## **IMPERIAL**



# **Generalised Spin Structures**

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#### Idea

Let M be a spin manifold.



## Example

If M admits a spin structure carrying a nowhere-vanishing parallel spinor, then M is Ricci-flat.

- Question: what if M is not spin?
- Idea: equip every orientable manifold with spin-like structures.

## **Spin structures I**

Let M<sup>n</sup> be an oriented Riemannian manifold

with bundle of oriented orthonormal frames FM.

A **spin structure** is a lift of the structure group of FM to the group  $\mathrm{Spin}(n)$  along the double covering

$$\lambda_n \colon \operatorname{Spin}(n) \to \operatorname{SO}(n).$$

In other words, it is a pair  $(P,\Phi)$  where

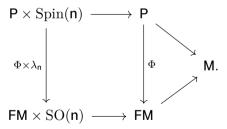
- P is a principal Spin(n)-bundle over M, and
- $\Phi \colon P \to FM$  is a  $\mathrm{Spin}(n)$ -equivariant bundle map covering the identity, where  $\mathrm{Spin}(n)$  acts on FM via  $\lambda_n$ .

## **Spin structures II**



In other words, it is a pair  $(P, \Phi)$  where

- ullet P is a principal  $\mathrm{Spin}(n)$ -bundle over M, and
- $\Phi \colon P \to FM$  is a  $\mathrm{Spin}(n)$ -equivariant bundle map covering the identity, where  $\mathrm{Spin}(n)$  acts on FM via  $\lambda_n$ .



## **Spin structures III**



Spin structures turn out not to depend on the orientation or the Riemannian metric:

#### **Theorem**

• M admits a spin structure if and only if the first two Stiefel-Whitney classes of M vanish:

$$\mathbf{w}_1(\mathsf{M}) = \mathbf{w}_2(\mathsf{M}) = 0.$$

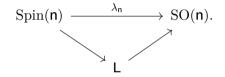
• In this case, spin structures are classified by the first cohomology  $H^1(M; \mathbb{Z}_2)$ .

## Spin<sup>r</sup> structures I



What can we do with non-spin manifolds?

Idea: enlarge the spin group:



## Example

$$\mathrm{Spin}^{\mathbb{C}}(\mathbf{n}) = \frac{\mathrm{Spin}(\mathbf{n}) \times \mathrm{U}(1)}{\langle (-1, -1) \rangle}, \qquad \mathrm{Spin}^{\mathbb{H}}(\mathbf{n}) = \frac{\mathrm{Spin}(\mathbf{n}) \times \mathrm{Sp}(1)}{\langle (-1, -1) \rangle}.$$

## Spin<sup>r</sup> structures II



Note that  $U(1) \cong Spin(2)$  and  $Sp(1) \cong Spin(3)$ .

#### Definition

$$\mathrm{Spin}^r(n) := \frac{\mathrm{Spin}(n) \times \mathrm{Spin}(r)}{\langle (-1,-1) \rangle}.$$

#### **Definition**

A **spin**<sup>r</sup> **structure** on an oriented Riemannian n-manifold is a lift of the structure group of the positively oriented orthonormal frame bundle FM to  $\mathrm{Spin}^r(n)$  along the composition

$$\begin{split} \lambda_n^r\colon \operatorname{Spin}^r(n) &\to \operatorname{SO}(n) \times \operatorname{SO}(r) \to \operatorname{SO}(n) \\ [a,b] &\mapsto (\lambda_n(a),\lambda_r(b)) \mapsto \lambda_n(a). \end{split}$$

## Spin<sup>r</sup> structures III



In other words, a **spin**<sup>r</sup> **structure** on M consists of the following data:

- a principal  $\operatorname{Spin}^{r}(n)$ -bundle P over M, and
- a  $\mathrm{Spin}^r(n)$ -equivariant bundle map  $\Phi \colon \mathsf{P} \to \mathsf{FM}$ , where  $\mathrm{Spin}^r(n)$  acts on FM through  $\lambda_n^r$ .

### Definition

The rank-r vector bundle associated to P along the composition

$$\mathrm{Spin}^r(n) \to \mathrm{SO}(n) \times \mathrm{SO}(r) \to \mathrm{SO}(r)$$

is called the **auxiliary bundle** of the spin<sup>r</sup> structure.

#### Characterisation



## Theorem (Albanese - Milivojević, 2021 [1])

The following are equivalent for an oriented Riemannian manifold M:

- 1. M is spin<sup>r</sup>;
- 2. there is an orientable rank-r real vector bundle  $\pi \colon \mathsf{E} \to \mathsf{M}$  such that  $\mathsf{TM} \oplus \mathsf{E}$  is spin, i.e.,  $\mathsf{w}_1(\mathsf{TM} \oplus \mathsf{E}) = \mathsf{w}_2(\mathsf{TM} \oplus \mathsf{E}) = 0$ ;
- 3. M embeds in a spin manifold with codimension r.

## A few examples

## **Examples**



1. Every oriented n-manifold M admits a spin<sup>n</sup> structure. Take E = TM, and note that

$$w_2(\mathsf{TM} \oplus \mathsf{E}) = w_2(\mathsf{TM}) + w_1(\mathsf{TM})w_1(\mathsf{E}) + w_2(\mathsf{E}) = 2w_2(\mathsf{TM}) = 0.$$

Every almost-complex manifold admits a spin<sup>2</sup> structure.
 Take E to be the anticanonical bundle of an almost-complex structure, and compute

$$\mathbf{w}_2(\mathsf{TM} \oplus \mathsf{E}) = \mathbf{w}_2(\mathsf{TM}) + \mathbf{w}_2(\mathsf{E}) = 2(\mathbf{c}_1(\mathsf{TM}) \bmod 2) = 0.$$

3. Every almost-quaternionic manifold admits a spin $^3$  structure. Take E to be the rank-3 subbundle of  ${\rm End}(TM)$  spanned by I, J, K.

## Proof of Albanese-Milivojević 1 ← 2



M is spin<sup>r</sup> ⇒ ∃ E such that TM ⊕ E is spin:
 Take E to be the auxiliary bundle of a spin<sup>r</sup> structure. Then, the frame bundle of TM ⊕ E lifts to Spin<sup>r</sup>(n) along

$$\mathrm{Spin}^r(n) \to \mathrm{Spin}(n+r) \to \mathrm{SO}(n+r).$$

In particular, it lifts to Spin(n + r).

∃ E such that TM ⊕ E is spin ⇒ M is spin<sup>r</sup>:
 This follows from the fact that the following is a pullback diagram in the categorical sense:

$$\begin{array}{ccc} \operatorname{Spin}^r(n) & \longrightarrow & \operatorname{Spin}(n+r) \\ \downarrow & & \downarrow \\ \operatorname{SO}(n) \times \operatorname{SO}(r) & \longrightarrow & \operatorname{SO}(n+r). \end{array}$$

## Proof of Albanese-Milivojević 2 ← 3

∃ E such that TM ⊕ E is spin ⇒ M embeds into a spin manifold with codimension r:
 M embeds with codimension r into the total space of E, which is spin because

$$\mathbf{w}_2(\mathsf{TE}) = \mathbf{w}_2(\pi^*(\mathsf{TM} \oplus \mathsf{E})) = \pi^*(\mathbf{w}_2(\mathsf{TM} \oplus \mathsf{E})) = 0.$$

M embeds into a spin manifold with codimension r ⇒ ∃ E such that TM ⊕ E is spin:
 Let ι: M ⇔ X be such an embedding, and take E to be the normal bundle of ι. Then,

$$0 = \iota^*(\mathsf{w}_2(\mathsf{TX})) = \mathsf{w}_2(\iota^*(\mathsf{TX})) = \mathsf{w}_2(\mathsf{TM} \oplus \mathsf{E}).$$



#### Invariance

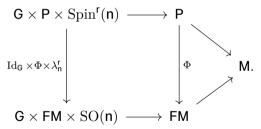


Let G be a connected Lie group acting smoothly on M by isometries.

Then, G acts naturally on FM by bundle isomorphisms.

#### **Definition**

A G-invariant spin structure on M is a spin structure (P,  $\Phi$ ) where both P and  $\Phi$  are G-equivariant.



## **Homogeneous Spaces**



- Let  $M^n$  be an oriented Riemannian homogeneous G-space, and fix  $o \in M$ .
- Then,  $M \cong G/H$ , where  $H = \operatorname{Stab}_{G}(o)$ .
- For every  $h \in H$ ,

$$(L_h)_* \colon T_oM \to T_{h \cdot o}M = T_oM.$$

### **Definition**

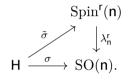
The **isotropy representation** is defined as

$$\sigma \colon H \to \mathrm{SO}(T_oM) \cong \mathrm{SO}(n)$$
 
$$h \mapsto (L_h)_*.$$

The positively oriented orthonormal frame bundle FM of M is isomorphic to  $G \times_{\sigma} SO(n)$ .

## Invariant spin<sup>r</sup> Structures on Homogeneous Spaces

Suppose  $\sigma$  lifts to  $\mathrm{Spin}^{\mathbf{r}}(\mathbf{n})$ :



Then, the pair  $(P, \Phi)$  where

- $P = G \times_{\tilde{\sigma}} \operatorname{Spin}^r(n)$ , and
- $\bullet \ \Phi \colon \mathsf{P} \to \mathsf{FM}, \quad [\mathsf{g},\mu] \mapsto [\mathsf{g},\lambda^{\mathsf{r}}_{\mathsf{n}}(\mu)]$

defines a G-invariant spin<sup>r</sup> structure on M.

#### Classification



## Theorem (A. - Lawn, 2023, [3])

Let G/H be an n-dimensional oriented Riemannian homogeneous space with H connected and isotropy representation  $\sigma\colon H\to \mathrm{SO}(n)$ . Then, there is a bijective correspondence between

- G-invariant spin<sup>r</sup> structures on G/H modulo G-equivariant equivalence, and
- Lie group homomorphisms  $\varphi \colon H \to \mathrm{SO}(r)$  such that  $\sigma \times \varphi \colon H \to \mathrm{SO}(n) \times \mathrm{SO}(r)$  lifts to  $\mathrm{Spin}^r(n)$  along  $\lambda_n^r$  modulo conjugation by an element of  $\mathrm{SO}(r)$ .

## **Invariant spin<sup>r</sup> structures on spheres**



Sphere	Acting group G	Minimal r for G-invariant spin <sup>r</sup> structure		
S <sup>n</sup>	SO(n+1)	r=n, if $n  eq 4$		
		r=3, if $n=4$		
$S^{2n+1}$	$\mathrm{U}(n+1)$	r=2		
$S^{2n+1}$	SU(n+1)	r = 1		
$S^{4n+3}$	$\operatorname{Sp}(\mathbf{n}+1)$	r = 1		
S <sup>4n+3</sup>	$\mathrm{Sp}(n+1)\cdot\mathrm{U}(1)$	r=1, if $n$ odd		
		r=2, if n even		
S <sup>4n+3</sup>	$\operatorname{Sp}(\mathbf{n}+1)\cdot\operatorname{Sp}(1)$	r=1, if $n$ odd		
		r=3, if n even		
$S^6$	$G_2$	r = 1		
$S^7$	$\operatorname{Spin}(7)$	r = 1		
$S^{15}$	Spin(9)	r = 1		

## Special invariant spin structures on spheres and holonomy lifts



## Theorem (A.-Lawn, 2023 [3])

Let G be the holonomy group of a simply connected irreducible non-symmetric Riemannian manifold of dimension n  $+1 \geq 4$ . Let H  $\leq$  G be a subgroup such that S<sup>n</sup>  $\cong$  G/H, which exists, by Berger's classification. Then, the following are equivalent:

- 1. There exists a homomorphic lift of the holonomy representation to  $\mathrm{Spin}^r(n+1)$ .
- 2. S<sup>n</sup> has a G-invariant spin<sup>r</sup> structure with strongly G-trivial auxiliary bundle.

## **Spinors**

The complex vector bundle  $\Sigma M o M$  associated to a spin structure via

$$\Delta_n \colon \operatorname{Spin}(n) \to \operatorname{End}_{\mathbb{C}}(\Sigma_n)$$

is called the **spinor bundle**: its sections are known as **spinors**.

Clifford multiplication: tangent vectors act fibrewise on spinors, satisfying

$$\mathbf{X}\cdot\mathbf{Y}\cdot\boldsymbol{\psi}+\mathbf{Y}\cdot\mathbf{X}\cdot\boldsymbol{\psi}=-2\mathbf{g}(\mathbf{X},\mathbf{Y})\boldsymbol{\psi}$$

for all vector fields X, Y  $\in \Gamma(TM)$  and spinors  $\psi \in \Gamma(\Sigma M)$ .

The Levi-Civita connection of M induces the **spin connection**  $\nabla$  on  $\Sigma$ M.

## **Generalised Killing spinors**



A spinor  $\psi$  is **generalised Killing** if it satisfies

$$\nabla_{\mathsf{X}}\psi = \mathsf{A}(\mathsf{X})\cdot\psi,$$

for all vector fields X, where A is a symmetric endomorphism of TM.

- If  $\mathbf{A}=0$ ,  $\psi$  is parallel;
- If  $A = \lambda \operatorname{Id}$  for some constant  $\lambda \in \mathbb{C}$ ,  $\psi$  is **Killing**.

The existence of these spinors is often related to curvature properties and G-structures.

## Spin<sup>r</sup> spinors

Let  $(P,\Phi)$  be a spin<sup>r</sup> structure. For odd m, the m-twisted spin<sup>r</sup> spinor bundle  $\Sigma^m_{n,r}M$  is one associated to P via the representation

$$\Delta_{n,r}^m := \Delta_n \otimes \Delta_r^{\otimes m}$$

of  $\mathrm{Spin}^r(n).$  Sections of  $\Sigma^m_{n,r}M$  are called  $\text{spin}^r$  spinors.

The Levi-Civita connection on M and a connection on the auxiliary bundle determine a connection on each  $\Sigma_{n,r}^m$ .

There is a characterisation of special holonomy in terms of **parallel twisted pure spinors** by Herrera - Santana, 2019 [4].

## Invariant spin<sup>r</sup> spinors on projective spaces



Jointly with Hofmann [2], we obtained the following:

М	G	r	m	$\dim \left(\Sigma_{*,r}^{m}\right)_{\mathrm{inv}}$	Special spinors	Geometry
$\mathbb{CP}^{n}$	SU(n+1)	2	1	2	pure, parallel	Kähler-Einstein
$\mathbb{CP}^{2n+1}$	$\operatorname{Sp}(\mathbf{n}+1)$	2, if n even	1	2	pure, parallel	Kähler-Einstein
		1, if n odd	1	2	generalised Killing	Einstein, nearly Kähler (n $= 1$ )
$\mathbb{HP}^{n}$	$\operatorname{Sp}(\mathbf{n}+1)$	3	n	1	pure, parallel	quaternionic Kähler

Table: For each compact, simple, and simply connected Lie group G acting transitively on M: the minimum values of r, m such that M admits a G-invariant spin<sup>r</sup> structure that carries a non-zero invariant m-twisted spin<sup>r</sup> spinor, the dimension of the space of such invariant spinors, and the geometric significance of these.

#### References

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- [3] D. Artacho and M.-A. Lawn. Generalised Spin<sup>r</sup> Structures on Homogeneous Spaces. 2023. DOI: 10.48550/ARXIV.2303.05433.
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# Thank you

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