# Introductory concepts for Term Structure Modeling

Diego Alvarez
diego.alvarez@colorado.edu

May 2021

## Author's note

This paper is meant to serve as a framework for future models as well as laying the groundwork for term structure models. Another reason for publishing this paper is showcase my skills regarding finance and mathematics. Please read through, and of course if any mistakes do appear feel free to contact me.

## Definitions

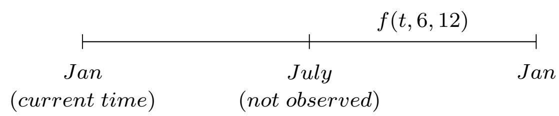
Throughout this paper there will be a series of different interest rate terms being thrown around. For ease of use and to avoid confusion the terms will be defined below.

- coupon - when a bond is issued it has coupons that are interest payments, they are also paid on a schedule.
- zero coupon bond - This is a bond that has no coupon and is bought at a discount value and when it matures it is worth \$1.
- spot rate - think of this as rate of return for without collecting coupons if you sell the bond immediately. For example if we had a \$1,000 zero coupon bond that had 2 years left until maturity and the current value is \$950 the spot rate is 2.59%. The spot rate is calculated as

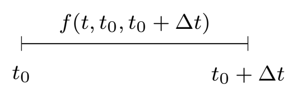
  $$\left(\frac{FV}{CV}\right)^{\frac{1}{\tau}} - 1$$

  For the most part we will commonly look at zero-coupon bonds. For those who are familiar with financial terms, the spot rate will be yield to maturity (YTM) of the zero-coupon bond. That is not the case for other coupon-bearing bonds.

- forward rate - the forward rate is an agreed upon interest rate for a bond that starts on a future date. For example if the it is January the 6-month forward rate for a maturity of 1 year is the interest rate for a 6 month bond starting in July.

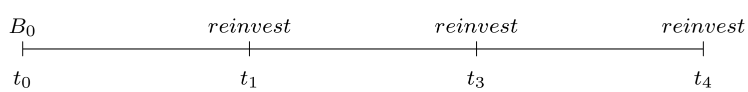



- instantaneous forward rate - an agreed upon rate for some future date date but in this case the settlement date is the next instant.




for an infinitesimally small $\Delta t$

- Bank account - This is a financial investment that we make that involves putting money into a bank account with some interest rate and then reinvesting the money over some interval.




- short rate - this refers to an interest rate that is set at some time $i$. Throughout this paper the short rate will be the instantaneous forward rate, this is done for ease of use with notation later on.

# Notation

This will help clarify some of the functions and independent variables that we use throughout this paper

- Standard notation for functions of 2 independent variables:

  For a function $g$ that will either model price or interest rate there are usually two independent variables associate with it. The function will usually take the form $g(t, T)$. $t$ will be the time variable, and $T$ will be the maturity.[1]

- Standard notation for a function of 3 independent variables:

  These functions will be used for the forward rate. It will usually be expressed as $f(t, t_i, t_{i+1})$. The first variable, $t$ is time variable, and the $t_i$ and $t_{i+1}$) are for the future date. Forward rates refer an interest rate that starts in the future. That would mean that for $f(t, t_i, t_{i+1})$ the interest rate payment starts on $t_i$ and ends on $t_{i+1}$ [2]

- The switch from 3 independent variables to 2 independent variables with the same function.

  Forward rates can reference a future rate. For example if we have a forward rate for the future interval $(t_1, t_2)$ the forward rate is expressed as $f(t, t_1, t_2)$. When we are at $t_1$ the forward rate becomes $f(t, t_2)$.

- $y(t, T)$: the spot rate

- $P(t, T)$: this is a pricing function. It models the price of a fixed-income security with maturity $T$.

- $B(t, T)$: This is our bank rate. Via definition of bank-rate stated above this is the interest rate that we would receive for depositing our money.

- $f(t, T)$: This is the forward rate for some maturity $T$

- $r(t)$ and $r_i$: This is the short rate. The short rate is the continuously compounded for an infinitesimally small period

$$r(t) = f(t, t)$$

[1] I like to think of $T$ as the variable that references the bond by maturity, other than the instantaneous forward rate its more of a variable that references the bond and doesn't change.

[2] In this case think of $t_1$ and $t_2$ as the referencing component of the security like how $T$ was the referencing security for the function of two independent variables.

# Motivations for commonly used practices

## Use of zero coupon bonds

The motivation for the zero coupon bond is that we know its final value. Throughout the paper the zero coupon bond will have the value of \$1.

$$P(T, T) = 1$$

Knowing the value at maturity allows us to circumvent the time aspect of zero coupon bonds. When working with other bonds the expected value at maturity is unknown. Using that feature of the zero coupon bond makes it possible to solve for other variables.

## The use of instantaneous forward rates

The use of the instantaneous forward rate is so we can build continuous interest rates within the bonds. This is done because we can discount the zero coupon bond as a series of forward rates and then decrease the time interval of each forward rate. The instantaneous forward rate also helps with finding other continuous interest rates, and is necessary for setting up differential equations with interest rates. When using a $\sum$ that is in reference to the simple forward rate, and when using the $\int$ that is in reference for instantaneous rate.

Something to keep in mind is that in application there is no such thing as an instantaneous forward rate because that would imply that we lend and receive interest rate payment and reinvest faster than the speed of light. Working out this problem of instantaneous forward rates will come up in later papers, specifically the LIBOR market model paper and the HJM Framework paper.

## Finding the instantaneous forward rate

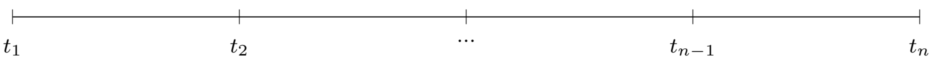
Start with the value of the zero coupon bond, knowing that the value of the bond at maturity, $T$ is worth \$1.

$$P(T,T) = 1$$

Then express the zero coupon bond at time $t$ through its spot value. Think of this more as representing the discounted value of the zero coupon bond.

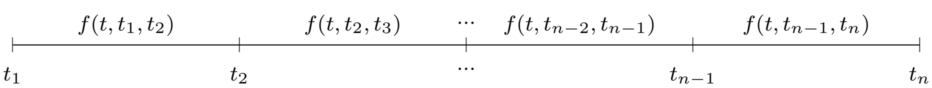
$$P(t,T) = e^{-y(t,T)(T-t)} \tag{1}$$

It is possible to translate the zero coupon bonds so that they are expressed in only forward rates. Let's break our time interval into a series of discrete interval from $t_0, ..., t_n$.




Now for each interval $(t_i, t_{i+1})$ make its associated forward rate. For an interval $(t_i, t_{i+1})$ the associated forward rate will be $f(t, t_i, t_{i+1})$

For all of the intervals the forward rates are




Then via the pricing function $P(t,T)$ the zero coupon bond will become all of the forward rates between the interval multiplied by the final price of the bond, \$1. In financial terms the zero coupon bond is being discounted to all of the forward rates of each interval. For a zero coupon bond with $n$ subintervals, it will have $n-1$ forward rates associated with it, that is because each forward rate represents an interval rather than a timestamp. Notice that the function for the forward rates are of two variables, that is because we observe each forward rate at their respective starting date. Also because all of our time intervals are the same length our difference in time $(T-t)$ from eq. 1 becomes $\Delta t$

Our pricing function where we discount the final value of the zero coupon bond \$1 to the each forward rate becomes

$$P(t,T) = e^{-f(t,t_0)\Delta t} e^{-f(t,t_1)\Delta t} ... e^{-f(t,t_{n-1})\Delta t} \cdot 1$$

Then represent those forward rates as a summation

$$P(t,T) = e^{-\sum_{i=0}^{n-1} f(t,t_i)\Delta t}$$

Then translate those into the instantaneous forward rate by making the interval infinitesimally small and taking the limit.[3]

$$P(t,T) = e^{-\int_t^T f(t,u)du} \tag{2}$$

Now setting the two sides equal and solving for the interest rate, $y(t,T)$ we get. (See Appendix A)

$$y(t,T) = \frac{1}{T-t}\int_t^T f(t,u)du \tag{3}$$

Now the spot interest rate is represented as the average of instantaneous forward rate. [4]

[3] Almost like reversing the definition of an integral $\int f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i*)\Delta x$

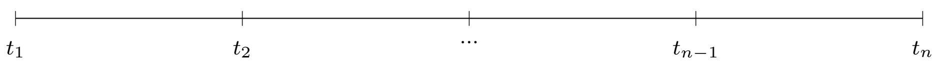
[4] In this case the referencing part is constantly changing which makes sense because the instantaneous forward rate the forward rate for an infinitesimally small time period.

# Finding the instantaneous rate for the bank account

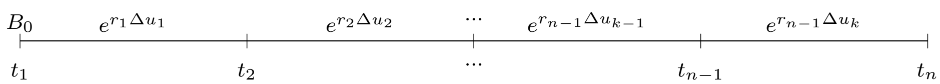
We start with \$1 at $t_0$ and we lend the money at an interest rate $r_i$ for interval $i$. The the standard bank rate formula is.

$$B_0 e^{r_i \Delta u_i} \tag{4}$$

For the instantaneous forward rate we split the interval into $n$ subintervals




Then we reinvest all the money over each interval and using $k$ as the index.




We can represent the bank account over $k$ intervals as

$$B_0 e^{r_1 \Delta u_1} e^{r_2 \Delta u_2} ... e^{r_k \Delta u_k}$$

Similar to finding the instantaneous forward rate we make a summation of interest rates

$$B_0 e^{r_1 \Delta u_1} e^{r_2 \Delta u_2} ... e^{r_k \Delta u_k} = B_0 e^{\sum_{i=0}^{k} r_i \Delta u_k}$$

Then integrate to get short rate

$$B_0 e^{\sum_{i=0}^{k} r_i \Delta u_k} = B_0 e^{\int_0^t r_u du}$$

The bank rate is[5]

$$B_t = e^{\int_0^t r_u du} \tag{5}$$

The value of the Bank account can't be expressed in terms of forward rates because the forward rate are determined ex-ante so we can't go back into time and observe the forward rates.

[5] We can drop the $B_0$ because it is \$1. In other words the $B_0$ doesn't change anything via identity property for multiplication

## The zero coupon bond in terms of the short rate

To find this we will need the future value of the short rate. We use the risk-neutral valuation. Via risk neutral valuation if we take the price of the zero coupon bond at maturity and scale it by the bank account at $T$, the expected value under our risk neutral measure $Q$ is the same as its current value. [6]

$$\frac{P(t,T)}{B(t)} = \mathbb{E}\left[\frac{P(T,T)}{B(T)}\middle|\mathscr{F}_t\right] \tag{6}$$

Now solve for $P(t,T)$ (see appendix B)

$$P(t,T) = \mathbb{E}\left[\frac{B(t)}{B(T)}\middle|\mathscr{F}_t\right] \tag{7}$$

We know that the bank rate at $T$ is

$$B_T = e^{\int_0^T r_u du}$$

The bank rate at $t$ is

$$B_t = e^{\int_0^t r_u du}$$

Knowing that $t < T$ (see appendix C)

$$\frac{B_T}{B_t} = e^{\int_t^T r_u du}$$

We can invert the equation above to get

$$\frac{B_t}{B_T} = e^{-\int_t^T r_u du}$$

Plugging that into the eq.7

$$P(t,T) = \mathbb{E}\left[e^{-\int_t^T r_u du}\right] \tag{8}$$

That gives us the value of the zero coupon bond in terms of the short rate.

[6] Using this risk neutral valuation is **very important** for the HJM model. It is also is important because it allows us to combine 2 equations together and start solving for parts of each.

## Finding the instantaneous forward for the zero coupon bond

The price of bond from eq.2 is

$$P(t,T) = e^{-\int_t^T f(t,u)du}$$

Now solve for the $f(t,T)$ to get (see appendix D)

$$\frac{-d}{dT} \ln P(t,T) = f(t,T) \tag{9}$$

Eq. 9 is the instantaneous forward rate for the zero coupon bond. We can also think of it as the limit of the simple forward rate. It can also represent the present value of zero coupon bond discounted by the forward rate as some $T+\Delta$

$$P(t,T,T+\Delta) = P(t,T)e^{-f(t,T,T+\Delta)\Delta} \tag{10}$$

Then when we isolate $f(t,T,T+\Delta)$ (see appendix E)

$$f(t,T,T+\Delta) = -\frac{\ln(P(t,T+\Delta) - \ln P(t,T)}{\Delta} \tag{11}$$

Then to find the instantaneous forward rate, (in this case it would look like $f(t,T,T)$ if we represented it in terms of 3 variables). We take the limit as $\Delta$ goes to 0.

$$\lim_{\Delta\to 0} f(t,T,T+\Delta) = -\lim_{\Delta\to 0} \frac{\ln P(t,T+\Delta) - \ln P(t,T)}{\Delta}$$

The equation above is the definition of a derivative which becomes

$$f(t,T,T) = f(t,T) = -\frac{d}{dT} \ln P(t,T)$$

## Comparing differentials of short rate and instantaneous forward rate

Let's call $r(t)$ our short rate and by definition it is also our instantaneous forward rate.

$$r(t) = \lim_{T\to t} f(t,T) = f(t,t) \tag{12}$$

Then we can find the differential of that (see appendix F)

$$dr(t) = df(t,T)\Big|_{T=t} + \frac{\partial f(t,T)}{\partial T}\Big|_{T=t} dt \tag{13}$$

This will be later used for two HJM model and then for short rate models.

## Appendix A: Solving for $y(t,T)$ with instantaneous rate with zero coupon bond

The value of the zero coupon bond from eq.1 is

$$P(t,T) = e^{-y(t,T)(T-t)}$$

And the value of the zero coupon bond in terms of instantaneous forward rate (which is eq.2)

$$P(t,T) = e^{-\int_t^T f(t,u)du}$$

Now set them equal

$$e^{-y(t,T)(T-t)} = e^{-\int_t^T f(t,u)du}$$

That becomes

$$y(t,T)(T-t) = \int_t^T f(t,u)du$$

Then solve for $y(t,T)$

$$y(t,T) = \frac{1}{T-t}\int_t^T f(t,u)du$$

## Appendix B: Solving for $P(t,T)$ under the risk neutral measure

We start with $P(t,T)$ scaled by $B(t)$ under the risk neutral measure

$$\frac{P(t,T)}{B(t)} = \mathbb{E}^Q\left[\frac{P(T,T)}{B(T)}\middle|\mathscr{F}_t\right]$$

Knowing that the value of $P(T,T) = 1$

$$\frac{P(t,T)}{B(t)} = \mathbb{E}^Q\left[\frac{1}{B(T)}\middle|\mathscr{F}_t\right]$$

And then solve for $P(t,T)$ by multiplying out $B(t)$ and passing it through the expected value

$$P(t,T) = \mathbb{E}^Q\left[\frac{B(t)}{B(T)}\middle|\mathscr{F}_t\right]$$

# Appendix C: Getting the proportion of the bank rate

Knowing that the bank rate is

$$B_t = e^{\int_0^t r_u du}$$

We can express $B_t$ and $B_T$ individually as

$$B_t = e^{\int_0^t r_u du} \text{ and } B_T = e^{\int_0^T r_u du}$$

We also assume that $t < T$ so when we set up this fraction we get

$$\frac{B_T}{B_t} = \frac{e^{\int_0^T r_u du}}{e^{\int_0^t r_u du}}$$

We can break up the upper integral

$$\int_0^T r_u du = \int_0^t r_u du + \int_t^T r_u du$$

Plugging that into the original fraction

$$\frac{B_T}{B_t} = \frac{e^{\left(\int_0^t r_u du + \int_t^T r_u du\right)}}{e^{\int_0^t r_u du}}$$

Then via exponent rules we get

$$\frac{B_T}{B_t} = \frac{e^{\int_0^t r_u du} \cdot e^{\int_t^T r_u du}}{e^{\int_0^t r_u du}}$$

Now reduce the fraction

$$\frac{B_T}{B_t} = e^{\int_t^T r_u du}$$

## Appendix D: Solving for the $f(t,T)$ for the zero coupon bond

Start with the price of the zero coupon bond that we found in eq.2

$$P(t,T) = e^{-\int_t^T f(t,u)du}$$

Take the log of both sides and move the negative sign over

$$-\ln P(t,T) = \int_t^T f(t,u)du$$

Then take a $T$-derivative

$$\frac{-d}{dT}\ln P(t,T) = \frac{d}{dT}\int_t^T f(t,u)du \tag{14}$$

Then via Leibniz Integral rule Rule which is (for some arbitrary function $f(x,z)$)

$$\frac{\partial}{\partial z}\int_a^b f(x,z)dx = \int_a^b \frac{\partial}{\partial z}f(x,z)dx + f(b,z)\frac{\partial b}{\partial z} - f(a,z)\frac{\partial a}{\partial z}$$

Applying Leibniz Integral rule to our integral (eq.14) we get

$$\frac{d}{dT}\int_t^T f(t,u)du = \int_t^T \frac{\partial}{\partial T}f(t,u)du + f(t,T)\frac{\partial}{\partial T}T - f(T,T)\frac{\partial}{\partial T}t \tag{15}$$

The partial derivative on right goes to 0.

$$\frac{d}{dT}\int_t^T f(t,u)du = \int_t^T \frac{\partial}{\partial T}f(t,u)du + f(t,T)\frac{\partial}{\partial T}T$$

Let's look at the integral on the right hand side. The derivative of a definite integral is 0.

$$\int_t^T \frac{\partial}{\partial T}f(t,u)du = 0$$

Working out the partial derivative on the right hand side we get

$$\frac{\partial}{\partial T}T = 1$$

That makes eq.15 become

$$\frac{d}{dT}\int_t^T f(t,u)du = f(t,T)$$

## Appendix E: finding the forward rate of the zero coupon bond for $T + \Delta$

from eq. 9 we have

$$P(t, T, T + \Delta) = P(t, T)e^{-f(t,T,T+\Delta)\Delta}$$

divide out

$$e^{f(t,T,T+\Delta)\Delta} = \frac{P(t, T)}{P(t, T + \Delta)}$$

Then take the log

$$f(t, T, T + \Delta)\Delta = \ln\left(\frac{P(t, T)}{P(t, T + \Delta)}\right)$$

Then divide out the $\Delta$ and use log rules

$$f(t, T, T + \Delta) = -\frac{\ln P(t, T, T + \Delta) - \ln P(t, T)}{\Delta}$$

## Appendix F: Finding the differential of the short rate

We first start with the short rate, which is the continuous forward rate. Starting with eq. 12

$$r(t) = \lim_{T \to t} f(t, T) = f(t, t)$$

When we want to find the differential of the short rate we are increasing the instantaneous forward rate for some time $\Delta t$. [7]

$$f(t + \Delta t, t + \Delta t)$$

That means that when we take find the differential for $r(t)$ we need to take derivative with respect to two independent variables of the forward rate above with respect to $T$ and evaluate at $t$

$$dr(t) = df(t, T)\Big|_{T=t} + \frac{\partial f(t, T)}{\partial T}\Big|_{T=t} dt$$

[7] In financial terms if we plot all of the forward rates for the timeline it will look like a staircase going from left to right top to bottom. The bank rate follows the leftmost diagonal (top to bottom), so to work that out in terms of forward rates we need to move down and to right to get to the next "step". That is why we move in 2 directions for $\Delta t$