Formulary

Algebra

Binomial theorem

$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

Trigonometric identities

Transformation

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

$\sin(\pi - \theta) = \sin\theta$

$\cos(\pi - \theta) = -\cos\theta$

$\tan(\pi - \theta) = -\tan\theta$

$\tan(\theta + \pi) = \tan\theta$

$$\begin{aligned} \cos\theta &= \sin\left(\tfrac{\pi}{2} - \theta\right) & \sin\theta &= \cos\left(\tfrac{\pi}{2} - \theta\right) \\ \cot\theta &= \tan\left(\tfrac{\pi}{2} - \theta\right) & \tan\theta &= \cot\left(\tfrac{\pi}{2} - \theta\right) \\ \csc\theta &= \sec\left(\tfrac{\pi}{2} - \theta\right) & \sec\theta &= \csc\left(\tfrac{\pi}{2} - \theta\right) \end{aligned}$$

Sum of angles

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$

$\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$\cos(2\theta) = 2\cos^2\theta - 1$

$\cos(2\theta) = 1 - 2\sin^2\theta$

Product to sum

$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$

$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$

$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$

$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$

Weird ones

$\sec^2\theta = 1 + \tan^2\theta$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

Limits

Definition

$\lim_{x\to a} f(x) = L$ if:

For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x-a| < \delta$ then:

$|f(x) - L| < \varepsilon$

Properties

$\lim_{x\to a} c = c$

$\lim_{x\to a} x = a$

Let $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ then:

- $\lim_{x\to a} cf(x) = cL$
- $\lim_{x\to a}[f(x) \pm g(x)] = L \pm M$
- $\lim_{x\to a}[f(x)g(x)] = LM$
- $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$ if $M \neq 0$
- $\lim_{x\to a}[f(x)]^n = L^n$

Common limits

$\lim_{x\to 0} \frac{\sin x}{x} = 1$

$\lim_{x\to 0} \frac{1-\cos x}{x} = 0$

$\lim_{x\to 0} \frac{\tan x}{x} = 1$

$\lim_{x\to 0} \frac{\arcsin x}{x} = 1$

$\lim_{x\to 0} \frac{\arctan x}{x} = 1$

$\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

$\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a$

$\lim_{x\to 0} \frac{\sinh x}{x} = 1$

$\lim_{x\to 0} \frac{\cosh x - 1}{x} = 0$

$\lim_{x\to 0} \frac{\tanh x}{x} = 1$

$\lim_{x\to \infty} \frac{\ln x}{x} = 0$

$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$

$\lim_{x\to \infty} e^{-x} = 0$

$\lim_{x\to \infty}(1+\frac{1}{x})^x = e$

$\lim_{x\to \infty}(1+\frac{k}{x})^x = e^k$

$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$

Continuity

Definition

f is continuous at $x = a$ if:

$\lim_{x\to a} f(x) = f(a)$

Strict definition

f is continuous at $x = a$ if:

$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in D(f)):$

$|x-a| < \delta \Rightarrow |f(x)-f(a)| < \varepsilon$

Properties

If f and g are continuous at $x = a$ then:

- $f+g$ is continuous at $x = a$
- $f-g$ is continuous at $x = a$
- $f \cdot g$ is continuous at $x = a$
- $\frac{f}{g}$ is continuous at $x = a$ if $g(a) \neq 0$

Intermediate value theorem

If f is continuous on $[a,b]$ and let M be any number between $f(a)$ and $f(b)$ then there exists a number c in $[a,b]$ such that $f(c) = M$

Derivatives

Definition

$\frac{d}{dx}f(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

Critical points

$X = c$ is a critical point of f if:

- $f'(c) = 0$
- $f'(c)$ is undefined

Concavity

f is convex on I if $f''(x) > 0$ for all $x \in I$

f is concave on I if $f''(x) < 0$ for all $x \in I$

Inflection points

$X = c$ is an inflection point of f if the concavity of f changes at c

Common derivatives

1. $\frac{d}{dx} \sin x = \cos x$

2. $\frac{d}{dx} \cos x = -\sin x$

3. $\frac{d}{dx} \tan x = \sec^2 x$

4. $\frac{d}{dx} \cot x = -\csc^2 x$

5. $\frac{d}{dx} \sec x = \sec x \tan x$

6. $\frac{d}{dx} \csc x = -\csc x \cot x$

7. $\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$

8. $\frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}}$

9. $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$

10. $\frac{d}{dx} \operatorname{arccot} x = -\frac{1}{1+x^2}$

11. $\frac{d}{dx} \sinh x = \cosh x$

12. $\frac{d}{dx} \cosh x = \sinh x$

13. $\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$

14. $\frac{d}{dx} \coth x = -\operatorname{csch}^2 x$

15. $\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$

16. $\frac{d}{dx} a^x = a^x \ln a$

Rolle's theorem

If f is continuous on $[a, b]$ and differentiable on (a, b) and $f(a) = f(b)$ then there exists a number c in (a, b) such that $f'(c) = 0$

Mean value theorem

If f is continuous on $[a, b]$ and differentiable on (a, b) then there exists a number c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

Extreme value theorem

If f is continuous on $[a, b]$ then f attains an absolute maximum and an absolute minimum on $[a, b]$

To find the absolute maximum and minimum of f on $[a, b]$:

1. Find the critical points of f in (a, b)
2. Evaluate f at the critical points and at the endpoints of $[a, b]$
3. The largest value is the absolute maximum and the smallest value is the absolute minimum

L'Hôpital's rule

If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty$ and $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists then:

$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

Taylor series

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

Maclaurin series

The Maclaurin series is a Taylor series centered at $a = 0$

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$

Common Maclaurin series

1. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$

2. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$

3.$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$

4. $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}x^n$

5.$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

6.$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$

7. $-\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$

8. $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$

9. $\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

10. $\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

11. $(1+x)^p = \sum_{n=0}^{\infty} \frac{p(p-1)\ldots(p-n+1)}{n!} x^n$

Series of real numbers

Definition

A series of real numbers is an expression of the form:

$a_1 + a_2 + a_3 + \ldots = \sum_{n=1}^{\infty} a_n$

where $\{a_n\}$ is a sequence of real numbers

$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$

$s_n = \sum_{k=1}^{n} a_k$

Convergence

A series $\sum_{n=1}^{\infty} a_n$ converges if $\lim_{n\to\infty} s_n$ exists

Necessary condition for convergence

$\lim_{n\to\infty} a_n = 0$

Convergence tests

Comparison test

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms and $a_n \leq b_n$ for all n then:

- If $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges
- If $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges

D'Alembert's ratio test

If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$ then:

- If $L < 1$ then $\sum_{n=1}^{\infty} a_n$ converges
- If $L > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges

Cauchy's root test

If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$ then:

- If $L < 1$ then $\sum_{n=1}^{\infty} a_n$ converges
- If $L > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges

Integral test

If f is a continuous, positive, decreasing function on $[1, \infty)$ and $a_n = f(n)$ then:

$\int_1^\infty f(x)dx$ converges if and only if $\sum_{n=1}^\infty a_n$ converges

Alternating series

An alternating series is a series of the form:

$a_1 - a_2 + a_3 - \ldots = \sum_{n=1}^\infty (-1)^{n+1} a_n$

where $\{a_n\}$ is a sequence of positive real numbers

Alternating series test (Leibniz's test)

If $\{a_n\}$ is a sequence of positive real numbers such that:

- $a_{n+1} \leq a_n$ for all n
- $\lim_{n\to\infty} a_n = 0$

then: $\sum_{n=1}^\infty (-1)^{n+1} a_n$ converges

Dirichlet's test

If $\{a_n\}$ and $\{b_n\}$ are sequences such that:

- $\{a_n\}$ is a decreasing sequence of positive real numbers
- $\lim_{n\to\infty} a_n = 0$
- $\sum_{n=1}^\infty b_n$ is bounded
  then: $\sum_{n=1}^\infty a_n b_n$ converges

Abel's test

Suppose the sequence $\{a_n\}$ is monotonic and the series $\sum_{n=1}^\infty b_n$ converges. Then the series $\sum_{n=1}^\infty a_n b_n$ converges.

Absolute convergence

A series $\sum_{n=1}^\infty a_n$ converges absolutely if $\sum_{n=1}^\infty |a_n|$ converges

Theorem

Every absolutely convergent series converges

Conditional convergence

A series $\sum_{n=1}^\infty a_n$ converges conditionally if it converges but does not converge absolutely

Power series

$\sum_{n=0}^\infty a_n (x - x_0)^n$

Radius of convergence

$R = \frac{1}{\lim_{n\to\infty} \sqrt[n]{|a_n|}}$

Interval of convergence

$[a - R, a + R]$

Theorems to check for convergence

1. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$ then the radius of convergence is $R = \frac{1}{L}$
2. If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$ then the radius of convergence is $R = \frac{1}{L}$

Differentiation and integration

A power series can be differentiated and integrated term by term.

$\frac{d}{dx} \sum_{n=0}^{\infty} a_n(x - x_0)^n = \sum_{n=1}^{\infty} na_n(x - x_0)^{n-1}$

$\int \sum_{n=0}^{\infty} a_n(x - x_0)^n dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x - x_0)^{n+1} + C$

The radius of convergence of the differentiated or integrated series is the same as the original series.

Integrals

Definition

$\int f(x)dx = F(x) + C$

Integral by parts

$\int u\, dv = uv - \int v\, du$

Basic forms

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$

$\int \frac{1}{ax+b} dx = \frac{1}{a} \ln|ax + b| + C$

Differential of an Integral

$\frac{\mathrm{d}}{\mathrm{d}\alpha} \int_{a(\alpha)}^{b(\alpha)} f(x, \alpha)\mathrm{d}x = f(b, \alpha)\frac{\mathrm{d}b}{\mathrm{d}\alpha} - f(a, \alpha)\frac{\mathrm{d}a}{\mathrm{d}\alpha} + \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial\alpha} f(x, \alpha)\mathrm{d}x$

Common integrals

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C$

2. $\int \frac{1}{x} dx = \ln|x| + C$

3. $\int e^{ax} \mathrm{d}x = \frac{1}{a} e^{ax} + c$

4. $\int a^x dx = \frac{a^x}{\ln a} + C$

5. $\int \ln x dx = x \ln x - x + C$

6. $\int \sin x dx = -\cos x + C$

7. $\int \cos x dx = \sin x + C$

8. $\int \tan x dx = -\ln|\cos x| + C$

9. $\int \cot x dx = \ln|\sin x| + C$

10. $\int \sec x dx = \ln|\sec x + \tan x| + C$

11. $\int \sec^2 x dx = \tan x + C$

12. $\int \csc^2 x dx = -\cot x + C$

13. $\int \sec x \tan x dx = \sec x + C$

14. $\int \frac{1}{\sqrt{a^2-x^2}} dx = \arcsin \frac{x}{a} + C$

15. $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \arctan \frac{x}{a} + C$

16. $\int \frac{1}{x\sqrt{x^2-a^2}} dx = \frac{1}{a} \operatorname{arcsec} \frac{|x|}{a} + C$

17. $\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left|x + \sqrt{x^2 + a^2}\right| + C$

18. $\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left|x + \sqrt{x^2 - a^2}\right| + C$

19. $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a} \ln\left|\frac{a+x}{a-x}\right| + C$