

IMPERIAL



Generalised Spin Structures

Diego Artacho
Brno, September 2025

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Let M be a spin manifold.



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spin
structure

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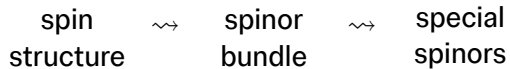
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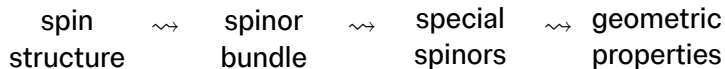
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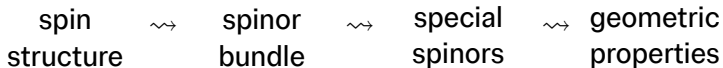
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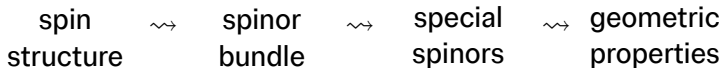
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If M admits a spin structure carrying a nowhere-vanishing parallel spinor, then M is Ricci-flat.

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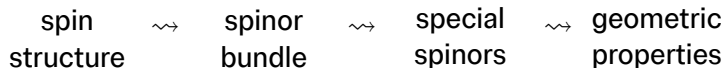
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If M admits a spin structure carrying a nowhere-vanishing parallel spinor, then M is Ricci-flat.

- **Question:** what if M is not spin?
- **Idea:** equip every orientable manifold with spin-like structures.

Spin structures I



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with bundle of oriented orthonormal frames FM .

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A **spin structure** is a lift of the structure group of FM to the group $\text{Spin}(n)$ along the double covering

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In other words, it is a pair (P, Φ) where

- P is a principal $\text{Spin}(n)$ -bundle over M , and
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Spin structures II



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$$\begin{array}{ccc} P \times \text{Spin}(n) & \longrightarrow & P \\ \downarrow \Phi \times \lambda_n & & \downarrow \Phi \\ \text{FM} \times \text{SO}(n) & \longrightarrow & \text{FM} \end{array} \quad \begin{array}{c} \nearrow \\ \searrow \end{array} \quad M.$$



Spin structures turn out not to depend on the orientation or the Riemannian metric:

Theorem

- M admits a spin structure if and only if the first two Stiefel-Whitney classes of M vanish:

$$w_1(M) = w_2(M) = 0.$$

- In this case, spin structures are classified by the first cohomology $H^1(M; \mathbb{Z}_2)$.

Spin^r structures I



What can we do with non-spin manifolds?

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Idea: enlarge the spin group:

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Definition

$$\text{Spin}^r(n) := \frac{\text{Spin}(n) \times \text{Spin}(r)}{\langle (-1, -1) \rangle}.$$

Definition

A **spin^r structure** on an oriented Riemannian n -manifold is a lift of the structure group of the positively oriented orthonormal frame bundle FM to $\text{Spin}^r(n)$ along the composition

$$\begin{aligned} \lambda_n^r: \text{Spin}^r(n) &\rightarrow \text{SO}(n) \times \text{SO}(r) \rightarrow \text{SO}(n) \\ [a, b] &\mapsto (\lambda_n(a), \lambda_r(b)) \mapsto \lambda_n(a). \end{aligned}$$



In other words, a **spin^r structure** on M consists of the following data:

- a principal $\text{Spin}^r(n)$ -bundle P over M , and
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Definition

The rank- r vector bundle associated to P along the composition

$$\text{Spin}^r(n) \rightarrow \text{SO}(n) \times \text{SO}(r) \rightarrow \text{SO}(r)$$

is called the **auxiliary bundle** of the spin^r structure.



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3. M embeds in a spin manifold with codimension r .

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Take E to be the anticanonical bundle of an almost-complex structure, and compute

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Take E to be the rank-3 subbundle of $\text{End}(TM)$ spanned by I, J, K .

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Take E to be the auxiliary bundle of a spin^r structure. Then, the frame bundle of $TM \oplus E$ lifts to $\text{Spin}^r(n)$ along

$$\text{Spin}^r(n) \rightarrow \text{Spin}(n+r) \rightarrow \text{SO}(n+r).$$

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- $\exists E$ such that $TM \oplus E$ is spin $\implies M$ is spin^r :

This follows from the fact that the following is a pullback diagram in the categorical sense:

$$\begin{array}{ccc} \text{Spin}^r(n) & \longrightarrow & \text{Spin}(n+r) \\ \downarrow & & \downarrow \\ \text{SO}(n) \times \text{SO}(r) & \longrightarrow & \text{SO}(n+r). \end{array}$$

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Let $\iota: M \hookrightarrow X$ be such an embedding, and take E to be the normal bundle of ι . Then,

$$0 = \iota^*(w_2(TX)) = w_2(\iota^*(TX)) = w_2(TM \oplus E).$$



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$$\begin{array}{ccc} G \times P \times \text{Spin}^r(n) & \longrightarrow & P \\ \downarrow \text{Id}_G \times \Phi \times \lambda_n^r & & \downarrow \Phi \\ G \times FM \times \text{SO}(n) & \longrightarrow & FM \end{array} \quad \begin{array}{c} \nearrow \\ \searrow \end{array} \quad M.$$

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The positively oriented orthonormal frame bundle FM of M is isomorphic to $G \times_\sigma \text{SO}(n)$.

Invariant spin^r Structures on Homogeneous Spaces



Suppose σ lifts to $\text{Spin}^r(n)$:

$$\begin{array}{ccc} & & \text{Spin}^r(n) \\ & \nearrow \tilde{\sigma} & \downarrow \lambda_n^r \\ \text{H} & \xrightarrow{\sigma} & \text{SO}(n). \end{array}$$

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- $P = G \times_{\tilde{\sigma}} \text{Spin}^r(n)$, and
- $\Phi: P \rightarrow \text{FM}, \quad [g, \mu] \mapsto [g, \lambda_n^r(\mu)]$

defines a G -invariant spin^r structure on M .



Theorem (A. - Lawn, 2023, [3])

Let G/H be an n -dimensional oriented Riemannian homogeneous space with H connected and isotropy representation $\sigma: H \rightarrow SO(n)$. Then, there is a bijective correspondence between

- G -invariant spin^r structures on G/H modulo G -equivariant equivalence, and
- Lie group homomorphisms $\varphi: H \rightarrow SO(r)$ such that $\sigma \times \varphi: H \rightarrow SO(n) \times SO(r)$ lifts to $\text{Spin}^r(n)$ along λ_n^r modulo conjugation by an element of $SO(r)$.



Invariant spin^r structures on spheres



Sphere	Acting group G	Minimal r for G -invariant spin^r structure
S^n	$SO(n+1)$	$r = n, \quad \text{if } n \neq 4$ $r = 3, \quad \text{if } n = 4$
S^{2n+1}	$U(n+1)$	$r = 2$
S^{2n+1}	$SU(n+1)$	$r = 1$
S^{4n+3}	$Sp(n+1)$	$r = 1$
S^{4n+3}	$Sp(n+1) \cdot U(1)$	$r = 1, \quad \text{if } n \text{ odd}$ $r = 2, \quad \text{if } n \text{ even}$
S^{4n+3}	$Sp(n+1) \cdot Sp(1)$	$r = 1, \quad \text{if } n \text{ odd}$ $r = 3, \quad \text{if } n \text{ even}$
S^6	G_2	$r = 1$
S^7	$\text{Spin}(7)$	$r = 1$
S^{15}	$\text{Spin}(9)$	$r = 1$



Theorem (A.-Lawn, 2023 [3])

Let G be the holonomy group of a simply connected irreducible non-symmetric Riemannian manifold of dimension $n + 1 \geq 4$. Let $H \leq G$ be a subgroup such that $S^n \cong G/H$, which exists, by Berger's classification. Then, the following are equivalent:



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1. There exists a homomorphic lift of the holonomy representation to $\text{Spin}^r(n + 1)$.
2. S^n has a G -invariant spin^r structure with strongly G -trivial auxiliary bundle. □

Spinors



The complex vector bundle $\Sigma M \rightarrow M$ associated to a spin structure via

$$\Delta_n: \text{Spin}(n) \rightarrow \text{End}_{\mathbb{C}}(\Sigma_n)$$

is called the **spinor bundle**: its sections are known as **spinors**.

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Clifford multiplication: tangent vectors act fibrewise on spinors, satisfying

$$X \cdot Y \cdot \psi + Y \cdot X \cdot \psi = -2g(X, Y)\psi$$

for all vector fields $X, Y \in \Gamma(TM)$ and spinors $\psi \in \Gamma(\Sigma M)$.

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The Levi-Civita connection of M induces the **spin connection** ∇ on ΣM .

Generalised Killing spinors



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The existence of these spinors is often related to curvature properties and G -structures.

Spin^r spinors



Let (P, Φ) be a spin^r structure. For odd m , the **m-twisted spin^r spinor bundle** $\Sigma_{n,r}^m M$ is the one associated to P via the representation

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of $\text{Spin}^r(n)$.

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There is a characterisation of special holonomy in terms of **parallel twisted pure spinors** by Herrera - Santana, 2019 [4].

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M	G	r	m	$\dim(\Sigma_{*,r}^m)_{\text{inv}}$	Special spinors	Geometry
\mathbb{CP}^n	$SU(n+1)$	2	1	2	pure, parallel	Kähler-Einstein
\mathbb{CP}^{2n+1}	$Sp(n+1)$	2, if n even	1	2	pure, parallel	Kähler-Einstein
		1, if n odd	1	2	generalised Killing	Einstein, nearly Kähler ($n=1$)
\mathbb{HP}^n	$Sp(n+1)$	3	n	1	pure, parallel	quaternionic Kähler

Table: For each compact, simple, and simply connected Lie group G acting transitively on M : the minimum values of r , m such that M admits a G -invariant spin^r structure that carries a non-zero invariant m -twisted spin^r spinor, the dimension of the space of such invariant spinors, and the geometric significance of these.

References

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Thank you

Generalised Spin Structures
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