Assignment #1

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1 Big step - call by name

Write the operational semantics rules for a big-step, call-by-name reduction for ULC. Write the semantically correct ones only, but write them all

$$\frac{}{v \Downarrow v} \quad \text{val}$$

$$\frac{t \Downarrow n' \quad t' \Downarrow n" \quad n' \oplus n" = n}{t \oplus t' \Downarrow n} \quad \text{bs-bop}$$

$$\frac{t \Downarrow \lambda x.t" \quad t"[t'/x] \Downarrow v}{t\ t' \Downarrow v} \quad \text{bs-app}$$

$$\frac{t_1 \Downarrow v_1 \quad t_2 \Downarrow v_2}{\langle t_1, t_2 \rangle \Downarrow \langle v_1, v_2 \rangle} \quad \text{pair}$$

$$\frac{t \Downarrow \langle v, v' \rangle}{t.1 \Downarrow v} \quad \text{first-projection}$$

$$\frac{t \Downarrow \langle v', v \rangle}{t.2 \Downarrow v} \quad \text{first-projection}$$

$$\frac{t \Downarrow v}{inL\ t \Downarrow inL\ v} \quad \text{inLeft}$$

$$\frac{t \Downarrow v}{inR\ t \Downarrow inR\ v} \quad \text{inRight}$$

$$\frac{t \Downarrow\ inL\ v' \quad t_1[v'/x_1] \Downarrow v}{\text{case } t \text{ of } \left| \begin{array}{l} inL\ x_1 \mapsto t_1 \\ inR\ x_2 \mapsto t_2 \end{array} \right. \Downarrow v} \quad \text{pattern matching L}$$

$$\frac{t \Downarrow\ inR\ v' \quad t_2[v'/x_2] \Downarrow v}{\text{case } t \text{ of } \left| \begin{array}{l} inL\ x_1 \mapsto t_1 \\ inR\ x_2 \mapsto t_2 \end{array} \right. \Downarrow v} \quad \text{pattern matching R}$$

2 Equivalence of SOS and COS

2.1 if $t \to t'$ then $t \rightsquigarrow t'$

Proof by induction on $\to$.
Inductive hypothesis: $\forall t_h, t'_h.if\ t_h \to t'_h\ then\ t_h \rightsquigarrow t'_h$

2.1.1 App-2

$t = (\lambda x.t_1)\ t_2$
$t' = (\lambda x.t_1)\ t_2'$

$$
\begin{array}{rll}
HP1 & (\lambda x.t_1)\ t_2 \to (\lambda x.t_1)\ t_2' & \\
HP2 & t_2 \to t_2' & \\
TH & (\lambda x.t_1)\ t_2 \rightsquigarrow (\lambda x.t_1)\ t_2' & \\
by\ IH\ with\ HP2\ w.h & t_2 \rightsquigarrow t_2' & HT1 \\
by\ inversion\ on\ HT1\ w.h & t_2 \equiv E[t_0] & HEI \\
 & t_2' \equiv E[t_0'] & HEI' \\
 & t_0 \rightsquigarrow^p t_0' & HPR \\
by\ HEI,\ HEI'\ t.s & (\lambda x.t_1)\ E[t_0] \rightsquigarrow (\lambda x.t_1)\ E[t_0'] & \\
by\ ctx\ with\ E' = (\lambda x.t_1)\ E & & \\
and\ HPR & (\lambda x.t_1)\ E[t_0] \equiv E'[t_0] \rightsquigarrow E'[t_0'] &
\end{array}
$$

3 Distinguish terms

$$
t \stackrel{def}{=} \lambda d : (\mathbb{N} \to \mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N} \to \mathbb{N}) \to \mathbb{N}
$$
$$
.d\ (\lambda a : \mathbb{N}.\lambda b : \mathbb{N}.b)\ (\lambda i : \mathbb{N}.\lambda j : \mathbb{N}.i)
$$

Reduction 1

$$
\begin{array}{ll}
(\lambda d.d\ (\lambda a.\lambda b.b)\ (\lambda i.\lambda j.i))\ (\lambda x.\lambda y.x\ 0\ (y\ 0\ 0)) & \rightsquigarrow \\
(\lambda x.\lambda y.x\ 0\ (y\ 0\ 0))\ (\lambda a.\lambda b.b)\ (\lambda i.\lambda j.i) & \rightsquigarrow \\
(\lambda y.(\lambda a.\lambda b.b)\ 0\ (y\ 0\ 0))\ (\lambda i.\lambda j.i) & \rightsquigarrow \\
(\lambda a.\lambda b.b)\ 0\ ((\lambda i.\lambda j.i)\ 0\ 0) & \rightsquigarrow \\
(\lambda b.b)\ ((\lambda i.\lambda j.i)\ 0\ 0) & \rightsquigarrow \\
(\lambda b.b)\ ((\lambda j.0)\ 0) & \rightsquigarrow \\
(\lambda b.b)\ 0 & \rightsquigarrow \\
0 &
\end{array}
$$

Reduction 2

$$
\begin{array}{ll}
(\lambda d.d\ (\lambda a.\lambda b.b)\ (\lambda i.\lambda j.i))\ (\lambda x.\lambda y.x\ 0\ (y\ 1\ 0)) & \rightsquigarrow \\
(\lambda x.\lambda y.x\ 0\ (y\ 1\ 0))\ (\lambda a.\lambda b.b)\ (\lambda i.\lambda j.i) & \rightsquigarrow \\
(\lambda y.(\lambda a.\lambda b.b)\ 0\ (y\ 1\ 0))\ (\lambda i.\lambda j.i) & \rightsquigarrow \\
(\lambda a.\lambda b.b)\ 0\ ((\lambda i.\lambda j.i)\ 1\ 0) & \rightsquigarrow \\
(\lambda b.b)\ ((\lambda i.\lambda j.i)\ 1\ 0) & \rightsquigarrow \\
(\lambda b.b)\ ((\lambda j.1)\ 0) & \rightsquigarrow \\
(\lambda b.b)\ 1 & \rightsquigarrow \\
1 &
\end{array}
$$

4 Safe untypable term

$$(\lambda x.35 * 12)\ (\lambda x.1\ x)$$

typing derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x:\mathbb{N}\vdash 35:\mathbb{N}}\,\text{nat} \qquad \dfrac{}{x:\mathbb{N}\vdash 35:\mathbb{N}}\,\text{nat}}{x:\tau\vdash 35*12:\mathbb{N}}\,\text{bop}}{\emptyset\vdash\lambda x.35*12:\tau\to\mathbb{N}}\,\text{lam} \qquad \dfrac{\dfrac{}{y:\tau'\vdash 1\ y:\tau}\,\text{none}}{\emptyset\vdash\lambda y.1\ y:\tau'\to\tau}\,\text{lam}}{\emptyset\vdash(\lambda x.35*12)\ (\lambda y.1\ y):}\,\text{app}$$

COS-SM-CBV

$$\begin{aligned}&(\lambda x.35 * 12)\ (\lambda x.1\ x) \leadsto\\ &35 * 12 \leadsto\\ &420\end{aligned}$$

5 Typing derivation

$$\dfrac{\dfrac{f:\mathbb{N}\to\mathbb{N}\in\Gamma}{\Gamma\vdash f:\mathbb{N}\to\mathbb{N}}\,\text{var} \qquad \dfrac{\dfrac{\dfrac{\dfrac{x:\mathbb{N}\in\Gamma'}{\Gamma'\vdash x:\mathbb{N}}\,\text{var} \qquad \dfrac{}{\Gamma'\vdash 2:\mathbb{N}}\,\text{nat}}{\Gamma'\left\{\begin{smallmatrix}f:\mathbb{N}\to\mathbb{N},\\ x:\mathbb{N}\end{smallmatrix}\right.\vdash x+2:\mathbb{N}}\,\text{op}}{\Gamma\vdash\lambda x.x+2:\mathbb{N}\to\mathbb{N}}\,\text{lam} \qquad \dfrac{}{\Gamma\vdash 4:\mathbb{N}}\,\text{nat}}{\Gamma\vdash(\lambda x.x+2)\ 4:\mathbb{N}}\,\text{app}}{\Gamma\,\{f:\mathbb{N}\to\mathbb{N}\vdash f\ ((\lambda x.x+2)\ 4):\mathbb{N}}\,\text{app}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x:\mathbb{N}\to\mathbb{N}\in\Gamma'}{\Gamma'\vdash x:\mathbb{N}\to\mathbb{N}}\,\text{var} \qquad \dfrac{y:\mathbb{N}\in\Gamma'}{\Gamma'\vdash y:\mathbb{N}}\,\text{var}}{\Gamma'\left\{\begin{smallmatrix}\Gamma,\\ x:\mathbb{N}\to\mathbb{N},\\ y:\mathbb{N}\end{smallmatrix}\right.\vdash x\ y:\mathbb{N}}\,\text{app}}{\begin{smallmatrix}\Gamma,\\ x:\mathbb{N}\to\mathbb{N}\end{smallmatrix}\vdash\lambda y.x\ y:\mathbb{N}\to\mathbb{N}}\,\text{lam}}{\Gamma\vdash\lambda x.\lambda y.x\ y:(\mathbb{N}\to\mathbb{N})\to\mathbb{N}\to\mathbb{N}}\,\text{lam} \qquad \dfrac{f:\mathbb{N}\to\mathbb{N}\in\Gamma}{\Gamma\vdash f:\mathbb{N}\to\mathbb{N}}\,\text{var}}{\Gamma\vdash(\lambda x.\lambda y.x\ y)\ f:\mathbb{N}\to\mathbb{N}}\,\text{app} \qquad \dfrac{}{\Gamma\vdash 3:\mathbb{N}}\,\text{nat}}{\Gamma\,\{f:\mathbb{N}\to\mathbb{N}\vdash((\lambda x.\lambda y.x\ y)\ f)\ 3:\mathbb{N}}\,\text{app}$$

6 Encoding

Sequencing

$$t ::= \cdots|t;t'$$
$$t;t' \equiv (\lambda x.t')\ t$$

(Assuming x is a free variable not used in t')

Let-in

$$t ::= \cdots | let\ x = t\ in\ t'$$
$$let\ x = t\ in\ t' \equiv (\lambda x.t')\ t$$

Arrays of Length 4

$$t ::= \cdots |[t, t, t, t]$$
$$v ::= \cdots |[v, v, v, v]$$
$$[t_1, t_2, t_3, t_4] \equiv \lambda a.a\ t_1\ t_2\ t_3\ t_4$$

Array field access

$$t ::= \cdots |t.i\ (i \in 0..3)$$
$$t.0 \equiv t\ (\lambda a.\lambda b.\lambda c.\lambda d.a)$$
$$t.1 \equiv t\ (\lambda a.\lambda b.\lambda c.\lambda d.b)$$
$$t.2 \equiv t\ (\lambda a.\lambda b.\lambda c.\lambda d.c)$$
$$t.3 \equiv t\ (\lambda a.\lambda b.\lambda c.\lambda d.d)$$

Array update

$$t ::= \cdots |t.i = t\ (i \in 0..3)$$
$$t.0 = t' \equiv t\ (\lambda a.\lambda b.\lambda c.\lambda d.\lambda z.z\ t'\ b\ c\ d)$$
$$t.1 = t' \equiv t\ (\lambda a.\lambda b.\lambda c.\lambda d.\lambda z.z\ a\ t'\ c\ d)$$
$$t.2 = t' \equiv t\ (\lambda a.\lambda b.\lambda c.\lambda d.\lambda z.z\ a\ b\ t'\ d)$$
$$t.3 = t' \equiv t\ (\lambda a.\lambda b.\lambda c.\lambda d.\lambda z.z\ a\ b\ c\ t')$$