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A typology of cutting and packing problems

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Abstract: Cutting and packing problems appear under various names in literature, e.g. cutting stock or trim loss problem, bin or strip packing problem, vehicle, pallet or container loading problem, nesting problem, knapsack problem etc. The paper develops a consistent and systematic approach for a comprehensive typology integrating the various kinds of problems. The typology is founded on the basic logical structure of cutting and packing problems. The purpose is to unify the different use of notions in the literature and to concentrate further research on special types of problems.

Keywords: Cutting, packing, production, distribution, engineering

1. Introduction

The topic of cutting and packing (abbreviated by C&P in the following) is characterized by the fact that problems of essentially the same logical structure appear under different names in literature, such as e.g.:

- cutting stock and trim loss problems,
- bin packing, dual bin packing, strip packing, vector packing, and knapsack (packing) problems,
- vehicle loading, pallet loading, container loading, and car loading problems,
- assortment, depletion, design, dividing, layout, nesting, and partitioning problems,
- (or even) capital budgeting, change making, line balancing, memory allocation, and multiprocessor scheduling problems.

This heterogeneous use is illustrated by the titles of a selection of English-language publications recorded in Table 1 by their author(s), year of publication, notion(s) used in the title, as well as the discipline corresponding to the name of the journal or book. These publications are chosen because they give certain surveys on the subject of C&P. There are some more surveys of this kind, e.g. the German-language book of Terno et al. (1987).

Table 1 also reflects the rapid development as well as the wide dispersion of research on this topic. With very few exceptions (Kantorovich, 1939; Brooks et al., 1940) scientific work started about thirty five years ago. Since then, there has been a fast growing number of papers dealing with various aspects and spread over various journals and proceedings of different nations with different languages and—last but not least—of different disciplines such as Management Science, Engineering Sciences, Information and Computer Sciences, Mathematics as well as Operational Research (as an interdisciplinary approach). First surveys appeared in the seventies, already realizing the strong relationship between cutting problems on the one hand and packing problems on the other (Brown, 1971; Golden, 1976). Table 1 shows that more than one paper containing some kind of survey has been published annually on average since 1980.

The surveys published up to now deal with particular aspects of certain special types of C&P-problems, mainly with algorithms and solution methods. There is, however, no survey on the whole subject of C&P which systematically integrates the various kinds of problems and notions. It is the purpose of the present paper. In order to achieve this aim a typology founded on the basic logical structure of C&P is developed. It allows identification of common properties of problems which, at first sight, seem to be unre-

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Table 1
Surveys on special aspects of C&P

Author(s)	Year	Notion(s)	Discipline
Brown	1971	Packing, depletion	Computer Science
Salkin/de Kluyver	1975	Knapsack	Logistics
Golden	1976	Cutting stock	Industrial Engineering
Hinxman	1980	Trim loss, assortment	Operational Research
Garey/Johnson	1981	Bin packing	Combinatorial Optimization
Israni/Sanders	1982	Cutting stock, layout	Manufacturing
Rayward-Smith/Shing	1983	Bin packing	Mathematics
Coffman et al.	1984	Bin packing	Computer Science
Dowsland	1985	Packing	Operational Research
Dyckhoff et al.	1985	Trim loss	Management
Israni/Sanders	1985	Parts nesting	Production
Berkey/Wang	1987	Bin packing	Operational Research
Dudzinski/Walukiewicz	1987	Knapsack	Operational Research
Martello/Toth	1987	Knapsack	Mathematics
Rode/Rosenberg	1987	Trim loss	Engineering/Production
Dyckhoff et al.	1988	Cutting stock	Production

lated. Conversely, differences between apparently similar problems become clearer by analysing their main characteristics. The characteristics can be used to distinguish various elementary and combined types. Hardly any of the single characteristics and elementary types discussed are totally new. Most of them have been used in literature before. It is the consistent and systematic approach for a comprehensive typology which is new (to the best of my knowledge). Perhaps this may be a basis for unifying the different use of notions in literature and for concentrating future research on special types of problems.

The next section describes the basic logical structure of C&P. It allows formulation of a first rough classification of C&P-problems. Moreover, it provides a scheme for Section 3 where various characteristics are systematically identified and analysed in more detail. Examples of some important elementary types are given. For the purpose of comparison with notions used in literature, Section 4 uses four characteristics to construct 96 combined types. As Section 5 demonstrates, some of these problem types can also be related to appropriate solution approaches proposed in literature. The last section contains a concluding remark.

2. Basic logical structure

The basic logical structure of C&P-problems becomes obvious by looking at some simple examples. Figure 1 illustrates a problem of cutting tubes for radiator heat units (described by Heicken and König, 1980). On the one hand there is an

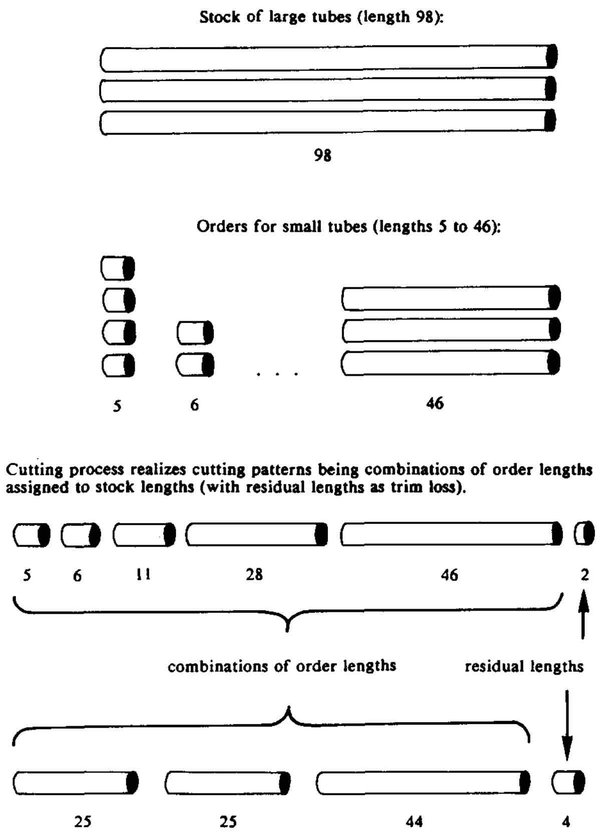

Figure 1. Cutting of tubes

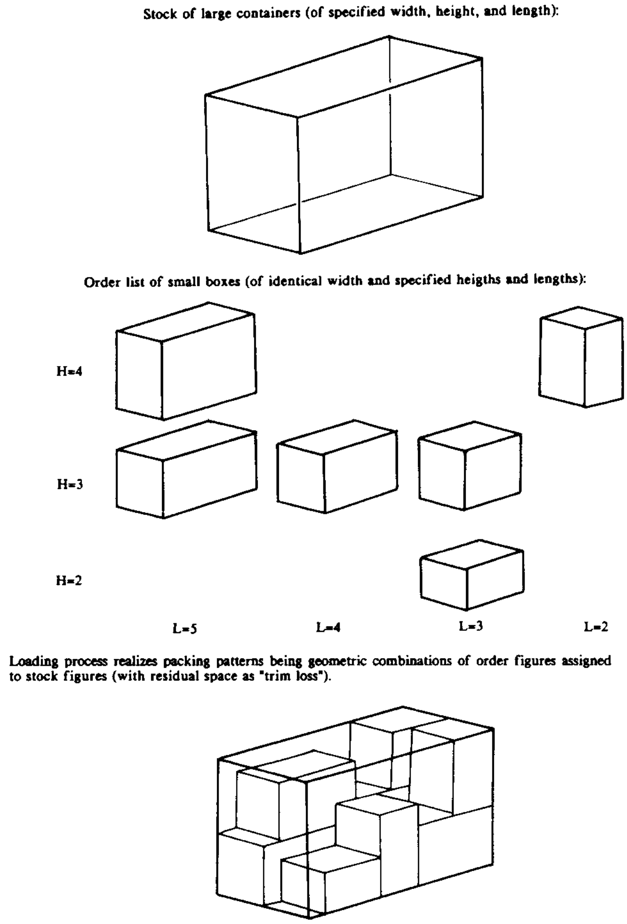

Figure 2. Container loading

unlimited stock of large tubes of length 98 used for producing smaller tubes. On the other hand there is a list of small tubes of lengths 5 to 46 that have to be produced to meet weekly demand. These two groups, the stock of large objects and the order book for small items, constitute the basic data of the cutting stock problem. Orders for small tubes are combined together forming cutting patterns that are assigned to large objects of the stock. The process has to obey certain objectives and constraints being specific for the problem at hand.

As a second example Figure 2 illustrates a container loading problem (for practical cases, cf. Gehring et al., 1990; Haessler and Talbot, 1990). This kind of problem also exhibits two groups of basic data, on the one hand a stock of large objects consisting of one or more containers, and on the other hand a list of smaller items that can be packed into the containers. Apart from certain individually given objectives and constraints, the principal aspect of container loading concerns the geometric combination of small items to packing patterns which can be assigned to containers of the stock.

Hence the common logical structure of C&P-problems may be determined as follows:

(a) There are two groups of basic data whose elements define geometric bodies of fixed shapes (later on called "figures") in a one- or more-dimensional space of real numbers:

– the stock of the so-called "(large) objects" on the one hand, and

– the list or order book of the so-called "(small) items" on the other hand.

(b) The cutting or packing process realizes patterns being geometric combinations of small items assigned to large objects. Residual pieces, i.e. fig-

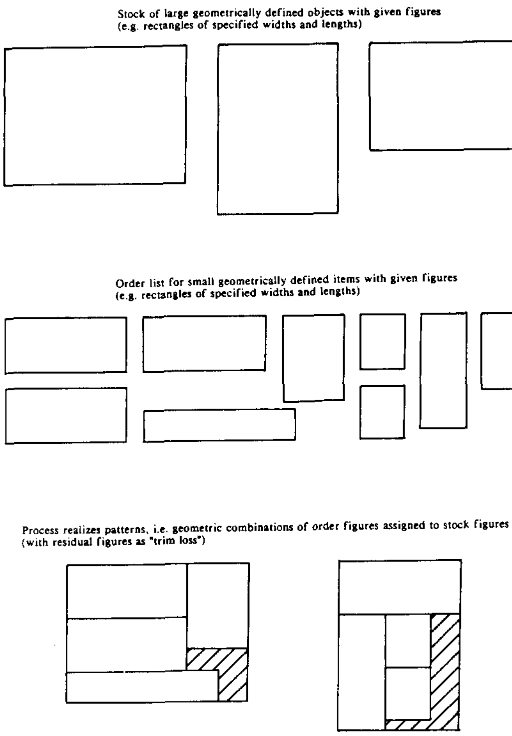

Figure 3. Basic structure of cutting and packing

ures occuring in patterns not belonging to small items, are usually treated as 'trim loss'.

In the example of Figure 3 both groups, i.e. the objects as well as the items, build rectangles of specified widths and lengths in two-dimensional space. This may be the case when cutting glass plates or loading pallets.

Because of the dominant role played by the patterns and their nature as geometric combinations one may say that C&P-problems belong to the field of 'geometric combinatorics'. In a narrow sense—as shown by Figure 4—C&P-problems are concerned with objects and items defined by one, two or all three spatial dimensions of the Euclidean space. In the cases of cutting (stock) problems the large objects are given by solid materials cut up into small items as pieces. Usual materials are paper and pulp, metal, glass, wood, plastics, leather and textiles. Since trim loss optimization is a main objective one also speaks of trim loss problems.

Packing and loading problems in the narrow sense are characterized by large objects defined as the empty, useful space of vehicles, cars, pallets, containers, bins, and so on. Packing small material

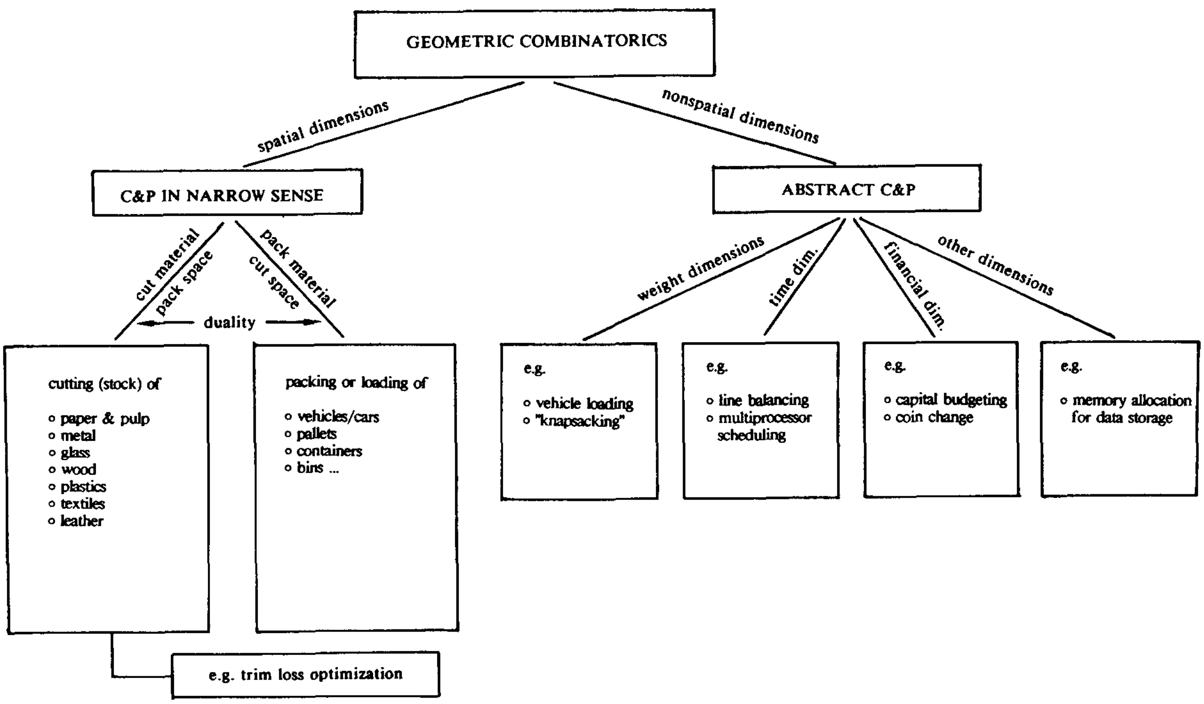

Figure 4. Phenomenology of cutting and packing problems

items into these objects may also be looked at as cutting the empty space of the large objects into parts of empty spaces some of which are occupied by small items, the other being 'trim loss'. Conversely, cutting stock problems may be looked at as packing the space occupied by small items into the space occupied by large objects. In other words: The strong relationship between cutting and packing results from the duality of material and space, i.e. the duality of a solid material body and the space occupied by it.

Furthermore, C&P can also be considered in an abstract, generalized sense taking place in non-spatial dimensions (cf. Figure 4). Examples with respect to dimensions of various natures are:

- knapsacking (Dantzig, 1957) and vehicle loading (Eilon and Christiofides, 1971) for the weight dimension,
- assembly line balancing (Wee and Magazine, 1982) and multiprocessor scheduling (Coffman et al., 1978) for the time dimension,
- capital budgeting (Lorie and Savage, 1955) and change making (Martello and Toth, 1980; Stehling, 1983) for financial dimensions, or
- computer memory allocation for data storage dimensions (cf. Garey and Johnson, 1981).

The example of assembly line balancing should suffice here in order to show the identity of the basic logical structure of material and abstract C&P-problems. As illustrated by Figure 5 the stock of large objects is defined by the work

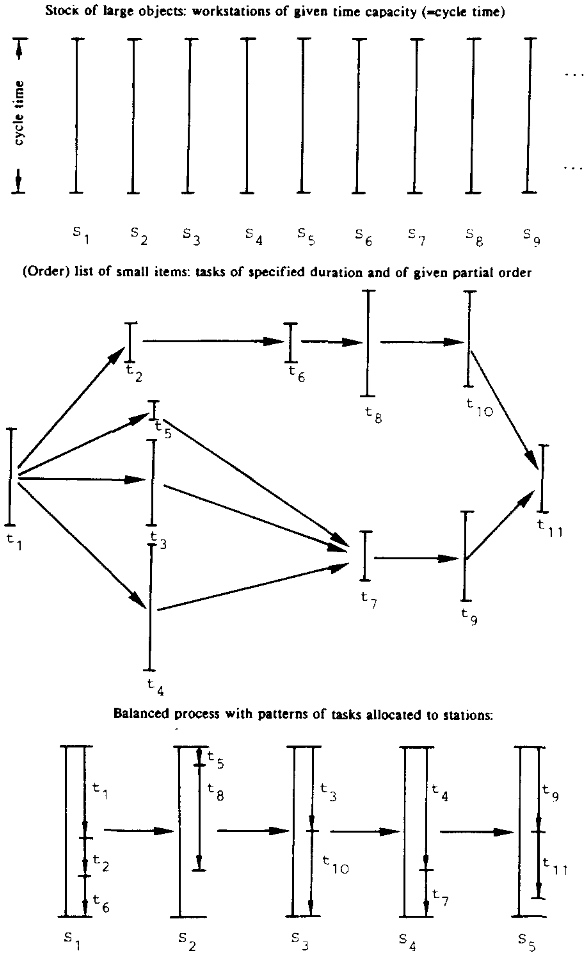

Figure 5. Assembly line balancing as an abstract C&P-problem

Table 2
Systematization on main characteristics

Characteristics of the ...	Geometrical characteristics	Combinatorial characteristics	Other characteristics
Large objects:	(Dimensionality) Shape of figures	Quantity measurement Assortment Availability	Objectives Status of information Variability
Small items:	In principle like those of the large objects		
Geometric combinations:	Dimensionality Pattern restrictions (admissible figures; kind of cuts; distances; orientation ...)	Pattern restrictions (number of cuts; kind, number, and combination of figures ...)	Objectives Status of information Variability
Assignments:	–	Restrictions for number of stages, order or frequency of patterns ...	Objectives Status of information Variability

stations of fixed time capacities, all being equal to the cycle time. The list of small items is given by the tasks of specified durations which have to be performed. Line balancing is achieved through patterns of tasks allocated to stations. Here, a special type of constraint has to be taken account of, namely the given partial order of tasks. Except for these additional constraints the logical structure is the same as for the classical bin packing problem. That is why line balancing has also been considered as 'generalized bin packing' (Wee and Magazine, 1982).

3. Characteristics and elementary types

The logical structure of C&P-problems provides a scheme for a more profound systematization. The rows of Table 2 differentiate main characteristics of C&P with respect to the question whether they refer to

- the stock of large objects,
- the list of small items,
- the patterns as geometric combinations of small figures for one large figure each, or to
- the assignments of small items to patterns as well as patterns to large objects.

The columns of Table 2 group main characteristics into geometrical ones, combinatorial ones and those which belong to both or other features.

3.1. Dimensionality

The most important characteristic is dimensionality. Instead of considering it separately for the large objects on the one hand and the small items on the other, it is attributed to the problem or more precisely to the patterns. Then dimensionality is the minimum number of dimensions of real numbers necessary to describe the geometry of the patterns. Elementary types are:

- one-dimensional,
- two-dimensional,
- three-dimensional, and
- multi-dimensional problems (more than three dimensions).

Four-dimensional problems might be obtained when three-dimensional packing problems in space have time as fourth dimension, e.g. when boxes have to be stored in a container for fixed periods of time without interruptions (or when baking bread in an oven). A very special type of a multi-dimensional problem is the so-called "vector-packing problem" (cf. Garey and Johnson, 1981), e.g. in the case of multi-period capital budgeting (Lorie and Savage, 1955).

Although it may seem to be straightforward classifying a problem with respect to its dimension this by far is not the case. For example, pallet loading is usually considered as two-dimensional. If, however, the height of the pallet built by the layers is restricted a third dimension is relevant (cf. Dowsland, 1985). Similarly, containers are often loaded by first building vertical stacks and then locating the stacks horizontally in the base of the container (Gehring et al., 1990; Haessler and Talbot, 1990). In both cases one might speak of dimensionality '2 + 1' instead of 3. Then, however, problems where glass plates are broken using only guillotine-cuts would have to be characterized as '1 + 1-dimensional'.

3.2. Quantity measurement

Another main characteristic is the way of measuring the quantity of the large objects and of the small items, respectively. Two cases can be distinguished (Gilmore, 1979):

- discrete (or 'integer') measurement, i.e. by natural numbers, and
- continuous (or 'fractional') measurement, i.e. by real numbers.

The first case refers to the frequency or number of objects or items of a certain shape. On the other hand, fractional quantities measure the whole length of several objects or items having the same shape ('figure') with respect to the relevant dimensions, the length of the objects or items being summed up with regard to a further dimension not being essential for the geometry of the patterns. Instead of the length it may also be the weight or the diameter, e.g. of rolls.

The combined type of one-dimensional problems with continuous measurement is often called "one-and-a-half-dimensional" (1.5-dimensional; cf. Dyckhoff et al., 1985).

3.3. Shape of figures

Another main characteristic of C&P-problems directly related to dimensionality is the shape of the figures of the large objects and the small items.

The figure of an object or an item is defined as its geometric representation in the space of relevant dimensions. Objects or items of the same figure possess the same geometric representation except for some translations within the space of relevant dimensions. Neglecting such translations a figure is uniquely determined by its

- form,
- size, and
- orientation.

Figures of the same form differ at most in their size or orientation (or position) in the relevant space. For more-dimensional problems an important question is whether the form of the figures is

- regular, or
- irregular.

Regular forms can be described by a few parameters. The vast majority of problems considered in literature is concerned with such regular forms, especially rectangular or block forms. Irregular forms with even non-convex, non-symmetric shapes are, however, typical for some industries (e.g. metal, textile, footwear, or woodworking industry).

Figures differing at most in their size can be made identical (by translations and) by changing the scale of measurement equally in all relevant dimensions. Depending on dimensionality, the size of a figure may be measured by its length, area, or volume. An important aspect for the ease or difficulty of solving a specific problem is often determined by the relative size of the objects and items (see e.g. Haessler and Talbot, 1990).

Figures of the same form and size differ at most with respect to their orientation (and position), i.e. they are congruent. Three important cases regarding orientation may be distinguished:

- If 'any orientation is allowed' then objects and items with congruent figures may not be distinguished for the problem at hand.
- If 'only 90 degree-turns are allowed' then only objects and items with respective figures are considered as identical.
- If the 'orientation is fixed' then objects and items with congruent figures are differentiated except those which can be made identical by translations.

With these definitions one can state that one-dimensional figures all have equal forms and orientations: They only differ in their size, i.e. their lengths. Examples of definite (regular) two-dimensional forms are circles and squares while rectangles with distinct ratios of width and length constitute different (regular) forms.

3.4. Assortment

Not only the shapes of figures principally permitted but also their particular assortment are important for characterizing a C&P-problem. The assortment is given by the shapes and number of permitted figures. One may distinguish as to whether different forms appear or all objects and items, respectively, have the same form. In both cases the figures may be different, be it many or only a few different figures. In the second case of an identical form differences between figures result from their size or their orientation. Important special types arise when the figures are congruent or even identical.

For example, pallet loading and bin packing usually refer to problems with large objects all having the same figure whereas the small items have congruent figures with pallet loading and very different figures with bin packing.

Another question is whether all permitted figures can be combined. 'Assortment problems' (Hinxman, 1980) or more precisely 'assortment selection problems' are concerned with the fact that only one or a few different figures principally permitted can actually be chosen. Examples are the produced width of paper machines and the package dimensions of goods to be loaded on pallets.

3.5. Availability

The assortment does not say anything about the quantity of objects and items considered. This is captured by the next main characteristic, the availability of objects or items of the permitted figures. It refers to

- lower and upper bounds on their quantity,
- their sequence or order, as well as
- the date when an object or item can be or has to be cut or packed.

One may distinguish between an infinite and a finite number of objects or items. In the second case there may be many or only a few objects or items. Classical bin packing and cutting stock problems are typically concerned with an infinite number of objects all of the same large figure; there are, however, only a few items for each small figure (of many different ones) with bin packing whereas with cutting stock there are a lot of items for each small figure (of relatively few different ones). Pallet loading or knapsack problems are usually characterized by only one large object.

Above, the number of objects and items given has been interpreted as an upper limit of availability for the cutting or packing process. In most cases there are lower limits, too, which may be identical with the upper ones. For example, the one large object of a pallet loading or knapsack problem has to be packed (with certain small items). Conversely, all given small items of a classical bin packing or cutting stock problem have to be packed (into bins) or cut (from the stock). Non-identical lower and upper limits may represent overrun and underrun tolerances with respect to the (order) items.

Further properties with respect to availability are obtained when there exists a certain order for the objects and items or when time plays an explicit role. A partial order of the small items has to be taken into account with assembly line balancing. Red-hot steel bars as large objects have to be cut not only in the sequence produced, but also without delay. Small items to be packed into containers may have different due dates that are important for shipping the containers.

3.6. Pattern restrictions

The characteristics discussed so far refer to properties of the large objects and to analogous ones of the small items, respectively. Most of them, however, have immediate impacts on the patterns as geometric combinations, as well as on the assignment of items to objects. In the first case the construction of the individual patterns is specified, resulting in geometrical or combinatorial restrictions. In the second case restrictions regarding the number, the order, or the combination of patterns have to be taken into account. These aspects are broadly discussed in the literature of C&P-problems and would require a separate survey on their own. In view of the purpose of the paper stated in the Introduction, only a rough overview is given here without going into much detail.

Four important groups of pattern restrictions can be distinguished which may have to be considered for the construction of geometric combinations of small figures for a large figure:

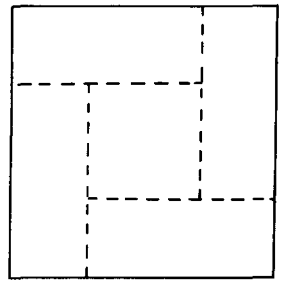

Figure 6. A two-dimensional, orthogonal, nested pattern

– Minimal or maximal distances between small figures or between cuts dividing large figures are often important, e.g. when breaking glass or loading containers.

– The orientation of the small figures relative to each other and/or to the large figure may have to be taken into account, e.g. when cutting patterned fabrics or when loading fragile goods on pallets.

– There may be restrictions with respect to the frequency of small items or figures in a pattern, especially regarding the combination or number of different small figures or the number of small items, be it in total or relative to certain figures. This includes, for example, maximum amounts of waste as well as maximum numbers of ranges of sizes or orders.

– The type and the number of permitted 'cuts' are essential, particularly if the objects and items are of rectangular or block form. Following this line non-orthogonal and orthogonal patterns are distinguished. The latter may be either nested patterns or guillotine patterns. The complexity of guillotine patterns depends on the number of changing cutting directions (stages) as well as the number of parallel cuts per stage. Figures 6 and 7

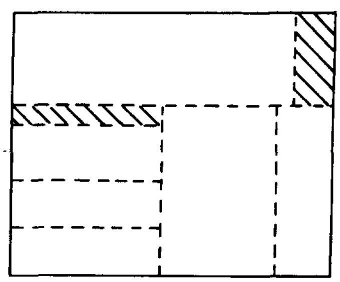

Figure 7. A two-dimensional, orthogonal, three-stage guillotine pattern

both show a two-dimensional orthogonal pattern: one nested pattern for 5 small items of 3 distinct figures and 2 different forms (Figure 6) and one three-stage guillotine pattern for 6 small items with 2 residual pieces (Figure 7). There may also be non-orthogonal patterns resulting from guillotine cuts. Because of weak stability, guillotine 'cuts' are usually of minor interest for pallet loading.

3.7. Assignment restrictions

The assignment of small items to large objects can be considered to take place in two succeeding —perhaps fictitious—steps, namely in first arranging small items into suitable patterns, and secondly assigning patterns to appropriate large objects. There may be restrictions regarding the

- kind of assignment,
- number of (assignment) stages,
- number, frequency, or sequence of patterns,
- dynamics of allocation.

An important property is the kind of assignment. For both, the given large objects and the given small items, it has to be clear whether only a selection or all of them have to be assigned to corresponding (non-trivial) patterns. In principle, this differentiation leads to four possible categories (being strongly connected with the availability of the objects and items discussed in Section 3.5). Two categories play a fundamental role in literature (which is why they are identified as main types in Section 4). Knapsack and dual bin packing problems are examples of the category where a selection of the given items has to be combined to patterns such that to each of the given objects a non-trivial pattern is assigned. In the opposite category, which is typical for classical cutting stock and bin packing problems, all items have to be assigned to a proper selection of the objects.

The number of (assignment) stages refers to the question as to whether small items are supposed to be cut from or packed into large objects simultaneously in one step or successively in several steps. In the first case only the large objects of the original problem data are admissible, in the second case residual (as well as small) figures of some patterns become 'large' figures of other patterns. Such multi-stage assignment processes can have an a priori limited or unlimited number of stages.

The order or sequence of patterns within the assignment process may be restricted, e.g. induced by the order or connection of objects or items or by the technology of the C&P-processes. Further restrictions refer to the frequency of patterns, be it the number of different types of patterns or the number of patterns of the same type.

The assignment of items to objects may be of static or dynamic nature. Static assignments might nevertheless be on-line processes where, in contrast to the off-line case, the objects or items are assigned successively without possible reallocation and without knowing the succeeding objects or items. In dynamic processes, different points or periods of time play an explicit role.

3.8. Objectives

It is not always possible to decide definitely whether a characteristic is a geometrical or a combinatorial one. Some include both aspects, some none. The objectives of C&P-problems often have geometrical as well as combinatorial aspects. Moreover, they can be attributed to the objects, items, patterns, or the assignment process (cf. Table 2). Here an 'objective' means a criterion to be maximized or minimized. Criteria to be satisfied with a certain level are treated like constraints of the cutting or packing process, e.g. the restriction that for handling purposes only small items belonging to the same customer's order should be contained in a pattern.

Similar to the last two subsections, only a very rough differentiation of objectives will be given here. Distinct kinds appear depending on whether they refer to the

- quantities of large objects or small and residual pieces assigned to patterns,
- geometry of the patterns (layout-optimization), or
- sequence, combination, or number of patterns.

Objective functions which are linear with respect to the quantities may value the objects or items by their size—e.g. trim loss or input minimization—or more generally by their prices—e.g. some cases of cost minimization. Non-linear objectives, for example, appear if the 'relative trim loss' is minimized or if fixed charges for the patterns have to be taken into account. It is, furthermore, typical for many C&P-problems that more than one objective has to be considered (cf. Wäscher, 1990).

3.9. Status of information and variability

The characteristics 'status of information' and 'variability' of Table 2 are relevant not only for C&P-problems. One has to state whether the problem data are deterministic, stochastic, or uncertain, and whether they are strict or may be variable in certain ranges. For example, rolls of plastic film taken from stock may have border defects of randomly fluctuating sizes. Another example is the continuous production of metal plates, slabs of precisely defined sizes are cut from ingots the sizes of which cannot be produced exactly. The demand for small items of the order list, usually assumed to be deterministic, may in fact be variable since certain deviations are accepted by the customers. The inexactness of measurement in practice may also be a reason for variability of data.

4. Combined types

In view of the great variety of real-world C&P-problems (cf. Hinxman, 1980; Dowsland, 1985; Dyckhoff et al., 1985) the characteristics considered so far do certainly not provide a complete list of all possible properties. Furthermore, some may overlap. For example, there is a strong relationship between the limits of availability, the kind of assignment, and those objectives which refer to quantities of objects and items. Nevertheless, the characteristics of Section 3 seem to be the most important ones, and the proposed scheme enables one to include further properties in a consistent and systematic manner.

As explained in the Introduction, however, going into details is not the purpose here. On the contrary, the paper aims at integrating the various kinds of C&P-problems into a global, comprehensive typology. In this sense, the approach developed in Sections 2 and 3 establishes a basis for defining important types of C&P-problems which future research might concentrate on.

As a first step into this direction, a simple system of 96 combined problem types which can be related to notions and solution approaches known from literature is proposed below. It is achieved by combining some main types of four important characteristics. Their importance stems from the fact that they have a decisive impact on the choice and the complexity of the solution approaches. Moreover, they have been chosen in such a manner that most of the 96 combined types make sense and then usually need individual solution methods applicable to the other types only with major revisions if at all. The four characteristics as well as their main types, denoted by appropriate symbols, are the following ones:

1. Dimensionality
   - (1) One-dimensional.
   - (2) Two-dimensional.
   - (3) Three-dimensional.
   - (N) N-dimensional with $N > 3$.
2. Kind of assignment
   - (B) All objects and a selection of items.
   - (V) A selection of objects and all items.
3. Assortment of large objects
   - (O) One object.
   - (I) Identical figure.
   - (D) Different figures.
4. Assortment of small items
   - (F) Few items (of different figures).
   - (M) Many items of many different figures.
   - (R) Many items of relatively few different (non-congruent) figures.
   - (C) Congruent figures.

With respect to the complexity involved by the geometry, it is clearly the dimensionality that has to be stated first (cf. Section 3.1). A further main role is played by the kind of assignment (cf. Section 3.7). The two prominent categories (of four principally possible ones) considered here are:

– A selection of small items has to be combined to patterns such that to each large object a non-trivial pattern is assigned ('Beladeproblem' in German).

– All small items have to be combined to patterns which then are assigned to a proper selection of large objects ('Verladeproblem' in German).

Three main types are distinguished with respect to the assortment of the large objects:

– There is only one large object (e.g. knapsacking or pallet loading).

– All large objects are of the same figure (e.g. bin packing).

– The large objects have different figures (e.g. cutting a stock of large objects containing residual pieces of prior periods).

Similarly, four main types regarding the assortment of the small items are pointed out:

– There are only relatively few small items usually of different figures (e.g. vehicle loading with about 10 items).

– There are many small items most of them having different figures (e.g. packing hundreds and more items of lengths distributed between 0 and 1 into bins of length 1).

– There are many small items but of relatively few different, non-congruent figures (e.g. cutting thousands and more items of less than 50 different figures).

– All small items are congruent (e.g. pallet loading).

By combining the distinguished main types of dimensionality, assignment, and assortment one obtains $4 \times 2 \times 3 \times 4 = 96$ different types of C&P-problems, shortly denoted by a fourth-tuple of respective symbols $\alpha/\beta/\gamma/\delta$. For example, 3/B/O/F denotes all three-dimensional C&P-problems where one large object has to be packed with a selection out of a few small items.

In order to examine the usefulness of this system, notions of special kinds of C&P-problems used in literature can be assigned to the respective combined type defined by its short notation. Table 3 contains a list of such notions. (A missing symbol for a characteristic means that all respective properties are possible.)

The classical knapsack problem belongs to a type characterized as 1-dimensional with one large object that has to be packed with a selection from the set of small items. The usual pallet loading problem differs in that it is 2-dimensional with congruent items. It can therefore be considered to be a special more-dimensional knapsack problem while dual bin packing problems are special cases of one-dimensional multi-knapsack problems.

The vehicle loading problem as originally analysed by Eilon and Christofides (1971) is one-dimensional. Only a few recent papers discuss three-dimensional loading problems, especially for containers as large objects (e.g. Gehring et al., 1990; Haessler and Talbot, 1990; cf. Dowsland, 1985).

Table 3
Notions assigned to corresponding combined types

Notion	Belongs to type
(Classical) knapsack problem	1/B/O/
Pallet loading problem	2/B/O/C
More-dimensional knapsack problem	/B/O/
Dual bin packing problem	1/B/O/M
Vehicle loading problem	1/V/I/F, or 1/V/I/M
Container loading problem	3/V/I/, or 3/B/O/
(Classical) bin packing problem	1/V/I/M
Classical cutting stock problem	1/V/I/R
2-dimensional bin packing problem	2/V/D/M
Usual 2-dimensional cutting stock problem	2/V/I/R
General cutting stock or trim loss problem	1/ / /, 2/ / /, or 3/ / /
Assembly line balancing problem	1/V/I/M
Multiprocessor scheduling problem	1/V/I/M
Memory allocation problem	1/V/I/M
Change making problem	1/B/O/R
Multi-period capital budgeting problem	n/B/O/

As becomes apparent by the notation, the classical vehicle loading, bin packing, and cutting stock problems belong to the same combined type regarding the first three characteristics. They, however, essentially differ in their assortment of small items. In the two-dimensional case a further difference exists with respect to the large objects. Two-dimensional bin packing is concerned with packing the items into one object of given width and of minimal length (or height). This is equivalent to an assortment selection problem where the stock is given by an infinite number of objects of this width and of all possible lengths and where only one object has to be chosen from stock, namely that of minimal length.

Furthermore, it becomes obvious that line balancing, multiprocessor scheduling, and memory allocation belong to the same combined type as the classical bin packing problem. Differences between them refer to characteristics not included in the basic typology defined here.

The last assertion shows a fundamental difference between this typology and others with apparently similar short notations. In particular, the notations used in scheduling of queueing theory determine very special problems as basic types each of which is uniquely described by a mathematical model. This means, any deviation from such a basic type, e.g. because of an additional restriction or a modified objective, leads to a new, different type. Here, the notation determines general types (of C&P-problems) each capturing a variety of still distinct problems. For example, nothing is stated about the shape of the objects and items in two- and more dimensional cases, e.g. whether they are regularly or irregularly formed. Even the classical cutting stock problems of type 1/V/I/R may be very different from each other, in particular with regard to pattern restrictions and objectives considered.

Hence, in order to more fully identify problems one has to extend the typology. It is straightforward to choose additional characteristics and main types from those discussed in Section 3. In view of the great variety of C&P-problems, however, a general extension does not make much sense. Depending such a second level of classification on the specific kind of problems under investigation seems to be better.

5. Solution approaches

As already stated before, the main purpose here is to develop a consistent and systematic approach to a comprehensive typology integrating all the various kinds of C&P-problems. Such a typology may form a basis for unifying the different use of notions in literature and for concentrating future research on particular types of problems. One important task then will be to connect problem types with solution approaches, i.e. to find the appropriate methods for each relevant problem type and conversely to identify problem types that can be solved by a certain method. A general type of solution method is called a "solution approach".

Analogously to the problems, a typology of solution methods may also be developed. In view of excellent existing surveys (cf. Tables 1 and 4), only a very rough typology is formulated here. Moreover, the discussion will concentrate on some of the above defined problem types in order to show how they are connected to certain solution approaches.

Table 4 distinguishes between object- or item-oriented approaches on the one hand and pattern-oriented approaches on the other hand. Methods of the first type immediately assign items to objects whereas pattern-oriented approaches first construct patterns and then assign large objects as well as small items to some of these patterns.

Table 4 does not provide a sharp classification. For example, the 'one pattern-type'—arising when pallets are loaded with identical products—can be considered as if only one large object exists. Methods of this type applicable in the one-dimensional case are well known combinatorial algorithms like those for the usual knapsack problem. In more dimensions the essential difficulties are geometrical ones, particularly in the cases of nonorthogonal patterns and non-rectangular or even irregular figures (cf. Dyckhoff, 1988). Although much of recent work has been devoted to the construction of two-dimensional patterns Table 1 contains no comprehensive survey of this subject. Papers on three-dimensional problems are still very rare, most of them written in the last few years.

The typology of solution approaches in Table 4 is based on combinatorial aspects of C&P-problems. It does explicitly not refer to the geometric properties. Hence the geometrical difficulties, which mainly result from the dimensionality and the shape of figures, are much the same as in the 'one pattern'-case. Therefore, in order to char-

Table 4
Solution approaches

Object- or item-oriented		Pattern-oriented	
Exact methods	Approximation algorithms	One pattern	Several patterns
E.g. branch and bound, dynamic programming (cf. Golden, 1976)	Bin packing algorithms (cf. Coffman et al., 1984)	Knapsack algorithms, various methods in more dimensions (cf. Hinxman, 1980; Dowsland, 1985; Terno et al., 1987; etc.)	LP-based and general heuristics (cf. Hinxman, 1980; Stadtler, 1988; Farley, 1988; etc.)

acterize the other approaches and to demonstrate their principal differences it suffices to consider one-dimensional problems, particularly three versions of the following type:

There are two lists S and D, called "stock" and "demand", consisting of an infinite number of large objects s_h of identical lengths L ($h = 1,2,\ldots$) and a finite number of small items d_k of lengths l_k ($k = 1,\ldots, K$), respectively. The demand has to be fulfilled, i.e. all items have to be cut from or packed into objects of the stock S. The objective is to minimize the number of large objects used.

(1) The list D consists of only a few small items, i.e. the number K is small, e.g. $K = 10$ (type 1/V/I/F, later on called "vehicle-loading"-type).

(2) The list D contains many small items, e.g. several hundreds, where most of the items have different lengths ('bin packing'-type 1/V/I/M).

(3) The list D contains a lot of small items, e.g. thousands, but the items are of only relatively few different lengths, e.g. less than fifty ('cutting stock'-type 1/V/I/R).

The three versions are special cases of the general problem types 1/V/I/F, 1/V/I/M, and 1/V/I/R (and can be considered as 'basic types' in the sense mentioned at the end of Section 4). The survey of Golden (1976) is still a good introduction to their principal solution approaches: object- and item-oriented exact or approximate algorithms in the first two cases, pattern-oriented heuristics in the third case. More recent papers concerning corresponding solution methods are noted in Table 4. Hence, a short overview should suffice here.

Except to restrictions regarding a possibly existing partial order of small items, the general mathematical model of all three versions is essentially the same as the integer programming model of assembly line balancing in the case of minimizing the number of established workstations. This is a hint for its complexity since additional restrictions lead to a smaller set of feasible solutions and hence usually to faster solutions with respect to decision tree methods. In fact, the problem in general is NP-complete (Garey and Johnson, 1979).

The vehicle loading-type (1) may nevertheless be solved exactly by methods of branch and bound or dynamic programming. Such methods are first discussed by Eilon and Christofides (1971) where the large objects are vehicles of standard size and the small items consignments of different weights, e.g. bundles of newspaper. Version (1) is, however, not of particular practical importance.

The other two versions can in general only be solved approximately, version (2) by fast algorithms called "bin packing-algorithms". An example is the FFD-algorithm ("First-Fit-Decreasing") with time complexity $O(K \log K)$. This algorithm arranges the small items in the order of decreasing length and then assigns them one by one to an appropriate large object using the following rule: An item is assigned to the first large object (bin) of the given list $S = (s_1, s_2, \ldots)$ into which it (still) fits. It can be proved (although with considerable difficulty; cf. Coffman et al., 1984) that this simple algorithm uses at most 22.3% objects more than the optimal solution, i.e. it never leads to an unnecessary trim loss larger than 18.2%. If the lengths of 200 small items are uniformly distributed between 0 and L (the length of the large objects) the FFD-algorithm has an average trim loss of only 1.9%. The average trim loss decreases to zero if the number of items, K, tends to infinity.

The structure of demand of version (3) is typical for cutting materials from stock. In this case many large objects are cut down in the same manner, i.e. by the same pattern. Hence it makes sense first to construct proper patterns and then to decide how many objects are cut by a certain pattern, i.e. how often the patterns are used. In view of these pattern-oriented approaches, the problem and its mathematical model can be reformulated. Items of an identical length l_i are no longer distinguished, but are put into the same group and are counted for their total demand b_i. (It is a sorting procedure similar to that of FFD.) A pattern j is described by the numbers a_{ij} of small items of lengths l_i ($i = 1,\ldots, m$) appearing in this pattern. Defining x_j as the number of large objects cut by pattern j ($j = 1,\ldots, n$) one obtains a usual linear programming formulation with cost coefficients equal to 1 and with integrality conditions for the variables x_j.

Two main approaches can be distinguished: heuristics and those being based on linear programming relaxations. Heuristics play an eminent role. They are very flexible allowing to take account of various additional restrictions and objectives appearing in practice, e.g. the costs arising through changing the pattern on a cutting machine. An example is the often used simple heuristic of 'repeated pattern exhaustion' (Haessler, 1971; cf. Pierce, 1964; Wäscher et al., 1985). The quality of this heuristic heavily depends on the problem-specific choice of 'good' patterns. When integrated into an interactive decision support system, such heuristics are able to take the experience of the user into account.

The LP-based approximation algorithms first solve the LP-relaxation of the pattern-oriented model and then search for an integer solution by more or less sophisticated strategies, usually by simply rounding up. More important is the fact that for practical problems there exists a very large number of possible patterns, often going into hundreds of millions (cf. Dyckhoff, 1988). In order to overcome this difficulty, one may use the classical method of 'delayed pattern generation', developed by Gilmore and Gomory (1961); (cf. Farley, 1988; Stadtler, 1988). In each step of the revised simplex method a new pattern corresponding to a new column improving the objective function value is generated. This is done by solving knapsack problems using the shadow prices of the demand conditions.

Besides the usual LP-formulation, called "multi-cut model", there is an alternative formulation, called "one-cut model", being equivalent in the sense that it allows for the same efficient C&P-possibilities and pursues the same objective (cf. Dyckhoff, 1988; Stadtler, 1988). In fact, it can be shown that there are a lot more equivalent models lying in between the two extreme ones (Dyckhoff, 1989). Because of its special structure the one-cut model can be solved by generalized network algorithms. Until now there is, however, no specialised algorithm that is faster than the Gilmore and Gomory-method for the multi-cut model (Stadtler, 1988). If this method is not available, the one-cut model can be helpful in practice because it consists of only hundreds or thousands of columns instead of millions although the number of rows increases more or less. Therefore it can often be solved by standard LP-algorithms.

6. Concluding remark

Sections 4 and 5 demonstrate how the approach for a comprehensive typology of C&P-problems developed in Sections 2 and 3 and based on their logical structure might be used for unifying the different notions in literature and for concentrating future research on particular problem types. The approach exhibits the personal view of the author and is surely not the ultimate answer to the questions posed in the Introduction. But, it is hoped that it can constitute a fruitful basis for further discussions.

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