

BY VFI

Diego Alvarez Flores

January 22, 2026

Utility-consumption

From the previous documents, I know that:

$$V_t = \left[(1 - \beta)C_t^{\frac{1-\gamma}{\theta}} + \beta (\mathbb{E}_t [V_{t+1}^{1-\gamma}])^{\frac{1}{\theta}} \right]^{\frac{\theta}{1-\gamma}}$$

Given the homogeneity of preferences and technology, optimal V_t and C_t take the form:

$$\begin{aligned} V_t &= \phi(S_t)W_t \equiv \phi_t W_t \\ C_t &= b(S_t)W_t \equiv b_t W_t \end{aligned}$$

Therefore:

$$V_t = \frac{\phi_t}{b_t} C_t$$

If I define $\nu_t = \frac{\phi_t}{b_t}$:

$$\begin{aligned} \Leftrightarrow \nu_t C_t &= \left[(1 - \beta)C_t^{\frac{1-\gamma}{\theta}} + \beta (\mathbb{E}_t [(\nu_{t+1} C_{t+1})^{1-\gamma}])^{\frac{1}{\theta}} \right]^{\frac{\theta}{1-\gamma}} \\ \Leftrightarrow (\nu_t C_t)^{\frac{1-\gamma}{\theta}} &= \left[(1 - \beta)C_t^{\frac{1-\gamma}{\theta}} + \beta (\mathbb{E}_t [(\nu_{t+1} C_{t+1})^{1-\gamma}])^{\frac{1}{\theta}} \right] \\ \Leftrightarrow \nu_t^{\frac{1-\gamma}{\theta}} &= \left[(1 - \beta) + \beta \left(\mathbb{E}_t \left[\left(\nu_{t+1} \frac{C_{t+1}}{C_t} \right)^{1-\gamma} \right] \right)^{\frac{1}{\theta}} \right] \end{aligned}$$

$$\Leftrightarrow \nu_t^{\frac{1-\gamma}{\theta}} = \left[(1 - \beta) + \beta \left(\mathbb{E}_t \left[\left(\nu_{t+1} e^{\ln(\frac{C_{t+1}}{C_t})} \right)^{1-\gamma} \right] \right)^{\frac{1}{\theta}} \right]$$

$$\Leftrightarrow \nu_t^{\frac{1-\gamma}{\theta}} = \left[(1 - \beta) + \beta \left(\mathbb{E}_t \left[(\nu_{t+1} e^{\Delta c_{t+1}})^{1-\gamma} \right] \right)^{\frac{1}{\theta}} \right]$$

Because $\Delta c_{t+1} = \mu + x_t + \sigma_t \eta_{t+1}$:

$$\Leftrightarrow \nu_t^{\frac{1-\gamma}{\theta}} = \left[(1 - \beta) + \beta \left(\mathbb{E}_t \left[(\nu_{t+1} e^{(\mu+x_t+\sigma_t \eta_{t+1})})^{1-\gamma} \right] \right)^{\frac{1}{\theta}} \right]$$

$$\Leftrightarrow \nu_t^{\frac{1-\gamma}{\theta}} = \left[(1 - \beta) + \beta \left(\mathbb{E}_t \left[(\nu_{t+1}^{1-\gamma} e^{(1-\gamma)(\mu+x_t+\sigma_t \eta_{t+1})}) \right] \right)^{\frac{1}{\theta}} \right]$$

$$\Leftrightarrow \nu_t^{\frac{1-\gamma}{\theta}} = \left[(1 - \beta) + \beta e^{(\frac{1-\gamma}{\theta})(\mu+x_t)} \left(\mathbb{E}_t \left[(\nu_{t+1}^{1-\gamma} e^{(1-\gamma)(\sigma_t \eta_{t+1})}) \right] \right)^{\frac{1}{\theta}} \right]$$

Since $\eta_{t+1} \sim N(0, 1)$ and is uncorrelated with other shocks (in particular ϵ_{t+1}):

$$\Leftrightarrow \nu_t^{\frac{1-\gamma}{\theta}} = \left[(1 - \beta) + \beta e^{(\frac{1-\gamma}{\theta})(\mu+x_t)} \left(\mathbb{E}_t \left[\nu_{t+1}^{1-\gamma} \right] \mathbb{E}_t \left[e^{(1-\gamma)(\sigma_t \eta_{t+1})} \right] \right)^{\frac{1}{\theta}} \right]$$

Finally:

$$\boxed{\nu_t^{\frac{1-\gamma}{\theta}} = \left[(1 - \beta) + \beta e^{(\frac{1-\gamma}{\theta})(\mu+x_t) + \frac{1}{2\theta}(1-\gamma)^2 \sigma_t^2} \left(\mathbb{E}_t \left[\nu_{t+1}^{1-\gamma} \right] \right)^{\frac{1}{\theta}} \right]}$$

Price-consumption

To get the price-consumption ratio, I start from the Euler equation:

$$\mathbb{E}_t \left[\beta^\theta \Delta C_{t+1}^{-\frac{\theta}{\psi}} (1 + R_{w,t+1})^{\theta-1} (1 + R_{i,t+1}) \right] = 1, \quad \forall i$$

In particular, it is satisfied for $1 + R_{w,t+1}$. Hence:

$$\Leftrightarrow \mathbb{E}_t \left[\beta^\theta \Delta C_{t+1}^{-\frac{\theta}{\psi}} (1 + R_{w,t+1})^\theta \right] = 1$$

$$\Leftrightarrow \mathbb{E}_t \left[\beta^\theta \Delta C_{t+1}^{-\frac{\theta}{\psi}} \left(\frac{P_{t+1} + C_{t+1}}{P_t} \right)^\theta \right] = 1$$

$$\Leftrightarrow \mathbb{E}_t \left[\beta^\theta \Delta C_{t+1}^{-\frac{\theta}{\psi}} \left(\frac{C_{t+1} \left(\frac{P_{t+1}}{C_{t+1}} + 1 \right)}{P_t} \right)^\theta \right] = 1$$

$$\Leftrightarrow \mathbb{E}_t \left[\beta^\theta \Delta C_{t+1}^{-\frac{\theta}{\psi}} \left(\frac{\frac{C_{t+1}}{C_{t+1}} \left(\frac{P_{t+1}}{C_{t+1}} + 1 \right)}{\frac{P_t}{C_t}} \right)^\theta \right] = 1$$

$$\Leftrightarrow \mathbb{E}_t \left[\beta^\theta \Delta C_{t+1}^{-\frac{\theta}{\psi}} \left(\frac{\Delta C_{t+1} \left(\frac{P_{t+1}}{C_{t+1}} + 1 \right)}{\frac{P_t}{C_t}} \right)^\theta \right] = 1$$

$$\Leftrightarrow \mathbb{E}_t \left[\beta^\theta \Delta C_{t+1}^{\theta - \frac{\theta}{\psi}} \left(\frac{P_{t+1}}{C_{t+1}} + 1 \right)^\theta \frac{P_t}{C_t}^{-\theta} \right] = 1$$

$$\Leftrightarrow \mathbb{E}_t \left[\beta^\theta \Delta C_{t+1}^{\theta - \frac{\theta}{\psi}} \left(\frac{P_{t+1}}{C_{t+1}} + 1 \right)^\theta \right] = \frac{P_t}{C_t}^\theta$$

$$\Leftrightarrow \left(\mathbb{E}_t \left[\beta^\theta \Delta C_{t+1}^{\theta - \frac{\theta}{\psi}} \left(\frac{P_{t+1}}{C_{t+1}} + 1 \right)^\theta \right] \right)^{\frac{1}{\theta}} = \frac{P_t}{C_t}$$

$$\Leftrightarrow \left(\mathbb{E}_t \left[\beta^\theta e^{\theta(1-\frac{1}{\psi})\Delta c_{t+1}} \left(\frac{P_{t+1}}{C_{t+1}} + 1 \right)^\theta \right] \right)^{\frac{1}{\theta}} = \frac{P_t}{C_t}$$

Because $\Delta c_{t+1} = \mu + x_t + \sigma_t \eta_{t+1}$:

$$\Leftrightarrow \left(\mathbb{E}_t \left[\beta^\theta e^{\theta(1-\frac{1}{\psi})(\mu+x_t+\sigma_t \eta_{t+1})} \left(\frac{P_{t+1}}{C_{t+1}} + 1 \right)^\theta \right] \right)^{\frac{1}{\theta}} = \frac{P_t}{C_t}$$

$$\Leftrightarrow \beta e^{(1-\frac{1}{\psi})(\mu+x_t)} \left(\mathbb{E}_t \left[e^{\theta(1-\frac{1}{\psi})\sigma_t \eta_{t+1}} \left(\frac{P_{t+1}}{C_{t+1}} + 1 \right)^\theta \right] \right)^{\frac{1}{\theta}} = \frac{P_t}{C_t}$$

Since $\eta_{t+1} \sim N(0, 1)$ and is uncorrelated with other shocks (in particular ϵ_{t+1}):

$$\Leftrightarrow \beta e^{(1-\frac{1}{\psi})(\mu+x_t)} \left(\mathbb{E}_t \left[e^{\theta(1-\frac{1}{\psi})\sigma_t \eta_{t+1}} \right] \mathbb{E}_t \left[\left(\frac{P_{t+1}}{C_{t+1}} + 1 \right)^\theta \right] \right)^{\frac{1}{\theta}} = \frac{P_t}{C_t}$$

Finally:

$$\boxed{\beta e^{(1-\frac{1}{\psi})(\mu+x_t)+\frac{1}{2}\theta(1-\frac{1}{\psi})^2\sigma_t^2} \left(\mathbb{E}_t \left[\left(\frac{P_{t+1}}{C_{t+1}} + 1 \right)^\theta \right] \right)^{\frac{1}{\theta}} = \frac{P_t}{C_t}}$$

Price-dividend

To get the price-dividend ratio, I start from the Euler equation again:

$$\mathbb{E}_t \left[\beta^\theta \Delta C_{t+1}^{-\frac{\theta}{\psi}} (1 + R_{w,t+1})^{\theta-1} (1 + R_{i,t+1}) \right] = 1, \quad \forall i$$

In particular, it is satisfied for $1 + R_{m,t+1}$. Hence:

$$\begin{aligned} & \Leftrightarrow \mathbb{E}_t \left[\beta^\theta \Delta C_{t+1}^{-\frac{\theta}{\psi}} (1 + R_{w,t+1})^{\theta-1} (1 + R_{m,t+1}) \right] = 1 \\ & \Leftrightarrow \mathbb{E}_t \left[\beta^\theta \Delta C_{t+1}^{-\frac{\theta}{\psi}} \left(\frac{P_{t+1} + C_{t+1}}{P_t} \right)^{\theta-1} \left(\frac{P_{t+1} + D_{t+1}}{P_t} \right) \right] = 1 \\ & \Leftrightarrow \mathbb{E}_t \left[\beta^\theta \Delta C_{t+1}^{-\frac{\theta}{\psi}} \left(\frac{C_{t+1}}{\textcolor{red}{C}_t} \frac{\frac{P_{t+1}}{C_{t+1}} + 1}{\frac{P_t}{C_t}} \right)^{\theta-1} \frac{D_{t+1}}{\textcolor{red}{D}_t} \left(\frac{\frac{P_{t+1}}{D_{t+1}} + 1}{\frac{P_t}{D_t}} \right) \right] = 1 \\ & \Leftrightarrow \mathbb{E}_t \left[\beta^\theta \Delta C_{t+1}^{-\frac{\theta}{\psi}} \left(\Delta C_{t+1} \frac{\frac{P_{t+1}}{C_{t+1}} + 1}{\frac{P_t}{C_t}} \right)^{\theta-1} \Delta D_{t+1} \left(\frac{\frac{P_{t+1}}{D_{t+1}} + 1}{\frac{P_t}{D_t}} \right) \right] = 1 \\ & \Leftrightarrow \mathbb{E}_t \left[\beta^\theta \Delta C_{t+1}^{\theta-1-\frac{\theta}{\psi}} \left(\frac{\frac{P_{t+1}}{C_{t+1}} + 1}{\frac{P_t}{C_t}} \right)^{\theta-1} \Delta D_{t+1} \left(\frac{P_{t+1}}{D_{t+1}} + 1 \right) \right] = \frac{P_t}{D_t} \\ & \Leftrightarrow \mathbb{E}_t \left[\beta^\theta e^{(\theta-1-\frac{\theta}{\psi}) \Delta c_{t+1}} \left(\frac{\frac{P_{t+1}}{C_{t+1}} + 1}{\frac{P_t}{C_t}} \right)^{\theta-1} e^{\Delta d_{t+1}} \left(\frac{P_{t+1}}{D_{t+1}} + 1 \right) \right] = \frac{P_t}{D_t} \end{aligned}$$

Because $\Delta c_{t+1} = \mu + x_t + \sigma_t \eta_{t+1}$ and $\Delta d_{t+1} = \mu_d + \phi x_t + \varphi_d \sigma_t u_{t+1}$:

$$\begin{aligned} & \Leftrightarrow \mathbb{E}_t \left[\beta^\theta e^{(\theta-1-\frac{\theta}{\psi})(\mu+x_t+\sigma_t \eta_{t+1})} \left(\frac{\frac{P_{t+1}}{C_{t+1}} + 1}{\frac{P_t}{C_t}} \right)^{\theta-1} e^{\mu_d + \phi x_t + \varphi_d \sigma_t u_{t+1}} \left(\frac{P_{t+1}}{D_{t+1}} + 1 \right) \right] = \frac{P_t}{D_t} \\ & \Leftrightarrow \beta^\theta e^{(\theta-1-\frac{\theta}{\psi})(\mu+x_t)+\mu_d+\phi x_t} \mathbb{E}_t \left[e^{(\theta-1-\frac{\theta}{\psi})(\sigma_t \eta_{t+1})} \left(\frac{\frac{P_{t+1}}{C_{t+1}} + 1}{\frac{P_t}{C_t}} \right)^{\theta-1} e^{\varphi_d \sigma_t u_{t+1}} \left(\frac{P_{t+1}}{D_{t+1}} + 1 \right) \right] = \frac{P_t}{D_t} \end{aligned}$$

Since $\eta_{t+1}, u_{t+1} \sim N(0, 1)$ and are uncorrelated with other shocks (in particular ϵ_{t+1}):

$$\Leftrightarrow \beta^\theta e^{(\theta-1-\frac{\theta}{\psi})(\mu+x_t)+\mu_d+\phi x_t} \mathbb{E}_t \left[e^{(\theta-1-\frac{\theta}{\psi})(\sigma_t \eta_{t+1})} \right] \mathbb{E}_t [e^{\varphi_d \sigma_t u_{t+1}}] \mathbb{E}_t \left[\left(\frac{\frac{P_{t+1}}{C_{t+1}} + 1}{\frac{P_t}{C_t}} \right)^{\theta-1} \left(\frac{P_{t+1}}{D_{t+1}} + 1 \right) \right] = \frac{P_t}{D_t}$$

$$\Leftrightarrow \beta^\theta e^{(\theta-1-\frac{\theta}{\psi})(\mu+x_t)+\mu_d+\phi x_t+\frac{1}{2}(\theta-1-\frac{\theta}{\psi})^2\sigma_t^2+\frac{1}{2}\varphi_d^2\sigma_t^2} \mathbb{E}_t \left[\left(\frac{\frac{P_{t+1}}{C_{t+1}} + 1}{\frac{P_t}{C_t}} \right)^{\theta-1} \left(\frac{P_{t+1}}{D_{t+1}} + 1 \right) \right] = \frac{P_t}{D_t}$$

Finally:

$$\boxed{\beta^\theta e^{(\theta-1-\frac{\theta}{\psi})(\mu+x_t)+\mu_d+\phi x_t+\frac{1}{2}(\theta-1-\frac{\theta}{\psi})^2\sigma_t^2+\frac{1}{2}\varphi_d^2\sigma_t^2} \left(\frac{P_t}{C_t} \right)^{1-\theta} \mathbb{E}_t \left[\left(\frac{P_{t+1}}{C_{t+1}} + 1 \right)^{\theta-1} \left(\frac{P_{t+1}}{D_{t+1}} + 1 \right) \right] = \frac{P_t}{D_t}}$$

Risk-free rate

For the risk-free rate:

$$1 + R_{f,t+1} = \frac{1}{\mathbb{E}_t(M_{t+1})}$$

$$\Leftrightarrow 1 + R_{f,t+1} = \frac{1}{\mathbb{E}_t(\beta^\theta \Delta C_{t+1}^{-\frac{\theta}{\psi}} (1 + R_{w,t+1})^{\theta-1})}$$

$$\Leftrightarrow 1 + R_{f,t+1} = \frac{1}{\mathbb{E}_t(\beta^\theta \Delta C_{t+1}^{-\frac{\theta}{\psi}} (\frac{P_{t+1} + C_{t+1}}{P_t})^{\theta-1})}$$

$$\Leftrightarrow 1 + R_{f,t+1} = \frac{1}{\mathbb{E}_t(\beta^\theta \Delta C_{t+1}^{-\frac{\theta}{\psi}} (\frac{C_{t+1}(\frac{P_{t+1}}{C_{t+1}} + 1)}{P_t})^{\theta-1})}$$

$$\Leftrightarrow 1 + R_{f,t+1} = \frac{1}{\mathbb{E}_t(\beta^\theta \Delta C_{t+1}^{-\frac{\theta}{\psi}} (\frac{\frac{C_{t+1}}{\textcolor{red}{C_t}} (\frac{P_{t+1}}{C_{t+1}} + 1)}{\frac{P_t}{\textcolor{red}{C_t}}})^{\theta-1})}$$

$$\Leftrightarrow 1 + R_{f,t+1} = \frac{1}{\mathbb{E}_t(\beta^\theta \Delta C_{t+1}^{\theta-1-\frac{\theta}{\psi}} (\frac{P_t}{C_t})^{1-\theta} (\frac{P_{t+1}}{C_{t+1}} + 1)^{\theta-1})}$$

$$\Leftrightarrow 1 + R_{f,t+1} = \frac{1}{\mathbb{E}_t(\beta^\theta e^{(\theta-1-\frac{\theta}{\psi})\Delta c_{t+1}} (\frac{P_t}{C_t})^{1-\theta} (\frac{P_{t+1}}{C_{t+1}} + 1)^{\theta-1})}$$

Because $\Delta c_{t+1} = \mu + x_t + \sigma_t \eta_{t+1}$:

$$\Leftrightarrow 1 + R_{f,t+1} = \frac{1}{\mathbb{E}_t(\beta^\theta e^{(\theta-1-\frac{\theta}{\psi})(\mu+x_t+\sigma_t \eta_{t+1})} (\frac{P_t}{C_t})^{1-\theta} (\frac{P_{t+1}}{C_{t+1}} + 1)^{\theta-1})}$$

$$\Leftrightarrow 1 + R_{f,t+1} = \frac{1}{\beta^\theta e^{(\theta-1-\frac{\theta}{\psi})(\mu+x_t)} \frac{P_t}{C_t}^{1-\theta} \mathbb{E}_t(e^{(\theta-1-\frac{\theta}{\psi})(\sigma_t \eta_{t+1})} (\frac{P_{t+1}}{C_{t+1}} + 1)^{\theta-1})}$$

Since $\eta_{t+1} \sim N(0, 1)$ and is uncorrelated with other shocks (in particular ϵ_{t+1}):

$$\Leftrightarrow 1 + R_{f,t+1} = \frac{1}{\beta^\theta e^{(\theta-1-\frac{\theta}{\psi})(\mu+x_t)} \frac{P_t}{C_t}^{1-\theta} \mathbb{E}_t(e^{(\theta-1-\frac{\theta}{\psi})(\sigma_t \eta_{t+1})}) \mathbb{E}_t((\frac{P_{t+1}}{C_{t+1}} + 1)^{\theta-1})}$$

Finally:

$$1 + R_{f,t+1} = \frac{1}{\beta^\theta e^{(\theta-1-\frac{\theta}{\psi})(\mu+x_t) + \frac{1}{2}(\theta-1-\frac{\theta}{\psi})^2 \sigma_t^2} \frac{P_t}{C_t}^{1-\theta} \mathbb{E}_t((\frac{P_{t+1}}{C_{t+1}} + 1)^{\theta-1})}$$