Corollary 2.9.1 Suppose in the context of Theorem 2.9.1 that the joint function $h = (h_1, h_2) : R^2 \to R^2$ defined by $h(x, y) = \big(h_1(x, y)\,,\ h_2(x, y)\big)$ is one-to-one, i.e., if $h_1(x_1, y_1) = h_1(x_2, y_2)$ and $h_2(x_1, y_1) = h_2(x_2, y_2)$, then $x_1 = x_2$ and $y_1 = y_2$. Then

$$p_{Z,W}(z, w) = p_{X,Y}(h^{-1}(z, w)),$$

where $h^{-1}(z, w)$ is the unique pair (x, y) such that $h(x, y) = (z, w)$.

EXAMPLE 2.9.1

Suppose X and Y have joint density function

$$p_{X,Y}(x, y) = \begin{cases} 1/6 & x = 2, y = 6 \\ 1/12 & x = -2, y = -6 \\ 1/4 & x = -3, y = 11 \\ 1/2 & x = 3, y = -8 \\ 0 & \text{otherwise.} \end{cases}$$

Let $Z = X + Y$ and $W = Y - X^2$. Then $p_{Z,W}(8, 2) = P(Z = 8, W = 2) = P(X = 2, Y = 6) + P(X = -3, Y = 11) = 1/6 + 1/4 = 5/12$. On the other hand, $p_{Z,W}(-5, -17) = P(Z = -5, W = -17) = P(X = 3, Y = -8) = \frac{1}{2}$. ∎

2.9.2 The Continuous Case (Advanced)

If X and Y are *continuous*, and the function $h = (h_1, h_2)$ is *one-to-one*, then it is again possible to compute a formula for the joint density of Z and W, as the following theorem shows. To state it, recall from multivariable calculus that, if $h = (h_1, h_2) : R^2 \to R^2$ is a differentiable function, then its *Jacobian derivative* J is defined by

$$J(x, y) = \det \begin{pmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_2}{\partial x} \\ \frac{\partial h_1}{\partial y} & \frac{\partial h_2}{\partial y} \end{pmatrix} = \frac{\partial h_1}{\partial x}\frac{\partial h_2}{\partial y} - \frac{\partial h_2}{\partial x}\frac{\partial h_1}{\partial y}.$$

Theorem 2.9.2 Let X and Y be jointly absolutely continuous, with joint density function $f_{X,Y}$. Let $Z = h_1(X, Y)$ and $W = h_2(X, Y)$, where $h_1, h_2 : R^2 \to R^1$ are differentiable functions. Define the joint function $h = (h_1, h_2) : R^2 \to R^2$ by

$$h(x, y) = (h_1(x, y), h_2(x, y))\,.$$

Assume that h is one-to-one, at least on the region $\{(x, y) : f(x, y) > 0\}$, i.e., if $h_1(x_1, y_1) = h_1(x_2, y_2)$ and $h_2(x_1, y_1) = h_2(x_2, y_2)$, then $x_1 = x_2$ and $y_1 = y_2$. Then Z and W are also jointly absolutely continuous, with joint density function $f_{Z,W}$ given by

$$f_{Z,W}(z, w) = f_{X,Y}(h^{-1}(z, w)) \,/\, |J(h^{-1}(z, w))|,$$

where J is the Jacobian derivative of h and where $h^{-1}(z, w)$ is the unique pair (x, y) such that $h(x, y) = (z, w)$.