ECONOMICS 2
Tutorial 7

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Questions: 6,7,10,14

http://personal.lse.ac.uk/BATTISTO/T7_slides.pdf

Question 6

- **Asset A**: Free risk and pays 2%
- **Asset B**: pays 10% with probability ½ and 0% with probability ½

a) Expected Returns and Variances

$$E(A) = 2\%$$

$$E(B) = \frac{1}{2}10\% + \frac{1}{2}0\% = 5\%$$

$$V(A) = 0$$

$$\begin{aligned} V(B) &= E(B^2) - E(B)^2 \\ &= 0.5\% - 0.25\% \\ &= 0.25\% \end{aligned}$$

Important Formula: $\boldsymbol{V(X) = E\left(X^2\right) - E(X)^2}$

Question 6

- **Asset A**: Free risk and pays 2%
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b) Portfolio: Invest 1/2 in each type

$$E(P) = \frac{1}{2}E(A) + \frac{1}{2}E(B)$$

$$= \frac{1}{2}2\% + \frac{1}{2}5\% = 3.5\%$$

$$V(P) = V\left(\frac{1}{2}A + \frac{1}{2}B\right) = \frac{1}{4}V(B) = \frac{1}{4}0.25\% = 0.0625\%$$

Important Formula: $\boldsymbol{V(a + bX) = b^2 V(X)}$

Question 6

- **Asset A**: Free risk and pays 2%
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c) Asset C pays: 10% when B pays 0%

0% when B pays 10% .

Portfolio: $\hat{P} = \frac{1}{2}B + \frac{1}{2}C$

$$E(\hat{P}) = \frac{1}{2}E(B) + \frac{1}{2}E(C) = 2.5\% + 2.5\% = 5\%$$

$$V(\hat{P}) = \frac{1}{4}\underbrace{V(B)}_{0.25\%} + \frac{1}{4}\underbrace{V(C)}_{0.25\%} + 2\frac{1}{4}\underbrace{Cov(B,C)}_{\underbrace{\boldsymbol{\rho}}_{-1}\sqrt{\boldsymbol{V(B)V(C)}}}$$

Question 6

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c) Asset C pays: 10% when B pays 0%
0% when B pays 10% .

Portfolio: $\hat{P} = \frac{1}{2}B + \frac{1}{2}C$

$$E(\hat{P}) = \frac{1}{2}E(B) + \frac{1}{2}E(C) = 2.5\% + 2.5\% = 5\%$$

$$V(\hat{P}) = \frac{1}{4}V(B) + \frac{1}{4}V(C) + 2\frac{1}{4}Cov(B,C)$$

$$= \frac{0.25\%}{4} + \frac{0.25\%}{4} - \frac{0.25\%}{2} = 0$$

Question 7

- Two risky assets A and B
- $C = xA + (1 - x)B$
- $r_C = xr_A + (1 - x)r_B$
- $\sigma_C^2 = x^2\sigma_A^2 + (1 - x)^2\sigma_B^2 + 2x(1 - x)Cov(A, B)$
- Initially $\rho = 1$

a) Simplify $\boldsymbol{\sigma_C^2}$ based on the definition of $\boldsymbol{\rho} = \frac{Cov(A,B)}{\sigma_A\sigma_B} = 1$

$$\sigma_C^2 = x^2\sigma_A^2 + (1 - x)^2\sigma_B^2 + 2x(1 - x)\sigma_A\sigma_B$$

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b) Simplify $\boldsymbol{\sigma_C}$

$$\sigma_C^2 = x^2\sigma_A^2 + (1-x)^2\sigma_B^2 + 2x(1-x)\sigma_A\sigma_B$$

$$\sigma_C^2 = [x\sigma_A + (1-x)\sigma_B]^2$$

$$\sigma_C = x\sigma_A + (1-x)\sigma_B$$

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c) Get x in terms of the σ's

$$\sigma_C = x\sigma_A + (1-x)\sigma_B$$

$$\downarrow$$

$$x = \frac{\sigma_C - \sigma_B}{\sigma_A - \sigma_B}$$

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d) Show that $\boldsymbol{r_C = constant + slope\ \sigma_C}$

Start from

$$r_C = xr_A + (1-x)r_B$$

$$= r_B + x(r_A - r_B)$$

$$= r_B + \frac{\sigma_C - \sigma_B}{\sigma_A - \sigma_B}(r_A - r_B)$$

$$\dots algebra \dots$$

$$= \frac{r_B\sigma_A - \sigma_B r_A}{\sigma_A - \sigma_B} + \frac{r_A - r_B}{\sigma_A - \sigma_B}\sigma_C$$

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Why did we make all this derivation?

Don't miss the big picture: We want to study how investors decide their portfolio

I need to know how the *risk-return trade-off* of the portfolio!

Budget Constraint!

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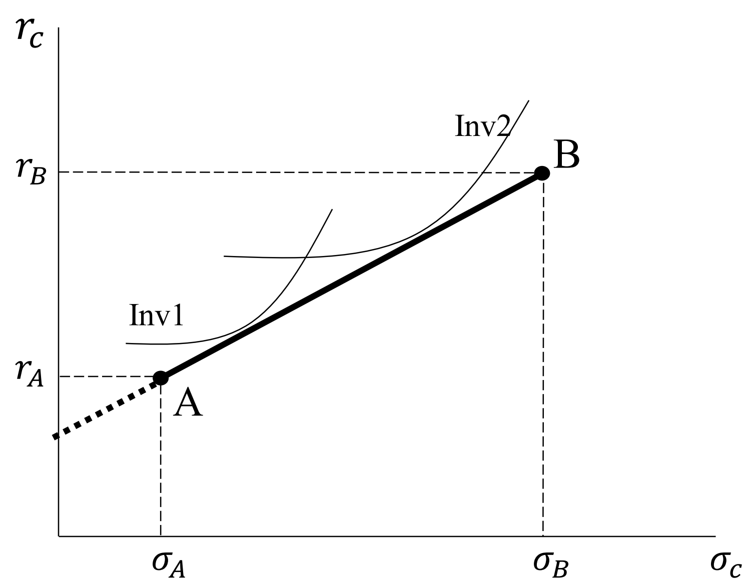
- Initially $\rho = 1$

e) Budget Constraint and Indifference Curves of Investor 1 and Investor 2



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- ~~Initially $\rho = 1$~~

e) Budget Constraint if $\rho = -1$

If you repeat steps **a)** and **b)** you get:

$$\sigma_C = x\sigma_A - (1 - x)\sigma_B$$

Note that there exist x^* such that

$$\sigma_C = x^*\sigma_A - (1 - x^*)\sigma_B = 0$$

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- ~~Initially $\rho = 1$~~

Key:

- If you move away from x^* <u>in any direction</u> you get more risk ($\sigma_C > 0$)
- If $x > x^*$, you get more of A (and lower return)
- If $x < x^*$, you get more of B (and higher return)

The graph can be constructed with this info

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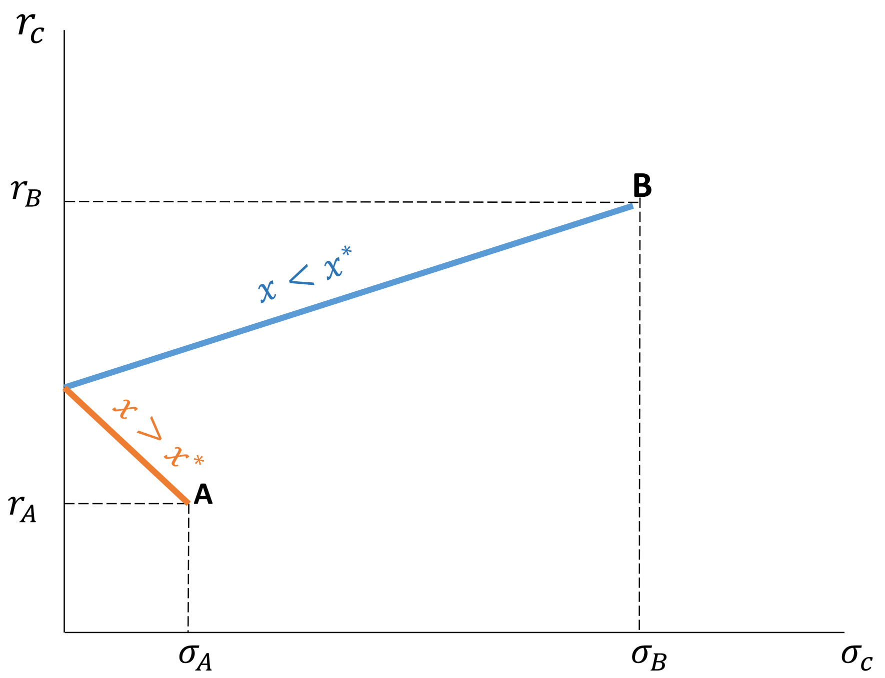
Assuming $r_A < r_B$



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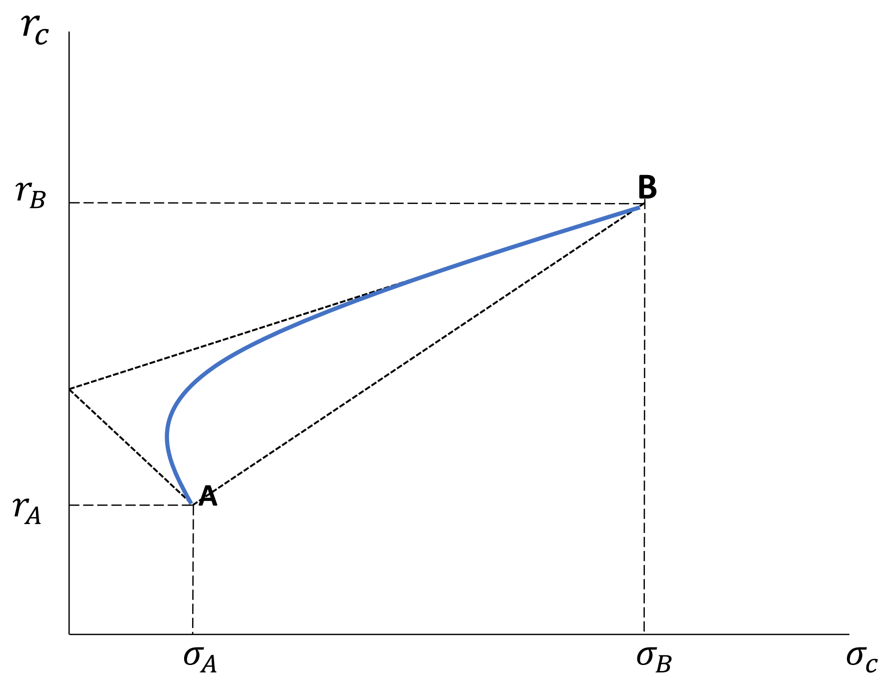
- Initially $\rho = 1$

f) Budget Constraint if $-1 < \rho < 1$



Question 10

- MC solar energy = 2
- P oil = 1.8
- MC oil = 0
- Oil can last 100 years at current use

a) If the real rate of interest is 0.05, what do you expect to happen to the current price of oil?

- Assume the usage remains the same
- Oil is an investment. After t periods, it should cost:

$$P_t = P_0(1 + 0.05)^t$$

- People will switch to Solar when $P_t = 2$

$$2 = 1.8(1 + 0.05)^t \Rightarrow \quad t = 2.16\ years$$

Question 10

- MC solar energy = 2
- P oil = 1.8
- MC oil = 0
- Oil can last 100 years at current use

But this is not an equilibrium situation!

- At current usage, there will be remaining oil for additional 97 years when P = 2
- Solar energy has infinite supply → Price should not rise → Max oil price = 2
- In 2 years, nobody will keep an asset which price will not increase anymore.

Price today must drop so it usages reaches 0 when P = 2

Question 10

- MC solar energy = 2
- P oil = 1.8
- MC oil = 0
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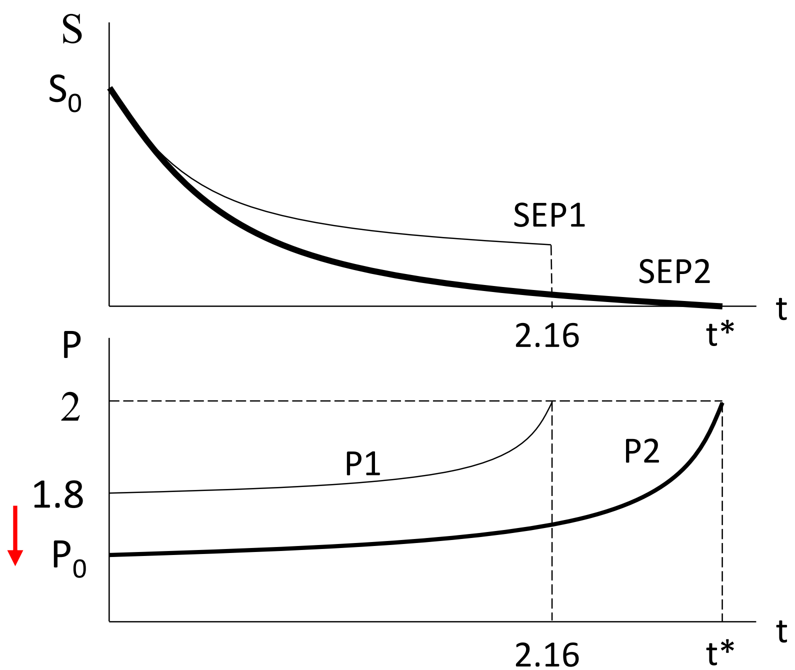
b) Path of usage and prices



MATRIX ALGEBRA:

- Determinant
- Inverse Matrix

Determinant:

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$$

$$|A| = 1 \times 2 \; - \; 3 \times (-1) = 5$$

Determinant:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & 3 \\ 4 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & 3 \\ 4 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} |\mathrm{A}| &= (1 \times 2 \times 0) + (1 \times 3 \times 4) + (-1 \times 2 \times 0) \\ &\quad -(-1 \times 2 \times 4) - (1 \times 3 \times 0) - (1 \times 2 \times 0) \\ &= 20 \end{aligned}$$

Matrix of Minors (to calculate inverse later)

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 4 & 0 & 0 \end{bmatrix}$$

Determinant of $\begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}$

$$\textbf{minor of A} = \begin{bmatrix} 0 & & \\ & & \\ & & \end{bmatrix}$$

Matrix of Minors (to calculate inverse later)

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & 3 \\ 4 & 0 & 0 \end{bmatrix}$$

Determinant of $\begin{bmatrix} 1 & -1 \\ 4 & 0 \end{bmatrix}$

$$\mathbf{minor\ of\ A} = \begin{bmatrix} 0 & & \\ & 4 & \\ & & \end{bmatrix}$$

Matrix of Minors (to calculate inverse later)

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & 3 \\ 4 & 0 & 0 \end{bmatrix}$$

$$\textbf{minor of A} = \begin{bmatrix} 0 & -12 & -8 \\ 0 & 4 & -4 \\ 5 & 5 & 0 \end{bmatrix}$$

Matrix of Cofactors (just change signs)

$$\textbf{minor of A} = \begin{bmatrix} 0 & -12 & -8 \\ 0 & 4 & -4 \\ 5 & 5 & 0 \end{bmatrix}$$

$$\downarrow$$

$$\textbf{cof of A} = \begin{bmatrix} 0 & 12 & 8 \\ 0 & -4 & 4 \\ -5 & -5 & 0 \end{bmatrix}$$

Inverse

$$\mathbf{A}^{-1} = \frac{1}{Det}(\text{cof of A})'$$

$$\boldsymbol{A}^{-1} = \frac{1}{20}\begin{bmatrix} 0 & 0 & -5 \\ 12 & -4 & -5 \\ 8 & 4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0.25 \\ 0.6 & 0.2 & -0.25 \\ -0.4 & 0.2 & 0 \end{bmatrix}$$