Efficient Frontier

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Author's note

This paper is meant to serve as a framework for future models as well as laying the groundwork for portfolio optimization and management. Another reason for publishing this paper is showcase my skills regarding finance and mathematics. Please read through, and of course if any mistakes do appear feel free to contact me.

Motivation for the Efficient Frontier

The efficient frontier is a graphical tool used to in portfolio management. It finds an optimal allocation strategy, which makes it an optimization problem. The plot of the efficient frontier finds the relationship between risk and return with different allocations of the same securities. The original motive for the efficient frontier was to find two portfolios of interest: the Markowitz Mean-Variance portfolio, and the minimum variance portfolio. More complex portfolios involve stochastic control and dynamic programming as well as solutions found using machine learning techniques. Like most financial models they usually aren't a silver bullet, there are many limitations and some problems which will be discussed at the end of this paper.

The model

Let's first start of by setting some of the initial restraints for the efficient frontier. We assume that we are working with investing in long equities and that there is no redundancy. In this case we will assume that all of the money will be invested. Although it is common practice to hold some cash as a buffer against risk, adding cash into the scope of this portfolio works somewhat against the idea of the evolution of prices. In the event that a practitioner would want to hold cash in the portfolio they can always find their desired allocation strategy and then apply that portfolio allocation model to a percentage of their portfolio.

We assume that we have N amount of risky investments to choose from. We also assume that the evolution of these prices are stochastic. The evolution a stock's price to be some stochastic function S with respect to time. For an investment $n \in N$ the price evolution will be

$$S_n(1), S_n(2), ..., S_n(T)$$

The returns of an individual security will be

$$R_n = \frac{S_n(T) - S_n(t)}{S_n(t)}$$

Now we need to classify a set of weights which will be our allocation strategy. The goal of this model is to find the optimal weights. Each weight will be the allocation size for each investment.

$$w = (w_1, ..., w_N)^T$$

Then we add this minimization constraint, because we are only allowed to invest at most 100% of our money.

$$\sum_{n=1}^{N} w_n = 1$$

The return of the portfolio can be represented as

$$R_P = \sum_{n=1}^{N} w_n R_n$$

This makes sense, our portfolio returns is each investment's return multiplied by their respective weight and then added up.

In this case we will be finding the Markowitz mean-variance portfolio and minimum-variance portfolio by looking at expected portfolio return and variance therefore we need to find are the first two moments of the portfolio.

$$\mu_P = E[R_p] = E\left[\sum_{n=1}^{N} w_n R_n\right]$$

Using the linearity of expectation and using the definition for expected returns we get

$$E\left[\sum_{n=1}^{N} w_n R_n\right] = \sum_{n=1}^{N} w_n E[R_n] = \sum_{n=1}^{N} w_n \bar{R_n}$$

Now let's find the second moment

$$\sigma_P^2 = var(R_P)$$

But we need to notice the R_P is more a collection of a series of returns therefore applying this property

$$cov(X, X) = var(X)$$

becomes

$$var(R_P) = cov(w_1 R_1, w_2 R_2) + cov(w_1 R_1, w_N R_N) + ... + cov(w_{N-1} R_{N-1}, w_N R_N)$$

Writing that as a summation we get

$$\sum_{i=1}^{N} \sum_{j=1}^{N} cov(w_i R_i, w_j R_j) = \sum_{i=1}^{N} w_i w_j cov(R_i, R_j) = \sum_{i=1}^{N} w_i w_j \sigma_{ij}$$

Essentially we can translate those covariances σ_{ij} into a covariance matrix. If we define Ω as a clearance matrix

$$\sigma_P^2 = w^T \Omega w$$

Two asset theorem

The ultimate goal of the efficient frontier is to find an optimal allocation strategy amongst a selection of investments. From a computational standpoint we plot a series of portfolios' standard deviation against their expected return. Then from there we probe to find "interesting" (in this case Markowitz mean-variance and minimum variance) portfolios. Taking a programming approach to this problem is relatively easy, it involves simulating a series of weights randomly and then using those series with historical returns and then finding the maximum or minimum of a portfolio's statistic. From a mathematical standpoint it is a bit harder but to understand how and why the shape of efficient frontier forms. To be understand that we need to look at an investment decision between two assets.

Let ρ be the correlation coefficient between the two assets and let the weight of the first asset be α which makes the second weight $1 - \alpha$

Plugging in our weights we get a usable definition for the first two moments of the portfolio

$$\mu_P = (1 - \alpha) R_1 + \alpha R_2$$

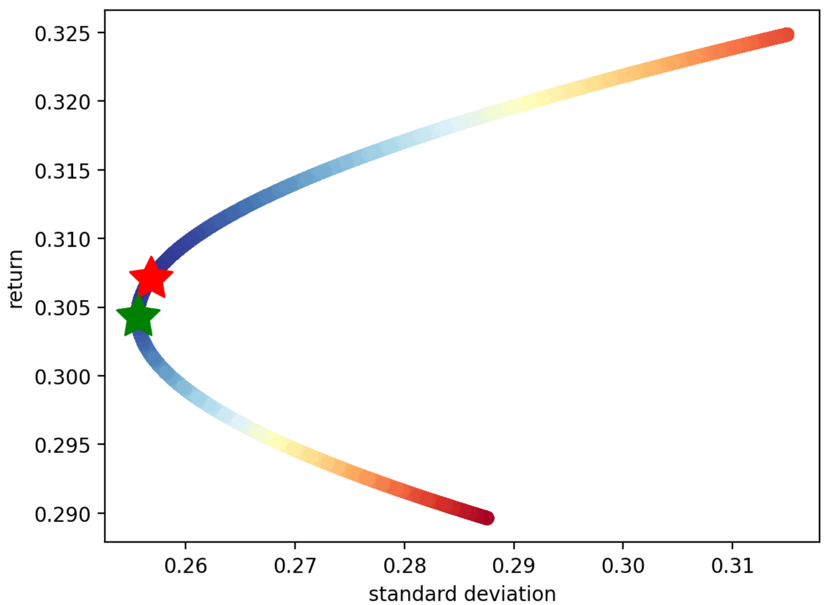
$$\sigma_P^2 = (1 - \alpha)\sigma_1^2 + 2\rho\alpha(1 - \alpha)\sigma_1\sigma_2 + \alpha^2\sigma_2^2$$

$$\alpha \in [0, 1]$$

(See appendix A for second moment).

Now we should start to discuss important points that outline the shape of the efficient frontier which makes a conic shape. We can find those by solving for specific values of ρ. If we work out the scenarios of $\rho = 1$ and $\rho = -1$.

This is a sample efficient frontier that I have generated using 10 years of returns from Apple and Amazon stock using mean returns and a covariance matrix with 100,000 simulations. The prices were pulled from yaho finance and Adjusted Close was used. This was created using a web-facing GUI hosted on my GitHub written in python code. Link for app in references.



Let's work out the scenario where $\rho = 1$

$$\sigma_P(\alpha; \rho = 1) = (1-\alpha)\sigma_1 + \alpha\sigma_2$$

(See appendix B)

Now we can work out these "edge cases" scenarios. Now we want to find the scenarios where we put all of our capital in one stock and one where we invest solely in the other. If we put the returns and standard deviation into a tuple we can work out these scenarios. From a mathematical sense we work out

$$\mu_P(\alpha = 0) = \lim_{\alpha \to 0} (1-\alpha)R_1 + \alpha R_2 = R_1$$

$$\sigma_P(\alpha = 0, \rho = 1) = \lim_{\alpha \to 0} (1-\alpha)\sigma_1 + \alpha\sigma_2 = \sigma_1$$

We get (σ_1, R_1) for when we let α go to 0. If we let α reach our maximum value we get

$$\mu_P(\alpha = 1) = \lim_{\alpha \to 1} (1-\alpha)R_1 + \alpha R_2 = R_2$$

$$\sigma_P(\alpha = 1, \rho = 1) \lim_{\alpha \to 1} (1-\alpha)\sigma_1 + \alpha\sigma_2 = \sigma_2$$

Which gives us (σ_2, R_2). Its obvious that we could've plugged in 0 and 1 for alpha and then connect the two points to a line using point-slope form. But the limits show that as α goes to its value it draws out a line that represents the trade off between the two securities. It is not always the case that the relationship will be linear between investments in real-world application, but for this model it is assumed.

Now let's work out the scenario of $\rho = -1$

$$\sigma_P(\alpha; \rho = 1) = |(1-\alpha)\sigma_1 - \alpha\sigma_2|$$

(See appendix C)

When we go to work out this edge case where we let α be a small number close to 0 we get

$$\lim_{\alpha \to 0} \sigma_P(\alpha; \rho = -1) = (1-\alpha)\sigma_1 - \alpha\sigma_2$$

If we set $\sigma_P = 0$ then we can solve for a specific α value

$$\alpha = \frac{\sigma_1}{\sigma_1 + \sigma_2}$$

Finding the Minimum Variance Portfolio

This portfolio is the allocation strategy with the smallest σ. If we have M portfolios to choose from the minimum variance portfolio can be found by

$$min(\sigma_1, \sigma_2, ..., \sigma_M)$$

Graphically it is the portfolio that is leftmost, and on the figure its the one with a green star. Financially it is the allocation strategy with the least amount of risk.

To mathematically find this we consider $\rho \in (0, 1)$. We are trying to find the best allocation with respect to α we need to take a partial derivative of σ_P^2 with respect to α

$$\frac{\partial \sigma_P^2}{\partial \alpha} = -2(1-\alpha)\sigma_1^2 + 2\alpha\sigma_2^2 + 2(1-2\alpha)\rho\sigma_1\sigma_2$$

Then to find the minimum variance we set the partial derivative to 0. Then solve for α.

$$\alpha = \frac{\sigma_1^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$$

Finding the Markowitz Mean-Variance Portfolio

This allocation strategy is the one with the best risk-to-reward profile. I would call it "the juice is worth the squeeze" portfolio. Computationally we can find this portfolio by finding the largest Sharpe ratio. The Sharpe ratio is a portfolio statistics calculated.

$$S = \frac{\mu_P}{\sigma_P}$$

If we have M portfolios to choose from the maximum Sharpe is

$$max(S_1, S_2, ..., S_M)$$

The motivation for finding the Markowitz mean-variance portfolio is finding the best risk-to-reward portfolio. The Sharpe ratio is portfolio statistic that measures the returns vs. the risk. Therefore finding the maximized sharpe is the same as finding the Markowitz mean-variance portfolio in this model.

Financially speaking the maximized Sharpe portfolio is the best compensation for the risk that is taken on. Points on the curve further to the right is taking on more risk than what the portfolio is rewarding, and points to the left is taking on to little risk for the reward.

Mathematically we can find that by

$$min\left(\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij}\right)$$

What makes this portfolio interesting is that it is an optimization problem, which means that there is a minimizing and maximizing constraints. Other research in allocation strategies are optimization problems with different minimizing and maximizing constraints.

In the Markowitz mean-variance portfolio we have these constraints

$$\sum_{i=1}^{N} w_i R_i = \mu_P$$

$$\sum_{i=1}^{N} w_i = 1$$

To work out this optimization problem we can use the Lagrangian method for constrained optimization. We can set up the equation as

$$L = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij} - \lambda_1\left(\sum_{i=1}^{N} w_i - 1\right) - \lambda_2\left(\sum_{i=1}^{N} w_i R_i - \mu_P\right)$$

Then take partial derivatives with respect to w_i, λ_1, and λ_2

$$\frac{\partial L}{\partial w_i} = \sum_{j=1}^{N} \sigma_{ij} w_j - \lambda_1 - \lambda_2 R_i$$

$$\frac{\partial L}{\partial \lambda_1} = \sum_{i=1}^{N} w_i - 1$$

$$\frac{\partial L}{\partial \lambda_2} = \sum_{i=1}^{N} w_i R_i - \mu_P$$

Then we set all of those equal to 0 and start solving for the weights. When we solve the first function gives us

$$\sum_{j=1}^{N} \sigma_{ij} w_j = \lambda_1 + \lambda_2 R_i$$

We can also take note that our covariance matrix can be constructed as

$$\sum_{i=1}^{N} \sigma_{ij} = \Omega$$

Which means that we when we solve for w we get

$$w = \Omega^{-1}(\bar{1}\lambda_1 + \lambda_2 \bar{\mu})$$

$\bar{1}$ is vector filled with ones

$$\bar{1} = (1\ 1\ ...\ 1)^T$$

And $\bar{\mu}$ is the expected value of the return

$$\bar{\mu} = (R_1\ R_2\ ...\ R_N)^T$$

Now looks look at our two constraints

$$\begin{cases} \frac{\partial L}{\partial \lambda_1} = 1 \\ \frac{\partial L}{\partial \lambda_2} = \mu_P \end{cases}$$

Then we want to apply those constraints to this

$$w = \Omega^{-1}(\bar{1}\lambda_1 + \lambda_2 \bar{\mu}) = \Omega^{-1}\bar{1}\lambda_1 + \Omega^{-1}\lambda_2 \bar{\mu}$$

For the first constraint $\frac{\partial L}{\partial \lambda_1} = 1$

$$1 = \lambda_1 \bar{1}^T \Omega^{-1} \bar{1} + \lambda_2 \bar{1}^T \Omega^{-1} \bar{\mu}$$

And for the second constraint $\frac{\partial L}{\partial \lambda_2} = \mu_P$

$$\mu_P = \lambda_1 \bar{\mu}^T \Omega^{-1} \bar{1} + \lambda_2 \bar{\mu}^T \Omega^{-1} \bar{\mu}$$

We can make the notation nice with

$$\begin{cases} \eta = \bar{1}^T \Omega^{-1} \bar{1} \\ \xi = \bar{1}^T \Omega^{-1} \bar{\mu} \\ \gamma = \bar{\mu}^T \Omega^{-1} \bar{\mu} \end{cases}$$

That makes the two equations we found above become

$$1 = \lambda_1 \bar{1}^T \Omega^{-1} \bar{1} + \lambda_2 \bar{1}^T \Omega^{-1} \bar{\mu} = \eta \lambda_1 + \xi \lambda_2$$

$$\mu_P = \lambda_1 \bar{\mu}^T \Omega^{-1} \bar{1} + \lambda_2 \bar{\mu}^T \Omega^{-1} \bar{\mu} = \xi \lambda_1 + \gamma \lambda_2$$

Then we can solve for λ_1 and λ_2

$$\lambda_1 = \frac{\gamma - \xi \mu_P}{\eta \gamma - \xi^2}, \quad \lambda_2 = \frac{\eta \mu_P - \xi}{\eta \gamma - \xi^2}$$

When we have a return for the portfolio μ_P we want to find the minimum variance for those returns which gives us the best risk to reward portfolio.

$$\sigma_P^2 = w^T \Omega (\lambda_1 \Omega^{-1} \bar{1} + \lambda_2 \Omega^{-1} \bar{\mu})$$

Then working that out we get

$$\lambda_1 + \lambda_2 \mu_P = \frac{\eta \mu_P^2 - 2\xi \mu_P + \gamma}{\eta \mu_P - \xi^2}$$

Then finding the weight w becomes.

$$w = \left(\frac{\gamma - \xi \mu_P}{\eta \gamma - \xi^2} \right) \Omega^{-1} \bar{1} + \left(\frac{\eta \mu_P - \xi}{\eta \gamma - \xi^2} \right) \Omega^{-1} \bar{\mu}$$

Limitations, Problems, and Future Areas of Research

The efficient frontier was first designed to find the best risk-to-reward profile. But there are many limitations of this model. The first problem arises with using mean returns and covariance. Alternative methods work with using different risk methods such as VaR (Value-at-Risk). There is also research into covariance matrices themselves such as finding more sparse matrices for better financial descriptors and computational ease.

Other things to look into is how the efficient frontier forms with different types of stocks or asset classes. I have noticed that the efficient frontier takes a different shape if you include more securities. I hope to publish my results later on my findings.

The efficient frontier also led to the creation of the Modern Portfolio Theory which is a contested theory. MDT makes an assumption that risk and return are a trade off which isn't always the case.

As previously stated a lot of the current research involves optimization problems which can include topics of stochastic control. Also some of the optimization problems can be solved using machine learning or convex optimization.

Final Note

I hope that this paper can showcase my mathematical and financial skills when it comes to quantitative models. Although many of the areas of future research are problems that can be tackled easily with programming, I am very wary to publish my work until I have a full understanding of the mathematics. Therefore future publications on future models will be withheld until I fully understand and publish all of the mathematics behind it.

Appendix A: Second Moment

$$(w_1 \; w_2) \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$(w_1\sigma_1 + w_2\sigma_{21} \quad w_1\sigma_{12} + w_2\sigma_{22}) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$w_1^2\sigma_1^2$$

Appendix B: σ_P for when $\rho = 1$

$$\sigma_P^2(\alpha; \rho = 1) = (1-\alpha)\sigma_1^2 + 2\alpha(1-\alpha)\sigma_1\sigma_2 + \alpha^2\sigma_2^2$$

$$\sigma_P^2(\alpha; \rho = 1) = ((1-\alpha)\sigma_1 + \alpha(1-\alpha)\sigma_2)^2$$

$$\sigma_P(\alpha; \rho = 1) = (1-\alpha)\sigma_1 + \alpha\sigma_2$$

Appendix C: σ_P for when $\rho = -1$

$$\sigma_P^2(\alpha; \rho = 1) = (1-\alpha)\sigma_1^2 - 2\alpha(1-\alpha)\sigma_1\sigma_2 + \alpha^2\sigma_2^2$$

$$\sigma_P^2(\alpha; \rho = 1) = ((1-\alpha)\sigma_1 - \alpha\sigma_2)^2$$

$$\sigma_P(\alpha; \rho = 1) = |(1-\alpha)\sigma_1 - \alpha\sigma_2|$$

Appendix D: solving for α when $\frac{\partial \sigma_P^2}{\partial \alpha}$

$$-2(1-\alpha)\sigma_1^2 + 2\alpha\sigma_2^2 + 2(1-2\alpha)\rho\sigma_1\sigma_2 = 0$$

$$\alpha = \frac{\sigma_1^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$$

References

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