Heath-Jarrow-Morton Framework

Diego Alvarez
diego.alvarez@colorado.edu

September 2021

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1 Introduction

The Heath-Jarrow-Morton (HJM) Model models a dynamics of an instantaneous forward. We focus on developing the model for a generic maturity T. We also assume no arbitrage. In this case a cornerstone to this model is finding the volatility term. The volatility term determines the type of stochastic process.

2 Motivations

Motivation for Risk Neutral Measure

This model is built upon a stochastic differential equation of the form

$$df(t,T) = \mu(t,T)dt + \sigma_f(t,T)dW_t$$

Later on we will find other expressions for the parameters in the SDE. That will be done by building them out through the term structure tools discussed in the older paper. Then from there we can bring the two sub models together under the risk neutral pricing.

The risk neutral measure that we found is

$$\frac{P(t,T)}{B(t)} = \mathbb{E}^Q\left[\frac{P(T,T)}{B(T)}\middle|\mathscr{F}_t\right]$$

From here if we knew of the volatility of an security's process then we can write the dynamics out by using the risk neutral measure. The risk neutral measure allows us to draw assumptions about how the security will perform at maturity. Under the risk neutral measure the expected return is the risk-free rate.

3 Initial Model

We first assume that there is no arbitrage

Start with a general stochastic differential equation of this form

$$df(t,T) = \mu(t,T)dt + \sigma_f(t,T)dW_t \tag{1}$$

The expected return under the risk free measure will be the instantaneous forward rate, but the instantaneous forward rate is not a tradable security. Instead we'll need the dynamics of the zero coupon of the same maturity. The SDE for the zero coupon bond is. (See appendix A for full calculation)

$$dP(t,T) = r_t P(t,T)dt + \sigma_P(t,T)P(t,T)dW_t \tag{2}$$

Recall that to *transfer* from the instantaneous forward rate to the zero coupon bond we get

$$f(t,T) = -\frac{d}{dT}\ln P(t,T)$$

If we assume that we can switch the derivative and differential and make this differential

$$df(t,T) = -\frac{d}{dT}d\ln P(t,T) \tag{3}$$

Then by Ito's lemma we get (See Appendix B for full calculation)

$$d\ln P(t,T) = \frac{1}{P(t,T)}dP(t,T) - \frac{1}{2}\cdot\frac{1}{P(t,T)^2}dP(t,T)^2 \tag{4}$$

We can find those fractions on the right hand side manipulating eq.2. Let's first start by dividing by $P(t,T)$.

$$\frac{dP(t,T)}{P(t,T)} = r_t dt + \sigma_P(t,T)dW_t \tag{5}$$

Now to find the fraction in the second term we take the square. Using the fact that $dt^2 = dt$ and $dW_t^2 = 0$, (results from Ito's lemma) we get (See appendix C for full calculation)

$$\frac{dP(t,T)^2}{P(t,T)} = \sigma_P(t,T)^2 dt \tag{6}$$

Now putting those fractions that we found from eq.5 and eq.6 we get

$$d\ln P(t,T) = r_t dt + \sigma_P(t,T)dW_t - \frac{1}{2}\cdot\sigma_P(t,T)^2 dt \tag{7}$$

Now take a derivative with respect to T

$$\frac{d}{dT}d\ln P(t,T) = \frac{d}{dT}\sigma_P(t,T)dW_t - \sigma_P(t,T)\frac{d}{dT}\sigma_P(t,T)dt$$

We can use the fact that the differential of the forward is negative of what we just found (eq.3) to get this

$$df(t,T) = -\frac{d}{dT}d\ln P(t,T) = \sigma_P(t,T)\frac{d}{dT}\sigma_P(t,T)dt - \frac{d}{dT}\sigma_P(t,T)dW_t$$

We know how the dynamics of the forward under the risk neutral measure

$$df(t,T) = \sigma_P(t,T)\frac{d}{dT}\sigma_P(t,T)dt - \frac{d}{dT}\sigma_P(t,T)dW_t \tag{8}$$

Now we can relate eq.8 to eq.1. We can do this because both of their differentials are in regards to the same variables. Therefore we can make this claim

$$\sigma_f = -\frac{d}{dT}\sigma_P(t,T), \quad \mu(t,T) = \sigma_P(t,T)\frac{d}{dT}\sigma_P(t,T)$$

We can rewrite the σ_P by undoing the derivative with via integration

$$\int_t^T \sigma_t(t,u)du + C = -\sigma_P(t,T)$$

We can use the fact that the volatility at maturity is 0 $\sigma_P(t,T) = 0$ therefore $C = 0$ leaving us with

$$\sigma_P(t,T) = -\int_0^T \sigma_f(t,u)du \tag{9}$$

Now working out the $\mu(t,T)$ for eq.1 and eq.8. We can then plug in the value of σ_P into our $\mu(t,T)$

$$\mu(t,T) = \sigma_P(t,T)\int_t^T \sigma_f(t,u)du$$

Now we have the dynamics of the instantaneous forward under the risk neutral measure

$$df(t,T) = \left(\sigma_f(t,T)\int_t^T \sigma_f(t,u)du\right)dt + \sigma_f(t,T)dW_t$$

Really what we can extract is that we needed to work out the volatility, without that we wouldn't have been able to translate eq.1 to eq.8 or work out. We were able to make our assumptions about the volatility because of the measure that we had set before we started making the model.

4 Under the T-Forward measure

Let's start by defining this new measure. This method is also called the change of Numeriare. It is used for pricing techniques that involve random discount factors using forward measures. We start off with this processing which is a martingale under the risk neutral.

$$\frac{V(0)}{B(0)} = \mathbb{E}^Q\left[\frac{V(t)}{B(t)}\right] \tag{1}$$

Instead of using the bank account we will use the price of zero coupon

$$\frac{V(0)}{P(0,T)} = \mathbb{E}^T\left[\frac{V(t)}{P(t,T)}\right] \tag{2}$$

Now we will find new ways to rewrite eq.1 and eq.2. We can do this by finding $V(0)$ for each

$$V(0) = \mathbb{E}\left[\frac{B(0)}{B(t)}V(t)\right], \quad V(0) = \mathbb{E}^T\left[\frac{P(0,T)}{P(T,T)}V(t)\right]$$

We can express then use the definition of expected value and keep in mind that that each equation is under their respective probability measure

$$V(0) = \int \frac{B(0)}{B(t)}V(t)dQ, \quad V(0) = \int \frac{P(0,T)}{P(t,T)}V(t)dP$$

They are both talking about the price of the asset therefore we can make this claim

$$\frac{B(0)}{B(t)}dQ = \frac{P(0,T)}{P(t,T)}dP^T$$

Now we can rewrite this as the derivative of the new probability measure with respect to the old one.

$$\frac{dP^T}{dQ} = \frac{P(t,T)}{P(0,T)} \cdot \frac{B(0)}{B(t)} \tag{3}$$

Now we can use the differential of the log price that we got using Ito's lemma in the last section. We know that the equation is (Section 1 eq.7)

$$d\ln P(t,T) = r_t dt + \sigma_P(t,T)dW_t - \frac{1}{2}\cdot\sigma_P(t,T)^2 dt$$

Then to undo the derivative we can integrate

$$\ln P(t,T) - \ln P(0,T) = \int_0^t \left(r_u - \frac{1}{2} \cdot \sigma_P(u,T)^2 \right) du + \int_0^t \sigma_P(u,T) dW_u$$

Now we are trying to solve for $\frac{P(0,T)}{P(t,T)}$ so we can get that by using logarithm rules and then undoing those logs to get

$$\frac{P(t,T)}{P(0,T)} = e^{\int_0^t \left(r_u - \frac{1}{2} \cdot \sigma_P(u,T)^2 \right) du + \int_0^t \sigma_P(u,T) dW_u}$$

The ratio that the bank account has is

$$B(t) = B(0) e^{\int_0^t r_u du} \Rightarrow \frac{B(0)}{B(T)} = e^{-\int_0^t r_u du}$$

Now we can put those ratios into eq.3

$$\frac{dP^T}{dQ} = \frac{P(t,T)}{P(0,T)} \cdot \frac{B(0)}{B(t)} = e^{-\frac{1}{2} \int_0^t \sigma_P(u,T)^2 du + \int_0^t \sigma_P(u,T) dW_u}$$

Then using the Radon-Nikodym derivative (see appendix D) we can work backwards to get

$$dW_t^T = dW_t - \sigma_P(t,T) dt$$

Now we know σ_P (from section 1 eq.9).

$$dW_t^T = dW_t + \int_t^T \sigma_f(t,u) du dt \tag{4}$$

Now we can put that into the dynamics of the instantaneous forward under our new measure. Let's start with the one we initially found (Section 1. eq.9)

$$df(t,T) = \left(\sigma_f(t,T) \int_t^T \sigma_f(t,u) du \right) + \sigma_f(t,T) dW_t$$

Then put in the differential that we found in eq.4 and rearrange it to get dW_t

$$df(t,T) = \left(\sigma_f(t,T) \int_t^T \sigma_f(t,u) du \right) dt + \sigma_f(t,T) \left(dW_t^T - \int_t^T \sigma_f(t,u) du dt \right)$$

Then the two integrals cancel out giving us this

$$df(t,T) = \sigma_f(t,T) dW_t^T$$

This means that the instantaneous forward is a martingale under the forward measure, this is done by modeling the instantaneous forward with a zero coupon bond of the same maturity.

5 Under the $T_f > T$-Forward Measure

Usually when implementing this model we want to model a bunch of forwards $T_1, T_2, ..., T_f$. And each of those would take we would have to solve a $df(t,T), ..., df(t,T_f)$.

Instead we will take an arbitrary $T_f > T$, and then under the risk neutral measure we can write out

$$V(0) = \mathbb{E}^Q\left[\frac{B(0)}{B(t)}X_t\right], \quad V(0) = \mathbb{E}^{T_f}\left[\frac{P(0,T_f)}{P(t,T_f)}\right]$$

Then finding the Radon-Nikodym derivative becomes

$$\frac{B(0)}{B(t)}dQ = \frac{P(0,T_f)}{P(t,T_f)}dP^{T_f}, \quad \frac{dP^{T_f}}{dQ} = \frac{P(t,T_f)}{P(0,T_f)} \cdot \frac{B(0)}{B(t)}$$

The Radon-Nikodym lets us sort of change lens between the risk neutral measure for T and the risk neutral measure for T_f. Let's look at the Brownian under the T_f We have to take into account that our bounds for time is longer than T therefore we make those adjustments in the integral

$$\begin{aligned} dW_t^{T_f} &= dW_t - \sigma_P(t,T_f) \\ &= dW_t + \int_t^{T_f} \sigma_f(t,u)dudt \end{aligned}$$

Now we can add that dW_t into the our $df(t,T)$ equation

$$df(t,T) = \left(\sigma_f(t,T)\int_t^T \sigma_f(t,u)du\right) + \sigma_f(t,T)\left(dW_t^{T_f} - \int_t^{T_f} \sigma_f(t,u)dudt\right)$$

We are able to cancel out the integrals because they are integrating over the same function $\sigma_f(t,u)$ with respect to the same variable u and $[t,T] \in [t,T_f]$ leaving us with

$$df(t,T) = -\sigma_f(t,u)\left(\int_t^{T_f} \sigma_f(t,u)du\right)dt + \sigma_f(t,T)dW_t^{T_f}$$

6 Markovian Volatility

Let's start by getting the instantaneous rate $df(t,T)$

$$df(t,T) = \left(\sigma_f(t,T)\int_t^T \sigma_f(t,u)du\right)dt + \sigma_f(t,T)dW_t$$

Now let's take the integral to undo $df(t,T)$

$$f(t,T) - f(0,T) = \int_0^t \left(\sigma_f(s,T)\int_s^T \sigma_f(s,u)du\right) + \int_0^t \sigma_f(s,T)dW_s$$

Let's assume that volatility is a deterministic function of only time and maturity $\sigma_f(t,T)$. When we look at the Brownian increments they are normally distributed (See appendix E). Now we want to make the volatility process Markovian.

To do this we need to make sure that distribution at the next interval only depends on the current distribution. This makes modelling a lot easier because we only need to look at the current distribution. Let's isolate the stochastic term

$$D(t) = \int_0^t \sigma_f(s,t)dW_s, \quad D(T) = \int_0^T \sigma_f(s,T)dW_s$$

To do this we need to check

$$D(T) - D(t) = \int_0^T \sigma_f(s,T)dW_s - \int_0^t \sigma_f(s,t)dW_s$$

We can regroup the functions by rewriting the integral but changing their bounds

$$D(T) - D(t) = \int_t^T \sigma_f(s,T)dW_s + \int_0^t (\sigma_f(s,T) - \sigma_f(s,t))dW_s \tag{1}$$

We used this trick because the first integral is a deterministic function with respect to the brownian

$$\int_t^T \sigma_f(s,T)dW_s = 0$$

Now to make the process Markov we have to make σ_f a separable function. Let's say $\sigma_f(s,T) = g(s)h(T)$. Then the eq.1 becomes

$$\begin{aligned} D(T) - D(t) &= \int_t^T h(T)g(s)dW_s + \int_0^t g(s)h(T) - g(s)h(t)dW_s \qquad (2) \\ &= \int_t^T g(s)dW_s + (h(T) - h(t)) \int_0^t g(s)dW_s \qquad (3) \end{aligned}$$

Let's take a step back and recall that when we make the change for $f(s,t) = g(s)h(t)$ that it would also change $D(t)$

$$D(t) = \int_0^t h(t)g(s)dW_s$$

Which we can rewrite as

$$\frac{D(t)}{h(t)} = \int_0^t g(s)dW_s$$

Now plug that into eq.3

$$D(T) - D(t) = h(t)\int_t^T g(s)dW_s + \frac{h(T) - h(t)}{h(t)}D(t)$$

The process is now Markov. If we were to go from $t \to T$ we would only need the information of the process at time t.

7 Under the lognormal distribution

Let's start with standard HJM under the risk neutral measure

$$df(t,T) = \left(\sigma_f(t,T)\int_t^T \sigma_f(t,u)du\right)dt + \sigma_f(t,T)dW_t \tag{1}$$

If we wanted to make the volatility a lognormal process then we would rewrite it as $\sigma_f(t,T) = \sigma f(t)$. Let's plug that into eq.1

$$\begin{aligned} df(t) &= \left(\sigma f(t)\int_t^T \sigma f(t)du\right)dt + \sigma f(t)dW_t \\ &= \sigma^2 f(t)^2 \int_t^T dudt + \sigma f(t)dW_t \\ &= \sigma^2 f(t)^2(T-t)dt + \sigma f(t)dW_t \end{aligned}$$

If we look at the deterministic portion $df(t) = \sigma^2 f(t)^2(T-t)dt$. We can solve that ODE via separation of variables

$$\int_0^t \frac{df(u)}{f(u)^2} = \sigma^2 \int_0^t (T-u)du$$

Then work out the integral

$$-\frac{1}{f(t)} + \frac{1}{f(0)} = -\frac{\sigma^2}{2}(T-t)^2 + \frac{\sigma^2}{2}T^2$$

Then solve for $f(t)$

$$f(t) = \frac{f(0)}{1-\sigma^2 t(T-\frac{t}{2})f(0)}$$

If we look specifically at the denominator we notice that there is a possibility that it can go to zero. As that happens the instantaneous forward $f(t)$ will go to infinity. If that was the case the price of the bond would be \$0 which which breaks the no-arbitrage constraint that we put on.

8 Discrete Setting

Let's start with the HJM dynamics under the risk neutral measure

$$df(t,T) = \left(\sigma_f(t,T)\int_t^T \sigma_f(t,u)du\right)dt + \sigma_f(t,T)dW_t$$

The HJM replicates the instantaneous forward for a fixed maturity. Think of the instantaneous forwards as this matrix

$$\begin{bmatrix} f(0,0) & f(0,1) & f(0,2) & f(0,3) & \dots \\ & f(1,1) & f(1,2) & f(1,3) & \dots \\ & & f(2,2) & f(2,3) & \dots \\ & & & \ddots & \dots \end{bmatrix}$$

At time t_0 we are modeling all of our instantaneous forwards $f(0,0), f(0,1), ..., f(0,n)$. When we go to t_1 all of our instantaneous forwards get updated to $f(1,1), f(1,2), ..., f(1,n)$. You can think of the matrix is moving down diagonally and each row is the instantaneous forwards that we use for the HJM.

Let's start by enumerating some of the forward rates, in this case we will use h for the intervals.

$$\begin{aligned}
t_1 &: B(t_1) = e^{f(0,0)h} \\
t_2 &: B(t_2) = e^{f(0,0)h} \cdot e^{f(1,1)h} \\
t_3 &: B(t_3) = e^{f(0,0)h} \cdot e^{f(1,1)h} \cdot e^{f(2,2)h} \\
&\vdots \\
t_n &: B(t_n) = e^{f(0,0)h} \cdot e^{f(1,1)h} \cdot \ldots \cdot e^{f(n-1,n-1)h}
\end{aligned}$$

We can write that out as a summation

$$B(t_i) = e^{\sum_{j=0}^{i} f(j,j)h} \tag{1}$$

Let's model a zero coupon bond that expires at t_4

$$P(t_0, t_4) = e^{-f(0,0)h - f(0,1)h - f(0,2)h - f(0,3)h} = e^{-\sum_{j=0}^{3} f(0,j)h}$$

With eq.1 we can write that as

$$\begin{aligned}
P(t_1, t_4) &= e^{-\sum_{j=1}^{3} f(1,j)h} \\
P(t_2, t_4) &= e^{-\sum_{j=2}^{3} f(2,j)h}
\end{aligned}$$

Now we can make a generic version of the equations above

$$P(t_i, T) = e^{-\sum_{j-1}^{n-1}} f(i,j)h$$

Now we can define the HJM under the risk neutral measure. We can use this identity again that we found under the risk neutral measure

$$\frac{P(t,T)}{B(t)} = \mathbb{E}^Q \left[\frac{P(S,T)}{B(S)} \middle| \mathscr{F}_t \right]$$

The discrete version of the the martingale under the risk neutral measure becomes

$$\frac{P(t_i,T)}{B(t_i)} = \mathbb{E}^Q \left[\frac{P(t_{i+1},T}{B(t_{i+1}} \middle| \mathscr{F}_i \right]$$

Now let's multiply each side by $\frac{B(t_i)}{P(t_i,T)}$ to get

$$1 = \mathbb{E}^Q \left[\frac{P(t_{i+1},T)}{P(t_i,T)} \cdot \frac{B(t_i)}{B(t_{i+1})} \middle| \mathscr{F}_i \right] \tag{2}$$

We already found $P(t_i, T)$ and $B(t_i)$. Now we need to find $P(t_{i+1}, T)$ and $B(t_{i+1})$ which we can do by slightly changing the summations that we get

$$B(t_{i+j}) = e^{\sum_{j=0}^{i} f(j,j)h}, \quad P(t_{i+1}, T) = e^{-\sum_{j=i+1}^{n-1}} f(i+1, j)h$$

Let's first look at $\frac{B(t_i)}{B(t_{i+1})}$. We can rewrite $B(t_{i+1})$ as

$$B(t_{i+1}) = e^{\sum_{j=0}^{i-1} f(j,j)h + f(i,i)h}$$

We've essentially separated the $B(t_i)$ from $B(t_{i+1})$ and the fraction reduces to

$$\frac{B(t_i)}{B(t_{i+1})} = e^{-f(i,i)h}$$

Now we need to solve for zero coupon using the same method. In this case we will get

$$P(t_i, T) = e^{-f(i,i)h - \sum_{j=i+1}^{n-1} f(i,j)h}$$

$$P(t_{i+1}, T) = e^{-\sum_{j=i+1}^{n-1} f(i+1,j)h}$$

When we put them into the fraction we get

$$\frac{P(t_{i+1}, T)}{P(t_i, T)} = e^{-\sum_{j=i+1}^{n-1}(f(i+1,j) - f(i,j))h + f(i,i)h}$$

Then don't cancel out because the forward curves are different. When we plug that into eq.2

$$1 = \mathbb{E}^Q\left[e^{-\sum_{j=i+1}^{n-1}(f(i+1,j)-f(i,j))h + f(i,i)h} \cdot e^{-f(i,i)h} \middle| \mathscr{F}_i\right] \tag{3}$$

$$= \mathbb{E}^Q\left[e^{-\sum_{j=i+1}^{n-1}(f(i+1,j)-f(i,j))h} \middle| \mathscr{F}_i\right] \tag{4}$$

If we look inside the summation we see that it is the sum of the increments. This is really a discretized of the differential $\Delta f(t, T)$. That differential is equal to a drift term and volatility term. Our $f(i+i, j) - f(i, j)$ would be

$$f(i+1, j) - f(i, j) = \mu_{i,j} h + \sigma_{i,j} \Delta W_{i+1}$$

Then we can plug all that into eq.4

$$1 = \mathbb{E}^Q\left[e^{-\sum_{j=i+1}^{n-1}(\mu_{i,j}h + \sigma_{i,j}\Delta W_{i+1})h} \middle| \mathscr{F}_i\right] \tag{5}$$

$$= \mathbb{E}^Q\left[e^{-\sum_{j=i+1}^{n-1} \mu_{i,j}h^2 - \sum_{j=i+1}^{n-1} \sigma_{i,j}h\Delta W_{i+1}} \middle| \mathscr{F}_i\right] \tag{6}$$

Now find the mean of the Random variable in the exponent

$$\mathbb{E}\left[-\sum_{j=i+1}^{n-1} \mu_{i,j}h^2 - \sum_{j=i+1}^{n-1} \sigma_{i,j}h\Delta W_{i+1}\right] = -\sum_{j=i+1}^{n-1} \mu_{i,j}h^2$$

For variance we have

$$\begin{aligned}\mathrm{Var}\left[-\sum_{j=i+1}^{n-1}\mu_{i,j}h^2-\sum_{j=i+1}^{n-1}\sigma_{i,j}h\Delta W_{i+1}\right]&=\left(\sum_{j=i+1}^{n-1}\sigma_{i,j}h\right)^2\mathrm{Var}[\Delta W_{i+1}]\\&=\left(\sum_{j=i+1}^{n-1}\sigma_{i,j}h\right)^2 h\end{aligned}$$

We can then plug all that into eq.4 and we are going to use this identity, where X is a random variables

$$\mathbb{E}[e^X]=e^{\mathbb{E}[X]+\frac{1}{2}\mathrm{Var}[X]}$$

Which makes eq.4 become

$$1=e^{-\sum_{j=i+1}^{n-1}\mu_{i,j}h^2}\cdot e^{\frac{1}{2}\left(\sum_{j=i+1}^{n-1}\sigma_{i,j}h\right)^2}$$

For the equation to be true we need to the exponents to sum to 0. Therefore we can make this statement

$$\begin{aligned}\sum_{j=i+1}^{n-1}\mu_{i,j}h^2&=\frac{1}{2}\left(\sum_{j=i+1}^{n-1}\sigma_{i,j}h\right)^2 h\\&=\sum_{j=i+1}^{n-1}\mu_{i,j}=\frac{1}{2h}\left(\sum_{j=i+1}^{n-1}\sigma_{i,j}h\right)^2\end{aligned}$$

We want to find out the increments so first shift the upper bound

$$\sum_{j=i+1}^{n-1}\mu_{i,j}=\frac{1}{2h}\left(\sum_{j=i+1}^{n-1}\sigma_{i,j}h\right)^2$$

$$\Downarrow$$

$$\sum_{j=i+1}^{n}\mu_{i,j}=\frac{1}{2h}\left(\sum_{j=i+1}^{n}\sigma_{i,j}h\right)^2$$

Then write out the increments

$$\mu_{i,n}=\frac{1}{2h}\left(\left(\sum_{j=i+1}^{n}\sigma_{i,j}h\right)^2-\left(\sum_{j=i+1}^{n-1}\sigma_{i,j}h\right)^2\right)$$

That becomes the drift of the instantaneous forward for t_n for the interval $[i,i+1]$. If we compare it

A. Refresher on zero coupon bond price

The value of the zero coupon bond in terms of the instantaneous forward rate is

$$P(t,T) = e^{-\int_t^T f(t,u)du}$$

B. finding $d\ln P(t,T)$ and refresher on Ito's lemma

We are using this differential which relates the instantaneous forward rate to the price of the zero coupon bond

$$df(t,T) = -\frac{d}{dT} d\ln P(t,T)$$

The *problem* at hand is the $d\ln P(t,T)$. To solve this we will have to use Ito's lemma. Recall that Ito's lemma says

$$df(X) = f_X dX + \frac{1}{2} f_{XX} dX^2$$

Then applying that to $\ln X$ and knowing that

$$\frac{d}{dX}\ln X = \frac{1}{X}$$

$$\frac{d^2}{dX^2}\ln X = \frac{1}{X^2}$$

We get

$$d\ln X = \frac{1}{X}dX - \frac{1}{2}\cdot\frac{1}{X^2}dX$$

Then plug in $P(t,T)$ for X

$$d\ln P(t,T) = \frac{1}{P(t,T)}dX - \frac{1}{2}\cdot\frac{1}{P(t,T)^2}dP(t,T)^2$$

C. Solving $\frac{dP(t,T)^2}{P(t,T)^2}$ and working with dt^2 and dW_t^2

When we take the square of

$$\frac{dP(t,T)}{P(t,T)} = r_t dt + \sigma_P(t,T)dW_t$$

we get

$$\frac{dP(t,T)^2}{P(t,T)^2} = r_t^2 dt^2 + \sigma_P(t,T)^2 dW_t^2$$

But we know that $dt^2 = 0$. We get this from Ito's lemma. The equation is really found by getting the solution of this integral

$$\int_0^s ds^2$$

Then by definition of an integral

$$\int_0^s ds^2 = \lim_{n \to \infty} \sum_{k=1}^{n} \Delta t_k^2$$

Here is the workaround for solving this integral. What we will do is solve it for the mean squared convergence which implies this

$$\lim_{n \to \infty} E\left[|X_n - X|^2\right] = 0$$

D. Girsanov's theorem and Radon-Nikodym Derivative

If we have a Brownian motion W_t under Q and we make this new process $Y(t)$.

$$Y(t) = \int_0^t y_u du$$

We can adjust the process W_t for $Y(t)$ which becomes

$$W_t^T = W_t - \int_0^t y_u du$$

This process is under a probability measure P^T. That probability measure comes from the Radon-Nikdoym derivative

$$\frac{dP^T}{dQ} = e^{-\frac{1}{2}\int_{-}^{t} y_u^2 du + \int_0^t y_u dW_u}$$

In differential form the two Brownian look like

$$dW_t^T = dW_t - y_t dt$$

Ultimately what we can do is that we are going to look at what we have and then translate that back to differential form via Radon-Nikodym derivative.

E. Showing that the volatility is Gaussian when volatility is a deterministic and a function of time and maturity

Let's start with our instantaneous rate

$$df(t,T) = \left(\sigma_f(t,T) \int_t^T \sigma_f(t,u) du\right) dt + \sigma_f(t,T) dW_t$$

then take the integral

$$f(t,T) - f(0,T) = \int_0^t \left(\sigma_f(s,T) \int_s^T \sigma_f(s,u)du \right) ds + \int_0^t \sigma_f(s,T)dW_s$$

this part is a linear combination of normals

$$\int_0^t \sigma_f(s,T)dW_s$$

And this part is deterministic

$$\int_0^t \left(\sigma_f(s,T) \int_s^T \sigma_f(s,u)du \right) ds$$

Really the whole thing becomes a linear transformation of deterministic functions that are Gaussian.