Computers & Operations Research 37 (2010) 991 – 1001

Contents lists available at ScienceDirect

Computers & Operations Research

journal homepage: www.elsevier.com/locate/cor

Arc-flow model for the two-dimensional guillotine cutting stock problem

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ARTICLE INFO

Available online 28 August 2009

Keywords:
Integer programming
Combinatorial optimization
Two-dimensional cutting stock problem
Guillotine constraints
Furniture industry

ABSTRACT

We describe an exact model for the two-dimensional cutting stock problem with two stages and the guillotine constraint. It is an integer linear programming (ILP) arc-flow model, formulated as a minimum flow problem, which is an extension of a model proposed by Valério de Carvalho for the one dimensional case. In this paper, we explore the behavior of this model when it is solved with a commercial software, explicitly considering all its variables and constraints. We also derive a new family of cutting planes and a new lower bound, and consider some variants of the original problem. The model was tested on a set of real instances from the wood industry, with very good results. Furthermore the lower bounds provided by the linear programming relaxation of the model compare favorably with the lower bounds provided by models based on assignment variables.

1. Introduction

Cutting stock problems (*CSP*) are combinatorial optimization problems, which occur in many real-world applications of business and industry, motivating several areas of research. Generally speaking, they consist of cutting a given set of small objects, called *items*, out of a given set of larger ones, called *stock sheets*, usually with the objective of minimizing the area of waste.

Due to the complexity and extensive nature of these problems, many different optimization formulations and solution approaches arise in the literature, depending on their dimension, application field and special constraints and requirements. Therefore, many researchers provided surveys and categorized bibliographies on this subject (Dowsland and Dowsland [7], Dyckhoff and Finke [9], Lodi et al. [15,16], Dyckhoff et al. [10], among others). Moreover, Dyckhoff [8] defined a formal typology for cutting and packing problems that systematically integrates its various kinds of problems and notions. This typology was improved by Wäscher et al. [21] with the definition of new categorization criteria.

In this paper, we propose a new model to solve exactly the two-dimensional cutting stock problem (2*D-CSP*), with two stages and the guillotine constraint. This model is an extension of the arc-flow model proposed by Valério de Carvalho [6]. According to the typology defined by Wäscher et al. [21], this problem can be categorized as a 2D regular SSSCSP (*single stock size cutting stock problem*).

The *CSP*, as well as its extensions and variants, is well known to be NP-hard. Exact solution approaches normally use branch-and-bound, column generation or dynamic programming. Given that our model has a pseudo-polynomial number of variables and constraints which is computationally practical for the instances addressed, we considered them all explicitly, and solved the problem with a commercial solver, in order to investigate its performance.

In Section 2, the two-dimensional cutting stock problem is defined, and some exact solution methods from the literature, concerning the 2*D-CSP* with the guillotine constraint, are briefly described. In Section 3, the proposed arc-flow model is presented, along with a new family of cutting planes, a new lower bound and some variants of the original model. Some computational results are reported in Section 4, and finally, in Section 5, some conclusions are drawn.

2. Two-dimensional cutting stock problem

The two-dimensional version of the *CSP* can be stated as follows: a given set of small items, each item $i \in \{1,\dots,m\}$ of width w_i, height h_i and demand of b_i pieces, has to be cut out of a virtually infinite supply of stock sheets of width W and height H (where $0 < w_i \le W$ and $0 < h_i \le H, \forall i \in \{1,\dots,m\}$), in order to minimize the number of stock sheets used.

The 2*D-CSP* is equivalent to the two-dimensional bin packing problem (2*D-BPP*), in the sense that they both have essentially the same logic structure. In the 2*D-BPP*, there are m rectangular items, each item $i \in \{1,\dots,m\}$ having a width w_i, a height h_i and a demand b_i, which have to be placed, without overlapping, in the minimum possible number of rectangular objects (bins) of width W and height H. The main difference between these two problems is that typically,

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doi:10.1016/j.cor.2009.08.005

in the *CSP*, the demands are high, whereas in the *BPP* the demands are very low.

The 2*D-CSP* can be further classified into several categories, depending on specific constraints. It can be *regular*, if the shapes of the items can be described by few parameters, or *irregular*, otherwise. Cutting irregular shapes is also known as *nesting*. Regular cuts can be *rectangular* or *non-rectangular*, for the cases where the items are rectangles or have a different shape, respectively. The rectangular cutting can be *oriented*, if an item of width w and height h is considered to be different from another one of width h and height w, or *non-oriented*, otherwise. If all cuts must be made straight from one edge to the opposite edge of the stock sheet (or of one of its already cut fragments), dividing it into two, the cutting patterns produced are of *guillotine* type. *Non-guillotine* patterns are not restricted by this rule, being the corresponding problems much harder to solve. A stage pattern is a guillotine pattern cut into pieces in a limited number of phases. The direction of the first stage cuts may be either horizontal or vertical (parallel to one side of the stock sheet), and the cuts of the same stage are in the same direction. The cut directions of any two adjacent stages must be perpendicular to each other. If the maximum number of stages is not allowed to exceed a value n, the problem is called *n-stage*. When there is no such restriction, the problem is called *non-stage*. Whenever a final stage for separating items from waste areas is allowed, the problem is called *non-exact*; otherwise, it is called *exact*.

This paper focuses on the exact solution of the two-stage, non-exact 2*D-CSP* with the guillotine constraint. Some exact solution methods were already described in the literature. Gilmore and Gomory [14] proposed the first model for the 2*D-CSP*, by extending their column generation approach to the 1*D-CSP* [12,13]. They solved the two-dimensional guillotine version of the problem [14] as a two-stage one-dimensional problem, with the first stage corresponding to the cutting of the stock sheets into strips, and the second stage corresponding to the cutting of those strips into the demanded items.

Amossen [2] described a branch-and-bound algorithm for the *dD-BPP* (where $d=2$ is a particular case), with the guillotine constraint. It is based on the exact branch-and-bound algorithm presented by Martello et al. [18] for the 3*BPP*, but generalized to all dimensions.

Puchinger and Raidl [19] presented two polynomial-size ILP models for the three-stage 2*D-BPP*: a restricted model, particularly useful for obtaining near-optimal solutions quickly, and the original model, computationally more expensive. These models were based on the model proposed by Lodi et al. [17] for the two-stage 2*D-BPP*. They also proposed a branch-and-price framework for a set covering formulation, based on a Dantzig–Wolfe decomposition of the previous ILP models, with a four level hierarchy of pricing algorithms. They applied dual subset inequalities in order to stabilize the column generation process.

Vanderbeck [20] proposed an approximate solution algorithm for the three-stage 2*D-CSP* based on a nested decomposition of the problem, with a recursive use of column generation. According to the author, the algorithm can be further adapted to generate exact solutions.

Cui [5] analyzed the problem of cutting circular blanks for the manufacturing of electric motors. He considered multi-segment patterns, which are guillotine two-stage patterns. Parts cut in the first stage are denoted as segments, and the ones cut in the second stage as strips. Each strip contains identical blanks that can appear in a given maximum number of rows. The linear programming model is solved with column generation, generating the multi-segment patterns with a knapsack algorithm. This problem could be approached as a two-stage 2*D-CSP* with the guillotine constraint and rectangular items if we considered as a different rectangular item each possible type of strip resulting from the combination of the heights of all the possible number of rows of each blank size with all their possible widths, calculated according to the author's definition of break points. For each different item, the type and number of circular blanks should be considered, in order to properly fulfill the demands.

Cintra et al. [3] proposed a column generation algorithm for the 2*D-CSP* with the guillotine constraint. They introduced two new Rectangular Knapsack algorithms to generate new columns, and described how to find integer solutions.

Alves et al. [1] presented an exact algorithm for the two-stage 2*D-CSP* with the guillotine constraint, and computational experiments with real-world instances from the furniture industry. It is a plain exact algorithm that does not use any heuristics or other strategies to accelerate its converge. It consists of a branch-and-price procedure whose branching scheme is compatible with the pricing subproblems.

3. Arc-flow model

Valério de Carvalho [6] proposed a ILP arc-flow model for the 1*D-CSP* (1)–(4), in which every cutting pattern corresponds to a path in an acyclic directed graph $G=(V,A)$, with $V=\{0,1,\ldots,W\}$ as its set of $W+1$ vertices, which define positions in the stock sheet, and $A=\{(a,b): 0\le a<b\le W \text{ and } b-a=w_i, \forall i=1,\ldots,m\}$ as its set of arcs. It is formulated as a minimum flow problem, where variables x_{ab} correspond to the flow in arc (a,b), i.e., the number of items of width $b-a$ placed at a distance of a units from the beginning of a given stock sheet, and variable z corresponds to the total flow through the graph, and can be seen as the return flow from vertex W to vertex 0.

$$\min \quad z \tag{1}$$

$$\text{s.t.} \quad \sum_{(a,b)\in A} x_{ab} - \sum_{(b,c)\in A} x_{bc} = \begin{cases} -z & \text{if } b=0 \\ 0 & \text{if } b=1,2,\ldots,W-1 \\ z & \text{if } b=W \end{cases} \tag{2}$$

$$\sum_{(c,c+w_i)\in A} x_{c,c+w_i} \ge b_i, \quad \forall i\in\{1,\ldots,m\} \tag{3}$$

$$x_{ab}\ge 0 \text{ and integer}, \quad \forall (a,b)\in A \tag{4}$$

Constraints (2) are related with flow conservation and constraints (3) ensure that demand is fulfilled. Valério de Carvalho [6] defined criteria to reduce the number of allowable arc-variables, reducing the size of the model and the symmetry of the solution space. Symmetry should be avoided, as much as possible, as it increases the computational effort, because the same physical solution can be explored in different nodes of the branching tree. For any pattern, the items are sorted by their decreasing widths and the arcs corresponding to its waste (unit arcs) always appear in the last positions of the stock sheet. Furthermore, the number of items of a given width in a cutting pattern cannot be greater than the demand of those items.

We extended this formulation for the two-stage 2*D-CSP* with the guillotine constraint. In the first variant of the problem, we consider that, in the first stage, the stock sheets are cut into horizontal strips, which are, in the second stage, cut into the demanded items. We handle the two dimensional problem as a set of two one-dimensional problems. In the first stage, we cut strips out of the stock sheets. This means that we only need to consider the item and the stock sheet heights. In the second stage, the strips generated are cut into the demanded items (with vertical cuts). Again, this means that we only have to consider the item and stock sheet widths, because we have defined the set of items that can be cut out of a given strip (the ones whose height is smaller than or equal to the strip height).

We consider $H^*=\{h_1^*,\ldots,h_{m_h}^*\}$ as the set of m_h different heights ordered by their increasing values, a graph $G^0=(V^0,A^0)$, with $V^0=\{0,1,\ldots,H\}$ and $A^0=\{(a,b): 0\le a<b\le H \text{ and } b-a=h_i^*, \forall i\in\{1,\ldots,m_h\}\}$, for the first stage, and a set of graphs $G^s=(V^s,A^s)$, with $V^s=\{0,1,\ldots,W\}$ and $A^s=\{(d,e): 0\le d<e\le W \text{ and } e-d=w_i, \forall i\in\{1,\ldots,m\}: h_i\le h_s\}$, for each of the m_h sets of patterns of the second

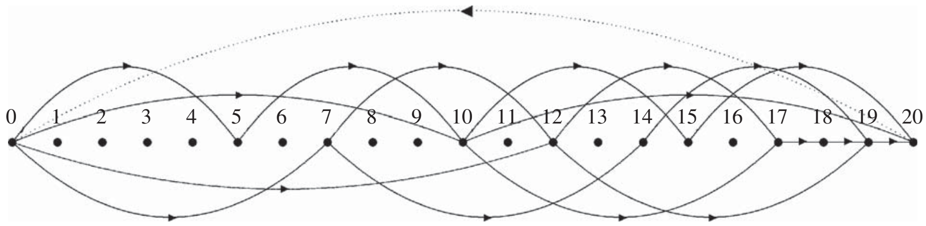

Fig. 1. Graph G_0 of the arc-flow model (Example 3.1).

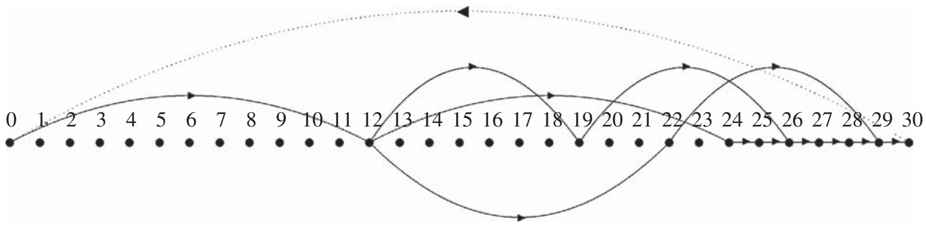

Fig. 2. Graph G_2 of the arc-flow model (Example 3.1).

stage. Every set A^s, $\forall s \in \{0,1,\ldots,m_h\}$, includes unit arcs that represent waste, which are strips of waste of width W when $s=0$, and waste within the strip when $s>0$.

$$\min \quad z^0 \tag{5}$$

$$\text{s.t.} \quad \sum_{(a,b)\in A^0} x^0_{ab} - \sum_{(b,c)\in A^0} x^0_{bc} = \begin{cases} -z^0 & \text{if } b=0 \\ 0 & \text{if } b=1,2,\ldots,H-1 \\ z^0 & \text{if } b=H \end{cases} \tag{6}$$

$$\sum_{(c,c+h^*_s)\in A^0} x^0_{c,c+h^*_s} - z^s = 0, \quad \forall s \in \{1,\ldots,m_h\} \tag{7}$$

$$\sum_{\substack{(d,e)\in A^s \\ h^*\in H^*}} x^s_{deh^*} - \sum_{\substack{(e,f)\in A^s \\ h^*\in H^*}} x^s_{efh^*} = \begin{cases} -z^s & \text{if } e=0, \\ 0 & \text{if } e=1,2,\ldots,W-1, \quad \forall s \in \{1,\ldots,m_h\} \\ z^s & \text{if } e=W, \end{cases} \tag{8}$$

$$\sum_{s=1}^{m_h} \sum_{(f,f+w_i)\in A^s} x^s_{f,f+w_i,h_i} \geq b_i, \quad \forall i \in \{1,\ldots,m\} \tag{9}$$

$$x^0_{ab} \geq 0 \text{ and integer}, \quad \forall (a,b) \in A^0, \tag{10}$$

$$x^s_{deh^*} \geq 0 \text{ and integer}, \quad \forall (d,e) \in A^s,\ \forall s \in \{1,\ldots,m_h\} \ \forall h^* \in H^* \tag{11}$$

In this formulation, variable z^0 represents the number of stock sheets used and variables z^s, $\forall s \in \{1,\ldots,m_h\}$, denote the number of strips of height h_s cut in the first stage. Variables x^0_{ab} represent the flow in arc (a,b), on graph G^0, and variables $x^s_{cdh^*}$ represent the flow in arc $(c,d) \in G^s$ corresponding to items of width $(d-c)$ and height $h^* \in H^*$. The third index h^* differentiates items with the same width but different heights within the same graph G^s. Constraints (7) make the connection between the two stages of the problem: the number of strips of height h^*_i, $\forall i \in \{1,\ldots,m_h\}$, cut in the first stage must be equal to the number of strips of height h^*_i cut in the second stage into the demanded items. Constraints (6) and (8) are related with flow conservation in the first and second stages, respectively, and constraints (9) ensure that all the demands are fulfilled.

3.1. Arc reduction

The model has a pseudo-polynomial number of variables and constraints. Variables represent arcs in the arc-flow model; reducing the number of arcs reduces the size of the model, increasing its efficiency. Three reduction criteria were applied in order to accomplish this:

(i) only maximal cutting patterns are considered: in the first stage, we only consider combinations of strips whose heights sum S_h is such that $H - S_h < h_{\min}$, being $h_{\min}$ the smallest item height, and in the second stage, we only consider combinations of items whose width sum S_w is such that $W - S_w < w_{\min}$, being $w_{\min}$ the smallest item width. This is possible because the demand constraints (9) are inequalities.
(ii) all the strips should contain at least one item with height equal to the strip height. We guarantee this by enforcing that all arcs leaving node 0, in all graphs of the second stage, correspond to items whose height is equal to the strip height.
(iii) in the first or the second stages, the strips and items, respectively, to some extent, are sorted by their decreasing values of size. This is accomplished by constructing the graphs in a way such that an arc (b,c), with $b \neq 0$ and taking into account reduction criterion (iii) for the graphs of the second stage, is only considered if there is another arc (a,b) such that $b-a \geq c-b$.

None of these criteria eliminates cutting patterns that cannot be replaced by equivalent ones, in what concerns finding the minimum number of used stock sheets.

Example 3.1. Consider an instance with stock sheets of height H=20 and width $W=30$ and a set of items $I = \{(h_i, w_i, b_i) : i \in \{1,\ldots,5\}\} = \{(5,7,4),(5,10,3),(7,12,5),(10,8,3),(12,10,5)\}$.

By applying the previous criteria, the graph corresponding to the first stage, G_0, and the graph of the second stage, corresponding to strips of height 7, G_2, would be the ones represented in Figs. 1 and 2, respectively.

3.2. Symmetry

In this model, symmetry occurs whenever different paths corresponding to an identical cutting pattern are allowed. Reduction criterion (iii), previously described, reduces some symmetry, although it does not totally prevent it. To reduce it further, additional constraints can be added to the model:

$$\sum_{(a,b)\in A^0} x^0_{ab}(b-a) \geq \sum_{(b,c)\in A^0} x^0_{bc}(c-b) \tag{12}$$

$$\sum_{\substack{(d,e)\in A^s:d>0\\ h^*\in H^*}} x^s_{deh^*}(e-d) \geq \sum_{\substack{(e,f)\in A^s\\ h^*\in H^*}} x^s_{efh^*}(f-e), \quad \forall s\in\{1,\ldots,m_h\} \tag{13}$$

Constraints (12) and (13) reinforce the reduction criterion (iii), in the first and second stage, respectively. At each node $k\in\{1,\ldots,W-1\}$, constraints (6) and (8) ensure that the flow into a node is equal to the flow out of it. With (12) and (13) we force the sum of the lengths of the arcs into a node $k\in\{1,\ldots,W-1\}$ to be greater than or equal to the sum of lengths of the arcs out of it, in order to avoid that any unit of flow goes from a given arc to another one with a greater length.

3.3. A new family of cutting planes

In order to strengthen the model, we derived a new family of cutting planes. They rely on the fact that any set of items with height greater than or equal to h_j must be cut out of strips with height greater than or equal to h_j. To compute a minimum number of strips needed to cut all items of set $I_j=\{i\in\{1,\ldots,m\}: h_i\geq h_j\}$, $\forall j\in\{1,\ldots,m_h\}$, two sets of lower bounds, $LB\_Strip\_1_j$ and $LB\_Strip\_2_j$ are considered. $LB\_Strip\_1_j$ is the continuous lower bound, i.e.,

$$LB\_Strip\_1_j=\left\lceil \frac{\sum_{i\in I_j} w_i b_i}{W}\right\rceil, \quad \forall j\in\{1,\ldots,m_h\} \tag{14}$$

This lower bound can be tightened using of dual feasible functions (see for example [11]).

To derive lower bound $LB\_Strip\_2_j$, we consider the maximum number of items that can be cut of a given strip. To compute this, we consider the item, from the set of items that could be cut from that strip, with minimum width.

$$LB\_Strip\_2_j=\left\lceil \frac{\sum_{i\in I_j} b_i}{\left\lfloor \dfrac{W}{\min_{i\in I_j} w_i}\right\rfloor}\right\rceil, \quad \forall j\in\{1,\ldots,m_h\} \tag{15}$$

The new family of cutting planes is, therefore, described as follows:

$$\sum_{l=j}^{m_h} z^l \geqslant \max(LB\_Strip\_1_j, LB\_Strip\_2_j), \quad \forall j\in\{1,\ldots,m_h\} \tag{16}$$

3.4. A new lower bound

We derived a new lower bound for the number of required stock sheets. This lower bound is based on the lower bounds described in Section 3.3, but now applied to the vertical dimension.

In order to obtain a lower bound for the sum of heights of all strips of a given solution, we use the lower bounds (14) and (15). Thus, let $lb\_aux_j=\max(LB\_Strip\_1_j, LB\_Strip\_2_j)$, $\forall j\in\{1,\ldots,m_h\}$. Remember that $H^*=\{h^*_1,\ldots,h^*_{m_h}\}$ is the set of m_h different heights ordered by their increasing values. Let us consider the maximum height, i.e., $h^*_{m_h}$. We know that at least $lb\_aux_{m_h}$ strips of this height will be used. We also know that at least $lb\_aux_{(m_h-1)}$ strips of heights $h^*_{m_h}$ or $h^*_{(m_h-1)}$ will be required. If we are considering the minimum possible sum of heights, we can say that we would have at least $lb\_aux_{m_h}$ heights $h^*_{m_h}$ and $(lb\_aux_{(m_h-1)}-lb\_aux_{m_h})$ heights $h^*_{(m_h-1)}$, and so on. Therefore, (17) is a lower bound for the sum of strip heights in the solution.

$$h\_strip=\sum_{j=1}^{m_h}\left(lb\_aux_j-\sum_{i\in I_j\setminus\{j\}} lb\_aux_i\right)\times h^*_j \tag{17}$$

To compute our lower bound LB, we consider two lower bounds,

$$LB_1=\left\lceil \frac{h_{strip}}{H}\right\rceil \quad \text{and} \quad LB_2=\left\lceil \frac{lb\_aux_1}{\left\lfloor \dfrac{H}{h^*_1}\right\rfloor}\right\rceil$$

being $LB=\max(LB_1, LB_2)$.

3.5. Variants of the problem

3.5.1. The non-oriented case

If the material of the stock sheets has any kind of pattern (for example, the wood grain), rotation of items is not generally allowed (*oriented* case). This means that an item of width w and height h is considered to be different from an item of width h and height w, as they are meant to have a different orientation. When this does not happen, and rotation is allowed, the number of feasible cutting patterns increases.

The previous model does not consider item rotation. However, it can easily be considered by defining a different item i' for each item i, such that $h_{i'}=w_i$, $w_{i'}=h_i$, $b_{i'}=0$ and the sum of items i and i' is equal to the demand of item i.

3.5.2. Orientation of the first stage's cuts

So far, we considered that, in the first stage, cuts are horizontal (generating horizontal strips) and, in the second stage, cuts are vertical. Of course, we may begin with vertical cuts and proceed with horizontal ones. Allowing the first cut to be either horizontal or vertical can obviously reduce the total number of stock sheets in the cutting plan, which may have some sheets with horizontal strips and others with vertical ones. Although this can potentially improve the optimal solution, it increases considerably the size of the model. Let us consider $H^*=\{h^*_1,\ldots,h^*_{m_h}\}$ and $W^*=\{w^*_1,\ldots,w^*_{m_v}\}$ as the set of m_h and m_v different heights and widths, respectively, ordered by their increasing values. We would have to consider two graphs for the first stage, $G^0_h=(V^0_h,A^0_h)$, with $V^0_h=\{0,1,\ldots,H\}$ and $A^0_h=\{(a,b): 0\leq a<b\leq H$ and $b-a=h^*_i, \forall i\in\{1,\ldots,m_h\}\}$ and $G^0_v=(V^0_v,A^0_v)$, with $V^0_v=\{0,1,\ldots,W\}$ and $A^0_v=\{(c,d): 0\leq c<d\leq W$ and $d-c=w^*_i, \forall i\in\{1,\ldots,m_v\}\}$, and two sets of graphs for the second stage, $G^s_h=(V^s_h,A^s_h)$, with $V^s_h=\{0,1,\ldots,W\}$ and $A^s_h=\{(e,f): 0\leq e<f\leq W$ and $f-e=w_i, \forall i\in\{1,\ldots,m\}: h_i\leq h_s\}, \forall s\in m_h$, and $G^u_v=(V^u_v,A^u_v)$, with $V^u_v=\{0,1,\ldots,H\}$ and $A^u_v=\{(g,h): 0\leq g<h\leq W$ and $h-g=h_i, \forall i\in\{1,\ldots,m\}: w_i\leq w_u\}, \forall u\in m_v$.

$$\min \quad z^0_h+z^0_v \tag{18}$$

$$\text{s. t.} \quad \sum_{(a,b)\in A^0_h} x^{h0}_{ab}-\sum_{(b,c)\in A^0_h} x^{h0}_{bc}=\begin{cases}-z^0_h & \text{if } b=0\\ 0 & \text{if } b=1,2,\ldots,H-1\\ z^0_h & \text{if } b=H\end{cases} \tag{19}$$

$$\sum_{(c,c+h^*_s)\in A^0_h} x^{h0}_{c,c+h^*_s}-z^s_h=0, \quad \forall s\in\{1,\ldots,m_h\} \tag{20}$$

$$\sum_{(g,k)\in A^0_v} x^{v0}_{gk}-\sum_{(k,l)\in A^0_v} x^{v0}_{kl}=\begin{cases}-z^0_v & \text{if } k=0\\ 0 & \text{if } k=1,2,\ldots,W-1\\ z^0_v & \text{if } k=W\end{cases} \tag{21}$$

$$\sum_{(l,l+w^*_u)\in A^0_v} x^{v0}_{l,l+w^*_u}-z^u_v=0, \quad \forall u\in\{1,\ldots,m_v\} \tag{22}$$

Table 1
Description of the set A of instances.

Name	n	n_t	a_i	W	H	a_s	a_{min}	a_{max}	a_{av}	b_{min}	b_{max}	b_{av}
A-1	2	24	7.45E+06	2750	1220	3.36E+06	5.33	21.03	9.25	6	18	12.00
A-2	4	38	7.51E+07	2750	1220	3.36E+06	3.49	65.69	58.90	1	28	9.50
A-3	2	17	1.63E+07	2750	1220	3.36E+06	5.33	29.97	28.52	1	16	8.50
A-4	2	16	7.31E+06	2750	1220	3.36E+06	0.87	26.35	13.61	8	8	8.00
A-5	8	138	5.73E+07	2470	2080	5.14E+06	2.28	44.73	8.08	8	32	17.25
A-6	2	17	3.96E+06	2750	1220	3.36E+06	6.80	9.11	6.94	1	16	8.50
A-7	5	58	3.15E+07	2750	1220	3.36E+06	6.02	32.92	16.19	8	16	11.60
A-8	1	16	2.41E+06	2750	1220	3.36E+06	4.49	4.49	4.49	16	16	16.00
A-9	30	770	2.93E+08	2550	2100	5.36E+06	0.70	35.32	7.11	6	68	25.67
A-10	3	44	9.88E+06	2550	2100	5.36E+06	0.70	8.63	4.19	12	16	14.67
A-11	20	724	2.32E+08	2550	2100	5.36E+06	2.11	23.22	5.98	8	140	36.20
A-12	3	44	5.82E+07	2550	2100	5.36E+06	8.63	34.49	24.68	8	20	14.67
A-13	8	304	6.35E+07	2550	2100	5.36E+06	0.70	17.03	3.90	8	120	38.00
A-14	31	809	3.38E+08	2550	2100	5.36E+06	1.41	25.14	7.80	8	52	26.10
A-15	12	339	1.88E+08	2550	2100	5.36E+06	2.87	25.14	10.34	8	80	28.25
A-16	27	744	4.17E+08	2550	2100	5.36E+06	0.66	25.14	10.48	4	96	27.56
A-17	3	135	2.39E+07	2550	2100	5.36E+06	2.96	3.67	3.30	28	57	45.00
A-18	20	559	3.16E+08	2550	2100	5.36E+06	0.66	25.14	10.56	5	71	27.95
A-19	27	507	2.73E+08	2550	2100	5.36E+06	1.08	37.44	10.05	3	50	18.78
A-20	10	215	1.20E+08	2550	2100	5.36E+06	2.19	37.44	10.43	3	60	21.50
A-21	21	450	1.36E+08	2550	2100	5.36E+06	2.10	14.69	5.63	4	70	21.43
A-22	1	24	1.04E+07	2550	2100	5.36E+06	8.06	8.06	8.06	24	24	24.00
A-23	8	248	6.32E+07	2550	2100	5.36E+06	2.93	9.78	4.76	8	60	31.00
A-24	107	217	1.65E+08	2550	2100	5.36E+06	1.24	42.32	14.24	1	12	2.03
A-25	75	156	8.54E+07	2550	2100	5.36E+06	1.21	40.06	10.22	1	7	2.08
A-26	34	61	3.42E+07	2550	2100	5.36E+06	1.07	33.52	10.47	1	8	1.79
A-27	79	180	9.62E+07	2550	2100	5.36E+06	0.66	29.62	9.98	1	15	2.28
A-28	54	106	5.30E+07	2550	2100	5.36E+06	1.41	23.66	9.35	1	6	1.96
A-29	82	218	1.35E+08	2550	2100	5.36E+06	0.52	42.82	11.60	1	18	2.66
A-30	24	39	1.76E+07	2550	2100	5.36E+06	1.76	19.80	8.43	1	7	1.63
A-31	36	64	3.64E+07	2550	2100	5.36E+06	3.01	41.47	10.62	1	6	1.78
A-32	99	184	1.30E+08	2550	2100	5.36E+06	1.45	57.49	13.17	1	9	1.86
A-33	134	309	1.75E+08	2550	2100	5.36E+06	1.05	52.68	10.56	1	15	2.31
A-34	26	46	2.46E+07	2550	2100	5.36E+06	1.50	41.35	9.99	1	10	1.77
A-35	68	144	8.09E+07	2550	2100	5.36E+06	1.45	51.65	10.49	1	14	2.12
A-36	16	52	3.48E+07	2550	2100	5.36E+06	1.29	56.28	12.48	1	8	3.25
A-37	8	78	2.22E+07	2550	2100	5.36E+06	1.96	49.54	5.31	1	40	9.75
A-38	42	160	1.08E+08	2550	2100	5.36E+06	1.07	56.28	12.61	1	40	3.81
A-39	11	22	1.46E+07	2550	2100	5.36E+06	1.64	30.98	12.42	1	4	2.00
A-40	40	163	4.81E+07	2750	1220	3.36E+06	1.56	61.47	8.79	1	20	4.08
A-41	32	71	4.73E+07	2750	1220	3.36E+06	1.10	69.69	19.87	1	9	2.22
A-42	8	13	1.48E+07	2750	1220	3.36E+06	10.18	61.47	33.95	1	3	1.63
A-43	11	22	1.59E+07	2750	1220	3.36E+06	1.43	65.69	21.54	1	4	2.00
Average	28.74	198.72	9.49E+07	2599.30	1874.42	4.84E+06	2.53	34.71	12.31	4.79	32.30	12.68

$$\sum_{\substack{(d,e)\in A_h^s\\ h^*\in H^*}} x^{hs}_{deh^*} - \sum_{\substack{(e,f)\in A_h^s\\ h^*\in H^*}} x^{hs}_{efh^*} = \begin{cases} -z_h^s & \text{if } e=0,\\ 0 & \text{if } e=1,2,\ldots,W-1,\\ z_h^s & \text{if } e=W,\end{cases} \quad \forall s\in\{1,\ldots,m_h\} \tag{23}$$

$$\sum_{\substack{(p,q)\in A_v^u\\ w^*\in W^*}} x^{vu}_{pqw^*} - \sum_{\substack{(q,r)\in A_w^u\\ w^*\in W^*}} x^{vu}_{qr\omega^*} = \begin{cases} -z_v^u & \text{if } q=0,\\ 0 & \text{if } q=1,2,\ldots,H-1,\\ z_v^u & \text{if } q=H,\end{cases} \quad \forall u\in\{1,\ldots,m_v\} \tag{24}$$

$$\sum_{s=1}^{m_h} \sum_{(f,f+w_i)\in A_h^s} x^{hs}_{f,f+w_i,h_i} + \sum_{u=1}^{m_v} \sum_{(r,r+h_i)\in A_v^u} x^{vu}_{r,r+h_i,w_i} \geq b_i, \quad \forall i\in\{1,\ldots,m\} \tag{25}$$

$$x^{h0}_{ab} \geq 0 \text{ and integer}, \quad \forall (a,b)\in A_h^0 \tag{26}$$

$$x^{hs}_{deh^*} \geq 0 \text{ and integer}, \quad \forall (d,e)\in A_h^s,\ \forall s\in\{1,\ldots,m_h\},\ \forall h^*\in H^* \tag{27}$$

$$x^{v0}_{gk} \geq 0 \text{ and integer}, \quad \forall (g,k)\in A_v^0 \tag{28}$$

$$x^{vu}_{pqw^*} \geq 0 \text{ and integer}, \quad \forall (p,q)\in A_v^u,\ \forall u\in\{1,\ldots,m_v\},\ \forall w^*\in W^* \tag{29}$$

This formulation is similar to formulation (5)–(11). Variables z_h^0 and z_v^0 represent the number of used stock sheets with horizontal and vertical first cuts, respectively, and variables z_h^s, $\forall s\in\{1,\ldots,m_h\}$ and z_v^u, $\forall u\in\{1,\ldots,m_v\}$, denote the number of horizontal and vertical strips of height h_s and width w_u, respectively, cut in the first stage. Variables x^{h0}_{ab} and x^{v0}_{cd} represent the flow that goes through arcs (a,b) and (c,d) on graphs G_h^0 and G_v^0, respectively. Variables $x^{hs}_{deh^*}$ and $x^{vu}_{pqv^*}$ represent the flow on graphs G_h^s and G_v^u that goes through arcs (d,e) and (p,q), corresponding to items of width $(e-d)$ and height $h^*\in H^*$ and height $(q-p)$ and width $w^*\in W^*$, respectively. The objective function (18) minimizes the total number of stock sheets used (the sum of stock sheets with horizontal cuts, z_h^0, and vertical cuts, z_v^0, in the first stage). Constraints (19) and (21) are related with flow conservation of the first stage and constraints (23) and (24) with flow conservation of the second stage. Constraints (20) and (22) make the connection of the two stages, ensuring that the number of horizontal or vertical strips cut in the first stage is the same as the number of horizontal or vertical strips used in the second stage. Finally, constraints (25) guarantee that all the demands are fulfilled.

Table 2
Description of the set B of instances.

Name	n	n_t	a_i	W	H	a_s	a_{min}	a_{max}	a_{av}	b_{min}	b_{max}	b_{av}
B-1	9	757	6.14E+08	4250	2120	9.01E+06	3.69	23.73	9.00	55	190	84.11
B-2	13	1040	1.35E+09	4250	2120	9.01E+06	3.69	34.16	14.41	28	165	80.00
B-3	4	1342	6.95E+08	4250	1860	7.91E+06	2.50	15.03	6.55	122	820	335.50
B-4	6	123	1.41E+08	4250	1860	7.91E+06	4.01	37.21	14.49	10	44	20.50
B-5	4	118	7.06E+07	4250	1860	7.91E+06	2.68	22.47	7.57	11	80	29.50
B-6	3	42	7.03E+07	4250	1860	7.91E+06	16.38	37.21	21.17	4	30	14.00
B-7	4	3000	2.22E+09	2440	1860	4.54E+06	3.38	23.15	16.34	600	1200	750.00
B-8	6	2545	1.46E+09	2800	1860	5.21E+06	4.54	24.02	11.03	225	1360	424.17
B-9	21	2629	6.45E+08	4250	1860	7.91E+06	1.19	5.42	3.11	28	330	125.19
B-10	21	2838	5.80E+08	4250	1860	7.91E+06	1.02	16.51	2.58	11	660	135.14
B-11	28	6134	2.04E+09	4250	1860	7.91E+06	1.49	27.70	4.22	27	1300	219.07
B-12	6	6570	2.89E+09	3760	1860	6.99E+06	1.94	16.71	6.30	110	2480	1095.00
B-13	11	5871	1.50E+09	3760	1860	6.99E+06	1.75	6.77	3.65	103	2060	533.73
B-14	7	318	2.67E+08	4250	2120	9.01E+06	3.41	14.42	9.31	23	84	45.43
B-15	5	106	1.13E+08	4250	2200	9.35E+06	5.22	21.65	11.38	5	38	21.20
B-16	8	316	2.03E+08	4250	2200	9.35E+06	3.28	15.61	6.87	15	74	39.50
B-17	2	57	6.84E+07	4250	2120	9.01E+06	11.11	15.04	13.32	25	32	28.50
B-18	3	147	1.08E+08	4250	2120	9.01E+06	6.02	11.11	8.13	44	55	49.00
B-19	11	529	4.06E+08	4250	2200	9.35E+06	3.28	15.62	8.20	28	97	48.09
B-20	6	1080	6.67E+08	4250	1860	7.91E+06	2.71	15.97	7.81	50	600	180.00
B-21	7	3000	1.50E+09	4250	1860	7.91E+06	2.26	15.97	6.33	110	1420	428.57
B-22	6	244	1.36E+08	4250	2120	9.01E+06	3.41	10.83	6.18	28	77	40.67
B-23	13	1403	5.67E+08	3770	1860	7.01E+06	2.11	48.63	5.76	9	220	107.92
B-24	2	165	6.54E+07	4250	1860	7.91E+06	2.20	10.65	5.01	55	110	82.50
B-25	5	708	1.94E+08	2600	1860	4.84E+06	3.21	11.71	5.67	22	334	141.60
B-26	8	3677	8.14E+08	3760	1860	6.99E+06	1.75	4.44	3.17	106	1260	459.63
B-27	5	3413	1.39E+09	3760	1860	6.99E+06	2.24	16.71	5.81	53	1260	682.60
B-28	6	880	2.84E+08	4250	1860	7.91E+06	2.11	8.62	4.09	110	220	146.67
B-29	5	440	1.34E+08	4250	1860	7.91E+06	2.82	5.61	3.86	55	110	88.00
B-30	14	1307	4.34E+08	4250	1860	7.91E+06	2.11	8.62	4.20	22	220	93.36
B-31	5	396	2.03E+08	4250	1860	7.91E+06	3.22	16.44	6.48	22	220	79.20
B-32	27	2600	1.38E+09	4250	1860	7.91E+06	1.62	21.07	6.72	20	240	96.30
B-33	6	689	3.54E+08	4250	1860	7.91E+06	2.27	18.76	6.50	55	220	114.83
B-34	14	1085	5.36E+08	4250	1860	7.91E+06	1.62	18.76	6.25	22	165	77.50
B-35	4	476	5.97E+08	3760	1860	6.99E+06	4.87	30.27	17.92	52	180	119.00
B-36	4	740	6.11E+08	3760	1860	6.99E+06	5.39	15.68	11.81	40	280	185.00
B-37	6	550	3.50E+08	4250	1860	7.91E+06	3.92	21.07	8.04	20	180	91.67
B-38	2	365	1.42E+08	2820	1870	5.27E+06	6.26	14.35	7.37	50	315	182.50
B-39	2	315	1.42E+08	2820	1870	5.27E+06	8.24	9.21	8.56	105	210	157.50
B-40	7	226	1.37E+08	4250	1860	7.91E+06	4.09	11.68	7.64	17	63	32.29
B-41	5	294	1.36E+08	4250	2200	9.35E+06	3.24	12.14	4.95	12	163	58.80
B-42	7	188	1.15E+08	4250	1860	7.91E+06	4.29	11.86	7.71	14	44	26.86
B-43	2	58	3.53E+07	4250	1860	7.91E+06	6.99	9.29	7.70	18	40	29.00
B-44	11	421	1.74E+08	4250	1860	7.91E+06	1.88	11.68	5.23	4	84	38.27
B-45	2	156	7.69E+07	4250	1860	7.91E+06	4.09	11.68	6.23	44	112	78.00
B-46	6	209	1.75E+08	4250	1860	7.91E+06	5.05	18.48	10.58	7	108	34.83
B-47	4	94	5.87E+07	4250	1860	7.91E+06	4.29	11.86	7.90	12	34	23.50
B-48	6	278	1.24E+08	4250	2200	9.35E+06	2.14	9.24	4.75	12	98	46.33
B-49	3	161	1.26E+08	4250	1860	7.91E+06	6.40	14.57	9.91	34	67	53.67
B-50	1	985	4.79E+08	4200	1860	7.81E+06	6.22	6.22	6.22	985	985	985.00
B-51	10	390	4.67E+08	4200	2130	8.95E+06	3.78	26.77	13.39	20	62	39.00
B-52	1	2390	8.90E+08	4250	2070	8.80E+06	4.23	4.23	4.23	2390	2390	2390.00
B-53	4	100	1.86E+07	4250	1860	7.91E+06	1.61	3.59	2.35	10	50	25.00
B-54	6	153	5.76E+07	2820	1870	5.27E+06	4.09	15.07	7.13	9	46	25.50
B-55	9	950	3.72E+08	4250	1860	7.91E+06	1.10	16.66	4.95	22	330	105.56
B-56	17	3781	1.42E+09	3760	1860	6.99E+06	1.80	18.83	5.38	32	1050	222.41
B-57	14	1144	3.30E+08	4250	1860	7.91E+06	1.17	6.58	3.65	22	275	81.71
B-58	6	465	1.42E+08	4250	1860	7.91E+06	1.82	5.31	3.86	60	165	77.50
B-59	2	231	7.21E+07	4250	1860	7.91E+06	2.88	6.99	3.95	60	171	115.50
B-60	4	242	8.89E+07	4250	1860	7.91E+06	2.88	8.82	4.65	29	109	60.50
B-61	2	148	3.71E+07	4250	1860	7.91E+06	2.36	5.34	3.17	40	108	74.00
B-62	4	349	1.29E+08	4250	1860	7.91E+06	2.88	8.82	4.67	38	142	87.25
B-63	8	394	2.69E+08	4250	1860	7.91E+06	3.19	17.40	8.62	12	80	49.25
B-64	9	295	1.45E+08	4250	1860	7.91E+06	1.74	32.29	6.23	10	83	32.78
B-65	20	2013	1.27E+09	4250	1860	7.91E+06	1.82	19.81	7.97	44	198	100.65
B-66	10	502	2.85E+08	4250	1860	7.91E+06	1.74	20.27	7.17	10	138	50.20
B-67	31	2500	1.87E+09	4250	1860	7.91E+06	1.82	20.88	9.44	9	280	80.65
B-68	17	710	3.65E+08	4250	1860	7.91E+06	1.74	32.29	6.50	10	130	41.76
B-69	3	209	1.44E+08	4250	1860	7.91E+06	1.84	14.25	8.74	55	88	69.67
B-70	16	1393	8.56E+08	4250	1860	7.91E+06	1.84	19.81	7.78	26	176	87.06
B-71	10	909	4.80E+08	4250	1860	7.91E+06	1.74	19.10	6.68	22	138	90.90
B-72	8	501	2.55E+08	4250	1860	7.91E+06	1.84	14.25	6.45	44	88	62.63
B-73	6	347	1.34E+08	4250	1860	7.91E+06	1.74	10.70	4.90	22	88	57.83
B-74	6	10185	2.56E+09	4050	2200	8.91E+06	1.63	6.01	2.82	165	6000	1697.50
B-75	8	3605	8.01E+08	3760	1860	6.99E+06	1.75	4.44	3.18	106	1260	450.63

Table 2 (continued).

Name	n	n_t	a_i	W	H	a_s	a_{min}	a_{max}	a_{av}	b_{min}	b_{max}	b_{av}
B-76	5	3413	1.39E + 09	3760	1860	6.99E + 06	2.24	16.71	5.81	53	1260	682.60
B-77	15	5359	1.16E + 09	2820	2070	5.84E + 06	1.68	6.82	3.71	158	841	357.27
B-78	6	1734	3.87E + 08	2820	2070	5.84E + 06	1.71	5.06	3.82	158	526	289.00
B-79	6	1052	2.37E + 08	2820	1870	5.27E + 06	2.47	6.43	4.26	105	316	175.33
B-80	7	1575	3.17E + 08	2820	2070	5.84E + 06	2.23	4.65	3.45	210	315	225.00
B-81	18	3802	2.94E + 09	4250	1860	7.91E + 06	1.90	19.55	9.78	40	770	211.22
B-82	10	1240	8.23E + 08	4250	2120	9.01E + 06	1.67	17.16	7.37	60	230	124.00
B-83	13	1160	7.07E + 08	4250	1860	7.91E + 06	1.90	16.33	7.71	50	150	89.23
B-84	12	1456	8.51E + 08	4250	2120	9.01E + 06	1.67	12.91	6.49	50	280	121.33
B-85	7	2065	8.04E + 08	4250	1860	7.91E + 06	2.12	10.17	4.92	70	1050	295.00
B-86	7	2065	8.04E + 08	4250	1860	7.91E + 06	2.12	10.17	4.92	70	1050	295.00
B-87	8	3678	8.15E + 08	3760	1860	6.99E + 06	1.75	4.44	3.17	106	1260	459.75
B-88	5	3627	1.43E + 09	3760	1860	6.99E + 06	2.24	16.71	5.65	267	1260	725.40
B-89	10	9090	2.30E + 09	3760	1860	6.99E + 06	1.75	6.77	3.62	320	3150	909.00
B-90	7	12813	5.23E + 09	3760	1860	6.99E + 06	2.24	16.71	5.83	155	3708	1830.43
B-91	8	6695	2.30E + 09	3760	2120	7.97E + 06	2.09	10.09	4.31	206	3090	836.88
B-92	7	3825	1.31E + 09	4250	1860	7.91E + 06	2.14	10.17	4.34	110	1900	546.43
B-93	8	4316	1.49E + 09	4250	1860	7.91E + 06	2.12	10.17	4.38	158	1890	539.50
B-94	4	430	8.15E + 07	4250	1860	7.91E + 06	2.12	4.25	2.40	25	200	107.50
B-95	3	1324	4.13E + 08	4250	1860	7.91E + 06	2.85	4.20	3.95	144	880	441.33
B-96	6	2090	7.67E + 08	3760	1860	6.99E + 06	2.40	11.52	5.25	110	660	348.33
B-97	2	1260	3.83E + 08	3760	1860	6.99E + 06	2.90	7.24	4.34	420	840	630.00
B-98	8	1188	4.53E + 08	4250	1860	7.91E + 06	2.64	7.06	4.83	55	440	148.50
B-99	4	4935	4.62E + 09	2700	1860	5.02E + 06	9.46	29.89	18.66	615	1860	1233.75
B-100	15	1218	4.49E + 08	4250	1860	7.91E + 06	1.42	12.02	4.66	13	300	81.20
B-101	17	1027	4.10E + 08	4250	1860	7.91E + 06	1.42	12.02	5.05	17	200	60.41
B-102	13	821	3.41E + 08	4250	1860	7.91E + 06	1.41	32.56	5.25	22	165	63.15
B-103	7	578	2.55E + 08	3760	1860	6.99E + 06	3.23	11.25	6.32	5	136	82.57
B-104	1	35	1.83E + 07	4250	2200	9.35E + 06	5.60	5.60	5.60	35	35	35.00
B-105	1	400	2.09E + 08	3760	1860	6.99E + 06	7.48	7.48	7.48	400	400	400.00
B-106	13	1222	5.11E + 08	4250	1860	7.91E + 06	2.71	12.99	5.29	27	206	94.00
B-107	31	5654	2.19E + 09	4250	1860	7.91E + 06	1.56	14.63	4.90	21	1628	182.39
B-108	24	1428	1.53E + 09	4250	2120	9.01E + 06	1.86	27.38	11.85	21	105	59.50
B-109	14	320	1.97E + 08	4250	2200	9.35E + 06	1.26	32.51	6.59	3	66	22.86
B-110	1	32	2.09E + 07	4250	2120	9.01E + 06	7.25	7.25	7.25	32	32	32.00
B-111	23	1213	4.04E + 08	4250	1860	7.91E + 06	1.56	14.63	4.21	8	381	52.74
B-112	10	169	6.46E + 07	4200	2200	9.24E + 06	1.61	5.67	4.14	3	104	16.90
B-113	7	10710	1.99E + 09	4250	2120	9.01E + 06	0.88	4.36	2.06	335	4405	1530.00
B-114	60	5499	1.89E + 09	3760	1860	6.99E + 06	1.15	25.29	4.92	6	1078	91.65
B-115	3	142	9.32E + 07	4250	2200	9.35E + 06	6.01	8.41	7.02	22	63	47.33
B-116	25	4470	1.76E + 09	4250	2200	9.35E + 06	1.60	10.24	4.21	27	526	178.80
B-117	1	53	3.78E + 07	4250	1860	7.91E + 06	9.03	9.03	9.03	53	53	53.00
B-118	4	444	2.98E + 08	4200	2200	9.24E + 06	4.32	15.79	7.27	22	296	111.00
B-119	13	2056	1.29E + 09	4250	2200	9.35E + 06	3.13	15.60	6.73	31	788	158.15
B-120	4	92	3.30E + 07	4250	1860	7.91E + 06	3.08	5.22	4.53	10	42	23.00
B-121	2	33	1.41E + 07	4200	2200	9.24E + 06	4.46	5.09	4.63	9	24	16.50
Average	8.92	1758.67	7.19E + 08	4029.26	1934.63	7.80E + 06	3.13	14.50	6.64	97.05	606.13	249.82

4. Computational results

The exact arc-flow model was tested with two sets of real instances from the furniture industry, set A and set B. The procedure for generating arcs was coded in C++, and the model was solved with ILOG CPLEX 10.2. The computational tests were run on a PC with a 1.87 GHz Intel Core Duo processor and a 2 GB RAM.

Tables 1 and 2 characterize the set A and set B instances, respectively. They show, for each instance, the number of different items, n, the total number of items, n_t, the area of all items, a_i, the width, W, and height, H, of the stock sheets, the area of one stock sheet, a_s, the minimum, a_{min}, maximum, a_{max}, and average, a_{av}, percentage of occupied area, in the stock sheet, by one item, and the minimum, b_{min}, maximum, b_{max}, and average, b_{av}, demand.

Tables 3 and 4 describe the computational results for the set A and set B instances, respectively. The columns present the results for the original version of the arc-flow model, without considering the cutting planes or lower bound, AF, and considering them, AF_{cut+lb}, respectively, and also for the ILP model proposed in [17], Lodi et al. and the branch-and-price algorithm proposed in [1], Alves et al., Z_{RL} and Z stand for the values of the linear relaxation and integer solutions, respectively, and t represents the total computational time, in seconds. An asterisk ($*$) represents an instance which was not solved exactly within the time limit of 7200 s.

As mentioned in Section 3.3, the cutting planes in AF_{cut+lb} can be improved using dual feasible functions. We computed them considering function $\phi^{(\varepsilon)}$, as described in [11,4]. This function, derived for the 1D-BPP, does not consider items that are smaller than a given parameter ε, and separates the remaining items in two different classes, according to whether they are larger than half of the bin size, or not. It gives a constant size to the first ones, and computes the others considering an estimation of the number of the first ones that could be placed in one bin with them.

The cutting planes and the lower bound applied to the arc-flow model improved its linear relaxations in 46.5% of the set A instances, and 32.2% of the set B instances. In three instances, for the set A, and four instances, for the set B, the lower bound provided by the linear relaxation increased one unit. With the cuts and lower bounds, only one instance of the set A and six instances of the set B have a lower bound that is not equal to the optimal integer solution. Although the

Table 3
Results for the set A of instances.

Name	Z	AF		AF_{cut+lb}		Lodi et al.		Alves et al.	
		Z_{LR}	$t(s)$	Z_{LR}	$t(s)$	Z_{LR}	$t(s)$	Z_{LR}	$t(s)$
A-1	4	3.30	0.20	3.30	0.17	2.22	0.06	3.30	0.02
A-2	36	36.00	0.33	36.00	0.36	22.42	0.02	36.00	0.02
A-3	8	8.00	0.16	8.00	0.16	4.86	0.02	8.00	0.05
A-4	3	2.67	0.19	3.00	0.22	2.19	0.02	2.67	0.00
A-5	13	12.53	0.75	12.71	0.75	11.16	32.00	12.53	0.75
A-6	2	1.89	0.09	2.00	0.09	1.18	0.02	1.89	0.02
A-7	14	13.13	0.64	13.13	0.69	9.39	*	13.13	0.03
A-8	2	1.07	0.08	2.00	0.09	1.00	0.05	1.07	0.01
A-9	61	60.67	27.34	60.67	228.94	54.74	*	60.67	*
A-10	3	1.97	0.34	3.00	0.30	1.84	48.13	2.00	0.06
A-11	46	45.76	7.72	45.79	224.17	43.33	*	45.76	54.80
A-12	14	14.00	0.33	14.00	0.38	10.86	0.14	14.00	0.01
A-13	14	13.53	0.91	13.53	0.86	11.86	*	13.53	0.75
A-14	67	66.82	183.53	66.82	52.11	63.08	*	66.82	*
A-15	39	38.94	1.11	38.94	1.16	35.06	*	38.94	3.03
A-16	83	82.26	12.03	82.26	108.95	77.94	*	82.26	15.61
A-17	5	4.70	0.34	4.73	0.28	4.46	13.83	4.70	*
A-18	65	64.39	2.97	64.65	3.38	59.34	*	64.68	*
A-19	58	57.23	17.33	57.23	11.39	50.93	*	57.23	15.58
A-20	27	26.10	1.22	26.10	1.11	22.42	*	26.10	1.28
A-21	28	27.24	57.63	27.29	23.38	25.33	*	27.24	10.08
A-22	3	2.40	0.24	3.00	0.09	1.93	0.05	2.40	0.01
A-23	14	12.92	0.77	13.12	0.78	11.80	*	12.92	2093.86
A-24	35	34.32	1012.61	*	*	30.94	*	34.36	1739.19
A-25	18	17.02	2595.83	17.29	*	16.26	*	17.33	*
A-26	8	7.05	19.98	7.09	59.61	6.44	0.83	7.08	24.38
A-27	20	19.04	3965.55	19.18	*	17.98	*	19.20	*
A-28	12	10.81	118.88	11.13	338.98	10.25	18.11	11.17	985.02
A-29	28	27.28	3155.83	27.28	*	25.32	*	27.29	*
A-30	4	3.59	15.50	3.66	8.81	3.41	0.22	3.68	14.08
A-31	8	7.47	23.39	7.47	62.20	6.84	2.33	7.49	81.97
A-32	27	26.85	83.91	26.85	702.58	24.32	*	26.90	2257.09
A-33	35	34.63	4793.70	34.63	*	32.71	*	34.67	*
A-34	6	5.20	3.13	5.24	2.63	4.63	0.20	5.22	6.99
A-35	17	16.36	44.72	16.37	126.81	15.17	*	16.40	358.70
A-36	9	8.88	1.72	8.88	1.73	6.55	0.38	8.88	0.70
A-37	5	4.63	0.86	4.63	0.88	4.17	16.48	4.63	*
A-38	23	22.11	11.83	22.11	17.64	20.21	*	22.12	54.73
A-39	4	3.02	1.00	3.03	1.03	2.78	0.05	3.06	0.38
A-40	17	15.81	79.20	15.81	663.89	14.39	*	15.90	*
A-41	19	18.50	4.72	18.50	4.06	14.18	0.28	18.50	11.97
A-42	8	7.17	0.95	7.38	0.99	4.53	0.00	7.17	0.08
A-43	7	6.38	1.00	6.38	1.08	4.80	0.03	6.42	5.03

linear relaxations improved considerably with the cutting planes and the lower bound, the computational times did not. We were able to solve one extra instance of the set B, but we could not solve five of the set A instances that were solved otherwise.

The ILP model proposed in [17] was not able to solve to optimality about 49% of the set A instances and about 83% of the set B instances. In fact, it did not even solve the linear relaxations of about 32% of the set B instances. Moreover, the lower bounds provided by its linear relaxations are considerably weaker than the ones provided by the arc-flow model.

The branch-and-price algorithm described in [1] solved 77% of the instances within the time limit of 7200 s, while the arc-flow model was able to solve all the instances, according to Table 3 (column AF). Moreover, considering the set of instances solved in both methods, the arc-flow model took 19% of the time spent by the branch-and-price algorithm. We also tested the algorithm proposed by Alves et al. using the set B instances. It solved 79% of the instances within the time limit of 7200 s, while the arc-flow model was only not able to solve two instances, according to Table 4 (column AF). Again, considering the set of instances that both methods solved, the arc-flow model took 6% of the time spent by the branch-and-price algorithm.

The set A and set B instances were also tested for the non-oriented case, AF_{Rot}, for the horizontal or vertical first cut case, $AF_{H/V}$, and finally, for the non-oriented case with horizontal or vertical first cut, $AF_{H/V+Rot}$. In what concerns these variants, AF_{Rot}, $AF_{H/V}$, $AF_{H/V+Rot}$, the arc-flow model solved, respectively, about 70%, 77% and 58% of the set A instances, and 93%, 98% and 92% of the set B instances, within the time limit of 7200 s. There was a reduction in the number of stock sheets used of about 4%, 2% and 7% for the set A instances, and 3%, 1% and 5% for the set B instances.

5. Conclusions

In this paper, we presented an exact arc-flow model for the two-dimensional cutting stock problem, with two stages and the guillotine constraint. We solved it with a commercial software (CPLEX), considering explicitly all its variables and constraints. Reduction criteria were applied to reduce the size and symmetry of the model, and increase its efficiency. We derived a new family of cutting planes and a new lower bound. We also explored some variants of the problem such as the rotation of items, and the possibility of considering the horizontal or vertical orientation for the first cut. Finally, we presented computational results based on real instances from the furniture industry. For the sets of tested instances, the model proved to be stronger and more efficient than other methods presented in the literature.

Table 4
Results for the set B of instances.

Name	Z	AF		AF_{cut+lb}		Lodi et al.		Alves et al.	
		Z_{LR}	$t(s)$	Z_{LR}	$t(s)$	Z_{LR}	$t(s)$	Z_{LR}	$t(s)$
B-1	76	74.95	0.59	74.95	0.64	68.16	*	74.95	8.98
B-2	171	170.79	1.08	170.79	1.11	149.82	*	170.79	0.22
B-3	110	108.91	0.24	109.06	0.22	87.95	*	108.91	0.11
B-4	21	20.70	0.98	20.70	1.00	17.82	334.58	20.70	0.11
B-5	11	10.53	0.45	10.53	0.47	8.94	8.42	10.53	0.14
B-6	10	9.67	0.38	10.00	0.42	8.89	0.08	9.67	0.01
B-7	537	536.25	0.34	536.25	0.39	*	*	536.25	0.01
B-8	334	333.89	0.30	333.89	0.30	*	*	333.89	0.13
B-9	85	84.19	63.14	84.19	89.36	*	*	84.19	*
B-10	78	77.74	388.45	77.80	244.70	*	*	77.75	*
B-11	271	270.05	20.11	270.05	82.70	*	*	*	*
B-12	465	464.96	0.34	464.96	0.36	*	*	464.96	0.08
B-13	230	229.11	1.30	229.11	0.69	*	*	257.95	*
B-14	35	34.33	0.78	34.33	0.83	29.60	*	34.33	0.98
B-15	16	15.44	1.42	15.44	1.42	12.06	15.92	15.44	0.31
B-16	24	23.25	0.85	23.25	0.88	21.71	*	23.25	0.94
B-17	12	11.13	0.17	11.13	0.17	7.59	0.25	11.13	0.08
B-18	14	13.10	0.27	13.21	0.30	11.95	10.80	13.10	*
B-19	47	46.21	1.09	46.21	1.13	43.37	*	46.21	2.17
B-20	95	95.00	0.84	95.00	0.83	84.37	*	95.00	0.03
B-21	229	228.33	0.42	228.33	0.50	*	*	228.33	0.11
B-22	17	16.50	0.61	16.50	0.62	15.08	*	16.50	0.23
B-23	85	84.33	1.11	84.33	1.13	80.82	*	84.33	3.11
B-24	9	8.83	0.20	9.00	0.25	8.27	*	8.83	0.11
B-25	47	46.75	0.34	46.75	0.36	40.11	*	46.75	0.03
B-26	128	127.71	0.66	127.71	0.70	*	*	127.71	*
B-27	221	220.51	0.34	220.51	0.34	*	*	220.51	0.16
B-28	38	36.98	0.30	37.00	0.35	35.99	*	36.98	0.16
B-29	19	18.24	0.39	18.26	0.41	16.97	*	18.24	0.41
B-30	57	56.77	16.52	56.77	17.64	54.93	*	56.77	5.69
B-31	30	29.33	0.61	29.33	0.63	25.64	*	29.33	0.05
B-32	192	191.07	3.56	191.07	2.50	*	*	191.07	*
B-33	52	50.89	0.75	50.89	0.78	44.80	*	50.89	0.78
B-34	83	82.67	1.61	82.67	1.63	67.85	*	82.67	1.64
B-35	96	95.83	0.39	95.83	0.42	85.32	226.53	95.83	0.08
B-36	96	95.83	0.41	95.83	0.44	87.39	*	95.83	0.03
B-37	53	52.22	0.39	52.22	0.42	44.23	*	52.22	0.03
B-38	35	34.58	0.17	34.63	0.20	26.89	705.75	34.58	0.00
B-39	35	35.00	0.19	35.00	0.22	26.96	35.88	35.00	0.00
B-40	19	18.64	2.47	18.64	2.53	17.27	*	18.64	0.06
B-41	17	16.67	4.22	16.67	4.61	14.55	*	16.67	0.66
B-42	16	15.39	0.28	15.39	0.28	14.50	*	15.39	0.09
B-43	6	5.33	0.69	5.39	0.70	4.47	0.56	5.33	0.55
B-44	24	23.32	12.38	23.32	12.69	22.02	*	23.32	*
B-45	11	10.17	2.88	10.17	2.91	9.72	*	10.17	0.11
B-46	28	26.99	1.06	26.99	1.09	22.12	*	26.99	0.03
B-47	8	7.92	0.13	8.00	0.19	7.43	4.06	7.92	0.06
B-48	14	13.45	10.41	13.45	11.13	13.22	*	13.45	1.31
B-49	21	20.07	0.44	20.14	0.45	15.96	1.31	20.07	0.03
B-50	71	70.36	0.11	71.00	0.13	61.28	5293.47	70.36	0.00
B-51	63	62.14	1.05	62.14	1.09	52.23	*	62.14	0.19
B-52	120	119.50	0.14	120.00	0.16	*	*	119.50	0.00
B-53	3	2.80	0.47	2.80	0.47	2.35	6.70	2.80	0.03
B-54	13	12.20	0.27	12.40	0.27	10.92	*	12.20	4.72
B-55	51	49.94	0.86	49.94	0.86	47.06	*	49.94	26.33
B-56	214	213.83	4.05	213.83	3.73	*	*	213.83	16.63
B-57	44	43.86	34.55	43.86	32.91	41.75	*	43.86	*
B-58	20	19.12	0.36	19.12	0.31	17.97	*	19.12	0.14
B-59	11	10.39	0.22	10.39	0.27	9.12	1648.48	10.39	0.01
B-60	13	12.17	0.53	12.19	0.56	11.25	*	12.17	0.09
B-61	6	5.33	0.24	5.33	0.23	4.69	*	5.33	0.01
B-62	18	17.55	0.53	17.57	0.55	16.30	*	17.55	0.06
B-63	38	37.76	1.06	37.76	1.08	33.98	*	37.76	0.31
B-64	20	19.28	1.16	19.28	1.27	18.38	*	19.28	2.78
B-65	172	171.56	3.17	171.56	2.45	*	*	171.56	10.17
B-66	41	40.00	1.17	40.05	1.30	36.00	*	40.00	1.52
B-67	262	260.96	21.86	261.01	18.03	*	*	260.96	3049.28
B-68	49	48.94	2.28	48.94	2.30	46.14	*	48.94	*
B-69	20	19.94	0.22	19.94	0.25	18.27	*	19.94	0.16
B-70	117	116.16	2.02	116.16	2.03	108.34	*	116.16	*
B-71	66	65.08	1.34	65.08	1.39	60.68	*	65.08	17.50
B-72	34	33.78	0.55	33.78	0.56	32.32	*	33.78	1.58
B-73	19	18.26	0.64	18.27	0.67	17.00	*	18.26	0.48
B-74	309	308.59	1.55	308.64	1.50	*	*	308.59	*

Table 4 (continued).

Name	Z	AF		AF_{cut+lb}		Lodi et al.		Alves et al.	
		Z_{LR}	$t(s)$	Z_{LR}	$t(s)$	Z_{LR}	$t(s)$	Z_{LR}	$t(s)$
B-75	126	125.86	0.70	125.86	0.70	*	*	125.86	1.36
B-76	221	220.51	0.33	220.51	0.38	*	*	220.51	0.16
B-77	208	207.00	1.61	207.02	1.41	*	*	207.00	67.67
B-78	73	72.53	0.28	72.54	0.27	*	*	72.53	0.13
B-79	50	49.53	0.50	49.53	0.58	44.86	*	49.53	2.81
B-80	57	56.91	0.50	56.91	0.66	*	*	56.91	1.24
B-81	425	424.04	1.83	424.04	1.84	*	*	424.04	62.45
B-82	105	104.83	1.19	104.84	1.23	91.38	*	104.83	4.44
B-83	97	96.55	1.64	96.59	2.56	89.38	*	96.55	9.70
B-84	103	102.17	1.27	102.23	1.33	94.45	*	110.50	*
B-85	106	105.05	0.58	105.05	0.52	*	*	105.05	0.53
B-86	106	105.05	0.56	105.05	0.50	*	*	105.05	0.53
B-87	128	127.73	0.67	127.73	0.67	*	*	127.73	0.94
B-88	226	225.62	0.33	225.62	0.36	*	*	225.62	0.61
B-89	353	352.44	0.66	352.44	0.72	*	*	352.44	4.14
B-90	841	840.42	0.50	840.42	0.50	*	*	840.42	0.13
B-91	317	316.50	0.81	316.50	0.88	*	*	316.50	*
B-92	176	175.24	0.34	175.26	0.39	*	*	175.24	0.25
B-93	201	200.07	0.78	200.11	0.80	*	*	200.07	4.70
B-94	12	11.08	0.44	11.08	0.44	10.30	*	11.08	0.05
B-95	60	59.60	0.34	59.66	0.39	52.31	*	59.60	*
B-96	124	123.75	0.63	123.75	0.66	*	*	123.75	*
B-97	60	60.00	0.20	60.00	0.23	*	*	60.00	0.01
B-98	68	67.02	1.27	67.02	0.45	57.34	*	67.02	3.28
B-99	1095	1095.00	0.78	1095.00	0.81	*	*	1095.00	0.01
B-100	59	58.57	15.86	58.60	4.19	56.81	*	58.57	13.64
B-101	59	58.34	3.81	58.55	8.14	51.90	*	58.34	*
B-102	46	45.22	2.13	45.22	2.17	43.11	*	45.22	*
B-103	41	39.86	3.27	40.02	3.27	36.52	*	39.86	*
B-104	3	2.33	0.14	3.00	0.16	1.96	0.05	2.33	0.02
B-105	34	33.33	0.11	34.00	0.13	29.92	73.41	33.33	0.00
B-106	71	70.18	2.05	70.18	2.02	64.67	*	70.18	*
B-107	300	299.00	586.72	299.00	417.27	*	*	299.00	*
B-108	185	183.72	*	183.72	16.73	169.27	*	183.72	*
B-109	26	25.14	4.61	25.14	3.44	21.09	*	25.14	9.23
B-110	4	3.20	0.13	4.00	0.14	2.32	0.08	3.20	0.00
B-111	56	55.17	79.60	55.17	1022.19	51.11	*	55.17	186.77
B-112	9	8.10	1.17	8.11	1.17	6.99	*	8.10	1.38
B-113	231	230.44	0.42	230.44	0.25	*	*	230.44	0.66
B-114	-	279.98	*	279.98	*	*	*	279.99	*
B-115	15	14.19	9.03	14.19	8.47	9.97	116.97	14.19	0.01
B-116	192	191.27	1765.78	191.28	6396.39	*	*	191.27	*
B-117	6	5.30	0.66	6.00	0.66	4.79	0.22	5.30	0.00
B-118	36	35.25	4.19	35.25	4.12	32.30	*	35.25	0.02
B-119	144	143.32	8.14	143.32	11.08	*	*	143.32	*
B-120	5	4.65	0.61	4.68	0.59	4.17	10.16	4.65	*
B-121	3	2.10	0.27	2.11	0.27	1.53	*	2.10	0.27

Acknowledgments

This work was partially supported by the Portuguese Science and Technology Foundation through the doctoral Grant SFRH/BD/39607/2007 for Rita Macedo and by the Project SCOOP (Sheet cutting and process optimization for furniture enterprises) (Contract no. COOP-CT-2006-032998), funded by the European Commission, 6th Framework Programme on Research, Technological Development and Demonstration, specific actions for SMEs, Cooperative Research Projects.

We thank the anonymous referees for their constructive comments, which led to a clearer presentation of the material.

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