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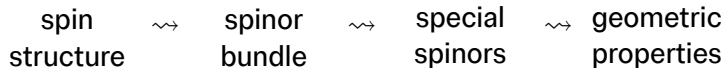
Generalised Spin Structures

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Idea

Let M be a spin manifold.



Example

If M admits a spin structure carrying a nowhere-vanishing parallel spinor, then M is Ricci-flat.

- **Question:** what if M is not spin?
- **Idea:** equip every orientable manifold with spin-like structures.

Spin structures I

Let M^n be an oriented Riemannian manifold with bundle of oriented orthonormal frames FM.

A **spin structure** is a lift of the structure group of FM to the group $\text{Spin}(n)$ along the double covering

$$\lambda_n: \text{Spin}(n) \rightarrow \text{SO}(n).$$

In other words, it is a pair (P, Φ) where

- P is a principal $\text{Spin}(n)$ -bundle over M , and
- $\Phi: P \rightarrow \text{FM}$ is a $\text{Spin}(n)$ -equivariant bundle map covering the identity, where $\text{Spin}(n)$ acts on FM via λ_n .

Spin structures II

In other words, it is a pair (P, Φ) where

- P is a principal $\text{Spin}(n)$ -bundle over M , and
- $\Phi: P \rightarrow FM$ is a $\text{Spin}(n)$ -equivariant bundle map covering the identity, where $\text{Spin}(n)$ acts on FM via λ_n .

$$\begin{array}{ccc}
\mathbf{P} \times \text{Spin}(\mathbf{n}) & \longrightarrow & \mathbf{P} \\
\downarrow \Phi \times \lambda_n & & \downarrow \Phi \\
\mathbf{FM} \times \text{SO}(\mathbf{n}) & \longrightarrow & \mathbf{FM}
\end{array}
\quad \begin{array}{c} \nearrow \\ \searrow \end{array} \quad \mathbf{M.}$$

Spin structures III

Spin structures turn out not to depend on the orientation or the Riemannian metric:

Theorem

- M admits a spin structure if and only if the first two Stiefel-Whitney classes of M vanish:

$$w_1(M) = w_2(M) = 0.$$

- In this case, spin structures are classified by first cohomology $H^1(M; \mathbb{Z}_2)$.

Spin^r structures I

What can we do with non-spin manifolds?

Idea: enlarge the spin group:

$$\begin{array}{ccc} \text{Spin}(\mathfrak{n}) & \xrightarrow{\lambda_n} & \text{SO}(\mathfrak{n}). \\ & \searrow & \nearrow \\ & \mathbf{L} & \end{array}$$

Example

$$\text{Spin}^{\mathbb{C}}(\mathfrak{n}) = \frac{\text{Spin}(\mathfrak{n}) \times \text{U}(1)}{\langle(-1, -1)\rangle}, \quad \text{Spin}^{\mathbb{H}}(\mathfrak{n}) = \frac{\text{Spin}(\mathfrak{n}) \times \text{Sp}(1)}{\langle(-1, -1)\rangle}.$$

Spin^r structures II

Note that $U(1) \cong \text{Spin}(2)$ and $\text{Sp}(1) \cong \text{Spin}(3)$.

Definition

$$\text{Spin}^r(\mathbf{n}) = \frac{\text{Spin}(\mathbf{n}) \times \text{Spin}(r)}{\langle (-1, -1) \rangle}.$$

Definition

A **spin^r structure** on an oriented Riemannian n -manifold is a reduction (lift) of the structure group of the positively oriented orthonormal frame bundle FM to $\text{Spin}^r(n)$ along the composition

$$\begin{aligned} \lambda_n^r: \text{Spin}^r(n) &\rightarrow \text{SO}(n) \times \text{SO}(r) \rightarrow \text{SO}(n) \\ [\mathbf{a}, \mathbf{b}] &\mapsto (\lambda_n(\mathbf{a}), \lambda_r(\mathbf{b})) \mapsto \lambda_n(\mathbf{a}). \end{aligned}$$

Spin^r structures III

In other words, a **spin^r structure** on M consists of the following data:

- a principal $\text{Spin}^r(n)$ -bundle P over M , and
- a $\text{Spin}^r(n)$ -equivariant bundle map $\Phi: P \rightarrow FM$, where $\text{Spin}^r(n)$ acts on FM through λ_n^r .

Definition

The rank- r vector bundle associated to P along the composition

$$\text{Spin}^r(n) \rightarrow \text{SO}(n) \times \text{SO}(r) \rightarrow \text{SO}(r)$$

is called the **auxiliary bundle** of the spin^r structure.

Characterisation

Theorem (Albanese - Milivojević, 2021 [1])

The following are equivalent for an oriented Riemannian manifold M :

1. M is spin^r ;
2. there is an orientable rank- r real vector bundle $\pi: E \rightarrow M$ such that $TM \oplus E$ is spin, i.e., $w_1(TM \oplus E) = w_2(TM \oplus E) = 0$;
3. M embeds in a spin manifold with codimension r .

Proof of Albanese-Milivojević 1 \iff 2

- M is $\text{spin}^r \implies \exists E$ such that $TM \oplus E$ is spin :

Take E to be the auxiliary bundle of a spin^r structure. Then, the frame bundle of $TM \oplus E$ lifts to $\text{Spin}^r(n)$ along

$$\text{Spin}^r(n) \rightarrow \text{Spin}(n+r) \rightarrow \text{SO}(n+r).$$

In particular, it lifts to $\text{Spin}(n+r)$.

- $\exists E$ such that $TM \oplus E$ is $\text{spin} \implies M$ is spin^r :

This follows from the fact that the following is a pullback diagram in the categorical sense:

$$\begin{array}{ccc} \text{Spin}^r(n) & \longrightarrow & \text{Spin}(n+r) \\ \downarrow & & \downarrow \\ \text{SO}(n) \times \text{SO}(r) & \longrightarrow & \text{SO}(n+r). \end{array}$$

Proof of Albanese-Milivojević 2 \iff 3

- $\exists E$ such that $TM \oplus E$ is spin \implies M embeds into a spin manifold with codimension r :
 M embeds with codimension r into the total space of E , which is spin because

$$w_2(TE) = w_2(\pi^*(TM \oplus E)) = \pi^*(w_2(TM \oplus E)) = 0.$$

- M embeds into a spin manifold with codimension $r \implies \exists E$ such that $TM \oplus E$ is spin:
Let $\iota: M \hookrightarrow X$ be such an embedding, and take E to be the normal bundle of ι . Then,

$$0 = \iota^*(w_2(TX)) = w_2(\iota^*(TX)) = w_2(TM \oplus E).$$



Invariance

Let G be a connected Lie group acting smoothly on M . Then, G acts naturally on FM by bundle isomorphisms.

Definition

A **G -invariant spin^r structure** on M is a spin^r structure (P, Φ) where both P and Φ are G -equivariant.

$$\begin{array}{ccc} G \times P \times \text{Spin}^r(n) & \longrightarrow & P \\ \downarrow \text{Id}_G \times \Phi \times \lambda_n^r & & \downarrow \Phi \\ G \times \text{FM} \times \text{SO}(n) & \longrightarrow & \text{FM} \end{array} \quad \begin{array}{c} \nearrow \\ \searrow \end{array} \quad M.$$

Homogeneous Spaces

- Let M^n be an oriented Riemannian homogeneous G -space, and fix $o \in M$.
- Then, $M \cong G/H$, where $H = \text{Stab}_G(o)$.
- For every $h \in H$,

$$(L_h)_*: T_o M \rightarrow T_{h \cdot o} M = T_o M.$$

Definition

The **isotropy representation** is defined as

$$\begin{aligned}\sigma: H &\rightarrow \text{SO}(T_o M) \cong \text{SO}(n) \\ h &\mapsto (L_h)_*.\end{aligned}$$

The positively oriented orthonormal frame bundle FM of M is isomorphic to $G \times_\sigma \text{SO}(n)$.

Invariant spin^r Structures on Homogeneous Spaces

Suppose σ lifts to $\text{Spin}^r(n)$:

$$\begin{array}{ccc} & & \text{Spin}^r(n) \\ & \nearrow \tilde{\sigma} & \downarrow \lambda_n^r \\ H & \xrightarrow{\sigma} & \text{SO}(n). \end{array}$$

Then, the pair (P, Φ) where

- $P = G \times_{\tilde{\sigma}} \text{Spin}^r(n)$, and
- $\Phi: P \rightarrow \text{FM}, \quad [g, \mu] \mapsto [g, \lambda_n^r(\mu)]$

defines a G -invariant spin^r structure on M .

Classification

Theorem (A. - Lawn, 2023, [3])

Let G/H be an n -dimensional oriented Riemannian homogeneous space with H connected and isotropy representation $\sigma: H \rightarrow SO(n)$. Then, there is a bijective correspondence between

- G -invariant spin^r structures on G/H modulo G -equivariant equivalence, and
- Lie group homomorphisms $\varphi: H \rightarrow SO(r)$ such that $\sigma \times \varphi: H \rightarrow SO(n) \times SO(r)$ lifts to $\text{Spin}^r(n)$ along λ_n^r modulo conjugation by an element of $SO(r)$.



Invariant spin^r structures on spheres

Sphere	Acting group G	Minimal r for G -invariant spin^r structure
S^n	$SO(n+1)$	$r = n, \quad \text{if } n \neq 4$ $r = 3, \quad \text{if } n = 4$
S^{2n+1}	$U(n+1)$	$r = 2$
S^{2n+1}	$SU(n+1)$	$r = 1$
S^{4n+3}	$Sp(n+1)$	$r = 1$
S^{4n+3}	$Sp(n+1) \cdot U(1)$	$r = 1, \quad \text{if } n \text{ odd}$ $r = 2, \quad \text{if } n \text{ even}$
S^{4n+3}	$Sp(n+1) \cdot Sp(1)$	$r = 1, \quad \text{if } n \text{ odd}$ $r = 3, \quad \text{if } n \text{ even}$
S^6	G_2	$r = 1$
S^7	$\text{Spin}(7)$	$r = 1$
S^{15}	$\text{Spin}(9)$	$r = 1$

Spinors

The complex vector bundle $\Sigma M \rightarrow M$ associated to a spin structure via

$$\Delta_n: \text{Spin}(n) \rightarrow \text{End}_{\mathbb{C}}(\Sigma_n)$$

is called the **spinor bundle**: its sections are known as **spinors**.

Clifford multiplication: tangent vectors act fibrewise on spinors via, and

$$X \cdot Y \cdot \psi + Y \cdot X \cdot \psi = -2g(X, Y)\psi$$

for all vector fields $X, Y \in \Gamma(TM)$ and spinors $\psi \in \Gamma(\Sigma M)$.

The Levi-Civita connection of M induces the **spin connection** ∇ on ΣM .

Generalised Killing spinors

A spinor ψ is **generalised Killing** if it satisfies

$$\nabla_X \psi = A(X) \cdot \psi,$$

for all vector fields X , where A is a symmetric endomorphism of TM .

- If $A = 0$, ψ is **parallel**;
- If $A = \lambda \text{Id}$ for some constant $\lambda \in \mathbb{C}$, ψ is **Killing**.

The existence of these spinors is often related to curvature properties and G -structures.

Spin^r spinors

Let (P, Φ) be a spin^r structure. The **m-twisted spin^r spinor bundle** $\Sigma_{n,r}^m M$ is the one associated to P via the representation

$$\Delta_{n,r}^m := \Delta_n \otimes \Delta_r^{\otimes m}$$

of $\text{Spin}^r(n)$. Sections of $\Sigma_{n,r}^m M$ are called **spin^r spinors**.

The Levi-Civita connection on M and a connection on the auxiliary bundle determine a connection on each $\Sigma_{n,r}^m$.

There is a characterisation of special holonomy in terms of **parallel twisted pure spinors** by Herrera - Santana, 2019 [4].

Invariant spin^r spinors on projective spaces

Jointly with Hofmann [2], we obtained the following:

M	G	r	m	$\dim(\Sigma_{*,r}^m)_{\text{inv}}$	Special spinors	Geometry
\mathbb{CP}^n	$SU(n+1)$	2	1	2	pure, parallel	Kähler-Einstein
\mathbb{CP}^{2n+1}	$Sp(n+1)$	2, if n even	1	2	pure, parallel	Kähler-Einstein
		1, if n odd	1	2	generalised Killing	Einstein, nearly Kähler (n = 1)
\mathbb{HP}^n	$Sp(n+1)$	3	n	1	pure, parallel	quaternionic Kähler

Table: For each compact, simple, and simply connected Lie group G acting transitively on M : the minimum values of r , m such that M admits a G -invariant spin^r structure that carries a non-zero invariant m -twisted spin^r spinor, the dimension of the space of such invariant spinors, and the geometric significance of these.

References

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- [3] D. Artacho and M.-A. Lawn. Generalised Spin^r Structures on Homogeneous Spaces. 2023. DOI: 10.48550/ARXIV.2303.05433.
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Thank you

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