Campbell–Shiller Return Approximation

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Starting from:

$$1 + R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t}$$

Taking logs:

$$\Leftrightarrow \ln(1 + R_{t+1}) = \ln(P_{t+1} + D_{t+1}) - \ln(P_t)$$

Let $r_{t+1} \equiv \ln(1 + R_{t+1})$, $p_t \equiv \ln(P_t)$, $d_t \equiv \ln(D_t)$, $z_t \equiv p_t - d_t$, $\Delta d_{t+1} \equiv d_{t+1} - d_t$. Hence:

$$\Leftrightarrow r_{t+1} = \ln(P_{t+1} + D_{t+1}) - p_t$$

$$\Leftrightarrow r_{t+1} = \ln\left(D_{t+1}\left(\frac{P_{t+1}}{D_{t+1}} + 1\right)\right) - p_t$$

$$\Leftrightarrow r_{t+1} = d_{t+1} + \ln\left(\frac{P_{t+1}}{D_{t+1}} + 1\right) - p_t$$

$$\Leftrightarrow r_{t+1} = d_{t+1} + \ln\left(e^{\ln\left(\frac{P_{t+1}}{D_{t+1}}\right)} + 1\right) - p_t$$

$$\Leftrightarrow r_{t+1} = d_{t+1} + \ln\left(e^{p_{t+1} - d_{t+1}} + 1\right) - p_t$$

$$\Leftrightarrow r_{t+1} = d_{t+1} + \ln\left(e^{z_{t+1}} + 1\right) - p_t$$

$$\Leftrightarrow r_{t+1} = d_{t+1} - d_t + \ln\left(e^{z_{t+1}} + 1\right) - p_t + d_t$$

$$\Leftrightarrow r_{t+1} = \Delta d_{t+1} + \ln\left(e^{z_{t+1}} + 1\right) - (p_t - d_t)$$

$$\Leftrightarrow r_{t+1} = \Delta d_{t+1} + \ln\left(e^{z_{t+1}} + 1\right) - z_t$$

Taking a first-order expansion of the last term around the mean:

$$\ln\left(e^{z_{t+1}} + 1\right) \approx \ln\left(e^{\bar{z}} + 1\right) + \frac{e^{\bar{z}}}{e^{\bar{z}} + 1}(z_{t+1} - \bar{z})$$

Therefore:

$$\Leftrightarrow r_{t+1} \approx \Delta d_{t+1} + \ln\left(e^{\bar{z}} + 1\right) + \frac{e^{\bar{z}}}{e^{\bar{z}} + 1}(z_{t+1} - \bar{z}) - z_t$$

$$\Leftrightarrow r_{t+1} \approx \ln\left(e^{\bar{z}} + 1\right) - \frac{e^{\bar{z}}}{e^{\bar{z}} + 1}\bar{z} + \frac{e^{\bar{z}}}{e^{\bar{z}} + 1}z_{t+1} - z_t + \Delta d_{t+1}$$

Let $\kappa_1 = \frac{e^{\bar{z}}}{e^{\bar{z}}+1}$. Then:

$$\Leftrightarrow r_{t+1} \approx \ln\left(e^{\bar{z}} + 1\right) - \kappa_1\bar{z} + \kappa_1 z_{t+1} - z_t + \Delta d_{t+1}$$

Let $\kappa_0 = \ln\left(e^{\bar{z}} + 1\right) - \kappa_1\bar{z}$. Finally:

$$\boxed{r_{t+1} \approx \kappa_0 + \kappa_1 z_{t+1} - z_t + \Delta d_{t+1}}$$