambda d.d\ (\lambda a.\lambda b.b)\ (\lambda i.\lambda j.i))\ (\lambda x.\lambda y.x\ 0\ (y\ 0\ 0)) & \rightsquigarrow \\
(\lambda x.\lambda y.x\ 0\ (y\ 0\ 0))\ (\lambda a.\lambda b.b)\ (\lambda i.\lambda j.i) & \rightsquigarrow \\
(\lambda y.(\lambda a.\lambda b.b)\ 0\ (y\ 0\ 0))\ (\lambda i.\lambda j.i) & \rightsquigarrow \\
(\lambda a.\lambda b.b)\ 0\ ((\lambda i.\lambda j.i)\ 0\ 0) & \rightsquigarrow \\
(\lambda b.b)\ ((\lambda i.\lambda j.i)\ 0\ 0) & \rightsquigarrow \\
(\lambda b.b)\ ((\lambda j.0)\ 0) & \rightsquigarrow \\
(\lambda b.b)\ 0 & \rightsquigarrow \\
0 &
\end{array}$$

Reduction 2

$$\begin{array}{ll}
(\lambda d.d\ (\lambda a.\lambda b.b)\ (\lambda i.\lambda j.i))\ (\lambda x.\lambda y.x\ 0\ (y\ 1\ 0)) & \rightsquigarrow \\
(\lambda x.\lambda y.x\ 0\ (y\ 1\ 0))\ (\lambda a.\lambda b.b)\ (\lambda i.\lambda j.i) & \rightsquigarrow \\
(\lambda y.(\lambda a.\lambda b.b)\ 0\ (y\ 1\ 0))\ (\lambda i.\lambda j.i) & \rightsquigarrow \\
(\lambda a.\lambda b.b)\ 0\ ((\lambda i.\lambda j.i)\ 1\ 0) & \rightsquigarrow \\
(\lambda b.b)\ ((\lambda i.\lambda j.i)\ 1\ 0) & \rightsquigarrow \\
(\lambda b.b)\ ((\lambda j.1)\ 0) & \rightsquigarrow \\
(\lambda b.b)\ 1 & \rightsquigarrow \\
1 &
\end{array}$$

4 Safe untypable term

$$(\lambda x.35 * 12)\ (\lambda y.1\ y)$$

typing derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x : \mathbb{N} \vdash 35 : \mathbb{N}}\text{nat} \qquad \dfrac{}{x : \mathbb{N} \vdash 12 : \mathbb{N}}\text{nat}}{x : \tau \vdash 35 * 12 : \mathbb{N}}\text{bop}}{\emptyset \vdash \lambda x.35 * 12 : \tau \to \mathbb{N}}\text{lam} \qquad \dfrac{\dfrac{}{y : \tau' \vdash 1\ y : \tau}\text{fail}}{\emptyset \vdash \lambda y.1\ y : \tau' \to \tau}\text{lam}}{\emptyset \vdash (\lambda x.35 * 12)\ (\lambda y.1\ y) :}\text{app}$$

COS-SM-CBV

$$\begin{array}{l}
(\lambda x.35 * 12)\ (\lambda x.1\ x) \rightsquigarrow \\
35 * 12 \rightsquigarrow \\
420
\end{array}$$

5 Typing derivation

$$\dfrac{\dfrac{f:\mathbb{N}\to\mathbb{N}\in\Gamma}{\Gamma\vdash f:\mathbb{N}\to\mathbb{N}}\text{var} \quad \dfrac{\dfrac{\dfrac{\dfrac{x:\mathbb{N}\in\Gamma'}{\Gamma'\vdash x:\mathbb{N}}\text{var}\quad \dfrac{}{\Gamma'\vdash 2:\mathbb{N}}\text{nat}}{\Gamma'\left\{\begin{smallmatrix}f:\mathbb{N}\to\mathbb{N},\\ x:\mathbb{N}\end{smallmatrix}\right.\vdash x+2:\mathbb{N}}\text{op}}{\Gamma\vdash\lambda x.x+2:\mathbb{N}\to\mathbb{N}}\text{lam}\quad\dfrac{}{\Gamma\vdash 4:\mathbb{N}}\text{nat}}{\Gamma\vdash(\lambda x.x+2)\ 4:\mathbb{N}}\text{app}}{\Gamma\left\{f:\mathbb{N}\to\mathbb{N}\right.\vdash f\ ((\lambda x.x+2)\ 4):\mathbb{N}}\text{app}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x:\mathbb{N}\to\mathbb{N}\in\Gamma'}{\Gamma'\vdash x:\mathbb{N}\to\mathbb{N}}\text{var}\quad\dfrac{y:\mathbb{N}\in\Gamma'}{\Gamma'\vdash y:\mathbb{N}}\text{var}}{\Gamma'\left\{\begin{smallmatrix}\Gamma,\\ x:\mathbb{N}\to\mathbb{N},\\ y:\mathbb{N}\end{smallmatrix}\right.\vdash x\ y:\mathbb{N}}\text{app}}{\begin{smallmatrix}\Gamma,\\ x:\mathbb{N}\to\mathbb{N}\end{smallmatrix}\vdash\lambda y.x\ y:\mathbb{N}\to\mathbb{N}}\text{lam}}{\Gamma\vdash\lambda x.\lambda y.x\ y:(\mathbb{N}\to\mathbb{N})\to\mathbb{N}\to\mathbb{N}}\text{lam}\quad\dfrac{f:\mathbb{N}\to\mathbb{N}\in\Gamma}{\Gamma\vdash f:\mathbb{N}\to\mathbb{N}}\text{var}}{\Gamma\vdash(\lambda x.\lambda y.x\ y)\ f:\mathbb{N}\to\mathbb{N}}\text{app}\quad\dfrac{}{\Gamma\vdash 3:\mathbb{N}}\text{nat}}{\Gamma\left\{f:\mathbb{N}\to\mathbb{N}\right.\vdash((\lambda x.\lambda y.x\ y)\ f)\ 3:\mathbb{N}}\text{app}$$

6 Encoding

6.1 Sequencing

$$t ::= \cdots|t;t'$$
$$t;t' \equiv (\lambda a:\tau.\lambda b:\tau'.b)\ t\ t'$$

reductions

$$\begin{array}{rll}
2+4;8+1 \equiv & (\lambda a.\lambda b.b)\ (2+4)\ (8+1) & \rightsquigarrow \\
(t \text{ gets evaluated first}) & (\lambda a.\lambda b.b)\ 6\ (8+1) & \rightsquigarrow \\
 & (\lambda b.b)\ (8+1) & \rightsquigarrow \\
(t' \text{ gets evaluated second}) & (\lambda b.b)\ 9 & \rightsquigarrow 9
\end{array}$$

Let-in

$$t ::= \cdots|let\ x = t\ in\ t'$$
$$let\ x = t\ in\ t' \equiv (\lambda x:\tau.t')\ t$$

reductions

$$\begin{array}{rll}
let\ x = (\lambda a.\lambda b.a*b)\ in\ x\ 11\ 13 \equiv & (\lambda x.x\ 11\ 13)\ (\lambda a.\lambda b.a*b) & \rightsquigarrow \\
 & (\lambda a.\lambda b.a*b)\ 11\ 13 & \rightsquigarrow 143
\end{array}$$

Arrays of Length 4

$$t ::= \cdots |[t, t, t, t]$$
$$v ::= \cdots |[v, v, v, v]$$

$$[t_1, t_2, t_3, t_4] \equiv (\lambda a : \tau_1.\lambda b : \tau_2.\lambda c : \tau_3.$$
$$\lambda d : \tau_4.\lambda x : \tau_1 \to \tau_2 \to \tau_3 \to \tau_4 \to \tau.x\ a\ b\ c\ d)\ t_1\ t_2\ t_3\ t_4$$

$$[v_1, v_2, v_3, v_4] \equiv \lambda x : \tau_1 \to \tau_2 \to \tau_3 \to \tau_4 \to \tau.x\ v_1\ v_2\ v_3\ v_4$$

reductions $[85, \lambda x.\lambda y.x + y, 29 + 44, 3]$

$$\begin{array}{ll}
(\lambda a.\lambda b.\lambda c.\lambda d.\lambda x.x\ a\ b\ c\ d)\ (85)\ (\lambda x.\lambda y.x + y)\ (29 + 44)\ 3 & \rightsquigarrow \\
(\lambda b.\lambda c.\lambda d.\lambda x.x\ 85\ b\ c\ d)\ (\lambda x.\lambda y.x + y)\ (29 + 44)\ 3 & \rightsquigarrow \\
(\lambda c.\lambda d.\lambda x.x\ 85\ (\lambda x.\lambda y.x + y)\ c\ d)\ (29 + 44)\ 3 & \rightsquigarrow \\
(\lambda c.\lambda d.\lambda x.x\ 85\ (\lambda x.\lambda y.x + y)\ c\ d)\ 73\ 3 & \rightsquigarrow \\
(\lambda d.\lambda x.x\ 85\ (\lambda x.\lambda y.x + y)\ 73\ d)\ 3 & \rightsquigarrow \\
\lambda x.x\ 85\ (\lambda x.\lambda y.x + y)\ 73\ 3 &
\end{array}$$

Array field access

$$t ::= \cdots |t.i\ (i \in 0..3)$$
$$t.0 \equiv t\ (\lambda a : \tau_1.\lambda b : \tau_2.\lambda c : \tau_3.\lambda d : \tau_4.a)$$
$$t.1 \equiv t\ (\lambda a : \tau_1.\lambda b : \tau_2.\lambda c : \tau_3.\lambda d : \tau_4.b)$$
$$t.2 \equiv t\ (\lambda a : \tau_1.\lambda b : \tau_2.\lambda c : \tau_3.\lambda d : \tau_4.c)$$
$$t.3 \equiv t\ (\lambda a : \tau_1.\lambda b : \tau_2.\lambda c : \tau_3.\lambda d : \tau_4.d)$$

reductions $[1, 1, 2, 3].0$

$$\begin{array}{ll}
((\lambda a.\lambda b.\lambda c.\lambda d.\lambda x.x\ a\ b\ c\ d)\ 1\ 1\ 2\ 3)\ (\lambda a.\lambda b.\lambda c.\lambda d.a) & \rightsquigarrow \\
((\lambda b.\lambda c.\lambda d.\lambda x.x\ 1\ b\ c\ d)\ 1\ 2\ 3)\ (\lambda a.\lambda b.\lambda c.\lambda d.a) & \rightsquigarrow \\
((\lambda c.\lambda d.\lambda x.x\ 1\ 1\ c\ d)\ 2\ 3)\ (\lambda a.\lambda b.\lambda c.\lambda d.a) & \rightsquigarrow \\
((\lambda d.\lambda x.x\ 1\ 1\ 2\ d)\ 3)\ (\lambda a.\lambda b.\lambda c.\lambda d.a) & \rightsquigarrow \\
(\lambda x.x\ 1\ 1\ 2\ 3)\ (\lambda a.\lambda b.\lambda c.\lambda d.a) & \rightsquigarrow \\
(\lambda a.\lambda b.\lambda c.\lambda d.a)\ 1\ 1\ 2\ 3 & \rightsquigarrow \\
(\lambda b.\lambda c.\lambda d.1)\ 1\ 2\ 3 & \rightsquigarrow \\
(\lambda c.\lambda d.1)\ 2\ 3 & \rightsquigarrow \\
(\lambda d.1)\ 3 & \rightsquigarrow \\
1 &
\end{array}$$

Array update

$$t ::= \cdots | t.i = t \ (i \in 0..3)$$

$$t.0 = t' \equiv t\ ((\lambda i : \tau'.\lambda a : \tau_1.\lambda b : \tau_2.\lambda c : \tau_3.\lambda d : \tau_4.\lambda z : \tau' \to \tau_2 \to \tau_3 \to \tau_4 \to \tau.z\ i\ b\ c\ d)\ t')$$
$$t.1 = t' \equiv t\ ((\lambda i : \tau'.\lambda a : \tau_1.\lambda b : \tau_2.\lambda c : \tau_3.\lambda d : \tau_4.\lambda z : \tau_1 \to \tau' \to \tau_3 \to \tau_4 \to \tau.z\ a\ i\ c\ d)\ t')$$
$$t.2 = t' \equiv t\ ((\lambda i : \tau'.\lambda a : \tau_1.\lambda b : \tau_2.\lambda c : \tau_3.\lambda d : \tau_4.\lambda z : \tau_1 \to \tau_2 \to \tau' \to \tau_4 \to \tau.z\ a\ b\ i\ d)\ t')$$
$$t.3 = t' \equiv t\ ((\lambda i : \tau'.\lambda a : \tau_1.\lambda b : \tau_2.\lambda c : \tau_3.\lambda d : \tau_4.\lambda z : \tau_1 \to \tau_2 \to \tau_3 \to \tau' \to \tau.z\ a\ b\ c\ i)\ t')$$

reductions $[3, 3, 3, 2].3 = 3$

$$\begin{array}{lr}
((\lambda a.\lambda b.\lambda c.\lambda d.\lambda x.x\ a\ b\ c\ d)\ 3\ 3\ 3\ 2)\ ((\lambda i.\lambda a.\lambda b.\lambda c.\lambda d.\lambda z.z\ a\ b\ c\ i)\ 3) & \rightsquigarrow \\
((\lambda b.\lambda c.\lambda d.\lambda x.x\ 3\ b\ c\ d)\ 3\ 3\ 2)\ ((\lambda i.\lambda a.\lambda b.\lambda c.\lambda d.\lambda z.z\ a\ b\ c\ i)\ 3) & \rightsquigarrow \\
((\lambda c.\lambda d.\lambda x.x\ 3\ 3\ c\ d)\ 3\ 2)\ ((\lambda i.\lambda a.\lambda b.\lambda c.\lambda d.\lambda z.z\ a\ b\ c\ i)\ 3) & \rightsquigarrow \\
((\lambda d.\lambda x.x\ 3\ 3\ 3\ d)\ 2)\ ((\lambda i.\lambda a.\lambda b.\lambda c.\lambda d.\lambda z.z\ a\ b\ c\ i)\ 3) & \rightsquigarrow \\
(\lambda x.x\ 3\ 3\ 3\ 2)\ ((\lambda i.\lambda a.\lambda b.\lambda c.\lambda d.\lambda z.z\ a\ b\ c\ i)\ 3) & \rightsquigarrow \\
(\lambda x.x\ 3\ 3\ 3\ 2)\ (\lambda a.\lambda b.\lambda c.\lambda d.\lambda z.z\ a\ b\ c\ 3) & \rightsquigarrow \\
(\lambda a.\lambda b.\lambda c.\lambda d.\lambda z.z\ a\ b\ c\ 3)\ 3\ 3\ 3\ 2 & \rightsquigarrow \\
(\lambda b.\lambda c.\lambda d.\lambda z.z\ 3\ b\ c\ 3)\ 3\ 3\ 2 & \rightsquigarrow \\
(\lambda c.\lambda d.\lambda z.z\ 3\ 3\ c\ 3)\ 3\ 2 & \rightsquigarrow \\
(\lambda d.\lambda z.z\ 3\ 3\ 3\ 3)\ 2 & \rightsquigarrow
\end{array}$$

$$\lambda z.z\ 3\ 3\ 3\ 3$$