Diffusion Equation

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2 Initial Model

Let's first start with the function $f(x,t)$ and to model the changes 1-dimensional dynamics. If we increase time by a small τ we get

$$f(x,t+\tau) = \int_{-\infty}^{\infty} f(x+\Delta,t)\phi(\Delta)d\Delta \tag{1}$$

Where Δ is the change in displacement that the particle makes. The function $\phi(\Delta)$ is the probability distribution of that particle's path. Essentially we are translating the future position of a particle is really the distance it will travel in time multiplied by the probability of that path being taken.

The cornerstone to this theory was representing both functions using taylor series.

We want to find the the taylor series for $f(x+\Delta,t)$

$$f(x+\Delta,t) = f(x,t) + \Delta \cdot \frac{\partial f(x,t)}{\partial x} + \frac{\Delta^2}{2!} \cdot \frac{\partial^2 f(x,t)}{\partial x^2} + ... + \frac{\Delta^n}{n!} \cdot \frac{\partial^n f(x,t)}{\partial x^n}$$

Then apply that to the definition of $f(x,t+\tau)$ that we had found in eq.1

$$\begin{aligned} f(x,t+\tau) =& f(x,t)\int_{-\infty}^{\infty}\phi(\Delta)d\Delta + \frac{\partial f(x,t)}{\partial x}\int_{-\infty}^{\infty}\Delta\phi(\Delta)d\Delta + \\ & \frac{\partial^2 f(x,t)}{\partial x^2}\int_{-\infty}^{\infty}\frac{\Delta^2}{2!}\phi(\Delta)d\Delta + ... + \frac{\partial^n f(x,t)}{\partial x^n}\int_{-\infty}^{\infty}\frac{\Delta^n}{n!}\phi(\Delta)d\Delta \end{aligned}$$

Let's look at the first term, we can see that the integral goes to 1.

$$f(x,t)\int_{-\infty}^{\infty}\phi(\Delta)d\Delta = f(x,t)\cdot 1$$

Then if we look at the the second term it goes to zero.

$$\frac{\partial f(x,t)}{\partial x}\int_{-\infty}^{\infty}\Delta\phi(\Delta)d\Delta = 0$$

The best way to think of this is that when we integrate the Δ we will get an even exponent and when we plug in the bound we'll end up subtracting the same value. This is the case for all even terms in the our taylor series.

Now we want to find show that this all goes to the diffusion equation. We'll also need the taylor series of $f(x,t+\tau)$

$$f(x,t+\tau) = f(x,t) + \tau \cdot \frac{\partial f(x,t)}{\partial t} + \frac{\tau^2}{2!} \cdot \frac{\partial^2 f(x,t)}{\partial t^2} + ... + \frac{\tau^n}{n!} \cdot \frac{\partial^n f(x,t)}{\partial t^n}$$

Here is the slick trick with the taylor series that we found for $f(x,t+\tau)$ and the taylor series for $f(x+\Delta,t)$ that we applied to our definition. Really beyond the 2nd terms they partials don't yield much. The way I like to think of it is by thinking of what the actual derivatives mean. As we keep taking more partial derivatives with respect to time we get velocity → acceleration → jerk → snap → crackle → pop. The movements of particles don't really exhibit acceleration, and therefore don't exhibit the other derivatives of position vector.

Applying that gives us

$$f(x,t+\tau) = f(x,t) + \frac{\partial f(x,t)}{\partial t}$$

Although we'll keep the other higher moments for the partial derivatives with respect to x

$$f(x,t) + \frac{\partial f(x,t)}{\partial t} = f(x,t) + \frac{\partial^2 f(x,t)}{\partial x^2}\int_{-\infty}^{\infty}\phi(\Delta)\frac{\Delta^2}{2!}d\Delta + \text{higher moments}$$

Now get rid of the $f(x,t)$ and divided by τ

$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial^2 f(x,t)}{\partial x^2} \int_{-\infty}^{\infty} \frac{\Delta^2}{2\tau} \cdot \phi(\Delta) d\Delta$$

Then if we let D be the mass diffusivity we get

$$\frac{\partial f(x,t)}{\partial t} = D \cdot \frac{\partial^2 f(x,t)}{\partial x^2}$$

3 Solution to Diffusion Equation

3.1 Difference Method

Let's start with a general diffusion equation

$$\frac{\partial f(x,t)}{\partial t} = D \frac{\partial^2 f(x,t)}{\partial x^2} \tag{1}$$

The goal is to find an invariant transformation that reduces the order of the PDE

$$v = \lambda x$$
$$u = \lambda^2 t$$

We know that the transformation is invariant (see Appendix A and B). In this case we will add on a constant to our transformation making the function $af(\lambda x, \lambda^2 t)$. We know that when we integrate our function we will end up with the number of particles.

$$\int_{mathbbR} af(\lambda x, \lambda^2 t) dx = n$$

The trick here is to switch integration. Let's recall that $v = \lambda x$ to switch integration to dv we need to take the derivative. Another way to think of it is by doing u-substitution but in this case our dummy variable is v. (See appendix C.)

$$\frac{a}{\lambda} \int_{\mathbb{R}} f(v, \lambda^2 t) dv = n$$

We know that this statement is true (integrating over the function gives us the number of particles)

$$\int_{\mathbb{R}} f(v, \lambda^2 t) dv = n$$

Therefore to make that statement true with the $\frac{a}{\lambda}$ we have to set that fraction equal to 1. Then from there we can solve for λ

$$\frac{a}{\lambda} = 1$$

Now the question becomes what should we set λ to. Let's look again at the diffusion equation which is

$$\frac{\partial f(x,t)}{\partial t} = D\frac{\partial^2 f(x,t)}{\partial x^2}$$

The left hand side represents

$$\frac{\partial f(x,t)}{\partial t} \Rightarrow \text{The change of particles over time}$$

The right hand side represents

$$D\frac{\partial^2 f(x,t)}{\partial x^2} \Rightarrow D \text{ times the change in particles over the change in area}$$

D (the mass diffusivity) is

$$D \Rightarrow \text{Area divided by time}$$

Now let's look back at our function $\lambda f(\lambda x, \lambda^2 t)$. Let's say we want to make $\lambda^2 t$ = 1 by picking a λ to make that statement true. If we chose $\lambda = \frac{1}{\sqrt{t}}$ we'll end up with

$$\frac{1}{\sqrt{t}} f\left(\frac{x}{\sqrt{t}}, 1\right)$$

Know the workaround this to get it all into a function of 1 variable (which is the ultimate goal) is to look at really what $\frac{x}{\sqrt{t}}$ means. If we take the square root of D we get length divided by square root time. So to get it all into one variable we set λ to be $\lambda = \frac{1}{\sqrt{Dt}}$. That gives us

$$\frac{1}{\sqrt{Dt}} f\left(\frac{x}{\sqrt{Dt}}, \frac{1}{D}\right)$$

Then if we set $z = \frac{x}{\sqrt{Dt}}$ we get

$$\lambda f(\lambda x, \lambda^2 t) = \frac{1}{\sqrt{Dt}}\bar{f}(z) \tag{2}$$

Now we need to take derivatives to set up the diffusion equation. Start by taking the derivative with respect to time. Let's take the inner derivative that we get from the chain rule

$$\frac{\partial z}{\partial t} = -\frac{x}{2\sqrt{D}} \cdot t^{-\frac{3}{2}}$$

Now for the whole derivative

$$\begin{aligned}
\frac{\partial \lambda f(\lambda x, \lambda^2 t)}{\partial t} &= \frac{\partial \bar{f}(z)}{\partial z} \cdot \frac{\partial z}{\partial t} \\
&= \frac{1}{\sqrt{Dt}} \cdot \frac{\partial \bar{f}(z)}{\partial t} + \bar{f}(z) \cdot \frac{\partial}{\partial t}\left(\frac{1}{\sqrt{Dt}}\right) \\
&= -\frac{1}{\sqrt{Dt}} \cdot \frac{\partial \bar{f}(z)}{\partial z} \cdot \frac{x}{2\sqrt{D}} \cdot t^{-\frac{3}{2}} - \bar{f}(z) \cdot \frac{1}{2\sqrt{(D)}} \cdot t^{-\frac{3}{2}} \\
&= -\frac{t^{-\frac{3}{2}}}{2\sqrt{D}}\left(z \cdot \frac{\partial \bar{f}(z)}{\partial z} + \bar{f}(z)\right)
\end{aligned}$$

Now let's take the derivative with respect to x. Let's start by finding that inner derivative first

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{Dt}}$$

Now for the whole derivative

$$\begin{aligned}
\frac{\partial \lambda f(\lambda x, \lambda^2 t)}{\partial t} &= \frac{1}{\sqrt{Dt}} \cdot \frac{\partial \bar{f}(z)}{\partial z} \cdot \frac{1}{\sqrt{Dt}} \\
&= \frac{1}{Dt} \cdot \frac{\partial \bar{f}(z)}{\partial z}
\end{aligned}$$

Now take the second derivative with respect to x

$$\frac{\partial^2 \lambda f(\lambda x, \lambda^2 t)}{\partial x^2} = \frac{1}{Dt} \cdot \frac{\partial^2 \bar{f}(z)}{\partial z^2} \cdot \frac{1}{\sqrt{Dt}}$$

Our new diffusion equation becomes

$$-\frac{t^{-\frac{3}{2}}}{2\sqrt{D}}\left(z \cdot \frac{\partial \bar{f}(z)}{\partial z} + \bar{f}(z)\right) = \frac{\partial^2 \bar{f}(z)}{\partial z^2} \cdot \frac{1}{\sqrt{D}} t^{-\frac{3}{2}}$$

Then we can cancel out D and t terms

$$-\frac{z}{2} \cdot \frac{\partial \bar{f}(z)}{\partial z} - \frac{1}{2}\bar{f}(z) = \frac{\partial^2 \bar{f}(z)}{\partial z^2}$$

Then we can rewrite it is in this form

$$\frac{d^2 \bar{f}(z)}{dz^2} + \frac{1}{2}z \cdot \frac{d\bar{f}(z)}{dz} + \frac{1}{2}\bar{f}(z) = 0$$

We can pack z and $\bar{f}(z)$ by using the multiplicative rule of derivatives in reverse

$$\frac{d^2 \bar{f}(z)}{dz^2} + \frac{1}{2}\frac{d}{dz}(z\bar{f}(z)) = 0$$

Now integrate to get

$$\int \frac{d^2 \bar{f}(z)}{dz^2} + \frac{1}{2}\frac{d}{dz}(z\bar{f}(z))dz \Rightarrow \frac{d\bar{f}(z)}{dz} + \frac{1}{2}z\bar{f}(z) = c$$

We are going to use a special trick. We know that $\bar{f}(z)$ is an even function, that is because it is equally likely to move in either direction which means that

$$\left.\frac{d\bar{f}(z)}{dz}\right|_{z=0} = 0$$

That means that $c = 0$ and our new equation becomes

$$\frac{d\bar{f}(z)}{\bar{f}(z)} + \frac{1}{2}zdz = 0$$

Then let's isolate the first fraction

$$\frac{d\bar{f}(z)}{\bar{f}(z)} = -\frac{1}{2}zdz$$

Notice that it is differential of the log of the function $\bar{f}(z)$

$$d\ln\bar{f}(z) = -\frac{1}{2}zdz$$

Then integrate to get

$$\int d\ln\bar{f}(z)dz \Rightarrow \ln\bar{f}(z) - \ln\bar{f}(0) = -\frac{z^2}{4}$$

Then exponentiate to get

$$\bar{f}(z) = \bar{f}(0)e^{-\frac{z^2}{4}} \tag{3}$$

Now we have the number of particles at z. Also keep in mind that we set

$$z = \frac{x}{\sqrt{Dt}}, \quad D = \frac{1}{2\tau}\int_{-\infty}^{\infty}\Delta^2\phi(\Delta)d\Delta$$

This says that equation 2 is really the strength of the particles moving.

Now we need to find $\bar{f}(0)$. We do this by applying in our initial conditions. If we consider that we have m particles and it is large enough for the statistics and probability to work out.

$$\bar{f}(0)\int_{\mathbb{R}} e^{-\frac{z^2}{4}dz} = m$$

Now we are going to use a trick, we want to use a u-substitution that will give us a normal density. If we set this for our substitution

$$z = \sqrt{2}u, \quad dz = \sqrt{2}du$$

Then we get

$$\sqrt{2}\bar{f}(0)\int_{\mathbb{R}} e^{-\frac{u^2}{2}}du = m$$

This look like a normal but it is missing the $\sqrt{2\pi}$ so we know that integral must give us

$$\sqrt{2}\bar{f}(0)\sqrt{2\pi} = m$$

Now solve for $\bar{f}(0)$

$$\bar{f}(0) = \frac{m}{\sqrt{4\pi}}$$

Now plug that into eq.3

$$\bar{f}(z) = \frac{m}{\sqrt{4\pi}} e^{-\frac{z^2}{4}}$$

Now we plug that back into eq.2

$$\lambda f(\lambda x, \lambda^2 t) = \frac{1}{\sqrt{Dt}} f\left(\frac{x}{\sqrt{Dt}}\right)$$

Now expand the function to get

$$\lambda f(\lambda x, \lambda^2 t) = \frac{m}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$$

This is giving us the number of particles at location x after t amount of time passes.

Now what we are going to do is use this functions at different x locations. In this case we will let x vary and call this function $\psi(x) = f(x, 0)$. Now we can aggregate all of the locations as

$$f(x,t) = \int_{-\infty}^{\infty} \psi(z) \frac{1}{\sqrt{4\pi Dt}} \cdot e^{-\frac{(x-z)^2}{4Dt}} dz$$

We have now found the solution. We give names to these functions. Everything in exponent is Green's Function and the $\psi(z)$ is the impulse function.

3.2 Analytical Solution

Now we will find the analytical solution to the diffusion equation. Let's start off with

$$\frac{\partial f(x,t)}{\partial t} = D \cdot \frac{\partial^2 f(x,t)}{\partial x^2}$$

Let's set theses initial conditions

$$f(x,0) = f(x), \;\; \forall x \in [0, L]$$

$$f(0,t) = f(L,t) = 0, \;\; \forall t > 0$$

Then we are going to use the separation of variables which assumes $f(x,t) = X(x)T(t)$. And let's plug that into the diffusion equation

$$\frac{1}{D} \cdot \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}$$

Then we can make these systems of equations

$$X''(x) + \lambda X(x) = 0 \tag{4}$$

$$T'(t) + D\lambda T(t) = 0 \tag{5}$$

When we solve the linear ODE and taking into consideration the initial conditions

$$X(x) = C_1 e^{\sqrt{-\lambda x}} + C_2 e^{-\sqrt{-\lambda x}}$$

Now we are going to break it up into how λ changes. For when $\lambda < 0$ we get $C_1 = C_2 = 0$.

Now for the $\lambda = 0$ we get

$$X(x) = C_1 x + C_2$$

Then for the $\lambda > 0$ case we get

$$X(x) = C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x)$$

When we plug in the boundary condition we get

$$X(0) = C_1 = 0$$

$$X(L) = C_2 \sin(\sqrt{\lambda} L) = 0$$

When we solve λ we get

$$\lambda_n = \left(\frac{\pi n}{L}\right)^2$$

Plug will give us a

$$X(x) = C_n \sin\left(\frac{\pi n}{L} x\right)$$

Now working out eq.5 we end up with

$$T'(t) + D\left(\frac{\pi n}{L}\right) T(t) = 0$$

That gives us

$$T(t) = B_n e^{-D\left(\frac{\pi n}{L}\right)^2 t}$$

Now lets put it all together

$$f(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{\pi n}{L} x\right) e^{-D\left(\frac{\pi n}{L}\right)^2 t}$$

Now we need to solve for A_n which becomes

$$A_n = \frac{2}{L} \int_0^L f(\delta) \sin\left(\frac{\pi n}{L} \delta\right) d\delta$$

Then all that becomes

$$f(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{L} \int_0^L \sin\left(\frac{\pi n}{L} \delta\right) d\delta\right) \sin\left(\frac{\pi n}{L} x\right) e^{-D\left(\frac{\pi n}{L}\right)^2 t}$$

4 Diffusion Convection Equation

This is more of an extension of the diffusion equation where we add in a convection / drift. Let's start with our diffusion equation

$$\frac{\partial f(x,t)}{\partial t} = D\frac{\partial^2 f(x,t)}{\partial x^2}$$

The diffusion convection equation becomes

$$\frac{\partial f(x,t)}{\partial t} = D\frac{\partial^2 f(x,t)}{\partial x^2} - \mu\frac{\partial f(x,t)}{\partial x}$$

We can think of this as a force or a flow, so they move randomly but they have a preferred direction. Normally when we thought of the diffusion equation back in section 1. We were finding the taylor series for $f(x+\Delta,t)$. But for this case $f(x+\Delta,t) \neq f(x-\Delta,t)$ that is because there is now a preference.

We can rewrite our diffusion equation to be

$$f(x,t+\tau)dx = dx\int_{-\infty}^{\infty} f(x+\Delta,t)\phi(-\Delta)d\Delta$$

The $\phi(-\Delta)$ happens because of the drift. Then we can get rid of the differentials.

$$f(x,t+\tau) = \int_{-\infty}^{\infty} f(x+\Delta,t)\phi(-\Delta)d\Delta \tag{1}$$

Now we can rewrite each side of the equation as

$$f(x,t+\tau) = f(x,t) + \frac{\partial f(x,t)}{\partial t}\tau$$

$$f(x+\Delta,t) = f(x,t) + \frac{\partial f(x,t)}{\partial x}\Delta + \frac{1}{2}\cdot\frac{\partial^2 f}{\partial x^2}\Delta^2$$

We can plug those into equation

$$f(x,t) + \frac{\partial f(x,t)}{\partial t}\tau = \int_{-\infty}^{\infty}\left(f(x,t) + \frac{\partial f(x,t)}{\partial x}\Delta + \frac{1}{2}\cdot\frac{\partial^2 f(x,t)}{\partial x^2}\Delta^2\right)\phi(-\Delta)d\Delta$$

Then we can break up the integral

$$f(x,t)+\frac{\partial f(x,t)}{\partial t}\tau = \int_{-\infty}^{\infty}\phi(-\Delta)d\Delta+\frac{\partial f(x,t)}{\partial x}\int_{-\infty}^{\infty}\Delta\phi(-\Delta)d\Delta)+\frac{1}{2}\cdot\frac{\partial^2 f(x,t)}{\partial x^2}\int_{-\infty}^{\infty}\Delta^2\phi(-\Delta)d\Delta$$

By rules of probability we know that

$$\int_{-\infty}^{\infty}\phi(-\Delta)d\Delta = 1$$

That gives us

$$f(x,t)+\frac{\partial f(x,t)}{\partial t}\tau = f(x,t)+\frac{1}{2}\cdot\frac{\partial^2 f(x,t)}{\partial x^2}\int_{-\infty}^{\infty}\Delta^2\phi(-\Delta)d\Delta+\frac{\partial^2 f(x,t)}{\partial x^2}\int_{-\infty}^{\infty}\Delta^2\phi(-\Delta)d\Delta$$

We can get rid of the $f(x,t)$ and get

$$\frac{\partial f(x,t)}{\partial t}\tau = \frac{1}{2}\cdot\frac{\partial^2 f(x,t)}{\partial x^2}\int_{-\infty}^{\infty}\Delta^2\phi(-\Delta)d\Delta + \frac{\partial^2 f(x,t)}{\partial x^2}\int_{-\infty}^{\infty}\Delta^2\phi(-\Delta)d\Delta$$

Then we can rewrite this as

$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial^2 f(x,t)}{\partial x^2}\cdot\frac{1}{2\tau}\cdot\int_{-\infty}^{\infty}\Delta^2\phi(-\Delta)d\Delta + \frac{\partial f(x,t)}{\partial x}\cdot\frac{1}{\tau}\int_{-\infty}^{\infty}\Delta\phi(-\Delta)d\Delta$$

Now we are going to use another u-substitution except we are doing this to change what we are integrating to respect with

$$\frac{\partial^2 f(x,t)}{\partial t} = -\frac{\partial =^2 f(x,t)}{\partial x^2}\cdot\frac{1}{2\tau}\int_{\infty}^{-\infty}\Delta^2\phi(\Delta)d\Delta + \frac{\partial f(x,t)}{\partial x}\cdot\frac{1}{\tau}\int_{\infty}^{-\infty}\Delta\phi(\Delta)d\Delta$$

Then to switch those integrating bounds we get

$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial^2 f(x,t)}{\partial x^2}\cdot\frac{1}{2\tau}\int_{-\infty}^{\infty}\Delta^2\phi(\Delta)d\Delta - \frac{\partial f(x,t)}{\partial x}\cdot\frac{1}{\tau}\int_{-\infty}^{\infty}\phi(\Delta)d\Delta$$

We can see that the first integral is the diffusion component

$$D = \frac{1}{2\tau}\int_{-\infty}^{\infty}\Delta^2\phi(\Delta)d\Delta$$

We can call the second integral the drift

$$\mu = \frac{1}{\tau}\int_{-\infty}^{\infty}\Delta\phi(\Delta)d\Delta$$

Now we get the diffusion convection equation

$$\frac{\partial f(x,t)}{\partial t} = D\frac{\partial f(x,t)}{\partial x^2} - \mu\frac{\partial f(x,t)}{\partial x}$$

Lets use this approximation method

$$\begin{aligned}\frac{f(x,t+\tau)-f(x,t)}{\tau} &= D\frac{f(x+\Delta,t)-2f(x,t)+f(x-\Delta,t)}{\Delta^2} - \mu\frac{f(x+\Delta,t)-f(x-\Delta,t)}{2\Delta}\\ &= \frac{2D}{\Delta^2}\left(\frac{f(x+\Delta,t)+f(x-\Delta,t)}{2} - f(x,t)\right) + \frac{\mu}{2\Delta}(f(x-\Delta,t)-f(x+\Delta,t))\end{aligned}$$

The first part shows that the particles diffuse from higher concentration to lower concentration. And the second part shows that location x will gain the particles from the left hand side and lose them to the right hand side at the same rate in accordance with the drift.

5 Diffusion Convection Equation solution

We have the diffusion equation and its solution

$$\frac{\partial f(x,t)}{\partial t} = D\frac{\partial^2 f(x,t)}{\partial x^2}, \quad f(x,t) = \frac{m}{\sqrt{2\pi Dt}}e^{-\frac{x^2}{4Dt}}$$

Now we want to solve the equation for the diffusion convection equation. We need to find the variable transformation, in this case we want the transformation to reflect convection

$$y = x - \mu t$$

Let's find the derivatives

$$\frac{\partial y}{\partial x} = 1, \quad \frac{\partial y}{\partial t} = -\mu$$

Now take the derivatives of $f(x,t)$ and keep in mind that we need the chain rule

$$\begin{aligned}\frac{\partial f(x,t)}{\partial x} &= \frac{\partial f(y,t)}{\partial y} \cdot \frac{\partial y}{\partial x} \\ &= \frac{\partial f(y,t)}{\partial y}\end{aligned}$$

Then take the second derivative

$$\frac{\partial^2 f(x,t)}{\partial x^2} = \frac{\partial^2 f(y,t)}{\partial y^2}$$

Now for the derivative with respect to t via total derivative

$$\begin{aligned}\frac{\partial f(x,t)}{\partial t} &= \frac{\partial f(y,t)}{\partial t} + \frac{\partial f(y,t)}{\partial t} \cdot \frac{\partial y}{\partial t} \\ &= \frac{\partial f(y,t)}{\partial t} - \mu\frac{\partial f(y,t)}{\partial y}\end{aligned}$$

Now plug that into our convection diffusion equation

$$\frac{\partial f(x,t)}{\partial t} = D\frac{\partial^2 f(x,t)}{\partial x^2} - \mu\frac{\partial f(x,t)}{\partial x}$$

$$\Downarrow$$

$$\frac{\partial f(y,t)}{\partial t} - \mu\frac{\partial f(y,t)}{\partial y} = D\frac{\partial^2 f(y,t)}{\partial y^2} - \mu\frac{\partial f(y,t)}{\partial y}$$

$$\Downarrow$$

$$\frac{\partial f(y,t)}{\partial t} = D\frac{\partial^2 f(y,t)}{\partial y^2}$$

Now to write out the solution we can use this fact $f(x,0) = f(y,0)$ so lets write out the solution in terms of y

$$f(y,t) = \frac{m}{\sqrt{4\pi Dt}}e^{-\frac{y^2}{4Dt}}$$

Then use the definition of y to get the final result

$$f(x,t) = \frac{m}{\sqrt{4\pi Dt}}e^{-\frac{(x-\mu t)^2}{4Dt}}$$

Appendix A: Showing that the transformation is invariant

To show that they are invariant we can apply u and v to $f(x,t)$ to make sure that they still hold. That gives us this function $f(\lambda x, \lambda^2 t)$. Now take the derivatives

$$\frac{du}{dx} = \lambda, \quad \frac{du}{dt} = \lambda^2$$

Now take partial derivatives of f and by chain rule

$$\begin{aligned}\frac{\partial f(v,u)}{\partial t} &= \frac{\partial f(v,u)}{\partial u} \cdot \frac{du}{dt} \\ &= \lambda^2 \cdot \frac{\partial f(v,u)}{\partial u}\end{aligned}$$

Now take the other partial derivative

$$\begin{aligned}\frac{\partial f(v,u)}{\partial x} &= \frac{\partial f(v,u)}{\partial v} \cdot \frac{dv}{du} \\ &= \lambda \cdot \frac{\partial f(v,u)}{\partial v}\end{aligned}$$

Now take the partial derivative with respect again for f_x to satisfy the heat equation

$$\begin{aligned}\frac{\partial^2 f(v,u)}{\partial x^2} &= \lambda \cdot \frac{\partial}{\partial x}\left(\frac{\partial f(v,u)}{\partial v}\right) \\ &= \lambda \cdot \frac{\partial^2 f(v,u)}{\partial v^2} \cdot \frac{dv}{du} \\ &= \lambda^2 \cdot \frac{\partial^2 f(v,u)}{\partial v^2}\end{aligned}$$

We are able to set up the diffusion equation under our transformation

$$\frac{\partial f(v,u)}{\partial u} = D \cdot \frac{\partial^2 f(v,u)}{\partial v^2}$$

Then if we plug in u and v we get

$$\frac{\partial f(\lambda x, \lambda^2 t)}{\partial t} = D \cdot \frac{\partial^2 f(\lambda x, \lambda^2 t)}{\partial x^2}$$

Appendix B: Finding the transformation

Let's try and find our transformation using an arbitrary one. If we scale each variable by some constants $a, b \in \mathbb{R}$ giving us

$$v = bx, \quad u = ct \Rightarrow f(v,u) = f(bx, ct)$$

Then we try working out the partial derivatives again, if we start with the t derivative

$$\begin{aligned}\frac{\partial f(v,u)}{\partial t} &= \frac{\partial f(v,u)}{\partial u} \cdot \frac{du}{dt} \\ &= \frac{\partial f(v,u)}{\partial u} \cdot c\end{aligned}$$

And for the x partial derivative which we will have to take twice

$$\begin{aligned}\frac{\partial f(v,u)}{\partial x} &= \frac{\partial f(v,u)}{\partial v} \cdot \frac{du}{dv} \\ &= \frac{\partial f(v,u)}{\partial v} \cdot b^2 \\ \frac{\partial^2 f(v,u)}{\partial x^2} &= b^2 \cdot \frac{\partial}{\partial x}\left(\frac{\partial f(v,u)}{\partial x}\right) \\ &= b^2 \cdot \frac{\partial^2 f(v,u)}{\partial x^2}\end{aligned}$$

Then if we set those equal to each other we see

$$c\frac{\partial f(v,u)}{\partial x} = b^2 \cdot \frac{\partial^2 f(v,u)}{\partial x^2}$$

Then we can see that $b^2 = c$. That is where we get the λ from the transformation.

C. Interchanging the integral (u-substitution)

We have this integral

$$\int_{\mathbb{R}} af(\lambda x, \lambda^2 t)dx$$

If we make this substitution $v = \lambda x$. Let's work it out like how we would work out a normal u-substitution

$$v = \lambda x$$

$$dv = \lambda dx$$

$$dx = \frac{dv}{\lambda}$$

Then plug it into integral

$$\int_{\mathbb{R}} af(v, \lambda^2 t)\frac{1}{\lambda}dv$$