# BY
# VFI

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From the previous documents, I know that:

$$V_t = \left[(1-\beta)C_t^{\frac{1-\gamma}{\theta}} + \beta\left(\mathbb{E}_t\left[V_{t+1}^{1-\gamma}\right]\right)^{\frac{1}{\theta}}\right]^{\frac{\theta}{1-\gamma}}$$

Given the homogeneity of preferences and technology, optimal $V_t$ and $C_t$ take the form:

$$V_t = \phi(S_t)W_t \equiv \phi_t W_t$$
$$C_t = b(S_t)W_t \equiv b_t W_t$$

Therefore:

$$V_t = \frac{\phi_t}{b_t}C_t$$

If I define $\nu_t = \frac{\phi_t}{b_t}$:

$$\Leftrightarrow \nu_t C_t = \left[(1-\beta)C_t^{\frac{1-\gamma}{\theta}} + \beta\left(\mathbb{E}_t\left[(\nu_{t+1}C_{t+1})^{1-\gamma}\right]\right)^{\frac{1}{\theta}}\right]^{\frac{\theta}{1-\gamma}}$$

$$\Leftrightarrow (\nu_t C_t)^{\frac{1-\gamma}{\theta}} = \left[(1-\beta)C_t^{\frac{1-\gamma}{\theta}} + \beta\left(\mathbb{E}_t\left[(\nu_{t+1}C_{t+1})^{1-\gamma}\right]\right)^{\frac{1}{\theta}}\right]$$

$$\Leftrightarrow \nu_t^{\frac{1-\gamma}{\theta}} = \left[(1-\beta) + \beta\left(\mathbb{E}_t\left[\left(\nu_{t+1}\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\right]\right)^{\frac{1}{\theta}}\right]$$

$$\Leftrightarrow \nu_t^{\frac{1-\gamma}{\theta}} = \left[(1-\beta) + \beta\left(\mathbb{E}_t\left[\left(\nu_{t+1}e^{\ln\left(\frac{C_{t+1}}{C_t}\right)}\right)^{1-\gamma}\right]\right)^{\frac{1}{\theta}}\right]$$

$$\Leftrightarrow \nu_t^{\frac{1-\gamma}{\theta}} = \left[(1-\beta) + \beta\left(\mathbb{E}_t\left[\left(\nu_{t+1}e^{\Delta c_{t+1}}\right)^{1-\gamma}\right]\right)^{\frac{1}{\theta}}\right]$$

Because $\Delta c_{t+1} = \mu + x_t + \sigma_t\eta_{t+1}$:

$$\Leftrightarrow \nu_t^{\frac{1-\gamma}{\theta}} = \left[(1-\beta) + \beta\left(\mathbb{E}_t\left[\left(\nu_{t+1}e^{(\mu+x_t+\sigma_t\eta_{t+1})}\right)^{1-\gamma}\right]\right)^{\frac{1}{\theta}}\right]$$

$$\Leftrightarrow \nu_t^{\frac{1-\gamma}{\theta}} = \left[(1-\beta) + \beta\left(\mathbb{E}_t\left[\left(\nu_{t+1}^{1-\gamma}e^{(1-\gamma)(\mu+x_t+\sigma_t\eta_{t+1})}\right)\right]\right)^{\frac{1}{\theta}}\right]$$

$$\Leftrightarrow \nu_t^{\frac{1-\gamma}{\theta}} = \left[(1-\beta) + \beta e^{(\frac{1-\gamma}{\theta})(\mu+x_t)}\left(\mathbb{E}_t\left[\left(\nu_{t+1}^{1-\gamma}e^{(1-\gamma)(\sigma_t\eta_{t+1})}\right)\right]\right)^{\frac{1}{\theta}}\right]$$

Since $\eta_{t+1} \sim N(0,1)$ and is uncorrelated with other shocks (in particular $\epsilon_{t+1}$):

$$\Leftrightarrow \nu_t^{\frac{1-\gamma}{\theta}} = \left[(1-\beta) + \beta e^{(\frac{1-\gamma}{\theta})(\mu+x_t)}\left(\mathbb{E}_t\left[\nu_{t+1}^{1-\gamma}\right]\mathbb{E}_t\left[e^{(1-\gamma)(\sigma_t\eta_{t+1})}\right]\right)^{\frac{1}{\theta}}\right]$$

Finally:

$$\boxed{\nu_t^{\frac{1-\gamma}{\theta}} = \left[(1-\beta) + \beta e^{(\frac{1-\gamma}{\theta})(\mu+x_t)+\frac{1}{2\theta}(1-\gamma)^2\sigma_t^2}\left(\mathbb{E}_t\left[\nu_{t+1}^{1-\gamma}\right]\right)^{\frac{1}{\theta}}\right]}$$

To get the price-consumption ratio, I start from the Euler equation:

$$\mathbb{E}_t\left[\beta^\theta \Delta C_{t+1}^{-\frac{\theta}{\psi}}(1+R_{w,t+1})^{\theta-1}(1+R_{i,t+1})\right] = 1, \quad \forall i$$

In particular, it is satisfied for $1+R_{w,t+1}$. Hence:

$$\Leftrightarrow \mathbb{E}_t\left[\beta^\theta \Delta C_{t+1}^{-\frac{\theta}{\psi}}(1+R_{w,t+1})^{\theta}\right] = 1$$

$$\Leftrightarrow \mathbb{E}_t\left[\beta^\theta \Delta C_{t+1}^{-\frac{\theta}{\psi}}(\frac{P_{t+1}+C_{t+1}}{P_t})^{\theta}\right] = 1$$

$$\Leftrightarrow \mathbb{E}_t\left[\beta^\theta \Delta C_{t+1}^{-\frac{\theta}{\psi}}(\frac{C_{t+1}(\frac{P_{t+1}}{C_{t+1}}+1)}{P_t})^{\theta}\right] = 1$$

$$\Leftrightarrow \mathbb{E}_t\left[\beta^\theta \Delta C_{t+1}^{-\frac{\theta}{\psi}}(\frac{\frac{C_{t+1}}{C_t}(\frac{P_{t+1}}{C_{t+1}}+1)}{\frac{P_t}{C_t}})^{\theta}\right] = 1$$

$$\Leftrightarrow \mathbb{E}_t \left[ \beta^\theta \Delta C_{t+1}^{-\frac{\theta}{\psi}} ( \frac{\Delta C_{t+1}(\frac{P_{t+1}}{C_{t+1}} + 1)}{\frac{P_t}{C_t}} )^\theta \right] = 1$$

$$\Leftrightarrow \mathbb{E}_t \left[ \beta^\theta \Delta C_{t+1}^{\theta-\frac{\theta}{\psi}} ( \frac{P_{t+1}}{C_{t+1}} + 1)^\theta \frac{P_t}{C_t}^{-\theta} \right] = 1$$

$$\Leftrightarrow \mathbb{E}_t \left[ \beta^\theta \Delta C_{t+1}^{\theta-\frac{\theta}{\psi}} ( \frac{P_{t+1}}{C_{t+1}} + 1)^\theta \right] = \frac{P_t}{C_t}^\theta$$

$$\Leftrightarrow \left( \mathbb{E}_t \left[ \beta^\theta \Delta C_{t+1}^{\theta-\frac{\theta}{\psi}} ( \frac{P_{t+1}}{C_{t+1}} + 1)^\theta \right] \right)^{\frac{1}{\theta}} = \frac{P_t}{C_t}$$

$$\Leftrightarrow \left( \mathbb{E}_t \left[ \beta^\theta e^{\theta(1-\frac{1}{\psi})\Delta c_{t+1}} ( \frac{P_{t+1}}{C_{t+1}} + 1)^\theta \right] \right)^{\frac{1}{\theta}} = \frac{P_t}{C_t}$$

Because $\Delta c_{t+1} = \mu + x_t + \sigma_t \eta_{t+1}$:

$$\Leftrightarrow \left( \mathbb{E}_t \left[ \beta^\theta e^{\theta(1-\frac{1}{\psi})(\mu + x_t + \sigma_t \eta_{t+1})} ( \frac{P_{t+1}}{C_{t+1}} + 1)^\theta \right] \right)^{\frac{1}{\theta}} = \frac{P_t}{C_t}$$

$$\Leftrightarrow \beta e^{(1-\frac{1}{\psi})(\mu + x_t)} \left( \mathbb{E}_t \left[ e^{\theta(1-\frac{1}{\psi})\sigma_t \eta_{t+1}} ( \frac{P_{t+1}}{C_{t+1}} + 1)^\theta \right] \right)^{\frac{1}{\theta}} = \frac{P_t}{C_t}$$

Since $\eta_{t+1} \sim N(0,1)$ and is uncorrelated with other shocks (in particular $\epsilon_{t+1}$):

$$\Leftrightarrow \beta e^{(1-\frac{1}{\psi})(\mu + x_t)} \left( \mathbb{E}_t \left[ e^{\theta(1-\frac{1}{\psi})\sigma_t \eta_{t+1}} \right] \mathbb{E}_t \left[ ( \frac{P_{t+1}}{C_{t+1}} + 1)^\theta \right] \right)^{\frac{1}{\theta}} = \frac{P_t}{C_t}$$

Finally:

$$\boxed{\beta e^{(1-\frac{1}{\psi})(\mu + x_t) + \frac{1}{2}\theta(1-\frac{1}{\psi})^2 \sigma_t^2} \left( \mathbb{E}_t \left[ ( \frac{P_{t+1}}{C_{t+1}} + 1)^\theta \right] \right)^{\frac{1}{\theta}} = \frac{P_t}{C_t}}$$

To get the price-dividend ratio, I start from the Euler equation again:

$$\mathbb{E}_t \left[ \beta^\theta \Delta C_{t+1}^{-\frac{\theta}{\psi}} (1 + R_{w,t+1})^{\theta-1} (1 + R_{i,t+1}) \right] = 1, \quad \forall i$$

In particular, it is satisfied for $1 + R_{m,t+1}$. Hence:

$$\Leftrightarrow \mathbb{E}_t \left[ \beta^\theta \Delta C_{t+1}^{-\frac{\theta}{\psi}} (1 + R_{w,t+1})^{\theta-1} (1 + R_{m,t+1}) \right] = 1$$

$$\Leftrightarrow \mathbb{E}_t \left[ \beta^\theta \Delta C_{t+1}^{-\frac{\theta}{\psi}} (\frac{P_{t+1} + C_{t+1}}{P_t})^{\theta-1} (\frac{P_{t+1} + D_{t+1}}{P_t}) \right] = 1$$

$$\Leftrightarrow \mathbb{E}_t \left[ \beta^\theta \Delta C_{t+1}^{-\frac{\theta}{\psi}} (\frac{C_{t+1}}{C_t} \frac{\frac{P_{t+1}}{C_{t+1}} + 1}{\frac{P_t}{C_t}})^{\theta-1} \frac{D_{t+1}}{D_t} (\frac{\frac{P_{t+1}}{D_{t+1}} + 1}{\frac{P_t}{D_t}}) \right] = 1$$

$$\Leftrightarrow \mathbb{E}_t \left[ \beta^\theta \Delta C_{t+1}^{-\frac{\theta}{\psi}} (\Delta C_{t+1} \frac{\frac{P_{t+1}}{C_{t+1}} + 1}{\frac{P_t}{C_t}})^{\theta-1} \Delta D_{t+1} (\frac{\frac{P_{t+1}}{D_{t+1}} + 1}{\frac{P_t}{D_t}}) \right] = 1$$

$$\Leftrightarrow \mathbb{E}_t \left[ \beta^\theta \Delta C_{t+1}^{\theta-1-\frac{\theta}{\psi}} (\frac{\frac{P_{t+1}}{C_{t+1}} + 1}{\frac{P_t}{C_t}})^{\theta-1} \Delta D_{t+1} (\frac{P_{t+1}}{D_{t+1}} + 1) \right] = \frac{P_t}{D_t}$$

$$\Leftrightarrow \mathbb{E}_t \left[ \beta^\theta e^{(\theta-1-\frac{\theta}{\psi})\Delta c_{t+1}} (\frac{\frac{P_{t+1}}{C_{t+1}} + 1}{\frac{P_t}{C_t}})^{\theta-1} e^{\Delta d_{t+1}} (\frac{P_{t+1}}{D_{t+1}} + 1) \right] = \frac{P_t}{D_t}$$

Because $\Delta c_{t+1} = \mu + x_t + \sigma_t \eta_{t+1}$ and $\Delta d_{t+1} = \mu_d + \phi x_t + \varphi_d \sigma_t u_{t+1}$:

$$\Leftrightarrow \mathbb{E}_t \left[ \beta^\theta e^{(\theta-1-\frac{\theta}{\psi})(\mu + x_t + \sigma_t \eta_{t+1})} (\frac{\frac{P_{t+1}}{C_{t+1}} + 1}{\frac{P_t}{C_t}})^{\theta-1} e^{\mu_d + \phi x_t + \varphi_d \sigma_t u_{t+1}} (\frac{P_{t+1}}{D_{t+1}} + 1) \right] = \frac{P_t}{D_t}$$

$$\Leftrightarrow \beta^\theta e^{(\theta-1-\frac{\theta}{\psi})(\mu + x_t) + \mu_d + \phi x_t} \mathbb{E}_t \left[ e^{(\theta-1-\frac{\theta}{\psi})(\sigma_t \eta_{t+1})} (\frac{\frac{P_{t+1}}{C_{t+1}} + 1}{\frac{P_t}{C_t}})^{\theta-1} e^{\varphi_d \sigma_t u_{t+1}} (\frac{P_{t+1}}{D_{t+1}} + 1) \right] = \frac{P_t}{D_t}$$

Since $\eta_{t+1}, u_{t+1} \sim N(0,1)$ and are uncorrelated with other shocks (in particular $\epsilon_{t+1}$):

$$\Leftrightarrow \beta^\theta e^{(\theta-1-\frac{\theta}{\psi})(\mu + x_t) + \mu_d + \phi x_t} \mathbb{E}_t \left[ e^{(\theta-1-\frac{\theta}{\psi})(\sigma_t \eta_{t+1})} \right] \mathbb{E}_t \left[ e^{\varphi_d \sigma_t u_{t+1}} \right] \mathbb{E}_t \left[ (\frac{\frac{P_{t+1}}{C_{t+1}} + 1}{\frac{P_t}{C_t}})^{\theta-1} (\frac{P_{t+1}}{D_{t+1}} + 1) \right] = \frac{P_t}{D_t}$$

$$\Leftrightarrow \beta^\theta e^{(\theta-1-\frac{\theta}{\psi})(\mu + x_t) + \mu_d + \phi x_t + \frac{1}{2}(\theta-1-\frac{\theta}{\psi})^2 \sigma_t^2 + \frac{1}{2}\varphi_d^2 \sigma_t^2} \mathbb{E}_t \left[ (\frac{\frac{P_{t+1}}{C_{t+1}} + 1}{\frac{P_t}{C_t}})^{\theta-1} (\frac{P_{t+1}}{D_{t+1}} + 1) \right] = \frac{P_t}{D_t}$$

Finally:

$$\boxed{\beta^\theta e^{(\theta-1-\frac{\theta}{\psi})(\mu + x_t) + \mu_d + \phi x_t + \frac{1}{2}(\theta-1-\frac{\theta}{\psi})^2 \sigma_t^2 + \frac{1}{2}\varphi_d^2 \sigma_t^2} (\frac{P_t}{C_t})^{1-\theta} \mathbb{E}_t \left[ (\frac{P_{t+1}}{C_{t+1}} + 1)^{\theta-1} (\frac{P_{t+1}}{D_{t+1}} + 1) \right] = \frac{P_t}{D_t}}$$