IMPERIAL

Generalised Spin Structures

Diego Artacho
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If M admits a spin structure carrying a nowhere-vanishing parallel spinor, then M is Ricci-flat.

- **Question**: what if M is not spin?
- **Idea**: equip every orientable manifold with spin-like structures.

Spin structures I

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with bundle of oriented orthonormal frames FM.

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$$\lambda_n\colon\ \mathrm{Spin}(n) \to \mathrm{SO}(n).$$

In other words, it is a pair (P, Φ) where

- P is a principal $\mathrm{Spin}(n)$-bundle over M, and
- $\Phi\colon\ P \to FM$ is a $\mathrm{Spin}(n)$-equivariant bundle map covering the identity, where $\mathrm{Spin}(n)$ acts on FM via λ_n.

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$$
\begin{tikzcd}
P \times \mathrm{Spin}(n) \arrow[r] \arrow[d, "\Phi\times\lambda_n"'] & P \arrow[d, "\Phi"] \arrow[dr] & \\
FM \times \mathrm{SO}(n) \arrow[r] & FM \arrow[r] & M.
\end{tikzcd}
$$

Spin structures III

Spin structures turn out not to depend on the orientation or the Riemannian metric:

Theorem

- M admits a spin structure if and only if the first two Stiefel-Whitney classes of M vanish:

$$\mathsf{w}_1(\mathsf{M}) = \mathsf{w}_2(\mathsf{M}) = 0.$$

- In this case, spin structures are classified by the first cohomology $\mathsf{H}^1(\mathsf{M}; \mathbb{Z}_2)$.

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$$\mathrm{Spin}^{\mathbb{C}}(\mathsf{n}) = \frac{\mathrm{Spin}(\mathsf{n}) \times \mathrm{U}(1)}{\langle(-1,-1)\rangle}$$

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$$\mathrm{Spin}^{\mathbb{C}}(n) = \frac{\mathrm{Spin}(n) \times \mathrm{U}(1)}{\langle(-1,-1)\rangle}, \qquad \mathrm{Spin}^{\mathbb{H}}(n) = \frac{\mathrm{Spin}(n) \times \mathrm{Sp}(1)}{\langle(-1,-1)\rangle}.$$

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$$\mathrm{Spin}^r(n) := \frac{\mathrm{Spin}(n) \times \mathrm{Spin}(r)}{\langle(-1,-1)\rangle}.$$

Definition

A **spin^r structure** on an oriented Riemannian n-manifold is a lift of the structure group of the positively oriented orthonormal frame bundle FM to $\mathrm{Spin}^r(n)$ along the composition

$$\begin{aligned} \lambda_n^r \colon \ \mathrm{Spin}^r(n) &\to \mathrm{SO}(n) \times \mathrm{SO}(r) \to \mathrm{SO}(n) \\ [a, b] &\mapsto (\lambda_n(a), \lambda_r(b)) \mapsto \lambda_n(a). \end{aligned}$$

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In other words, a **spin^r structure** on M consists of the following data:

- a principal $\mathrm{Spin}^r(n)$-bundle P over M, and
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Definition

The rank-r vector bundle associated to P along the composition

$$\mathrm{Spin}^r(n) \to \mathrm{SO}(n) \times \mathrm{SO}(r) \to \mathrm{SO}(r)$$

is called the **auxiliary bundle** of the spin^r structure.

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Theorem (Albanese - Milivojević, 2021 [1])

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2. there is an orientable rank-r real vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$ such that $\mathsf{TM} \oplus \mathsf{E}$ is spin, i.e., $w_1(\mathsf{TM} \oplus \mathsf{E}) = w_2(\mathsf{TM} \oplus \mathsf{E}) = 0$;

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2. there is an orientable rank-r real vector bundle $\pi\colon E \to M$ such that $TM \oplus E$ is spin, i.e., $w_1(TM \oplus E) = w_2(TM \oplus E) = 0$;
3. M embeds in a spin manifold with codimension r.

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   Take E to be the anticanonical bundle of an almost-complex structure, and compute

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3. Every almost-quaternionic manifold admits a spin^3 structure.
   Take E to be the rank-3 subbundle of $\mathrm{End}(TM)$ spanned by I, J, K.

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- M is spin^r $\implies$ $\exists$ E such that $TM \oplus E$ is spin:
  Take E to be the auxiliary bundle of a spin^r structure. Then, the frame bundle of $TM \oplus E$ lifts to $\mathrm{Spin}^r(n)$ along

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- $\exists$ E such that TM $\oplus$ E is spin $\implies$ M is spin^r:
  This follows from the fact that the following is a pullback diagram in the categorical sense:
  $$\begin{tikzcd} \mathrm{Spin}^r(n) \arrow[r] \arrow[d] & \mathrm{Spin}(n+r) \arrow[d] \\ \mathrm{SO}(n) \times \mathrm{SO}(r) \arrow[r] & \mathrm{SO}(n+r). \end{tikzcd}$$

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  M embeds with codimension r into the total space of E, which is spin because

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- M embeds into a spin manifold with codimension r $\implies$ $\exists$ E such that TM $\oplus$ E is spin:
  Let $\iota \colon$ M $\hookrightarrow$ X be such an embedding, and take E to be the normal bundle of ι. Then,

$$0 = \iota^*(\mathrm{w}_2(\mathrm{TX})) = \mathrm{w}_2(\iota^*(\mathrm{TX})) = \mathrm{w}_2(\mathrm{TM} \oplus \mathrm{E}).$$

$\square$

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$$\begin{tikzcd}
\mathsf{G} \times \mathsf{P} \times \mathrm{Spin}^{\mathsf{r}}(\mathsf{n}) \arrow[r] \arrow[d, "\mathrm{Id}_{\mathsf{G}} \times \Phi \times \lambda^{\mathsf{r}}_{\mathsf{n}}"'] & \mathsf{P} \arrow[d, "\Phi"] \arrow[dr] & \\
& & \mathsf{M}. \\
\mathsf{G} \times \mathsf{FM} \times \mathrm{SO}(\mathsf{n}) \arrow[r] & \mathsf{FM} \arrow[ur] &
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The **isotropy representation** is defined as

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The positively oriented orthonormal frame bundle FM of M is isomorphic to $G \times_\sigma \mathrm{SO}(n)$.

Invariant spin$^{\mathsf{r}}$ Structures on Homogeneous Spaces

Suppose σ lifts to $\mathrm{Spin}^{\mathsf{r}}(\mathsf{n})$:

$$\begin{tikzcd} & \mathrm{Spin}^{\mathsf{r}}(\mathsf{n}) \arrow[d, "\lambda^{\mathsf{r}}_{\mathsf{n}}"] \\ \mathsf{H} \arrow[ur, "\tilde{\sigma}"] \arrow[r, "\sigma"] & \mathrm{SO}(\mathsf{n}). \end{tikzcd}$$

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Then, the pair (P, Φ) where

- $\mathsf{P} = \mathsf{G} \times_{\tilde{\sigma}} \mathrm{Spin}^{\mathsf{r}}(\mathsf{n})$, and
- $\Phi \colon \mathsf{P} \to \mathsf{FM}, \quad [\mathsf{g}, \mu] \mapsto [\mathsf{g}, \lambda^{\mathsf{r}}_{\mathsf{n}}(\mu)]$

defines a G-invariant spin$^{\mathsf{r}}$ structure on M.

Classification

Theorem (A. - Lawn, 2023, [3])

Let G/H be an n-dimensional oriented Riemannian homogeneous space with H connected and isotropy representation $\sigma\colon H \to \mathrm{SO}(n)$. Then, there is a bijective correspondence between

- G-invariant spin^r structures on G/H modulo G-equivariant equivalence, and
- Lie group homomorphisms $\varphi\colon H \to \mathrm{SO}(r)$ such that $\sigma \times \varphi\colon H \to \mathrm{SO}(n) \times \mathrm{SO}(r)$ lifts to $\mathrm{Spin}^r(n)$ along λ^r_n modulo conjugation by an element of $\mathrm{SO}(r)$.

$\square$

Invariant spin^r structures on spheres

Sphere	Acting group G	Minimal r for G-invariant spin^r structure
S^n	$SO(n+1)$	$r = n, \quad \text{if } n \neq 4$ $r = 3, \quad \text{if } n = 4$
S^{2n+1}	$U(n+1)$	$r = 2$
S^{2n+1}	$SU(n+1)$	$r = 1$
S^{4n+3}	$Sp(n+1)$	$r = 1$
S^{4n+3}	$Sp(n+1) \cdot U(1)$	$r = 1, \quad \text{if } n \text{ odd}$ $r = 2, \quad \text{if } n \text{ even}$
S^{4n+3}	$Sp(n+1) \cdot Sp(1)$	$r = 1, \quad \text{if } n \text{ odd}$ $r = 3, \quad \text{if } n \text{ even}$
S^6	G_2	$r = 1$
S^7	$Spin(7)$	$r = 1$
S^{15}	$Spin(9)$	$r = 1$

Special invariant spin^r structures on spheres and holonomy lifts

Theorem (A.-Lawn, 2023 [3])

Let G be the holonomy group of a simply connected irreducible non-symmetric Riemannian manifold of dimension $n + 1 \geq 4$. Let $H \leq G$ be a subgroup such that $S^n \cong G/H$, which exists, by Berger's classification. Then, the following are equivalent:

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1. There exists a homomorphic lift of the holonomy representation to $\mathrm{Spin}^r(n+1)$.
2. S^n has a G-invariant spin^r structure with strongly G-trivial auxiliary bundle. □

Spinors

The complex vector bundle $\Sigma M \to M$ associated to a spin structure via

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Clifford multiplication: tangent vectors act fibrewise on spinors, satisfying

$$X \cdot Y \cdot \psi + Y \cdot X \cdot \psi = -2g(X, Y)\psi$$

for all vector fields $X, Y \in \Gamma(TM)$ and spinors $\psi \in \Gamma(\Sigma M)$.

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The Levi-Civita connection of M induces the **spin connection** ∇ on ΣM.

Generalised Killing spinors

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The existence of these spinors is often related to curvature properties and G-structures.

Spin^r spinors

Let (P, Φ) be a spin^r structure. For odd m, the m-**twisted spin^r spinor bundle** $\Sigma^m_{n,r}M$ is the one associated to P via the representation

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of $\mathrm{Spin}^r(n)$.

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There is a characterisation of special holonomy in terms of **parallel twisted pure spinors** by Herrera - Santana, 2019 [4].

Invariant spin^r spinors on projective spaces

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M	G	r	m	$\dim\left(\Sigma^{m}_{*,r}\right)_{\mathrm{inv}}$	Special spinors	Geometry
$\mathbb{CP}^n$	$\mathrm{SU}(n+1)$	2	1	2	pure, parallel	Kähler-Einstein
$\mathbb{CP}^{2n+1}$	$\mathrm{Sp}(n+1)$	2, if n even 1, if n odd	1 1	2 2	pure, parallel generalised Killing	Kähler-Einstein Einstein, nearly Kähler ($n=1$)
$\mathbb{HP}^n$	$\mathrm{Sp}(n+1)$	3	n	1	pure, parallel	quaternionic Kähler

Table: For each compact, simple, and simply connected Lie group G acting transitively on M: the minimum values of r, m such that M admits a G-invariant spin^r structure that carries a non-zero invariant m-twisted spin^r spinor, the dimension of the space of such invariant spinors, and the geometric significance of these.

References

[1] M. Albanese and A. Milivojević. "Spin^h and further generalisations of spin". In: J. Geom. Phys. 164 (2021), Paper No. 104174, 13. ISSN: 0393-0440. DOI: 10.1016/j.geomphys.2021.104174.

[2] D. Artacho and J. Hofmann. "The geometry of generalised spin^r spinors on projective spaces". In: SIGMA Symmetry Integrability Geom. Methods Appl. 21 (2025), Paper No. 017, 32. ISSN: 1815-0659. DOI: 10.3842/SIGMA.2025.017.

[3] D. Artacho and M.-A. Lawn. Generalised Spin^r Structures on Homogeneous Spaces. 2023. DOI: 10.48550/ARXIV.2303.05433.

[4] R. Herrera and N. Santana. "Spinorially twisted Spin structures. II: twisted pure spinors, special Riemannian holonomy and Clifford monopoles". In: SIGMA Symmetry Integrability Geom. Methods Appl. 15 (2019), Paper No. 072, 48. ISSN: 1815-0659. DOI: 10.3842/SIGMA.2019.072.

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Thank you

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