BY
Replication

Diego Alvarez Flores

December 23, 2025

SDF Derivation[1]

The maximization problem is:

$$U_t = \max_{C_t, \alpha_t} \left\{ (1-\beta) C_t^{\frac{1-\gamma}{\theta}} + \beta \left(\mathbb{E}_t \left[U_{t+1}^{1-\gamma} \right] \right)^{\frac{1}{\theta}} \right\}^{\frac{\theta}{1-\gamma}}$$

subject to:

$$\begin{aligned} W_{t+1} &= (W_t - C_t)\, \alpha_t'(\iota + R_{t+1}) \\ \alpha_t'\iota &= 1 \\ C_t &\leq D_{w,t} \end{aligned}$$

where $U_t \equiv U(W_t, S_t)$ and S_t is a vector of exogenous state variables.

Given the homogeneity of preferences and technology, optimal U_t and C_t take the form:

$$\begin{aligned} U_t &= \phi(S_t) W_t \equiv \phi_t W_t \\ C_t &= b(S_t) W_t \equiv b_t W_t \end{aligned}$$

Substituting the conjectured solution into the program gives:

$$\phi_t W_t = \max_{C_t, \alpha_t} \left\{ (1-\beta) C_t^{\frac{1-\gamma}{\theta}} + \beta \left(\mathbb{E}_t \left[(\phi_{t+1} W_{t+1})^{1-\gamma} \right] \right)^{\frac{1}{\theta}} \right\}^{\frac{\theta}{1-\gamma}} \tag{1}$$

Using the budget constraint $W_{t+1} = (W_t - C_t)\alpha_t'(\iota + R_{t+1})$ and defining the portfolio return on wealth:

$$1 + R_{w,t+1} \equiv \alpha_t'(1 + R_{t+1})$$

[1] From Howard Kung's PhD notes.

we can rewrite (1) as:

$$\phi_t W_t = \max_{C_t, \alpha_t} \left\{ (1-\beta) C_t^{\frac{1-\gamma}{\theta}} + \beta (W_t - C_t)^{\frac{1-\gamma}{\theta}} \left(\mathbb{E}_t \left[(\phi_{t+1}(1+R_{w,t+1}))^{1-\gamma} \right] \right)^{\frac{1}{\theta}} \right\}^{\frac{\theta}{1-\gamma}} \tag{2}$$

The first-order condition with respect to C_t is:

$$\left[\begin{array}{c} \frac{\theta}{1-\gamma} \left\{ (1-\beta) C_t^{\frac{1-\gamma}{\theta}} + \beta (W_t - C_t)^{\frac{1-\gamma}{\theta}} \left(\mathbb{E}_t \left[(\phi_{t+1}(1+R_{w,t+1}))^{1-\gamma} \right] \right)^{\frac{1}{\theta}} \right\}^{\frac{\theta}{1-\gamma}-1} \\ \left(\frac{1-\gamma}{\theta} (1-\beta) C_t^{\frac{1-\gamma}{\theta}-1} - \frac{1-\gamma}{\theta} \beta (W_t - C_t)^{\frac{1-\gamma}{\theta}-1} y_t^\star \right) \end{array} \right] = 0$$

where:

$$y_t^\star \equiv \left(\mathbb{E}_t \left[(\phi_{t+1}(1+R_{w,t+1}))^{1-\gamma} \right] \right)^{\frac{1}{\theta}}$$

Since

$$\frac{\theta}{1-\gamma} \left\{ (1-\beta) C_t^{\frac{1-\gamma}{\theta}} + \beta (W_t - C_t)^{\frac{1-\gamma}{\theta}} \left(\mathbb{E}_t \left[(\phi_{t+1}(1+R_{w,t+1}))^{1-\gamma} \right] \right)^{\frac{1}{\theta}} \right\}^{\frac{\theta}{1-\gamma}-1} \neq 0$$

the FOC simplifies to:

$$\left(\frac{1-\gamma}{\theta} (1-\beta) C_t^{\frac{1-\gamma}{\theta}-1} - \frac{1-\gamma}{\theta} \beta (W_t - C_t)^{\frac{1-\gamma}{\theta}-1} y_t^\star \right) = 0$$

Using $C_t = b_t W_t$, this condition can be rewritten as:

$$(1-\beta) b_t^{\frac{1-\gamma}{\theta}-1} = \beta (1-b_t)^{\frac{1-\gamma}{\theta}-1} y_t^\star \tag{3}$$

On the other hand, at the optimum, the Bellman equation (2) can also be written in terms of b_t as:

$$\phi_t^{\frac{1-\gamma}{\theta}} = (1-\beta) b_t^{\frac{1-\gamma}{\theta}} + \beta (1-b_t)^{\frac{1-\gamma}{\theta}} y_t^\star \tag{4}$$

Combining (3) and (4), we obtain:

$$\frac{\phi_t^{\frac{1-\gamma}{\theta}} - (1-\beta) b_t^{\frac{1-\gamma}{\theta}}}{1-b_t} = (1-\beta) b_t^{\frac{1-\gamma}{\theta}-1}$$

$$\Leftrightarrow \phi_t^{\frac{1-\gamma}{\theta}} - (1-\beta) b_t^{\frac{1-\gamma}{\theta}} = (1-\beta) b_t^{\frac{1-\gamma}{\theta}-1} (1-b_t)$$

$$\Leftrightarrow \phi_t^{\frac{1-\gamma}{\theta}} = (1-\beta) b_t^{\frac{1-\gamma}{\theta}} + (1-\beta) b_t^{\frac{1-\gamma}{\theta}-1} (1-b_t)$$

$$\Leftrightarrow \phi_t^{\frac{1-\gamma}{\theta}} = (1-\beta) b_t^{\frac{1-\gamma}{\theta}-1} [b_t + 1 - b_t]$$

$$\Leftrightarrow \phi_t^{\frac{1-\gamma}{\theta}} = (1-\beta) b_t^{\frac{1-\gamma}{\theta}-1}$$

$$\Leftrightarrow \phi_t^{\frac{1-\gamma}{\theta}} = (1-\beta) b_t^{\frac{1-\gamma-\theta}{\theta}}$$

$$\Leftrightarrow \phi_t = (1-\beta)^{\frac{\theta}{1-\gamma}} b_t^{\frac{1-\gamma-\theta}{1-\gamma}}$$

$$\Leftrightarrow \phi_t = (1-\beta)^{\frac{\theta}{1-\gamma}} \left(\frac{C_t}{W_t}\right)^{\frac{1-\gamma-\theta}{1-\gamma}}$$

$$\Leftrightarrow \phi_t = (1-\beta)^{\frac{1}{1-\frac{1}{\psi}}} \left(\frac{C_t}{W_t}\right)^{\frac{1-\gamma-\frac{1-\gamma}{1-\frac{1}{\psi}}}{1-\gamma}}$$

$$\Leftrightarrow \phi_t = (1-\beta)^{\frac{1}{\frac{\psi-1}{\psi}}} \left(\frac{C_t}{W_t}\right)^{\frac{1-\gamma-\frac{1-\gamma}{\frac{\psi-1}{\psi}}}{1-\gamma}}$$

$$\Leftrightarrow \phi_t = (1-\beta)^{\frac{\psi}{\psi-1}} \left(\frac{C_t}{W_t}\right)^{\frac{\frac{(1-\gamma)(\psi-1)-\psi(1-\gamma)}{\psi-1}}{1-\gamma}}$$

$$\Leftrightarrow \phi_t = (1-\beta)^{\frac{\psi}{\psi-1}} \left(\frac{C_t}{W_t}\right)^{\frac{(1-\gamma)(\psi-1)-\psi(1-\gamma)}{(\psi-1)(1-\gamma)}}$$

$$\Leftrightarrow \phi_t = (1-\beta)^{\frac{\psi}{\psi-1}} \left(\frac{C_t}{W_t}\right)^{\frac{(\psi-1)-\psi}{\psi-1}}$$

$$\Leftrightarrow \phi_t = (1-\beta)^{\frac{\psi}{\psi-1}} \left(\frac{C_t}{W_t}\right)^{\frac{1}{1-\psi}}$$

Thus, maximized lifetime utility depends only on the consumption–wealth ratio.

Rearranging (3), we get:

$$(1-\beta) b_t^{\frac{1-\gamma}{\theta}-1} = \beta(1-b_t)^{\frac{1-\gamma}{\theta}-1} y_t^{\star}$$

$$\Leftrightarrow (1-\beta)\left(\frac{b_t}{1-b_t}\right)^{\frac{1-\gamma}{\theta}-1} = \beta y_t^\star$$

$$\Leftrightarrow (1-\beta)\left(\frac{b_t}{1-b_t}\right)^{\frac{1-\gamma-\theta}{\theta}} = \beta y_t^\star$$

$$\Leftrightarrow (1-\beta)^\theta\left(\frac{b_t}{1-b_t}\right)^{1-\gamma-\theta} = \beta^\theta y_t^{\star\theta}$$

$$\Leftrightarrow (1-\beta)^\theta\left(\frac{b_t}{1-b_t}\right)^{1-\gamma-\theta} = \beta^\theta \mathbb{E}_t\left[(\phi_{t+1}(1+R_{w,t+1}))^{1-\gamma}\right]$$

$$\Leftrightarrow (1-\beta)^\theta\left(\frac{b_t}{1-b_t}\right)^{\theta(1-\frac{1}{\psi})-\theta} = \beta^\theta \mathbb{E}_t\left[(\phi_{t+1}(1+R_{w,t+1}))^{1-\gamma}\right]$$

$$\Leftrightarrow (1-\beta)^\theta\left(\frac{b_t}{1-b_t}\right)^{\theta(1-\frac{1}{\psi}-1)} = \beta^\theta \mathbb{E}_t\left[(\phi_{t+1}(1+R_{w,t+1}))^{1-\gamma}\right]$$

Hence:

$$(1-\beta)^\theta\left(\frac{b_t}{1-b_t}\right)^{-\frac{\theta}{\psi}} = \beta^\theta\, \mathbb{E}_t\left[(\phi_{t+1}(1+R_{w,t+1}))^{1-\gamma}\right] \tag{5}$$

Using the solution for ϕ_{t+1}, multiplying by $(1+R_{w,t+1})$, and substituting in the normalized budget constraint, we obtain:

$$\phi_t = (1-\beta)^{\frac{\psi}{\psi-1}}\left(\frac{C_t}{W_t}\right)^{\frac{1}{1-\psi}}$$

$$\Leftrightarrow \phi_{t+1} = (1-\beta)^{\frac{\psi}{\psi-1}}\left(\frac{C_{t+1}}{W_{t+1}}\right)^{\frac{1}{1-\psi}}$$

$$\Leftrightarrow \phi_{t+1}(1+R_{w,t+1}) = (1-\beta)^{\frac{\psi}{\psi-1}}\left(\frac{C_{t+1}}{W_{t+1}}\right)^{\frac{1}{1-\psi}}(1+R_{w,t+1})$$

$$\Leftrightarrow \phi_{t+1}(1+R_{w,t+1}) = (1-\beta)^{\frac{\psi}{\psi-1}}\left(\frac{C_{t+1}}{(1+R_{w,t+1})(W_t-C_t)}\right)^{\frac{1}{1-\psi}}(1+R_{w,t+1})$$

$$\Leftrightarrow \phi_{t+1}(1+R_{w,t+1}) = (1-\beta)^{\frac{\psi}{\psi-1}} \left(\frac{C_{t+1}C_t}{(1+R_{w,t+1})(W_t-C_t)C_t}\right)^{\frac{1}{1-\psi}} (1+R_{w,t+1})$$

$$\Leftrightarrow \phi_{t+1}(1+R_{w,t+1}) = (1-\beta)^{\frac{\psi}{\psi-1}} \left(\frac{C_{t+1}}{(1+R_{w,t+1})C_t}\right)^{\frac{1}{1-\psi}} \left(\frac{C_t}{W_t-C_t}\right)^{\frac{1}{1-\psi}} (1+R_{w,t+1})$$

$$\Leftrightarrow \phi_{t+1}(1+R_{w,t+1}) = (1-\beta)^{\frac{\psi}{\psi-1}} \left(\frac{C_{t+1}}{(1+R_{w,t+1})C_t}\right)^{\frac{1}{1-\psi}} \left(\frac{bW_t}{W_t-bW_t}\right)^{\frac{1}{1-\psi}} (1+R_{w,t+1})$$

$$\Leftrightarrow \phi_{t+1}(1+R_{w,t+1}) = (1-\beta)^{\frac{\psi}{\psi-1}} \left(\frac{b_t}{1-b_t}\right)^{\frac{1}{1-\psi}} \left(\frac{\Delta C_{t+1}}{1+R_{w,t+1}}\right)^{\frac{1}{1-\psi}} (1+R_{w,t+1})$$

where $\Delta C_{t+1} \equiv C_{t+1}/C_t$.

Rearranging:

$$\Leftrightarrow [\phi_{t+1}(1+R_{w,t+1})]^{1-\gamma} = (1-\beta)^{\frac{\psi(1-\gamma)}{\psi-1}} \left(\frac{b_t}{1-b_t}\right)^{\frac{1-\gamma}{1-\psi}} \left(\frac{\Delta C_{t+1}}{1+R_{w,t+1}}\right)^{\frac{1-\gamma}{1-\psi}} (1+R_{w,t+1})^{1-\gamma}$$

$$\Leftrightarrow \beta^{\theta}[\phi_{t+1}(1+R_{w,t+1})]^{1-\gamma} = \beta^{\theta}(1-\beta)^{\frac{\psi(1-\gamma)}{\psi-1}} \left(\frac{b_t}{1-b_t}\right)^{\frac{1-\gamma}{1-\psi}} \left(\frac{\Delta C_{t+1}}{1+R_{w,t+1}}\right)^{\frac{1-\gamma}{1-\psi}} (1+R_{w,t+1})^{1-\gamma}$$

$$\Leftrightarrow \beta^{\theta}[\phi_{t+1}(1+R_{w,t+1})]^{1-\gamma} = \beta^{\theta}(1-\beta)^{\frac{(1-\gamma)}{\frac{\psi-1}{\psi}}} \left(\frac{b_t}{1-b_t}\right)^{\frac{\theta(1-\frac{1}{\psi})}{1-\psi}} \Delta C_{t+1}^{\frac{\theta(1-\frac{1}{\psi})}{1-\psi}} (1+R_{w,t+1})^{1-\gamma-\frac{\theta(1-\frac{1}{\psi})}{1-\psi}}$$

$$\Leftrightarrow \beta^{\theta}[\phi_{t+1}(1+R_{w,t+1})]^{1-\gamma} = \beta^{\theta}(1-\beta)^{\frac{(1-\gamma)}{1-\frac{1}{\psi}}} \left(\frac{b_t}{1-b_t}\right)^{\frac{\theta(\frac{\psi-1}{\psi})}{1-\psi}} \Delta C_{t+1}^{\frac{\theta(\frac{\psi-1}{\psi})}{1-\psi}} (1+R_{w,t+1})^{1-\gamma-\frac{\theta(\frac{\psi-1}{\psi})}{1-\psi}}$$

$$\Leftrightarrow \beta^{\theta}[\phi_{t+1}(1+R_{w,t+1})]^{1-\gamma} = \beta^{\theta}(1-\beta)^{\theta} \left(\frac{b_t}{1-b_t}\right)^{\theta(\frac{\psi-1}{\psi(1-\psi)})} \Delta C_{t+1}^{\theta(\frac{\psi-1}{\psi(1-\psi)})} (1+R_{w,t+1})^{1-\gamma-\theta(\frac{\psi-1}{\psi(1-\psi)})}$$

$$\Leftrightarrow \beta^{\theta}[\phi_{t+1}(1+R_{w,t+1})]^{1-\gamma} = \beta^{\theta}(1-\beta)^{\theta} \left(\frac{b_t}{1-b_t}\right)^{-\theta(\frac{1-\psi}{\psi(1-\psi)})} \Delta C_{t+1}^{-\theta(\frac{1-\psi}{\psi(1-\psi)})} (1+R_{w,t+1})^{1-\gamma+\theta(\frac{1-\psi}{\psi(1-\psi)})}$$

$$\Leftrightarrow \beta^\theta[\phi_{t+1}(1+R_{w,t+1})]^{1-\gamma} = \beta^\theta(1-\beta)^\theta \left(\frac{b_t}{1-b_t}\right)^{-\frac{\theta}{\psi}} \Delta C_{t+1}^{-\frac{\theta}{\psi}}(1+R_{w,t+1})^{1-\gamma+\frac{\theta}{\psi}}$$

$$\Leftrightarrow \beta^\theta[\phi_{t+1}(1+R_{w,t+1})]^{1-\gamma} = \beta^\theta(1-\beta)^\theta \left(\frac{b_t}{1-b_t}\right)^{-\frac{\theta}{\psi}} \Delta C_{t+1}^{-\frac{\theta}{\psi}}(1+R_{w,t+1})^{\theta(1-\frac{1}{\psi})+\frac{\theta}{\psi}}$$

$$\Leftrightarrow \beta^\theta[\phi_{t+1}(1+R_{w,t+1})]^{1-\gamma} = \beta^\theta(1-\beta)^\theta \left(\frac{b_t}{1-b_t}\right)^{-\frac{\theta}{\psi}} \Delta C_{t+1}^{-\frac{\theta}{\psi}}(1+R_{w,t+1})^{\theta(\frac{\psi-1}{\psi})+\frac{\theta}{\psi}}$$

$$\Leftrightarrow \beta^\theta[\phi_{t+1}(1+R_{w,t+1})]^{1-\gamma} = \beta^\theta(1-\beta)^\theta \left(\frac{b_t}{1-b_t}\right)^{-\frac{\theta}{\psi}} \Delta C_{t+1}^{-\frac{\theta}{\psi}}(1+R_{w,t+1})^{\theta(\frac{\psi-1+1}{\psi})}$$

$$\Leftrightarrow \beta^\theta[\phi_{t+1}(1+R_{w,t+1})]^{1-\gamma} = \beta^\theta(1-\beta)^\theta \left(\frac{b_t}{1-b_t}\right)^{-\frac{\theta}{\psi}} \Delta C_{t+1}^{-\frac{\theta}{\psi}}(1+R_{w,t+1})^{\theta}$$

Taking expectations and substituting into (5) yields:

$$\Leftrightarrow \beta^\theta \mathbb{E}_t[(\phi_{t+1}(1+R_{w,t+1}))^{1-\gamma}] = \mathbb{E}_t\left[\beta^\theta(1-\beta)^\theta \left(\frac{b_t}{1-b_t}\right)^{-\frac{\theta}{\psi}} \Delta C_{t+1}^{-\frac{\theta}{\psi}}(1+R_{w,t+1})^{\theta}\right]$$

$$\Leftrightarrow (1-\beta)^\theta \left(\frac{b_t}{1-b_t}\right)^{-\frac{\theta}{\psi}} = \mathbb{E}_t\left[\beta^\theta(1-\beta)^\theta \left(\frac{b_t}{1-b_t}\right)^{-\frac{\theta}{\psi}} \Delta C_{t+1}^{-\frac{\theta}{\psi}}(1+R_{w,t+1})^{\theta}\right]$$

Finally:

$$1 = \mathbb{E}_t\left[\beta^\theta \Delta C_{t+1}^{-\frac{\theta}{\psi}}(1+R_{w,t+1})^{\theta-1}(1+R_{w,t+1})\right] \tag{6}$$

Since the Euler equation holds for every return:

$$\mathbb{E}_t\left[\beta^\theta \Delta C_{t+1}^{-\frac{\theta}{\psi}}(1+R_{w,t+1})^{\theta-1}(1+R_{i,t+1})\right] = 1, \quad \forall i \tag{7}$$

where:

$$M_{t+1} = \beta^\theta \Delta C_{t+1}^{-\frac{\theta}{\psi}}(1+R_{w,t+1})^{\theta-1} \tag{8}$$

is the stochastic discount factor.

This is the asset pricing Euler equation for the model. The term $(1+R_{w,t+1})^{\theta-1}$ captures how the agent values long-term growth prospects.

A's (Consumption)

The Epstein-Zin preferences are:

$$U_t = \max_{C_t,\alpha_t} \left\{ (1-\beta) C_t^{\frac{1-\gamma}{\theta}} + \beta \left(\mathbb{E}_t \left[U_{t+1}^{1-\gamma} \right] \right)^{\frac{1}{\theta}} \right\}^{\frac{\theta}{1-\gamma}}$$

with $\theta = \frac{1-\gamma}{1-\frac{1}{\psi}}$. The log consumption growth is given by:

$$\begin{aligned}
x_{t+1} &= \rho x_t + \varphi_e \sigma_t \epsilon_{t+1} \\
\Delta c_{t+1} &= \mu + x_t + \sigma_t \eta_{t+1} \\
\sigma_{t+1}^2 &= \sigma^2 + \nu_1 (\sigma_t^2 - \sigma^2) + \sigma_w w_{t+1}
\end{aligned}$$

And the log dividend growth is given by:

$$\Delta d_{t+1} = \mu_d + \phi x_t + \varphi_d \sigma_t u_{t+1}$$

with the shocks $\eta_t, \epsilon_t, u_t, w_t$ i.i.d. $N(0,1)$ and uncorrelated. Now, let $z_t = \ln(\frac{P_t}{C_t})$. The Campbell-Shiller approximation for returns is:

$$r_{w,t+1} = \kappa_0 + \kappa_1 z_{t+1} - z_t + \Delta c_{t+1}$$

Since there are two state variables, I conjecture that the price-to-consumption ratio is:

$$z_t = A_0 + A_1 x_t + A_2 \sigma_t^2$$

The Euler equation is:

$$\mathbb{E}_t \left[\beta^\theta \, \Delta C_{t+1}^{-\frac{\theta}{\psi}} (1 + R_{w,t+1})^{\theta-1} (1 + R_{i,t+1}) \right] = 1, \quad \forall i$$

In particular, the equation is satisfied for $1 + R_{i,t+1} = 1 + R_{w,t+1}$. Hence:

$$\mathbb{E}_t \left[\beta^\theta \, \Delta C_{t+1}^{-\frac{\theta}{\psi}} (1 + R_{w,t+1})^{\theta} \right] = 1$$

$$\Leftrightarrow \mathbb{E}_t \left[e^{\ln\left(\beta^\theta \, \Delta C_{t+1}^{-\frac{\theta}{\psi}} (1 + R_{w,t+1})^{\theta} \right)} \right] = 1$$

$$\Leftrightarrow \mathbb{E}_t \left[e^{\theta \ln(\beta) - \frac{\theta}{\psi} \Delta c_{t+1} + \theta r_{w,t+1}} \right] = 1$$

Plugging in Δc_{t+1} and the Campbell-Shiller approximation:

$$\Leftrightarrow \mathbb{E}_t\left[e^{\theta\ln(\beta)-\frac{\theta}{\psi}(\mu+x_t+\sigma_t\eta_{t+1})+\theta(\kappa_0+\kappa_1 z_{t+1}-z_t+\Delta c_{t+1})}\right]=1$$

Plugging in Δc_{t+1} again and z_t, z_{t+1}:

$$\Leftrightarrow \mathbb{E}_t\left[e^{\theta\ln(\beta)-\frac{\theta}{\psi}(\mu+x_t+\sigma_t\eta_{t+1})+\theta(\kappa_0+\kappa_1(A_0+A_1x_{t+1}+A_2\sigma^2_{t+1})-A_0-A_1x_t-A_2\sigma_t^2+\mu+x_t+\sigma_t\eta_{t+1})}\right]=1$$

Plugging in x_{t+1} and σ^2_{t+1}:

$$\Leftrightarrow \mathbb{E}_t\left[e^{\begin{bmatrix}\theta\ln(\beta)-\frac{\theta}{\psi}(\mu+x_t+\sigma_t\eta_{t+1})\\+\theta(\kappa_0+\kappa_1(A_0+A_1[\rho x_t+\varphi_e\sigma_t\epsilon_{t+1}]+A_2[\sigma^2+\nu_1(\sigma_t^2-\sigma^2)+\sigma_w w_{t+1}])\\-A_0-A_1x_t-A_2\sigma_t^2+\mu+x_t+\sigma_t\eta_{t+1})\end{bmatrix}}\right]=1$$

$$\Leftrightarrow \mathbb{E}_t\left[e^{\begin{bmatrix}\theta\ln(\beta)-\frac{\theta}{\psi}\mu-\frac{\theta}{\psi}x_t-\frac{\theta}{\psi}\sigma_t\eta_{t+1}\\+\theta\kappa_0+\theta\kappa_1(A_0+A_1[\rho x_t+\varphi_e\sigma_t\epsilon_{t+1}]+A_2[\sigma^2+\nu_1(\sigma_t^2-\sigma^2)+\sigma_w w_{t+1}])\\-\theta A_0-\theta A_1x_t-\theta A_2\sigma_t^2+\theta\mu+\theta x_t+\theta\sigma_t\eta_{t+1}\end{bmatrix}}\right]=1$$

$$\Leftrightarrow \mathbb{E}_t\left[e^{\begin{bmatrix}\theta\ln(\beta)-\frac{\theta}{\psi}\mu-\frac{\theta}{\psi}x_t-\frac{\theta}{\psi}\sigma_t\eta_{t+1}\\+\theta\kappa_0+\theta\kappa_1A_0+\theta\kappa_1A_1\rho x_t+\theta\kappa_1A_1\varphi_e\sigma_t\epsilon_{t+1}\\+\theta\kappa_1A_2\sigma^2+\theta\kappa_1A_2\nu_1(\sigma_t^2-\sigma^2)+\theta\kappa_1A_2\sigma_w w_{t+1}\\-\theta A_0-\theta A_1x_t-\theta A_2\sigma_t^2+\theta\mu+\theta x_t+\theta\sigma_t\eta_{t+1}\end{bmatrix}}\right]=1$$

Rearranging:

$$\Leftrightarrow \mathbb{E}_t\left[e^{\begin{bmatrix}\theta\ln(\beta)-\frac{\theta}{\psi}\mu-\frac{\theta}{\psi}x_t+\theta\kappa_0+\theta\kappa_1A_0\\+\theta\kappa_1A_1\rho x_t+\theta\kappa_1A_2\sigma^2+\theta\kappa_1A_2\nu_1(\sigma_t^2-\sigma^2)\\-\theta A_0-\theta A_1x_t-\theta A_2\sigma_t^2+\theta\mu+\theta x_t\\-\frac{\theta}{\psi}\sigma_t\eta_{t+1}+\theta\kappa_1A_1\varphi_e\sigma_t\epsilon_{t+1}+\theta\kappa_1A_2\sigma_w w_{t+1}+\theta\sigma_t\eta_{t+1}\end{bmatrix}}\right]=1$$

Since $\epsilon_{t+1}, \eta_{t+1}, w_{t+1} \sim N(0,1)$ and are uncorrelated:

$$\Leftrightarrow \begin{bmatrix} \theta \ln(\beta) - \frac{\theta}{\psi}\mu - \frac{\theta}{\psi}x_t + \theta\kappa_0 + \theta\kappa_1 A_0 + \theta\kappa_1 A_1 \rho x_t + \theta\kappa_1 A_2 \sigma^2 + \theta\kappa_1 A_2 \nu_1 (\sigma_t^2 - \sigma^2) \\ -\theta A_0 - \theta A_1 x_t - \theta A_2 \sigma_t^2 + \theta\mu + \theta x_t + \frac{1}{2}\sigma_t^2([\theta - \frac{\theta}{\psi}]^2 + (\theta\kappa_1 A_1 \varphi_e)^2) + \frac{1}{2}\sigma_w^2(\theta\kappa_1 A_2)^2 \end{bmatrix} = 0$$

Since this equation must be satisfied for all values of x_t:

$$-\frac{\theta}{\psi} + \theta\kappa_1 A_1 \rho - \theta A_1 + \theta = 0$$

$$\Leftrightarrow \theta - \frac{\theta}{\psi} = \theta A_1 - \theta\kappa_1 A_1 \rho$$

$$\Leftrightarrow 1 - \frac{1}{\psi} = A_1(1 - \kappa_1 \rho)$$

Hence:

$$\boxed{A_1 = \frac{1 - \frac{1}{\psi}}{1 - \kappa_1 \rho}}$$

Since this equation must be satisfied for all values of σ_t^2:

$$\theta\kappa_1 A_2 \nu_1 - \theta A_2 + \frac{1}{2}([\theta - \frac{\theta}{\psi}]^2 + (\theta\kappa_1 A_1 \varphi_e)^2) = 0$$

$$\Leftrightarrow A_2 \theta(1 - \kappa_1 \nu_1) = \frac{1}{2}([\theta - \frac{\theta}{\psi}]^2 + (\theta\kappa_1 A_1 \varphi_e)^2)$$

Hence:

$$\boxed{A_2 = \frac{\frac{1}{2}([\theta - \frac{\theta}{\psi}]^2 + (\theta\kappa_1 A_1 \varphi_e)^2)}{\theta(1 - \kappa_1 \nu_1)}}$$

And A_0 absorbs the constant terms:

$$\theta \ln(\beta) - \frac{\theta}{\psi}\mu + \theta\kappa_0 + \theta\kappa_1 A_0 + \theta\kappa_1 A_2 \sigma^2 - \theta\kappa_1 A_2 \nu_1 \sigma^2 - \theta A_0 + \theta\mu + \frac{1}{2}\sigma_w^2(\theta\kappa_1 A_2)^2 = 0$$

$$\theta \ln(\beta) - \frac{\theta}{\psi}\mu + \theta\kappa_0 + \theta\kappa_1 A_2 \sigma^2 - \theta\kappa_1 A_2 \nu_1 \sigma^2 + \theta\mu + \frac{1}{2}\sigma_w^2(\theta\kappa_1 A_2)^2 = \theta A_0 - \theta\kappa_1 A_0$$

$$\ln(\beta) - \frac{1}{\psi}\mu + \kappa_0 + \kappa_1 A_2 \sigma^2 - \kappa_1 A_2 \nu_1 \sigma^2 + \mu + \frac{1}{2}\sigma_w^2 \theta(\kappa_1 A_2)^2 = A_0(1 - \kappa_1)$$

Hence:

$$\boxed{A_0 = \frac{\ln(\beta) + \mu(1 - \frac{1}{\psi}) + \kappa_0 + \kappa_1 A_2 \sigma^2 (1 - \nu_1) + \frac{1}{2}\sigma_w^2 \theta(\kappa_1 A_2)^2}{1 - \kappa_1}}$$

Risk Premium (Consumption)

For the conditional risk premium on the consumption claim ($r_{c,t} = r_{w,t}$), I calculate:

$$\ln(M_{t+1}) - \mathbb{E}_t(\ln(M_{t+1})) = \theta\ln(\beta) - \frac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)r_{c,t+1} - \theta\ln(\beta) + \frac{\theta}{\psi}\mathbb{E}_t(\Delta c_{t+1}) - (\theta-1)\mathbb{E}_t(r_{c,t+1})$$

Plugging in Δc_{t+1} and the Campbell-Shiller approximation:

$$\Leftrightarrow \ln(M_{t+1}) - \mathbb{E}_t(\ln(M_{t+1})) = \begin{bmatrix} -\frac{\theta}{\psi}(\mu + x_t + \sigma_t\eta_{t+1}) + (\theta-1)(\kappa_0 + \kappa_1 z_{t+1} - z_t + \Delta c_{t+1}) \\ +\frac{\theta}{\psi}\mathbb{E}_t(\mu + x_t + \sigma_t\eta_{t+1}) - (\theta-1)\mathbb{E}_t(\kappa_0 + \kappa_1 z_{t+1} - z_t + \Delta c_{t+1}) \end{bmatrix}$$

$$\Leftrightarrow \ln(M_{t+1}) - \mathbb{E}_t(\ln(M_{t+1})) = \begin{bmatrix} -\frac{\theta}{\psi}\sigma_t\eta_{t+1} + (\theta-1)(\kappa_1 z_{t+1} + \Delta c_{t+1}) \\ -(\theta-1)\mathbb{E}_t(\kappa_1 z_{t+1} + \Delta c_{t+1}) \end{bmatrix}$$

Plugging in Δc_{t+1} again and z_{t+1}:

$$\Leftrightarrow \ln(M_{t+1}) - \mathbb{E}_t(\ln(M_{t+1})) = \begin{bmatrix} -\frac{\theta}{\psi}\sigma_t\eta_{t+1} + (\theta-1)(\kappa_1(A_0 + A_1 x_{t+1} + A_2\sigma_{t+1}^2) \\ +\mu + x_t + \sigma_t\eta_{t+1}) \\ -(\theta-1)\mathbb{E}_t(\kappa_1(A_0 + A_1 x_{t+1} + A_2\sigma_{t+1}^2) \\ +\mu + x_t + \sigma_t\eta_{t+1}) \end{bmatrix}$$

$$\Leftrightarrow \ln(M_{t+1}) - \mathbb{E}_t(\ln(M_{t+1})) = \begin{bmatrix} -\frac{\theta}{\psi}\sigma_t\eta_{t+1} + (\theta-1)(\kappa_1 A_1 x_{t+1} + \kappa_1 A_2\sigma_{t+1}^2 + \sigma_t\eta_{t+1}) \\ -(\theta-1)\mathbb{E}_t(\kappa_1 A_1 x_{t+1} + \kappa_1 A_2\sigma_{t+1}^2 + \sigma_t\eta_{t+1}) \end{bmatrix}$$

Plugging in the state variables in $t+1$:

$$\Leftrightarrow \ln(M_{t+1}) - \mathbb{E}_t(\ln(M_{t+1})) = \begin{bmatrix} -\frac{\theta}{\psi}\sigma_t\eta_{t+1} + (\theta-1)(\kappa_1 A_1(\rho x_t + \varphi_e\sigma_t\epsilon_{t+1}) \\ +\kappa_1 A_2(\sigma^2 + \nu_1(\sigma_t^2 - \sigma^2) + \sigma_w w_{t+1}) + \sigma_t\eta_{t+1}) \\ -(\theta-1)\mathbb{E}_t(\kappa_1 A_1(\rho x_t + \varphi_e\sigma_t\epsilon_{t+1}) \\ +\kappa_1 A_2(\sigma^2 + \nu_1(\sigma_t^2 - \sigma^2) + \sigma_w w_{t+1}) + \sigma_t\eta_{t+1}) \end{bmatrix}$$

$$\Leftrightarrow \ln(M_{t+1}) - \mathbb{E}_t(\ln(M_{t+1})) = -\frac{\theta}{\psi}\sigma_t\eta_{t+1} + (\theta-1)(\kappa_1 A_1\varphi_e\sigma_t\epsilon_{t+1} + \kappa_1 A_2\sigma_w w_{t+1} + \sigma_t\eta_{t+1})$$

Additionally, I calculate:

$$r_{c,t+1} - \mathbb{E}_t(r_{c,t+1}) = \kappa_0 + \kappa_1 z_{t+1} - z_t + \Delta c_{t+1} - \mathbb{E}_t(\kappa_0 + \kappa_1 z_{t+1} - z_t + \Delta c_{t+1})$$

Plugging in Δc_{t+1} and z_{t+1}:

$$\Leftrightarrow r_{c,t+1} - \mathbb{E}_t(r_{c,t+1}) = \begin{bmatrix} \kappa_1(A_0 + A_1 x_{t+1} + A_2\sigma_{t+1}^2) + \mu + x_t + \sigma_t\eta_{t+1} \\ -\mathbb{E}_t(\kappa_1(A_0 + A_1 x_{t+1} + A_2\sigma_{t+1}^2) + \mu + x_t + \sigma_t\eta_{t+1}) \end{bmatrix}$$

$$\Leftrightarrow r_{c,t+1} - \mathbb{E}_t(r_{c,t+1}) = \begin{bmatrix} \kappa_1 A_1 x_{t+1} + \kappa_1 A_2 \sigma_{t+1}^2 + \sigma_t \eta_{t+1} \\ -\mathbb{E}_t(\kappa_1 A_1 x_{t+1} + \kappa_1 A_2 \sigma_{t+1}^2) \end{bmatrix}$$

Plugging in the state variables in $t+1$:

$$\Leftrightarrow r_{c,t+1} - \mathbb{E}_t(r_{c,t+1}) = \begin{bmatrix} \kappa_1 A_1(\rho x_t + \varphi_e \sigma_t \epsilon_{t+1}) + \kappa_1 A_2(\sigma^2 + \nu_1(\sigma_t^2 - \sigma^2) + \sigma_w w_{t+1}) + \sigma_t \eta_{t+1} \\ -\mathbb{E}_t(\kappa_1 A_1(\rho x_t + \varphi_e \sigma_t \epsilon_{t+1}) + \kappa_1 A_2(\sigma^2 + \nu_1(\sigma_t^2 - \sigma^2) + \sigma_w w_{t+1})) \end{bmatrix}$$

$$\Leftrightarrow r_{c,t+1} - \mathbb{E}_t(r_{c,t+1}) = \left[\kappa_1 A_1 \varphi_e \sigma_t \epsilon_{t+1} + \kappa_1 A_2 \sigma_w w_{t+1} + \sigma_t \eta_{t+1}\right]$$

To combine these two derivations, I use that $\mathbb{E}_t(r_{c,t+1} - r_f) + \frac{1}{2}\mathbb{V}ar_t(r_{c,t+1}) = -\mathbb{C}ov_t(\ln(M_{t+1}), r_{c,t+1})$. First, for $\mathbb{V}ar_t(r_{c,t+1})$:

$$\mathbb{V}ar_t(r_{c,t+1}) = \mathbb{E}_t([r_{c,t+1} - \mathbb{E}_t(r_{c,t+1})]^2)$$

From the previous calculations:

$$\Leftrightarrow \mathbb{V}ar_t(r_{c,t+1}) = \mathbb{E}_t([\kappa_1 A_1 \varphi_e \sigma_t \epsilon_{t+1} + \kappa_1 A_2 \sigma_w w_{t+1} + \sigma_t \eta_{t+1}]^2)$$

Since the shocks are independent, the cross products are zero. Hence:

$$\boxed{\mathbb{V}ar_t(r_{c,t+1}) = \sigma_t^2(1 + \kappa_1^2 A_1^2 \varphi_e^2) + \kappa_1^2 A_2^2 \sigma_w^2}$$

Secondly, for $\mathbb{C}ov_t(\ln(M_{t+1}), r_{c,t+1})$:

$$\mathbb{C}ov_t(\ln(M_{t+1}), r_{c,t+1}) = \mathbb{E}_t([r_{c,t+1} - \mathbb{E}_t(r_{c,t+1})][\ln(M_{t+1}) - \mathbb{E}_t(\ln(M_{t+1}))])$$

From the previous calculations:

$$\Leftrightarrow \mathbb{C}ov_t(\ln(M_{t+1}), r_{c,t+1}) = \mathbb{E}_t \begin{pmatrix} [\kappa_1 A_1 \varphi_e \sigma_t \epsilon_{t+1} + \kappa_1 A_2 \sigma_w w_{t+1} + \sigma_t \eta_{t+1}] \\ [-\frac{\theta}{\psi}\sigma_t \eta_{t+1} + (\theta - 1)(\kappa_1 A_1 \varphi_e \sigma_t \epsilon_{t+1} + \kappa_1 A_2 \sigma_w w_{t+1} + \sigma_t \eta_{t+1})] \end{pmatrix}$$

$$\Leftrightarrow \mathbb{C}ov_t(\ln(M_{t+1}), r_{c,t+1}) = \mathbb{E}_t \begin{pmatrix} [\kappa_1 A_1 \varphi_e \sigma_t \epsilon_{t+1} + \kappa_1 A_2 \sigma_w w_{t+1} + \sigma_t \eta_{t+1}] \\ [-\frac{\theta}{\psi}\sigma_t \eta_{t+1} + (\theta - 1)\kappa_1 A_1 \varphi_e \sigma_t \epsilon_{t+1} + (\theta - 1)\kappa_1 A_2 \sigma_w w_{t+1} + (\theta - 1)\sigma_t \eta_{t+1})] \end{pmatrix}$$

Since the shocks are independent, the cross products are zero. Hence:

$$\Leftrightarrow \mathbb{C}ov_t(\ln(M_{t+1}), r_{c,t+1}) = \kappa_1 A_1 \varphi_e \sigma_t (\theta - 1)\kappa_1 A_1 \varphi_e \sigma_t + \kappa_1 A_2 \sigma_w (\theta - 1)\kappa_1 A_2 \sigma_w + \sigma_t \sigma_t (\theta - 1 - \tfrac{\theta}{\psi})$$

$$\Leftrightarrow \mathbb{C}ov_t(\ln(M_{t+1}), r_{c,t+1}) = (\theta - 1)\kappa_1^2 A_1^2 \varphi_e^2 \sigma_t^2 + (\theta - 1)\kappa_1^2 A_2^2 \sigma_w^2 + \sigma_t^2(\theta - 1 - \tfrac{\theta}{\psi})$$

Putting it all together:

$$\mathbb{E}_t(r_{c,t+1} - r_f) + \frac{1}{2}\mathbb{V}ar_t(r_{c,t+1}) = -\mathbb{C}ov_t(\ln(M_{t+1}), r_{c,t+1})$$

Hence the conditional risk premium on the consumption claim is:

$$\boxed{\mathbb{E}_t(r_{c,t+1} - r_f) = \begin{bmatrix} -(\theta - 1)\kappa_1^2 A_1^2 \varphi_e^2 \sigma_t^2 - (\theta - 1)\kappa_1^2 A_2^2 \sigma_w^2 - \sigma_t^2(\theta - 1 - \frac{\theta}{\psi}) \\ -\frac{1}{2}(\sigma_t^2(1 + \kappa_1^2 A_1^2 \varphi_e^2) + \kappa_1^2 A_2^2 \sigma_w^2) \end{bmatrix}}$$

Or without Jensen's correction term:

$$\boxed{\mathbb{E}_t(r_{c,t+1} - r_f) + \frac{1}{2}\mathbb{V}ar_t(r_{c,t+1}) = -(\theta - 1)\kappa_1^2 A_1^2 \varphi_e^2 \sigma_t^2 - (\theta - 1)\kappa_1^2 A_2^2 \sigma_w^2 - \sigma_t^2(\theta - 1 - \frac{\theta}{\psi})}$$

A's (Dividend)

Similarly, I conjecture for the log price-dividend ratio $z_{m,t} \equiv \ln(\frac{P_t}{D_t})$:

$$z_{m,t} = A_{0,m} + A_{1,m}x_t + A_{2,m}\sigma_t^2$$

Using the Campbell-Shiller approximation for returns:

$$r_{m,t+1} = \kappa_{0,m} + \kappa_{1,m}z_{m,t+1} - z_{m,t} + \Delta d_{t+1}$$

The Euler equation is:

$$\mathbb{E}_t\left[\beta^\theta\,\Delta C_{t+1}^{-\frac{\theta}{\psi}}(1+R_{w,t+1})^{\theta-1}(1+R_{i,t+1})\right] = 1, \quad \forall i$$

In particular, the equation is satisfied for $1+R_{i,t+1} = 1+R_{m,t+1}$. Hence:

$$\mathbb{E}_t\left[\beta^\theta\,\Delta C_{t+1}^{-\frac{\theta}{\psi}}(1+R_{w,t+1})^{\theta-1}(1+R_{m,t+1})\right] = 1$$

$$\Leftrightarrow \mathbb{E}_t\left[e^{\ln\left(\beta^\theta\,\Delta C_{t+1}^{-\frac{\theta}{\psi}}(1+R_{w,t+1})^{\theta-1}(1+R_{m,t+1})\right)}\right] = 1$$

$$\Leftrightarrow \mathbb{E}_t\left[e^{\theta\ln(\beta)-\frac{\theta}{\psi}\Delta c_{t+1}+(\theta-1)r_{w,t+1}+r_{m,t+1}}\right] = 1$$

Plugging in Δc_{t+1} and the Campbell-Shiller approximations:

$$\Leftrightarrow \mathbb{E}_t\left[e^{\theta\ln(\beta)-\frac{\theta}{\psi}(\mu+x_t+\sigma_t\eta_{t+1})+(\theta-1)(\kappa_0+\kappa_1 z_{t+1}-z_t+\Delta c_{t+1})+\kappa_{0,m}+\kappa_{1,m}z_{m,t+1}-z_{m,t}+\Delta d_{t+1}}\right] = 1$$

Plugging in Δc_{t+1} again, z_t, z_{t+1}, $z_{m,t}, z_{m,t+1}$, and Δd_{t+1}:

$$\Leftrightarrow \mathbb{E}_t\left[e^{\left[\begin{array}{c}\theta\ln(\beta)-\frac{\theta}{\psi}\mu-\frac{\theta}{\psi}x_t-\frac{\theta}{\psi}\sigma_t\eta_{t+1}+(\theta-1)\kappa_0+(\theta-1)\kappa_1(A_0+A_1x_{t+1}+A_2\sigma_{t+1}^2)\\ -(\theta-1)(A_0+A_1x_t+A_2\sigma_t^2)+(\theta-1)(\mu+x_t+\sigma_t\eta_{t+1})+\kappa_{0,m}+\\ \kappa_{1,m}(A_{0,m}+A_{1,m}x_{t+1}+A_{2,m}\sigma_{t+1}^2)-A_{0,m}-A_{1,m}x_t-A_{2,m}\sigma_t^2+\mu_d+\phi x_t+\varphi_d\sigma_t u_{t+1}\end{array}\right]}\right] = 1$$

Plugging in x_{t+1} and σ^2_{t+1}:

$$\Leftrightarrow \mathbb{E}_t\left[e^{\left[\begin{array}{c}\theta\ln(\beta)-\frac{\theta}{\psi}\mu-\frac{\theta}{\psi}x_t-\frac{\theta}{\psi}\sigma_t\eta_{t+1}+(\theta-1)\kappa_0\\ +(\theta-1)\kappa_1(A_0+A_1[\rho x_t+\varphi_e\sigma_t\epsilon_{t+1}]+A_2[\sigma^2+\nu_1(\sigma_t^2-\sigma^2)+\sigma_w w_{t+1}])\\ -(\theta-1)(A_0+A_1x_t+A_2\sigma_t^2)+(\theta-1)(\mu+x_t+\sigma_t\eta_{t+1})+\kappa_{0,m}+\\ \kappa_{1,m}(A_{0,m}+A_{1,m}[\rho x_t+\varphi_e\sigma_t\epsilon_{t+1}]+A_{2,m}[\sigma^2+\nu_1(\sigma_t^2-\sigma^2)+\sigma_w w_{t+1}])\\ -A_{0,m}-A_{1,m}x_t-A_{2,m}\sigma_t^2+\mu_d+\phi x_t+\varphi_d\sigma_t u_{t+1}\end{array}\right]}\right]=1$$

$$\Leftrightarrow \mathbb{E}_t\left[e^{\left[\begin{array}{c}\theta\ln(\beta)-\frac{\theta}{\psi}\mu-\frac{\theta}{\psi}x_t-\frac{\theta}{\psi}\sigma_t\eta_{t+1}+(\theta-1)\kappa_0\\ +(\theta-1)\kappa_1A_0+(\theta-1)\kappa_1A_1\rho x_t+(\theta-1)\kappa_1A_1\varphi_e\sigma_t\epsilon_{t+1}+(\theta-1)\kappa_1A_2\sigma^2\\ +(\theta-1)\kappa_1A_2\nu_1(\sigma_t^2-\sigma^2)+(\theta-1)\kappa_1A_2\sigma_w w_{t+1}\\ -(\theta-1)A_0-(\theta-1)A_1x_t-(\theta-1)A_2\sigma_t^2+(\theta-1)\mu+(\theta-1)x_t+(\theta-1)\sigma_t\eta_{t+1}\\ +\kappa_{0,m}+\kappa_{1,m}A_{0,m}+\kappa_{1,m}A_{1,m}\rho x_t+\kappa_{1,m}A_{1,m}\varphi_e\sigma_t\epsilon_{t+1}\\ +\kappa_{1,m}A_{2,m}\sigma^2+\kappa_{1,m}A_{2,m}\nu_1(\sigma_t^2-\sigma^2)+\kappa_{1,m}A_{2,m}\sigma_w w_{t+1}\\ -A_{0,m}-A_{1,m}x_t-A_{2,m}\sigma_t^2+\mu_d+\phi x_t+\varphi_d\sigma_t u_{t+1}\end{array}\right]}\right]=1$$

Since $\epsilon_{t+1},\eta_{t+1},w_{t+1},u_{t+1}\sim N(0,1)$ and are uncorrelated:

$$\left[\begin{array}{c}\theta\ln(\beta)-\frac{\theta}{\psi}\mu-\frac{\theta}{\psi}x_t+(\theta-1)\kappa_0+(\theta-1)\kappa_1A_0+(\theta-1)\kappa_1A_1\rho x_t\\ +(\theta-1)\kappa_1A_2\sigma^2+(\theta-1)\kappa_1A_2\nu_1(\sigma_t^2-\sigma^2)-(\theta-1)A_0-(\theta-1)A_1x_t\\ -(\theta-1)A_2\sigma_t^2+(\theta-1)\mu+(\theta-1)x_t+\kappa_{0,m}+\kappa_{1,m}A_{0,m}+\kappa_{1,m}A_{1,m}\rho x_t\\ +\kappa_{1,m}A_{2,m}\sigma^2+\kappa_{1,m}A_{2,m}\nu_1(\sigma_t^2-\sigma^2)-A_{0,m}-A_{1,m}x_t-A_{2,m}\sigma_t^2+\mu_d+\phi x_t\\ +\frac{1}{2}\sigma_t^2[(\theta-1-\frac{\theta}{\psi})^2+((\theta-1)\kappa_1A_1\varphi_e+\kappa_{1,m}A_{1,m}\varphi_e)^2+\varphi_d^2]\\ +\frac{1}{2}\sigma_w^2((\theta-1)\kappa_1A_2+\kappa_{1,m}A_{2,m})^2\end{array}\right]=0$$

Since this equation must be satisfied for all values of x_t:

$$-\frac{\theta}{\psi}+(\theta-1)\kappa_1A_1\rho-(\theta-1)A_1+(\theta-1)+\kappa_{1,m}A_{1,m}\rho-A_{1,m}+\phi=0$$

From the previous part, $-\frac{\theta}{\psi}+\theta\kappa_1A_1\rho-\theta A_1+\theta=0$. Therefore:

$$\Leftrightarrow -\kappa_1A_1\rho+A_1-1+\kappa_{1,m}A_{1,m}\rho-A_{1,m}+\phi=0$$

$$\Leftrightarrow A_1(1-\kappa_1\rho) - 1 + \kappa_{1,m}A_{1,m}\rho - A_{1,m} + \phi = 0$$

From the previous part, $A_1 = \frac{1-\frac{1}{\psi}}{1-\kappa_1\rho}$. Hence:

$$\Leftrightarrow \frac{1-\frac{1}{\psi}}{1-\kappa_1\rho}(1-\kappa_1\rho) - 1 + \kappa_{1,m}A_{1,m}\rho - A_{1,m} + \phi = 0$$

$$\Leftrightarrow -\frac{1}{\psi} + \kappa_{1,m}A_{1,m}\rho - A_{1,m} + \phi = 0$$

$$\Leftrightarrow A_{1,m} - \kappa_{1,m}A_{1,m}\rho = \phi - \frac{1}{\psi}$$

$$\Leftrightarrow A_{1,m}(1-\kappa_{1,m}\rho) = \phi - \frac{1}{\psi}$$

Hence:

$$\boxed{A_{1,m} = \frac{\phi - \frac{1}{\psi}}{1-\kappa_{1,m}\rho}}$$

Since this equation must be satisfied for all values of σ_t^2:

$$\begin{pmatrix} (\theta-1)\kappa_1 A_2\nu_1 - (\theta-1)A_2 + \kappa_{1,m}A_{2,m}\nu_1 - A_{2,m} \\ +\frac{1}{2}[(\theta-1-\frac{\theta}{\psi})^2 + ((\theta-1)\kappa_1 A_1\varphi_e + \kappa_{1,m}A_{1,m}\varphi_e)^2 + \varphi_d^2] \end{pmatrix} = 0$$

$$\Leftrightarrow \begin{pmatrix} A_2(\theta-1)(\kappa_1\nu_1 - 1) \\ +\frac{1}{2}[(\theta-1-\frac{\theta}{\psi})^2 + ((\theta-1)\kappa_1 A_1\varphi_e + \kappa_{1,m}A_{1,m}\varphi_e)^2 + \varphi_d^2] \end{pmatrix} = A_{2,m}(1-\kappa_{1,m}\nu_1)$$

$$\Leftrightarrow A_{2,m}(1-\kappa_{1,m}\nu_1) = \begin{pmatrix} A_2(1-\theta)(1-\kappa_1\nu_1) \\ +\frac{1}{2}[(\theta-1-\frac{\theta}{\psi})^2 + ((\theta-1)\kappa_1 A_1\varphi_e + \kappa_{1,m}A_{1,m}\varphi_e)^2 + \varphi_d^2] \end{pmatrix}$$

Hence:

$$\boxed{A_{2,m} = \frac{A_2(1-\theta)(1-\kappa_1\nu_1) + \frac{1}{2}[(\theta-1-\frac{\theta}{\psi})^2 + ((\theta-1)\kappa_1 A_1\varphi_e + \kappa_{1,m}A_{1,m}\varphi_e)^2 + \varphi_d^2]}{(1-\kappa_{1,m}\nu_1)}}$$

And $A_{0,m}$ absorbs the constant terms:

$$\begin{bmatrix} \theta\ln(\beta) - \frac{\theta}{\psi}\mu + (\theta-1)\kappa_0 + (\theta-1)\kappa_1 A_0 + (\theta-1)\kappa_1 A_2\sigma^2 - (\theta-1)\kappa_1 A_2\nu_1\sigma^2 \\ -(\theta-1)A_0 + (\theta-1)\mu + \kappa_{0,m} + \kappa_{1,m}A_{0,m} + \kappa_{1,m}A_{2,m}\sigma^2 - \kappa_{1,m}A_{2,m}\nu_1\sigma^2 - A_{0,m} + \mu_d \\ +\frac{1}{2}\sigma_w^2((\theta-1)\kappa_1 A_2 + \kappa_{1,m}A_{2,m})^2 \end{bmatrix} = 0$$

$$\Leftrightarrow \begin{bmatrix} \theta\ln(\beta) - \frac{\theta}{\psi}\mu + (\theta-1)\kappa_0 + (\theta-1)\kappa_1 A_0 + (\theta-1)\kappa_1 A_2\sigma^2 - (\theta-1)\kappa_1 A_2\nu_1\sigma^2 \\ -(\theta-1)A_0 + (\theta-1)\mu + \kappa_{0,m} + \kappa_{1,m}A_{2,m}\sigma^2 - \kappa_{1,m}A_{2,m}\nu_1\sigma^2 + \mu_d \\ +\frac{1}{2}\sigma_w^2((\theta-1)\kappa_1 A_2 + \kappa_{1,m}A_{2,m})^2 \end{bmatrix} = A_{0,m}(1-\kappa_{1,m})$$

$$\Leftrightarrow A_{0,m}(1-\kappa_{1,m}) = \begin{bmatrix} \theta\ln(\beta) + \mu(\theta-1-\frac{\theta}{\psi}) + (\theta-1)[\kappa_0 + A_0(\kappa_1-1)] + (\theta-1)\kappa_1 A_2\sigma^2[1-\nu_1] \\ +\kappa_{0,m} + \kappa_{1,m}A_{2,m}\sigma^2[1-\nu_1] + \mu_d + \frac{1}{2}\sigma_w^2((\theta-1)\kappa_1 A_2 + \kappa_{1,m}A_{2,m})^2 \end{bmatrix}$$

Finally:

$$\boxed{A_{0,m} = \frac{\begin{bmatrix} \theta\ln(\beta) + \mu(\theta-1-\frac{\theta}{\psi}) + (\theta-1)[\kappa_0 + A_0(\kappa_1-1)] + (\theta-1)\kappa_1 A_2\sigma^2[1-\nu_1] \\ +\kappa_{0,m} + \kappa_{1,m}A_{2,m}\sigma^2[1-\nu_1] + \mu_d + \frac{1}{2}\sigma_w^2((\theta-1)\kappa_1 A_2 + \kappa_{1,m}A_{2,m})^2 \end{bmatrix}}{1-\kappa_{1,m}}}$$

Risk Premium (Dividend)

I already derived $\ln(M_{t+1}) - \mathbb{E}_t(\ln(M_{t+1}))$. Now, I calculate:

$$r_{m,t+1} - \mathbb{E}_t(r_{m,t+1}) = \kappa_{0,m} + \kappa_{1,m}z_{m,t+1} - z_{m,t} + \Delta d_{t+1} - \mathbb{E}_t(\kappa_{0,m} + \kappa_{1,m}z_{m,t+1} - z_{m,t} + \Delta d_{t+1})$$

Plugging in Δd_{t+1} and $z_{m,t+1}$:

$$\Leftrightarrow r_{m,t+1} - \mathbb{E}_t(r_{m,t+1}) = \begin{bmatrix} \kappa_{1,m}(A_{0,m} + A_{1,m}x_{t+1} + A_{2,m}\sigma_{t+1}^2) + \mu_d + \phi x_t + \varphi_d\sigma_t u_{t+1} \\ -\mathbb{E}_t(\kappa_{1,m}(A_{0,m} + A_{1,m}x_{t+1} + A_{2,m}\sigma_{t+1}^2) + \mu_d + \phi x_t + \varphi_d\sigma_t u_{t+1}) \end{bmatrix}$$

$$\Leftrightarrow r_{m,t+1} - \mathbb{E}_t(r_{m,t+1}) = \begin{bmatrix} \kappa_{1,m}A_{1,m}x_{t+1} + \kappa_{1,m}A_{2,m}\sigma_{t+1}^2 + \varphi_d\sigma_t u_{t+1} \\ -\mathbb{E}_t(\kappa_{1,m}A_{1,m}x_{t+1} + \kappa_{1,m}A_{2,m}\sigma_{t+1}^2) \end{bmatrix}$$

Plugging in the state variables in $t+1$:

$$\Leftrightarrow r_{m,t+1} - \mathbb{E}_t(r_{m,t+1}) = \begin{bmatrix} \kappa_{1,m}A_{1,m}(\rho x_t + \varphi_e\sigma_t\epsilon_{t+1}) + \kappa_{1,m}A_{2,m}(\sigma^2 + \nu_1(\sigma_t^2 - \sigma^2) + \sigma_w w_{t+1}) + \varphi_d\sigma_t u_{t+1} \\ -\mathbb{E}_t(\kappa_{1,m}A_{1,m}(\rho x_t + \varphi_e\sigma_t\epsilon_{t+1}) + \kappa_{1,m}A_{2,m}(\sigma^2 + \nu_1(\sigma_t^2 - \sigma^2) + \sigma_w w_{t+1})) \end{bmatrix}$$

$$\Leftrightarrow r_{m,t+1} - \mathbb{E}_t(r_{m,t+1}) = \left[\kappa_{1,m}A_{1,m}\varphi_e\sigma_t\epsilon_{t+1} + \kappa_{1,m}A_{2,m}\sigma_w w_{t+1} + \varphi_d\sigma_t u_{t+1}\right]$$

To combine these two derivations, I use that $\mathbb{E}_t(r_{m,t+1} - r_f) + \frac{1}{2}\mathbb{V}ar_t(r_{m,t+1}) = -\mathbb{C}ov_t(\ln(M_{t+1}), r_{m,t+1})$. First, for $\mathbb{V}ar_t(r_{m,t+1})$:

$$\mathbb{V}ar_t(r_{m,t+1}) = \mathbb{E}_t([r_{m,t+1} - \mathbb{E}_t(r_{m,t+1})]^2)$$

From the previous calculations:

$$\Leftrightarrow \mathbb{V}ar_t(r_{m,t+1}) = \mathbb{E}_t([\kappa_{1,m}A_{1,m}\varphi_e\sigma_t\epsilon_{t+1} + \kappa_{1,m}A_{2,m}\sigma_w w_{t+1} + \varphi_d\sigma_t u_{t+1}]^2)$$

Since the shocks are independent, the cross products are zero. Hence:

$$\boxed{\mathbb{V}ar_t(r_{m,t+1}) = \sigma_t^2(\varphi_d^2 + \kappa_{1,m}^2 A_{1,m}^2 \varphi_e^2) + \kappa_{1,m}^2 A_{2,m}^2 \sigma_w^2}$$

Secondly, for $\mathbb{C}ov_t(\ln(M_{t+1}), r_{m,t+1})$:

$$\mathbb{C}ov_t(\ln(M_{t+1}), r_{m,t+1}) = \mathbb{E}_t([r_{m,t+1} - \mathbb{E}_t(r_{m,t+1})][\ln(M_{t+1}) - \mathbb{E}_t(\ln(M_{t+1}))])$$

From the previous calculations:

$$\Leftrightarrow \mathbb{C}ov_t(\ln(M_{t+1}), r_{m,t+1}) = \mathbb{E}_t \begin{pmatrix} [\kappa_{1,m} A_{1,m} \varphi_e \sigma_t \epsilon_{t+1} + \kappa_{1,m} A_{2,m} \sigma_w w_{t+1} + \varphi_d \sigma_t u_{t+1}] \\ [-\frac{\theta}{\psi} \sigma_t \eta_{t+1} + (\theta - 1)(\kappa_1 A_1 \varphi_e \sigma_t \epsilon_{t+1} + \kappa_1 A_2 \sigma_w w_{t+1} + \sigma_t \eta_{t+1})] \end{pmatrix}$$

$$\Leftrightarrow \mathbb{C}ov_t(\ln(M_{t+1}), r_{m,t+1}) = \mathbb{E}_t \begin{pmatrix} [\kappa_{1,m} A_{1,m} \varphi_e \sigma_t \epsilon_{t+1} + \kappa_{1,m} A_{2,m} \sigma_w w_{t+1} + \varphi_d \sigma_t u_{t+1}] \\ [-\frac{\theta}{\psi} \sigma_t \eta_{t+1} + (\theta - 1)\kappa_1 A_1 \varphi_e \sigma_t \epsilon_{t+1} + (\theta - 1)\kappa_1 A_2 \sigma_w w_{t+1} + (\theta - 1)\sigma_t \eta_{t+1})] \end{pmatrix}$$

Since the shocks are independent, the cross products are zero. Hence:

$$\Leftrightarrow \mathbb{C}ov_t(\ln(M_{t+1}), r_{m,t+1}) = \kappa_{1,m} A_{1,m} \varphi_e \sigma_t (\theta - 1)\kappa_1 A_1 \varphi_e \sigma_t + \kappa_{1,m} A_{2,m} \sigma_w (\theta - 1)\kappa_1 A_2 \sigma_w$$

$$\Leftrightarrow \mathbb{C}ov_t(\ln(M_{t+1}), r_{m,t+1}) = \kappa_{1,m} A_{1,m} (\theta - 1)\kappa_1 A_1 \varphi_e^2 \sigma_t^2 + \kappa_{1,m} A_{2,m} (\theta - 1)\kappa_1 A_2 \sigma_w^2$$

Putting it all together:

$$\mathbb{E}_t(r_{m,t+1} - r_f) + \frac{1}{2}\mathbb{V}ar_t(r_{m,t+1}) = -\mathbb{C}ov_t(\ln(M_{t+1}), r_{m,t+1})$$

Hence the conditional risk premium on the dividend claim is:

$$\boxed{\mathbb{E}_t(r_{m,t+1} - r_f) = \begin{bmatrix} -\kappa_{1,m} A_{1,m} (\theta - 1)\kappa_1 A_1 \varphi_e^2 \sigma_t^2 - \kappa_{1,m} A_{2,m} (\theta - 1)\kappa_1 A_2 \sigma_w^2 \\ -\frac{1}{2}(\sigma_t^2(\varphi_d^2 + \kappa_{1,m}^2 A_{1,m}^2 \varphi_e^2) + \kappa_{1,m}^2 A_{2,m}^2 \sigma_w^2) \end{bmatrix}}$$

Or without Jensen's correction term:

$$\boxed{\mathbb{E}_t(r_{m,t+1} - r_f) + \frac{1}{2}\mathbb{V}ar_t(r_{m,t+1}) = -\kappa_{1,m} A_{1,m} (\theta - 1)\kappa_1 A_1 \varphi_e^2 \sigma_t^2 - \kappa_{1,m} A_{2,m} (\theta - 1)\kappa_1 A_2 \sigma_w^2}$$

Unconditional Market Return Volatility

I calculate:

$$r_{m,t+1} - \mathbb{E}(r_{m,t+1}) = \kappa_{0,m} + \kappa_{1,m} z_{m,t+1} - z_{m,t} + \Delta d_{t+1} - \mathbb{E}(\kappa_{0,m} + \kappa_{1,m} z_{m,t+1} - z_{m,t} + \Delta d_{t+1})$$

Plugging in Δd_{t+1}, $z_{m,t}$,and $z_{m,t+1}$:

$$\Leftrightarrow r_{m,t+1} - \mathbb{E}(r_{m,t+1}) = \begin{bmatrix} \kappa_{1,m}(A_{0,m} + A_{1,m}x_{t+1} + A_{2,m}\sigma_{t+1}^2) - A_{0,m} \\ -A_{1,m}x_t - A_{2,m}\sigma_t^2 + \mu_d + \phi x_t + \varphi_d \sigma_t u_{t+1} \\ -\mathbb{E}(\kappa_{1,m}(A_{0,m} + A_{1,m}x_{t+1} + A_{2,m}\sigma_{t+1}^2) - A_{0,m} \\ -A_{1,m}x_t - A_{2,m}\sigma_t^2 + \mu_d + \phi x_t + \varphi_d \sigma_t u_{t+1}) \end{bmatrix}$$

$$\Leftrightarrow r_{m,t+1} - \mathbb{E}(r_{m,t+1}) = \begin{bmatrix} \kappa_{1,m}A_{1,m}x_{t+1} + \kappa_{1,m}A_{2,m}\sigma_{t+1}^2 - A_{1,m}x_t - A_{2,m}\sigma_t^2 + \phi x_t + \varphi_d \sigma_t u_{t+1} \\ -\mathbb{E}(\kappa_{1,m}A_{1,m}x_{t+1} + \kappa_{1,m}A_{2,m}\sigma_{t+1}^2 - A_{1,m}x_t - A_{2,m}\sigma_t^2 + \phi x_t + \varphi_d \sigma_t u_{t+1}) \end{bmatrix}$$

Plugging in the state variables in $t+1$:

$$\Leftrightarrow r_{m,t+1} - \mathbb{E}(r_{m,t+1}) = \begin{bmatrix} \kappa_{1,m}A_{1,m}(\rho x_t + \varphi_e \sigma_t \epsilon_{t+1}) + \kappa_{1,m}A_{2,m}(\sigma^2 + \nu_1(\sigma_t^2 - \sigma^2) + \sigma_w w_{t+1}) \\ -A_{1,m}x_t - A_{2,m}\sigma_t^2 + \phi x_t + \varphi_d \sigma_t u_{t+1} \\ -\mathbb{E}(\kappa_{1,m}A_{1,m}(\rho x_t + \varphi_e \sigma_t \epsilon_{t+1}) + \kappa_{1,m}A_{2,m}(\sigma^2 + \nu_1(\sigma_t^2 - \sigma^2) + \sigma_w w_{t+1}) \\ -A_{1,m}x_t - A_{2,m}\sigma_t^2 + \phi x_t + \varphi_d \sigma_t u_{t+1}) \end{bmatrix}$$

$$\Leftrightarrow r_{m,t+1} - \mathbb{E}(r_{m,t+1}) = \begin{bmatrix} \kappa_{1,m}A_{1,m}\rho x_t + \kappa_{1,m}A_{1,m}\varphi_e \sigma_t \epsilon_{t+1} + \kappa_{1,m}A_{2,m}\nu_1\sigma_t^2 + \kappa_{1,m}A_{2,m}\sigma_w w_{t+1} \\ -A_{1,m}x_t - A_{2,m}\sigma_t^2 + \phi x_t + \varphi_d \sigma_t u_{t+1} \\ -\mathbb{E}(\kappa_{1,m}A_{1,m}\rho x_t + \kappa_{1,m}A_{1,m}\varphi_e \sigma_t \epsilon_{t+1} + \kappa_{1,m}A_{2,m}\nu_1\sigma_t^2 + \kappa_{1,m}A_{2,m}\sigma_w w_{t+1} \\ -A_{1,m}x_t - A_{2,m}\sigma_t^2 + \phi x_t + \varphi_d \sigma_t u_{t+1}) \end{bmatrix}$$

Since the shocks are independent:

$$\Leftrightarrow r_{m,t+1} - \mathbb{E}(r_{m,t+1}) = \begin{bmatrix} \kappa_{1,m}A_{1,m}\rho x_t + \kappa_{1,m}A_{1,m}\varphi_e \sigma_t \epsilon_{t+1} + \kappa_{1,m}A_{2,m}\nu_1\sigma_t^2 + \kappa_{1,m}A_{2,m}\sigma_w w_{t+1} \\ -A_{1,m}x_t - A_{2,m}\sigma_t^2 + \phi x_t + \varphi_d \sigma_t u_{t+1} \\ -\mathbb{E}(\kappa_{1,m}A_{1,m}\rho x_t + \kappa_{1,m}A_{2,m}\nu_1\sigma_t^2 - A_{1,m}x_t - A_{2,m}\sigma_t^2 + \phi x_t) \end{bmatrix}$$

Notice that $\mathbb{E}(x_t) = 0$. Therefore:

$$\Leftrightarrow r_{m,t+1} - \mathbb{E}(r_{m,t+1}) = \begin{bmatrix} \kappa_{1,m}A_{1,m}\rho x_t + \kappa_{1,m}A_{1,m}\varphi_e \sigma_t \epsilon_{t+1} + \kappa_{1,m}A_{2,m}\nu_1\sigma_t^2 + \kappa_{1,m}A_{2,m}\sigma_w w_{t+1} \\ -A_{1,m}x_t - A_{2,m}\sigma_t^2 + \phi x_t + \varphi_d \sigma_t u_{t+1} - \mathbb{E}(\kappa_{1,m}A_{2,m}\nu_1\sigma_t^2 - A_{2,m}\sigma_t^2) \end{bmatrix}$$

$$\Leftrightarrow r_{m,t+1} - \mathbb{E}(r_{m,t+1}) = \begin{bmatrix} x_t(\kappa_{1,m}A_{1,m}\rho - A_{1,m} + \phi) + \kappa_{1,m}A_{1,m}\varphi_e \sigma_t \epsilon_{t+1} + \kappa_{1,m}A_{2,m}\sigma_w w_{t+1} \\ +A_{2,m}\sigma_t^2(\kappa_{1,m}\nu_1 - 1) + \varphi_d \sigma_t u_{t+1} - A_{2,m}\mathbb{E}(\sigma_t^2)(\kappa_{1,m}\nu_1 - 1) \end{bmatrix}$$

$$\Leftrightarrow r_{m,t+1} - \mathbb{E}(r_{m,t+1}) = \begin{bmatrix} x_t(A_{1,m}(\kappa_{1,m}\rho - 1) + \phi) + \kappa_{1,m}A_{1,m}\varphi_e \sigma_t \epsilon_{t+1} + \kappa_{1,m}A_{2,m}\sigma_w w_{t+1} \\ +A_{2,m}(\kappa_{1,m}\nu_1 - 1)(\sigma_t^2 - \mathbb{E}(\sigma_t^2)) + \varphi_d \sigma_t u_{t+1} \end{bmatrix}$$

From the previous part:

$$A_{1,m} = \frac{\phi - \frac{1}{\psi}}{1 - \kappa_{1,m}\rho} = \frac{\frac{1}{\psi} - \phi}{\kappa_{1,m}\rho - 1}$$

Hence:

$$\Leftrightarrow r_{m,t+1} - \mathbb{E}(r_{m,t+1}) = \begin{bmatrix} x_t(\frac{\frac{1}{\psi}-\phi}{\kappa_{1,m}\rho-1}(\kappa_{1,m}\rho - 1) + \phi) + \kappa_{1,m}A_{1,m}\varphi_e\sigma_t\epsilon_{t+1} + \kappa_{1,m}A_{2,m}\sigma_w w_{t+1} \\ +A_{2,m}(\kappa_{1,m}\nu_1 - 1)(\sigma_t^2 - \mathbb{E}(\sigma_t^2)) + \varphi_d\sigma_t u_{t+1} \end{bmatrix}$$

$$\Leftrightarrow r_{m,t+1} - \mathbb{E}(r_{m,t+1}) = \begin{bmatrix} \frac{x_t}{\psi} + \kappa_{1,m}A_{1,m}\varphi_e\sigma_t\epsilon_{t+1} + \kappa_{1,m}A_{2,m}\sigma_w w_{t+1} \\ +A_{2,m}(\kappa_{1,m}\nu_1 - 1)(\sigma_t^2 - \mathbb{E}(\sigma_t^2)) + \varphi_d\sigma_t u_{t+1} \end{bmatrix}$$

Consequently, the unconditional variance is:

$$\mathbb{V}ar(r_{m,t+1}) = \mathbb{E}([r_{m,t+1} - \mathbb{E}(r_{m,t+1})]^2)$$

From the previous calculations:

$$\Leftrightarrow \mathbb{V}ar(r_{m,t+1}) = \mathbb{E}\left(\begin{bmatrix} \frac{x_t}{\psi} + \kappa_{1,m}A_{1,m}\varphi_e\sigma_t\epsilon_{t+1} + \kappa_{1,m}A_{2,m}\sigma_w w_{t+1} \\ +A_{2,m}(\kappa_{1,m}\nu_1 - 1)(\sigma_t^2 - \mathbb{E}(\sigma_t^2)) + \varphi_d\sigma_t u_{t+1} \end{bmatrix}^2\right)$$

Since the shocks are independent, the cross products are zero. Additionally, $\mathbb{E}(\sigma_t^2) = \sigma^2$. Hence:

$$\boxed{\mathbb{V}ar(r_{m,t+1}) = \frac{\mathbb{V}ar(x_t)}{\psi^2} + \sigma^2(\kappa_{1,m}^2A_{1,m}^2\varphi_e^2 + \varphi_d^2) + A_{2,m}^2(\kappa_{1,m}\nu_1 - 1)^2\mathbb{V}ar(\sigma_t^2) + \kappa_{1,m}^2A_{2,m}^2\sigma_w^2}$$

The unconditional variance of $z_{m,t}$ (the price–dividend ratio for the market portfolio) is:

$$\mathbb{V}ar(z_{m,t}) = A_{1,m}^2\mathbb{V}ar(x_t) + A_{2,m}^2\mathbb{V}ar(\sigma_t^2)$$

Finally:

$$\mathbb{V}ar_{t+1}(r_{m,t+2}) - \mathbb{E}_t(\mathbb{V}ar_{t+1}(r_{m,t+2})) = \begin{bmatrix} \sigma_{t+1}^2(\varphi_d^2 + \kappa_{1,m}^2A_{1,m}^2\varphi_e^2) + \kappa_{1,m}^2A_{2,m}^2\sigma_w^2 \\ -\mathbb{E}_t(\sigma_{t+1}^2(\varphi_d^2 + \kappa_{1,m}^2A_{1,m}^2\varphi_e^2) + \kappa_{1,m}^2A_{2,m}^2\sigma_w^2) \end{bmatrix}$$

$$\Leftrightarrow \mathbb{V}ar_{t+1}(r_{m,t+2}) - \mathbb{E}_t(\mathbb{V}ar_{t+1}(r_{m,t+2})) = \left[\sigma_{t+1}^2(\varphi_d^2 + \kappa_{1,m}^2A_{1,m}^2\varphi_e^2) - \mathbb{E}_t(\sigma_{t+1}^2(\varphi_d^2 + \kappa_{1,m}^2A_{1,m}^2\varphi_e^2))\right]$$

Plugging in σ_{t+1}^2:

$$\Leftrightarrow \mathbb{V}ar_{t+1}(r_{m,t+2}) - \mathbb{E}_t(\mathbb{V}ar_{t+1}(r_{m,t+2})) = \begin{bmatrix} (\sigma^2 + \nu_1(\sigma_t^2 - \sigma^2) + \sigma_w w_{t+1})(\varphi_d^2 + \kappa_{1,m}^2A_{1,m}^2\varphi_e^2) \\ -\mathbb{E}_t((\sigma^2 + \nu_1(\sigma_t^2 - \sigma^2) + \sigma_w w_{t+1})(\varphi_d^2 + \kappa_{1,m}^2A_{1,m}^2\varphi_e^2)) \end{bmatrix}$$

Finally:

$$\boxed{\mathbb{V}ar_{t+1}(r_{m,t+2}) - \mathbb{E}_t(\mathbb{V}ar_{t+1}(r_{m,t+2})) = \sigma_w w_{t+1}(\varphi_d^2 + \kappa_{1,m}^2A_{1,m}^2\varphi_e^2)}$$

Risk-Free Rate

For the risk-free rate:

$$1 + R_{f,t+1} = \frac{1}{\mathbb{E}_t(M_{t+1})}$$

$$\Leftrightarrow r_{f,t+1} = -\ln(\mathbb{E}_t(M_{t+1}))$$

$$\Leftrightarrow r_{f,t+1} = -\ln(\mathbb{E}_t(e^{\ln(M_{t+1})}))$$

$$\Leftrightarrow r_{f,t+1} = -\ln(\mathbb{E}_t(e^{\theta\ln(\beta)-\frac{\theta}{\psi}\Delta c_{t+1}+(\theta-1)r_{w,t+1}}))$$

Because shocks are normally distributed:

$$\Leftrightarrow r_{f,t+1} = -\ln(e^{\theta\ln(\beta)-\frac{\theta}{\psi}\mathbb{E}_t(\Delta c_{t+1})+(\theta-1)\mathbb{E}_t(r_{w,t+1})+\frac{1}{2}\mathbb{V}ar_t(-\frac{\theta}{\psi}\Delta c_{t+1}+(\theta-1)r_{w,t+1})})$$

$$\Leftrightarrow r_{f,t+1} = -\theta\ln(\beta) + \frac{\theta}{\psi}\mathbb{E}_t(\Delta c_{t+1}) - (\theta-1)\mathbb{E}_t(r_{w,t+1}) - \frac{1}{2}\mathbb{V}ar_t(-\frac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)r_{w,t+1})$$

$$\Leftrightarrow 0 = r_{f,t+1} + \theta\ln(\beta) - \frac{\theta}{\psi}\mathbb{E}_t(\Delta c_{t+1}) + (\theta-1)\mathbb{E}_t(r_{w,t+1}) + \frac{1}{2}\mathbb{V}ar_t(-\frac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)r_{w,t+1}) \tag{9}$$

Similarly, for $r_{w,t+1}$:

$$\mathbb{E}_t\left[e^{\theta\ln(\beta)-\frac{\theta}{\psi}\Delta c_{t+1}+\theta r_{w,t+1}}\right] = 1$$

$$\Leftrightarrow \ln\left(\mathbb{E}_t\left[e^{\theta\ln(\beta)-\frac{\theta}{\psi}\Delta c_{t+1}+\theta r_{w,t+1}}\right]\right) = 0$$

Because shocks are normally distributed:

$$\Leftrightarrow \ln(e^{\theta\ln(\beta)-\frac{\theta}{\psi}\mathbb{E}_t(\Delta c_{t+1})+\theta\mathbb{E}_t(r_{w,t+1})+\frac{1}{2}\mathbb{V}ar_t(-\frac{\theta}{\psi}\Delta c_{t+1}+\theta r_{w,t+1})}) = 0$$

$$\Leftrightarrow \theta\ln(\beta) - \frac{\theta}{\psi}\mathbb{E}_t(\Delta c_{t+1}) + \theta\mathbb{E}_t(r_{w,t+1}) + \frac{1}{2}\mathbb{V}ar_t(-\frac{\theta}{\psi}\Delta c_{t+1} + \theta r_{w,t+1}) = 0$$

$$\Leftrightarrow \ln(\beta) - \frac{1}{\psi}\mathbb{E}_t(\Delta c_{t+1}) + \mathbb{E}_t(r_{w,t+1}) + \frac{\theta}{2}\mathbb{V}ar_t(-\frac{1}{\psi}\Delta c_{t+1} + r_{w,t+1}) = 0$$

$$\Leftrightarrow (\theta-1)\ln(\beta) - \frac{(\theta-1)}{\psi}\mathbb{E}_t(\Delta c_{t+1}) + (\theta-1)\mathbb{E}_t(r_{w,t+1}) + \frac{\theta(\theta-1)}{2}\mathbb{V}ar_t(-\frac{1}{\psi}\Delta c_{t+1} + r_{w,t+1}) = 0 \tag{10}$$

From 9 and 10, I have:

$$\begin{gathered} r_{f,t+1} + \theta\ln(\beta) - \tfrac{\theta}{\psi}\mathbb{E}_t(\Delta c_{t+1}) + (\theta-1)\mathbb{E}_t(r_{w,t+1}) + \tfrac{1}{2}\mathbb{V}ar_t(-\tfrac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)r_{w,t+1}) = \\ (\theta-1)\ln(\beta) - \tfrac{(\theta-1)}{\psi}\mathbb{E}_t(\Delta c_{t+1}) + (\theta-1)\mathbb{E}_t(r_{w,t+1}) + \tfrac{\theta(\theta-1)}{2}\mathbb{V}ar_t(-\tfrac{1}{\psi}\Delta c_{t+1} + r_{w,t+1}) \end{gathered}$$

$$\Leftrightarrow r_{f,t+1} = -\ln(\beta) + \tfrac{1}{\psi}\mathbb{E}_t(\Delta c_{t+1}) + \tfrac{\theta(\theta-1)}{2}\mathbb{V}ar_t(-\tfrac{1}{\psi}\Delta c_{t+1} + r_{w,t+1}) - \tfrac{1}{2}\mathbb{V}ar_t(-\tfrac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)r_{w,t+1})$$

Plugging in Δc_{t+1} and $r_{w,t+1}$:

$$\Leftrightarrow r_{f,t+1} = \begin{bmatrix} -\ln(\beta) + \tfrac{1}{\psi}\mathbb{E}_t(\Delta c_{t+1}) + \tfrac{\theta(\theta-1)}{2}\mathbb{V}ar_t(-\tfrac{1}{\psi}(\mu + x_t + \sigma_t\eta_{t+1}) + \kappa_0 + \kappa_1 z_{t+1} - z_t + \Delta c_{t+1}) \\ -\tfrac{1}{2}\mathbb{V}ar_t(-\tfrac{\theta}{\psi}(\mu + x_t + \sigma_t\eta_{t+1}) + (\theta-1)(\kappa_0 + \kappa_1 z_{t+1} - z_t + \Delta c_{t+1})) \end{bmatrix}$$

$$\Leftrightarrow r_{f,t+1} = \begin{bmatrix} -\ln(\beta) + \tfrac{1}{\psi}\mathbb{E}_t(\Delta c_{t+1}) + \tfrac{\theta(\theta-1)}{2}\mathbb{V}ar_t(-\tfrac{1}{\psi}\sigma_t\eta_{t+1} + \kappa_1 z_{t+1} + \Delta c_{t+1}) \\ -\tfrac{1}{2}\mathbb{V}ar_t(-\tfrac{\theta}{\psi}\sigma_t\eta_{t+1} + (\theta-1)(\kappa_1 z_{t+1} + \Delta c_{t+1})) \end{bmatrix}$$

Plugging in Δc_{t+1} and z_{t+1}:

$$\Leftrightarrow r_{f,t+1} = \begin{bmatrix} -\ln(\beta) + \tfrac{1}{\psi}\mathbb{E}_t(\Delta c_{t+1}) \\ +\tfrac{\theta(\theta-1)}{2}\mathbb{V}ar_t(-\tfrac{1}{\psi}\sigma_t\eta_{t+1} + \kappa_1(A_0 + A_1 x_{t+1} + A_2\sigma_{t+1}^2) + \mu + x_t + \sigma_t\eta_{t+1}) \\ -\tfrac{1}{2}\mathbb{V}ar_t(-\tfrac{\theta}{\psi}\sigma_t\eta_{t+1} + (\theta-1)(\kappa_1(A_0 + A_1 x_{t+1} + A_2\sigma_{t+1}^2) + \mu + x_t + \sigma_t\eta_{t+1})) \end{bmatrix}$$

$$\Leftrightarrow r_{f,t+1} = \begin{bmatrix} -\ln(\beta) + \tfrac{1}{\psi}\mathbb{E}_t(\Delta c_{t+1}) + \tfrac{\theta(\theta-1)}{2}\mathbb{V}ar_t((1 - \tfrac{1}{\psi})\sigma_t\eta_{t+1} + \kappa_1(A_1 x_{t+1} + A_2\sigma_{t+1}^2)) \\ -\tfrac{1}{2}\mathbb{V}ar_t((\theta - 1 - \tfrac{\theta}{\psi})\sigma_t\eta_{t+1} + (\theta-1)(\kappa_1(A_1 x_{t+1} + A_2\sigma_{t+1}^2))) \end{bmatrix}$$

Plugging in x_{t+1} and σ_{t+1}^2:

$$\Leftrightarrow r_{f,t+1} = \begin{bmatrix} -\ln(\beta) + \tfrac{1}{\psi}\mathbb{E}_t(\Delta c_{t+1}) \\ +\tfrac{\theta(\theta-1)}{2}\mathbb{V}ar_t((1 - \tfrac{1}{\psi})\sigma_t\eta_{t+1} + \kappa_1(A_1(\rho x_t + \varphi_e\sigma_t\epsilon_{t+1}) \\ +A_2(\sigma^2 + \nu_1(\sigma_t^2 - \sigma^2) + \sigma_w w_{t+1}))) \\ -\tfrac{1}{2}\mathbb{V}ar_t((\theta - 1 - \tfrac{\theta}{\psi})\sigma_t\eta_{t+1} + (\theta-1)(\kappa_1(A_1(\rho x_t + \varphi_e\sigma_t\epsilon_{t+1}) \\ +A_2(\sigma^2 + \nu_1(\sigma_t^2 - \sigma^2) + \sigma_w w_{t+1})))) \end{bmatrix}$$

$$\Leftrightarrow r_{f,t+1} = \begin{bmatrix} -\ln(\beta) + \tfrac{1}{\psi}\mathbb{E}_t(\Delta c_{t+1}) \\ +\tfrac{\theta(\theta-1)}{2}\mathbb{V}ar_t((1 - \tfrac{1}{\psi})\sigma_t\eta_{t+1} + \kappa_1 A_1\varphi_e\sigma_t\epsilon_{t+1} + \kappa_1 A_2\sigma_w w_{t+1}) \\ -\tfrac{1}{2}\mathbb{V}ar_t((\theta - 1 - \tfrac{\theta}{\psi})\sigma_t\eta_{t+1} + (\theta-1)(\kappa_1 A_1\varphi_e\sigma_t\epsilon_{t+1} + \kappa_1 A_2\sigma_w w_{t+1})) \end{bmatrix}$$

Since $\epsilon_{t+1}, \eta_{t+1}, w_{t+1} \sim N(0,1)$ and are uncorrelated:

$$\Leftrightarrow r_{f,t+1} = \begin{bmatrix} -\ln(\beta) + \frac{1}{\psi}\mathbb{E}_t(\Delta c_{t+1}) \\ +\frac{\theta(\theta-1)}{2}((1-\frac{1}{\psi})^2\sigma_t^2 + (\kappa_1 A_1 \varphi_e)^2\sigma_t^2 + (\kappa_1 A_2)^2\sigma_w^2) \\ -\frac{1}{2}((\theta - 1 - \frac{\theta}{\psi})^2\sigma_t^2 + (\theta-1)^2(\kappa_1 A_1\varphi_e)^2\sigma_t^2 + (\theta-1)^2(\kappa_1 A_2)^2\sigma_w^2) \end{bmatrix}$$

$$\Leftrightarrow r_{f,t+1} = \begin{bmatrix} -\ln(\beta) + \frac{1}{\psi}\mathbb{E}_t(\Delta c_{t+1}) \\ +\frac{\theta(\theta-1)}{2}(1-\frac{1}{\psi})^2\sigma_t^2 + \frac{\theta-1}{2}(\kappa_1 A_1\varphi_e)^2\sigma_t^2 + \frac{\theta-1}{2}(\kappa_1 A_2)^2\sigma_w^2 - \frac{1}{2}(\theta - 1 - \frac{\theta}{\psi})^2\sigma_t^2 \end{bmatrix}$$

$$\Leftrightarrow r_{f,t+1} = \begin{bmatrix} -\ln(\beta) + \frac{1}{\psi}\mathbb{E}_t(\Delta c_{t+1}) \\ +\frac{1}{2}(\theta^2-\theta)(1-\frac{2}{\psi}+\frac{1}{\psi^2})\sigma_t^2 + \frac{\theta-1}{2}(\kappa_1 A_1\varphi_e)^2\sigma_t^2 + \frac{\theta-1}{2}(\kappa_1 A_2)^2\sigma_w^2 \\ -\frac{1}{2}(\theta^2 - 2\theta + 1 - \frac{2\theta^2}{\psi} + \frac{\theta^2}{\psi^2} + \frac{2\theta}{\psi})\sigma_t^2 \end{bmatrix}$$

$$\Leftrightarrow r_{f,t+1} = \begin{bmatrix} -\ln(\beta) + \frac{1}{\psi}\mathbb{E}_t(\Delta c_{t+1}) \\ +\frac{1}{2}(\theta^2 - \theta - \frac{2\theta^2}{\psi} + \frac{2\theta}{\psi} + \frac{\theta^2}{\psi^2} - \frac{\theta}{\psi^2})\sigma_t^2 + \frac{\theta-1}{2}(\kappa_1 A_1\varphi_e)^2\sigma_t^2 + \frac{\theta-1}{2}(\kappa_1 A_2)^2\sigma_w^2 \\ -\frac{1}{2}(\theta^2 - 2\theta + 1 - \frac{2\theta^2}{\psi} + \frac{\theta^2}{\psi^2} + \frac{2\theta}{\psi})\sigma_t^2 \end{bmatrix}$$

$$\Leftrightarrow r_{f,t+1} = \begin{bmatrix} -\ln(\beta) + \frac{1}{\psi}\mathbb{E}_t(\Delta c_{t+1}) \\ +\frac{1}{2}(\theta - 1 - \frac{\theta}{\psi^2})\sigma_t^2 + \frac{\theta-1}{2}(\kappa_1 A_1\varphi_e)^2\sigma_t^2 + \frac{\theta-1}{2}(\kappa_1 A_2)^2\sigma_w^2 \end{bmatrix}$$

Plugging in Δc_{t+1}:

$$\Leftrightarrow r_{f,t+1} = \begin{bmatrix} -\ln(\beta) + \frac{1}{\psi}\mathbb{E}_t(\mu + x_t + \sigma_t\eta_{t+1}) \\ +\frac{1}{2}(\theta - 1 - \frac{\theta}{\psi^2})\sigma_t^2 + \frac{\theta-1}{2}(\kappa_1 A_1\varphi_e)^2\sigma_t^2 + \frac{\theta-1}{2}(\kappa_1 A_2)^2\sigma_w^2 \end{bmatrix}$$

Hence:

$$\boxed{r_{f,t+1} = -\ln(\beta) + \frac{1}{\psi}\mu + \frac{\theta-1}{2}(\kappa_1 A_2)^2\sigma_w^2 + \frac{1}{\psi}x_t + \frac{1}{2}(\theta - 1 - \frac{\theta}{\psi^2})\sigma_t^2 + \frac{\theta-1}{2}(\kappa_1 A_1\varphi_e)^2\sigma_t^2}$$

From my previous calculations:

$$\Leftrightarrow r_{f,t+1} = -\theta\ln(\beta) + \frac{\theta}{\psi}\mathbb{E}_t(\Delta c_{t+1}) - (\theta-1)\mathbb{E}_t(r_{w,t+1}) - \frac{1}{2}\mathbb{V}ar_t(-\frac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)r_{w,t+1})$$

$$\Leftrightarrow r_{f,t+1} = -\theta\ln(\beta) + \frac{\theta}{\psi}\mathbb{E}_t(\Delta c_{t+1}) + (1-\theta)\mathbb{E}_t(r_{w,t+1}) - \frac{1}{2}\mathbb{V}ar_t(-\frac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)r_{w,t+1})$$

Subtracting $(1-\theta)r_{f,t+1}$ from both sides:

$$\Leftrightarrow \theta r_{f,t+1} = -\theta\ln(\beta) + \frac{\theta}{\psi}\mathbb{E}_t(\Delta c_{t+1}) + (1-\theta)(\mathbb{E}_t(r_{w,t+1}) - r_{f,t+1}) - \frac{1}{2}\mathbb{V}ar_t(-\frac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)r_{w,t+1})$$

Dividing both sides by θ:

$$\Leftrightarrow r_{f,t+1} = -\ln(\beta) + \frac{1}{\psi}\mathbb{E}_t(\Delta c_{t+1}) + \frac{(1-\theta)}{\theta}(\mathbb{E}_t(r_{w,t+1}) - r_{f,t+1}) - \frac{1}{2\theta}\mathbb{V}ar_t(-\frac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)r_{w,t+1})$$

Since I already computed everything except for $\mathbb{V}ar_t(-\frac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)r_{w,t+1})$:

$$\mathbb{V}ar_t(-\frac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)r_{w,t+1}) = \mathbb{V}ar_t(\ln(M_{t+1}))$$

$$\mathbb{V}ar_t(-\frac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)r_{w,t+1}) = \mathbb{E}_t([\ln(M_{t+1}) - \mathbb{E}_t(\ln(M_{t+1}))]^2)$$

From my previous calculations:

$$\ln(M_{t+1}) - \mathbb{E}_t(\ln(M_{t+1})) = -\frac{\theta}{\psi}\sigma_t\eta_{t+1} + (\theta-1)(\kappa_1 A_1\varphi_e\sigma_t\epsilon_{t+1} + \kappa_1 A_2\sigma_w w_{t+1} + \sigma_t\eta_{t+1})$$

Therefore:

$$\mathbb{V}ar_t(-\frac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)r_{w,t+1}) = \mathbb{E}_t([-\frac{\theta}{\psi}\sigma_t\eta_{t+1} + (\theta-1)(\kappa_1 A_1\varphi_e\sigma_t\epsilon_{t+1} + \kappa_1 A_2\sigma_w w_{t+1} + \sigma_t\eta_{t+1})]^2)$$

Since the shocks are independent, the cross products are zero. Hence:

$$\mathbb{V}ar_t(-\frac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)r_{w,t+1}) = (\theta-1-\frac{\theta}{\psi})^2\sigma_t^2 + (\theta-1)^2(\kappa_1 A_1\varphi_e)^2\sigma_t^2 + (\theta-1)^2(\kappa_1 A_2)^2\sigma_w^2$$

For the unconditional expectation of the risk-free rate, notice that $\mathbb{E}(x_t) = 0$ and $\mathbb{E}(\sigma_t^2) = \sigma^2$. Hence:

$$\boxed{\mathbb{E}(r_{f,t+1}) = -\ln(\beta) + \frac{1}{\psi}\mu + \frac{\theta-1}{2}(\kappa_1 A_2)^2\sigma_w^2 + \frac{1}{2}(\theta-1-\frac{\theta}{\psi^2})\sigma^2 + \frac{\theta-1}{2}(\kappa_1 A_1\varphi_e)^2\sigma^2}$$

Alternatively:

$$\mathbb{E}(r_{f,t+1}) = -\ln(\beta) + \frac{1}{\psi}\mathbb{E}(\Delta c_{t+1}) + \frac{(1-\theta)}{\theta}\mathbb{E}((r_{w,t+1}) - r_{f,t+1}) - \frac{1}{2\theta}\mathbb{E}(\mathbb{V}ar_t(-\frac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)r_{w,t+1}))$$

$$\Leftrightarrow \mathbb{E}(r_{f,t+1}) = \begin{bmatrix} -\ln(\beta) + \frac{1}{\psi}\mathbb{E}(\Delta c_{t+1}) + \frac{(1-\theta)}{\theta}\mathbb{E}((r_{w,t+1}) - r_{f,t+1}) \\ -\frac{1}{2\theta}\mathbb{E}((\theta-1-\frac{\theta}{\psi})^2\sigma_t^2 + (\theta-1)^2(\kappa_1 A_1\varphi_e)^2\sigma_t^2 + (\theta-1)^2(\kappa_1 A_2)^2\sigma_w^2) \end{bmatrix}$$

$$\Leftrightarrow \mathbb{E}(r_{f,t+1}) = \begin{bmatrix} -\ln(\beta) + \frac{1}{\psi}\mathbb{E}(\Delta c_{t+1}) + \frac{(1-\theta)}{\theta}\mathbb{E}((r_{w,t+1}) - r_{f,t+1}) \\ -\frac{1}{2\theta}((\theta - 1 - \frac{\theta}{\psi})^2\mathbb{E}(\sigma_t^2) + (\theta-1)^2(\kappa_1 A_1 \varphi_e)^2\mathbb{E}(\sigma_t^2) + (\theta-1)^2(\kappa_1 A_2)^2\sigma_w^2) \end{bmatrix}$$

Finally, the unconditional variance of the risk-free rate is:

$$\boxed{\mathbb{V}ar(r_{f,t+1}) = \frac{1}{\psi^2}\mathbb{V}ar(x_t) + \frac{1}{4}\mathbb{V}ar(\sigma_t^2)\left[(\theta - 1 - \frac{\theta}{\psi^2}) + (\theta-1)(\kappa_1 A_1 \varphi_e)^2\right]^2}$$