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Generalised Spin Structures

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Idea

Let M be a spin manifold.



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spin → spinor → special → geometric structure bundle spinors properties
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Examples

- If M admits a spin structure carrying a nowhere-vanishing parallel spinor, then M is Ricci-flat.
- Wang characterised Riemannian holonomies of spin manifolds in terms of the space of parallel spinors [13].
- Question: what if M is not spin?
- Idea: equip every orientable manifold with spin-like structures.

Spin structures I

Let Mⁿ be an oriented Riemannian manifold

with bundle of oriented orthonormal frames FM.

A **spin structure** is a lift of the structure group of FM to the group $\mathrm{Spin}(n)$ along the double covering

$$\lambda_n \colon \operatorname{Spin}(n) \to \operatorname{SO}(n).$$

In other words, it is a pair (P, Φ) where

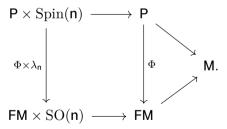
- P is a principal Spin(n)-bundle over M, and
- $\Phi \colon P \to FM$ is a $\mathrm{Spin}(n)$ -equivariant bundle map covering the identity, where $\mathrm{Spin}(n)$ acts on FM via λ_n .

Spin structures II



In other words, it is a pair (P, Φ) where

- ullet P is a principal $\mathrm{Spin}(n)$ -bundle over M, and
- $\Phi \colon P \to FM$ is a $\mathrm{Spin}(n)$ -equivariant bundle map covering the identity, where $\mathrm{Spin}(n)$ acts on FM via λ_n .



Spin structures III



Spin structures turn out not to depend on the orientation or the Riemannian metric:

Theorem

• M admits a spin structure if and only if the first two Stiefel-Whitney classes of M vanish:

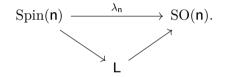
$$\mathbf{w}_1(\mathbf{M}) = \mathbf{w}_2(\mathbf{M}) = 0.$$

• In this case, spin structures are classified by the first cohomology $H^1(M; \mathbb{Z}_2)$.

Spin^r structures I

What can we do with non-spin manifolds?

Idea: enlarge the spin group:



Example

$$\mathrm{Spin}^{\mathbb{C}}(\mathbf{n}) = \frac{\mathrm{Spin}(\mathbf{n}) \times \mathrm{U}(1)}{\langle (-1, -1) \rangle}, \qquad \mathrm{Spin}^{\mathbb{H}}(\mathbf{n}) = \frac{\mathrm{Spin}(\mathbf{n}) \times \mathrm{Sp}(1)}{\langle (-1, -1) \rangle}.$$

Spin^r structures II



Note that $U(1) \cong Spin(2)$ and $Sp(1) \cong Spin(3)$.

Definition

$$\mathrm{Spin}^r(n) := \frac{\mathrm{Spin}(n) \times \mathrm{Spin}(r)}{\langle (-1,-1) \rangle}.$$

Definition

A **spin**^r **structure** on an oriented Riemannian n-manifold is a lift of the structure group of the positively oriented orthonormal frame bundle FM to $\mathrm{Spin}^r(n)$ along the composition

$$\begin{split} \lambda_n^r\colon \operatorname{Spin}^r(n) &\to \operatorname{SO}(n) \times \operatorname{SO}(r) \to \operatorname{SO}(n) \\ [a,b] &\mapsto (\lambda_n(a),\lambda_r(b)) \mapsto \lambda_n(a). \end{split}$$

Spin^r structures III



In other words, a **spin**^r **structure** on M consists of the following data:

- ullet a principal $\mathrm{Spin}^r(n)$ -bundle P over M, and
- a $\mathrm{Spin}^r(n)$ -equivariant bundle map $\Phi \colon \mathsf{P} \to \mathsf{FM}$, where $\mathrm{Spin}^r(n)$ acts on FM through λ_n^r .

Definition

The rank-r vector bundle associated to P along the composition

$$\mathrm{Spin}^r(n) \to \mathrm{SO}(n) \times \mathrm{SO}(r) \to \mathrm{SO}(r)$$

is called the **auxiliary bundle** of the spin^r structure.

 One could twist by other Lie groups – see Avis - Isham [5], Friedrich - Trautman [8], Lazaroiu - Shahbazi [10, 12, 11].

Characterisation



Theorem (Albanese - Milivojević, 2021 [2])

The following are equivalent for an oriented Riemannian manifold M:

- 1. M is spin^r;
- 2. there is an orientable rank-r real vector bundle $\pi \colon \mathsf{E} \to \mathsf{M}$ such that $\mathsf{TM} \oplus \mathsf{E}$ is spin, i.e., $\mathsf{w}_1(\mathsf{TM} \oplus \mathsf{E}) = \mathsf{w}_2(\mathsf{TM} \oplus \mathsf{E}) = 0$;
- 3. M embeds in a spin manifold with codimension r.

A few examples



Examples

1. Every oriented n-manifold M admits a spinⁿ structure. Take E = TM, and note that

$$w_2(TM \oplus E) = w_2(TM) + w_1(TM)w_1(E) + w_2(E) = 2w_2(TM) = 0.$$

Every almost-complex manifold admits a spin² structure.
 Take E to be the anticanonical bundle of an almost-complex structure, and compute

$$\mathsf{w}_2(\mathsf{TM} \oplus \mathsf{E}) = \mathsf{w}_2(\mathsf{TM}) + \mathsf{w}_2(\mathsf{E}) = 2(\mathsf{c}_1(\mathsf{TM}) \bmod 2) = 0.$$

Proof of Albanese-Milivojević 1 ← 2



M is spin^r ⇒ ∃ E such that TM ⊕ E is spin:
 Take E to be the auxiliary bundle of a spin^r structure. Then, the frame bundle of TM ⊕ E lifts to Spin^r(n) along

$$\mathrm{Spin}^r(n) \to \mathrm{Spin}(n+r) \to \mathrm{SO}(n+r).$$

In particular, it lifts to Spin(n + r).

∃ E such that TM ⊕ E is spin ⇒ M is spin^r:
 This follows from the fact that the following is a pullback diagram in the categorical sense:

$$\begin{array}{ccc} \operatorname{Spin}^r(n) & \longrightarrow & \operatorname{Spin}(n+r) \\ \downarrow & & \downarrow \\ \operatorname{SO}(n) \times \operatorname{SO}(r) & \longrightarrow & \operatorname{SO}(n+r). \end{array}$$

Proof of Albanese-Milivojević 2 ← 3

∃ E such that TM ⊕ E is spin ⇒ M embeds into a spin manifold with codimension r:
 M embeds with codimension r into the total space of E, which is spin because

$$\mathbf{w}_2(\mathsf{TE}) = \mathbf{w}_2(\pi^*(\mathsf{TM} \oplus \mathsf{E})) = \pi^*(\mathbf{w}_2(\mathsf{TM} \oplus \mathsf{E})) = 0.$$

M embeds into a spin manifold with codimension r ⇒ ∃ E such that TM ⊕ E is spin:
 Let ι: M ⇔ X be such an embedding, and take E to be the normal bundle of ι. Then,

$$0 = \iota^*(\mathsf{w}_2(\mathsf{TX})) = \mathsf{w}_2(\iota^*(\mathsf{TX})) = \mathsf{w}_2(\mathsf{TM} \oplus \mathsf{E}).$$



Invariance

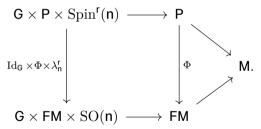


Let G be a connected Lie group acting smoothly on M by isometries.

Then, G acts naturally on FM by bundle isomorphisms.

Definition

A G-invariant spin structure on M is a spin structure (P, Φ) where both P and Φ are G-equivariant.



Homogeneous Spaces

- Let M^n be an oriented Riemannian homogeneous G-space, and fix $o \in M$.
- Then, $M \cong G/H$, where $H = \operatorname{Stab}_{G}(o)$.
- $G \to G/H \cong M$ is a principal H-bundle.
- For every $h \in H$,

$$(L_h)_* \colon T_oM \to T_{h \cdot o}M = T_oM.$$

Definition

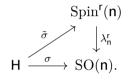
The **isotropy representation** is defined as

$$\begin{split} \sigma \colon H &\to \mathrm{SO}(T_o M) \cong \mathrm{SO}(n) \\ h &\mapsto (L_h)_*. \end{split}$$

FM is isomorphic to $G \times_{\sigma} SO(n)$.

Invariant spin^r Structures on Homogeneous Spaces

Suppose σ lifts to $\mathrm{Spin}^{\mathbf{r}}(\mathbf{n})$:



Then, the pair (P, Φ) where

- $P = G \times_{\tilde{\sigma}} \operatorname{Spin}^r(n)$, and
- $\bullet \ \Phi \colon \mathsf{P} \to \mathsf{FM}, \quad [\mathsf{g},\mu] \mapsto [\mathsf{g},\lambda^{\mathsf{r}}_{\mathsf{n}}(\mu)]$

defines a G-invariant spin^r structure on M.

Classification



Theorem (A. - Lawn, 2023, [4])

Let G/H be an n-dimensional oriented Riemannian homogeneous space with H connected and isotropy representation $\sigma \colon H \to \mathrm{SO}(n)$. Then, there is a bijective correspondence between

- G-invariant spin^r structures on G/H modulo G-equivariant equivalence, and
- Lie group homomorphisms $\varphi \colon H \to \mathrm{SO}(r)$ such that $\sigma \times \varphi \colon H \to \mathrm{SO}(n) \times \mathrm{SO}(r)$ lifts to $\mathrm{Spin}^r(n)$ along λ_n^r modulo conjugation by an element of $\mathrm{SO}(r)$.

Special invariant spin structures on spheres and holonomy lifts



Theorem (A.-Lawn, 2023 [4])

Let G be the holonomy group of a simply connected irreducible non-symmetric Riemannian manifold of dimension n + 1 \geq 4. Let H \leq G be a subgroup such that Sⁿ \cong G/H, which exists, by Berger's classification. Then, the following are equivalent:

- 1. There exists a homomorphic lift of the holonomy representation to $\mathrm{Spin}^r(n+1)$.
- 2. Sⁿ has a G-invariant spin^r structure with strongly G-trivial auxiliary bundle.

Invariant spin^r structures on spheres



Sphere	Acting group G	Minimal r for G-invariant spin ^r structure		
S ⁿ	SO(n+1)	r=n,ifn eq 4		
		r=3, if $n=4$		
S^{2n+1}	$\mathrm{U}(n+1)$	r = 2		
S^{2n+1}	SU(n+1)	r = 1		
S^{4n+3}	$\operatorname{Sp}(n+1)$	r = 1		
S ⁴ⁿ⁺³	$\mathrm{Sp}(n+1)\cdot\mathrm{U}(1)$	r=1, if n odd		
		r=2, if n even		
S ⁴ⁿ⁺³	$\operatorname{Sp}(\mathbf{n}+1)\cdot\operatorname{Sp}(1)$	r=1, if n odd		
		r=3, if n even		
S^6	G_2	r = 1		
S ⁷	Spin(7)	r = 1		
S^{15}	Spin(9)	r = 1		

Spinors

The complex vector bundle $\Sigma M o M$ associated to a spin structure via

$$\Delta_n \colon \operatorname{Spin}(n) \to \operatorname{End}_{\mathbb{C}}(\Sigma_n)$$

is called the **spinor bundle**: its sections are known as **spinors**.

Clifford multiplication: tangent vectors act fibrewise on spinors, satisfying

$$\mathbf{X}\cdot\mathbf{Y}\cdot\boldsymbol{\psi}+\mathbf{Y}\cdot\mathbf{X}\cdot\boldsymbol{\psi}=-2\mathbf{g}(\mathbf{X},\mathbf{Y})\boldsymbol{\psi}$$

for all vector fields X, Y $\in \Gamma(TM)$ and spinors $\psi \in \Gamma(\Sigma M)$.

The Levi-Civita connection of M induces the **spin connection** ∇ on Σ M.

Generalised Killing spinors



A spinor ψ is **generalised Killing** if it satisfies

$$\nabla_{\mathsf{X}}\psi = \mathsf{A}(\mathsf{X})\cdot\psi,$$

for all vector fields X, where A is a symmetric endomorphism of TM.

- If A = 0, ψ is **parallel** see Wang [13];
- If A = $\lambda \operatorname{Id}$ for some constant $\lambda \in \mathbb{C}$, ψ is **Killing** see Bär [6].

The existence of these spinors is often related to curvature properties and G-structures – see Conti - Salamon [7], Agricola - Friedrich [1].

Spin^r spinors

Let (P, Φ) be a spin^r structure.

For odd m, the m-twisted spin spinor bundle $\Sigma_{n,r}^m M$ is the one associated to P via the representation

$$\Delta_{n,r}^m := \Delta_n \otimes \Delta_r^{\otimes m}$$

of $\mathrm{Spin}^r(n)$. Sections of $\Sigma^m_{n,r}M$ are called **spin**^r **spinors**.

The Levi-Civita connection on M and a connection on the auxiliary bundle determine a connection on each $\Sigma^m_{n,r}M$.

There is a characterisation of special holonomy in terms of **parallel twisted pure spinors** by Herrera - Santana, 2019 [9].

Invariant spin^r spinors on projective spaces



Jointly with Hofmann [3], we obtained the following:

М	G	r	m	$\dim \left(\Sigma_{*,r}^{m}\right)_{\mathrm{inv}}$	Special spinors	Geometry
\mathbb{CP}^{n}	SU(n+1)	2	1	2	pure, parallel	Kähler-Einstein
\mathbb{CP}^{2n+1}	$\operatorname{Sp}(\mathbf{n}+1)$	2, if n even	1	2	pure, parallel	Kähler-Einstein
		1, if n odd	1	2	generalised Killing	Einstein, nearly Kähler (n $= 1$)
\mathbb{HP}^{n}	$\operatorname{Sp}(\mathbf{n}+1)$	3	n	1	pure, parallel	quaternionic Kähler

Table: For each compact, simple, and simply connected Lie group G acting transitively on M: the minimum values of r, m such that M admits a G-invariant spin^r structure that carries a non-zero invariant m-twisted spin^r spinor, the dimension of the space of such invariant spinors, and the geometric significance of these.

References I

- [1] I. Agricola et al. "Spinorial description of $\mathrm{SU}(3)$ and G_2 -manifolds". In: J. Geom. Phys. 98 (2015), pp. 535–555. ISSN: 0393-0440,1879-1662. DOI: $10.1016/\mathrm{j.geomphys.2015.08.023.}$ URL: $\mathrm{https://doi.org/10.1016/j.geomphys.2015.08.023.}$
- [2] M. Albanese and A. Milivojević. "Spinh and further generalisations of spin". In: J. Geom. Phys. 164 (2021), Paper No. 104174, 13. ISSN: 0393-0440. DOI: 10.1016/j.geomphys.2021.104174.
- [3] D. Artacho and J. Hofmann. "The geometry of generalised spin^r spinors on projective spaces". In: SIGMA Symmetry Integrability Geom. Methods Appl. 21 (2025), Paper No. 017, 32. ISSN: 1815-0659. DOI: 10.3842/SIGMA.2025.017.
- [4] D. Artacho and M.-A. Lawn. Generalised Spin^r Structures on Homogeneous Spaces. To appear in Diff. Geom. Appl. 2025. DOI: 10.48550/ARXIV.2303.05433.

References II

- [5] S. J. Avis and C. J. Isham. "Generalized Spin structures on four-dimensional space-times". In: Comm. Math. Phys. 72.2 (1980), pp. 103–118. ISSN: 0010-3616,1432-0916. URL: http://projecteuclid.org/euclid.cmp/1103907653.
- [6] Christian Bär. "Real Killing spinors and holonomy". In: Comm. Math. Phys. 154.3 (1993), pp. 509–521. ISSN: 0010-3616,1432-0916. URL: http://projecteuclid.org/euclid.cmp/1104253076.
- [7] D. Conti and S. Salamon. "Generalized Killing spinors in dimension 5". In: Trans. Amer. Math. Soc. 359.11 (2007), pp. 5319–5343. ISSN: 0002-9947,1088-6850. DOI: 10.1090/S0002-9947-07-04307-3. URL: https://doi.org/10.1090/S0002-9947-07-04307-3.

References III

- [8] Th. Friedrich and A. Trautman. "Spin spaces, Lipschitz groups, and spinor bundles". In: vol. 18. 3-4. Special issue in memory of Alfred Gray (1939–1998). 2000, pp. 221–240. DOI: 10.1023/A:1006713405277. URL: https://doi.org/10.1023/A:1006713405277.
- [9] R. Herrera and N. Santana. "Spinorially twisted Spin structures. II: twisted pure spinors, special Riemannian holonomy and Clifford monopoles". In: SIGMA Symmetry Integrability Geom. Methods Appl. 15 (2019), Paper No. 072, 48. ISSN: 1815-0659. DOI: 10.3842/SIGMA.2019.072.
- [10] C.I. Lazaroiu and C.S. Shahbazi. "Complex Lipschitz structures and bundles of complex Clifford modules". In: Differential Geom. Appl. 61 (2018), pp. 147–169. ISSN: 0926-2245,1872-6984. DOI: 10.1016/j.difgeo.2018.08.006. URL: https://doi.org/10.1016/j.difgeo.2018.08.006.

References IV

- [11] C.I. Lazaroiu and C.S. Shahbazi. "Dirac operators on real spinor bundles of complex type". In: Differential Geom. Appl. 80 (2022), Paper No. 101849, 53. ISSN: 0926-2245,1872-6984. DOI: 10.1016/j.difgeo.2022.101849. URL: https://doi.org/10.1016/j.difgeo.2022.101849.
- [12] C.I. Lazaroiu and C.S. Shahbazi. "Real pinor bundles and real Lipschitz structures". In: Asian J. Math. 23.5 (2019), pp. 749–836. ISSN: 1093-6106,1945-0036. DOI: 10.4310/AJM.2019.v23.n5.a3. URL: https://doi.org/10.4310/AJM.2019.v23.n5.a3.
- [13] M.Y. Wang. "Parallel spinors and parallel forms". In: Ann. Global Anal. Geom. 7.1 (1989), pp. 59–68. ISSN: 0232-704X. DOI: 10.1007/BF00137402. URL: https://doi.org/10.1007/BF00137402.

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Thank you

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