BY
Replication
One State Variable

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A's (Consumption)

The Epstein-Zin preferences are:

$$U_t = \max_{C_t, \alpha_t} \left\{ (1-\beta) C_t^{\frac{1-\gamma}{\theta}} + \beta \left(\mathbb{E}_t \left[U_{t+1}^{1-\gamma} \right] \right)^{\frac{1}{\theta}} \right\}^{\frac{\theta}{1-\gamma}}$$

with $\theta = \frac{1-\gamma}{1-\frac{1}{\psi}}$. The log consumption growth is given by:

$$\begin{aligned} x_{t+1} &= \rho x_t + \varphi_e \sigma \epsilon_{t+1} \\ \Delta c_{t+1} &= \mu + x_t + \sigma \eta_{t+1} \end{aligned}$$

And the log dividend growth is given by:

$$\Delta d_{t+1} = \mu_d + \phi x_t + \varphi_d \sigma u_{t+1}$$

with the shocks η_t, ϵ_t, u_t i.i.d. $N(0,1)$ and uncorrelated. Now, let $z_t = \ln(\frac{P_t}{C_t})$. The Campbell-Shiller approximation for returns is:

$$r_{w,t+1} = \kappa_0 + \kappa_1 z_{t+1} - z_t + \Delta c_{t+1}$$

Since there is one state variable, I conjecture that the price-to-consumption ratio is:

$$z_t = A_0 + A_1 x_t$$

The Euler equation is:

$$\mathbb{E}_t \left[\beta^\theta \, \Delta C_{t+1}^{-\frac{\theta}{\psi}} (1 + R_{w,t+1})^{\theta - 1} (1 + R_{i,t+1}) \right] = 1, \quad \forall i$$

In particular, the equation is satisfied for $1+R_{i,t+1} = 1+R_{w,t+1}$. Hence:

$$\mathbb{E}_t\left[\beta^\theta\,\Delta C_{t+1}^{-\frac{\theta}{\psi}}(1+R_{w,t+1})^\theta\right] = 1$$

$$\Leftrightarrow \mathbb{E}_t\left[e^{\ln\left(\beta^\theta\,\Delta C_{t+1}^{-\frac{\theta}{\psi}}(1+R_{w,t+1})^\theta\right)}\right] = 1$$

$$\Leftrightarrow \mathbb{E}_t\left[e^{\theta\ln(\beta)-\frac{\theta}{\psi}\Delta c_{t+1}+\theta r_{w,t+1}}\right] = 1$$

Plugging in Δc_{t+1} and the Campbell-Shiller approximation:

$$\Leftrightarrow \mathbb{E}_t\left[e^{\theta\ln(\beta)-\frac{\theta}{\psi}(\mu+x_t+\sigma\eta_{t+1})+\theta(\kappa_0+\kappa_1 z_{t+1}-z_t+\Delta c_{t+1})}\right] = 1$$

Plugging in Δc_{t+1} again and z_t, z_{t+1}:

$$\Leftrightarrow \mathbb{E}_t\left[e^{\theta\ln(\beta)-\frac{\theta}{\psi}(\mu+x_t+\sigma\eta_{t+1})+\theta(\kappa_0+\kappa_1(A_0+A_1x_{t+1})-A_0-A_1x_t+\mu+x_t+\sigma\eta_{t+1})}\right] = 1$$

Plugging in x_{t+1}:

$$\Leftrightarrow \mathbb{E}_t\left[e^{\begin{bmatrix}\theta\ln(\beta)-\frac{\theta}{\psi}(\mu+x_t+\sigma\eta_{t+1})\\ +\theta(\kappa_0+\kappa_1(A_0+A_1[\rho x_t+\varphi_e\sigma\epsilon_{t+1}])\\ -A_0-A_1x_t+\mu+x_t+\sigma\eta_{t+1})\end{bmatrix}}\right] = 1$$

$$\Leftrightarrow \mathbb{E}_t\left[e^{\begin{bmatrix}\theta\ln(\beta)-\frac{\theta}{\psi}\mu-\frac{\theta}{\psi}x_t-\frac{\theta}{\psi}\sigma\eta_{t+1}\\ +\theta\kappa_0+\theta\kappa_1(A_0+A_1[\rho x_t+\varphi_e\sigma\epsilon_{t+1}])\\ -\theta A_0-\theta A_1x_t+\theta\mu+\theta x_t+\theta\sigma\eta_{t+1}\end{bmatrix}}\right] = 1$$

$$\Leftrightarrow \mathbb{E}_t\left[e^{\begin{bmatrix}\theta\ln(\beta)-\frac{\theta}{\psi}\mu-\frac{\theta}{\psi}x_t-\frac{\theta}{\psi}\sigma\eta_{t+1}\\ +\theta\kappa_0+\theta\kappa_1A_0+\theta\kappa_1A_1\rho x_t+\theta\kappa_1A_1\varphi_e\sigma\epsilon_{t+1}\\ -\theta A_0-\theta A_1x_t+\theta\mu+\theta x_t+\theta\sigma\eta_{t+1}\end{bmatrix}}\right] = 1$$

Rearranging:

$$\Leftrightarrow \mathbb{E}_t \left[e^{\begin{bmatrix} \theta \ln(\beta) - \frac{\theta}{\psi}\mu - \frac{\theta}{\psi}x_t + \theta\kappa_0 + \theta\kappa_1 A_0 + \theta\kappa_1 A_1 \rho x_t \\ -\theta A_0 - \theta A_1 x_t + \theta\mu + \theta x_t \\ -\frac{\theta}{\psi}\sigma\eta_{t+1} + \theta\kappa_1 A_1 \varphi_e \sigma\epsilon_{t+1} + \theta\sigma\eta_{t+1} \end{bmatrix}} \right] = 1$$

Since $\epsilon_{t+1}, \eta_{t+1} \sim N(0,1)$ and are uncorrelated:

$$\Leftrightarrow \begin{bmatrix} \theta \ln(\beta) - \frac{\theta}{\psi}\mu - \frac{\theta}{\psi}x_t + \theta\kappa_0 + \theta\kappa_1 A_0 + \theta\kappa_1 A_1 \rho x_t \\ -\theta A_0 - \theta A_1 x_t + \theta\mu + \theta x_t + \frac{1}{2}\sigma^2([\theta - \frac{\theta}{\psi}]^2 + (\theta\kappa_1 A_1 \varphi_e)^2) \end{bmatrix} = 0$$

Since this equation must be satisfied for all values of x_t:

$$-\frac{\theta}{\psi} + \theta\kappa_1 A_1 \rho - \theta A_1 + \theta = 0$$

$$\Leftrightarrow \theta - \frac{\theta}{\psi} = \theta A_1 - \theta\kappa_1 A_1 \rho$$

$$\Leftrightarrow 1 - \frac{1}{\psi} = A_1(1 - \kappa_1 \rho)$$

Hence:

$$\boxed{A_1 = \frac{1 - \frac{1}{\psi}}{1 - \kappa_1 \rho}}$$

And A_0 absorbs the constant terms:

$$\theta \ln(\beta) - \frac{\theta}{\psi}\mu + \theta\kappa_0 + \theta\kappa_1 A_0 - \theta A_0 + \theta\mu + \frac{1}{2}\sigma^2([\theta - \frac{\theta}{\psi}]^2 + (\theta\kappa_1 A_1 \varphi_e)^2) = 0$$

$$\Leftrightarrow \theta \ln(\beta) - \frac{\theta}{\psi}\mu + \theta\kappa_0 + \theta\mu + \frac{1}{2}\sigma^2([\theta - \frac{\theta}{\psi}]^2 + (\theta\kappa_1 A_1 \varphi_e)^2) = \theta A_0 - \theta\kappa_1 A_0$$

$$\Leftrightarrow \ln(\beta) - \frac{1}{\psi}\mu + \kappa_0 + \mu + \frac{1}{2}\sigma^2\theta([1 - \frac{1}{\psi}]^2 + (\kappa_1 A_1 \varphi_e)^2) = A_0 - \kappa_1 A_0$$

Hence:

$$\boxed{A_0 = \frac{\ln(\beta) - \frac{1}{\psi}\mu + \kappa_0 + \mu + \frac{1}{2}\sigma^2\theta([1 - \frac{1}{\psi}]^2 + (\kappa_1 A_1 \varphi_e)^2)}{1 - \kappa_1}}$$

Risk Premium (Consumption)

For the conditional risk premium on the consumption claim ($r_{c,t} = r_{w,t}$), I calculate:

$$\ln(M_{t+1}) - \mathbb{E}_t(\ln(M_{t+1})) = \theta\ln(\beta) - \frac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)r_{c,t+1} - \theta\ln(\beta) + \frac{\theta}{\psi}\mathbb{E}_t(\Delta c_{t+1}) - (\theta-1)\mathbb{E}_t(r_{c,t+1})$$

Plugging in Δc_{t+1} and the Campbell-Shiller approximation:

$$\Leftrightarrow \ln(M_{t+1}) - \mathbb{E}_t(\ln(M_{t+1})) = \begin{bmatrix} -\frac{\theta}{\psi}(\mu + x_t + \sigma\eta_{t+1}) + (\theta-1)(\kappa_0 + \kappa_1 z_{t+1} - z_t + \Delta c_{t+1}) \\ +\frac{\theta}{\psi}\mathbb{E}_t(\mu + x_t + \sigma\eta_{t+1}) - (\theta-1)\mathbb{E}_t(\kappa_0 + \kappa_1 z_{t+1} - z_t + \Delta c_{t+1}) \end{bmatrix}$$

$$\Leftrightarrow \ln(M_{t+1}) - \mathbb{E}_t(\ln(M_{t+1})) = \begin{bmatrix} -\frac{\theta}{\psi}\sigma\eta_{t+1} + (\theta-1)(\kappa_1 z_{t+1} + \Delta c_{t+1}) \\ -(\theta-1)\mathbb{E}_t(\kappa_1 z_{t+1} + \Delta c_{t+1}) \end{bmatrix}$$

Plugging in Δc_{t+1} again and z_{t+1}:

$$\Leftrightarrow \ln(M_{t+1}) - \mathbb{E}_t(\ln(M_{t+1})) = \begin{bmatrix} -\frac{\theta}{\psi}\sigma\eta_{t+1} + (\theta-1)(\kappa_1(A_0 + A_1 x_{t+1}) \\ +\mu + x_t + \sigma\eta_{t+1}) \\ -(\theta-1)\mathbb{E}_t(\kappa_1(A_0 + A_1 x_{t+1}) \\ +\mu + x_t + \sigma\eta_{t+1}) \end{bmatrix}$$

$$\Leftrightarrow \ln(M_{t+1}) - \mathbb{E}_t(\ln(M_{t+1})) = \begin{bmatrix} -\frac{\theta}{\psi}\sigma\eta_{t+1} + (\theta-1)(\kappa_1 A_1 x_{t+1} + \sigma\eta_{t+1}) \\ -(\theta-1)\mathbb{E}_t(\kappa_1 A_1 x_{t+1} + \sigma\eta_{t+1}) \end{bmatrix}$$

Plugging in the state variable in $t+1$:

$$\Leftrightarrow \ln(M_{t+1}) - \mathbb{E}_t(\ln(M_{t+1})) = \begin{bmatrix} -\frac{\theta}{\psi}\sigma\eta_{t+1} + (\theta-1)(\kappa_1 A_1(\rho x_t + \varphi_e\sigma\epsilon_{t+1}) + \sigma\eta_{t+1}) \\ -(\theta-1)\mathbb{E}_t(\kappa_1 A_1(\rho x_t + \varphi_e\sigma\epsilon_{t+1}) + \sigma\eta_{t+1}) \end{bmatrix}$$

$$\Leftrightarrow \ln(M_{t+1}) - \mathbb{E}_t(\ln(M_{t+1})) = -\frac{\theta}{\psi}\sigma\eta_{t+1} + (\theta-1)(\kappa_1 A_1\varphi_e\sigma\epsilon_{t+1} + \sigma\eta_{t+1})$$

Additionally, I calculate:

$$r_{c,t+1} - \mathbb{E}_t(r_{c,t+1}) = \kappa_0 + \kappa_1 z_{t+1} - z_t + \Delta c_{t+1} - \mathbb{E}_t(\kappa_0 + \kappa_1 z_{t+1} - z_t + \Delta c_{t+1})$$

Plugging in Δc_{t+1} and z_{t+1}:

$$\Leftrightarrow r_{c,t+1} - \mathbb{E}_t(r_{c,t+1}) = \begin{bmatrix} \kappa_1(A_0 + A_1 x_{t+1}) + \mu + x_t + \sigma\eta_{t+1} \\ -\mathbb{E}_t(\kappa_1(A_0 + A_1 x_{t+1}) + \mu + x_t + \sigma\eta_{t+1}) \end{bmatrix}$$

$$\Leftrightarrow r_{c,t+1} - \mathbb{E}_t(r_{c,t+1}) = \left[\kappa_1 A_1 x_{t+1} + \sigma\eta_{t+1} - \mathbb{E}_t(\kappa_1 A_1 x_{t+1})\right]$$

Plugging in the state variable in $t+1$:

$$\Leftrightarrow r_{c,t+1} - \mathbb{E}_t(r_{c,t+1}) = \left[\kappa_1 A_1(\rho x_t + \varphi_e \sigma \epsilon_{t+1}) + \sigma \eta_{t+1} - \mathbb{E}_t(\kappa_1 A_1(\rho x_t + \varphi_e \sigma \epsilon_{t+1}))\right]$$

$$\Leftrightarrow r_{c,t+1} - \mathbb{E}_t(r_{c,t+1}) = \left[\kappa_1 A_1 \varphi_e \sigma \epsilon_{t+1} + \sigma \eta_{t+1}\right]$$

To combine these two derivations, I use that $\mathbb{E}_t(r_{c,t+1} - r_f) + \frac{1}{2}\mathbb{V}ar_t(r_{c,t+1}) = -\mathbb{C}ov_t(\ln(M_{t+1}), r_{c,t+1})$. First, for $\mathbb{V}ar_t(r_{c,t+1})$:

$$\mathbb{V}ar_t(r_{c,t+1}) = \mathbb{E}_t([r_{c,t+1} - \mathbb{E}_t(r_{c,t+1})]^2)$$

From the previous calculations:

$$\Leftrightarrow \mathbb{V}ar_t(r_{c,t+1}) = \mathbb{E}_t([\kappa_1 A_1 \varphi_e \sigma \epsilon_{t+1} + \sigma \eta_{t+1}]^2)$$

Since the shocks are independent, the cross products are zero. Hence:

$$\boxed{\mathbb{V}ar_t(r_{c,t+1}) = \sigma^2(1 + \kappa_1^2 A_1^2 \varphi_e^2)}$$

Secondly, for $\mathbb{C}ov_t(\ln(M_{t+1}), r_{c,t+1})$:

$$\mathbb{C}ov_t(\ln(M_{t+1}), r_{c,t+1}) = \mathbb{E}_t([r_{c,t+1} - \mathbb{E}_t(r_{c,t+1})][\ln(M_{t+1}) - \mathbb{E}_t(\ln(M_{t+1}))])$$

From the previous calculations:

$$\Leftrightarrow \mathbb{C}ov_t(\ln(M_{t+1}), r_{c,t+1}) = \mathbb{E}_t\begin{pmatrix} [\kappa_1 A_1 \varphi_e \sigma \epsilon_{t+1} + \sigma \eta_{t+1}] \\ [-\frac{\theta}{\psi}\sigma \eta_{t+1} + (\theta - 1)(\kappa_1 A_1 \varphi_e \sigma \epsilon_{t+1} + \sigma \eta_{t+1})] \end{pmatrix}$$

$$\Leftrightarrow \mathbb{C}ov_t(\ln(M_{t+1}), r_{c,t+1}) = \mathbb{E}_t\begin{pmatrix} [\kappa_1 A_1 \varphi_e \sigma \epsilon_{t+1} + \sigma \eta_{t+1}] \\ [-\frac{\theta}{\psi}\sigma \eta_{t+1} + (\theta - 1)\kappa_1 A_1 \varphi_e \sigma \epsilon_{t+1} + (\theta - 1)\sigma \eta_{t+1})] \end{pmatrix}$$

Since the shocks are independent, the cross products are zero. Hence:

$$\Leftrightarrow \mathbb{C}ov_t(\ln(M_{t+1}), r_{c,t+1}) = \kappa_1 A_1 \varphi_e \sigma (\theta - 1)\kappa_1 A_1 \varphi_e \sigma + \sigma^2(\theta - 1 - \tfrac{\theta}{\psi})$$

$$\Leftrightarrow \mathbb{C}ov_t(\ln(M_{t+1}), r_{c,t+1}) = (\theta - 1)\kappa_1^2 A_1^2 \varphi_e^2 \sigma^2 + +\sigma^2(\theta - 1 - \tfrac{\theta}{\psi})$$

Putting it all together:

$$\mathbb{E}_t(r_{c,t+1} - r_f) + \frac{1}{2}\mathbb{V}ar_t(r_{c,t+1}) = -\mathbb{C}ov_t(\ln(M_{t+1}), r_{c,t+1})$$

Hence the conditional risk premium on the consumption claim is:

$$\boxed{\mathbb{E}_t(r_{c,t+1} - r_f) = \left[-(\theta - 1)\kappa_1^2 A_1^2 \varphi_e^2 \sigma^2 - \sigma^2(\theta - 1 - \tfrac{\theta}{\psi}) - \tfrac{1}{2}\sigma^2(1 + \kappa_1^2 A_1^2 \varphi_e^2)\right]}$$

Or without Jensen's correction term:

$$\boxed{\mathbb{E}_t(r_{c,t+1} - r_f) + \frac{1}{2}\mathbb{V}ar_t(r_{c,t+1}) = -(\theta - 1)\kappa_1^2 A_1^2 \varphi_e^2 \sigma^2 - \sigma^2(\theta - 1 - \frac{\theta}{\psi})}$$

A's (Dividend)

Similarly, I conjecture for the log price-dividend ratio $z_{m,t} \equiv \ln(\frac{P_t}{D_t})$:

$$z_{m,t} = A_{0,m} + A_{1,m} x_t$$

Using the Campbell-Shiller approximation for returns:

$$r_{m,t+1} = \kappa_{0,m} + \kappa_{1,m} z_{m,t+1} - z_{m,t} + \Delta d_{t+1}$$

The Euler equation is:

$$\mathbb{E}_t \left[\beta^\theta \, \Delta C_{t+1}^{-\frac{\theta}{\psi}} (1 + R_{w,t+1})^{\theta - 1} (1 + R_{i,t+1}) \right] = 1, \quad \forall i$$

In particular, the equation is satisfied for $1 + R_{i,t+1} = 1 + R_{m,t+1}$. Hence:

$$\mathbb{E}_t \left[\beta^\theta \, \Delta C_{t+1}^{-\frac{\theta}{\psi}} (1 + R_{w,t+1})^{\theta - 1} (1 + R_{m,t+1}) \right] = 1$$

$$\Leftrightarrow \mathbb{E}_t \left[e^{\ln\left(\beta^\theta \, \Delta C_{t+1}^{-\frac{\theta}{\psi}} (1+R_{w,t+1})^{\theta-1} (1+R_{m,t+1}) \right)} \right] = 1$$

$$\Leftrightarrow \mathbb{E}_t \left[e^{\theta \ln(\beta) - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1) r_{w,t+1} + r_{m,t+1}} \right] = 1$$

Plugging in Δc_{t+1} and the Campbell-Shiller approximations:

$$\Leftrightarrow \mathbb{E}_t \left[e^{\theta \ln(\beta) - \frac{\theta}{\psi}(\mu + x_t + \sigma \eta_{t+1}) + (\theta - 1)(\kappa_0 + \kappa_1 z_{t+1} - z_t + \Delta c_{t+1}) + \kappa_{0,m} + \kappa_{1,m} z_{m,t+1} - z_{m,t} + \Delta d_{t+1}} \right] = 1$$

Plugging in Δc_{t+1} again, z_t, z_{t+1}, $z_{m,t}, z_{m,t+1}$, and Δd_{t+1}:

$$\Leftrightarrow \mathbb{E}_t \left[e^{\begin{bmatrix} \theta \ln(\beta) - \frac{\theta}{\psi}\mu - \frac{\theta}{\psi} x_t - \frac{\theta}{\psi}\sigma\eta_{t+1} + (\theta-1)\kappa_0 + (\theta-1)\kappa_1(A_0 + A_1 x_{t+1}) \\ -(\theta-1)(A_0 + A_1 x_t) + (\theta-1)(\mu + x_t + \sigma\eta_{t+1}) + \kappa_{0,m} + \\ \kappa_{1,m}(A_{0,m} + A_{1,m} x_{t+1}) - A_{0,m} - A_{1,m} x_t + \mu_d + \phi x_t + \varphi_d \sigma u_{t+1} \end{bmatrix}} \right] = 1$$

Plugging in x_{t+1}:

$$\Leftrightarrow \mathbb{E}_t\left[e^{\begin{bmatrix} \theta\ln(\beta) - \frac{\theta}{\psi}\mu - \frac{\theta}{\psi}x_t - \frac{\theta}{\psi}\sigma\eta_{t+1} + (\theta-1)\kappa_0 \\ +(\theta-1)\kappa_1(A_0 + A_1[\rho x_t + \varphi_e\sigma\epsilon_{t+1}]) \\ -(\theta-1)(A_0 + A_1 x_t) + (\theta-1)(\mu + x_t + \sigma\eta_{t+1}) + \kappa_{0,m} + \\ \kappa_{1,m}(A_{0,m} + A_{1,m}[\rho x_t + \varphi_e\sigma\epsilon_{t+1}]) \\ -A_{0,m} - A_{1,m}x_t + \mu_d + \phi x_t + \varphi_d\sigma u_{t+1} \end{bmatrix}}\right] = 1$$

$$\Leftrightarrow \mathbb{E}_t\left[e^{\begin{bmatrix} \theta\ln(\beta) - \frac{\theta}{\psi}\mu - \frac{\theta}{\psi}x_t - \frac{\theta}{\psi}\sigma\eta_{t+1} + (\theta-1)\kappa_0 \\ +(\theta-1)\kappa_1 A_0 + (\theta-1)\kappa_1 A_1\rho x_t + (\theta-1)\kappa_1 A_1\varphi_e\sigma\epsilon_{t+1} \\ -(\theta-1)A_0 - (\theta-1)A_1 x_t + (\theta-1)\mu + (\theta-1)x_t + (\theta-1)\sigma\eta_{t+1} \\ +\kappa_{0,m} + \kappa_{1,m}A_{0,m} + \kappa_{1,m}A_{1,m}\rho x_t + \kappa_{1,m}A_{1,m}\varphi_e\sigma\epsilon_{t+1} \\ -A_{0,m} - A_{1,m}x_t + \mu_d + \phi x_t + \varphi_d\sigma u_{t+1} \end{bmatrix}}\right] = 1$$

Since $\epsilon_{t+1}, \eta_{t+1}, u_{t+1} \sim N(0,1)$ and are uncorrelated:

$$\begin{bmatrix} \theta\ln(\beta) - \frac{\theta}{\psi}\mu - \frac{\theta}{\psi}x_t + (\theta-1)\kappa_0 + (\theta-1)\kappa_1 A_0 + (\theta-1)\kappa_1 A_1\rho x_t \\ -(\theta-1)A_0 - (\theta-1)A_1 x_t + (\theta-1)\mu + (\theta-1)x_t + \kappa_{0,m} + \kappa_{1,m}A_{0,m} \\ +\kappa_{1,m}A_{1,m}\rho x_t - A_{0,m} - A_{1,m}x_t + \mu_d + \phi x_t \\ +\frac{1}{2}\sigma^2[(\theta - 1 - \frac{\theta}{\psi})^2 + ((\theta-1)\kappa_1 A_1\varphi_e + \kappa_{1,m}A_{1,m}\varphi_e)^2 + \varphi_d^2] \end{bmatrix} = 0$$

Since this equation must be satisfied for all values of x_t:

$$-\frac{\theta}{\psi} + (\theta-1)\kappa_1 A_1\rho - (\theta-1)A_1 + (\theta-1) + \kappa_{1,m}A_{1,m}\rho - A_{1,m} + \phi = 0$$

From the previous part, $-\frac{\theta}{\psi} + \theta\kappa_1 A_1\rho - \theta A_1 + \theta = 0$. Therefore:

$$\Leftrightarrow -\kappa_1 A_1\rho + A_1 - 1 + \kappa_{1,m}A_{1,m}\rho - A_{1,m} + \phi = 0$$

$$\Leftrightarrow A_1(1 - \kappa_1\rho) - 1 + \kappa_{1,m}A_{1,m}\rho - A_{1,m} + \phi = 0$$

From the previous part, $A_1 = \frac{1-\frac{1}{\psi}}{1-\kappa_1\rho}$. Hence:

$$\Leftrightarrow \frac{1-\frac{1}{\psi}}{1-\kappa_1\rho}(1-\kappa_1\rho) - 1 + \kappa_{1,m}A_{1,m}\rho - A_{1,m} + \phi = 0$$

$$\Leftrightarrow -\frac{1}{\psi} + \kappa_{1,m}A_{1,m}\rho - A_{1,m} + \phi = 0$$

$$\Leftrightarrow A_{1,m} - \kappa_{1,m}A_{1,m}\rho = \phi - \frac{1}{\psi}$$

$$\Leftrightarrow A_{1,m}(1-\kappa_{1,m}\rho) = \phi - \frac{1}{\psi}$$

Hence:

$$\boxed{A_{1,m} = \frac{\phi - \frac{1}{\psi}}{1-\kappa_{1,m}\rho}}$$

And $A_{0,m}$ absorbs the constant terms:

$$\begin{bmatrix} \theta\ln(\beta) - \frac{\theta}{\psi}\mu + (\theta-1)\kappa_0 + (\theta-1)\kappa_1 A_0 \\ -(\theta-1)A_0 + (\theta-1)\mu + \kappa_{0,m} + \kappa_{1,m}A_{0,m} - A_{0,m} + \mu_d \\ +\frac{1}{2}\sigma^2[(\theta-1-\frac{\theta}{\psi})^2 + ((\theta-1)\kappa_1 A_1\varphi_e + \kappa_{1,m}A_{1,m}\varphi_e)^2 + \varphi_d^2] \end{bmatrix} = 0$$

$$\Leftrightarrow \begin{bmatrix} \theta\ln(\beta) - \frac{\theta}{\psi}\mu + (\theta-1)\kappa_0 + (\theta-1)\kappa_1 A_0 \\ -(\theta-1)A_0 + (\theta-1)\mu + \kappa_{0,m} + \mu_d \\ +\frac{1}{2}\sigma^2[(\theta-1-\frac{\theta}{\psi})^2 + ((\theta-1)\kappa_1 A_1\varphi_e + \kappa_{1,m}A_{1,m}\varphi_e)^2 + \varphi_d^2] \end{bmatrix} = A_{0,m}(1-\kappa_{1,m})$$

$$\Leftrightarrow A_{0,m}(1-\kappa_{1,m}) = \begin{bmatrix} \theta\ln(\beta) + \mu(\theta-1-\frac{\theta}{\psi}) + (\theta-1)[\kappa_0 + A_0(\kappa_1-1)] + \kappa_{0,m} + \mu_d \\ +\frac{1}{2}\sigma^2[(\theta-1-\frac{\theta}{\psi})^2 + ((\theta-1)\kappa_1 A_1\varphi_e + \kappa_{1,m}A_{1,m}\varphi_e)^2 + \varphi_d^2] \end{bmatrix}$$

Finally:

$$\boxed{A_{0,m} = \frac{\begin{bmatrix} \theta\ln(\beta) + \mu(\theta-1-\frac{\theta}{\psi}) + (\theta-1)[\kappa_0 + A_0(\kappa_1-1)] + \kappa_{0,m} + \mu_d \\ +\frac{1}{2}\sigma^2[(\theta-1-\frac{\theta}{\psi})^2 + ((\theta-1)\kappa_1 A_1\varphi_e + \kappa_{1,m}A_{1,m}\varphi_e)^2 + \varphi_d^2] \end{bmatrix}}{1-\kappa_{1,m}}}$$

Risk Premium (Dividend)

I already derived $\ln(M_{t+1}) - \mathbb{E}_t(\ln(M_{t+1}))$. Now, I calculate:

$$r_{m,t+1} - \mathbb{E}_t(r_{m,t+1}) = \kappa_{0,m} + \kappa_{1,m} z_{m,t+1} - z_{m,t} + \Delta d_{t+1} - \mathbb{E}_t(\kappa_{0,m} + \kappa_{1,m} z_{m,t+1} - z_{m,t} + \Delta d_{t+1})$$

Plugging in Δd_{t+1} and $z_{m,t+1}$:

$$\Leftrightarrow r_{m,t+1} - \mathbb{E}_t(r_{m,t+1}) = \begin{bmatrix} \kappa_{1,m}(A_{0,m} + A_{1,m} x_{t+1}) + \mu_d + \phi x_t + \varphi_d \sigma u_{t+1} \\ -\mathbb{E}_t(\kappa_{1,m}(A_{0,m} + A_{1,m} x_{t+1}) + \mu_d + \phi x_t + \varphi_d \sigma u_{t+1}) \end{bmatrix}$$

$$\Leftrightarrow r_{m,t+1} - \mathbb{E}_t(r_{m,t+1}) = \begin{bmatrix} \kappa_{1,m} A_{1,m} x_{t+1} + \varphi_d \sigma u_{t+1} \\ -\mathbb{E}_t(\kappa_{1,m} A_{1,m} x_{t+1}) \end{bmatrix}$$

Plugging in the state variable in $t+1$:

$$\Leftrightarrow r_{m,t+1} - \mathbb{E}_t(r_{m,t+1}) = \begin{bmatrix} \kappa_{1,m} A_{1,m}(\rho x_t + \varphi_e \sigma \epsilon_{t+1}) + \varphi_d \sigma u_{t+1} \\ -\mathbb{E}_t(\kappa_{1,m} A_{1,m}(\rho x_t + \varphi_e \sigma \epsilon_{t+1})) \end{bmatrix}$$

$$\Leftrightarrow r_{m,t+1} - \mathbb{E}_t(r_{m,t+1}) = \left[\kappa_{1,m} A_{1,m} \varphi_e \sigma \epsilon_{t+1} + \varphi_d \sigma u_{t+1}\right]$$

To combine these two derivations, I use that $\mathbb{E}_t(r_{m,t+1} - r_f) + \frac{1}{2}\mathbb{V}ar_t(r_{m,t+1}) = -\mathbb{C}ov_t(\ln(M_{t+1}), r_{m,t+1})$. First, for $\mathbb{V}ar_t(r_{m,t+1})$:

$$\mathbb{V}ar_t(r_{m,t+1}) = \mathbb{E}_t([r_{m,t+1} - \mathbb{E}_t(r_{m,t+1})]^2)$$

From the previous calculations:

$$\Leftrightarrow \mathbb{V}ar_t(r_{m,t+1}) = \mathbb{E}_t([\kappa_{1,m} A_{1,m} \varphi_e \sigma \epsilon_{t+1} + \varphi_d \sigma u_{t+1}]^2)$$

Since the shocks are independent, the cross products are zero. Hence:

$$\boxed{\mathbb{V}ar_t(r_{m,t+1}) = \sigma^2(\varphi_d^2 + \kappa_{1,m}^2 A_{1,m}^2 \varphi_e^2)}$$

Secondly, for $\mathbb{C}ov_t(\ln(M_{t+1}), r_{m,t+1})$:

$$\mathbb{C}ov_t(\ln(M_{t+1}), r_{m,t+1}) = \mathbb{E}_t([r_{m,t+1} - \mathbb{E}_t(r_{m,t+1})][\ln(M_{t+1}) - \mathbb{E}_t(\ln(M_{t+1}))])$$

From the previous calculations:

$$\Leftrightarrow \mathbb{C}ov_t(\ln(M_{t+1}), r_{m,t+1}) = \mathbb{E}_t \begin{pmatrix} [\kappa_{1,m} A_{1,m} \varphi_e \sigma \epsilon_{t+1} + \varphi_d \sigma u_{t+1}] \\ [-\frac{\theta}{\psi}\sigma \eta_{t+1} + (\theta - 1)(\kappa_1 A_1 \varphi_e \sigma \epsilon_{t+1} + \sigma \eta_{t+1})] \end{pmatrix}$$

$$\Leftrightarrow \mathbb{C}ov_t(\ln(M_{t+1}), r_{m,t+1}) = \mathbb{E}_t \begin{pmatrix} [\kappa_{1,m}A_{1,m}\varphi_e\sigma\epsilon_{t+1} + \varphi_d\sigma u_{t+1}] \\ [-\frac{\theta}{\psi}\sigma\eta_{t+1} + (\theta-1)\kappa_1 A_1\varphi_e\sigma\epsilon_{t+1} + (\theta-1)\sigma\eta_{t+1})] \end{pmatrix}$$

Since the shocks are independent, the cross products are zero. Hence:

$$\Leftrightarrow \mathbb{C}ov_t(\ln(M_{t+1}), r_{m,t+1}) = \kappa_{1,m}A_{1,m}\varphi_e\sigma(\theta-1)\kappa_1 A_1\varphi_e\sigma$$

$$\Leftrightarrow \mathbb{C}ov_t(\ln(M_{t+1}), r_{m,t+1}) = \kappa_{1,m}A_{1,m}(\theta-1)\kappa_1 A_1\varphi_e^2\sigma^2$$

Putting it all together:

$$\mathbb{E}_t(r_{m,t+1} - r_f) + \frac{1}{2}\mathbb{V}ar_t(r_{m,t+1}) = -\mathbb{C}ov_t(\ln(M_{t+1}), r_{m,t+1})$$

Hence the conditional risk premium on the dividend claim is:

$$\boxed{\mathbb{E}_t(r_{m,t+1} - r_f) = -\kappa_{1,m}A_{1,m}(\theta-1)\kappa_1 A_1\varphi_e^2\sigma^2 - \frac{1}{2}\sigma^2(\varphi_d^2 + \kappa_{1,m}^2 A_{1,m}^2\varphi_e^2)}$$

Or without Jensen's correction term:

$$\boxed{\mathbb{E}_t(r_{m,t+1} - r_f) + \frac{1}{2}\mathbb{V}ar_t(r_{m,t+1}) = -\kappa_{1,m}A_{1,m}(\theta-1)\kappa_1 A_1\varphi_e^2\sigma^2}$$

Unconditional Market Return Volatility

I calculate:

$$r_{m,t+1} - \mathbb{E}(r_{m,t+1}) = \kappa_{0,m} + \kappa_{1,m}z_{m,t+1} - z_{m,t} + \Delta d_{t+1} - \mathbb{E}(\kappa_{0,m} + \kappa_{1,m}z_{m,t+1} - z_{m,t} + \Delta d_{t+1})$$

Plugging in Δd_{t+1}, $z_{m,t}$, and $z_{m,t+1}$:

$$\Leftrightarrow r_{m,t+1} - \mathbb{E}(r_{m,t+1}) = \begin{bmatrix} \kappa_{1,m}(A_{0,m} + A_{1,m}x_{t+1}) - A_{0,m} - A_{1,m}x_t + \mu_d + \phi x_t + \varphi_d\sigma u_{t+1} \\ -\mathbb{E}(\kappa_{1,m}(A_{0,m} + A_{1,m}x_{t+1}) - A_{0,m} - A_{1,m}x_t + \mu_d + \phi x_t + \varphi_d\sigma u_{t+1}) \end{bmatrix}$$

$$\Leftrightarrow r_{m,t+1} - \mathbb{E}(r_{m,t+1}) = \begin{bmatrix} \kappa_{1,m}A_{1,m}x_{t+1} - A_{1,m}x_t + \phi x_t + \varphi_d\sigma u_{t+1} \\ -\mathbb{E}(\kappa_{1,m}A_{1,m}x_{t+1} - A_{1,m}x_t + \phi x_t + \varphi_d\sigma u_{t+1}) \end{bmatrix}$$

Plugging in the state variable in $t+1$:

$$\Leftrightarrow r_{m,t+1} - \mathbb{E}(r_{m,t+1}) = \begin{bmatrix} \kappa_{1,m}A_{1,m}(\rho x_t + \varphi_e\sigma\epsilon_{t+1}) - A_{1,m}x_t + \phi x_t + \varphi_d\sigma u_{t+1} \\ -\mathbb{E}(\kappa_{1,m}A_{1,m}(\rho x_t + \varphi_e\sigma\epsilon_{t+1}) - A_{1,m}x_t + \phi x_t + \varphi_d\sigma u_{t+1}) \end{bmatrix}$$

$$\Leftrightarrow r_{m,t+1} - \mathbb{E}(r_{m,t+1}) = \begin{bmatrix} \kappa_{1,m}A_{1,m}\rho x_t + \kappa_{1,m}A_{1,m}\varphi_e\sigma\epsilon_{t+1} - A_{1,m}x_t + \phi x_t + \varphi_d\sigma u_{t+1} \\ -\mathbb{E}(\kappa_{1,m}A_{1,m}\rho x_t + \kappa_{1,m}A_{1,m}\varphi_e\sigma\epsilon_{t+1} - A_{1,m}x_t + \phi x_t + \varphi_d\sigma u_{t+1}) \end{bmatrix}$$

$$\Leftrightarrow r_{m,t+1} - \mathbb{E}(r_{m,t+1}) = \begin{bmatrix} \kappa_{1,m}A_{1,m}\rho x_t + \kappa_{1,m}A_{1,m}\varphi_e\sigma\epsilon_{t+1} - A_{1,m}x_t + \phi x_t + \varphi_d\sigma u_{t+1} \\ -\mathbb{E}(\kappa_{1,m}A_{1,m}\rho x_t - A_{1,m}x_t + \phi x_t) \end{bmatrix}$$

Notice that $\mathbb{E}(x_t) = 0$. Therefore:

$$\Leftrightarrow r_{m,t+1} - \mathbb{E}(r_{m,t+1}) = \kappa_{1,m}A_{1,m}\rho x_t + \kappa_{1,m}A_{1,m}\varphi_e\sigma\epsilon_{t+1} - A_{1,m}x_t + \phi x_t + \varphi_d\sigma u_{t+1}$$

$$\Leftrightarrow r_{m,t+1} - \mathbb{E}(r_{m,t+1}) = x_t(\kappa_{1,m}A_{1,m}\rho - A_{1,m} + \phi) + \kappa_{1,m}A_{1,m}\varphi_e\sigma\epsilon_{t+1} + \varphi_d\sigma u_{t+1}$$

$$\Leftrightarrow r_{m,t+1} - \mathbb{E}(r_{m,t+1}) = x_t(A_{1,m}(\kappa_{1,m}\rho - 1) + \phi) + \kappa_{1,m}A_{1,m}\varphi_e\sigma\epsilon_{t+1} + \varphi_d\sigma u_{t+1}$$

From the previous part:

$$A_{1,m} = \frac{\phi - \frac{1}{\psi}}{1 - \kappa_{1,m}\rho} = \frac{\frac{1}{\psi} - \phi}{\kappa_{1,m}\rho - 1}$$

Hence:

$$\Leftrightarrow r_{m,t+1} - \mathbb{E}(r_{m,t+1}) = x_t(\frac{\frac{1}{\psi} - \phi}{\kappa_{1,m}\rho - 1}(\kappa_{1,m}\rho - 1) + \phi) + \kappa_{1,m}A_{1,m}\varphi_e\sigma\epsilon_{t+1} + \varphi_d\sigma u_{t+1}$$

$$\Leftrightarrow r_{m,t+1} - \mathbb{E}(r_{m,t+1}) = \frac{x_t}{\psi} + \kappa_{1,m}A_{1,m}\varphi_e\sigma\epsilon_{t+1} + \varphi_d\sigma u_{t+1}$$

Consequently, the unconditional variance is:

$$\mathbb{V}ar(r_{m,t+1}) = \mathbb{E}([r_{m,t+1} - \mathbb{E}(r_{m,t+1})]^2)$$

From the previous calculations:

$$\Leftrightarrow \mathbb{V}ar(r_{m,t+1}) = \mathbb{E}\left(\left[\tfrac{x_t}{\psi} + \kappa_{1,m}A_{1,m}\varphi_e\sigma\epsilon_{t+1} + \varphi_d\sigma u_{t+1}\right]^2\right)$$

Since the shocks are independent, the cross products are zero. Hence:

$$\boxed{\mathbb{V}ar(r_{m,t+1}) = \frac{\mathbb{V}ar(x_t)}{\psi^2} + \sigma^2(\kappa_{1,m}^2A_{1,m}^2\varphi_e^2 + \varphi_d^2)}$$

The unconditional variance of $z_{m,t}$ (the price–dividend ratio for the market portfolio) is:

$$\mathbb{V}ar(z_{m,t}) = A_{1,m}^2\mathbb{V}ar(x_t)$$

Finally:

$$\mathbb{V}ar_{t+1}(r_{m,t+2}) - \mathbb{E}_t(\mathbb{V}ar_{t+1}(r_{m,t+2})) = \sigma^2(\varphi_d^2 + \kappa_{1,m}^2A_{1,m}^2\varphi_e^2) - \mathbb{E}_t(\sigma^2(\varphi_d^2 + \kappa_{1,m}^2A_{1,m}^2\varphi_e^2))$$

Consequently:

$$\boxed{\mathbb{V}ar_{t+1}(r_{m,t+2}) - \mathbb{E}_t(\mathbb{V}ar_{t+1}(r_{m,t+2})) = 0}$$

Risk-Free Rate

For the risk-free rate:

$$1 + R_{f,t+1} = \frac{1}{\mathbb{E}_t(M_{t+1})}$$

$$\Leftrightarrow r_{f,t+1} = -\ln(\mathbb{E}_t(M_{t+1}))$$

$$\Leftrightarrow r_{f,t+1} = -\ln(\mathbb{E}_t(e^{\ln(M_{t+1})}))$$

$$\Leftrightarrow r_{f,t+1} = -\ln(\mathbb{E}_t(e^{\theta\ln(\beta) - \frac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)r_{w,t+1}}))$$

Because shocks are normally distributed:

$$\Leftrightarrow r_{f,t+1} = -\ln(e^{\theta\ln(\beta) - \frac{\theta}{\psi}\mathbb{E}_t(\Delta c_{t+1}) + (\theta-1)\mathbb{E}_t(r_{w,t+1}) + \frac{1}{2}\mathbb{V}ar_t(-\frac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)r_{w,t+1})})$$

$$\Leftrightarrow r_{f,t+1} = -\theta\ln(\beta) + \frac{\theta}{\psi}\mathbb{E}_t(\Delta c_{t+1}) - (\theta-1)\mathbb{E}_t(r_{w,t+1}) - \frac{1}{2}\mathbb{V}ar_t(-\frac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)r_{w,t+1})$$

$$\Leftrightarrow 0 = r_{f,t+1} + \theta\ln(\beta) - \frac{\theta}{\psi}\mathbb{E}_t(\Delta c_{t+1}) + (\theta-1)\mathbb{E}_t(r_{w,t+1}) + \frac{1}{2}\mathbb{V}ar_t(-\frac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)r_{w,t+1}) \tag{1}$$

Similarly, for $r_{w,t+1}$:

$$\mathbb{E}_t\left[e^{\theta\ln(\beta) - \frac{\theta}{\psi}\Delta c_{t+1} + \theta r_{w,t+1}}\right] = 1$$

$$\Leftrightarrow \ln\left(\mathbb{E}_t\left[e^{\theta\ln(\beta) - \frac{\theta}{\psi}\Delta c_{t+1} + \theta r_{w,t+1}}\right]\right) = 0$$

Because shocks are normally distributed:

$$\Leftrightarrow \ln(e^{\theta\ln(\beta) - \frac{\theta}{\psi}\mathbb{E}_t(\Delta c_{t+1}) + \theta\mathbb{E}_t(r_{w,t+1}) + \frac{1}{2}\mathbb{V}ar_t(-\frac{\theta}{\psi}\Delta c_{t+1} + \theta r_{w,t+1})}) = 0$$

$$\Leftrightarrow \theta\ln(\beta) - \frac{\theta}{\psi}\mathbb{E}_t(\Delta c_{t+1}) + \theta\mathbb{E}_t(r_{w,t+1}) + \frac{1}{2}\mathbb{V}ar_t(-\frac{\theta}{\psi}\Delta c_{t+1} + \theta r_{w,t+1}) = 0$$

$$\Leftrightarrow \ln(\beta) - \frac{1}{\psi}\mathbb{E}_t(\Delta c_{t+1}) + \mathbb{E}_t(r_{w,t+1}) + \frac{\theta}{2}\mathbb{V}ar_t(-\frac{1}{\psi}\Delta c_{t+1} + r_{w,t+1}) = 0$$

$$\Leftrightarrow (\theta-1)\ln(\beta) - \frac{(\theta-1)}{\psi}\mathbb{E}_t(\Delta c_{t+1}) + (\theta-1)\mathbb{E}_t(r_{w,t+1}) + \frac{\theta(\theta-1)}{2}\mathbb{V}ar_t(-\frac{1}{\psi}\Delta c_{t+1} + r_{w,t+1}) = 0 \tag{2}$$

From 1 and 2, I have:

$$\begin{gathered} r_{f,t+1} + \theta\ln(\beta) - \tfrac{\theta}{\psi}\mathbb{E}_t(\Delta c_{t+1}) + (\theta-1)\mathbb{E}_t(r_{w,t+1}) + \tfrac{1}{2}\mathbb{V}ar_t(-\tfrac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)r_{w,t+1}) = \\ (\theta-1)\ln(\beta) - \tfrac{(\theta-1)}{\psi}\mathbb{E}_t(\Delta c_{t+1}) + (\theta-1)\mathbb{E}_t(r_{w,t+1}) + \tfrac{\theta(\theta-1)}{2}\mathbb{V}ar_t(-\tfrac{1}{\psi}\Delta c_{t+1} + r_{w,t+1}) \end{gathered}$$

$$\Leftrightarrow r_{f,t+1} = -\ln(\beta) + \tfrac{1}{\psi}\mathbb{E}_t(\Delta c_{t+1}) + \tfrac{\theta(\theta-1)}{2}\mathbb{V}ar_t(-\tfrac{1}{\psi}\Delta c_{t+1} + r_{w,t+1}) - \tfrac{1}{2}\mathbb{V}ar_t(-\tfrac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)r_{w,t+1})$$

Plugging in Δc_{t+1} and $r_{w,t+1}$:

$$\Leftrightarrow r_{f,t+1} = \begin{bmatrix} -\ln(\beta) + \tfrac{1}{\psi}\mathbb{E}_t(\Delta c_{t+1}) + \tfrac{\theta(\theta-1)}{2}\mathbb{V}ar_t(-\tfrac{1}{\psi}(\mu + x_t + \sigma\eta_{t+1}) + \kappa_0 + \kappa_1 z_{t+1} - z_t + \Delta c_{t+1}) \\ -\tfrac{1}{2}\mathbb{V}ar_t(-\tfrac{\theta}{\psi}(\mu + x_t + \sigma\eta_{t+1}) + (\theta-1)(\kappa_0 + \kappa_1 z_{t+1} - z_t + \Delta c_{t+1})) \end{bmatrix}$$

$$\Leftrightarrow r_{f,t+1} = \begin{bmatrix} -\ln(\beta) + \tfrac{1}{\psi}\mathbb{E}_t(\Delta c_{t+1}) + \tfrac{\theta(\theta-1)}{2}\mathbb{V}ar_t(-\tfrac{1}{\psi}\sigma\eta_{t+1} + \kappa_1 z_{t+1} + \Delta c_{t+1}) \\ -\tfrac{1}{2}\mathbb{V}ar_t(-\tfrac{\theta}{\psi}\sigma\eta_{t+1} + (\theta-1)(\kappa_1 z_{t+1} + \Delta c_{t+1})) \end{bmatrix}$$

Plugging in Δc_{t+1} and z_{t+1}:

$$\Leftrightarrow r_{f,t+1} = \begin{bmatrix} -\ln(\beta) + \tfrac{1}{\psi}\mathbb{E}_t(\Delta c_{t+1}) \\ +\tfrac{\theta(\theta-1)}{2}\mathbb{V}ar_t(-\tfrac{1}{\psi}\sigma\eta_{t+1} + \kappa_1(A_0 + A_1 x_{t+1}) + \mu + x_t + \sigma\eta_{t+1}) \\ -\tfrac{1}{2}\mathbb{V}ar_t(-\tfrac{\theta}{\psi}\sigma\eta_{t+1} + (\theta-1)(\kappa_1(A_0 + A_1 x_{t+1}) + \mu + x_t + \sigma\eta_{t+1})) \end{bmatrix}$$

$$\Leftrightarrow r_{f,t+1} = \begin{bmatrix} -\ln(\beta) + \tfrac{1}{\psi}\mathbb{E}_t(\Delta c_{t+1}) + \tfrac{\theta(\theta-1)}{2}\mathbb{V}ar_t((1-\tfrac{1}{\psi})\sigma\eta_{t+1} + \kappa_1 A_1 x_{t+1}) \\ -\tfrac{1}{2}\mathbb{V}ar_t((\theta-1-\tfrac{\theta}{\psi})\sigma\eta_{t+1} + (\theta-1)(\kappa_1 A_1 x_{t+1})) \end{bmatrix}$$

Plugging in x_{t+1}:

$$\Leftrightarrow r_{f,t+1} = \begin{bmatrix} -\ln(\beta) + \tfrac{1}{\psi}\mathbb{E}_t(\Delta c_{t+1}) \\ +\tfrac{\theta(\theta-1)}{2}\mathbb{V}ar_t((1-\tfrac{1}{\psi})\sigma\eta_{t+1} + \kappa_1 A_1(\rho x_t + \varphi_e\sigma\epsilon_{t+1})) \\ -\tfrac{1}{2}\mathbb{V}ar_t((\theta-1-\tfrac{\theta}{\psi})\sigma\eta_{t+1} + (\theta-1)(\kappa_1 A_1(\rho x_t + \varphi_e\sigma\epsilon_{t+1}))) \end{bmatrix}$$

$$\Leftrightarrow r_{f,t+1} = \begin{bmatrix} -\ln(\beta) + \tfrac{1}{\psi}\mathbb{E}_t(\Delta c_{t+1}) \\ +\tfrac{\theta(\theta-1)}{2}\mathbb{V}ar_t((1-\tfrac{1}{\psi})\sigma\eta_{t+1} + \kappa_1 A_1\varphi_e\sigma\epsilon_{t+1}) \\ -\tfrac{1}{2}\mathbb{V}ar_t((\theta-1-\tfrac{\theta}{\psi})\sigma\eta_{t+1} + (\theta-1)(\kappa_1 A_1\varphi_e\sigma\epsilon_{t+1})) \end{bmatrix}$$

Since $\epsilon_{t+1}, \eta_{t+1} \sim N(0,1)$ and are uncorrelated:

$$\Leftrightarrow r_{f,t+1} = \begin{bmatrix} -\ln(\beta) + \frac{1}{\psi}\mathbb{E}_t(\Delta c_{t+1}) + \frac{\theta(\theta-1)}{2}((1-\frac{1}{\psi})^2\sigma^2 + (\kappa_1 A_1 \varphi_e)^2\sigma^2) \\ -\frac{1}{2}((\theta - 1 - \frac{\theta}{\psi})^2\sigma^2 + (\theta-1)^2(\kappa_1 A_1 \varphi_e)^2\sigma^2) \end{bmatrix}$$

$$\Leftrightarrow r_{f,t+1} = \begin{bmatrix} -\ln(\beta) + \frac{1}{\psi}\mathbb{E}_t(\Delta c_{t+1}) \\ +\frac{\theta(\theta-1)}{2}(1-\frac{1}{\psi})^2\sigma^2 + \frac{\theta-1}{2}(\kappa_1 A_1 \varphi_e)^2\sigma^2 - \frac{1}{2}(\theta - 1 - \frac{\theta}{\psi})^2\sigma^2 \end{bmatrix}$$

$$\Leftrightarrow r_{f,t+1} = \begin{bmatrix} -\ln(\beta) + \frac{1}{\psi}\mathbb{E}_t(\Delta c_{t+1}) \\ +\frac{1}{2}(\theta^2 - \theta)(1 - \frac{2}{\psi} + \frac{1}{\psi^2})\sigma^2 + \frac{\theta-1}{2}(\kappa_1 A_1 \varphi_e)^2\sigma^2 \\ -\frac{1}{2}(\theta^2 - 2\theta + 1 - \frac{2\theta^2}{\psi} + \frac{\theta^2}{\psi^2} + \frac{2\theta}{\psi})\sigma^2 \end{bmatrix}$$

$$\Leftrightarrow r_{f,t+1} = \begin{bmatrix} -\ln(\beta) + \frac{1}{\psi}\mathbb{E}_t(\Delta c_{t+1}) \\ +\frac{1}{2}(\theta^2 - \theta - \frac{2\theta^2}{\psi} + \frac{2\theta}{\psi} + \frac{\theta^2}{\psi^2} - \frac{\theta}{\psi^2})\sigma^2 + \frac{\theta-1}{2}(\kappa_1 A_1 \varphi_e)^2\sigma^2 \\ -\frac{1}{2}(\theta^2 - 2\theta + 1 - \frac{2\theta^2}{\psi} + \frac{\theta^2}{\psi^2} + \frac{2\theta}{\psi})\sigma^2 \end{bmatrix}$$

$$\Leftrightarrow r_{f,t+1} = \begin{bmatrix} -\ln(\beta) + \frac{1}{\psi}\mathbb{E}_t(\Delta c_{t+1}) \\ +\frac{1}{2}(\theta - 1 - \frac{\theta}{\psi^2})\sigma^2 + \frac{\theta-1}{2}(\kappa_1 A_1 \varphi_e)^2\sigma^2 \end{bmatrix}$$

Plugging in Δc_{t+1}:

$$\Leftrightarrow r_{f,t+1} = \begin{bmatrix} -\ln(\beta) + \frac{1}{\psi}\mathbb{E}_t(\mu + x_t + \sigma\eta_{t+1}) \\ +\frac{1}{2}(\theta - 1 - \frac{\theta}{\psi^2})\sigma^2 + \frac{\theta-1}{2}(\kappa_1 A_1 \varphi_e)^2\sigma^2 \end{bmatrix}$$

Hence:

$$\boxed{r_{f,t+1} = -\ln(\beta) + \frac{1}{\psi}\mu + \frac{1}{\psi}x_t + \frac{1}{2}(\theta - 1 - \frac{\theta}{\psi^2})\sigma^2 + \frac{\theta-1}{2}(\kappa_1 A_1 \varphi_e)^2\sigma^2}$$

From my previous calculations:

$$\Leftrightarrow r_{f,t+1} = -\theta\ln(\beta) + \frac{\theta}{\psi}\mathbb{E}_t(\Delta c_{t+1}) - (\theta-1)\mathbb{E}_t(r_{w,t+1}) - \frac{1}{2}\mathbb{V}ar_t(-\frac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)r_{w,t+1})$$

$$\Leftrightarrow r_{f,t+1} = -\theta\ln(\beta) + \frac{\theta}{\psi}\mathbb{E}_t(\Delta c_{t+1}) + (1-\theta)\mathbb{E}_t(r_{w,t+1}) - \frac{1}{2}\mathbb{V}ar_t(-\frac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)r_{w,t+1})$$

Subtracting $(1-\theta)r_{f,t+1}$ from both sides:

$$\Leftrightarrow \theta r_{f,t+1} = -\theta\ln(\beta) + \frac{\theta}{\psi}\mathbb{E}_t(\Delta c_{t+1}) + (1-\theta)(\mathbb{E}_t(r_{w,t+1}) - r_{f,t+1}) - \frac{1}{2}\mathbb{V}ar_t(-\frac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)r_{w,t+1})$$

Dividing both sides by θ:

$$\Leftrightarrow r_{f,t+1} = -\ln(\beta) + \frac{1}{\psi}\mathbb{E}_t(\Delta c_{t+1}) + \frac{(1-\theta)}{\theta}(\mathbb{E}_t(r_{w,t+1}) - r_{f,t+1}) - \frac{1}{2\theta}\mathbb{V}ar_t(-\frac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)r_{w,t+1})$$

Since I already computed everything except for $\mathbb{V}ar_t(-\frac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)r_{w,t+1})$:

$$\mathbb{V}ar_t(-\frac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)r_{w,t+1}) = \mathbb{V}ar_t(\ln(M_{t+1}))$$

$$\mathbb{V}ar_t(-\frac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)r_{w,t+1}) = \mathbb{E}_t([\ln(M_{t+1}) - \mathbb{E}_t(\ln(M_{t+1}))]^2)$$

From my previous calculations:

$$\ln(M_{t+1}) - \mathbb{E}_t(\ln(M_{t+1})) = -\frac{\theta}{\psi}\sigma\eta_{t+1} + (\theta-1)(\kappa_1 A_1 \varphi_e \sigma\epsilon_{t+1} + \sigma\eta_{t+1})$$

Therefore:

$$\mathbb{V}ar_t(-\frac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)r_{w,t+1}) = \mathbb{E}_t([-\frac{\theta}{\psi}\sigma\eta_{t+1} + (\theta-1)(\kappa_1 A_1 \varphi_e \sigma\epsilon_{t+1} + \sigma\eta_{t+1})]^2)$$

Since the shocks are independent, the cross products are zero. Hence:

$$\mathbb{V}ar_t(-\frac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)r_{w,t+1}) = (\theta - 1 - \frac{\theta}{\psi})^2\sigma^2 + (\theta-1)^2(\kappa_1 A_1 \varphi_e)^2\sigma^2$$

For the unconditional expectation of the risk-free rate, notice that $\mathbb{E}(x_t) = 0$. Hence:

$$\boxed{\mathbb{E}(r_{f,t+1}) = -\ln(\beta) + \frac{1}{\psi}\mu + \frac{1}{2}(\theta - 1 - \frac{\theta}{\psi^2})\sigma^2 + \frac{\theta-1}{2}(\kappa_1 A_1 \varphi_e)^2\sigma^2}$$

Alternatively:

$$\mathbb{E}(r_{f,t+1}) = -\ln(\beta) + \frac{1}{\psi}\mathbb{E}(\Delta c_{t+1}) + \frac{(1-\theta)}{\theta}\mathbb{E}((r_{w,t+1}) - r_{f,t+1}) - \frac{1}{2\theta}\mathbb{E}(\mathbb{V}ar_t(-\frac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)r_{w,t+1}))$$

$$\Leftrightarrow \mathbb{E}(r_{f,t+1}) = \begin{bmatrix} -\ln(\beta) + \frac{1}{\psi}\mathbb{E}(\Delta c_{t+1}) + \frac{(1-\theta)}{\theta}\mathbb{E}((r_{w,t+1}) - r_{f,t+1}) \\ -\frac{1}{2\theta}\mathbb{E}((\theta - 1 - \frac{\theta}{\psi})^2\sigma^2 + (\theta-1)^2(\kappa_1 A_1 \varphi_e)^2\sigma^2) \end{bmatrix}$$

$$\Leftrightarrow \mathbb{E}(r_{f,t+1}) = \begin{bmatrix} -\ln(\beta) + \frac{1}{\psi}\mathbb{E}(\Delta c_{t+1}) + \frac{(1-\theta)}{\theta}\mathbb{E}((r_{w,t+1}) - r_{f,t+1}) \\ -\frac{1}{2\theta}((\theta - 1 - \frac{\theta}{\psi})^2\sigma^2 + (\theta-1)^2(\kappa_1 A_1 \varphi_e)^2\sigma^2) \end{bmatrix}$$

Finally, the unconditional variance of the risk-free rate is:

$$\boxed{\mathbb{V}ar(r_{f,t+1}) = \frac{1}{\psi^2}\mathbb{V}ar(x_t)}$$