An algorithm for computing the volume of intersection between simplices in $\mathbb{R}^d$

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Contents

1 Theoretical framework

In this section we establish the key notions of affine spaces, convex sets and convex polytopes and simplices. We also present and show their main properties.

1.1 Affine spaces

In this section we review some of the basic properties of affine spaces as well as basic results on intersction between affine spaces.

1.1.1 Definition and properties

Definition 1 (**Affine linear combination).** *Given a (real) vector space V and $\{p_1, \cdots, p_m\} \subset V$ the affine linear combination of these elements is the set*

$$\operatorname{affine}\{p_1, \cdots, p_m\} := \left\{ \sum_{i=1}^{m} \alpha_i \, p_i \middle| \sum_i \alpha_i = 1 \right\} . \tag{1}$$

Definition 2 (**Affine space).** *Let V be a vector space, $F \subseteq V$ a vector sub space and $p \in V$. The affine space generated by F and passing through p is*

$$\Pi(p, F) := \{p + v \,|\, v \in F\} . \tag{2}$$

The following properties hold for every affine space $\Pi(p, F)$:

1. *For all $q, m \in \Pi(p, F)$, it follows immediately from the above definition that $q - m \in F$.*

2. *For any $q \in \Pi(p, F)$, $\Pi(q, F) = \Pi(p, F)$.*

   Indeed, for all $r \in \Pi(p, F)$ there is $v \in F$ such that $r = q + v = p + (q - p) + v \in \Pi(q, F)$ and thus $\Pi(p, F) \subseteq \Pi(q, F)$. The reverse inclusion follows with the same argument.

3. *For every collection $\{p_0, \cdots, p_r\} \subset \Pi(p, F)$, $\operatorname{affine}\{p_0, \cdots, p_r\} \subseteq \Pi(p, F)$.*

Every $q \in \text{affine}\{p_0, \cdots, p_r\}$ can be expressed as $q = \lambda_0 p_0 + \sum_{j\geq 1} \lambda_j p_j = \left(\sum_j \lambda_j\right) p_0 + \sum_{j\geq 1} \lambda_j (p_j - p_0) = p_0 + \sum_{j\geq 1} \lambda_j (p_j - p_0) \in \Pi(p_0, F) = \Pi(p, F)$.

4. *If* $F = \text{span}\langle p_1 - p_0, \cdots, p_r - p_0\rangle$ *then* $\text{affine}\{p_0, \cdots, p_r\} = \Pi(p, F)$.

   The previous property already establishes that $\text{affine}\{p_0, \cdots, p_r\} \subseteq \Pi(p, F)$. Conversely, every $q \in \Pi(p, F) = \Pi(p_0, F)$ can be expressed as $q = p_0 + \sum_{i\geq 1} \lambda_i (p_i - p_0)$ (since $F = \text{span}\langle p_1 - p_0, \cdots, p_r - p_0\rangle$) and thus $q = \left(1 - \sum_{i\geq 1} \lambda_i\right) p_0 + \sum_{i\geq 1} \lambda_i p_i \in \text{affine}\{p_0, \cdots, p_r\}$.

Definition 3 (**System of generators).** *A collection* $\{p_0, \cdots, p_r\} \subset \Pi(p, F)$ *such that* $\text{affine}\{p_0, \cdots, p_r\} = \Pi(p, F)$ *is said to be a system of generators.*

Lemma 1. *Every affine space,* $\Pi(p, F)$*, admits a system of generators.*

Proof. Let $\{f_1, \cdots, f_n\}$ be a basis of F. Then $\text{affine}\{p, p + f_1, \cdots, p + f_n\} = \Pi(p, F)$. Indeed, the inclusion $\text{affine}\{p, p + f_1, \cdots, p + f_n\} \subseteq \Pi(p, F)$ is guaranteed by property 3 above. Coversely, every $q \in \Pi(p, F)$ is expressed as $q = p + f$ for some $f = \mu_1 f_1 + \cdots + \mu_n f_n \in F$ or, equivalently, $q = (1 - \sum_i \mu_i)\, p + \sum_j \mu_j (p + f_j)$.

□

Lemma 2. *For every finite collection* $P := \{p_0, \cdots, p_r\} \subset V$*,* $\text{affine}\{P\} = \Pi(p_0, \text{span}\langle p_1 - p_0, \cdots, p_r - p_0\rangle)$

Proof. Indeed, for every $q \in \text{affine}\{P\}$ it holds that $q = \sum_{i=0}^{r} \lambda_i p_i = p_0 + \sum_{i=1}^{r} \lambda_i (p_i - p_0) \in \Pi(p_0, \text{span}\langle p_1 - p_0, \cdots, p_r - p_0\rangle)$. Conversely, for every $m \in \Pi(p_0, \text{span}\langle p_1 - p_0, \cdots, p_r - p_0\rangle)$ it follows that $m = p_0 + \sum_{j=1}^{r} \mu_j (p_j - p_0) = \left(1 - \sum_{j=1}^{r} \mu_j\right) p_0 + \sum_{i=1}^{r} \mu_i p_i \in \text{affine}\{P\}$.

□

Definition 4 (**Dimension of an affine space).** *The dimension of* $\Pi(p, F)$ *is defined as that of* F*. Equivalently, the dimension of* $\text{affine}\{p_0, \cdots, p_r\}$ *is* $\dim \text{span}\langle p_1 - p_0, \cdots, p_r - p_0\rangle$.

Proposition 1. *Let* $\{\Pi(p_i, F_i)\}_{i\in I}$ *be a family of affine spaces with* $\cap_{i\in I}\Pi(p_i, F_i) \neq \emptyset$*. Then* $\cap_{i\in I}\Pi(p_i, F_i) = \Pi(q, \cap_{i\in I} F_i)$*, where* $q \in \cap_{i\in I}\Pi(p_i, F_i)$.

Proof. Let $q \in \cap_{i\in I}\Pi(p_i, F_i)$. For every $m \in \cap_{i\in I}\Pi(p_i, F_i)$, $m - q \in F_i$ for all $i \in I$ and thus $m - q \in \cap_{i\in I}F_i$ and $m = q + (m - q) \in \Pi\left(q, \cap_{i\in I}F_i\right)$. Conversely, for every $v \in \cap_{i\in I}F_i$, it holds that $q + v \in \Pi(p_i, F_i)$ for all $i \in I$ and therefore $q + v \in \cap_{i\in I}\Pi(p_i, F_i)$.

□

Definition 5 (**Least dimensional affine space**). *Let V be a vector space and $A \subseteq V$ an arbitrary (non empty) set. The least dimensional affine space containing A is defined as the intersection between all the affine spaces that contain A as a subset.*

Since $A \subseteq V$ and V is an affine space generated by itself, the above intersection does always exist.

Lemma 3. *Let $P \subset V$ be non empty and finite. Then* $\text{affine}\{P\}$ *is the least dimensional affine space containing P.*

Proof. Let $P := \{p_0, \cdots, p_r\}$ and let Π denote the least dimensonal affine space containing P. Since $P \subset \text{affine}\{P\}$ the inclusion $\Pi \subseteq \text{affine}\{P\}$ follows from the definition of least dimensional affine space. Conversely, because $P \subset \Pi$, then $\text{affine}\{P\} \subseteq \Pi$ (property 3) and the equality follows.

□

1.1.2 Affine morphisms and isomorphic affine spaces

Definition 6. *A map $f : \Pi(p, F) \to \Pi(q, G)$ is said to be an affine morphism if for all $x, y \in \Pi(p, F)$ and for all $\lambda, \mu \in \mathbb{R}$ with $\lambda + \mu = 1$ it holds that $f(\lambda\, x + \mu\, y) = \lambda\, f(x) + \mu\, f(y)$.*

Lemma 4. *Let $f : \Pi(p, F) \to \Pi(q, G)$ be an affine morphism and let $\{x_1, \cdots, x_n\} \subset \Pi(p, F)$ and $\{\lambda_1, \cdots, \lambda_n\} \subset \mathbb{R}$ with $\sum_{i=1}^{n} \lambda_i = 1$.*
Then $f\left(\sum_{i=1}^{n} \lambda_i x_i\right) = \sum_{i=1}^{n} \lambda_i f(x_i)$.

Proof. By induction over the number of points n.
For $n = 2$, the statement is true by definition of affine morphism. Suppose it is true for $n \geq 2$ and let $\{x_1, \cdots, x_n, x_{n+1}\} \subset \Pi(p, F)$ and $\{\lambda_1, \cdots, \lambda_{n+1}\} \subset \mathbb{R}$ with $\sum_{i=1}^{n+1} \lambda_i = 1$. Obviously, not all λ_i can be 1. Suppose, w.l.o.g., that $\lambda_{n+1} \neq 1$, then $\sum_{i=1}^{n+1} \lambda_i x_i = \lambda_{n+1} x_{n+1} + (1 - \lambda_{n+1}) \sum_{i=1}^{n} \dfrac{\lambda_i}{1 - \lambda_{n+1}}\, x_i$ where $y := \sum_{i=1}^{n} \dfrac{\lambda_i}{1 - \lambda_{n+1}}\, x_i \in \Pi(p, F)$ (property 3) and therefore $f\left(\sum_{i=1}^{n+1} \lambda_i x_i\right) =$

$\lambda_{n+1} f(x_{n+1}) + (1-\lambda_{n+1}) f(y)$. By the induction hypothesis, it then follows that $f(y) = \sum_{i=1}^{n} \dfrac{\lambda_i}{1-\lambda_{n+1}} f(x_i)$ and therefore $f\left(\sum_{i=1}^{n+1} \lambda_i x_i\right) = \sum_{i=1}^{n+1} \lambda_i f(x_i)$.

□

Lemma 5. *If $f : \Pi(p,F) \to \Pi(q,G)$ is an affine morphism, then the map $f_* : F \to G$, befined by $f_*(u) = f(p+u) - f(p)$, is linear. Conversely, given any linear map $h : F \to G$, the map $h^* : \Pi(p,F) \to \Pi(q,G)$, given by $h^*(x) = q + h(x-p)$, is an affine morphism.*

Proof. Suppose $f : \Pi(p,F) \to \Pi(q,G)$ is an affine morphism. Let $u, v \in F$ it holds that $f_*(u+v) := f(p+u+v) - f(p) = f\left((p+u)+(p+v)-p\right) - f(p)$ and since f is an affine morphism, it follows that $f\left((p+u)+(p+v)-p\right) = f(p+u)+f(p+v)-f(p)$ and thus $f_*(u+v) = (f(p+u)-f(p)) + (f(p+v) - f(p)) = f_*(u) + f_*(v)$. Furthermore, fro every $\lambda \in \mathbb{R}$, $f_*(\lambda\, u) = f(p+\lambda\, u) - f(p) = f\left(\lambda\,(p+u)+(1-\lambda)p\right) - f(p) = \lambda\, f(p+u) + (1-\lambda) f(p) - f(p) = \lambda\, \left(f(p+u) - f(p)\right) = \lambda\, f_*(u)$.
Conversely, suppose $h : F \to G$ is linear. Let $x, y \in \Pi(p,F)$ arbitrary and $\lambda, \mu \in \mathbb{R}$, with $\lambda + \mu = 1$. Then $h^*(\lambda\, x + \mu\, y) = q + h(\lambda\, x + \mu\, y - p) = q + h\left(\lambda\,(x-p) + \mu\,(y-p)\right) = q + \lambda\, h(x-p) + \mu\, h(y-p) = \lambda\, \left(q + h(x-p)\right) + \mu\, \left(q + h(y-p)\right) = \lambda\, h^*(x) + \mu\, h^*(y)$.

□

Definition 7 (Affine isomorphism). *An affine morphism $f : \Pi(p,F) \to \Pi(q,G)$ is said to be an affine isomorphism if f is a bijection.*

Definition 8 (Isomorphic affine spaces). *. Two affine spaces $\Pi(p,F)$ and $\Pi(q,G)$ are said to be affinely isomorphic, denoted as $\Pi(p,F) \cong_{aff} \Pi(q,G)$, if there is an affine isomorphism $f : \Pi(p,F) \to \Pi(q,G)$.*

Proposition 2. *For every affine isomorphism $f : \Pi(p,F) \to \Pi(q,G)$, its inverse map $f^{-1} : \Pi(q,G) \to \Pi(p,F)$ is an affine morphism (and hence also an isomorphism).*

Proof. Let $x, y \in \Pi(q,G)$ and $\lambda, \mu \in \mathbb{R}$ with $\lambda + \mu = 1$. Let $z, w, t \in \Pi(p,F)$ be (the unique) elements such that $f(z) = x$, $f(w) = y$ and $f(t) = \lambda\, x + \mu\, y$. Since $f(t) = f(\lambda\, z + \mu\, w)$ and f is injective, it follows that $t = \lambda\, z + \mu\, w$. Then, by definition of inverse map, it follows that $f^{-1}(\lambda\, x + \mu\, y) = t = \lambda\, f^{-1}(x) + \mu\, f^{-1}(y)$.

□

Proposition 3. $\Pi(p,F) \cong_{aff} \Pi(q,G)$ *if and only if* $F \cong G$.

Proof. Suppose $f : \Pi(p,F) \to \Pi(q,G)$ is an affine isomorphism, we claim that $f_* : F \to G$ is an isomorphism of vector spaces. By lemma 5 it holds that f_* is linear so we only need to show that it is a bijection. Indeed, if $f_*(u) = f_*(v)$ then $f(p+u) - f(p) = f(p+v) - f(p)$ and because f is injective it must hold $p+u = p+v$ and then $u = v$. In addition, for every $v \in G$, $f(p) + v \in \Pi(q,G)$ and therefore there is (a unique) $x \in \Pi(p,F)$ such that $f(x) = f(p) + v$ which states that for $x - p \in F$, it holds $f_*(x-p) = f(p+x-p) - f(p) = f(x) - f(p) = v$.
Conversely, assume $h : F \to G$ is an isomorphism of vector spaces. Then the map $h^* : \Pi(p,F) \to \Pi(q,G)$, as defined in lemma 5, is an affine morphism, so we only need to show it is a bijection. Indeed, if $h^*(x) = h^*(y)$ then $q + h(x-p) = q + h(y-p)$ or equivalently $h(x-p) = h(y-p)$ and from the injectivity of h it follows that $y = x$. Moreover, for every $y \in \Pi(q,G)$, $y - q \in G$ and thus there is $u \in F$ such that $h(u) = y - q$ but this precisely states that $y = q + h(u) = q + h(p+u-p) = h^*(p+u)$.

$\square$

1.1.3 Affinely independent generators

Proposition 4. *Let* $G := \{g_0, \cdots, g_r\} \subset \Pi(p,F)$ *be a system of generators. The following statements are equivalent:*

1. *Any point* $q \in \Pi(p,F)$ *is expressed as a unique affine linear combination of the elements in* G.

2. *The system of linear equations*

$$\sum_i \alpha_i g_i = 0 \tag{3}$$

$$\sum_j \alpha_j = 0 \tag{4}$$

   admits only the trivial solution.

3. $\{g_1 - g_0, \cdots, g_r - g_0\}$ *is a basis of* F.

Proof. We will show the equivalence between the statements cyclicly.

- $1 \to 2$
  Let α be any solution to the system $\sum_i \alpha_i g_i = 0$ and $\sum_j \alpha_j = 0$. For every $q \in \Pi(p, F)$, it then holds that $q = \sum_i \beta_i g_i = \sum_j (\beta_j + \alpha_j) g_j$. However, statement *1* requires that $\beta_i = \beta_i + \alpha_i$ for all i and therefore α to be the trivial solution.

- $2 \to 3$
  Let λ be any solution to $\sum_{j \geq 1} \lambda_j (g_j - g_0) = 0$. It can be written as $-\left(\sum_j \lambda_j\right) g_0 + \sum_{l \geq 1} \lambda_l g_l = 0$. Statement *2* then requires $\lambda_i = 0$, for all i. This shows the linearly independence of the set $\{g_1 - g_0, \cdots, g_r - g_0\}$ and since, by assumption, they generate F, it is a basis.

- $3 \to 1$
  Given $q \in \Pi(p, F)$, arbitrary, let α be any solution to $q = \sum_i \alpha_i g_i$ and $\sum_j \alpha_j = 1$. It can be re expressed as $q = g_0 + \sum_{l \geq 1} \alpha_l (g_l - g_0)$ or equivalently as $q - g_0 = \sum_{l \geq 1} \alpha_l (g_l - g_0)$. Since $q - g_0 \in F$ and $\{g_1 - g_0, \cdots, g_r - g_0\}$ is a basis of F (by assumption), it follows that the last equality is true for a unique set of values for $\alpha_1, \cdots, \alpha_r$, which uniquely determines α_0. Statement *1* thus follows.

□

A system verifying any of these properties will be called an affinely independent system of generators or, interchangeably, a minimal system of generators. In what follows we will drop F and p from the notation of the affine spaces, unless there is room for confusion.

Proposition 5. *Let $\{p_0, \cdots, p_r\} \subset \Pi$ be affinely independent. Then, for every $q \in \Pi$, $\{p_0 - q, \cdots, p_r - q\}$ are also affinely independent.*

Proof. Indeed, the equation $0 = \sum_i \alpha_i (p_i - q) = \sum_i \alpha_i p_i - \left(\sum_j \alpha_j\right) q$ together with the constraint $\sum_l \alpha_l = 0$, both require $\alpha_0 = \cdots = \alpha_r = 0$.

□

Proposition 6. *Let $\tilde{\Pi} \subseteq \Pi$ be two affine spaces. Then every minimal system of generators of $\tilde{\Pi}$ may be extended to a (minimal) system of generators of Π.*

Proof. Let $\{p_0, \cdots, p_d\}$ be an affinely independent system of generators of Π. We will prove the statement by induction over the number of generators of $\tilde{\Pi}$.
Suppose $\tilde{\Pi} = \text{affine}\{q\}$. In particular $q = \sum_{i=0}^{d} \alpha_i p_i$. Assume, without loss of generality (w.l.o.g.), that $\alpha_0 \neq 0$. Then for all $p \in \Pi$ we have that

$$p = \sum_{i=0}^{d} \beta_i p_i = \sum_{j=1}^{d} \left(\beta_j - \frac{\beta_0}{\alpha_0} \alpha_j \right) p_j + \frac{\beta_0}{\alpha_0} q \tag{5}$$

and thus $\Pi \subseteq \text{affine}\{q, p_2, \cdots, p_{m+1}\}$.
Since it already holds that $\text{affine}\{q, p_2, \cdots, p_{m+1}\} \subseteq \Pi$ then we find $\Pi = \text{affine}\{q, p_2, \cdots, p_{m+1}\}$.

Suppose the statement is true for any affine space $\tilde{\Pi} \subseteq \Pi$ minimally generated by $r \geq 1$ generators. Let $\Pi' = \text{affine}\{q_0, \cdots, q_r\}$. The induction hypothesis may be applied to the affine space generated by the first r generators and, w.l.o.g., one finds $\Pi = \text{affine}\{q_0, \cdots, q_{r-1}, p_r, \cdots, p_d\}$. On the other hand, $q_r = \sum_{l=0}^{r-1} \alpha_l q_l + \sum_{j \geq r} \beta_j p_j$ with not all β_j vanishing, otherwise the collection $\{q_0, \cdots, q_r\}$ could not be minimal. Assume, w.l.o.g., $\beta_r \neq 0$. For every $p \in \Pi$, it holds that

$$\begin{aligned} p &= \sum_{i=0}^{r-1} \gamma_i q_i + \sum_{j=r}^{d} \gamma_j p_j \\ &= \sum_{i=0}^{r-1} \left(\gamma_i - \frac{\gamma_r}{\beta_r} \alpha_i \right) q_i + \sum_{j=r+1}^{d} \left(\gamma_j - \frac{\gamma_r}{\beta_r} \beta_j \right) p_j + \frac{\gamma_r}{\beta_r} q_r \end{aligned} \tag{6}$$

and therefore $\Pi \subseteq \text{affine}\{q_0, \cdots, q_r, p_{r+1}, \cdots, p_d\}$ and from this it follows that $\Pi = \text{affine}\{q_0, \cdots, q_r, p_{r+1}, \cdots, p_d\}$.

□

1.1.4 Intersection between affine spaces

In this section we briefly review the basic results on the intersection between affine spaces. In particular, we will be interested in determining when two affine spaces intersect uniquely.

Let V be a real vector space and $\Pi_1 = \text{affine}\{p_1, \cdots, p_m\} \subset V$ and $\Pi_2 = \text{affine}\{q_1, \cdots, q_r\} \subset V$ be any two affine spaces. Consider the sets

$$\mathcal{S} = \left\{ (\alpha, \beta) \in \mathbb{R}^{m+r} \,\middle|\, \begin{array}{l} \sum_i \alpha_i p_i = \sum_j \beta_j q_j \\ \sum_i \alpha_i = \sum_j \beta_j \end{array} \right\}, \tag{7}$$

$$\mathcal{S}_A = \left\{ (\alpha, \beta) \in S \,\middle|\, \sum_i \alpha_i = A \right\}, \; A \in \mathbb{R}. \tag{8}$$

Proposition 7. *The following statements are true*

1. *$\mathcal{S}$ and $\mathcal{S}_0$ are vector subspaces of $\mathbb{R}^{m+r}$ and $\dim\mathcal{S} - \dim\mathcal{S}_0$ is either 0 or 1.*
2. *For all $A \in \mathbb{R}$, $\mathcal{S}_A$ is an affine space generated by $\mathcal{S}_0$*

Proof. To see the statement *1*, consider the linear maps $f : \mathbb{R}^{m+r} \to V \times \mathbb{R}$ and $g : \mathbb{R}^{m+r} \to V \times \mathbb{R}^2$ defined as

$$f(\alpha, \beta) = \left(\sum_i \alpha_i - \sum_j \beta_j \,,\, \sum_i \alpha_i p_i - \sum_j \beta_j q_j \right) \tag{9}$$

$$g(\alpha, \beta) = \left(\sum_i \alpha_i \,,\, \sum_i \alpha_i - \sum_j \beta_j \,,\, \sum_i \alpha_i p_i - \sum_j \beta_j q_j \right) \tag{10}$$

It is clear that $\mathcal{S} = \text{Ker}(f)$ and $\mathcal{S}_0 = \text{Ker}(g)$ and therefore they are vector sub spaces. In addition, $\dim \text{Ker}(f) = m + r - \text{rank}(f)$ and $\dim \text{Ker}(g) = m + r - \text{rank}(g)$ and thus $\dim \text{Ker}(f) - \dim \text{Ker}(g) = \text{rank}(g) - \text{rank}(f)$. Let $B = \{b_1, \cdots, b_n\}$ be a basis for V. Then $\left\{ \{0 \times b_j\}_{j=1}^n, 1 \times \mathbf{0} \right\}$ is a basis for $\mathbb{R} \times V$ and $\{\{0 \times 0 \times b_i\}_{i=1}^n, 1 \times 0 \times \mathbf{0}, 0 \times 1 \times \mathbf{0}\}$ is a basis for $\mathbb{R}^2 \times V$. In these bases, the matrix of g is constructed from that of f by adding an extra row. This means that its rank is either the same as that of f or gets increased just by 1 and therefore $\dim\mathcal{S} - \dim\mathcal{S}_0 \in \{0, 1\}$.

To see the second statement, suppose that $\dim\mathcal{S} = \dim\mathcal{S}_0$, then $\mathcal{S} = \mathcal{S}_0$ and there is no $\mathcal{S}_A$ other than $\mathcal{S}_0$ which trivially is an affine space generated

by $\mathcal{S}_0$.
Otherwise, consider the quotient space $\mathcal{S}/\mathcal{S}_0$ which has dimension $\dim\mathcal{S} - \dim\mathcal{S}_0 = 1$. For any $s \in \mathcal{S}$ denote by $[s] \in \mathcal{S}/\mathcal{S}_0$, its equivalence class. Let $\sigma \in \mathcal{S} \setminus \mathcal{S}_0$. For every $s \in \mathcal{S}$ there is a unique $\lambda \in \mathbb{R}$ such that $[s] = [\lambda\sigma]$. Or equivalently, any $s \in \mathcal{S}$ can be expressed as $s = \lambda\sigma + s_0$, for a unique $\lambda \in \mathbb{R}$ and a unique $s_0 \in \mathcal{S}_0$. Say that $\sigma = (\alpha, \beta)$ and take $a = \sum_i \alpha_i \neq 0$. Given $A \in \mathbb{R}$ take $\lambda = A/a$. It follows that

$$\left\{ \frac{A}{a}\sigma + s_0 \middle| s_0 \in \mathcal{S}_0 \right\} \subset \mathcal{S}_A \,.$$

Conversely, for any $s \in \mathcal{S}_A$, $\sum s = 2A$ and therefore $s = \dfrac{A}{a}\sigma + s_0$ for a (unique) $s_0 \in \mathcal{S}_0$. This shows that

$$\mathcal{S}_A = \left\{ \frac{A}{a}\sigma + s_0 \middle| s_0 \in \mathcal{S}_0 \right\} \,.$$

□

Proposition 8. $\Pi_1 \cap \Pi_2 \cong_{aff} \mathcal{S}_1$.

Proof. We claim that the map $F : \mathcal{S}_1 \to \Pi_1 \cap \Pi_2$, defined as $F(\alpha, \beta) = \sum_i \alpha_i p_i$, is an affine isomorphism. To see it is an affine morphism let $a_1 := (\alpha_1, \beta_1), a_2 := (\alpha_2, \beta_2) \in \mathcal{S}_1$ and $\lambda, \mu \in \mathbb{R}$ with $\lambda + \mu = 1$. It holds that $F(\lambda\, a_1 + \mu\, a_2) = \sum_i (\lambda\, (\alpha_1)_i + \mu\, (\alpha_2)_i) p_i = \lambda\, F(a_1) + \mu\, F(a_2)$. We now show it is a bijection. Indeed, for any $p \in \Pi_1 \cap \Pi_2$ there is $(\alpha, \beta) \in \mathbb{R}^{m+r}$ such that $p = \sum_i \alpha_i p_i = \sum_j \beta_j q_j$ and $\sum_i \alpha_i = \sum_j \beta_j = 1$, so that $(\alpha, \beta) \in \mathcal{S}_1$ and $F(\alpha, \beta) = p$. This shows that F is surjective.
To see that it is also injective, suppose that $F(\alpha, \beta) = F(\alpha', \beta')$, that is: $\sum_i \alpha_i p_i = \sum_l \alpha'_l p_l$. In addition, since $(\alpha, \beta), (\alpha', \beta') \in \mathcal{S}_1$, it must also hold $\sum_i \beta_j q_j = \sum_k \beta'_k q_k$ and because these sets of generators are minimal, it follows that $\alpha = \alpha'$ and $\beta = \beta'$.

□

We now show the main result of this section.

Theorem 1. *The following statements are true:*

1. $\Pi_1 \cap \Pi_2 = \emptyset$ *if and only if* $\dim\mathcal{S} = \dim\mathcal{S}_0$.

2. *If* $\dim\mathcal{S} > \dim\mathcal{S}_0$ *the affine spaces intersect in an affine space of dimension* $\dim\mathcal{S}_0$*.*

3. **(Shared directions)** *If* F_1 *and* F_2 *are the vector spaces generating* Π_1 *and* Π_2*, respectively, then* $F_1 \cap F_2 \cong \mathcal{S}_0$*.*

Proof. 1. If $\dim\mathcal{S} = \dim\mathcal{S}_0$, then $\mathcal{S} = \mathcal{S}_0$ and therefore $\mathcal{S}_1 = \emptyset$, proposition 8 thus requires $\Pi_1 \cap \Pi_2 = \emptyset$. Conversely, if $\Pi_1 \cap \Pi_2 = \emptyset$ again by proposition 8 it must hold $\mathcal{S}_1 = \emptyset$. Suppose however that there is some $(\alpha, \beta) \in \mathcal{S} \setminus \mathcal{S}_0$, then $a = \sum_i \alpha_i \neq 0$ and $\frac{1}{a}(\alpha, \beta) \in \mathcal{S}_1$, which is a contradiction.

2. $\dim S > \dim S_0$
   By the previous proposition, $\Pi_1 \cap \Pi_2 \cong_{aff} \mathcal{S}_1$ and the last is an affine space generated by $\mathcal{S}_0$.

3. (**Shared directions**) From propositions 1, 3 and 8 it follows that $F_1 \cap F_2 \cong \mathcal{S}_0$.

□

1.2 Convex sets

In this section we will review some of the basic properties of convex sets. Most of the results presented here can be found in standard text books on convex geometry, see for instance Grünbaum (2003).

Definition 9 (Convex set). *A set* $K \subset \mathbb{R}^n$ *is said to be convex if for every* $p, q \in K$ *the segment joining both points,* $\overline{pq} = \{\lambda p + (1-\lambda)q \,|\, 0 \leq \lambda \leq 1\}$*, is entirely contained in* K*.*

The empty set is taken to be convex.

Lemma 6. *The intersection of any family of convex sets is convex.*

Proof. Let $\{K_i\}_{i\in I}$ be an arbitrary family of convex sets. If $\cap_{i\in I} K_i = \emptyset$ it is convex. Suppose then, that $K := \cap_{i\in I} K_i \neq \emptyset$ and let $p, q \in K$ arbitrary. It follows that the segment $\overline{pq}$ is contained in K_i for every $i \in I$ and hence $\overline{pq} \subset K$.

□

This enables the existence of the smallest convex set containing any given set, as we review in the next section.

1.2.1 Convex hull and convex polytopes

Proposition 9 (**Convex hull**). *Let $A \subset \mathbb{R}^n$, the intersection of all the convex sets containing A is the smallest convex set containg A. Such a set is called the convex hull of A and it is denoted as $ch(A)$.*

Proof. Let $\mathcal{C}(A) = \{K \subseteq \mathbb{R}^n \,|\, A \subseteq K \,, K \text{ is convex}\}$. Notice that $\mathbb{R}^n \in \mathcal{C}(A)$. The above lemma guarantees that $\cap \mathcal{C}(A)$ is convex and it clearly holds that $A \subseteq \cap \mathcal{C}(A)$. In addition, every convex set $K \supseteq A$ is an element of $\mathcal{C}(A)$ and therefore $\cap \mathcal{C}(A) \subseteq K$.

□

Definition 10 (**Convex polytope**). *Given $V = \{v_1, \cdots, v_m\} \subset \mathbb{R}^n$, the convex polytope generated by this set of points is defined as*

$$conv(V) := \left\{ \sum_{i=1}^{m} \alpha_i v_i \,\middle|\, \alpha_j \geq 0, \, \sum_{i=1}^{m} \alpha_i = 1 \right\}$$

For $x \in conv(V)$, the equality $x = \alpha_1 v_1 + \cdots + \alpha_m v_m$ is called a convex expansion of x in the points V.

Proposition 10. *$conv(V)$ is convex.*

Proof. Indeed, let $p, q \in conv(V)$ and $0 \leq \lambda \leq 1$. Since $p = \sum_i \alpha_i v_i$ and $q = \sum_i \beta_i v_i$, with $\alpha_i, \beta_j \geq 0$ and $\sum_i \alpha_i = \sum_j \beta_j = 1$, it clearly holds that $\lambda p + (1 - \lambda) q = \sum_j [\lambda \alpha_j + (1 - \lambda) \beta_j] \, v_j \in conv(V)$.

□

Theorem 2. *Let $K \subset \mathbb{R}^n$ convex and $V = \{v_1, \cdots, v_m\} \subset K$. Then $conv(V) \subset K$.*

Proof. We will proof the statement by induction over the number of points. Since K is convex, the statement is true for $m = 2$.
Suppose the statement is true for any set of m points in K and let $v_1, \cdots, v_{m+1} \in K$. Consider any collection of $m+1$ numbers, $\alpha_1, \cdots, \alpha_{m+1}$, non negative and adding up to 1. If only one of them is non vanishing, say the i-th parameter, then we must have $\alpha_i = 1$ and thus $\sum_j \alpha_j v_j = v_i \in K$. If at least two of them are non vanishing assume, with no loss of generality, that $\alpha_1 \, \alpha_{m+1} > 0$ and define $\lambda = \sum_{i=1}^m \alpha_i > 0$. It holds that $\lambda + \alpha_{m+1} = 1$. Then $\sum_{j=1}^{m+1} \alpha_j v_j = \lambda p + (1-\lambda) v_{m+1}$ where $p = \sum_{i=1}^m \frac{1}{\lambda} \alpha_i v_i$ and $\sum_{i=1}^m \frac{1}{\lambda} \alpha_i = 1$. Thus $p \in K$ by induction hypothesis. The convexity of K implies then $\sum_{j=1}^{m+1} \alpha_j v_j \in K$, which proves the statement.

□

As a corollary of this theorem we have that

Corollary 1. *Let $V \subset \mathbb{R}^n$ be a finite set of points. Then $conv(V) = ch(V)$.*

Proof. Since $V \subset conv(V)$, which is convex itself, it must hold that $ch(V) \subseteq conv(V)$. On the other hand, by theorem 2, it also holds that $conv(V) \subseteq ch(V)$.

□

Lemma 7. *Let $V := \{v_1, \cdots, v_k\}$ not affinely independent. Then for every $x \in conv(V)$ there is $v_x \in V$ such that $x \in conv\,(V \setminus \{v_x\})$.*

Proof. Let $x = \sum_{i=1}^{k+1} \lambda_i \, v_i$. If some $\lambda_i = 0$, then $x \in conv\,(V \setminus \{v_i\})$ and the statement trivially follows. Assume then that all $\lambda_i > 0$. Since the points in V are not affinely independent, the equalities $\sum_i \alpha_i v_i = 0$ and $\sum_i \alpha_i = 0$ have non trivial solutions. Let (α_i) be any such non vanishing solution. There must be at least one strictly positive value among (α_i). Let $\dfrac{\lambda_r}{\alpha_r} := \min \left\{ \dfrac{\lambda_i}{\alpha_i} \,\middle|\, \alpha_i > 0 \right\} > 0$. It follows that $x = \sum_i \lambda_i \left(1 - \dfrac{\lambda_r}{\alpha_r}\dfrac{\alpha_i}{\lambda_i}\right) v_i$, where $\sum_i \lambda_i \left(1 - \dfrac{\lambda_r}{\alpha_r}\dfrac{\alpha_i}{\lambda_i}\right) = \sum_i \lambda_i = 1$ and $1 - \dfrac{\lambda_r}{\alpha_r}\dfrac{\alpha_i}{\lambda_i} > 0$ if $\alpha_i \leq 0$ while $1 - \dfrac{\lambda_r}{\alpha_r}\dfrac{\alpha_i}{\lambda_i} \geq 1 - \dfrac{\lambda_i}{\alpha_i}\dfrac{\alpha_i}{\lambda_i} = 0$ for $\alpha_i > 0$. Furthermore, $1 - \dfrac{\lambda_r}{\alpha_r}\dfrac{\alpha_i}{\lambda_i}$ vanishes for $i = r$. This shows that $x \in conv\,(V \setminus \{v_r\})$.

□

Corollary 2. *A finite set of points V is affinely independent if and only if there is $x \in conv(V)$ such that for every non empty subset $U \subseteq V$, $x \notin conv\,(V \setminus U)$.*

Proof. Let $V := \{v_1, \cdots, v_k\}$. If these points are affinely independent, then $x = \dfrac{1}{k} \sum_{i=1}^{k} v_i$ admits no other convex expansion.
Conversely, suppose there is $x \in V$ verifying the above property and yet V is affinely dependent. Lemma 7 then yields a contradiction.

□

Theorem 3 (Carathéodory). *Let $V \subset \mathbb{R}^n$ be a non empty finite set and let $d := \dim \operatorname{affine} \{V\}$. Then every $x \in conv(V)$ may be expressed as a convex expansion of at most $d+1$ points in V.*

Proof. Let $k = \sharp(V)$. If $k \leq d+1$ the statement is obvious. Suppose $k > d+1$. Then V can not be affinely independent (statement 3 in proposition 4) and lemma 7 assures that for some $v_x \in V$, $x \in conv\,(V \setminus \{v_x\})$. Whenever $\sharp\,(V \setminus \{v_x\}) > d+1$, the corresponding points are affinely dependent and lemma 7 may be applied again. The iterated process obviously stops after a finite number of steps and the number of points in the final set is no bigger than $d+1$.

□

Definition 11. *The dimension of a convex set K is defined as the dimension of the least dimensional affine space containing K.*

Corollary 3. *$conv(V)$ has dimension d if and only if every $x \in conv(V)$ expresses as a convex expansion of at most $d+1$ points from V and there is $y \in conv(V)$ that admits no convex expansion on less than $d+1$ points in V. Furthermore, a point x lies in the relative interior of $conv(V)$ if and only if it expresses as a convex expansion of no less than $d+1$ points in V.*

Proof. If $\operatorname{affine}\{V\}$ has dimension d, then V contains exactly $d+1$ affinely indepenent points, $\{p_0, \cdots, p_d\} \subseteq V$, and $\operatorname{affine}\{V\} = \operatorname{affine}\{p_0, \cdots, p_d\}$. Then $y = \dfrac{1}{d+1} \sum_{i=0}^{d} p_i$ does not admit an affine expansion on less than $d+1$ points. In addition, to every $x \in conv(V)$ we may apply the Carathéodory's theorem to conclude that x can be expressed as a convex expansion on at

most $d+1$ points. Conversely, if there is $x \in conv(V)$ not admiting a convex expansion on less than $d+1$ points from V, then corollary 2 implies that there are $d+1$ affinely independent points in V. Furthermore if no other $x(V)$ expresses as a convex expansion on more than $d+1$ elements in V, the number of affinely independent points in V is exactly $d+1$.
To see the second statement, suppose x lies in the relative interior of $conv(V)$. The Carathéodory's theorem guarantees that there are affinely independent points $v_0, \cdots, v_k \in V$, with $k \leq d$, such that $x = \sum_{i=0}^{k} \alpha_i \, v_i$ with all $\alpha_i > 0$. Let $B_\epsilon(x)$ denote the d-dimensional open ball centered at x and contained in affine$\{V\}$. If $k < d$ assume, w.l.o.g., that $\{v_0, \cdots, v_k\}$ are affinely independent and consider the unit vector $u = \dfrac{1}{\|v_{k+1} - v_0\|}\,(v_{k+1} - v_0)$. It follows that the point $x - \dfrac{\epsilon}{2} u$ belongs to $B_\epsilon(x)$ but lies outside $conv(V)$, for every $\epsilon > 0$. Indeed, $\|y - x\| = \epsilon/2$ while $y = -\dfrac{\epsilon}{2\ \|v_{k+1} - v_0\|}\, v_{k+1} + \cdots$. This contradicts that x lies in the relative interior of $conv(V)$.
Conversely, suppose that $x = \sum_{i=0}^{d} \alpha_i \, v_i$ with all $\alpha_i > 0$ and not admitting a convex expansion on less points. Again, by lemma 7, $v_0, \cdots, v_d$ are affinely independent. For every $y \in B_\epsilon(x)$, it holds that $y = x + \sum_{i=1}^{d} \gamma_i \,(v_i - v_0)$ such that $\|\sum_{i=1}^{d} \gamma_i \,(v_i - v_0)\| < \epsilon$. It follows that for $\epsilon > 0$, small enough, $\alpha_i + \gamma_i > 0$ and $\alpha_0 - \sum_j \gamma_j > 0$. This shows that x lies in the relative interior of $conv(V)$.

□

1.2.2 Extreme points

Definition 12 (Extreme points). *A point x in a convex set K is said to be an extreme point if the equality $x = \lambda\, y + (1 - \lambda)\, z$ for $y, z \in K$ and $0 < \lambda < 1$, implies $x = y = z$. The set of extreme points of K is denoted as $ext(K)$.*

Proposition 11. *Let F be a vector space and $K \subseteq F$ be a convex set. The following statements are equivalent:*

1. *x is an extreme point of K.*

2. *For every non vanishing $u \in F$ and every $\epsilon > 0$, there is $|\lambda| < \epsilon$ such that $x + \lambda\, u \notin K$.*

Proof. Let us first see the implication $1 \to 2$. Indeed, let $u \in F$, non vanishing, and suppose there is $\epsilon > 0$ such that $x + \lambda\, u \in K$, for all $|\lambda| < \epsilon$. In particular one has that $y = x + \dfrac{\epsilon}{2}\, u \in K$ and $z = x - \dfrac{\epsilon}{2}\, u \in K$ while $x = \dfrac{1}{2}(y+z)$, contradicting that x is an extreme point.
To see the reverse implication, let $x \in K$ verify the property of the second statement and suppose that there are $y, z \in K$ and $0 < \lambda_0 < 1$ such that $x = \lambda_0\, y + (1-\lambda_0)\, z$. If $y = z$ the equality already yields $x = y$. Suppose then that $y \neq z$, which also implies that $x \neq y$ and $x \neq z$. To see this suppose $x = z$ then the above equality states that $z = \lambda_0\, y + (1-\lambda_0) z$ which requires $\lambda_0\, y = \lambda_0\, z$ and, because $\lambda_0 > 0$, $y = z$. The distinction $x \neq z$ follows using the same argument and the fact that $\lambda_0 < 1$. Taking $\epsilon = \min\{\lambda_0, 1-\lambda_0\}$, it follows that for all $|\mu| < \epsilon$, $x + \mu\,(y - z) = (\lambda_0 + \mu)\, y + (1 - \lambda_0 - \mu)\, z$ where $\lambda_0 + \mu \geq \lambda_0 - \epsilon \geq 0$ and $\lambda_0 + \mu \leq \lambda_0 + \epsilon \leq 1$. Thus, for all $|\mu| < \epsilon$, $x + \mu\,(y - z) \in K$, which contradicts the property in the statement *2*. This shows that the equality $x = \lambda_0\, y + (1-\lambda_0)\, z$, with $0 < \lambda_0 < 1$, requires $x = y = z$ and therefore $x \in ext(K)$.

$\square$

Lemma 8. *Given a (non empty) convex set K, for every $x \in K$ there are: an affine space $\Pi(x, F(x))$, $\epsilon > 0$ and a basis of $F(x)$, $\{u_1, \cdots, u_k\}$, such that $x + \sum_{i=1}^{k} \lambda_i u_i \in K$ for all $|\lambda_i| < \epsilon$. In addition, $F(x) = \{\mathbf{0}\}$ if and only if $x \in ext(K)$. Furthermore, for every $y \in K$ with $y - x \in F(x)$, it holds that $F(y) \subseteq F(x)$.*

Proof. Let F be the vector space where K is contained. By proposition 11, $x \in ext(K)$ if and only if $\mathbf{0}$ is the only vector verifying that $x + \lambda\mathbf{0} \in K$, for all $|\lambda|$ small enough, and therefore $F(x) = \{\mathbf{0}\}$. Assume then, that $x \in K \setminus ext(K)$ and let $F(x) := \{u \in F \,|\, \exists\, \epsilon > 0 :\, x + \lambda\, u \in K, \forall\, |\lambda| < \epsilon\}$. We claim that $F(x)$ is a subspace of F. Indeed, $\mathbf{0} \in F(x)$ since $x + \lambda\,\mathbf{0} \in K$ for all λ. In addition, given $u \in F(x)$ and $\lambda \in \mathbb{R}$ it follows that, for $\epsilon < \epsilon_u / |\lambda|$ and for all $|\mu| < \epsilon$, $x + \mu\,(\lambda \cdot u) = x + \mu\lambda\, u \in K$, since $|\mu\lambda| < \epsilon_u$, and thus $\lambda \cdot u \in F(x)$. Moreover, for $u, v \in F(x)$, take $\epsilon = \min\{\epsilon_u, \epsilon_v\}/2$. Then, for all $|\lambda| < \epsilon$ it holds that $x + \lambda\,(u + v) = \dfrac{1}{2}\,(x + 2\lambda\, u) + \dfrac{1}{2}\,(x + 2\lambda\, v) \in K$, since $x + 2\lambda\, u \in K$ and $x + 2\lambda\, v \in K$, for $|2\lambda| < \min\{\epsilon_u, \epsilon_v\}$ and K is convex. Since at least there is one non vanishing vector in $F(x)$, it follows that $\dim F(x) \geq 1$. Moreover, let $B = \{u_1, \cdots, u_k\}$ be a basis of $F(x)$ and take $\delta = \min\{\epsilon_{u_1}, \cdots, \epsilon_{u_k}\}$. We claim that

$$\Pi(x, \operatorname{span}\langle B\rangle; \delta) := \left\{ x + \sum_{i=1}^{k} \lambda_i u_i \;\middle|\; \sum_{i=1}^{k} |\lambda_i| < \delta \right\} \subset K\,.$$

Indeed, every $p = x + \sum_{i=1}^{k} a_i\, u_i \in \Pi(x, \operatorname{span}\langle B\rangle; \delta)$ can be expressed as $p = \sum_{i=1}^{k} \dfrac{|a_i|}{\sum_{j=1}^{k} |a_j|} \left(x + \operatorname{sign}(a_i) \left(\sum_{j=1}^{k} |a_j|\right) u_i \right)$.
Because $\sum_{j=1}^{k} |a_j| < \delta \leq \epsilon_{u_i}$, then $x + \operatorname{sign}(a_i) \left(\sum_{j=1}^{k} |a_j|\right) u_i \in K$ and then it follows that $p \in K$. Taking $\epsilon = \delta/k$, it holds that $x + \sum_{i=1}^{k} \lambda_i u_i \in \Pi(x, \operatorname{span}\langle B\rangle; \delta) \subset K$ for all $|\lambda_i| < \epsilon$.
To see the last statement, let $y \in K$ such that $y - x \in F(x)$. If $y \in ext(K)$, then $F(y) = \{\mathbf{0}\} \subseteq F(x)$. Assume then that $y \in K \setminus ext(K)$ and suppose that there is (a non vanishing) $u \in F(y)$ not in $F(x)$. Since, $y - x \in F(x)$, the segment $\overline{xy}$ can be extended to $x' = x - \lambda(y - x) \in K$, for some $\lambda > 0$. In addition the points $z = y + \mu\, u$ and $w = y - \mu\, u$ belong to K for $|\mu| > 0$, small enough. Thus x belongs to the interior of $conv(\{x', z, w\}) \subset K$, which contains the direction u. This yields a contradiction with the assumption that $u \notin F(x)$. Thus, $F(y) \subseteq F(x)$.

□

From this lemma we have the following result (which is a key result for the next section):

Theorem 4 (**Minkowski).** *Every compact convex set is the convex hull of its extreme points.*

Proof. If $K = \emptyset$ then $ext(K) = \emptyset$ and $ch(ext(K)) = \emptyset$. Similarly, if $K = \{p\}$ then $ext(K) = K = ch(ext(K))$. Let us see the statement for a compact convex set of dimension 1, which is a closed and finite segment, S. Clearly, S has only two extreme points, p and q (the two ends of the segment) and for every $x \in S$ it clearly holds that $x = \mu\, p + (1 - \mu)\, q$ for $0 \leq \mu \leq 1$ and therefore $S = ch(\{p, q\})$.
Assume that K is a compact convex set other than the previous trivial cases. Since $ext(K) \subset K$ (convex) the inclusion $ch(ext(K)) \subseteq K$ follows immediately. Conversely, let $x \in K$ arbitrary. If $x \in ext(K)$, then obviously $x \in ch(ext(K))$. Assume that $x \in K \setminus ext(K)$ and let $\Pi(x, F(x))$ be the affine space from the previous lemma. For any (no vanishing) $u \in F(x)$ the set

$S := \Pi(x, \mathrm{span}\langle u\rangle) \cap K$ is convex, compact and has dimension 1. Therefore $x \in ch(\{p,q\})$, with $\{p,q\} = ext(S)$. Moreover, since $\{p-x, q-x\} \subset F(x)$, $F(p) \subseteq F(x)$ and $F(q) \subseteq F(x)$ whereas neither $u \notin F(p)$ nor $u \notin F(q)$. This implies that $\dim F(p) \leq \dim F(x) - 1$ and the same for $F(q)$. If $F(p)$ (respectively, $F(q)$) is not the trivial vector space, then p (respectively, q) is not an extreme point and the same argument may be applied to p (respectively, q). Because at each iteration the dimension of the newly found vector space is at least one unit less that the previous one, the process ends after a finite number of steps and the final set of points, $\{p_1, \cdots, p_m\}$ contains only extreme points. In addition, $x \in ch(\{p_1, \cdots, p_m\}) \subseteq ch(ext(K))$. This shows that $K \subseteq ch(ext(K))$ and the equality follows.

□

Remark 1. *Not every compact and convex set has a finite set of extreme points. For instance, the solid disc in* $\mathbb{R}^2$, $D := \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leq 1\}$, *is compact and convex but* $ext(D) = S^1 := \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}$

1.3 Simplices

An n-simplex or just simplex is an n-dimensional polytope generated by $n+1$ affinely independent vertices. More precisely, if the simplex is generated by the vertices $\{v_1, \cdots, v_{n+1}\}$, we denote it as $S = \langle v_1 \cdots v_{n+1}\rangle$, and by choosing a particular order in the generating vertices, way say that the simplex gets an orientation.

A boundary of S of dimension $0 \leq k \leq n+1$ is a simplex generated by $k+1$ vertices picked from the set $\{v_1, \cdots, v_{n+1}\}$. If the order is respected we say that the boundary inherits the orientation of the simplex. For completeness, we have also included the whole simplex as a boundary. A boundary of dimension $n-1$ is called a face of the simplex and is denoted as $\partial_i S$, indicating that it is generated by all the vertices except for v_i. The boundaries of dimension one are called edges. If B is a boundary generated by the vertices $\{v_{\alpha_1}, \cdots, v_{\alpha_b}\}$ then it is contained in $n+1-b$ faces given by $\partial_j S$ with $j \notin \{\alpha_1, \cdots, \alpha_b\}$. And clearly it holds that $B = \cap_{j \notin \{\alpha_1, \cdots, \alpha_b\}} \partial_j S$.

1.3.1 Interior and exterior of a simplex

Let $S = \langle v_0, \cdots, v_n \rangle$ be a simplex in $\mathbb{R}^n$. The quantity

$$\mathcal{E}(v_0, \cdots, v_n) := \det \begin{pmatrix} 1 & (v_0)_1 & \cdots & (v_0)_n \\ \vdots & \vdots & \dots & \vdots \\ 1 & (v_n)_1 & \cdots & (v_n)_n \end{pmatrix}, \tag{11}$$

is defined as the orientation of the simplex. This quantity is non vanishing, because the vertices are affinely independent, and totally anti symmetric in its entries, that is: $\mathcal{E}\left(v_{\sigma_1}, \cdots, v_{\sigma_{n+1}}\right) = \mathrm{sign}(\sigma)\, \mathcal{E}\left(v_1, \cdots, v_{n+1}\right)$, for every permutation, σ, of $\{0, \cdots, n\}$. Using the properties of the determinant, it is easy to check that $\mathcal{E}(v_0, \cdots, v_n) = \det\left(u_1, \cdots, u_n\right)$, with $u_i := v_i - v_0$. Any point in $\mathbb{R}^n$ is uniquely expanded as $p = \sum_{i=0}^n \alpha_i v_i$, and it belongs to the simplex if and only if all $\alpha_k \geq 0$. One easily checks that

$$\alpha_k = \frac{\mathcal{E}\left(v_0, \cdots, p, \cdots, v_n\right)}{\mathcal{E}\left(v_0, \cdots, v_n\right)}, \tag{12}$$

where, in the denominator, the k-th entry of $\mathcal{E}$ is fed with the point p. To see this, assume first $k \geq 1$.
The point p may be expressed as $p - v_0 = \sum_{i=1}^n \alpha_i\ (v_i - v_0)$ and $\mathcal{E}\left(v_0, \cdots, p, \cdots, v_n\right) = \det\left(u_1, \cdots, \sum_{i=1}^n \alpha_i\, u_i, \cdots, u_n\right) = \alpha_k \det\left(u_1, \cdots, u_n\right) = \alpha_k\, \mathcal{E}\left(v_0, \cdots, v_n\right)$. For $k = 0$, $\mathcal{E}\left(p, \cdots, v_n\right) = -\mathcal{E}\left(v_1, p, \cdots, v_{n+1}\right)$ and $p - v_1 = \alpha_0\left(v_0 - v_1\right) + \sum_{i \geq 2} \alpha_i\left(v_i - v_1\right)$. Using the same argument one finds $\mathcal{E}\left(p, v_1, \cdots, v_n\right) = -\mathcal{E}\left(v_1, p, \cdots, v_n\right) = -\alpha_0\, \mathcal{E}\left(v_1, v_0, \cdots, v_n\right) = \alpha_0\, \mathcal{E}\left(v_0, \cdots, v_n\right)$.

Lemma 9. *A point p lies in the interior of the simplex S, denoted as $p \in \mathring{S}$, if and only if $p = \sum_{i=0}^n \alpha_i\, v_i$, with all $\alpha_i > 0$.*

Proof. Suppose $p \in \mathring{S}$ and yet $p = \sum_{i=0}^n \alpha_i\, v_i$ with some $\alpha_i = 0$. Assume, w.l.o.g., that $p = \sum_{i=0}^k \alpha_i\, v_i$, with $k < n$. Since $p \in \mathring{S}$, there is $\epsilon > 0$ such that the open ball centered at p with radius ϵ, $B_\epsilon(p)$, lies entirely in S. However, for every $0 < \delta < \epsilon / \left\|v_n - p\right\|$, the point $q = p - \delta\left(v_n - p\right) \in B_\epsilon(p)$ but $q = -\delta\, v_n + \sum_{i=0}^k (1+\delta)\alpha_i\, v_i \notin S$, which is a contradiction.
Conversely, suppose that $p = \sum_{i=0}^n \alpha_i\, v_i$ with all $\alpha_i > 0$ and consider the maps $f_k : \mathbb{R}^n \to \mathbb{R}$ defined by $f_k(x) := \dfrac{\mathcal{E}\left(v_0, \cdots, x, \cdots, v_n\right)}{\mathcal{E}\left(v_0, \cdots, v_n\right)}$. They are clearly continuous and since $f_k(p) = \alpha_k > 0$ for all k, it follows that for some $\epsilon > 0$, $f_k(y) > 0$ for all $y \in B_\epsilon(p)$ and thus $B_\epsilon(p) \subset S$.

$\square$

Remark 2. *The volume of a simplex* $S := \langle v_0, \cdots, v_n \rangle$ *is the quantity*

$$vol(S) := \frac{1}{n!} \, |\mathcal{E}\,(v_0, \cdots, v_n)| \tag{13}$$

1.3.2 Intersection between simplices

We now show the main result on which we base our method for computing the volume of the intersection between simplices.

Theorem 5. *Let* $S_1, S_2 \subset \mathbb{R}^n$ *be two simplices of dimensions* $n \geq m \geq 1$*, respectively, and with* $S_1 \cap S_2 \neq \emptyset$*. Let* $\mathcal{I}$ *be the set of points in* $\mathbb{R}^n$ *constructed as follows:* $p \in \mathcal{I}$ *if* $p \in \mathring{B}_1 \cap \mathring{B}_2$*, where* B_1 *and* B_2 *are boundaries of* S_1 *and* S_2*, respectively, and not supporting any common direction. Then it holds that* $S_1 \cap S_2 = ch(\mathcal{I})$*.*

Proof. We claim that $\mathcal{I} = ext\,(S_1 \cap S_2)$. Indeed, let $\{p_0, \cdots, p_n\}$ be the vertices of S_1 and $\{q_0, \cdots, q_m\}$ be the vertices of S_2 and let $x \in \mathring{B}_1 \cap \mathring{B}_2$ with B_1 and B_2, boundaries verifying the properties required in theorem 2. Let $u \in \mathbb{R}^n$ be an arbitrary (non vanishing) vector and assume, w.l.o.g., that the direction u is not supported by Π_1 (the least dimensional affine space supporting B_1). Further assume, w.l.o.g., that $\{p_0, \cdots, p_r\}$ are the vertices of the boundary B_1. Then $u = \sum_{i=1}^{r} \gamma_i (p_i - p_0) + \sum_{j=r+1}^{n} \mu_j (p_j - p_0)$ with not all μ_j vanishing. Assume, w.l.o.g., that $\mu_{r+1} \neq 0$. Therefore, $x + \lambda\, u = \lambda\, \mu_{r+1}\, p_{r+1} + \sum_{i=0}^{r} \alpha_i(\lambda)\, p_i + \sum_{j > r+1} \beta_j(\lambda)\, p_j$. Given $\epsilon > 0$ arbitrary, take $\lambda = -\text{sign}\,(\mu_{r+1})\, \dfrac{\epsilon}{2}$, it then holds that $x + \lambda\, u = -\,|\mu_{r+1}|\, \dfrac{\epsilon}{2}\, p_{r+1} + \cdots$ and therefore $x + \lambda\, u \notin S_1$. This shows that x is an extreme point of $S_1 \cap S_2$ and since $x \in \mathcal{I}$ was arbitrary, it follows that $\mathcal{I} \subset ext\,(S_1 \cap S_2)$.
To see the reverse inclusion, let $x \in ext\,(S_1 \cap S_2)$ arbitrary. Assume, w.l.o.g., that $x = \sum_{i=0}^{r} \alpha_i p_i = \sum_{j=0}^{s} \beta_i q_i$ with all $\alpha_i > 0$ and all $\beta_j > 0$. Call Π_r the least dimensional affine space containing the vertices $\{p_0, \cdots, p_r\}$ and Π_s the affine space with the same property with respect to the vertices $\{q_0, \cdots, q_s\}$. It holds that Π_r and Π_s do not support any common direction. Indeed, suppose u is a (non vanishing) vector along a direction supported by both Π_r and Π_s then, having that all the coefficients α_i and β_j are strictly positive,

it follows that for some $\epsilon > 0$ small enough, $x + \lambda\, u \in S_1 \cap S_2$, for all $|\lambda| < \epsilon$, contradicting that x is an extreme point. This shows that $\mathcal{I} = ext\,(S_1 \cap S_2)$. Given that both S_1 and S_2 are compact and convex sets, so it is $S_1 \cap S_2$ and the proof is completed by using theorem 4.

□

Notice that for any pair of boundaries B_1 and B_2 with unique intersection, $B_1 \cap B_2 = \{p\}$, there are boundaries $B_i' \subseteq B_i$ that do not support any common direction and $p \in \mathring{B}_1' \cap \mathring{B}_2'$. Indeed, let $B_1 := \langle p_0, \cdots, p_r \rangle$ and $B_2 := \langle q_0, \cdots, q_s \rangle$ and assume, w.l.o.g., that $p = \sum_{i=0}^{r'} \alpha_i p_i = \sum_{i=0}^{s'} \beta_j q_j$, with $r' \leq r$, $s' \leq s$ and all $\alpha_i, \beta_j > 0$. Then, for $B_1' := \langle p_0, \cdots, p_{r'} \rangle \subseteq B_1$ and $B_2' := \langle q_0, \cdots, q_{s'} \rangle \subseteq B_2$, it follows that $p \in \mathring{B}_1' \cap \mathring{B}_2'$ is the only common point and, since all of the convex coefficients of p in the vertices of these boundaries are strictly positive, their intersection can be unique only if B_1' and B_2' do not support and common direction. Thus, restricting the definition of $\mathcal{I}$ to boundaries sharing no direction does not leave out any unique intersection between boundaries.

Proposition 12. *Let $P = S_1 \cap S_2$. The following statements are true:*

1. *Any convex boundary of P is contained in some face of S_1 or S_2.*

2. *A convex boundary of P is a proper face if and only if it is not contained in more than one face of the same simplex.*

Proof. Let us see the first statement. Indeed

$$\begin{aligned} \partial P &= (\partial S_1 \cap S_2) \cup (S_1 \cap \partial S_2) \\ &= \cup_{i=0}^{n} \left(F_i^{(1)} \cap S_2 \right) \cup_{j=0}^{n} \left(F_j^{(2)} \cap S_1 \right) \end{aligned}$$

where $F_i^{(a)}$ denotes the face of S_a opposite to its i-th vertex. So each (non empty) set $F_i^{(1)} \cap S_2$ or $F_j^{(2)} \cap S_1$ is a convex boundary of P. Among them, the proper faces of P are those with dimensionality $n-1$.
As for the second statement, With no loss of generality assume that two of these faces are $\langle u_0, \cdots u_{n-1} \rangle$ and $\langle u_1, \cdots u_n \rangle$. Then $F \subset \langle u_1, \cdots, u_{n-1} \rangle$ which

implies that F belongs to an affine space of dimension at most $n-2$, and thus it can not be a proper face.

□

Corollary 4. *$\mathcal{I} = \cup_{a=1,2\,;\,0\le i\le n}\mathcal{I}_i^{(a)}$, with $\mathcal{I}_i^{(a)} := \mathcal{I}\cap F_i^{(a)}$.*
In addition, call $\mathcal{F}$ the set of the unique elements in $\left\{\mathcal{I}_i^{(a)}\right\}_{a=1,2\,;\,0\le i\le n}$ that are non empty and appear only once for $a=1$ or only once for $a=2$. In the case an element appears once for $a=1$ and once for $a=2$ it is taken to belong to $\mathcal{F}$ only if the corresponding faces that contain the element are parallel.
Then for each $I\in\mathcal{F}$, $ch(I)$ is a face of $S_1\cap S_2$ and conversely, each face of $S_1\cap S_2$ is the convex hull of a unique $I\in\mathcal{F}$.

Proof. By construction, all the points in $\mathcal{I}$ lie in some face of some of the simplices and thus $\cup_{i,a}\mathcal{I}_i^{(a)} = \mathcal{I}$. In addition, for any face F of P, assume, w.l.o.g., that $F_i^{(1)}$ is the unique face of S_1 containing F. Then $F = F_i^{(1)}\cap S_2$ and from theorem 5, applied to $F_i^{(1)}$ and S_2, it follows that $ch\left(\mathcal{I}_i^{(1)}\right) = F$. Conversely, let $I\in\mathcal{F}$ and assume, w.l.o.g., that $F_i^{(1)}$ is the unique face of S_1 containing I. Again from theorem 5 it follows that $F_i^{(1)}\cap S_2$, which is a proper face of P, is the convex hull of I.

□

2 A method for finding $ext(S_1\cap S_2)$

In these section we give a detailed description of the method to find the set of extreme points of the polytope $S_1\cap S_2$ based on theorem 5. We will apply the results found in section 1.1.4, restricted to boundaries of two simplices.

2.1 Intersection between boundaries

Suppose that $S_1 = \langle p_1,\cdots,p_{n+1}\rangle$ and $S_2 = \langle q_1,\cdots,q_{n+1}\rangle$ are two simplices in $\mathbb{R}^n$ and that $B_1 = \langle p_{\rho_1},\cdots,p_{\rho_r}\rangle$ and $B_2 = \langle q_{\sigma_1},\cdots,q_{\sigma_s}\rangle$ are two boundaries for which we want to compute their intersection. Let $\Pi_1 = \text{affine}\{p_{\rho_1},\cdots,p_{\rho_r}\}$ and $\Pi_2 = \text{affine}\{q_{\sigma_1},\cdots,q_{\sigma_s}\}$. Assume, w.l.o.g., that

$r \geq s$. Let ρ' and σ' be permutations of $\{1, \cdots, n+1\}$ such that $\rho'_i = \rho_i$ for all $1 \leq i \leq r$ and $\sigma'_j = \sigma_j$ for all $1 \leq j \leq s$, and define the new vertices

$$u_a := p_{\rho'_{a+1}} - p_{\rho_1}\,, \qquad 0 \leq a \leq n\,, \tag{14}$$

$$v_b := q_{\sigma'_{b+1}} - p_{\rho_1}\,, \qquad 0 \leq b \leq n\,. \tag{15}$$

According to proposition 5, both $\{u_0, \cdots, u_n\}$ and $\{v_0, \cdots, v_n\}$ are minimal systems of generators. In addition, $\{u_1, \cdots, u_n\}$ is a basis of $\mathbb{R}^n$. The statement that there is a point common to Π_1 and Π_2 is that the system

$$\sum_{i=1}^{r} \alpha_i p_{\rho_i} = \sum_{j=1}^{s} \beta_j q_{\sigma_j}\,, \tag{16}$$

$$\sum_{i=1}^{r} \alpha_i = \sum_{j=1}^{s} \beta_j\,, \tag{17}$$

has a non trivial solution. Given that $\sum_{i=1}^{r} \alpha_i\, p_{\rho_i} = \left(\sum_{i=1}^{r} \alpha_i\right) p_{\rho_1} + \sum_{i=1}^{r} \alpha_i\, u_{i-1}$, the same statement is equivalent to requiring the system

$$\sum_{i=0}^{r-1} a_i\, u_i = \sum_{j=0}^{s-1} b_j\, v_j\,, \tag{18}$$

$$\sum_{i=0}^{r-1} a_i = \sum_{j=0}^{s-1} b_j\,, \tag{19}$$

to have non trivial solution. Furthermore, the vertices $\{v_0, \cdots, v_n\}$ are expressed as $v_i = \sum_{j=0}^{n} \gamma_{ij} u_j$, with $\sum_j \gamma_{ij} = 1$, and therefore, in the basis $\{u_1, \cdots, u_n\}$, these vertices have components $v_i := \sum_{l=1}^{n} (v_i)_l\, u_l$, with $(v_i)_l := \gamma_{il}$. Also in this basis, the linear maps in equations (9) and (10), whose kernels equal $\mathcal{S}$ and $\mathcal{S}_0$, respectively, are represented by the matrices

$$M = \begin{pmatrix} 1_r & -1_s \\ D & \Gamma \end{pmatrix}$$

$$M_0 = \begin{pmatrix} 1_r & -1_s \\ 1_r & 0_s \\ D & \Gamma \end{pmatrix}$$

respectively. Here D is a $n \times r$ matrix given by

$$D = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Furthermore Γ is the $n \times s$ matrix given by $\Gamma_{ab} := -\gamma_{a,b-1}$ and 1_a and 0_a denote a-dimensional row vectors formed entirely by ones and zeros, respectively.

Lemma 10. *It holds that* $\operatorname{rank} M = r + \operatorname{rank} \tilde{\Gamma}$*, where* $\tilde{\Gamma}$ *be the* $(n-r+1) \times s$ *matrix containing the last* $n-r+1$ *rows of the submatrix* Γ *in* M*.*

Proof. Let $k := \operatorname{rank} \tilde{\Gamma}$ and, w.l.o.g., assume that $\tilde{c}_1, \cdots, \tilde{c}_k$ are linearly independent columns of $\tilde{\Gamma}$. Consider the first $r \times r$ minor of M, that is

$$D_r = \begin{pmatrix} 1 & 1_{r-1} \\ 0 & Id_{r-1} \end{pmatrix}$$

The matrix D_r is non singular. To see this, notice that a linear combination of its rows equated to zero reads $(\alpha, \beta_1 + \alpha, \cdots, \beta_{r-1} + \alpha) = 0$, which requires $\alpha = \beta_1 = \cdots = \beta_{r-1} = 0$. Call $\{d_1, \cdots, d_r\}$ the columns of D_r. They form a basis of $\mathbb{R}^r$. On the other hand, the first $r + k$ columns of M are of the form

$$C_l = \begin{pmatrix} d_l \\ 0 \end{pmatrix}, \ 1 \leq l \leq r$$

$$C_{r+j} = \begin{pmatrix} \delta_j \\ \tilde{c}_j \end{pmatrix}, \ 1 \leq j \leq k$$

To see that they are linearly independent consider the equality

$$\sum_{l=1}^{r+k} \alpha_l C_l = 0$$

which translates into the equations

$$\sum_{j=1}^{k} \alpha_{r+j}\tilde{c}_j = 0$$

$$\sum_{i=1}^{r} \alpha_i d_i + \sum_{j=1}^{k} \alpha_{r+j}\delta_j = 0$$

The first equation requires $\alpha_{r+1} = \cdots = \alpha_{r+k} = 0$ and therefore the second equality requires the remaining parameters to vanish. To see that any other column can be expressed as a linear combination of this set, let $C = \begin{pmatrix} \delta \\ \tilde{c} \end{pmatrix}$ be any of the last $s-k$ columns of M. Since $\tilde{c}$ is a column of $\tilde{\Gamma}$, it clearly holds that $\tilde{c} = \sum_j \lambda_j \tilde{c}_j$. On the other hand, $\delta_j = \sum_i \mu_i^{(j)} d_i$ and $\delta = \sum_i \mu_i d_i$. Then,

$$\begin{aligned}
\sum_{j=1}^{s} \lambda_j C_{r+j} &= \begin{pmatrix} \sum_j \lambda_j \delta_j \\ \tilde{c} \end{pmatrix} = \begin{pmatrix} \sum_i \sum_j \lambda_j \mu_i^{(j)} d_i \\ \tilde{c} \end{pmatrix} \\
&= \begin{pmatrix} \delta + \left(\sum_i \sum_j \lambda_j \mu_i^{(j)} d_i - \delta\right) \\ \tilde{c} \end{pmatrix} \\
&= \begin{pmatrix} \delta \\ \tilde{c} \end{pmatrix} + \sum_i \begin{pmatrix} \left(\sum_j \lambda_j \mu_i^{(j)} - \mu_i\right) d_i \\ 0 \end{pmatrix}
\end{aligned}$$

and therefore

$$\begin{pmatrix} \delta \\ \tilde{c} \end{pmatrix} = \sum_{j=1}^{s} \lambda_j C_{r+j} + \sum_{i=1}^{r} \left(\mu_i - \sum_j \lambda_j \mu_i^{(j)}\right) C_i$$

This completes the proof.

□

Now we can state the necessary and sufficient conditions for the planes supporting the boundaries to intersect in a single point

Proposition 13. *The affine spaces* Π_1 *and* Π_2 *intersect in a single point if* *and only if*

$$\operatorname{rank}\tilde{\Gamma} = s - 1 \tag{20}$$

$$\operatorname{rank}\begin{pmatrix} \tilde{\Gamma} \\ 1_s \end{pmatrix} = s \tag{21}$$

Proof. Theorem 1 establishes that these planes intersect in a single point if and only if $\dim \mathcal{S}_0 = 0$ and $\dim \mathcal{S} = 1$. Using that $\operatorname{rank} M = r + \operatorname{rank}\tilde{\Gamma}$, it follows that if $\operatorname{rank}\tilde{\Gamma} \leq s-2$ then $\dim \mathcal{S} = \dim \operatorname{Ker}(M) = r+s-\operatorname{rank} M \geq 2$, in which case, Π_1 and Π_2 intersect in an affine space of dimension at least 1. If $\operatorname{rank}\tilde{\Gamma} = s$, then $\dim \mathcal{S} = r + s - \operatorname{rank} M = 0$ and $\Pi_1 \cap \Pi_2 = \emptyset$. If $\operatorname{rank}\tilde{\Gamma} = s - 1$, then $\dim \mathcal{S} = 1$. In addition, $\dim \mathcal{S}_0 = 0$ if ond only if $\operatorname{rank} M_0 = r+s = \operatorname{rank} M + 1$. This completely depends on whether the row vector $(1_r, 0_s)$ can be expressed as a linear combination of the rows in M. That is:

$$1_r = (\alpha, \beta_1 + \alpha, \cdots, \beta_{r-1} + \alpha)$$

$$0_s = \alpha 1_s + \sum_{i=1}^{n} \beta_i \Gamma_i$$

with Γ_i denoting the i-th row of Γ. The first equation requires $\alpha = 1$ and $\beta_1 = \cdots = \beta_{r-1} = 0$. The second equation requires

$$\sum_{j=r}^{n} \beta_j \Gamma_j = -1_s$$

Therefore the rank of M_0 is equal to that of M if and only if the row vector 1_s can be expressed as a linear combination of the rows of $\tilde{\Gamma}$.

□

Remark 3. *If* $r+s \geq n+3$, *then* $\dim \mathcal{S}_0 = r+s-\operatorname{rank} M_0 \geq n+3-\operatorname{rank} M_0 \geq 1$, *given that* $\operatorname{rank} M_0 \leq n+2$. *Therefore, in that case, the boundaries intersect either emptily or along an affine space of dimension at least* 1.

To summarize, let $\tilde{c}_s = \sum_{j=1}^{s-1} \lambda_j \tilde{c}_j$ be a linearly dependent column of $\tilde{\Gamma}$. Then, there are coefficients $\mu_i, \mu_i^{(j)}$ such that

$$\sum_{i=1}^{r} \left(\mu_i - \sum_{j=1}^{s-1} \lambda_j \mu_i^{(j)} \right) C_i + \sum_{j=1}^{s-1} \lambda_j C_{r+j} - C_s = 0\,,$$

where C_l are defined as in lemma 10. An element of $\mathcal{S} \setminus \mathcal{S}_0$ is given by $(\alpha, -\beta)$ with

$$\begin{aligned}
\alpha_i &= \mu_i - \sum_{j=1}^{s-1} \lambda_j \mu_i^{(j)} \,, \; 1 \leq i \leq r \\
-\beta_j &= \lambda_j \,, \; 1 \leq j \leq s-1 \\
\beta_s &= 1
\end{aligned}$$

Notice that since we are in the case $\mathcal{S}_0 = \{\mathbf{0}\}$ and $\beta_s \neq 0$ it is guaranteed $A := \sum_i \beta_i \neq 0$, and the coefficients of the affine expansions of the intersecting point are $a = \alpha/A$ and $b = \beta/A$. Explicitly,

$$\begin{aligned}
b_j &= -\frac{\lambda_j}{1 - \sum_{l=1}^{s-1} \lambda_l} \,, \qquad 1 \leq j \leq s-1 \\
b_s &= \frac{1}{1 - \sum_{j=1}^{s-1} \lambda_j} \\
a_i &= b_s \mu_i + \sum_{j=1}^{s-1} b_j \mu_i^{(j)} \,, \qquad 1 \leq i \leq r
\end{aligned}$$

Remark 4. *The constants a_i and b_i are the coefficients of the affine expansion of the unique intersection between the original affine spaces Π_1 and Π_2 in the vertices $\{p_{\rho_1}, \cdots, p_{\rho_r}\}$ and $\{q_{\sigma_1}, \cdots, q_{\sigma_s}\}$, respectively. To see this, notice that* affine $\{u_0, \cdots, u_{r-1}\} = \Pi_1 - p_{\rho_1}$ *and* affine $\{v_0, \cdots, v_{s-1}\} = \Pi_2 - p_{\rho_1}$. *Thus the corresponding unique intersection between Π_1 and Π_2 is $p = q + p_{\rho_1} = p_{\rho_1} + \sum_{i=1}^{s} b_i \, (q_{\sigma_i} - p_{\rho_1}) = \sum_{i=1}^{s} b_i \, q_{\sigma_i}$. Similarly, $p = p_{\rho_1} + \sum_{i=1}^{r} a_i \, (p_{\rho_i} - p_{\rho_1}) = \sum_{i=1}^{r} a_i \, p_{\rho_i}$. Only when all $a_m > 0$ and all $b_k > 0$, $\mathring{B}_1 \cap \mathring{B}_2 = \{p\}$ and thus $p \in \mathcal{I}$.*

2.2 Simplices sharing a face

Let $S_1 = \langle u_1 \cdots u_n v \rangle$ and $S_2 = \langle u_1 \cdots u_n w \rangle$, be two simplices with

$$\begin{aligned}
v &= \sum_{i=1}^{n} \alpha_i u_i + \alpha_{n+1} w \,, \; \alpha_{n+1} \neq 0 \\
w &= -\frac{1}{\alpha_{n+1}} \sum_{i=1}^{n} \alpha_i u_i + \frac{1}{\alpha_{n+1}} v
\end{aligned}$$

Let $B_1 = \langle u_{\rho_1} \cdots u_{\rho_{r-1}} v \rangle$ and $B_2 = \langle u_{\sigma_1} \cdots u_{\sigma_{s-1}} w \rangle$ be two boundaries. Defining the sets

$$\begin{aligned}
&\rho = \{\rho_1, \cdots, \rho_{r-1}\} \ , \ \sigma = \{\sigma_1, \cdots, \sigma_{s-1}\} \\
&R = \{1, \cdots, n\} \setminus \rho \cup \sigma
\end{aligned}$$

we have the following result.

Lemma 11. *B_1 and B_2 intersect in a single point (other than a vertex) if and only if the following requirements are met*

1. $\rho \cap \sigma = \emptyset$
2. $\alpha_R = 0$
3. $\alpha_\rho \leq 0$
4. $\alpha_{n+1} > 0$
5. $\alpha_\sigma \geq 0$

Proof. Suppose B_1 and B_2 intersect in single point (other than a shared vertex). Then B_1 and B_2 can not share any vertex, for then the intersection would be precisely the shared vertex. As a consequence, for the boundaries to have non trivial intersection, it is required that $\rho \cap \sigma = \emptyset$. Notice that this is possible only if $r - 1 + s - 1 \leq n$. The matrix M takes a particularly simple form in this case.

$$M = \begin{pmatrix} 1 & 1_{r-1} & 1_{s-1} & 1 \\ 0 & Id_{r-1} & 0 & \frac{\tilde{\alpha}_\rho}{\alpha_{n+1}} \\ 0 & 0 & Id_{s-1} & -\frac{\alpha_\sigma}{\alpha_{n+1}} \\ 0 & 0 & 0 & -\frac{\alpha_R}{\alpha_{n+1}} \end{pmatrix}$$

$$\tilde{\Gamma} = \begin{pmatrix} Id_{s-1} & -\frac{\alpha_\sigma}{\alpha_{n+1}} \\ 0 & -\frac{\alpha_R}{\alpha_{n+1}} \end{pmatrix}$$

with $\tilde{\alpha}_\rho = \left\{-\alpha_{\rho_2}, \cdots, -\alpha_{\rho_{r-1}}, 1\right\}$. Notice that the first $s - 1$ columns of $\tilde{\Gamma}$ are linearly independent, thus it is required that $\alpha_R = 0$. Furthermore, for

$\mathrm{rank}\mathrm{M}_0 = r + s$ one needs 1_s to be linearly independent of the rows of $\tilde{\Gamma}$. More specifically,

$$\begin{aligned} 1_s &= \beta_1 \left(1, 0, \cdots, 0, -\frac{\alpha_{\sigma_1}}{\alpha_{n+1}}\right) + \cdots + \beta_{s-1} \left(0, \cdots, 0, 1, -\frac{\alpha_{\sigma_{s-1}}}{\alpha_{n+1}}\right) \\ &= \left(\beta_1, \cdots, \beta_{s-1}, -\frac{1}{\alpha_{n+1}} \sum_{i=1}^{s-1} \beta_i \alpha_i\right) \end{aligned}$$

which requires $\beta_1 = \cdots = \beta_{s-1} = 1$. Therefore for $\mathrm{rank} \begin{pmatrix} \tilde{\Gamma} \\ 1_s \end{pmatrix} = s$, one needs $-\frac{1}{\alpha_{n+1}} \sum_{i=1}^{s-1} \alpha_{\sigma_i} \neq 1$, or equivalently $\sum_{i=1}^{s-1} \alpha_{\sigma_i} + \alpha_{n+1} \neq 0$. In terms of the quantity $\Sigma = \frac{1}{\alpha_{n+1}} \sum_j \alpha_{\sigma_j}$, this last condition is just $1 + \Sigma \neq 0$. Therefore the planes supporting the boundaries intersect uniquely (and in a point not being a vertex) only if

1. $\alpha_R = 0$
2. $1 + \Sigma \neq 0$
3. $\rho \cap \sigma = \emptyset$

More specifically we find

$$\begin{aligned} \tilde{c}_s &= -\frac{1}{\alpha_{n+1}} \begin{pmatrix} \alpha_{\sigma_1} \\ \vdots \\ \alpha_{\sigma_{s-1}} \end{pmatrix} \\ \tilde{c}_s &= \frac{-\alpha_{\sigma_1}}{\alpha_{n+1}} \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \frac{-\alpha_{\sigma_{s-1}}}{\alpha_{n+1}} \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} \\ \lambda_s &= -1, \lambda_1 = -\frac{\alpha_{\sigma_1}}{\alpha_{n+1}}, \cdots, \lambda_{s-1} = -\frac{\alpha_{\sigma_{s-1}}}{\alpha_{n+1}} \end{aligned}$$

and the intersecting point has convex expansions in the vertices of B_2 and

B_1, respectively, given by:

$$
\begin{aligned}
b_i &= \frac{\alpha_{\sigma_i}}{\alpha_{n+1} + \sum_{j=1}^{s-1} \alpha_{\sigma_j}} \\
b_s &= \frac{\alpha_{n+1}}{\alpha_{n+1} + \sum_{j=1}^{s-1} \alpha_{\sigma_j}} \\
a_i &= b_s \mu_i + \sum_{j=1}^{s-1} b_j \mu_i^{(j)}\,, \;\; 1 \leq i \leq r
\end{aligned}
$$

with

$$
\mu_i^{(j)} = \delta_{ij}, \qquad \mu = \begin{pmatrix} -\frac{\alpha_{\rho_1}}{\alpha_{n+1}} - \Sigma \\ -\frac{\alpha_{\rho_2}}{\alpha_{n+1}} \\ \vdots \\ -\frac{\alpha_{\rho_{r-1}}}{\alpha_{n+1}} \\ \frac{1}{\alpha_{n+1}} \end{pmatrix}
$$

Notice that $1 + \frac{1}{\alpha_{n+1}} \left(\sum_{l=2}^{r-1} \alpha_{\rho_l} - 1 \right) = 1 - \frac{1}{\alpha_{n+1}} \left(\alpha_{n+1} + \alpha_{\rho_1} + \sum_{j=1}^{s-1} \alpha_{\sigma_j} \right) = -\frac{\alpha_{\rho_1}}{\alpha_{n+1}} - \Sigma$. Thus we have

$$
\begin{aligned}
b_i &= \frac{1}{1+\Sigma} \frac{\alpha_{\sigma_i}}{\alpha_{n+1}} \\
b_s &= \frac{1}{1+\Sigma}
\end{aligned}
$$

$$
\begin{aligned}
a_1 &= \frac{-\alpha_{\rho_1}/\alpha_{n+1} - \Sigma}{1+\Sigma} + \frac{1}{1+\Sigma} \sum_{j=1}^{s-1} \frac{\alpha_{\sigma_j}}{\alpha_{n+1}} = -\frac{1}{1+\Sigma} \frac{\alpha_{\rho_1}}{\alpha_{n+1}} \\
a_i &= -\frac{1}{1+\Sigma} \frac{\alpha_{\rho_i}}{\alpha_{n+1}}\,, \;\; 2 \leq i \leq r-1 \\
a_r &= \frac{1}{\alpha_{n+1}(1+\Sigma)}
\end{aligned}
$$

The intersecting point does belong to both boundaries only if these coefficients are non negative. From $b_s \geq 0$ it follows that $1 + \Sigma > 0$. Then, $a_r \geq 0$, requires that $\alpha_{n+1} > 0$ and from this, $b_j \geq 0 \to \alpha_{\sigma_j} \geq 0$. Finally, $a_i \geq 0$ requires $\alpha_{\rho_i} \leq 0$. These, together with $\alpha_R = 0$ and $\rho \cap \sigma = \emptyset$, are the requirements listed in the statement of the lemma.

Conversely, if these requirements are met, the intersection is unique ($\alpha_R = 0$) and $\Sigma = \frac{1}{\alpha_{n+1}} \sum_j \alpha_{\sigma_j} \geq 0$ and thus all the coefficients are non negative.

□

Remark 5. *If any of the boundaries does not contain the non common vertex, then the problem reduces to the intersection between two boundaries of the same simplex not sharing any direction, that is: a vertex of the corresponding simplex.*

From this lemma we have the following result

Theorem 6. *If two simplices share a face, their intersection is just the shared face or a (possibly) different simplex.*

Proof. Let $S_1 = \langle u_1 \cdots u_n v\rangle$ and $S_2 = \langle u_1 \cdots u_n w\rangle$ with $v = \sum_i \alpha_i u_i + \alpha_{n+1} w$. Consider the sets

$$
\begin{aligned}
I_+ &= \{1 \leq i \leq n+1 \,|\, \alpha_i > 0\} \\
I_- &= \{1 \leq i \leq n+1 \,|\, \alpha_i < 0\} \\
I_0 &= \{1 \leq i \leq n+1 \,|\, \alpha_i = 0\}
\end{aligned}
$$

We can readily consider the following three trivial situations:

1. $n + 1 \notin I_+$.
   The intersection is just the shared face because either the vertices u and w lie in opposite sides of the shared face, or the simplex S_1 is singular.

2. $I_- = \emptyset$.
   The vertex v belongs to S_2 and thus the intersection is just the simplex S_1.

3. $I_+ = \{n + 1\}$.
   In this case it is the vertex w which belongs to S_1 and thus the intersection is just the simplex S_2.

Let then $I_+ = \{\sigma_1, \cdots, \sigma_{s-1}, n+1\}$ and $I_- = \{\rho_1, \cdots, \rho_{r-1}\}$. It is clear, after lemma 11, that the boundaries $B_\rho = \langle u_{\rho_1} \cdots u_{\rho_{r-1}} v\rangle$ and $B_\sigma = \langle u_{\sigma_1} \cdots u_{\sigma_{s-1}} w\rangle$, do not support any common direction and intersect uniquely in a point lying in the interior of both boundaries.

Let $B_{\rho'} = \langle u_{\rho'} v \rangle$ and $B_{\sigma'} = \langle u_{\sigma'} w \rangle$ be any pair of boundaries intersecting uniquely. Lemma 11, then requires $\rho' \cap \sigma' = \emptyset$ and, w.l.o.g., $\rho' \subseteq I_- \cup I_0$ and $\sigma' \subseteq I_+ \cup I_0$ and, in addition, $R' := \{1, \cdots, n\} \setminus \rho' \cup \sigma' \subseteq I_0$. It also holds that $I_+ \subseteq \sigma'$ and $I_- \subseteq \rho'$. Indeed, since $\{1, \cdots, n\} = \rho' \cup \sigma' \cup R'$, if there is some $i \in I_+ \setminus \sigma'$ then it must happen that $i \in \rho' \cup R'$ which requires $\alpha_i \leq 0$ and therefore $i \notin I_+$, in the first place. The same argument shows that $I_- \subseteq \rho'$. Thus it must hold that $\rho' = I_- \cup R'_-$, $\sigma' = I_+ \cup R'_+$, with $R'_\pm \subset I_0$ and, precisely because the extra coefficients $\alpha_{R'_\pm}$ are just zero, lemma 11 guarantees that these boundaries intersect in the same point, p, in which B_ρ and B_σ do. It just happens that p does not belong the the interior of the boundaries $B_{\rho'}$ and $B_{\sigma'}$. Thus, only the pair of boundaries B_ρ and B_σ produces an intersecting point not being a shared vertex. As a consequence, the intersection between both simplices is generated by the n common vertices plus the new point p, which is a new simplex.

□

3 Algorithm for computing the volume of intersection between simplices

Let $S_1 := \langle u_1 \cdots u_{n+1} \rangle$ and $S_2 := \langle v_1 \cdots v_{n+1} \rangle$ be the two simplices for which we want to compute the volume of $S_1 \cap S_2$.

Let $c_1 := \dfrac{1}{n+1} \sum_{i=1}^{n+1} u_i$ and $c_2 := \dfrac{1}{n+1} \sum_{i=1}^{n+1} v_i$, be the corresponding centroids. Let $r_1 := \max \{ \|u_i - c_1\| \mid 1 \leq i \leq n+1 \}$ denote the radius of the circumsphere of S_1 (the smallest solid ball centered at c_1 that contains S_1) and similarly for S_2, $r_2 := \max \{ \|v_i - c_2\| \mid 1 \leq i \leq n+1 \}$.

The algorithm we propose for the computation of $vol(S_1 \cap S_2)$ can be outlined as follows:

Step 1: Check that the simplices are not too far apart to intersect, i.e., whether $\|c_1 - c_2\| < r_1 + r_2$. Otherwise $S_1 \cap S_2$ is either empty or at most one common vertex.

Step 2: Check that at least one of the circumspheres contains vertices of the other simplex.

That is, the sets $V_{1,2} := \{u_i \mid \|u_i - c_2\| \leq r_2\}$ and $V_{2,1} := \{v_j \mid \|v_j - c_1\| \leq r_1\}$ are not both empty. Otherwise, the circumspheres intersect but the simplices do not.

Step 3: Compute the affine expansions of the vertices of each simplex in terms of the vertices of the other (using, for example, equation (12)). For each simplex, identify the vertices that belong to the other simplex. Let $u_i = \sum_{j=1}^{n+1} U_{ij} v_j$ and $v_i = \sum_{j=1}^{n+1} V_{ij} u_j$, be such affine expansions.

Step 4: Let n_{com} be the number of common vertices.

(a) If $n_{com} = n + 1$, the simplices coincide and the volume of the intersection is $vol(S_1)$.

(b) If $n_{com} = n$, the simplices share a face.

Relabel the vertices of each simplex such that the the n common vertices appear first and in the same order in each simplex, that is: $S_1 = \langle u_{a_1}, \cdots, u_{a_n}, u_{a_{n+1}} \rangle$ and $S_2 = \langle v_{b_1}, \cdots, v_{b_n}, v_{b_{n+1}} \rangle$, with $v_{b_i} = u_{a_i}$, for all $1 \leq i \leq n$. According to theorem 6, we need to check the following instances:

i. If $U_{a_{n+1}, b_{n+1}} \leq 0$, the simplices intersect just along the shared face and therefore, $vol(S_1 \cap S_2) = 0$.

ii. Else, if $U_{a_{n+1}, b_i} \geq 0$, for all $1 \leq i \leq n$, then $S_1 \subset S_2$ and thus $vol(S_1 \cap S_2) = vol(S_1)$.

iii. Else, if $U_{a_{n+1}, b_i} \leq 0$, for all $1 \leq i \leq n$, then $S_2 \subset S_1$ and therefore $vol(S_1 \cap S_2) = vol(S_2)$.

iv. Else, let $I_+ := \left\{ i \,|\, U_{a_{n+1}, b_i} > 0 \right\}$. From lemma 11, it follows that $S_1 \cap S_2 = \langle u_{a_1}, \cdots, u_{a_n}, p \rangle$ with

$$p = \frac{1}{A} \sum_{i \in I_+} U_{a_{n+1}, b_i} v_{b_i} , \tag{22}$$

$$A = \sum_{i \in I_+} U_{a_{n+1}, b_i} , \tag{23}$$

and thus, $vol(S_1 \cap S_2) = vol\,(\langle u_1, \cdots, u_n, p\rangle)$.

(c) If $n_{com} < n$, proceed to step **5**.

Step 5: Let B denote the set of all pairs of boundaries, (B_1, B_2), with $B_1 \subseteq S_1 \backslash S_2$ and $B_2 \subseteq S_2 \backslash S_1$. From B construct the subset $\mathcal{B}$ as: $(B_1, B_2) \in B$ is an element of $\mathcal{B}$ if B_1 and B_2 do not share any vertex and $a+b \leq n+2$, where a and b are the number of vertices in B_1 and B_2, respectively.

For each pair $(B_1, B_2) \in \mathcal{B}$, ($B_1 := \langle u_{i_1}, \cdots, u_{i_a}\rangle$ and $B_2 := \langle v_{j_1}, \cdots, v_{j_b}\rangle$) apply the following procedure :

(a) If $a \geq b$, define $B_\rho := B_1$ and $B_\sigma := B_2$, otherwise, define $B_\rho := B_2$ and $B_\sigma := B_1$.
Generically denote $B_\rho = \langle p_{\rho_1}, \cdots, p_{\rho_r}\rangle$ and $B_\sigma = \langle q_{\sigma_1}, \cdots, q_{\sigma_s}\rangle$, where $r \geq s$.

(b) Let ρ' be a permutation of $\{1, \cdots, n+1\}$ such that $\rho'_i = \rho_i$, for all $1 \leq i \leq r$.

(c) If $a \geq b$ define $W := V$, otherwise, define $W := U$ (where U and V are the matrices of affine expansion coefficients computed in step **3**) and construct the matrix $\Gamma_{ij} := W_{\sigma_j \rho'_{r+i}}$, $1 \leq i \leq n+1-r$ and $1 \leq j \leq s$.

(d) Check that $\operatorname{rank}\Gamma = s-1$ and $\operatorname{rank}\begin{pmatrix}\Gamma \\ 1_s\end{pmatrix} = s$. If they are otherwise, quit the procedure to analyze another pair in $\mathcal{B}$.

(e) Find any non vanishing vector $\Lambda \in \mathbb{R}^s$ such that $\Gamma \cdot \Lambda = 0$ and re scale it as $\lambda := -\dfrac{1}{|\Lambda_{l_0}|}\Lambda$, where Λ_{l_0} is any non vanishing component of Λ.

(f) Compute the coefficients

$$\alpha_i = \frac{1}{A}\sum_{j=1}^{s} \lambda_j \mu_i^{(j)}\,, \qquad 1 \leq i \leq r$$
$$\beta_j = \frac{1}{A}\,\lambda_j\,, \qquad 1 \leq j \leq s$$

with

$$\mu_i^{(l)} := W_{\sigma_l \rho_i}\,, \qquad 1 \le l \le s\,,\ 2 \le i \le r$$
$$\mu_1^{(l)} := 1 - \sum_{i=2}^{r} W_{\sigma_l \rho_i}\,, \qquad 1 \le l \le s$$
$$A := \sum_{m=1}^{s} \lambda_m$$

(g) If $\alpha_i > 0$ and $\beta_j > 0$ for all $1 \le i \le r$ and $1 \le j \le s$, store the point $p := \sum_{j=1}^{s} \beta_j\, q_{\sigma_j}$ (because $p \in ext(S_1 \cap S_2)$).
Otherwise, proceed to analyze another pair of boundaries in $\mathcal{B}$.

Step 6: The set $ext(S_1 \cap S_2)$ is formed by the vertices of each simplex belonging to the other simplex (identified in step **3**) and all the points found in step **5**.

(a) If $K := \sharp\,\{ext(S_1 \cap S_2)\} \le n$, then $vol(S_1 \cap S_2) = 0$.

(b) Else, if $K = n+1$, then $S_1 \cap S_2$ is a simplex and therefore $vol(S_1 \cap S_2) = vol\,(ext(S_1 \cap S_2))$.

(c) Else ($K > n+1$), we apply the decomposition of $ext(S_1 \cap S_2)$ as in corollary 4 to identify the faces of the polytope and then triangulate them (some of the faces might already be simplices). Let $\{\langle s_1 \rangle, \cdots, \langle s_m \rangle\}$ be the set of all the simplices in the triangulations of the faces of the polytope and compute $c := \dfrac{1}{K} \sum_{p_i \in ext(S_1 \cap S_2)} p_i$ (the centroid of $S_1 \cap S_2$). It is easy to show that $\{\langle c, s_1 \rangle, \cdots, \langle c, s_m \rangle\}$ is a triangulation of $S_1 \cap S_2$ and therefore $vol(S_1 \cap S_2) = \sum_{i=1}^{m} vol(\langle c, s_i \rangle)$.

End of the algorithm.

References

[Grünbaum 2003] Grünbaum, Branko: *Convex polytopes, volume 221 of Graduate Texts in Mathematics.* 2003