Accounting for Uncertainty in the Markowitz World

An Empirical Study

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Abstract

This thesis studies the effect of accounting for uncertainty in the Markowitz asset allocation. The out-of-sample performance of the asset allocation shows a high sensitivity to errors in the parameter estimates. Several scholars have developed approaches to account for uncertainty in different ways, such as Ledoit and Wolf (2004), Garlappi et al. (2007) and Kan and Zhou (2007).

The empirical study performed, tests the expected out-of-sample performance and the real out-of-sample performance of the different techniques and compares them to alternative allocations.

Strategies accounting for uncertainty are found to improve the Markowitz framework. Furthermore, we can not observe superior performance in the out-of-sample application, when considering longer sample sizes. Also, adding a short-sale constraint is found to improve the original portfolio. Lastly, we observe that simple asset allocations, such as the equally weighted allocation show a respectable performance.

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5.2 Out-of-sample results of Garlappi et al. (2007) 135

Table of Abbreviations

Abbreviation	Explanation
Eq.	Equation
Sec.	Section
Tab.	Table
Fig.	Figure
MV	Mean-Variance
CARA	Constant Absolute Risk Aversion
FOC	First Order Condition
SR	Sharpe Ratio
TO	Turnover
CER	Certainty Equivalent Return

1 Introduction

"Let every man divide his money into three parts, and invest a third in land, a third in business, and a third let him keep by him in reserve."

This quote stems from the more than 1500-year-old Talmud, a central piece of religious literature compiling rabbinic discussions. It might nowadays be considered as one of the first asset allocation strategies established.

The term asset allocation simply describes the way how an investor divides his wealth among numerous assets. Many researchers and professionals tried to find the optimal asset allocation. With more than 1,000 books on amazon.com, that tackle the topic, the result is a broad collection of philosophies about the issue.

When publishing his research about the optimal asset allocation of risky assets in 1952, Markowitz set a milestone. Assuming that the risk of an asset is described by the standard deviation and that covariances depict the interrelations among assets' returns, he found an optimal solution on how to allocate wealth. In this setting, an optimal portfolio consists of a combination of assets, such that it represents the allocation with the lowest risk among all portfolios yielding the same expected return.

However, the framework shows a high sensitivity towards its inputs. In order to apply it, individuals have to rely on estimates of expected returns, variances, and covariances. Estimating these parameters with sufficient precision is challenging, such that the sensitivity represents an issue. Thus, despite being optimized in theory, the model does not necessarily imply good performance in a real-life application. This problem of imprecise estimates is also referred to, by the term parameter uncertainty or ambiguity.

This uncertainty poses a series of questions:

- How can an individual account for the parameter uncertainty in the Markowitz framework?
- Does accounting for uncertainty improve the performance?
- How do optimized allocations compare to alternative strategies?
- Does an increase in data used for estimation, improve the performance?
- Is it possible to reliably predict the performance?
- Does imposing constraints on the allocation improve performance?

In our thesis, we provide an overview of selected asset allocation strategies, that aim to enhance the original, accounting for parameter uncertainty. These alternatives are compared with respect to the original model, as well as to other asset allocation strategies, such as the 1 over N strategy.

While a consistent body of research has been focusing on either the expected performance of an investment strategy or the actual out-of-sample performance, we conduct both analyses, in order to provide a detailed comparison. In terms of expected performance, our thesis also introduces the short-sale constraint on assets, which has so far not been examined in research. Also, we include another extension to a model accounting for parameter uncertainty. Lastly, in order to verify findings, it should be highlighted that the two analyses are conducted on two different sets of data.

The thesis is structured in three parts. In the *first part*, the theoretical framework of the Markowitz model is established and critically evaluated. Then, extensions to the model aiming to produce more robust allocations are being discussed.

The *second part* contains the empirical analysis. First, the methodology and the data used is described and then the results are presented. The analysis evaluates the different asset allocation strategies, firstly in a simulated environment and secondly by using real-world data.

Lastly, the *third part* covers the discussion of the analyses and their findings, as well as the conclusion.

Part I

The Mean-Variance Model, Criticism and its Extensions

2 The Asset Allocation According to Markowitz (1952)

In this section we will introduce the general one-period asset allocation problem and the well known solution offered by Markowitz (1952).

The classic asset allocation problem deals with an investor who wishes to invest her initial wealth W_0 in the financial markets for a given period time. She has access to trade in N risky assets, whose returns are assumed to be normally distributed. The assets' returns can be characterized by the expected rate of return of $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_N)^\top$ and the variance-covariance matrix ($\boldsymbol{\Sigma}$) of returns:[1]

$$\boldsymbol{\Sigma} = \begin{pmatrix} \text{cov}_{1,1} & \ldots & \text{cov}_{1,N} \\ \vdots & \ddots & \vdots \\ \text{cov}_{N,1} & \cdots & \text{cov}_{N,N} \end{pmatrix} \tag{2.1}$$

If we let $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_N)^\top$ denote the vector of weights initially invested in each asset, the expected terminal wealth W of the investor is given by:

$$W = W_0 \left[1 + \boldsymbol{\pi}^\top \boldsymbol{\mu}\right] \tag{2.2}$$

The problem an investor generally faces is how to allocate her wealth among the assets, hence how to choose $\boldsymbol{\pi}$. A solution for the investor can only be found if we know the preferences of her. Preferences are represented by a utility function $U(W)$, which depends on the terminal wealth. In the model of Markowitz, investors are assumed to be both greedy and risk averse. This means that the investor has monotonous preferences which are characterized by her preferring higher wealth over lower and rejecting any risky gamble where the expected payoff is negative or zero. It is assumed that the investor's preferences are represented by the so-called Constant Absolute Risk Aversion (CARA) Utility function, such as the following

$$U(W) = -e^{-kW} \tag{2.3}$$

with first- and second-order derivative,

[1] In the following, note that vectors are depicted with uncapitalized bold letters, and matrices with capitalized bold letters. Vectors are column vectors, if not stated otherwise.

$$U'(W) = ke^{-kW}, \quad U''(W) = -k^2 e^{-kW}$$

the absolute risk aversion is defined as

$$\text{ARA}(W) = -\frac{U''(W)}{U'(W)} = k \tag{2.4}$$

where k is a constant independent of the wealth, hence the name constant absolute risk aversion.

For $k > 0$ the slope of $U(W)$ is positive and decreases in wealth. Thus, fulfilling the upper mentioned assumptions.

With normally distributed returns it can be shown that the terminal wealth W is normally distributed. With its mean $\mu(\boldsymbol{\pi})$ and variance $\sigma^2(\boldsymbol{\pi})$, being defined as

$$\begin{aligned} \mu(\boldsymbol{\pi}) &= \boldsymbol{\pi}^\top \boldsymbol{\mu} = \sum_{i=1}^{N} \pi_i \mu_i \\ \sigma^2(\boldsymbol{\pi}) &= \boldsymbol{\pi}^\top \boldsymbol{\Sigma} \boldsymbol{\pi} = \sum_{i=1}^{N} \sum_{j=1}^{N} \pi_i \pi_j \Sigma_{ij} \end{aligned} \tag{2.5}$$

.

In general, for:[2]

$$x \sim N\left(m, s^2\right)$$

$$E(e^{-kx}) = e^{\left(-km + \frac{1}{2}k^2 s^2\right)}$$

such that if

$$W \sim \mathcal{N}\left(W_0\left(1+\mu\right), {W_0}^2 \sigma^2\right) \tag{2.6}$$

it follows that:

[2] As shown by Munk (2015)

$$E\left[U\left(W\right)\right] = -e^{-kW_0\left(1+\mu-\frac{1}{2}kW_0\sigma^2\right)} \tag{2.7}$$

with k as defined in eq. (2.4).

Given that the CARA function is of the form $-e^{-kW_0(1+x)}$, it is increasing in x. A rational investor then aims to maximize her expected utility. With $kW_0 \equiv \gamma$ the the following maximization problem results:

$$\max_{\boldsymbol{\pi}} E[r] - \frac{\gamma}{2}\sigma^2 \tag{2.8}$$

with $E[r] = \boldsymbol{\pi}^\top \boldsymbol{\mu}$ and $\sigma^2 = \boldsymbol{\pi}^\top \boldsymbol{\Sigma}\boldsymbol{\pi}$, we can write the following optimization problem:

$$\begin{aligned} \max_{\boldsymbol{\pi}} \quad & \boldsymbol{\pi}^\top \boldsymbol{\mu} - \frac{\gamma}{2}\boldsymbol{\pi}^\top \boldsymbol{\Sigma}\boldsymbol{\pi} \\ s.t. \quad & \boldsymbol{\pi}^\top \mathbf{1} = 1 \end{aligned} \tag{2.9}$$

This maximization in eq. (2.8) is also referred to by the term *mean-variance criterion*, where the risk aversion is now given by the parameter γ. The γ represents an investor-dependent level of risk aversion. A higher aversion corresponds to a higher γ, hence a larger penalty for the variance. In the following section, we first introduce the most important results of this theory in a one-fund setting, meaning that investors have the possibility to invest only in risky assets. In section 2.4, a risk-free security will be added to the analysis, extending the model into a two-fund setting.

2.1 Mean-Variance Efficient Portfolios in a One-Fund setting

When investors do not have access to a risk-free asset, Markowitz (1952) introduced an optimal solution to the mean-variance (MV) framework. This solution defines the *efficient frontier*. In a diagram, representing expected portfolio return and standard deviation on the y- and x-axis, respectively, known as risk-return diagram, the frontier can be exemplified as

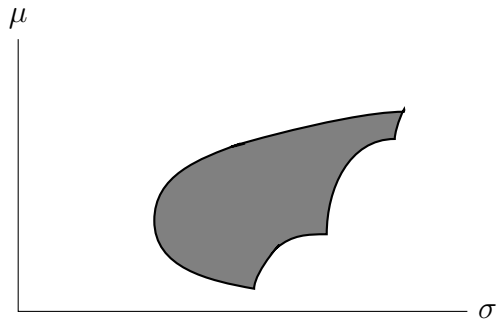


Figure 2.1: Efficient frontier of risky assets

This bullet-shaped curve represents all the combinations of assets that show the lowest possible risk for a given return $\bar{\mu}$. It is obtained by mapping the standard deviation of the mean-variance efficient portfolio against the expected portfolio return.

The model exploits the aggregation of not perfectly correlated assets, which contributes to lowering the standard deviation of the portfolio, as the impact of idiosyncratic / asset-specific risk fades out and also finds an optimal weight for each asset in order to find the lowest possible risk.[3] For a given return $\bar{\mu}$, the portfolio weight vector $\boldsymbol{\pi}$, minimizing the portfolio risk will solve the following constrained optimization problem:

$$\begin{aligned} \min_{\boldsymbol{\pi}} \quad & \boldsymbol{\pi}^\top \boldsymbol{\Sigma} \boldsymbol{\pi} \\ s.t \quad & \boldsymbol{\pi}^\top \boldsymbol{\mu} = \bar{\mu} \\ & \boldsymbol{\pi}^\top \mathbf{1} = 1 \end{aligned} \tag{2.10}$$

where $\boldsymbol{\pi}^\top \mathbf{1} = 1$ describes the constraint of the portfolio weights summing to one, with $\mathbf{1}$ being a unit-vector of dimension $N \times 1$.

Using the method of Lagrange multipliers, the Lagrange function is given by:

$$\mathcal{L}(\boldsymbol{\pi}, \alpha, \beta) = \boldsymbol{\pi}^\top \boldsymbol{\Sigma} \boldsymbol{\pi} + \alpha(\bar{\mu} - \boldsymbol{\pi}^\top \boldsymbol{\mu}) + \beta(1 - \boldsymbol{\pi}^\top \mathbf{1}) \tag{2.11}$$

In order to determine the optimal investment strategy, we need to find $\boldsymbol{\pi}^*$ fulfilling the following first order conditions (FOC).

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\pi}} = 0, \quad \frac{\partial \mathcal{L}}{\partial \alpha} = 0 \quad \frac{\partial \mathcal{L}}{\partial \beta} = 0 \tag{2.12}$$

[3]The correlation ρ is calculated as $\rho_{i,j} = \frac{Cov_{i,j}}{\sigma_i \sigma_j}$

Defining the following constants A, B, C, D as:

$$A = \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}, \quad B = \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1} = \mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}, \quad C = \mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1} \tag{2.13}$$

and

$$D = AC - B^2$$

we obtain the mean-variance efficient portfolio with expected return $\bar{\mu}$ as:

$$\boldsymbol{\pi}(\bar{\mu}) = \frac{C\bar{\mu} - B}{D} \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} + \frac{A - B\bar{\mu}}{D} \boldsymbol{\Sigma}^{-1} \mathbf{1} \tag{2.14}$$

That if substituted into (2.5), results in the portfolio variance:[4]

$$\sigma^2(\bar{\mu}) = \boldsymbol{\pi}(\bar{\mu})^\top \boldsymbol{\Sigma} \boldsymbol{\pi}(\bar{\mu}) = \frac{C\bar{\mu}^2 - B\bar{\mu} + A}{D} \tag{2.15}$$

The $\boldsymbol{\pi}$ vector of weights stated in (2.14), is said to be mean-variance efficient. It is the least risky allocation among all possible portfolios yielding the same expected return $\bar{\mu}$.

2.2 Optimal Portfolio under CARA Utility in a One-Fund Setting

The optimization problem that an investor described by CARA utility faces (as defined in (2.9)), is solved as follows. First setting up the Lagrangian function

$$\mathcal{L} = \boldsymbol{\pi}^\top \boldsymbol{\mu} - \frac{\gamma}{2} \boldsymbol{\pi}^\top \boldsymbol{\Sigma} \boldsymbol{\pi} + \alpha \left(1 - \boldsymbol{\pi}^\top \mathbf{1}\right) \tag{2.16}$$

Inducing the first order conditions

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\pi}} = \boldsymbol{\mu} - \gamma \boldsymbol{\pi} \boldsymbol{\Sigma} - \alpha \mathbf{1} = 0 \tag{2.17}$$

and

$$\frac{\partial \mathcal{L}}{\partial \alpha} = 1 - \boldsymbol{\pi}^\top \mathbf{1} = 0 \tag{2.18}$$

[4]See appendix A.1 for a complete derivation.

solving for $\boldsymbol{\pi}$ yields:[5]

$$\boldsymbol{\pi} = \frac{1}{\gamma}\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu} - \frac{B-\gamma}{C}\mathbf{1}\right) \tag{2.19}$$

In addition to the efficient frontier, the CARA optimizing asset allocations can now be included in the risk-return diagram. The risk and return parameters as well as the efficient frontier were determined based on the monthly return series, that are later being used for the empirical analysis in Part II. With $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ as follows

	$\boldsymbol{\mu}$	Variance-covariance matrix of annualized returns $\boldsymbol{\Sigma}$							
	annualized	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5	Asset 6	Asset 7	Asset 8
Asset 1	2.30%	6.81%	3.03%	2.36%	2.32%	1.79%	2.71%	3.60%	2.02%
Asset 2	6.50%	3.03%	4.96%	2.96%	2.41%	2.04%	2.70%	3.72%	1.99%
Asset 3	8.22%	2.36%	2.96%	3.35%	2.01%	1.75%	2.44%	2.88%	1.85%
Asset 4	5.79%	2.32%	2.41%	2.01%	3.94%	2.30%	2.52%	2.56%	1.59%
Asset 5	6.64%	1.79%	2.04%	1.75%	2.30%	2.40%	2.10%	2.10%	1.21%
Asset 6	5.14%	2.71%	2.70%	2.44%	2.52%	2.10%	4.40%	3.07%	1.92%
Asset 7	6.15%	3.60%	3.72%	2.88%	2.56%	2.10%	3.07%	5.14%	2.13%
Asset 8	7.32%	2.02%	1.99%	1.85%	1.59%	1.21%	1.92%	2.13%	4.50%

Table 2.1: Annualized $\boldsymbol{\mu}$-vector and $\boldsymbol{\Sigma}$ matrix of eight assets chosen

and assuming two investors with CARA utility as described in sec. 2 the optimal allocations depicted in tab. 2.2 result, γ being 1 and 2 respectively.

Asset i	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
π_i, $\gamma = 1$	-92.35%	-27.61%	173.13%	-32.26%	96.97%	-75.67%	12.57%	45.22%
π_i, $\gamma = 2$	-43.50%	-15.75%	97.92%	-17.50%	82.01%	-38.93%	1.91%	33.83%

Table 2.2: Asset allocation in a one-fund setting

	$\gamma = 1$	$\gamma = 2$
$E[r]$	15.07%	11.05%
σ	31.71%	20.04%

Table 2.3: Portfolio characteristics in a one-fund setting; $\gamma = 1, 2$

From these allocations we can already see that the composition changes substantially for two different values of gamma; however, both portfolios represent mean-variance efficient

[5]For detailed derivation, see appendix A.2

portfolios, as they offer the lowest volatility for the given expected return. The optimized portfolios, efficient frontier and the single assets' characteristics are depicted in the risk-return-diagram (fig. 2.2).

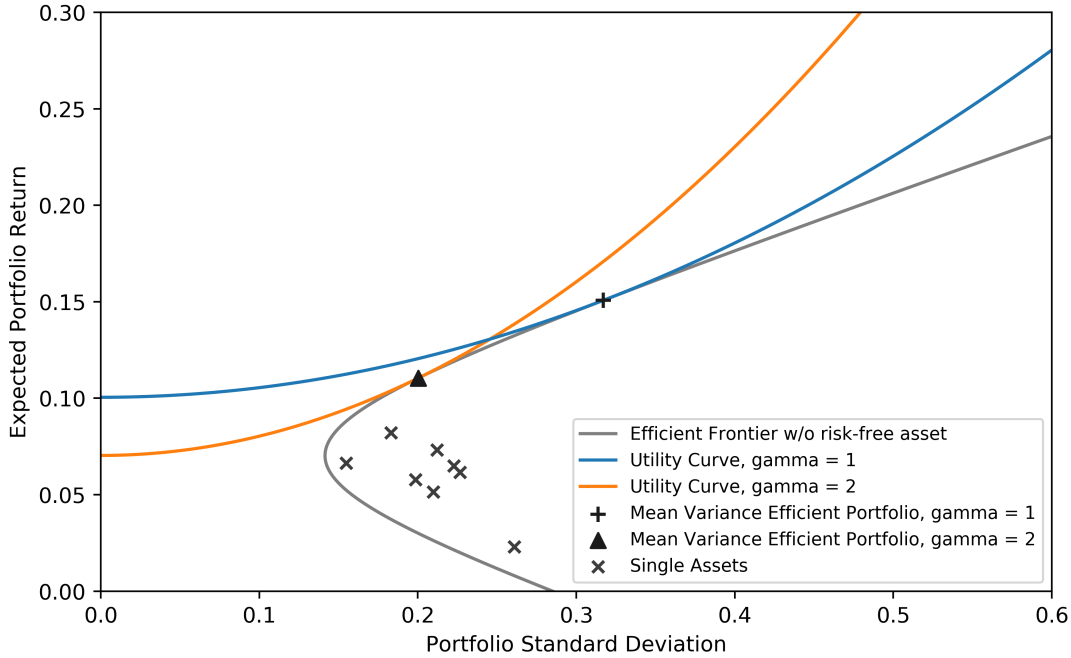


Figure 2.2: Efficient frontier and MV-optimized allocations for $\gamma = 1, 2$

In the diagram we can see, that for both investors the upper calculated allocations represent investments, optimal with respect to their preferences. They are optimal as both utility functions are tangent to the efficient frontier, hence can not be shifted further up. In addition to that, we can see that characteristics of single assets all fall into the inefficient area below the efficient frontier, as an investment in a single asset does not take advantage of diversification.

2.3 The Global Minimum Variance Portfolio

On the efficient frontier there are some portfolios, which are commonly considered due to their special features. The first one is the minimum variance portfolio. As the name implies, it represents the allocation with lowest variance among all portfolios. The problem is depicted by the following optimization:

$$\begin{aligned} \min_{\boldsymbol{\pi}} \quad & \boldsymbol{\pi}^\top \boldsymbol{\Sigma} \boldsymbol{\pi} \\ s.t. \quad & \boldsymbol{\pi}^\top \mathbf{1} = 1 \end{aligned} \tag{2.20}$$

First minimizing the variance in (2.15) and substituting the resulting expected portfolio return into equation (2.14), yields that $\boldsymbol{\pi}_{\min}$ can be written as[6]

$$\boldsymbol{\pi}_{\min} = \frac{1}{C}\boldsymbol{\Sigma}^{-1}\mathbf{1} = \frac{1}{1^\top \boldsymbol{\Sigma}^{-1} 1}\boldsymbol{\Sigma}^{-1}\mathbf{1} \tag{2.21}$$

A special characteristic of the minimum variance portfolio is that it does not depend on the expected return vector. The only one parameter that is critical for the computation, is the variance-covariance matrix. Assets with a low standard deviation and/or those having a low correlation with other assets will have a higher weight than assets that show opposite characteristics. The expected return and variance of the minimum variance allocation can be calculated as

$$\bar{\mu}_{\min} = \frac{B}{C}, \qquad \sigma^2_{\min} = \frac{1}{C} \tag{2.22}$$

Using the same parameters of tab. 2.1, we obtain the following weights for the minimum variance allocation.

Asset i	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
π_i	5.35%	-3.90%	22.72%	-2.74%	67.05%	-2.19%	-8.74%	22.45%

Table 2.4: Minimum variance portfolio weights

Yielding an expected return of 7.02% and a portfolio standard deviation of 14.15%.

Plotting the risk-return-diagram, we find that for a risk aversion of $\gamma = 1,2$ the decision to invest into the minimum variance portfolio is sub-optimal, as higher levels of utility could be achieved, in any allocation within the grey and the dark grey area for $\gamma = 1$ and $\gamma = 2$, respectively. This also holds for all other levels of risk aversion in a CARA utility setting, as the magnitude of the efficient frontier at the minimum variance allocation is undefined

[6]For complete derivation see app. A.3.

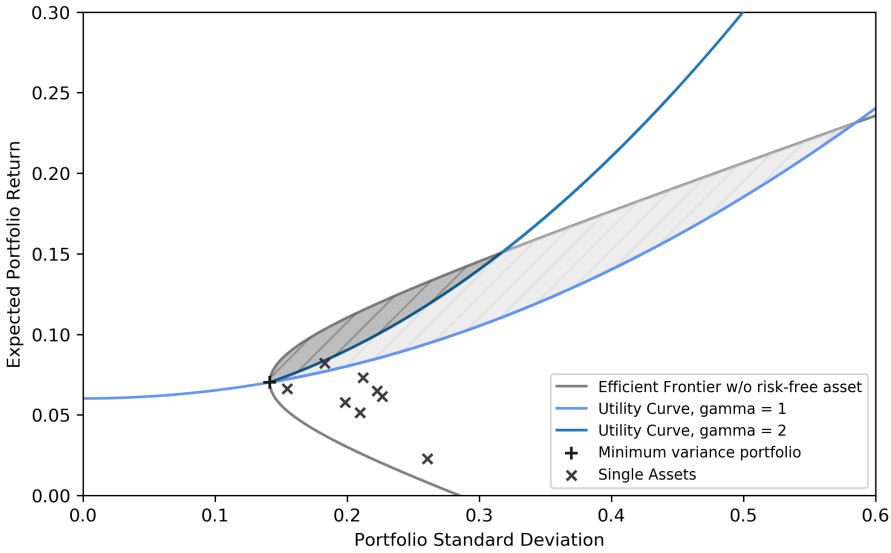


Figure 2.3: Risk-return diagram depicting the minimum variance portfolio

2.4 Mean-Variance Efficient Portfolios in a Two-Fund setting

The original Markowitz approach only considers a combination of risky assets that would minimize the overall volatility for a given expected portfolio return $\bar{\mu}$. When investors are allowed to allocate their wealth to a risk free security with a return r_f, the wealth at the end of the period is then given by[7]

$$W = W_0 \left[1 + r_f + \boldsymbol{\pi}^\top \boldsymbol{\mu}\right] \tag{2.23}$$

The optimal asset allocation then determines the fraction invested in the risk free asset, and the remaining fraction invested in the the risky portfolio or asset. In the following the optimal weight ω_p in the risky portfolio is established.

If we consider an investor with CARA preferences, she intents to optimize the *mean-variance criterion* defined in eq. (2.8) A utility-maximizer invests the amount ω_p in the risky assets and $(1 - \omega_p)$ in the risk free asset. Such that $E[r]$ and σ^2 can be defined as

$$E[r] = \omega_p r_p + (1 - \omega_p)\, r_f, \quad \sigma^2 = \omega_p^2 \sigma_p^2 \tag{2.24}$$

With r_p and σ_p being the return and standard deviation of the risky portfolio respectively.

[7] By modifying eq. (2.2)

The value of ω_p^* that maximizes the expected utility can be found by maximizing the following problem

$$\max_{\omega_p} \quad \omega_p r_p + (1 - \omega_p) r_f - \frac{\gamma}{2} \omega_p^2 \sigma_p^2 \tag{2.25}$$

We find the FOC given by:

$$\frac{\partial f(\omega)}{\partial \omega} = r_p - r_f - \gamma \omega \sigma_p^2 = 0 \quad \Leftrightarrow \omega_p^* = \frac{r_p - r_f}{\gamma \sigma_p^2} \tag{2.26}$$

Due to its convexity, ω_p^* represents the the weight in the risky portfolio for which the utility function is maximized.[8]

The formula shows, that the amount invested in the risky asset is decreasing in the risk aversion coefficient γ and increasing in the excess returns of the risky assets over the risk-free return.

Using data from tab. 2.1, the following linear combinations of risky and risk-free assets result.

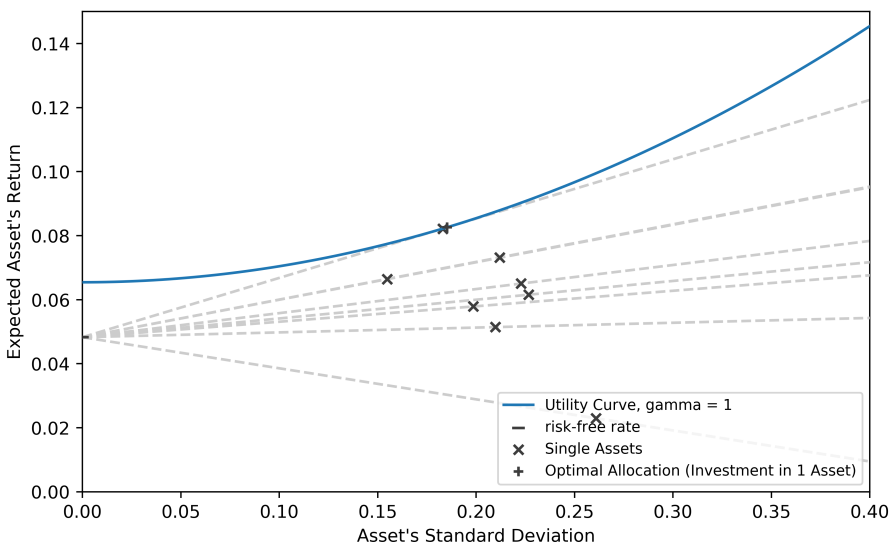


Figure 2.4: Risk-return diagram of two-fund combinations

[8]Second order condition for optimum being maximum:

$$\frac{\partial f(\omega)}{\partial \omega^2} = -\gamma {\sigma_p}^2 < 0$$

The dashed lines represent multiple possible combinations of each risky asset with the risk-free asset. In the hypothetical case where an investor would be restricted to invest in only one of those alternatives, she would invest according to the tangent of her utility with the "highest" dashed line as depicted by the blue curve. This point represents the highest level of utility she could achieve. In this specific case, it means investing 101% in the risky asset and -1% (hence borrowing) in the risk-free asset.

2.4.1 The Global Maximum Sharpe Ratio Portfolio / Tangency Portfolio

Going from the restricted case — depicted in the previous figure (2.4) — to a case considering the mean-variance optimizing setting, there is one portfolio that is of major importance: the tangency portfolio. It represents the allocation into risky assets that in an investor-dependent combination with the risk-free asset maximizes utility for all investors.

The fraction to be invested in the tangency portfolio and the risk free asset is determined as described in equation (2.26). A feature of the tangency portfolio is, that it is the portfolio with the highest sharpe ratio of all portfolios. The Sharpe ratio as defined by Sharpe (1994), is an indicator of risk-adjusted return, used in order to evaluate the return performance of an asset. It measures the excess return of an asset, per unit of risk, measured in terms of standard deviation. It is defined as follows:

$$\mathrm{SR} = \frac{r_i - r_f}{\sigma} \tag{2.27}$$

where r_i is the return of asset i, r_f the risk-free rate and σ the standard deviation of return.

It is depicted by the portfolio that maximizes the slope of the straight line going through the point $(0, r_f)$, which is tangent to the efficient frontier of risky asset, hence the name tangency portfolio. The slope of this line equals the Sharpe ratio.

The weights of the portfolio of risky assets can then be determined as follows:[9]

$$\boldsymbol{\pi}_{\tan} = \frac{1}{B - Cr_f} \boldsymbol{\Sigma}^{-1} \left(\boldsymbol{\mu} - r_f \mathbf{1} \right) \tag{2.28}$$

[9]Note that the composition can be calculated without assuming any utility characteristics of an investor

The first term of the equation $\frac{1}{B-Cr_f}$ is a scalar, whereas the second term $\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f\mathbf{1})$ is an $N \times 1$ vector.

Assets with a high Sharpe ratio most likely have a higher weight in this portfolio, as indicated by the second term $\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f\mathbf{1})$, which represents a somewhat similar expression as the Sharpe ratio; however, it also accounts for the correlation structure among assets.

As already outlined above, an investor with CARA preferences invests ω_p as defined in equation (2.26) in $\boldsymbol{\pi}_{tan}$. Thus, the weight vector including risk aversion can be expressed as

$$\boldsymbol{\pi}^*_{tan} = \omega^* \boldsymbol{\pi}_{\tan} = \frac{\mu_{\tan} - r_f}{\gamma\sigma^2} \cdot \frac{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f\mathbf{1})}{B - Cr_f} \tag{2.29}$$

which can be reformulated as

$$\boldsymbol{\pi}^*_{tan} = \frac{1}{\gamma}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f\mathbf{1}) \tag{2.30}$$

and the weight $(1 - \omega_p) = 1 - \mathbf{1}^\top \boldsymbol{\pi}^*_{tan}$ of the risk-free asset.[10]

Assuming $r_f = 4.81\%$ (average risk-free rate from 1970 to 2018), the following weights for the tangency portfolio composition were obtained

Asset i	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
π_i	-83.74%	-25.52%	159.87%	-29.66%	94.34%	-69.19%	10.69%	43.22%

Table 2.5: Tangency portfolio weights $\boldsymbol{\pi}_{tan}$

For two exemplary investors with risk aversion 1 and 2 the optimal investment would have the following characteristics:

	$\gamma = 1$	$\gamma = 2$
ω_p	109.66%	54.83%
$E[r_p]$	15.29%	10.06%
σ_p	32.34%	16.17%

Table 2.6: Optimal two-fund allocation for $\gamma = 1, 2$

[10] see Appendix A.4 for a detailed derivation

A weight of ω_p is invested into the tangency portfolio and the remainder in the risk-free asset.

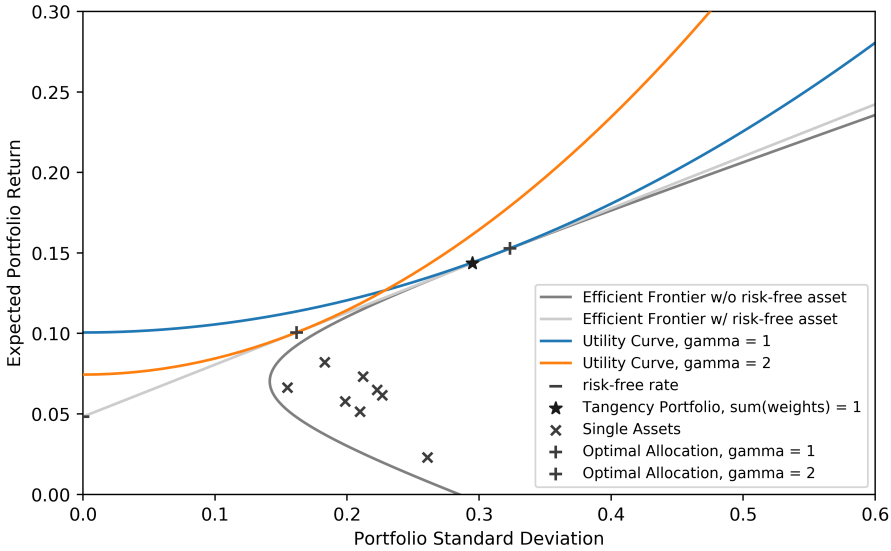


Figure 2.5: Risk-return diagram including capital allocation line

The diagram illustrates the new efficient frontier incorporating the risk-free asset. The weight of 109.66% and 54.83% in the risky asset for $\gamma = 1$ and 2 respectively, can also be seen in the plot. Where the utility function for $\gamma = 1$ is tangent right of the tangency portfolio $\boldsymbol{\pi}_{tan}$ and for $\gamma = 2$ to the left of it.

When comparing this figure to fig. 2.4, we can see that the tangency line represents the highest level of utility any investor could achieve. When comparing to the previous efficient frontier in the one-fund setting (see fig. 2.2), we can see that the efficient frontier including a risk-free assets shows a higher level.

2.5 Critics of Markowitz' Portfolio Theory

The mean-variance optimization according to Markowitz provides an optimal setting that according to Michaud (1989) and Kolm et al. (2014)

- provides a convenient framework to consider important portfolio constraints for the asset allocation
- makes use of the portfolio diversification based on the correlation structure of portfolio constituents

- can be used to control the investor's exposure to risk
- optimally uses information, considering that simplistic ad-hoc allocations — the prevalent method by then — might fail given available information
- enables to calculate the asset allocations quickly, hence being able to adapt to new market conditions in a timely manner

However, this optimum assumes the knowledge of the true input parameters, whose determination with sufficient precision seems to be a challenging if not impossible task. Despite its advantages, the reliance on estimates of the input parameters, therefore, makes the classical model impractical to apply and constituted to a lack of acceptance among investment professionals (Green and Hollifield, 1992; Kolm et al., 2014; Tu and Zhou, 2011). Especially, the bad out-of-sample performance of the mean-variance model gave motivation for research on why the poor performance is apparent as well as how to improve the model and make it more robust.

The sensitivity of the MV-model towards its parameter inputs and the resulting suboptimal allocations have been examined by several researchers. Michaud (1989) came up with the term of MV-portfolios being "estimation-error maximizer". This, according to Fisher and Statman (1997) might serve as an explanation for the puzzle why the MV-model might not be applied in practice.

As previously explained in sec. 2, the calculation of mean-variance efficient portfolios relies solely on the estimates of the expected return vector $\hat{\boldsymbol{\mu}}$ and the variance-covariance matrix of asset returns $\hat{\boldsymbol{\Sigma}}$. While some scholars focused on the impact of parameter uncertainty on the mean-variance efficient portfolio composition such as Michaud (1989) and Best and Grauer (1991), other researchers, such as Jobson and Korkie (1981) and Demiguel et al. (2009), focused on the financial impact with respect to investment results achieved in comparison with other asset allocation strategies.

Another field focused on the expected utility loss induced by using estimators (Chopra and Ziemba, 1993; Kan and Zhou, 2007). In the following a summary of the previously named research will be given.

2.5.1 Elementary Reasons Against the MV-Model

Apart from the problems caused by parameter uncertainty Michaud (1989) found also more simple reasons not to make use of the MV-Model.

He claims that, as portfolios are evaluated against a benchmark, they are not evaluated by which assets they include but rather by which assets are not included as compared to the benchmark.

Moreover, investment professionals who tried to implement the MV-model halted their efforts after realizing, that the resulting strategies were either counter-intuitive or without obvious investment value, and in addition to that not marketable (due to liquidity, short-selling/leverage restrictions).

Nonetheless, he argues that the most influential reason against MV is probably of political nature. According to Michaud, traditional portfolio managers would give away responsibility to quantitatively-oriented specialists in order to implement this rather quantitative investment process. Thus, giving away responsibilities and key roles of the investment process to different people.

In addition to that, the MV-model works with assumptions, that do not necessarily hold in reality, as noted by Munk (2016) and Officer (1972). They especially refer to the following two issues:

- Normally distributed returns rarely exist in reality as return distributions usually show higher moments (Skew, Kurtosis)
- The negative exponential utility function seems unrealistic, as wealthy individuals are willing to give up the same amount as poor individuals, to avoid a risky gamble

Lastly, it should be noted that the MV-model assumes, that a certain level of wealth corresponds to a certain utility, however an investor typically gets utility from consumption over time (Munk, 2016).

While those reasons also contribute to the low acceptance of the mean-variance setting, the focus of this thesis will be on the parameter uncertainty.

2.5.2 Impact of Parameter Uncertainty on the MV-Model

This subsection highlights the frequently referenced literature examining the sensitivity of the MV-model weights with respect to its parameter inputs.

The single optimal solution of the MV-model can be highly misleading. Especially, when taking into account parameter uncertainty this single optimum might have several equivalent portfolios as exemplified by the grey confidence region in the figure below.[11]

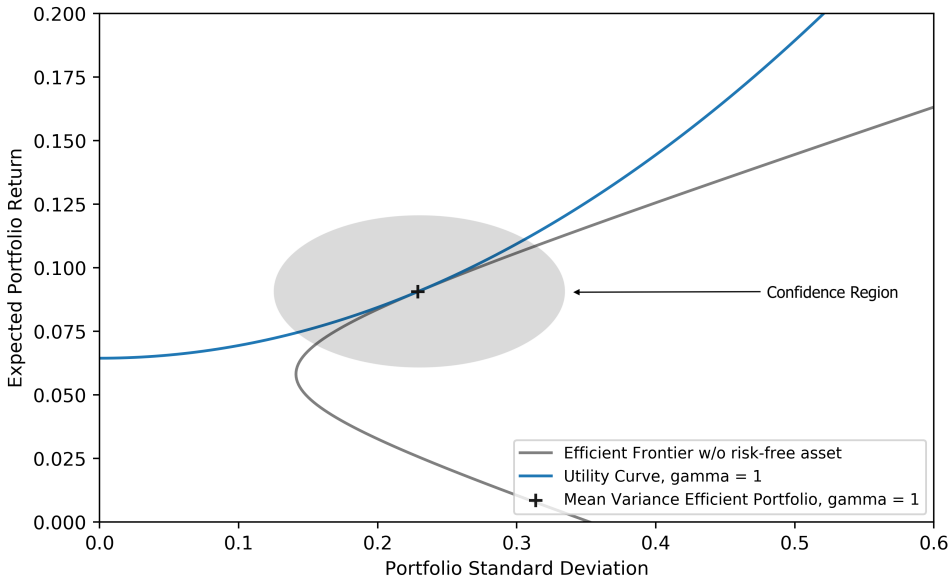


Figure 2.6: Confidence region of the "optimal" portfolio

In addition to the simple reasons to not apply the MV-model provided by Michaud (1989), he also analyzed the issue more broadly. The, according to him, "estimation-error optimizing" MV-Model over-weights those assets that show a high expected return as well as a low variance and covariance with other assets. On the other hand, it under-weights those assets with inverted characteristics. He claims, that especially those boundary cases are also usually those where the estimates are most likely to be faulty. Thus, leads to portfolio weights and expected portfolio characteristics, that significantly understate the actual portfolio risk and overstated the expected return. Where, a large contribution to that, is the use of the sample mean as an estimator for the expected asset returns.

Sensitivity towards the Expected Return Vector Best and Grauer (1991) focused on the impact of the expected return vector on the optimal portfolio weights. For portfolios that have no constraint other than the weight constraint $\boldsymbol{\pi}^\top \mathbf{1} = 1$, they found extremely sensitive portfolio weights with regards to changes in expected returns. When extending the analysis by a non-negativity constraint for portfolio weights, they found that even small changes in the expected return estimate of one asset can lead to the exclusion of several other assets from the portfolio, as those seem less attractive. This implies

[11] compare Michaud (1989)

that there is higher concentration of wealth on fewer assets, decreasing the advantage of diversification.

Starting form 10 assets, Best and Grauer (1991) then go on and analyze how the impact of the expected return figure changes, when more than 10 assets are considered. For 20, 50 and 100 assets they found that the average elasticity of weights increases sharply from 5.79 for the 10 asset case to 218 for the 100 asset case. In other words, an increase in the number of assets leads to highly unstable solutions.

As a way to circumvent the issues related with parameter sensitivity of the model, they propose binding weight constraints as the rule for real-world portfolio application.

In the empirical analysis, a comparison of different ways of incorporating the uncertainty caused by the expected return estimate, is presented.

Sensitivity towards the Variance-Covariance Matrix López de Prado (2016) and Michaud (1989) provide a reasoning in their research why the MV-model performs badly in case of high values for single components of $\hat{\mathbf{\Sigma}}$. Those high covariances — independent of whether they are positive or negative — lead to a very unstable estimate of the inverse $\hat{\mathbf{\Sigma}}^{-1}$. Unstable hereby means, that only small changes in components of $\hat{\mathbf{\Sigma}}$ have a signification impact on the resulting allocation.

Since the solution to an MV-investment decision requires the determination of $\hat{\mathbf{\Sigma}}^{-1}$, as can be seen in sec. 2, the model becomes highly unstable, and portfolio weights strongly vary in case of small changes of components of $\hat{\mathbf{\Sigma}}$. A measure of stability of $\hat{\mathbf{\Sigma}}$ is given by its condition number. It is defined as the ratio of the maximum and minimum eigenvalue of $\hat{\mathbf{\Sigma}}$. The higher this condition number, the less stable $\hat{\mathbf{\Sigma}}$, hence, the more prone the MV-model is to fail in the out-of-sample application. A $\hat{\mathbf{\Sigma}}$ with a high condition number is characterized as "ill-conditioned".

This issue becomes especially critical, when the returns used calculate $\hat{\mathbf{\Sigma}}$ are based on an insufficient amount of historical data. Then, two estimates of $\hat{\mathbf{\Sigma}}$, where the time period considered has only shifted by one period can lead to extreme changes. Introducing a shrinkage estimator aims to obtain a more robust estimate of the variance-covariance matrix. This shrinkage is also applied in the empirical analysis shown in part II.

Another contribution in this area of research is the paper of Kan and Zhou (2007), who found that even though the main source of sensitivity stems from the expected return

estimate, the impact of the variance-covariance matrix increases sharply in the number of assets, especially for small estimation windows.

2.5.3 Illustrating Parameter Uncertainty

In order to visualize the findings with respect to the uncertainty and the impact of either the expected return vector $\boldsymbol{\mu}$, or the variance-covariance matrix $\boldsymbol{\Sigma}$, we perform the following Monte Carlo analysis.

Setting up Monte Carlo Simulation of Stock Return Series Simulating returns implies to establish a set of assumptions about their characteristics. First, it is assumed that stock returns follow a normal distribution with an expected return μ and a standard deviation σ. Random asset returns can then be simulated as

$$r = \mu \Delta t + \sigma \epsilon \sqrt{\Delta t} \tag{2.31}$$

where Δt is the time period over which the return is simulated and ϵ the random shock drawn from a standard normal distribution (Hull, 2012). For independent returns over assets, we could just simulate N-Assets' return series based on their expected return and standard deviation. However, as we consider correlated assets, it has to be taken into account for the construction of the simulation. For several correlated assets the random returns therefore have to be drawn from a multivariate normal distribution with $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ as input parameters

$$\boldsymbol{E} \sim \mathcal{N}_k(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \tag{2.32}$$

Sampling from the multivariate distribution can be done in the following way (Hull, 2012; Wicklin, 2012). First, a matrix $\boldsymbol{X}$ of standard normal independently and identically distributed (IID) numbers is generated.

$$\boldsymbol{X} = \begin{pmatrix} x_{1,1} & \cdots & x_{N,1} \\ \vdots & \ddots & \vdots \\ x_{T,1} & \cdots & x_{T,N} \end{pmatrix} \sim \mathcal{N}(0,1) \tag{2.33}$$

With T being the length of the time series to be simulated and N the number of assets considered. In order to induce correlation into the IID matrix, it has to be multiplied with the Cholesky decomposition of the variance-covariance matrix $\boldsymbol{\Sigma}$.[12] The Cholesky decomposition of $\boldsymbol{\Sigma}$ is defined as the matrix $\boldsymbol{C}$, which multiplied with its transpose will result $\boldsymbol{\Sigma}$

$$\boldsymbol{C}\boldsymbol{C}^{\top} = \boldsymbol{\Sigma} \tag{2.34}$$

The matrix of correlated random numbers can then be written as

$$\boldsymbol{E} = \boldsymbol{X}\boldsymbol{C}^{\top} \tag{2.35}$$

such that correlated returns for T periods and N assets can be expressed by the expected return and a correlated stochastic component $\boldsymbol{E}$ as defined above

$$\boldsymbol{r} = \boldsymbol{\mu} + \boldsymbol{E} \tag{2.36}$$

This way the return of assets is depending on the expected return of assets and a stochastic component, as in the case of a single asset defined in (2.31).

Visualization of Portfolios Based on Simulated Return Data Assuming that we know the true distribution of returns for a given number of assets we will determine the certain optimal portfolio weights $\boldsymbol{\pi}^*$ (compare Munk (2015); Kan and Zhou (2007); Jobson and Korkie (1981)). Those will represent the weights an investor would choose in a two-fund setting investing in the risk free rate and the tangency portfolio. With a "true"/certain $\boldsymbol{\mu}^*$-vector and true $\boldsymbol{\Sigma}^*$-matrix,

$$\boldsymbol{\mu}^* = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_N \end{pmatrix}, \qquad \boldsymbol{\Sigma}^* = \begin{pmatrix} \text{cov}_{1,1} & \dots & \text{cov}_{1,N} \\ \vdots & \ddots & \vdots \\ \text{cov}_{N,1} & \cdots & \text{cov}_{N,N} \end{pmatrix} \tag{2.37}$$

[12]The Cholesky decomposition is "a method of sampling from a multivariate normal distribution" (Hull, 2012).

the certain efficient frontier and tangency portfolio $\boldsymbol{\pi}^*$ in a two-fund setting are being calculated as described in Section 2.4.1.

In a next step, correlated monthly returns of 8 assets, over 60 month are drawn from the multivariate normal distribution, calibrated based on the certain parameters $\boldsymbol{\mu}^*$ and $\boldsymbol{\Sigma}^*$.[13]

Those correlated return series are then used to determine the input parameters for the tangency portfolio weights. The calculation of input parameters follows the plain vanilla approach of determining the unbiased estimators

$$\hat{\boldsymbol{\mu}} = \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{r_t}, \qquad \hat{\boldsymbol{\Sigma}} = \frac{1}{T-1}\sum_{t=1}^{T} \left(\boldsymbol{r_t} - \hat{\boldsymbol{\mu}}\right)\left(\boldsymbol{r_t} - \hat{\boldsymbol{\mu}}\right)^\top$$

Based on these estimates, the weights $\hat{\boldsymbol{\pi}}$ of the tangency portfolio are calculated. Also, two different figures of portfolio return r_p and standard deviation σ_p are calculated. The first pair numbers describe the portfolio characteristics calculated based on estimated weights and true parameters,

$$\begin{aligned} \tilde{r}_p\left(\hat{\boldsymbol{\pi}}, \boldsymbol{\mu}^*\right) &= \hat{\boldsymbol{\pi}}^\top \boldsymbol{\mu}^* \\ \tilde{\sigma}_p\left(\hat{\boldsymbol{\pi}}, \boldsymbol{\Sigma}^*\right) &= \sqrt{\hat{\boldsymbol{\pi}}^\top \boldsymbol{\Sigma}^* \hat{\boldsymbol{\pi}}} \end{aligned} \tag{2.38}$$

while the second pair of numbers instead, bases its calculation on the estimated parameters

$$\begin{aligned} \hat{r}_p\left(\hat{\boldsymbol{\pi}}, \hat{\boldsymbol{\mu}}\right) &= \hat{\boldsymbol{\pi}}^\top \hat{\boldsymbol{\mu}} \\ \hat{\sigma}_p\left(\hat{\boldsymbol{\pi}}, \hat{\boldsymbol{\Sigma}}\right) &= \sqrt{\hat{\boldsymbol{\pi}}^\top \hat{\boldsymbol{\Sigma}} \hat{\boldsymbol{\pi}}} \end{aligned} \tag{2.39}$$

Thus, the first figure is telling us what the actual (true) characteristics of the estimated asset allocation are. The latter, on the other hand, provides us with the parameters, that an investor assumes to achieve, when investing according to her calculated allocation and estimation.

In the following risk-return plot, we show the results for those two figures, where yellow dots depict $\hat{r}_p$ and $\hat{\sigma}_p$ and the turquoise dots depict $\tilde{r}_p$ and $\tilde{\sigma}_p$. Also, the "true" efficient frontier, with (grey) and without (black) the risk-free asset.

[13] Parameters used as the inputs for the simulation, are the same as have been used for fig. 2.2

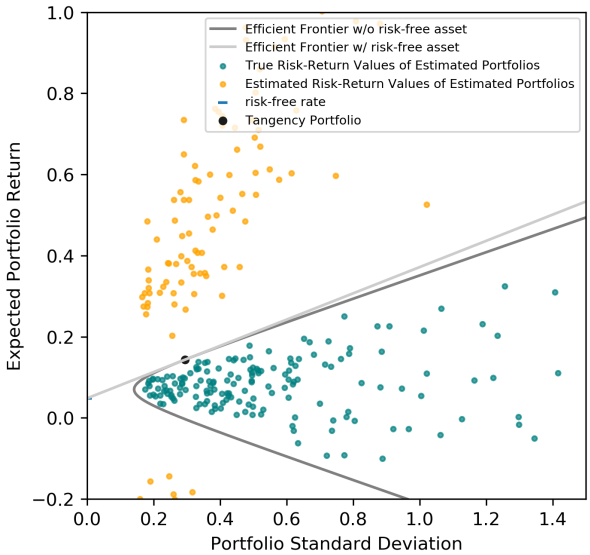


Figure 2.7: Inefficient asset allocations resulting from estimation error

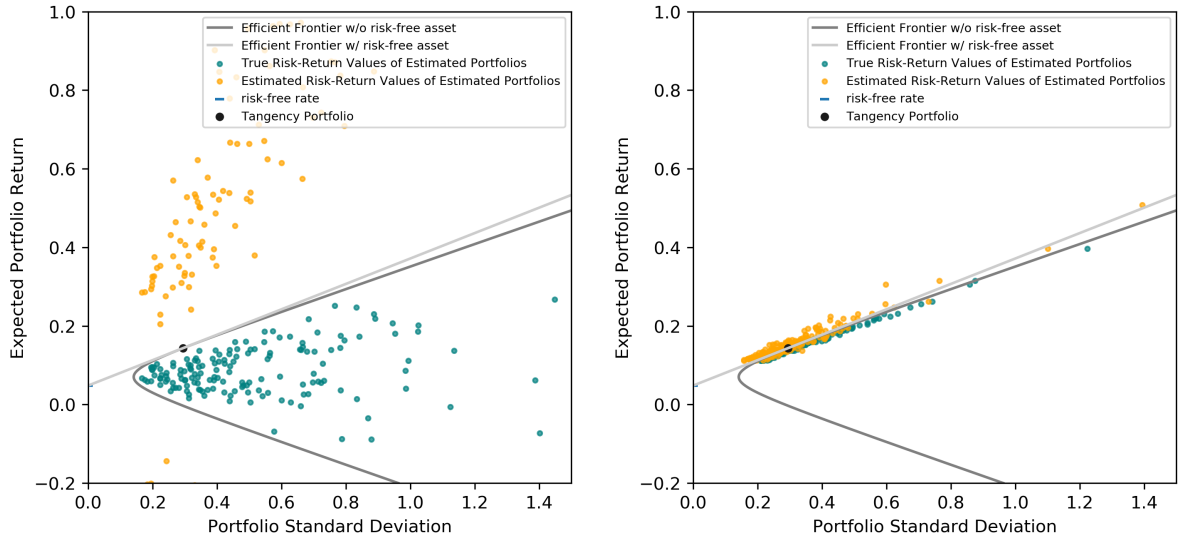
We can see that the portfolios based on the estimated parameters would make investors believe in much more optimal investment decisions than they actually are. Estimated risk and return figures of optimal portfolios significantly overstate expected portfolio returns and understate the portfolio's risk. Considering the true parameters, they obviously all represent inefficient allocations, as they fall in the inefficient area below the efficient frontier.

Another result of the simulation confirms research about the impact of each of the parameters ($\hat{\boldsymbol{\mu}}$, $\hat{\boldsymbol{\Sigma}}$). In fig. 2.8a the portfolios weights are calculated as $\bar{\boldsymbol{\pi}}\left(\hat{\boldsymbol{\mu}}, \boldsymbol{\Sigma}^*\right)$, hence with estimated expected returns and certain variance-covariance matrix.

$$\begin{aligned} \bar{r}_p\left(\bar{\boldsymbol{\pi}}, \hat{\boldsymbol{\mu}}\right) &= \bar{\boldsymbol{\pi}}^\top \hat{\boldsymbol{\mu}} \\ \bar{\sigma}_p^*\left(\bar{\boldsymbol{\pi}}, \boldsymbol{\Sigma}^*\right) &= \sqrt{\bar{\boldsymbol{\pi}}^\top \boldsymbol{\Sigma} \bar{\boldsymbol{\pi}}} \end{aligned} \tag{2.40}$$

Portfolios computed in that way tend to be much further away from the efficient frontier and the tangency portfolio, as compared to the second case (fig. 2.8b), where $\boldsymbol{\mu}$ is known and $\boldsymbol{\Sigma}$ unknown, hence, $\check{\boldsymbol{\pi}}\left(\boldsymbol{\mu}^*, \hat{\boldsymbol{\Sigma}}\right)$ with portfolio characteristics are calculated as follows.

$$\begin{aligned} \check{r}_p\left(\check{\boldsymbol{\pi}}, \boldsymbol{\mu}^*\right) &= \check{\boldsymbol{\pi}}^\top \boldsymbol{\mu}^* \\ \check{\sigma}_p\left(\check{\boldsymbol{\pi}}, \hat{\boldsymbol{\Sigma}}\right) &= \sqrt{\check{\boldsymbol{\pi}}^\top \hat{\boldsymbol{\Sigma}} \check{\boldsymbol{\pi}}} \end{aligned} \tag{2.41}$$



(a) $\bar{r}_p$, $\bar{\sigma}_p$ **(b)** $\check{r}_p$, $\check{\sigma}_p$

Figure 2.8: Comparison of estimation efficiency holding one parameter constant

This small simulation already shows the significant impact that the estimation of the expected return vector has on the optimal asset allocation, and provides motivation to improve the parameter estimate or to include this uncertainty into the estimation of the optimal asset allocation.

2.5.4 Underperformance of MV-Model Out-of-Sample

In the previously mentioned research, the focus was mainly put on the portfolio composition and related issues resulting from estimation errors. However, also the financial impact that such sub-optimal portfolios can have, was examined.

In 1985, Jobson and Korkie already showed that a simple asset allocation strategy, such as an equally-weighted portfolio, can have a higher Sharpe ratio than the mean-variance optimized portfolio. In addition to that, they found that the hypothesis of equal performance of both strategies can not be rejected.

Similar findings were made by Demiguel et al. (2009), who showed that the out-of-sample Sharpe Ratio of a $1/N$-Strategy — meaning an equally weighted asset strategy — can have a higher out-of-sample Sharpe ratio. Concluding further that the advantages of "perfect" diversification are completely eroded by the errors induced by estimating the expected return vector and the variance-covariance matrix. In addition to that, they could not find a portfolio being able to significantly outperform the $1/N$-Allocation.

While Kolm et al. (2014) on the other hand, did not perform a study on the underperformance they still criticize that the model does not take care of transaction costs. Such that even though the model would be based on parameter inputs of good quality, it might still underperform when including transaction costs, because of eventual too high turnover rates.[14]

2.5.5 Expected Utility Loss due to Parameter Estimation

The resulting utility loss due to suboptimal portfolios was examined by Chopra and Ziemba (1993); Fabozzi et al. (2010) and Kan and Zhou (2007). Kan and Zhou and Fabozzi et al. showed that the utility loss must be strictly positive in the case of using estimators. Whereas, Chopra and Ziemba (1993) measured the effect of errors in the estimation of both $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\Sigma}}$ in terms of the mean percentage cash-equivalent utility loss, which stems from the sub-optimal investment decision. This loss characterizes the lower utility achieved, by investing in an optimal portfolio, where the parameter inputs show an error and is a measure of the suboptimal portfolios under uncertainty such as the ones visualized before in sec. 2.5.3. With increasing error of the estimation, they found larger losses with respect to the expected return, than for variances or covariances (see fig. 2.9). This result is also coherent with the visualization shown before.

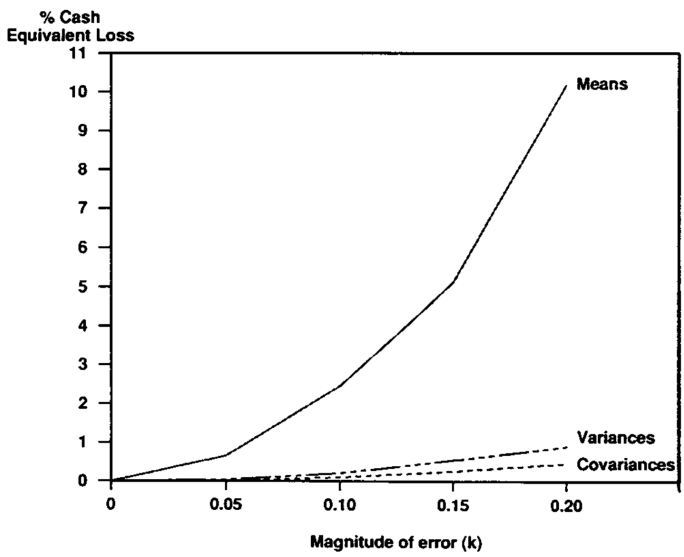


Figure 2.9: Percentage cash-equivalent utility loss (Chopra and Ziemba, 1993)

[14]The turnover of a strategy is a measure of the changes in the portfolio weights due to rebalancing. For the empirical analysis it is defined in sec. 4.7.1.

Kan and Zhou (2007) derive mathematical proof for the utility loss. Confirming the findings that the maximum likelihood estimators $\hat{\boldsymbol{\mu}}$ for $\boldsymbol{\mu}$ induces greater utility loss than the risk estimator $\hat{\boldsymbol{\Sigma}}$ for $\boldsymbol{\Sigma}$ for most of the cases, as can be seen in fig. 2.10.

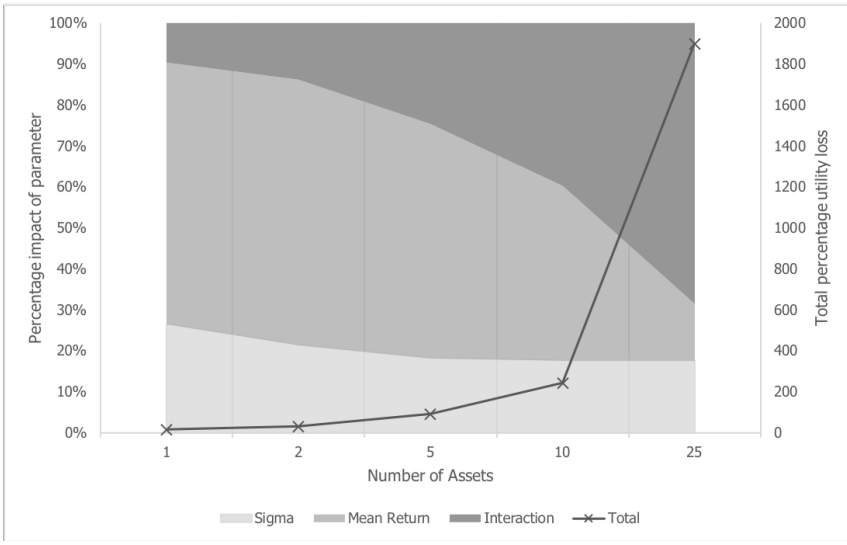


Figure 2.10: Utility loss caused by estimation error

With an increasing number of assets considered, the number of parameters to be estimated is clearly higher. Therefore, the relative impact of the estimation error in the expected return vector decreases up until it is lower than the relative impact of the variance-covariance matrix. Also, we can see that the total percentage utility loss grows exponentially with the number of assets.

3 Improving the Mean-Variance Setting

After discussing the disadvantages of the MV-model and the need for a more robust setting, different approaches are introduced. These approaches are then evaluated in the empirical analysis.

Since the initial development of the mean-variance setting, an extensive amount of research has been conducted on how to deal with the issue of uncertainty. However, these approaches differ among researchers. Early techniques on an improvement of parameter estimates were already established in the 1980s.

Scholars, such as Jobson and Korkie (1981); Frost and Savarino (1986); Jorion (1986); Ledoit and Wolf (2004) and Kan and Zhou (2007) tried to find different estimators, that lead to a better out-of-sample performance.

Harlow (1991) shows, that alternative measurements of risk, which are coherent with a different perception of risk by investors — such as lower partial moments — might lead to a better out-of-sample performance. In particular, the risk in his setting is represented by the standard deviation of returns measure which only takes the downside risk of an asset into account.

A different approach was developed by Garlappi et al. (2007). Their improvement of the original mean-variance model extends the setting by introducing a parameter of aversion towards uncertainty. Thus, actively considering uncertainty in the model. In this setting, an optimal solution is established.

In the following we explain the three different approaches

3.1 Improving Parameter Estimation

The maximum likelihood estimators for both parameters represent good estimators from a statistical point of view. Nonetheless, research showed that they perform rather poor in an out-of-sample application.[15] We will therefore give an overview about research that aims at improving the input parameters.

[15] see Section 2.5.4

3.1.1 The Traditional Bayesian Approach

The following explanation of the traditional Bayesian approach is based on the theoretical foundations provided by Garlappi et al. (2007) and Kan and Zhou (2007).

The classic framework to derive expected utility can be depicted as follows:[16]

$$E\left[U\left(r|\,\Theta\right)\right] = \int U\left(r\right) g\left(r|\,\Theta\right) dr \tag{3.1}$$

Thus, the expected utility is an integral over the product of the utility, given return r, and the probability of the realization of r. Where, the probability of realization depends on a set of given parameters Θ. With uncertainty however, Θ is unknown. The classical Bayesian approach, established by Savage (1954), assumes that the investor has a prior belief about the parameters Θ. This prior belief can be interpreted as an assumption about what distribution returns generally follow. In a next step, the investor derives a conditional prior $p\left(\Theta|X\right)$, including historical returns $X = (r_1, \ldots, r_T)$. Thus, the expected utility given the realizations X, can be expressed as:

$$E\left[U\left(r|\,X\right)\right] = \int U\left(r\right)\left(g\left(r|\,\Theta\right) p\left(\Theta|X\right)\right) dr \tag{3.2}$$

In the classical Bayes estimation, the probability of an event is not just determined by its frequency over time, but rather it is interpreted as a level of trust in the occurrence of an event. The main feature of the Bayesian approach to predictions is to combine information of historical data with the investor's prior beliefs about the distribution of assets returns.

This predictive distribution will influence the investor's way of allocating wealth. It is interesting to note, that while there is estimation risk (represented in the above equation by the likelihood $p\left(\Theta|X\right)$), the investor is still neutral to parameter uncertainty. In other words, the investor's expected utility is not affected by whether the uncertainty comes from the distribution of returns conditional to parameters $g\left(r|\,\Theta\right)$ or from the likelihood of these parameters given the data $p\left(\Theta|X\right)$.

[16] The expected utility defined in eq. (2.7), is the analytic solution to this depiction.

3.1.2 Kan Zhou Application of Bayesian Estimation

In addition to finding analytic proof for the expected utility loss induced by using parameter estimates, Kan and Zhou (2007) also showed how different estimators lead to lower expected utility losses. The expected utility loss function, also called the risk function is defined as

$$\rho\left(\boldsymbol{\pi}^{*}, \hat{\boldsymbol{\pi}}\right) = E\left[L\left(\boldsymbol{\pi}^{*}, \hat{\boldsymbol{\pi}}\right)\right] = U\left(\boldsymbol{\pi}^{*}\right) - E\left[\tilde{U}\left(\hat{\boldsymbol{\pi}}\right)\right] \tag{3.3}$$

with $\boldsymbol{\pi}^{*}$ being the true weight vector and $\hat{\boldsymbol{\pi}}$ the estimated weight vector and $U\left(\boldsymbol{\pi}\right)$ being defined as the utility function of the following shape:

$$\mathrm{U}\left(\boldsymbol{\pi}\right) = E[r] - \frac{\gamma}{2}\sigma^{2} \tag{3.4}$$

This definition of the utility is at the same time the expression of the certainty equivalent return, which will later be explained in sec. 4.6. It allows for a comparison of losses among investors with different risk aversion, as it is not a utility-function in a classical sense, such as CARA utility defined in sec. 2.

The weights, to be invested in the risky assets, are set according to the tangency portfolio rule.[17] For the case of estimating the parameters based on their maximum likelihood (ML) estimators, it follows that $\hat{\boldsymbol{\pi}}$ is the ML estimator of $\boldsymbol{\pi}^{*}$.[18]

Mean-Variance Efficient Portfolios in a Three-Fund setting The strategies presented so far have considered two cases, when only risky assets alone are available (one-fund setting), or when both risky assets and a risk-free security are available for investment (two-fund setting). The main criteria has been to find an asset allocation that a priori maximizes the expected utility of the investor.

This leads to the problem previously described in sec. 2.5.5, that the expected utility arising from the obtained allocation is lower than the utility, obtained when the weights

[17] compare Equation (2.28)

[18] ML estimators of mean excess returns and excess return variance-covariance matrix:

$$\boldsymbol{r}_{t}^{e} = \boldsymbol{r}_{t} - \mathbf{1} r_{f,t} \quad \hat{\boldsymbol{\mu}} = \frac{1}{T}\sum_{t=1}^{\top} \boldsymbol{r}_{\boldsymbol{t}}^{\boldsymbol{e}}, \qquad \hat{\boldsymbol{\Sigma}} = \frac{1}{T}\sum_{t=1}^{\top} \left(\boldsymbol{r}_{\boldsymbol{t}}^{\boldsymbol{e}} - \hat{\boldsymbol{\mu}}\right)\left(\boldsymbol{r}_{\boldsymbol{t}}^{\boldsymbol{e}} - \hat{\boldsymbol{\mu}}\right)^{\top}$$

are derived based on certain parameters. Kan and Zhou (2007) define the difference between utilities as utility loss.

The three-fund-allocation combines an additional risky portfolio with the two fund setting. It is presented as a possible alternative in order to mitigate the problem of uncertain parameter estimates. The additional portfolio is the global minimum variance portfolio as introduced in sec. 2.3.

A somewhat comparing approach was already established by Jorion in 1986, who combined bayesian estimation of the parameters in a three fund setting. According to Kan and Zhou (2007), the solution reduces uncertainty due to non perfect correlation among the estimation errors in each of the risky portfolios.

The weights of the resulting portfolio are a function of two new variables defined as c and d, which are constants to be chosen optimally, such that the expected certainty equivalent return is maximized as

$$E\left[U\left(\hat{\boldsymbol{\pi}}\left(c,d\right)\right)\right] = E[\hat{\boldsymbol{\pi}}\left(c,d\right)]^{\top}\boldsymbol{\mu} - \frac{\gamma}{2}E\left[\hat{\boldsymbol{\pi}}(c,d)^{\top}\boldsymbol{\Sigma}\hat{\boldsymbol{\pi}}\left(c,d\right)\right] \tag{3.5}$$

The three fund asset allocation according to Kan and Zhou is set up as

$$\hat{\boldsymbol{\pi}} = \hat{\boldsymbol{\pi}}\left(c,d\right) = \frac{1}{\gamma}\left(c\hat{\boldsymbol{\Sigma}}^{-1}\left(\hat{\boldsymbol{\mu}} - r_f\mathbf{1}^{\top}\right) + d\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}\right) \tag{3.6}$$

where $\hat{\boldsymbol{\mu}}$ is the expected return estimate. The weight of $\omega_p = \hat{\boldsymbol{\pi}}^{\top}\mathbf{1}$ is then invested in the risky assets and the weight $1 - \omega_p$ in the risk free asset.

After defining the optimal c and d Kan and Zhou (2007) show that eq. (3.6) can be written as[19]

$$\hat{\boldsymbol{\pi}} = \frac{c_3}{\gamma}\left[\left(\frac{\hat{\psi}_a^2}{\hat{\psi}_a^2 + \frac{N}{T}}\right)\hat{\boldsymbol{\Sigma}}^{-1}\left(\hat{\boldsymbol{\mu}} - r_f\mathbf{1}\right) + \left(\frac{\frac{N}{T}}{\hat{\psi}_a^2 + \frac{N}{T}}\right)\hat{\mu}_g\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}\right] \tag{3.7}$$

From the numerator of the second fraction, it can be seen that the total position in the minimum variance portfolio is increasing in N and decreasing in T. This implies that if either the number of assets considered increases, or the estimation window length decreases, the allocation shrinks towards the minimum variance portfolio.

[19] see Appendix C for definition of c_3, $\hat{\psi}_a^2$, and $\hat{\mu}_g$

3.1.3 Shrinking the Variance-Covariance Matrix

Another bayesian approach was developed by Ledoit and Wolf (2004). While the main source of sensitivity is caused by the expected return vector, also the variance-covariance matrix induces parameter uncertainty into MV-Setting. This uncertainty induced by the estimate of $\boldsymbol{\Sigma}$ gave motivation to improve the estimate.

The simple maximum likelihood estimator of $\hat{\boldsymbol{\Sigma}}$ is consistent. This means that its expected value is equal to the true covariance matrix of the population. However, it is not necessarily a robust estimator. This stems from the number of degrees of freedom of the covariance matrix, as suggested by Michaud (1989). The consequence is, that applying the MV-model in order to find an optimal asset allocation leads to unstable results.

Ledoit and Wolf, found that their proposed shrinkage improves performance measures, such as reducing the tracking error and increasing the information ratio. A shrinkage approach therefore helps making estimates of the variance-covariance matrix more robust, meaning less sensitive towards extreme single components of the matrix as those are being "shrunk".

With regard to the covariance matrix, the extreme values are pulled towards the central diagonal. By denoting the sample covariance matrix with $\hat{\boldsymbol{\Sigma}}$ and a structured estimator by $\boldsymbol{F}$, also referred to as target the shrunk matrix $\hat{\boldsymbol{\Sigma}}_{Shrunk}$ can be mathematically described as follows:

$$\hat{\boldsymbol{\Sigma}}_{Shrunk} = \delta \boldsymbol{F} + (1 - \delta)\hat{\boldsymbol{\Sigma}} \tag{3.8}$$

where $0 < \delta < 1$ is the shrinkage coefficient. This technique requires to chose the structured estimator and the shrinkage constant. The structured estimator, that Ledoit and Wolf (2004) use is the constant correlation model. This setting assumes that all assets show the same correlation with each other. The elements of the target matrix are defined as follows:

$$f_{ii} = \sigma_{ij}, \qquad f_{ij} = \bar{\rho}\sqrt{\sigma_{ii}\sigma_{jj}} \tag{3.9}$$

where $\bar{\rho}$ is the average sample correlation

$$\bar{\rho} = \frac{2}{(N-1)N} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \rho_{ij}, \quad \rho_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}} \tag{3.10}$$

The choice of the optimal shrinkage constant $\hat{\delta}^*$ is the one that minimizes the expected distance between the shrinkage estimator and the true covariance matrix.[20]

This minimization problem is set up as follows

$$\begin{aligned} &\min_{\delta} E[||\boldsymbol{\Sigma}^* - \hat{\boldsymbol{\Sigma}}||^2] \\ &s.t. \quad \boldsymbol{\Sigma}^* = \delta \boldsymbol{F} + (1-\delta)\hat{\boldsymbol{\Sigma}} \end{aligned} \tag{3.11}$$

The optimal shrinkage resulting from the optimization is then expressed as[21]

$$\hat{\boldsymbol{\Sigma}}_{Shrunk} = \hat{\delta}^* \boldsymbol{F} + \left(1 - \hat{\delta}^*\right) \hat{\boldsymbol{\Sigma}} \tag{3.12}$$

3.2 Asset Allocation in a Downside Risk Setting

Among the several notions of risk, the results of Harlow (1991) show that the widely used measures such as variance and standard deviation, that are popular in the traditional mean-variance setting, present flaws in presence of multiple constraints, as the assumptions that support the variance as an adequate measure of risk are restrictive.

In the mean-variance setting, the returns are assumed to be normally distributed. This implies that the only dispersion measure that is of interest to the investor is the variance.

However, some investors might be more interested in the dispersion of the returns around a set target return that is different from the mean return as is assumed in the mean-variance setting. In other words, the variance does not properly represent a measure of risk for the investor.

An asymmetric measure of risk could be a more adequate way to measure the investor's view on risk. This implies not focusing on the whole distribution of returns, but only on a selected portion of it, which in a downside-risk setting is the left tail. The resulting

[20] The measure of distance is defined as the squared Frobenius norm. It is the sum of squared elements of a matrix

[21] For solution and proof, see Ledoit and Wolf (2004)

measures of risk are known as *lower partial moments* (LPM). As shown by Libby and Fishburnt (1977) and Kahneman and Tversky (1979), individuals weight losses stronger than gains.[22]

Constructing an efficient portfolio in this setting means to improve the risk/return trade-off within a downside-risk framework. Generally, the n-lower partial moments are defined as:

$$LPM = \sum_{r_p<\tau}^{\tau} p_p(\tau - r_p)^n \tag{3.13}$$

with τ being the target rate, r_p being the returns of the portfolio and p_p the probability of occurrence.[23] For $n = 0, 1, 2$ we obtain the following

- For $n = 0$, the probability of the return being lower than the target rate τ
- For $n = 1$ the target shortfall, which is the expected value of the deviation of the returns below the target τ
- For $n = 2$, the target semivariance is obtained, i.e. the expected value of the squared deviation of returns below the target.

By measuring returns below a target, the LPM can account for higher moments of the return distribution.

The optimization problem is comparable to the Markowitz mean-variance setting, with the difference that here the measure of risk that is minimized is the target semivariance LPM_2. With τ being the selected target, the portfolio optimization problem can be then defined in the following way

$$\begin{aligned} \min_{\boldsymbol{\pi}} \sum_{r_p<\tau}^{\tau} p_p(\tau - r_p)^2 \\ \boldsymbol{\pi}^\top \mathbf{1} = 1 \end{aligned} \tag{3.14}$$

Since there is no closed form solution to the problem, we need to keep in mind that it relies on numerical solvers in order to determine $\boldsymbol{\pi}^*$.

[22]This general result also known as prospect theory explains for which Kahneman was awarded the Nobel memorial prize in economics.

[23]The probability of occurrence is assumed to be $1/T$, with T being the length of the estimation window

3.3 Introducing Ambiguity Aversion into the MV-Setting

A different way to take into account parameter uncertainty in the mean-variance setting is established by Garlappi et al. (2007). The main motivation to develop an extension, is to obtain a model that accounts for an aversion towards parameter uncertainty.

Building on the classic Mean-Variance model, Garlappi et al. develop a setting in which one can account for an ambiguity aversion of an investor towards this parameter uncertainty, focusing mainly on the estimation of expected returns.

First the theoretical concepts of ambiguity and ambiguity aversion are laid out before introducing the extension presented by Garlappi et al. (2007).

3.3.1 Ambiguity and Ambiguity Aversion

Ambiguity or uncertainty describe a situation, where an individual does not know the probability distribution of an outcome. An individual is said to be ambiguity averse, when she expresses the preference to know the distribution over not knowing the distribution.

According to Ellsberg (1961) even in a situation where an individual does not know the probability distribution of an uncertainty she faces, the individual behaves as if she tried to assign numerical probabilities or "degrees of belief", such as is done in the Bayesian estimation. However, she usually prefers the situation, where she can calculate the probability, rather than guessing it. In its research Ellsberg explained it with the following example:

An individual is given the choice to place a bet on either an urn of 100 balls where he knows that 50 balls are red and 50 black, or an urn containing 100 red and black balls, however, entirely not knowing the ratio between the two different kinds.

When being asked about whether to place a bet on a color of the ball drawn in the first or the second urn, the preference is usually towards the first urn. This preference is called ambiguity aversion.

Nonetheless, we should strictly distinguish between ambiguity aversion and risk aversion. While the former is the aversion against unknown distribution, the latter is the aversion towards the actual risk associated with a decision. In the setting of a bet, the risk aversion the reason for an individual to ask for a risk premium, when taking risk. This means that she requires a potential gain, which is higher than the expected value of the bet.

3.3.2 Model of Garlappi et al. (2007)

Ambiguity aversion in the context of Garlappi et al. (2007) is interpreted as the aversion an investor has towards uncertain parameter estimates. In other words, the more an investor distrusts the computation and validity of expected returns, the more ambiguity averse she is.

This requires a new optimization problem, which depends on the ambiguity aversion. In addition to the already developed general mean-variance criterion for asset allocation, described in section 2.4.1, the authors include two new elements to the problem. The extension is written as the following optimization problem:

$$\max_{\boldsymbol{\pi}} \min_{\boldsymbol{\mu}} \quad \boldsymbol{\pi}^\top \boldsymbol{\mu} - \frac{\gamma}{2} \boldsymbol{\pi}^\top \boldsymbol{\Sigma} \boldsymbol{\pi} \tag{3.15}$$

subject to

$$f\left(\boldsymbol{\mu}, \hat{\boldsymbol{\mu}}, \boldsymbol{\Sigma}\right) \leq \boldsymbol{\epsilon} \tag{3.16}$$

$$\boldsymbol{\pi}^\top \mathbf{1} = 1 \tag{3.17}$$

Where the newly introduced constraint $f\left(\boldsymbol{\mu}, \hat{\boldsymbol{\mu}}, \boldsymbol{\Sigma}\right)$ can be depicted as the following function:[24]

$$f\left(\boldsymbol{\mu}, \hat{\boldsymbol{\mu}}, \boldsymbol{\Sigma}\right) = \begin{pmatrix} \frac{\left(\mu_1 - \hat{\mu}_1\right)^2}{\sigma_1^2 / T_1} \\ \vdots \\ \frac{\left(\mu_N - \hat{\mu}_N\right)^2}{\sigma_N^2 / T_N} \end{pmatrix} \leq \begin{pmatrix} \epsilon_1 \\ \vdots \\ \epsilon_N \end{pmatrix} \tag{3.18}$$

To understand why the constraint above can be interpreted as a confidence interval, we start from the following equality

$$\sqrt{\frac{\left(\mu_j - \hat{\mu}_j\right)^2}{\sigma_j^2 / T_j}} = \pm \frac{\left(\mu_j - \hat{\mu}_j\right)}{\sigma_j / \sqrt{T_j}}$$

[24] This expression of the constraint changes for the case, when a single value of ϵ is assumed for all assets.

from which we can see, that the square root of the constraint follows a normal distribution as shown by Bewley (1988); Garlappi et al. (2007).

The new optimization can therefore be interpreted as:

- First, imposing a constraint on the estimated expected return that restricts the value of the true expected return to lie within a confidence interval around the estimated expected return. Thereby including the possibility of estimation error into the problem.
- Second, maximizing the mean-variance criterion. Among thes "ambiguity-optimal" allocation choices determined by the minimization, the one that also maximizes the mean-variance criterion is chosen.

Separate Representation about Uncertainty of Expected Returns The model is first established by optimizing the single assets' weights separately, as to provide a general understanding for the solution to their model. The Garlappi et al. constraint (3.16) as defined above takes the following shape for each asset j.

$$\frac{(\mu_j - \hat{\mu}_j)^2}{\sigma_j^2/T_j} \leq \epsilon_j, \quad j = 1, \ldots, N \tag{3.19}$$

ϵ_j expresses a product of the ambiguity aversion and the ambiguity. On one hand, ambiguity aversion is assumed to be common across all assets, since generally it describes the aversion against not knowing the distribution. On the other hand, the ambiguity itself can be considered being asset-specific. This means that ϵ_j can vary among different assets. The interpretation of ambiguity aversion as a confidence interval, allows to express a probabilistic statement. This statement enables us to determine the probability of the event that all returns actually lie within the given confidence intervals. Setting I_j to be the confidence interval of asset j, the statement can be written as follows

$$P\left(\mu_1 \in I_1, \ldots, \mu_N \in I_N\right) = 1 - p \tag{3.20}$$

which in case of independent returns can be computed as:

$$1 - p = (1 - p_1)(1 - p_2)\ldots(1 - p_N) \tag{3.21}$$

The result is the probability of all returns falling into the N intervals. The problem set up in formula (3.15) is solved as follows: First, it is minimized over $\boldsymbol{\mu}$ subject to the Garlappi constraint. Each element of this minimization depends hereby on the weight given to the specific asset, the expected return associated with it, as well as, its standard error, and ambiguity aversion. Due to it's interpretation as confidence interval

$$\frac{(\mu_j - \hat{\mu}_j)}{\sigma_j/\sqrt{T_j}} \leq \sqrt{\epsilon_j}$$

the problem is solved, when each value $\pi_j \mu_j$ is minimized within its interval. Therefore it is optimized depending on the sign of π_j as shown in the following:

$$\min_{\mu_j} \pi_j \mu_j = \begin{cases} \hat{\mu}_j - \sigma_j \dfrac{\sqrt{\epsilon_j}}{\sqrt{T}} & \pi_j > 0 \\ \hat{\mu}_j + \sigma_j \dfrac{\sqrt{\epsilon_j}}{\sqrt{T}} & \pi_j < 0 \end{cases}, \quad j = 1, \ldots, N \tag{3.22}$$

which can also be written as[25]

$$\mu_j = \hat{\mu}_j - sign\left(\pi_j\right) \sigma_j \frac{\sqrt{\epsilon_j}}{\sqrt{T}}, \quad j = 1, \ldots, N \tag{3.23}$$

This means that negative weights will lead to an upward adjustment of the corresponding asset's return and positive weights will lead to a downward adjustment. Having established the above solution, the problem can be written as the following maximization: [26]

$$\max_{\boldsymbol{\pi}} \left\{ \boldsymbol{\pi}^\top \left(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu}^{adj} \right) - \frac{\gamma}{2} \boldsymbol{\pi}^\top \boldsymbol{\Sigma} \boldsymbol{\pi} \right\} \tag{3.24}$$

with $\boldsymbol{\mu}^{adj}$ being

$$\boldsymbol{\mu}^{adj} \equiv \left(sign\left(\pi_1\right) \frac{\sigma_1}{\sqrt{T}} \sqrt{\varepsilon_1}, \ldots, sign\left(\pi_N\right) \frac{\sigma_N}{\sqrt{T}} \sqrt{\varepsilon_N} \right)^\top \tag{3.25}$$

[25] The $sign(\cdot)$-function is evaluated as $+1$ for positive values and -1 for negative values of π_j.
[26] See Proposition 1 in Garlappi et al. (2007)

Nonetheless, this solution does not offer an analytic way to solve it and also includes the rather unrealistic assumption of independent returns. Although it is not applied later in the empirical analysis, it provides an framework to understand the representation in the upcoming paragraph.

Joint Representation about Uncertainty of Expected Returns In order to take into account that returns are not independent, a more elegant way to express the optimization problem under ambiguity aversion is set up. This setup allows offers an analytic solution. Instead of formulating the confidence intervals for each asset separately, it is done for all assets jointly, reducing the ϵ-vector to a single constant.[27] The constraint $f(\,\cdot\,) \leq \epsilon$ is then reformulated as follows.

$$\frac{T\,(T-N)}{(T-1)\,N}(\hat{\boldsymbol{\mu}}-\boldsymbol{\mu})^{\top}\boldsymbol{\Sigma}^{-1}\,(\hat{\boldsymbol{\mu}}-\boldsymbol{\mu}) \leq \epsilon \tag{3.26}$$

For normally distributed returns, Garlappi et al. show, that the left part of the constraint follows a χ^2-distribution with N degrees of freedom. Hence, the whole expression can again be interpreted as a confidence interval corresponding to the following probabilistic statement

$$P\left[\frac{T\,(T-N)}{(T-1)\,N}(\hat{\boldsymbol{\mu}}-\boldsymbol{\mu})^{\top}\boldsymbol{\Sigma}^{-1}\,(\hat{\boldsymbol{\mu}}-\boldsymbol{\mu}) \leq \epsilon\right] = 1-p \tag{3.27}$$

Given the new joint constraint (3.26) the initial max-min problem set up in equation (3.15) can be rewritten as a maximization problem, for which an analytic solution can be found.

First, the "inner" minimization is derived

$$\min_{\boldsymbol{\mu}} \boldsymbol{\pi}^{\top}\boldsymbol{\mu} - \frac{\gamma}{2}\boldsymbol{\pi}^{\top}\boldsymbol{\Sigma}\boldsymbol{\pi} \tag{3.28}$$

subject to:

$$(\hat{\boldsymbol{\mu}}-\boldsymbol{\mu})^{\top}\boldsymbol{\Sigma}^{-1}\,(\hat{\boldsymbol{\mu}}-\boldsymbol{\mu}) \leqslant \varepsilon \tag{3.29}$$

[27] Nevertheless, this does not allow for setting different levels of the ambiguity aversion parameter ϵ for different assets.

$$\boldsymbol{\pi}^\top \mathbf{1} = 1 \tag{3.30}$$

With $\varepsilon \equiv \epsilon \frac{(T-1)N}{T(T-N)}$ and $0 \leqslant \varepsilon \leqslant 1$, the Langrangian function solves for (Proof in Appendix B.1.1)

$$\mathcal{L}\left(\boldsymbol{\mu}^*, \lambda\right) = \boldsymbol{\pi}^\top \hat{\boldsymbol{\mu}} - \left(\frac{1}{4\lambda} - \frac{\gamma}{2}\right) \boldsymbol{\pi}^\top \boldsymbol{\Sigma} \boldsymbol{\pi} - \lambda\varepsilon \tag{3.31}$$

Hence, the initial $\max - \min$ problem in equation (3.15) is simplified to the following maximization problem, which depends on the Lagrange parameter λ:

$$\max_{\boldsymbol{\pi},\lambda} \boldsymbol{\pi}^\top \hat{\boldsymbol{\mu}} - \left(\frac{1}{4\lambda} - \frac{\gamma}{2}\right) \boldsymbol{\pi}^\top \boldsymbol{\Sigma} \boldsymbol{\pi} - \lambda\varepsilon \tag{3.32}$$

Solving for λ, we then get[28]

$$\lambda = \frac{1}{2}\sqrt{\frac{\boldsymbol{\pi}^\top \boldsymbol{\Sigma} \boldsymbol{\pi}}{\varepsilon}} \tag{3.33}$$

and substituting λ into eq. (3.32), we end up with the following maximization problem:[29]

$$\max_{\boldsymbol{\pi}} \ \boldsymbol{\pi}^\top \hat{\boldsymbol{\mu}} - \frac{\gamma}{2} \boldsymbol{\pi}^\top \boldsymbol{\Sigma} \boldsymbol{\pi} \left(1 + \frac{2\sqrt{\varepsilon}}{\gamma\sqrt{\boldsymbol{\pi}^\top \boldsymbol{\Sigma} \boldsymbol{\pi}}}\right) \tag{3.34}$$

Which solves for[30][31]

$$\boldsymbol{\pi}^* = \frac{1}{\gamma} \boldsymbol{\Sigma}^{-1} \left(\frac{1}{1 + \frac{\sqrt{\varepsilon}}{\gamma\sigma_p^*}}\right) \left[\hat{\boldsymbol{\mu}} - \frac{B - \gamma\left(1 + \frac{\sqrt{\varepsilon}}{\gamma\sigma_p^*}\right)}{C} \mathbf{1}\right] \tag{3.35}$$

[28] Derivation in Appendix B.1.2
[29] Derivations are shown in Appendix B.1.3
[30] Compare Proposition 2 in Garlappi et al. (2007)
[31] Constants as defined in sec. 2 will be used.

$$A = \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}, \quad B = \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1} = \mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}, \quad C = \mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}$$

where σ_p^* is the unique positive root to the following polynomial:[32]

$$0 = C\gamma^2\sigma_p^4 + 2C\gamma\sqrt{\varepsilon}\sigma_p^3 + \left(C\varepsilon - AC + B^2 - \gamma^2\right)\sigma_p^2 - 2\gamma\sqrt{\varepsilon}\sigma_p - \varepsilon \tag{3.36}$$

From the result in eq. 3.35 we can see, that for $\varepsilon \to 0$, hence $\epsilon \to 0$ or $T \to +\infty$, $\boldsymbol{\pi}^*$ converges to the following expression

$$\boldsymbol{\pi}^* = \frac{1}{\gamma}\boldsymbol{\Sigma}^{-1}\left[\hat{\boldsymbol{\mu}} - \frac{B-\gamma}{C}\mathbf{1}\right] \tag{3.37}$$

This equation is the same as in the regular one-fund MV-Setting, which is explained in sec. 2.2.

In the following figure, we can see what allocations result for different values of epsilon.[33]

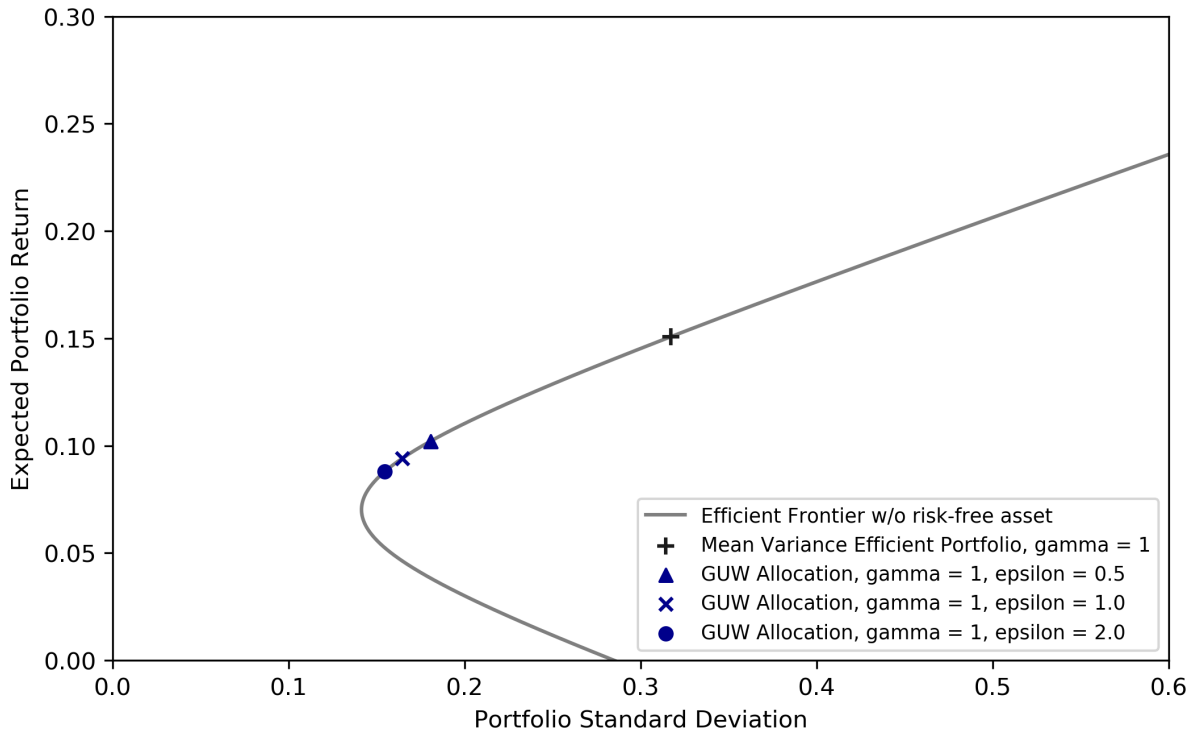


Figure 3.1: Garlappi et al. portfolios for different values of ϵ; $T = 60$; $N = 8$

As can be seen from the graph, the application of the model shows a shrinkage behavior. The asset allocation shrinks towards the minimum variance portfolio for increasing values

[32] Proof in Appendix B.1.3

[33] Allocations computed based on the same data as in sec. 2

of ϵ. This relationship was also further examined in the paper of Garlappi et al. (2007). They found, that for the model of ambiguity aversion, we can write the shrinkage of portfolio weights as

$$\boldsymbol{\pi}_{AA}(\epsilon) = \phi_{AA}(\epsilon)\,\boldsymbol{\pi}_{\min} + (1 - \phi_{AA}(\epsilon))\,\boldsymbol{\pi}_{MV} \tag{3.38}$$

where $\boldsymbol{\pi}_{\min}$ and $\boldsymbol{\pi}_{MV}$ are the weight vectors of the minimum variance and the mean-variance portfolio respectively. Here, the shrinkage factor $\phi_{AA}(\epsilon)$ is defined as:

$$\phi_{AA}(\epsilon) = \frac{\sqrt{\epsilon\frac{(T-1)N}{T(T-N)}}}{\gamma{\sigma_p}^* + \sqrt{\epsilon\frac{(T-1)N}{T(T-N)}}} \tag{3.39}$$

Such that the shrinkage factor $\phi_{AA}(\epsilon)$ increases in ϵ, whereas it decreases in T, γ and σ_p^*.

3.3.3 Extending the Model of Garlappi et al. (2007)

Garlappi et al. (2007) assume that an investor is characterized by a certain level of ambiguity aversion, which is assumed to be constant over time. However, we could also think of an investor whose trust in historical varies over time. This implies that her ambiguity aversion changes over time. We therefore propose to take a measure for uncertainty into account.

Assuming an investor who believes that the well known capital asset pricing model holds. Then he believes in the following relationship to hold for every asset:

$$E[r_i] - r_f = \beta_i\,(E[r_m] - r_f) \tag{3.40}$$

This can be interpreted as the expected excess return of an asset to be explained by the asset's beta and the market's excess return. The beta is determined in a regression of the following form

$$r_{it+1} - r_{f,t} = \alpha_i + \beta_i(r_{mt+1} - r_{r,t}) + \varepsilon_{it+1} \tag{3.41}$$

If the CAPM holds, the value for the regression constant α will be 0. In our model, we will therefore assume that an investor's uncertainty aversion depends on the degree that α diverges from 0. While for our analysis α represents a measure of uncertainty, it also has another meaning according to Jensen (1968). He initially interpreted the measure as an indicator for abnormal return. Where assets showing a positive α have a higher return than implied by the CAPM model and vice versa.

However, it might also stem from poor estimation data and the resulting poor regression model. Assuming that an investor is more ambiguity averse, when regressions yield higher values of α, we can express ϵ as depending on α. This will have no impact on the analytic solution of the initially developed model; nonetheless, it will have an impact on the application of the model over time. For this purpose, we will define a function of the mean absolute deviation (MAD) of the assets' α's from 0. Where the MAD is defined as

$$\mathrm{MAD}\left(\boldsymbol{\alpha}\right) = \frac{1}{N}\sum_{i=1}^{N}|\alpha_i| \tag{3.42}$$

and the function determining ϵ is assumed to be of linear relation

$$\epsilon\left(\mathrm{MAD}(\%)\right) = \kappa \cdot \mathrm{MAD}(\%) \tag{3.43}$$

where κ is the parameter of sensitivity towards non-zero values of α, and MAD is given in percentage terms. This extension allows to take into account, that an investor might believe in the estimates obtained to a larger extent if the CAPM model seems to hold. Implementing the model of Garlappi et al. (2007) then implies to estimate the general mean-variance parameters $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\Sigma}}$, and to determine the degree of uncertainty aversion based on the upper explained model.

Part II

Empirical Analysis

4 Methodology and Data

The empirical analysis to be performed follows a deductive approach. According to the definition of Bryman and Bell (2011), the approach takes into account what is known from previous research, then, based on that deducts hypotheses tests these in an empirical analysis.

Generally, our empirical analysis aims to evaluate the performance of the previously introduced asset allocations under uncertainty. This evaluation is performed in a quantitative analysis, which is structured in two pillars. The *first pillar* is represented by an analysis of expected performance. It aims to evaluate a strategy's expected performance based on the common assumptions for portfolio construction. In order to analyze this expected performance, we conduct a Monte Carlo simulation. Whereas, the *second pillar* is represented by an analysis of out-of-sample performance of asset allocations, that were determined based on real market return data.

Hereby, we aim to follow the three important principles of *reliability*, *replication* and *validity* (Bryman and Bell, 2011).

Reliability describes, whether the results of a study are repeatable. The concept of reliability is tested by using a control data set on which the same analysis is performed.

Replication deals with the issue if other researchers are able to replicate each others' experiments. A failure to do so would raise issues about the credibility of the findings.

Validity is concerned with the question, whether the measure chosen is really measuring the concept chosen to examine.

4.1 Hypotheses

A hypothesis refers to:

> "A proposition or principle put forth or stated (without any reference to its correspondence with fact) merely as a basis for reasoning or argument, or as a premiss from which to draw a conclusion; a supposition."[34]

For our analysis the hypotheses put forth and being tested are the following

[34] Compare Oxford English Dictionary (2018)

1. Models that account for parameter uncertainty, outperform the original tangency portfolio
2. Utility-optimized portfolios better relative to benchmark allocations
3. Larger estimation windows lead to more accurate parameter estimates (less uncertainty) and therefore better performance
4. The performance of the out-of-sample analysis follows the expected performance (CER)
5. Short-sale constrained portfolios perform better under uncertainty than non-constrained asset allocations

4.2 Benchmark Portfolios

In addition to evaluating the portfolios established before, we will also consider benchmark allocations, that were not optimized in a utility setting. Those benchmark allocations are the minimum-variance allocation, as defined in sec. 2.3, the 1/N, risk parity, and the lower partial moments allocation as described in sec. 3.2. Below, the 1/N and the risk parity allocation are established.

4.2.1 1/N-Asset Allocation

The first benchmark strategy outside the Markowitz setting is the equal weight strategy, which is also known as the Talmudian investment strategy. Including the strategy as a benchmark is motivated by the research of Demiguel et al. (2009) and Benartzi and Thaler (2001).

Demiguel et al. evaluate the naïve equal-weight (1/N) asset allocation strategy in comparison to several extensions of the classic mean variance setting. They are not arguing in favor of implementing a 1/N-Strategy, however, they use it as a benchmark to evaluate the several "optimized" portfolios.

Also, it was found found that the usage of the 1/N portfolio is not just a hypothetical application used in science, but also appears to be applied in practice (Benartzi and Thaler, 2001). They found that some people tend to spread their contributions evenly among assets irrespective of the choice of investment assets given.

As the 1/N asset allocation approach does not take into account any historical measures for risk or return, the performance of this strategy highly depends on the choice of assets considered for investment and their underlying correlation structure.

The portfolio weights for each asset are then determined as:

$$\pi_i = \frac{1}{N}, \quad \forall i \in \{i \mid i \text{ is integer} \wedge 1 \leq i \leq N\} \tag{4.1}$$

4.2.2 Risk Parity Allocation

The risk parity or equally-weighted risk contribution allocation was developed, in order to overcome the issues of the MV-setting. The allocation does not consider expected returns. Instead, it focuses solely on the risk associated with a certain asset being added to the portfolio. It can be characterized as a risk-focused diversification strategy.The portfolio weights, are set in a way such that the risk contribution of each asset is equal. Following Chaves et al. (2012) an equally weighted risk contribution can be achieved, following the recipe below.

Firstly, the concepts of marginal risk contribution (MRC) and total risk contribution (TRC) have to be defined. MRC is depicted as

$$MRC_i = \frac{\partial \sigma_p}{\partial \pi_i} = \sum_{j=1}^{N} \pi_j \sigma_{ij} = \text{cov}\left(r_i, r_p\right) \tag{4.2}$$

where π_i represents the weight of an asset i, σ_p the portfolio standard deviation as defined in equation (2.5) and σ_{ij} the covariance of asset i and asset j. Hence, MRC is the partial derivative of the portfolio standard deviation with respect to the weight of asset i. The MRC therefore represents the marginal risk added to the portfolio risk when infinitesimally increasing the asset's weight π_i.

The TRC on the other hand shows the contribution of an asset towards the portfolio risk σ_p. Thus, portfolio risk can be split up into its contributors. It is defined as:

$$TRC_i = \pi_i \frac{\partial \sigma_p}{\partial \pi_i} = \sum_{j=1}^{N} \pi_i \pi_j \sigma_{ij} = \pi_i \text{cov}\left(r_i, r_p\right) \tag{4.3}$$

Assuming an investor wants to achieve an equally-weighted risk contribution portfolio the following needs to hold

$$\pi_i \frac{\partial \sigma_p}{\partial \pi_i} = \pi_j \frac{\partial \sigma_p}{\partial \pi_j} = \lambda, \quad \forall i, j \tag{4.4}$$

This describes that the TRC of asset i has to equal the TRC of any other asset included in the portfolio of assets.

Before optimization, the final composition of the portfolio is not known and also the total portfolio's risk is unknown. Hence, λ is a constant depending on the final allocation of weights. While there is a closed form solution to the MV-Setting, there is only one with regards to RP in very restricted cases. Hence, when applying the approach it usually requires numerical optimization methods. The optimization problem to be solved can be expressed as follows

$$\begin{aligned} &\min \sum_{i=1}^{N} \sum_{j=1}^{N} (TRC_i - TRC_j)^2 \\ &s.t. \quad \boldsymbol{\pi}^\top \mathbf{1} = 1 \end{aligned} \tag{4.5}$$

This expression describes the minimization of the squared difference in TRC between asset i and asset j subject to the "sum-to-one" weight constraint.

4.3 Overview of Portfolios Considered in Analysis

The portfolios considered for analysis can be categorized in two major groups. The **first group** takes preferences into account, as implied by the use of γ. The **second group** instead, does not consider the risk aversion for portfolio construction. Another characteristic that distinguishes the strategies considered, is the availability of a risk-free security. The following table gives an overview about the portfolios considered, their allocation rule and whether a risk-free asset is implied.

Abbreviation	Description of Asset Allocation Strategy	Criterion for asset allocation	r_f considered?
	Portfolio rules considering risk aversion coefficient γ		
Tangency	Tangency / Maximum Sharpe Ratio (sec. 2.4.1)	$\max \boldsymbol{\pi}^\top \boldsymbol{\mu} - \frac{\gamma}{2}\boldsymbol{\pi}^\top \boldsymbol{\Sigma}\boldsymbol{\pi}$	*Yes*
LW-Tangency	Adjusted Tangency with $\hat{\boldsymbol{\Sigma}}_{LW}$ (sec. 3.1.3)	$\max \boldsymbol{\pi}^\top \boldsymbol{\mu} - \frac{\gamma}{2}\boldsymbol{\pi}^\top \hat{\boldsymbol{\Sigma}}_{LW}\boldsymbol{\pi}$	*Yes*
Mean-Variance	Mean-Variance Portfolio in One-Fund-Setting (sec. 2.1)	$\max \boldsymbol{\pi}^\top \boldsymbol{\mu} - \frac{\gamma}{2}\boldsymbol{\pi}^\top \boldsymbol{\Sigma}\boldsymbol{\pi}$	*No*
GUW	Uncertainty Aversion (sec. 3.3.2)	$\max \boldsymbol{\pi}^\top \boldsymbol{\mu} - \frac{\gamma}{2}\boldsymbol{\pi}^\top \boldsymbol{\Sigma}\boldsymbol{\pi}\left(1 + \frac{2\sqrt{\varepsilon}}{\gamma\sqrt{\boldsymbol{\pi}^\top \boldsymbol{\Sigma}\boldsymbol{\pi}}}\right)$	*No*
GUW ext.	Uncertainty Aversion Extended (sec. 3.3.3)	$\max \boldsymbol{\pi}^\top \boldsymbol{\mu} - \frac{\gamma}{2}\boldsymbol{\pi}^\top \boldsymbol{\Sigma}\boldsymbol{\pi}\left(1 + \frac{2\sqrt{\varepsilon^*}}{\gamma\sqrt{\boldsymbol{\pi}^\top \boldsymbol{\Sigma}\boldsymbol{\pi}}}\right)$	*No*
TF - Kan Zhou	Three Fund (sec. 3.1.2)	$\max \hat{\boldsymbol{\pi}}(c,d)^\top \boldsymbol{\mu} - \frac{\gamma}{2}\hat{\boldsymbol{\pi}}(c,d)^\top \boldsymbol{\Sigma}\hat{\boldsymbol{\pi}}\,(c,d)$	*Yes*
	*Portfolio rules **not** considering risk aversion coefficient γ*		
Min-Variance	Minimum Variance Portfolio (sec. 2.3)	$\min \sigma_p^2$	*No*
$1/N$	Equally-Weighted (sec. 4.2.1)	$\pi_i = \frac{1}{N}, \quad \forall i$	*No*
RP	Risk Parity (sec. 4.2.2)	$\pi_i \dfrac{\partial \sigma_p}{\partial \pi_i} = \pi_j \dfrac{\partial \sigma_p}{\partial \pi_j} = \lambda, \quad \forall i,j$	*No*
LPM	Lower Partial Moments (sec. 3.2)	$\min \mathrm{LPM}_2$	*No*

Table 4.1: Asset allocation strategies

4.4 Introducing the Short Sale Constraint

So far, the practice of short selling of risky assets was not restricted. However, short selling is not available to all investors. And also the model can yield non-implementable large negative positions depending on the parameter inputs. It is therefore interesting to take a look at the (expected) performance in such a restricted case. In both analyses, we will also consider the constrained case. The additional condition to be fulfilled, when short sales are not allowed, is to have all portfolio weights π_i non-negative. This is expressed as the following.

$$\pi_i \geq 0, \quad \forall i \tag{4.6}$$

To solve the optimization problem defined for each strategy, we have to rely on numerical methods as no closed form solutions are given, except for $1/N$.[35] Restricting the three fund implementation of Kan and Zhou is strenuous. Therefore, it is not considered in the no short-sale constrained analyses.

4.5 Discussion of Data Considered

The main set of asset returns used as input for the Monte Carlo simulation, as well as for the out-of-sample performance analysis, follows the selection of Garlappi et al. (2007). This enables to compare their results with our findings.[36]

Countries	Ticker	Abbreviation	Currency
Italy	MSITALL	ITA	EUR (ITL)
Germany	MSGERML	GER	EUR (DEM)
Switzerland	MSSWITL	SWI	CHF
Canada	MSCNDAL	CAN	CAD
USA	MSUSAML	USA	USD
UK	MSUTDKL	UK	GBP
France	MSFRNCL	FRA	EUR (FF)
Japan	MSJPANL	JPN	JPY
World	*MSWRLD*	*WRLD*	USD

Table 4.2: Indices considered for analysis

[35]The 1/N allocation only considers positive weights by construction.
[36]The data was obtained from Thomson Reuters Datastream 5.1, accessed on 23. April 2018

In addition to the assets chosen by them, also an index for the market is chosen. This is required by our extension to the original model. However, the time period of the return series will be extended, also covering data from July 2001 to February 2018. Resulting in 577 Months of return data instead of 379.

As can be seen in tab. 4.2, the indices are quoted in the local currency. In order to obtain USD denominated return series, the values were converted to USD, according to the exchange rate holding on the day of the return observation. In the following, the abbreviations as defined above will be used when referring to the assets.

4.5.1 Risk-Free Rate

In addition to the indices' return series, a risk free rate has to be selected. As proposed by Damodaran (2008), the risk free rate should represent the return one can expect of an investment, that shows no risk of bankruptcy and no reinvestment risk over the term of the investment. We assume, that these characteristics are given by an investment in the 3-month US T-Bill. The market rate therefore serves as a proxy for the risk-free rate used for the calculations. During the time period 01/01/1970 - 01/02/2018 the annualized rate evolved as follows.

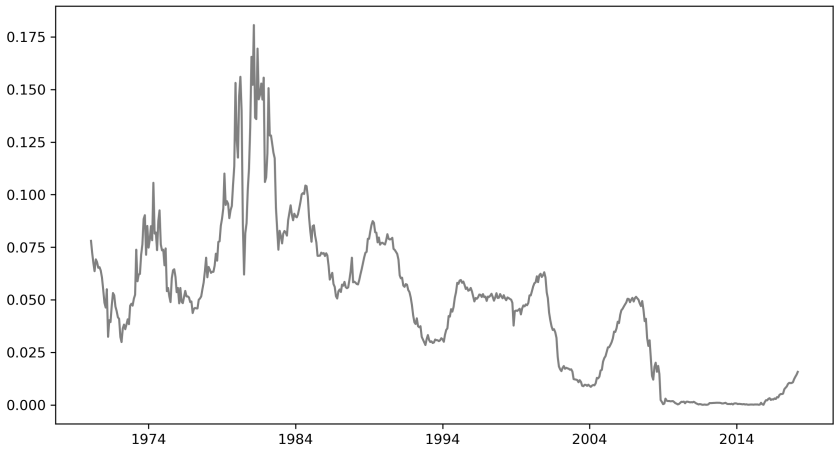


Figure 4.1: Annualized risk-free yield (TB3MS)

In the graph, we can see that the risk-free rate substantially fluctuated over the last 48 years. It should be noted, that in the case of the tangency allocations and the three-fund allocation, to be examined these values can have a substantial impact on the investment allocation. This is due to the fact that both types of strategies rely on a risk-free asset.

If the risk-free rate increases, ceteris paribus, the weight in the risky assets will decrease, and vice versa.

4.5.2 Frequency of the Data Observation

The frequency with which the returns are being determined and considered for the model can be of different character. While some analysts might focus on intra-day or daily data, others evaluate less frequent data such as weekly, monthly or even annual. Which frequency to use depends on multiple reasons.

First and foremost, the investment strategy and the frequency of readjustment has to be taken into account. If an individual invests over a one month time horizon she will be interested in the risk and expected return over the next month. Another reason to use monthly data is, that the availability of daily data might be restricted and not be given for each asset considered for a long time period. On the other hand, more frequent data allows for more estimation accuracy, due to more observations. For comparability, we choose to use monthly returns, as was done by Demiguel et al. (2009); Garlappi et al. (2007); Ledoit and Wolf (2004) and Kan and Zhou (2007).

4.5.3 Estimation Period

As Damodaran (1999) elaborates on the estimation for the beta coefficient, the choice of the estimation window length is of higher complexity. Considering a too short period means, that only a low number of individual observations is being considered. Hence, single extreme values have a higher relative impact on the parameter estimate. When shifting the estimation window by one period, the parameter estimates can change substantially.

In order to circumvent this problem, one could consider a longer time frame. However, a too long period might not truly reflect the parameter's behaviour anymore. The reason is the change in the risk profile of assets over time. Applying long estimation windows, therefore might lead to biased parameter estimates.

Consider a company that undertakes a change in its strategy. Hence, the risk might be affected, which could result in a changing correlation with other market participants or the market. While Damodaran focuses on the issue of estimating the beta of a company, the problem is not different from the estimation of expected returns and variance/covari-

ance. He therefore recommends to adapt the estimation length, based on the company considered.

Other scholars, such as Jobson and Korkie (1981) recommend a time frame of 48 to 84 months, when using monthly data in order to obtain a good tradeoff between having sufficient data and neglecting irrelevant data. On the other hand, Garlappi et al. (2007) use an estimation window of 120 months to examine their proposed model.

For our analysis, we therefore examine different lengths of the window from 48 - 120 months.[37] This will let us analyze the impact of the estimation window on expected performance as well as out-of-sample performance of each asset allocation.

4.5.4 Summary Statistics

The return series of assets considered in tab. 4.3, as described before show the following summary statistics.

Asset	$\hat{\mu}$	$\hat{\sigma}$	min	max	Skew	Kurtosis
ITA	0.191%	7.53%	-52.91%	26.39%	-0.68	4.57
GER	0.542%	6.43%	-24.92%	29.67%	-0.44	2.02
SWI	0.685%	5.28%	-19.14%	23.31%	-0.36	1.73
CAN	0.482%	5.73%	-30.05%	20.72%	-0.78	3.19
USA	0.553%	4.47%	-24.36%	16.17%	-0.71	2.95
UK	0.429%	6.06%	-24.90%	44.01%	0.35	5.47
FRA	0.513%	6.54%	-31.35%	22.05%	-0.55	1.83
JPN	0.610%	6.12%	-28.11%	22.20%	-0.02	1.28
WRLD	0.521%	4.37%	-20.84%	14.14%	-0.70	2.29

Table 4.3: Summary statistics of return series data used

As can be seen in the table above, all indices show a positive mean return over the time period considered (1970 - 2018). However, considering the long time period we can see that indices developed quite differently. While the Swiss Index gained 4172%, the Italian index only gained 203% over the whole period. Showing a higher risk profile than the Swiss Index. Hence, the risk adjusted returns, among the securities selected are substantially different.

Skewness and kurtosis of the return series indicate that returns are not normally distributed. Stock returns tend to show a negative skew; however, this is not the case for the

[37] Examined estimation lengths: 48, 60, 72, 96, 120 months.

UK Index. This means, that for all the risk assets, except the UK index, securities tend to have more frequent small positive movements, and on the other hand large downward movements that are less frequent than implied by a normal distribution. In the UK this is the opposite.[38]

Another observation refers to the positive values of Kurtosis, implying fatter tails of the return distributions. This means that "extreme" returns are more likely than implied by a normal distribution. Again, the UK Index stands out from the crowd, where a high value of Kurtosis is observed.

One way to visualize a divergence from the normal distribution, is to plot a QQ-diagram. This diagram is a scatter diagram, which maps the theoretical quantiles implied by a normal distribution against the quantiles of the empirical distribution Ford (2015). Showing the theoretical quantiles of a normal distribution on the horizontal axis and the sample quantiles on the vertical axis. It therefore allows to compare the true quantiles of a normal distribution against those obtained by the historical returns. As can be seen in Figure 4.2 the indication given by the statistical parameters are confirmed. Deviations from the normal distribution especially when approaching the tails of the distribution are observed.

In addition to the summary statistics and the QQ-Plot, also a test for normality is performed. According to Razali and Wah (2011), the Shapiro-Wilk test has the highest power for significance levels of $\alpha = 5\%$ and $\alpha = 10\%$, which is why it was chosen as an appropriate test for evaluation. Rejecting the null hypothesis, that the return series considered is not of normally distributed.

In table 4.4, we can see that for all single index return series as well as for the combined data set the assumption of normally distributed returns does not hold with considerably high confidence level.[39]

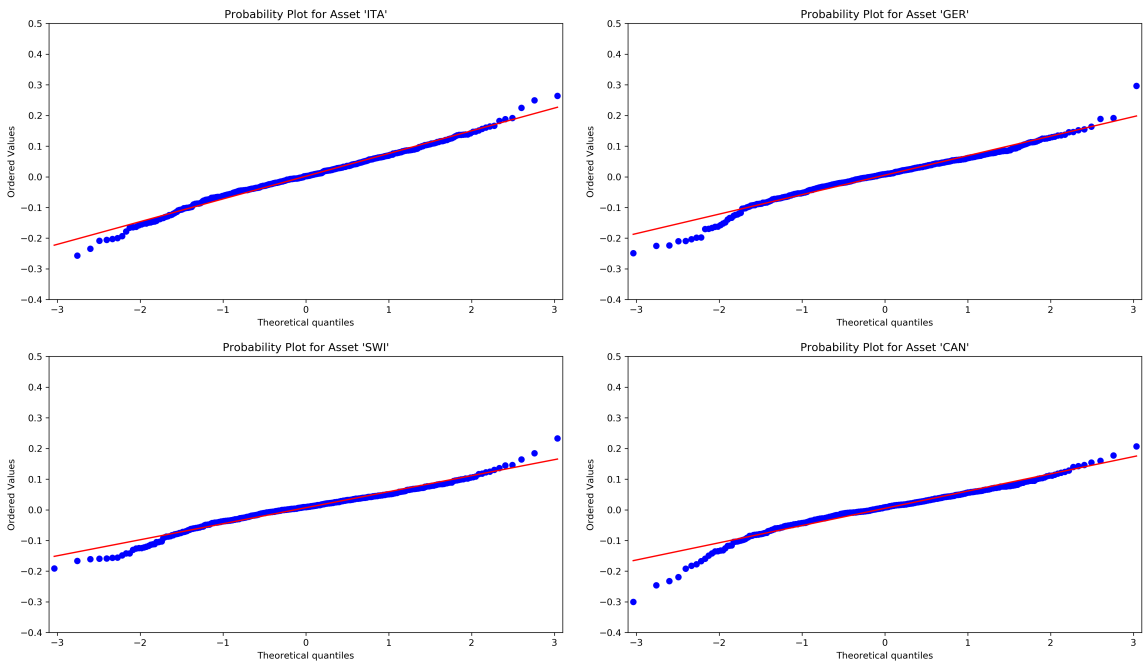
[38] This implies the following relationship: Mean > Median > Mode

[39] Similar results were found for tests on weekly and daily return data.

Assets	W	p-value	Reject
ITA	0.96	1.3E-10	Yes
GER	0.97	2.8E-09	Yes
SWI	0.97	6.7E-09	Yes
CAN	0.96	4.2E-12	Yes
USA	0.96	5.8E-11	Yes
UK	0.95	1.2E-12	Yes
FRA	0.98	4.7E-08	Yes
JPN	0.99	2.8E-04	Yes
WRLD	0.9625	5.4E-11	Yes
All Assets	0.9661	2.5E-33	Yes

Table 4.4: Shapiro-Wilk test statistics and p-values; main data set

While this is contrary to the assumptions set in most asset allocations, the out-of-sample analysis helps to identify how real performance deviates from expected performance or if the model is still applicable.



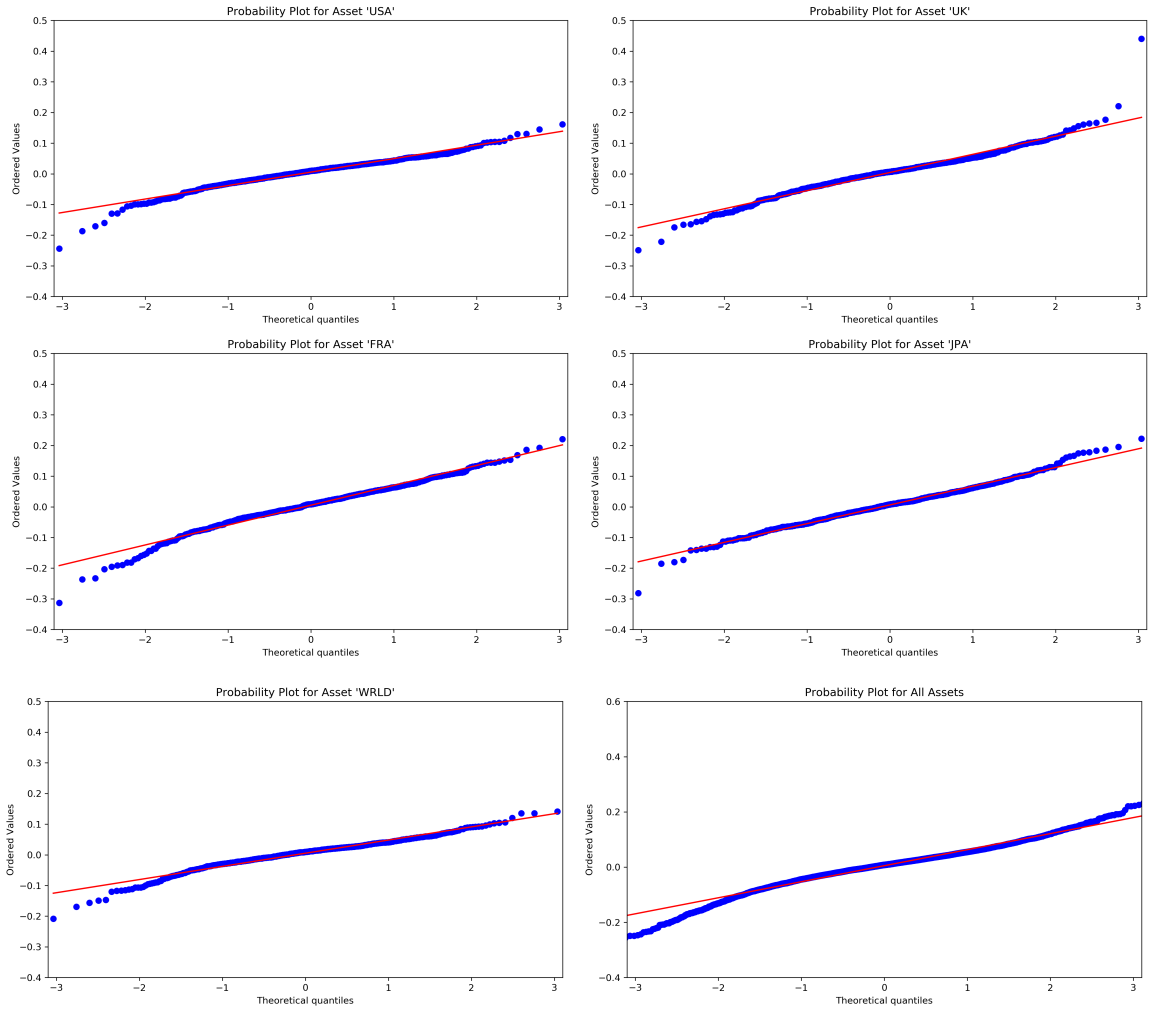


Figure 4.2: QQ-plots of asset returns

4.6 Empirical Analysis of Expected Certainty Equivalent Return (CER)

In the following, we describe the setup of our analysis of expected performance of allocations depicted in tab. 4.1.

Assuming, that investor preferences are represented by a certain utility function, each asset allocation strategy resulting in an uncertain future value of wealth W, yields a level of expected utility. However, the utility level is only a measure of preference that, allows to conclude which option an individual would prefer over another. Utility functions are therefore of ordinal character (Varian, 1992). Thus, the measure does not allow for comparison in cardinal terms.

In order to translate a certain level of expected utility in an interpretable number, we have to rely on other measures. Particular attention is put on the certainty equivalent return (CER), defined as the risk-free return that would make an investor indifferent to investing in the risky alternative. The CER allows for cross-comparability among different different levels of risk aversion.

In the following analysis, we calculate the expected CER of asset allocation strategies, for which the input parameters have been determined from samples of a Monte Carlo simulation.

The reason to conduct a Monte Carlo analysis, is to incorporate randomness into the model (Johnson, 2013) and to determine statistical values, for which analytical solution is available (Cuthbertson and Nitzsche, 2004). This randomness and its impact on the parameters, estimated to calculate portfolio weights make it a suitable model. The methodology of the Monte Carlo analysis is described in the following.

Monte Carlo Simulation Firstly, a set of certain expected return, variance and covariance parameters are chosen for the 8 assets and the market. The set of parameters $\boldsymbol{\mu}^*$ and $\boldsymbol{\Sigma}^*$ is calculated as the sample mean and variance-covariance estimators of the eight assets previously introduced and is used to calibrate the Monte Carlo Analysis.[40] Thus, we get

[40] Covering return observations from 02/1970-02/2018

Asset	ITA	GER	SWI	CAN	USA	UK	FRA	JPN	*WRLD*
ITA	6.81%	3.03%	2.36%	2.32%	1.79%	2.71%	3.60%	2.02%	2.29%
GER	3.03%	4.96%	2.96%	2.41%	2.04%	2.70%	3.72%	1.99%	2.45%
SWI	2.36%	2.96%	3.35%	2.01%	1.75%	2.44%	2.88%	1.85%	2.07%
CAN	2.32%	2.41%	2.01%	3.94%	2.30%	2.52%	2.56%	1.59%	2.32%
USA	1.79%	2.04%	1.75%	2.30%	2.40%	2.10%	2.10%	1.21%	2.08%
UK	2.71%	2.70%	2.44%	2.52%	2.10%	4.40%	3.07%	1.92%	2.44%
FRA	3.60%	3.72%	2.88%	2.56%	2.10%	3.07%	5.14%	2.13%	2.54%
JPN	2.02%	1.99%	1.85%	1.59%	1.21%	1.92%	2.13%	4.50%	2.14%
WRLD	2.29%	2.45%	2.07%	2.32%	2.08%	2.44%	2.54%	2.14%	2.29%
$\boldsymbol{\mu}^*$	2.30%	6.50%	8.22%	5.79%	6.64%	5.14%	6.15%	7.32%	6.25%

Table 4.5: $\boldsymbol{\Sigma}^*$ and $\boldsymbol{\mu}^*$; annualized figures

Expected Certainty Equivalent Return Calculation Next, the certain parameter asset allocation, maximizing the utility of a CARA investor is calculated based on $\boldsymbol{\mu}^*$ and $\boldsymbol{\Sigma}^*$. This optimal allocation consists of an investment in the tangency portfolio and the risk-free asset. [41]

Then, the returns all assets and the market are simulated following the recipe described in sec. 2.5.3. Thus, obtaining returns, drawn from a multivariate normal distribution:

$$\boldsymbol{r}\left(\boldsymbol{\mu}^*, \boldsymbol{\Sigma}^*\right) = \begin{pmatrix} r_{1,1} & \cdots & r_{1,N} \\ \vdots & \ddots & \vdots \\ r_{T,1} & \cdots & r_{T,N} \end{pmatrix} \tag{4.7}$$

Their length T will cover time spans of 48, 60, 72, 96 and 120 months corresponding to the desired estimation window length of each simulation.

The portfolio weights $\hat{\boldsymbol{\pi}}_i\left(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\Sigma}}\right)$ of each asset allocation i defined in table 4.1, are then determined. In case, a risk-free asset is considered for an allocation the optimal weight in the risky assets $\hat{\omega}_p^*$ is taken into account.[42]

[41] Optimal allocation as described in sec. 2.4.1

[42] $\hat{\omega}_p^*$ is determined either as described in 2.4 or if the model defines the $\boldsymbol{\pi}$ as not summing to one as $\hat{\omega}_p^* = \boldsymbol{\pi}^\top \mathbf{1}$. The risk-free rate is represented by the sample mean of the 3 month US T-Bill rate from 01/1970 to 01/2018.

As we assume the hypothetical knowledge of the certain parameters of assets' expected returns and covariance structure of returns, we can now calculate the expected portfolio return $E[r_i]$ and standard deviation σ_i for each portfolio strategy i

$$E[r_i] = \hat{\boldsymbol{\pi}}_i^\top \boldsymbol{\mu} \quad \sigma_i = \sqrt{\hat{\boldsymbol{\pi}}^\top \boldsymbol{\Sigma} \hat{\boldsymbol{\pi}}} \tag{4.8}$$

In a last step, the CER as defined in Demiguel et al. (2009) is calculated for the optimal asset allocation based on true returns parameters, and also for all portfolios with the parameters $E[r_i]$ and σ_i as defined above.

The CER according to Demiguel et al. (2009) can be derived as follows:

For the definition of the certainty equivalent (CE) as the terminal wealth W^*, for which the following holds

$$U\left(W^*\right) = E\left[U\left(W\right)\right] \tag{4.9}$$

As defined above, the CE represents the risk-free wealth for which the investor is indifferent between investing in the asset offering a terminal wealth equal to the CE, or the risky allocation. The relationship depicted in a graph looks as follows

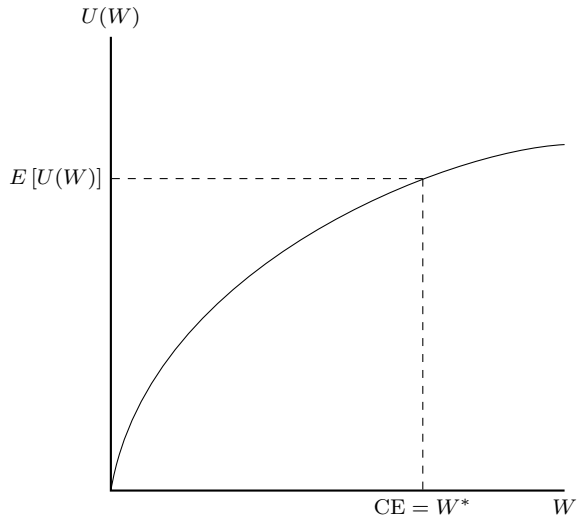


Figure 4.3: CE and expected utility

With the utility depicted on the y-axis and the corresponding level of CER on the x-axis.

In the CARA utility setting W^* can be derived as follows[43]

$$U(W^*) = E\left[U\left(W\right)\right] = -e^{-kW_0\left(1+r-\frac{1}{2}kW_0\sigma^2\right)} \tag{4.10}$$

$$\Leftrightarrow -e^{-kW^*} = -e^{-kW_0\left(1+r-\frac{1}{2}kW_0\sigma^2\right)} \tag{4.11}$$

$$\Leftrightarrow W^* = W_0\left(1 + r - \frac{1}{2}kW_0\sigma^2\right) \tag{4.12}$$

with $\gamma = kW_0$, it follows that,[44]

$$W^* = \text{CE} = W_0\left(1 + r - \frac{1}{2}\gamma\sigma^2\right) \tag{4.13}$$

such that the certainty equivalent return is the simple discrete return based on W^* and W_0:[45]

$$\text{CER} = \frac{W^*}{W_0} - 1 = r - \frac{1}{2}\gamma\sigma^2 \tag{4.14}$$

In our analysis, we compare the CER of the certain tangency allocation, with the expected CER of portfolios constructed under uncertainty. As one simulation only depicts one single draw of returns, this process is repeated 10,000 times. The expected CER results from the simulations' sample mean.

This allows to evaluate the expected performance of an allocation including the uncertainty induced by estimation.

4.7 Empirical Analysis of Out-of-Sample Performance

Out-of-sample performance refers to the performance obtained, when actively trading the investment strategies proposed previously. The same allocations as before are considered.

[43] Compare Equation (2.7)

[44] As defined in sec. 2.

[45] r is the expected return of the asset allocation and σ, the portfolio's standard deviation calculated as defined in eq. (2.5)

Once, a window for the estimation of parameters has been chosen, the return data of that window is used to estimate the input parameters used to calculate the portfolio weights. This procedure is depicted in fig. 4.4. The same estimation window lengths chosen previously, are also considered for this analysis (48, 60, 72, 96, 120 months).[46]

Once a portfolio is traded, it is readjusted on a monthly basis. The updated portfolio weights are then determined based on the estimation window ahead of the readjustment date, up until the strategy is implemented the last time and sold at $t = E$ as depicted in fig. 4.5.

The period in which the portfolios are traded comprises the time from $t =$ January 1980 to $E =$ January 2018. The $1/N$-allocation, by construction being constant over time, represents the exception in this case, as no data is needed to estimate the allocation itself.

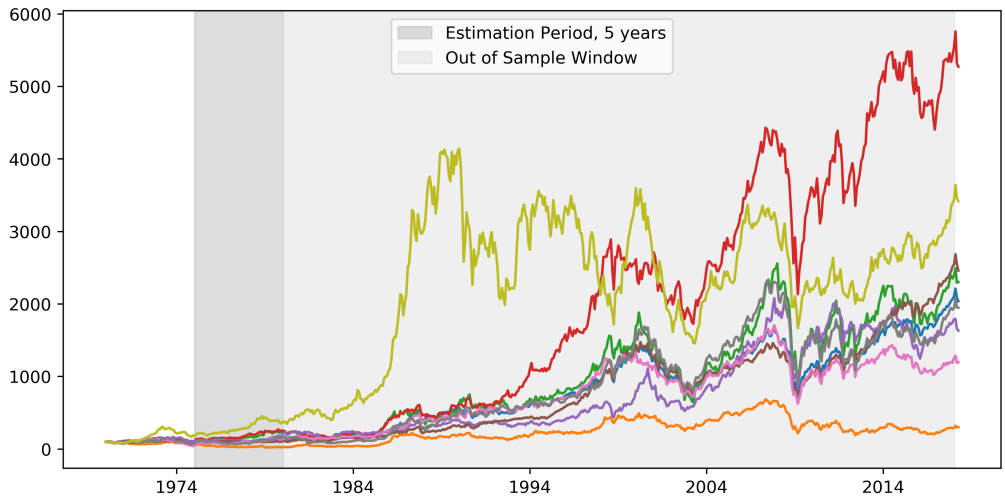


Figure 4.4: Illustration of first estimation window and out-of-sample window

[46]The data used comprises the same data as for the previous analysis, defined in sec. 4.5

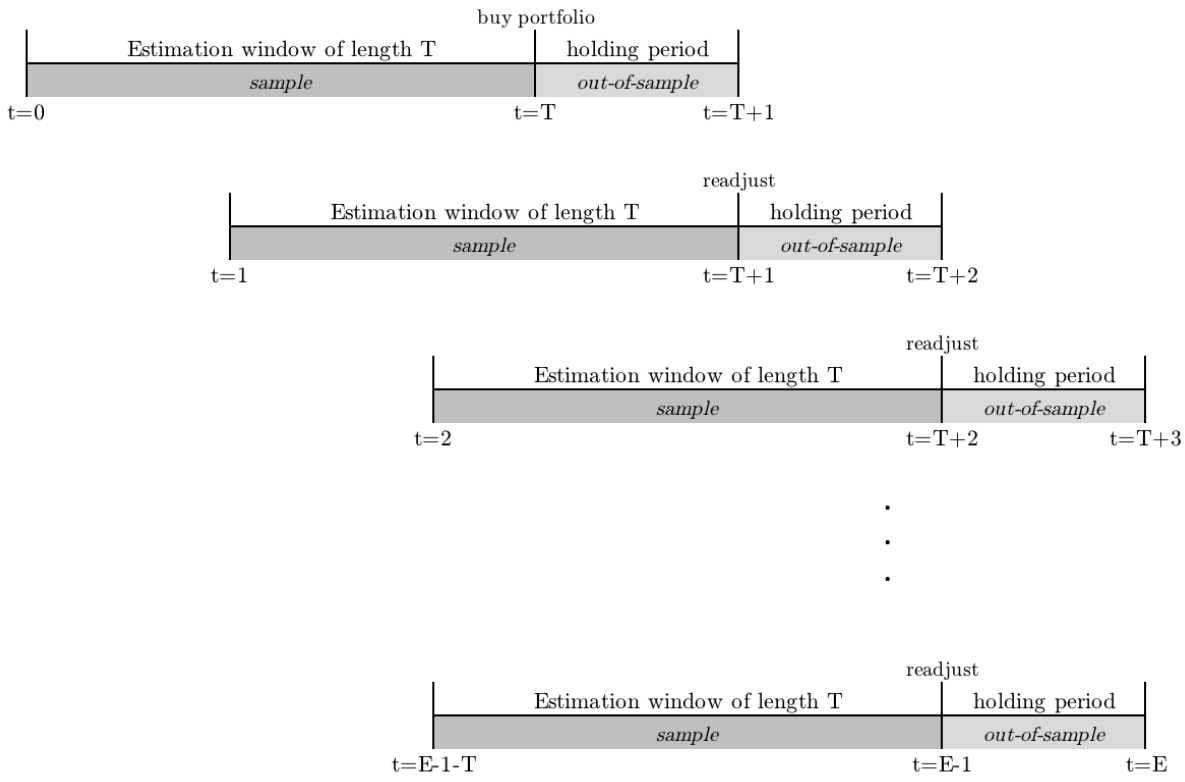


Figure 4.5: Out-of-sample methodology

4.7.1 Performance Measures

Performance, as a concept is evaluated based on different **measures**. In the first analysis the sole measure of expected performance is the expected certainty equivalent return as already defined in sec 4.6. This measure is also computed in the out-of-sample analysis, where it is calculated based on the mean and standard deviation of returns, resulting from the implementation of a certain strategy. In the following, we will explain further performance measures, that are used in order to compare the different allocation strategies.

Sharpe Ratio The Sharpe ratio is a very common tool in order to evaluate an investment. It is a measure of risk-adjusted return.[47] Hence, it can be seen as a tool to evaluate whether the return of an investment was just generated due to excessive risk taking or by "smart" investment decisions.

[47]In the following, the term risk-adjust return refers to the Sharpe ratio

While the Sharpe ratio mentioned in section 2.4.1 is defined, in order to determine the best allocation to invest in, the measure can also be used to evaluate investment strategies ex post.

Following the definition of Sharpe (1994), the excess returns of asset i in period t

$$r_{i,t}^{e} = r_{i,t} - r_{f,t} \tag{4.15}$$

and their corresponding mean and standard deviation

$$r_i^e = \frac{1}{T}\sum_{t=1}^{\top} r_{i,t}^e, \quad \sigma_i^e = \sqrt{\frac{\sum_{t=1}^{\top}\left(r_{i,t}^e - r_i^e\right)^2}{T-1}} \tag{4.16}$$

the Sharpe ratio (SR) can be expressed as follows:

$$SR_i = \frac{r_i^e}{\sigma_i^e} \tag{4.17}$$

Turnover The turnover shows the amount of assets that need to be traded on average while implementing a certain trading strategy. This includes all rebalancing made to the portfolio over the period considered. As defined by Demiguel et al. (2009) it is calculated as the average sum of the absolute values of the trades across all assets.

$$\text{TO} = \frac{1}{M}\sum_{t=1}^{M}\sum_{i=1}^{N}\left(|\omega_{t+1}\pi_{i,t+1} - \omega_t\pi_{i,t^+}|\right) \tag{4.18}$$

With M and N being the length of the out-of-sample window and the number of assets, respectively.

While an investor allocates his wealth to according to a certain strategy at the beginning of a month, the composition changes until the next rebalancing. The weight of asset i at the end of period t is therefore depicted as $\omega_t\pi_{i,t^+}$, whereas the desired weight in the period $t+1$ is $\omega_{t+1}\pi_{i,t+1}$.

The turnover can be used as an indicator for the trading costs, which are to some degree proportional to the wealth traded.

5 Results of the Data Analysis

In the following, the results of both empirical analyses are presented. In each analysis, we describe results, for both cases, including and excluding a short-sale constraint on risky assets, and risk aversion of $\gamma = 1$. In addition to that, we compare the level of CER for different values of γ (keeping T constant), and using a control data set. Thus, providing a holistic overview, whilst verifying the robustness of our results.

First, the expected performance, in terms of CER, is being evaluated. Then, the focus will be put on the out-of-sample performance of the different strategies.[48]

5.1 Expected Certainty Equivalent Return Analysis

In the following, the results of the Monte Carlo analysis of expected certainty equivalent return are presented.

5.1.1 Expected CER Short Sale Allowed

Allowing for short sales, yields the following portfolio weights of the certain allocation

Asset	π_i
ITA	-92.24%
GER	-30.19%
SWI	177.78%
CAN	-33.88%
USA	116.82%
UK	-78.78%
FRA	11.37%
JPN	50.10%
r_f-Asset	-20.98%

Table 5.1: Portfolio weights; short sale allowed; $\gamma = 1$

implying an expected monthly return of 1.32% and a monthly portfolio standard deviation of 9.64%. These values yield a certain CER of 0.855%. Since the certain allocation maximizes the expected utility, any other allocation, will imply a lower CER.

[48]The extended GUW model was calculated assuming $\kappa = 0.25$.

The Monte Carlo analysis allowing for short sales yielded results depicted in tab. 5.2 and fig. 5.1.

	Estimation window length T				
Utility-Optimized Portolios	48	60	72	96	120
Certain Tangency	0.855%	0.855%	0.855%	0.855%	0.855%
Tangency	-14.603%	-9.856%	-7.274%	-4.690%	-3.334%
LW-Tangency	-12.776%	-9.056%	-6.835%	-4.540%	-3.271%
Mean-Variance	-11.542%	-8.047%	-5.912%	-3.863%	-2.691%
GUW $\epsilon = 0.5$	-0.966%	-0.528%	-0.233%	0.016%	0.166%
GUW $\epsilon = 1$	0.137%	0.251%	0.336%	0.401%	0.443%
GUW $\epsilon = 1.5$	0.397%	0.434%	0.466%	0.491%	0.508%
GUW $\epsilon = 2$	0.462%	0.482%	0.500%	0.515%	0.526%
GUW $\epsilon = 3$	0.493%	0.504%	0.517%	0.526%	0.534%
GUW ext.	0.433%	0.438%	0.452%	0.457%	0.466%
TF - Kan Zhou	-1.016%	-0.704%	-0.475%	-0.227%	-0.064%
Benchmark Portfolios					
Min-Variance	0.492%	0.496%	0.500%	0.501%	0.502%
1 / N	0.393%	0.393%	0.393%	0.393%	0.393%
RP	0.408%	0.408%	0.409%	0.408%	0.409%
LPM (6.2%)	0.436%	0.434%	0.430%	0.415%	0.402%

Table 5.2: Expected monthly CER $\gamma = 1$; short sale allowed

Hypothesis 1: *Models that account for parameter uncertainty, outperform the original tangency portfolio*

Firstly, we should note that the tangency portfolio shows a notably lower performance under uncertainty, than under certain parameters. It ranges between (-14.60%) and (-3.33%) depending on the observation length.

As the LW-tangency, GUW and the three-fund allocation outperform the tangency allocation under uncertainty, we find support for the hypothesis.

Whereas the LW-tangency only slightly improves performance, the GUW allocations and the three-fund allocation, instead show a large improvement. In case of GUW, the performance increases in ϵ, hence when it invests more in the minimum variance allocation. It is interesting that even though the GUW allocation does not intend to optimize performance, it actually shows improves.

Nonetheless, with increasing estimation window length, the difference in expected performance decreases. Whereas GUW performs $\sim 15\%$ better than the tangency allocation

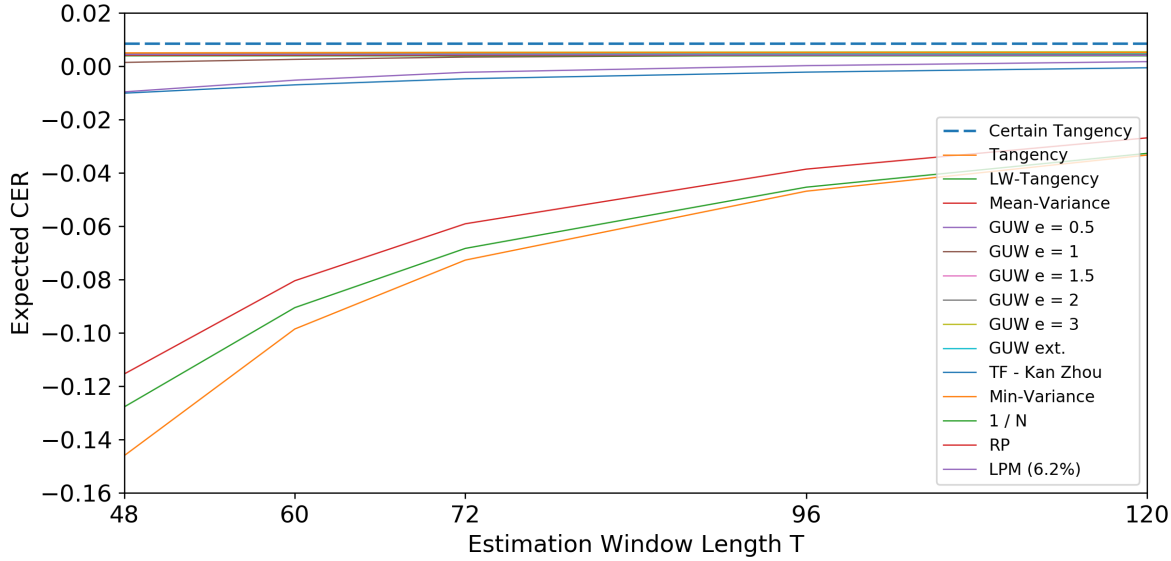


Figure 5.1: Expected CER; $\gamma = 1$; short sale allowed

for $T = 48$, the difference is only $\sim 4\%$ for $T = 120$. Thus, the positive effect decreases in T.

Hypothesis 2: *Utility-optimized portfolios perform better relative to benchmark allocations*

The analysis shows contradicting findings with respect to the hypothesis. Firstly, among the utility optimized models, the GUW and the three-fund portfolios show the best performance. However, the tangency, LW-tangency and mean-variance allocation perform rather modest.

Secondly, we see that all benchmark portfolios show a positive expected CER. Where the minimum variance portfolio shows a comparable performance to the GUW allocations with $\epsilon = 1.5, 2, 3$. Hence, we can not conclude that considering the risk aversion in the portfolio construction leads to better performance.

Hypothesis 3: *Larger estimation windows lead to more accurate parameter estimates (less uncertainty) and therefore better performance*

This pattern can be confirmed for all utility-optimized portfolios, as well as the minimum-variance portfolio. However, the pattern can not be observed for the rest of the portfolios, which show a rather stable or even negative trend, such as in the LPM setting.

For those portfolios, where longer estimation windows impact performance positively, we can also see that the positive effect decreases in T. Whereas the marginal increase was 0.4% per month between 48 and 60 months, the ratio decreased to 0.05% per month between 96 and 120 month (Tangency). Thus suggesting, that further increasing estimation length has a decreasing positive impact on expected performance improvement. This converging behaviour can also be seen in fig. 5.1.

Additional Findings:

Performing a Welch test (according to Welch (1947)) on the difference in performance of GUW $\epsilon = 3$ and the minimum variance allocation, we can see that for T=48, we can not reject the hypothesis of equal performance of the GUW and the minimum variance allocation. However, for larger values of T the hypothesis is rejected on a very high confidence level.

Estimation window length T	48	60	72	96	120
p-value	0.3900	0.0004	0.0000	0.0000	0.0000

Table 5.3: Welch test for equal CER; $\gamma = 1$

Also a t-test for CER=0 was performed. Here, results show that each of the resulting CERs is different from 0 with high confidence (>99.99%).[49]

Another finding is that the extions to GUW ranks between the GUW model with $\epsilon = 1.5$ and $\epsilon = 2$ for short estimation windows and then between $\epsilon = 1$ and $\epsilon = 1.5$ for larger estimation windows.

5.1.2 Expected CER: Short Sale Constrained

Any additional constraint, that affects the optimal allocation, decreases the CER. Including a short sell constraint necessarily results in a lower certain CER, as the optimal asset allocation in the unconstrained case showed negative weights on risky assets. The certain optimal allocation with a short sale constraint is as follows.

[49] See appendix 4.3

Asset	π_i
ITA	0.00%
GER	0.00%
SWI	91.03%
CAN	0.00%
USA	8.85%
UK	0.00%
FRA	0.00%
JPN	19.93%
r_f-Asset	-19.81%

Table 5.4: Portfolio weights; no short sale; $\gamma = 1$

We can see that the constraint affects portfolio weights considerably. Assets, whose weights are negative in the unconstrained case are now excluded from the allocation. Also french index "FRA" is excluded. The expected return decreases to 0.72% and the portfolio standard deviation decreases to 5.75%. This results in a CER of 0.555%. Performing the same analysis as before, yields results depicted in tab. 5.5 and fig. 5.2.

	Estimation window length T				
Utility-Optimized Portolios	48	60	72	96	120
Certain Tangency	0.555%	0.555%	0.555%	0.555%	0.555%
Tangency	-1.670%	-1.132%	-0.865%	-0.466%	-0.271%
LW-Tangency	-1.652%	-1.126%	-0.864%	-0.467%	-0.272%
Mean-Variance	0.361%	0.374%	0.381%	0.397%	0.410%
GUW $\epsilon = 0.5$	0.444%	0.448%	0.452%	0.457%	0.462%
GUW $\epsilon = 1$	0.457%	0.460%	0.463%	0.469%	0.473%
GUW $\epsilon = 1.5$	0.463%	0.466%	0.470%	0.474%	0.478%
GUW $\epsilon = 2$	0.466%	0.470%	0.473%	0.478%	0.482%
GUW $\epsilon = 3$	0.471%	0.474%	0.478%	0.482%	0.485%
GUW ext.	0.462%	0.464%	0.466%	0.469%	0.472%
TF - Kan Zhou	-	-	-	-	-
Benchmark Portfolios					
Min-Variance	0.474%	0.478%	0.481%	0.485%	0.487%
1 / N	0.393%	0.393%	0.393%	0.393%	0.393%
RP	0.408%	0.408%	0.409%	0.408%	0.408%
LPM (6.2%)	0.372%	0.362%	0.346%	0.323%	0.306%

Table 5.5: Expected monthly CER $\gamma = 1$; no short sale

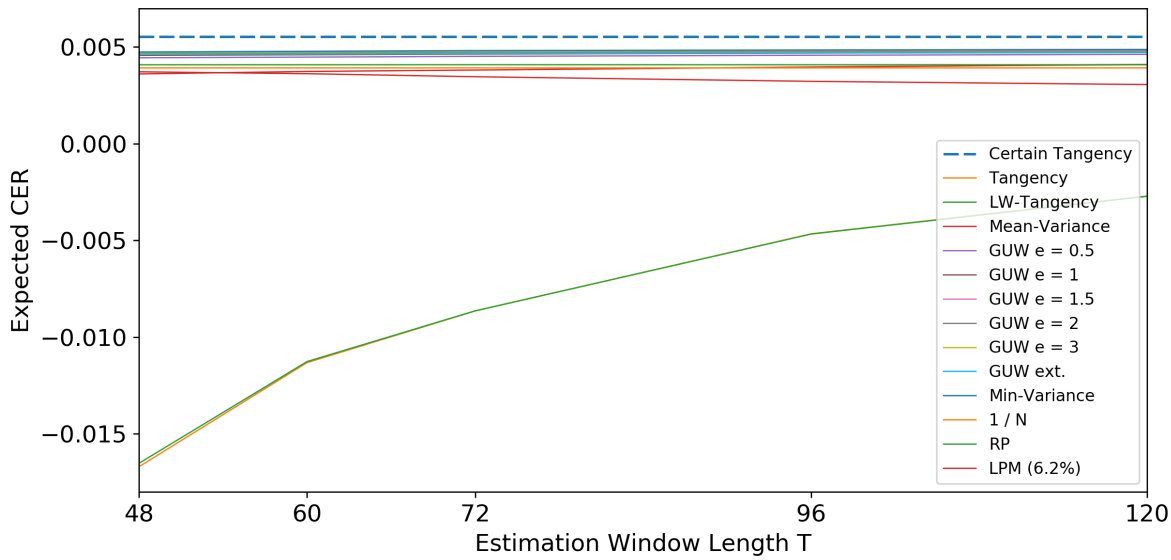


Figure 5.2: Expected CER; $\gamma = 1$; short sale not allowed

Hypothesis 1: *Models that account for parameter uncertainty, outperform the original tangency portfolio*

The GUW allocations still clearly outperform the tangency allocation for all levels of epsilon. However, the advantage decreased notably compared to the unrestricted case.

In addition to that, the positive impact of the shrunk covariance matrix faded out due to the constraint, up until the point, where the tangency and the LW-tangency perform equally well (for $T = 72 - 120$).

Hypothesis 2: *Utility-optimized portfolios perform better relative to benchmark allocations*

As in the non-constrained case, we can not see a better performance of utility-optimized allocations. Especially, the good performance of the minimum variance portfolio, which is not dependent on γ, contradicts the hypothesis.

Hypothesis 3: *Larger estimation windows lead to more accurate parameter estimates (less uncertainty) and therefore better performance*

The short sell constrained portfolios show a similar behaviour with respect to the length of the estimation window as in the non-constrained case. Here also utility-optimized portfolios and the minimum variance show a better performance with longer estimation windows.

Hypothesis 5: *Short-sale constrained portfolios perform better under uncertainty than non-constrained asset allocations*

On one hand, considering a short sale constraint especially improves expected performance of the original tangency allocation and the mean-variance allocation. On the other hand, portfolios considering uncertainty as well as the minimum variance portfolio perform slightly worse.

Additional findings In addition to the hypotheses outlined, we also see the following results of interest:

- Whereas, in the previous case without constraint, the GUW allocation yielded the best results, we can now see that it is outperformed by the minimum variance allocation.
- The GUW model is not expected perform substantially better than the mean-variance allocation without a risk-free asset.
- The performance of all strategies is less dispersed than in the non-restricted case and closer to the certain CER.
- The 1/N strategy, which by construction only considers positive weights, shows the same result as in the unconstrained case.

5.1.3 Different Values of γ

In order to see whether results are different for investors being characterized by a risk aversion different from $\gamma = 1$, we also computed results for different values of gamma ($\gamma = 0.5, 1, 1.5, 2, 3$). In this case, and considering a fixed estimation window length of $T = 60$, we observe the behaviour as depicted in tab. 5.6.

The first notable observation is, that the more risk averse an investor is, the better the expected performance of the tangency and the mean-variance allocation under uncertainty. This means, that an investor with a high risk aversion, is more satisfied when investing following these strategies, than a less risk averse investor. Also, the expected performance for a highly risk averse investor is much less dispersed than for an investor with lower risk aversion.

Another finding is, that there is no common pattern among asset allocations considering uncertainty. While the LW-tangency and the three-fund allocation show an increasing

	Risk aversion coefficient γ				
Utility-Optimized Portolios	0.5	1.0	1.5	2.0	3.0
Certain Tangency	1.320%	0.855%	0.700%	0.622%	0.545%
Tangency	-20.102%	-9.856%	-6.441%	-4.733%	-2.165%
LW-Tangency	-18.501%	-9.056%	-5.907%	-4.333%	-2.019%
Mean-Variance	-16.552%	-8.047%	-5.247%	-3.871%	-1.823%
GUW $\epsilon = 0.5$	-1.339%	-0.528%	-0.307%	-0.230%	-0.074%
GUW $\epsilon = 1$	0.203%	0.251%	0.219%	0.174%	0.136%
GUW $\epsilon = 1.5$	0.518%	0.434%	0.361%	0.295%	0.213%
GUW $\epsilon = 2$	0.564%	0.482%	0.409%	0.343%	0.249%
GUW $\epsilon = 3$	0.573%	0.504%	0.440%	0.378%	0.281%
GUW ext.	0.533%	0.438%	0.360%	0.291%	0.192%
TF - Kan Zhou	-1.798%	-0.704%	-0.340%	-0.158%	0.025%
Benchmark Portfolios					
Min-Variance	0.543%	0.496%	0.449%	0.402%	0.315%
1 / N	0.448%	0.393%	0.339%	0.284%	0.175%
RP	0.461%	0.408%	0.356%	0.304%	0.199%
LPM (6.2%)	0.500%	0.434%	0.367%	0.300%	0.166%

Table 5.6: Expected monthly CER varying γ; T= 60; short sale allowed

expected CER in γ, the GUW shows a different pattern. Depending on the ambiguity aversion ϵ we can see increasing expected CER for $\epsilon = 0.5$; however, for higher values of ϵ and the extension, this trend reverses.

Interestingly, the best performance is given by the minimum-variance allocation, for a risk aversion from $\gamma = 1.5$ onward, even though decreasing in γ, as for the other benchmark portfolios.[50]

5.1.4 Expected CER Using Control Data

The following table depicts the expected CER with using a control data set for $\gamma = 1$ and short sale allowed.[51]

[50]Considering different estimation window lengths, we obtain results, confirming the trends outlined above. See appendix D.2.2

[51]For no short sale allowed see appendix D.2.3

	Estimation window length T				
Utility-Optimized Portolios	48	60	72	96	120
Certain Tangency	2.256%	2.256%	2.256%	2.256%	2.256%
Tangency	-21.044%	-13.485%	-9.742%	-5.500%	-3.505%
LW-Tangency	-16.090%	-11.247%	-8.469%	-4.995%	-3.239%
Mean-Variance	-17.493%	-11.415%	-8.538%	-5.266%	-3.630%
GUW $\epsilon = 0.5$	-1.224%	-0.484%	-0.148%	0.242%	0.447%
GUW $\epsilon = 1$	0.470%	0.626%	0.701%	0.786%	0.841%
GUW $\epsilon = 1.5$	0.824%	0.861%	0.881%	0.905%	0.926%
GUW $\epsilon = 2$	0.893%	0.912%	0.923%	0.935%	0.947%
GUW $\epsilon = 3$	0.924%	0.935%	0.942%	0.949%	0.957%
GUW ext.	0.875%	0.893%	0.902%	0.906%	0.923%
TF - Kan Zhou	0.036%	0.352%	0.654%	0.964%	1.187%
Benchmark Portfolios					
Min-Variance	0.931%	0.936%	0.938%	0.940%	0.941%
1 / N	0.911%	0.911%	0.911%	0.911%	0.911%
RP	0.917%	0.917%	0.917%	0.917%	0.917%
LPM (6.2%)	0.836%	0.834%	0.822%	0.805%	0.780%

Table 5.7: Expected monthly CER $\gamma = 1$; short sale allowed

Hypothesis 1: *Models that account for parameter uncertainty, outperform the original tangency portfolio*

The hypothesis finds further support, when considering the results of the control data set. Firstly, the LW-tangency allocation shows a better expected performance than the tangency allocation under uncertainty.

Secondly, we see a similar behaviour with regards to the GUW and the three-fund allocation. Both, perform significantly better than the tangency portfolio.

Hypothesis 2: *Utility-optimized portfolios perform better relative to benchmark allocations*

As for the main data set, the hypothesis, can not be supported by the control analysis. Only allocations considering uncertainty, perform well among the ones being utility-optimized.

Moreover, the fact that benchmark portfolios perform rather well, contradicts the hypothesis. This result can also be found for the main data set.

Hypothesis 3: *Larger estimation windows lead to more accurate parameter estimates (less uncertainty) and therefore better performance*

Also for the control data set, we can see a similar pattern in the results. For all utility-optimized portfolios and the minimum variance allocation this relation holds, whereas it does not for the remainder (1/N, RP, LPM)

Hypothesis 5: *Short-sale constrained portfolios perform better under uncertainty than non-constrained asset allocations*

The results obtained show a behaviour consistent with the main data set. When comparing the constrained and the unconstrained case, we see that the results are much less dispersed and that especially those portfolios, that perform rather modest in the unconstrained case, now perform well.

Additional Findings The application of the analysis on a control data set yielded one interesting result. While previously, the GUW with $\epsilon = 3$ was performing best in the unconstrained case, we now see that the three-fund solution performs best for long estimation windows (96, 120 months).

It is worth to mention that the LW-tangency allocations now also shows a better performance than the mean-variance allocation in the main data, hence, the marginal improvement seems to depend on the parameters used to calibrate the analysis.

5.2 Out-of-Sample Perfomance Analysis

As introduced before, the out-of-sample performance analysis evaluates the different asset allocations based on real market data. Each asset allocation implemented is rebalanced on a monthly basis during the 38-year period from 01/1980 to 01/2018. In order to compare the results with the previous analysis, the certainty equivalent return of each strategy is reported. Other performance metrics, such as the Sharpe ratio and the Turnover (reported in parentheses) are also reported. Each of the utility-optimized portfolios and the CER figures refer to a risk aversion coefficient of $\gamma = 1$.

5.2.1 Out-of-sample Performance Short Sale Allowed

	Estimation window length T				
Utility-Optimized Portolios	48	60	72	96	120
Tangency	-0.950%	-0.464%	-0.291%	-0.434%	-0.485%
LW-Tangency	-0.819%	-0.077%	-0.169%	-0.127%	-0.222%
Mean-Variance	-12.237%	-6.669%	-5.877%	-3.531%	-2.910%
GUW $\epsilon = 0.5$	-0.958%	-0.084%	0.056%	0.225%	-0.038%
GUW $\epsilon = 1$	0.157%	0.482%	0.501%	0.539%	0.361%
GUW $\epsilon = 1.5$	0.457%	0.618%	0.588%	0.604%	0.487%
GUW $\epsilon = 2$	0.550%	0.670%	0.613%	0.622%	0.533%
GUW $\epsilon = 3$	0.589%	0.677%	0.618%	0.625%	0.564%
GUW ext.	0.556%	0.729%	0.605%	0.551%	0.441%
TF - Kan Zhou	-1.160%	-1.400%	-1.373%	-0.296%	-0.686%
Benchmark Portfolios					
Min-Variance	0.525%	0.546%	0.565%	0.569%	0.572%
1 / N	0.452%	0.452%	0.452%	0.452%	0.452%
RP	0.504%	0.525%	0.492%	0.468%	0.451%
LPM (0%)	0.904%	0.309%	0.448%	0.467%	0.251%

Table 5.8: Monthly CER out-of-sample; $\gamma = 1$; short sale allowed

Hypothesis 1: *Models that account for parameter uncertainty, outperform the original tangency portfolio*

In table 5.8, we can see that the GUW portfolios perform better than, the tangency allocation, with the CER increasing in ϵ. Also the extended GUW allocation yields a good performance. For $T = 60$, the GUW ext. portfolio represents the best performing allocation among all portfolios.

	Estimation Length T				
	48	60	72	96	120
Utility-Optimized Portfolios					
Tangency	-0.075	-0.042	-0.023	-0.022	-0.012
	(1.91)	*(1.69)*	*(1.34)*	*(1.17)*	*(1.06)*
LW-Tangency	-0.051	0.018	0.009	0.019	-0.009
	(1.88)	*(0.77)*	*(1.45)*	*(1.24)*	*(3.54)*
Mean-Variance	0.042	0.018	0.027	0.044	0.009
	(1.95)	*(1.68)*	*(12.31)*	*(0.05)*	*(0.55)*
GUW $\epsilon = 0.5$	0.044	0.044	0.044	0.045	0.024
	(0.05)	*(0.05)*	*(0.68)*	*(6.36)*	*(1.20)*
GUW $\epsilon = 1$	0.048	0.050	0.044	0.047	0.029
	(0.08)	*(0.06)*	*(0.05)*	*(0.10)*	*(1.20)*
GUW $\epsilon = 1.5$	0.049	0.053	0.050	0.048	0.029
	(3.62)	*(18.71)*	*(0.06)*	*(0.62)*	*(2.96)*
GUW $\epsilon = 2$	0.051	0.060	0.053	0.050	0.039
	(1.13)	*(0.11)*	*(0.09)*	*(3.81)*	*(0.71)*
GUW $\epsilon = 3$	0.055	0.064	0.057	0.051	0.044
	(0.11)	*(1.52)*	*(5.54)*	*(0.06)*	*(0.09)*
GUW ext.	0.061	0.066	0.061	0.056	0.044
	(0.31)	*(0.25)*	*(2.23)*	*(1.52)*	*(0.05)*
TF - Kan Zhou	0.062	0.067	0.062	0.063	0.048
	(0.78)	*(2.69)*	*(1.25)*	*(0.88)*	*(0.59)*
Benchmark Portfolios					
Min-Variance	0.064	0.072	0.068	0.064	0.049
	(1.09)	*(0.93)*	*(0.79)*	*(0.67)*	*(0.05)*
1 / N	0.066	0.083	0.071	0.071	0.052
	(9.19)	*(0.66)*	*(0.85)*	*(0.59)*	*(0.49)*
RP	0.071	0.087	0.071	0.073	0.060
	(0.56)	*(0.97)*	*(0.21)*	*(0.16)*	*(0.39)*
LPM (6.2%)	0.127	0.089	0.077	0.079	0.073
	(1.51)	*(0.48)*	*(0.43)*	*(0.34)*	*(0.13)*

Table 5.9: SR and TO (in parentheses); $\gamma = 1$; monthly figures

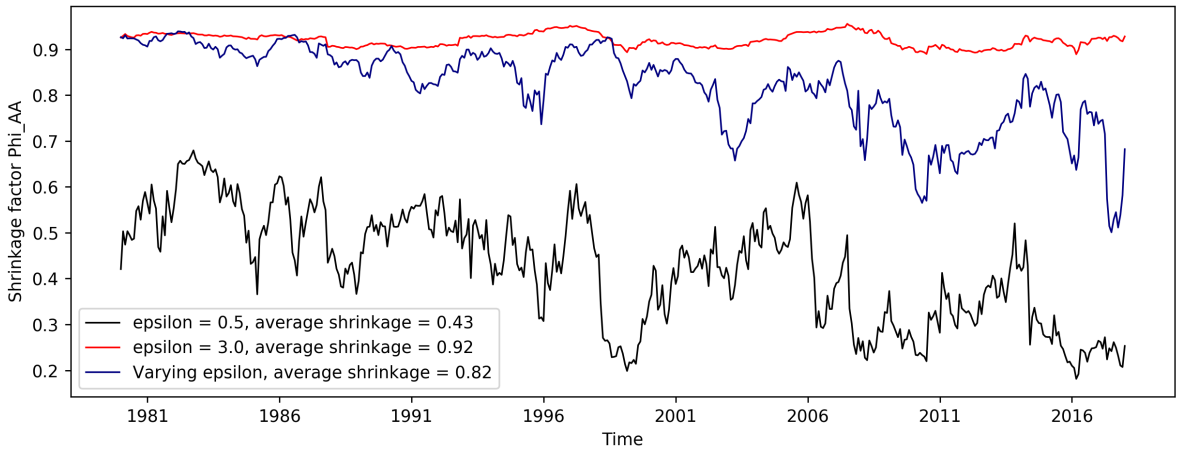


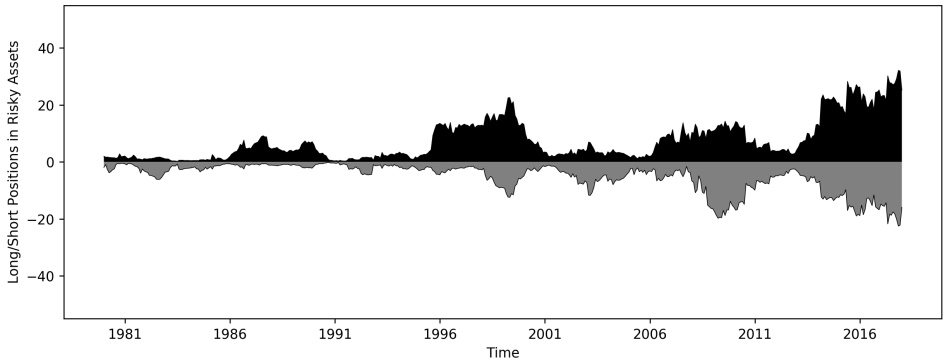
Figure 5.3: ϕ_{AA} factor at different values of ϵ; $T = 60$; out-of-sample analysis

Fig. 5.3 shows that by construction the portfolio exposure to the minimum-variance portfolio increases in ϵ.

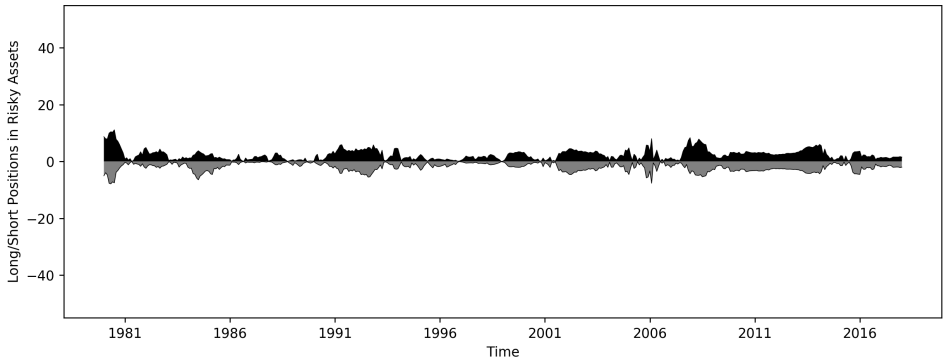
Another portfolio accounting for uncertainty, the LW-tangency allocation, performs better than the simple tangency allocation. However, the increase in performance is rather modest and not constant in T.

The three-fund allocation yields a lower CER than both the tangency and the minimum variance allocation for short estimation windows. This is surprising, as those portfolios are the ones from which it was originated.

The results shown in table 5.9, comprises the other performance metrics analyzed. We can see supporting and contradicting results with respect to the hypothesis. While GUW, LW-Tangency and TF-Kan Zhou all show higher Sharpe ratios than the tangency allocation, we see that from $\epsilon = 1.5$ for T=48 and from $\epsilon = 1$ for T=60 onward, the turnover of the GUW allocation is better (lower) than for the tangency allocation. However, the three fund allocation shows a much higher TO, which can also be seen by comparing the two previously established fig. 5.4a and fig. 5.4b. Here it should be noted, that when referring to the Turnover, an investor prefers to trade as few as possible.



(a) TF-Kan Zhou



(b) Tangency

Figure 5.4: Weights of selected allocations in risky assets over time; T=60; $\gamma = 1$

Hypothesis 2: *Utility-optimized portfolios perform better relative to benchmark allocations*

We see contradicting results. As we could see in the findings for hypothesis 1, the GUW allocations perform quite well, whereas the other utility-optimized portfolios perform rather poorly. Also, we can see that the benchmark optimized portfolios show a good performance. Hence, we do not see clear findings in order to support the hypothesis.

The pattern is confirmed when performance is assessed based on the Sharpe ratio. Benchmark portfolios overall show quite high Sharpe ratios. Also, the turnover contradicts the hypothesis. Especially, the utility-optimized portfolios show high turnover rates, whereas the benchmark portfolios (except for the LPM) show a low TO.

Hypothesis 3: *Larger estimation windows lead to more accurate parameter estimates (less uncertainty) and therefore better performance*

The hypothesis can not be confirmed for all the utility-optimized portfolios. Apart from the mean-variance, these portfolios do not improve in T. We see, that the marginal improvement in CER due to more data taken into account present different patterns depending on the portfolio strategies.

The Sharpe ratios of the mean-variance, GUW and the three fund allocation are decreasing in T.

In a one fund setting instead, the Sharpe ratios for both the it appears that a longer estimation window, such as when $T = 120$ result in a low SR. While $T = 48$ and $T = 60$ we can see higher SRs. In terms of TO, we can see a decreasing pattern in T, with only few exceptions.

Hypothesis 4: *The performance of the out-of-sample analysis follows the expected performance (CER)*

The majority of asset allocations show a somewhat similar pattern as in the expected CER analysis, apart from the following exceptions:

- The tangency allocation outperforms the mean variance allocations
- Longer estimation windows do not necessarily lead to improved performance
- The estended GUW allocation was able to outperform the other allocations

Additional Findings:

In the out-of-sample analysis, the tangency allocation performs notably better than predicted, while the performance of the mean-variance allocation is still rather modest. It can be observed, that the mean-variance allocation shows rather high long and short positions.

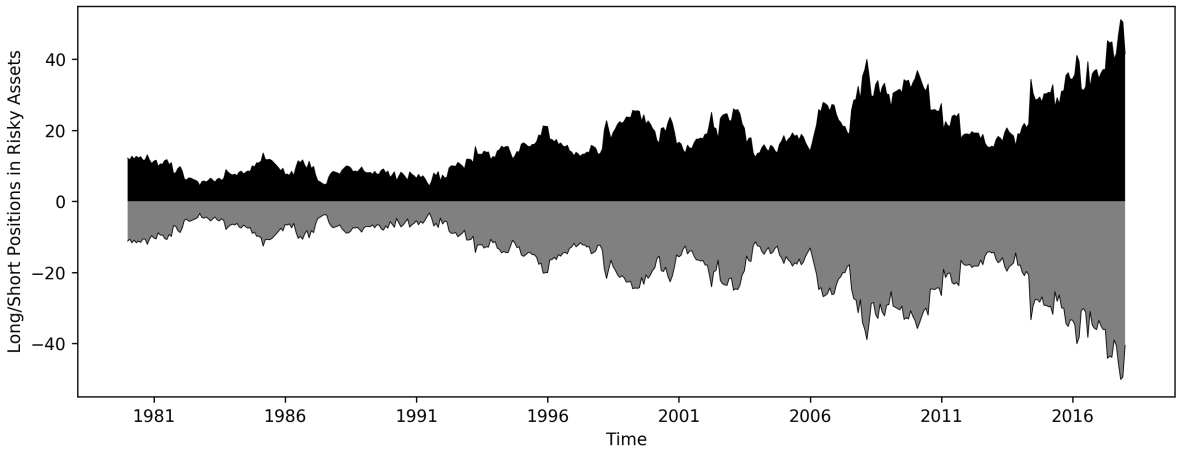


Figure 5.5: Weights in risky assets: "mean-variance"-allocation

However, it should be noted that the performance of some benchmark portfolios is as good as of the portfolios considering uncertainty.

5.2.2 Short sale not allowed

As in the out-of-sample analysis, the negative weights reduce to zero.

	Estimation window length T				
Utility-Optimized Portolios	48	60	72	96	120
Tangency	-0.704%	0.111%	-0.323%	0.162%	0.587%
LW-Tangency	-0.641%	0.117%	-0.291%	0.205%	0.569%
Mean-Variance	0.457%	0.532%	0.527%	0.466%	0.558%
GUW $\epsilon = 0.5$	0.495%	0.552%	0.510%	0.509%	0.550%
GUW $\epsilon = 1$	0.475%	0.539%	0.520%	0.527%	0.542%
GUW $\epsilon = 1.5$	0.469%	0.537%	0.530%	0.537%	0.536%
GUW $\epsilon = 2$	0.474%	0.533%	0.536%	0.546%	0.535%
GUW $\epsilon = 3$	0.473%	0.515%	0.537%	0.555%	0.538%
GUW ext.	0.463%	0.542%	0.539%	0.564%	0.535%
TF - Kan Zhou	-	-	-	-	-
Benchmark Portfolios					
Min-Variance	0.462%	0.488%	0.521%	0.530%	0.539%
1 / N	0.452%	0.452%	0.452%	0.452%	0.452%
RP	0.457%	0.460%	0.466%	0.463%	0.462%
LPM (0%)	0.456%	0.240%	0.225%	0.547%	0.278%

Table 5.10: Monthly CER out-of-sample; $\gamma = 1$; no short sale

	Estimation Length T				
	48	60	72	96	120
Utility-Optimized Portfolios					
Tangency	0.030	0.070	-0.037	0.012	0.069
	(0.93)	*(0.95)*	*(0.45)*	*(0.36)*	*(0.33)*
LW-Tangency	0.037	0.071	-0.029	0.017	0.066
	(0.91)	*(0.96)*	*(0.46)*	*(0.34)*	*(0.33)*
Mean-Variance	0.046	0.059	0.058	0.048	0.064
	(0.33)	*(0.29)*	*(0.23)*	*(0.16)*	*(0.18)*
GUW $\epsilon = 0.5$	0.053	0.065	0.056	0.056	0.064
	(0.22)	*(0.20)*	*(0.19)*	*(0.16)*	*(0.15)*
GUW $\epsilon = 1$	0.049	0.063	0.058	0.060	0.063
	(0.21)	*(0.18)*	*(0.17)*	*(0.14)*	*(0.13)*
GUW $\epsilon = 1.5$	0.048	0.063	0.061	0.063	0.062
	(0.19)	*(0.16)*	*(0.15)*	*(0.12)*	*(0.12)*
GUW $\epsilon = 2$	0.049	0.062	0.062	0.065	0.062
	(0.18)	*(0.15)*	*(0.14)*	*(0.11)*	*(0.11)*
GUW $\epsilon = 3$	0.049	0.058	0.063	0.067	0.063
	(0.17)	*(0.15)*	*(0.13)*	*(0.10)*	*(0.10)*
GUW ext.	0.046	0.064	0.063	0.069	0.062
	(0.19)	*(0.16)*	*(0.15)*	*(0.13)*	*(0.13)*
TF - Kan Zhou	-	-	-	-	-
	-	-	-	-	-
Benchmark Portfolios					
Min-Variance	0.046	0.052	0.060	0.062	0.065
	(0.15)	*(0.12)*	*(0.11)*	*(0.10)*	*(0.09)*
1 / N	0.044	0.044	0.044	0.044	0.044
	(0.05)	*(0.05)*	*(0.05)*	*(0.05)*	*(0.05)*
RP	0.045	0.046	0.047	0.046	0.046
	(0.05)	*(0.05)*	*(0.05)*	*(0.05)*	*(0.05)*
LPM (6.2%)	0.047	0.014	0.007	0.062	0.021
	(0.65)	*(0.53)*	*(0.63)*	*(0.61)*	*(0.45)*

Table 5.11: SR and TO (in parentheses); $\gamma = 1$; no short sale; monthly figures

Hypothesis 1: *Models that account for parameter uncertainty, outperform the original tangency portfolio*

GUW portfolios still outperform the tangency allocation, for all levels of uncertainty aversion. It appears however, that increasing the ϵ does not lead to better performance. The relative advantage that GUW had on a mean variance portfolio without a risk free asset is decreased heavily such that there is no clear winner between these two strategies. Shrinking the covariance matrix in presence of a constraint on the short sales lead to a less considerable benefit than when short selling is allowed. This is also reflected in higher Sharpe ratios for GUW portfolios. The LW-tangency also shows a higher Sharpe ratios than the tangency allocation does.

Hypothesis 2: *Utility-optimized portfolios perform better relative to benchmark allocations*

As in the unconstrained analysis, there is no clear out-performance of the utility-optimized models over the other portfolios. Although mean-variance, GUW and three fund allocation all outperform the benchmark portfolios, this does not hold for the tangency and the LW-tangency allocation. Among the benchmark portfolios, the minimum-variance strategy represents the best in terms of CER.

Except for the tangency and the LW-tangency allocation, all utility-optimized portfolios show higher Sharpe ratios than the benchmark portfolios.

Hypothesis 3: *Larger estimation windows lead to more accurate parameter estimates (less uncertainty) and therefore better performance*

Under constraint, we observe the same trend as in the unconstrained analysis. The portfolios overall show an improvement in CER, when increasing the window length.

Hypothesis 4: *The performance of the out-of-sample analysis follows the expected performance (CER)*

While generally we can confirm the patterns observed in the expected CER analysis, we also observe contradicting findings. A surprising result is that the tangency allocation performs the best in terms of CER for T=120, which was not given in any other analysis so far.

Hypothesis 5: *Short-sale constrained portfolios perform better under uncertainty than non-constrained asset allocations*

Again, the CER is less dispersed, as was already observed in the Monte Carlo Analysis. This hypothesis seems to hold especially for portfolios, that performed poorly in the non-restricted analysis, such as tangency, LW-tangency and mean-variance, as they show a stark increase in performance. However, the other allocations perform slightly worse, except for 1/N which by construction only allows positive weights. Also, it should be mentioned that the extended GUW allocation performs best among all GUW allocations for $T = 72$ and $T = 96$.

For the other performance metrics, we can find somewhat similar patterns as for the CER results described right above. The portfolios that reported poor results in absence of short sales constraints, now show notably better results.

5.2.3 Different Values of γ

The main purpose here is to check whether the same patterns are found when assuming a different risk aversion.

	Risk aversion coefficient γ				
Utility-Optimized Portolios	0.5	1	1.5	2.0	3.0
Tangency	-1.288%	-0.464%	-0.190%	-0.052%	0.085%
LW-Tangency	-0.510%	-0.077%	0.068%	0.140%	0.212%
Mean-Variance	-13.804%	-6.669%	-4.320%	-3.166%	-2.056%
GUW ϵ =0.5	-0.700%	-0.084%	0.070%	0.115%	0.103%
GUW ϵ =1	0.349%	0.482%	0.471%	0.431%	0.330%
GUW ϵ =1.5	0.568%	0.618%	0.575%	0.518%	0.401%
GUW ϵ =2	0.713%	0.670%	0.609%	0.547%	0.429%
GUW ϵ =3	0.740%	0.677%	0.616%	0.557%	0.445%
GUW ext.	0.809%	0.729%	0.646%	0.571%	0.436%
TF - Kan Zhou	-3.160%	-1.400%	-0.813%	-0.519%	-0.226%
Benchmark Portfolios					
Min-Variance	0.588%	0.546%	0.503%	0.460%	0.375%
1 / N	0.509%	0.452%	0.396%	0.339%	0.226%
RP	0.579%	0.525%	0.471%	0.416%	0.308%
LPM (0%)	0.380%	0.309%	0.238%	0.167%	0.025%

Table 5.12: Monthly CER out-of-sample; varying γ; T= 60; short sale allowed

As in the Monte Carlo Analysis, the CER of tangency and mean-variance portfolios increases in γ. In other words, a more risk averse investor is better off by investing in these portfolios than a less risk averse investor. For GUW portfolios, a higher γ does not lead to improvements in the allocation.

The benchmark portfolios all show a decreasing CER when γ increases.

5.2.4 Out-of-Sample Performance Analysis Using Control Data

The previous analysis is also performed with the control data set. Tables 5.13 and 5.14 show the average CER in the out-of-sample, together with average return, Sharpe ratio and the turnover.[52]

	Estimation window length T				
Utility-Optimized Portolios	48	60	72	96	120
Tangency	-7.815%	-1.115%	0.131%	0.051%	0.508%
LW-Tangency	0.136%	0.498%	1.318%	0.866%	0.950%
Mean-Variance	0.859%	0.962%	0.951%	0.913%	0.788%
GUW ϵ =0.5	-2.600%	-1.201%	-0.482%	-0.117%	0.231%
GUW ϵ =1	-0.029%	0.490%	0.640%	0.633%	0.737%
GUW ϵ =1.5	0.650%	0.874%	0.868%	0.826%	0.868%
GUW ϵ =2	0.867%	0.961%	0.935%	0.888%	0.917%
GUW ϵ =3	0.994%	1.003%	0.973%	0.932%	0.956%
GUW ext.	1.048%	0.965%	0.968%	0.906%	0.953%
TF - Kan Zhou	-0.693%	0.297%	1.262%	2.571%	2.719%
Benchmark Portfolios					
Min-Variance	1.038%	1.038%	1.005%	1.009%	1.034%
1 / N	0.988%	0.991%	0.991%	0.991%	0.991%
RP	0.998%	1.003%	1.000%	1.001%	1.001%
LPM (6.2%)	0.904%	0.930%	0.686%	0.681%	0.821%

Table 5.13: Monthly CER out-of-sample; $\gamma = 1$; control data set; short sale allowed

Hypothesis 1: *Models that account for parameter uncertainty, outperform the original tangency portfolio*

The hypothesis finds support, as for almost all allocations we see a better performance in terms of CER. However, one exception to this is the GUW allocation with $\epsilon = 0.5$.

[52]for results when short sales are not allowed, see tables 4.22 and 4.23 in appendix.

	Estimation Length T				
	48	60	72	96	120
Utility-Optimized Portfolios					
Tangency	-0.025	0.038	0.041	0.019	0.071
	(8.83)	*(4.72)*	*(2.23)*	*(1.56)*	*(1.45)*
LW-Tangency	0.107	0.138	0.141	0.103	0.110
	(1.48)	*(1.54)*	*(0.58)*	*(0.39)*	*(0.41)*
Mean-Variance	0.114	0.131	0.128	0.123	0.103
	(0.36)	*(0.30)*	*(0.28)*	*(0.24)*	*(0.20)*
GUW $\epsilon = 0.5$	0.030	0.031	0.037	0.025	0.038
	(5.80)	*(4.28)*	*(3.27)*	*(2.14)*	*(1.60)*
GUW $\epsilon = 1$	0.046	0.067	0.077	0.076	0.094
	(2.78)	*(2.04)*	*(1.56)*	*(1.07)*	*(0.85)*
GUW $\epsilon = 1.5$	0.078	0.111	0.118	0.118	0.129
	(1.67)	*(1.22)*	*(0.95)*	*(0.69)*	*(0.58)*
GUW $\epsilon = 2$	0.115	0.142	0.144	0.140	0.149
	(1.14)	*(0.86)*	*(0.71)*	*(0.54)*	*(0.46)*
GUW $\epsilon = 3$	0.163	0.169	0.166	0.161	0.168
	(0.74)	*(0.61)*	*(0.53)*	*(0.42)*	*(0.36)*
GUW ext.	0.176	0.160	0.171	0.155	0.168
	(0.76)	*(0.63)*	*(0.52)*	*(0.41)*	*(0.35)*
TF - Kan Zhou	0.135	0.147	0.168	0.212	0.218
	(54.89)	*(36.93)*	*(27.35)*	*(16.77)*	*(12.23)*
Benchmark Portfolios					
Min-Variance	0.212	0.215	0.208	0.209	0.215
	(0.40)	*(0.33)*	*(0.28)*	*(0.21)*	*(0.18)*
1 / N	0.177	0.177	0.177	0.177	0.177
	(0.04)	*(0.04)*	*(0.04)*	*(0.04)*	*(0.04)*
RP	0.188	0.189	0.188	0.188	0.188
	(0.05)	*(0.04)*	*(0.04)*	*(0.04)*	*(0.04)*
LPM (6.2%)	0.127	0.139	0.090	0.087	0.109
	(1.51)	*(1.34)*	*(1.22)*	*(1.01)*	*(1.00)*

Table 5.14: SR and TO (in parentheses); $\gamma = 1$; control dataset; monthly figures

The shrinkage adds value to the allocations as it increases the Sharpe ratios. Indeed the LW-Tangency portfolio outperforms the Tangency for all values of T.

Hypothesis 2: *Utility-optimized portfolios perform better relative to benchmark allocations*

The control data set shows even more contradicting results.

Among the benchmark portfolios, the minimum-variance allocation again shows one of the highest performance. Also other performance metrics pinpoint towards a rejection of the hypothesis.

Hypothesis 3: *Larger estimation windows lead to more accurate parameter estimates (less uncertainty) and therefore better performance*

While, widening the estimation windows does have a positive effects on the majority of the utility-optimized portfolio, whereas it does not affect benchmark portfolios. This, as well as the other performance metrics confirm previous findings.

Hypothesis 4: *The performance of the out-of-sample analysis follows the expected performance (CER)*

As in the expected CER analysis, we can see that the best performance for long estimation windows is given by the TF-Kan Zhou allocation. However, the trend of increasing CER in T is not visible in the out-of-sample application. Thus, the hypothesis can not be confirmed.

Hypothesis 5: *Short-sale constrained portfolios perform better under uncertainty than non-constrained asset allocations*

As for the main data set, the performance in terms of CER is less dispersed.[53].

Additional Findings:

Another interesting results is the good performance of the mean-variance allocation and the strong improvement of the uncertainty adjustment by the LW-Tangency allocation, which both show consistent positive results. These findings could not be observed in the analysis with the main data set.

[53]Results of no-short sale in Appendix tab. 4.22 and 4.23

Part III

Discussion and Conclusion

6 Discussion

Conducting an analysis of expected performance and out-of-sample performance enabled us to assess the applicability of Markowitz' investment model and its alternatives.

6.1 Summary of Main Findings

Hypothesis 1: *Models that account for parameter uncertainty, outperform the original tangency portfolio*

In the analysis of expected performance, the results support the hypothesis, for all portfolios taking uncertainty into account. This pattern is seen in both, the constrained, and the unconstrained case (LW-tangency, GUW, TF-Kan Zhou). This, suggests that the hypothesis holds.

In the out-of-sample analysis however, we find the following contradicting results:

- The LW-tangency allocation performs better out-of-sample than the tangency allocation in almost all cases
- The GUW allocations show a substantially better performance over almost all values of epsilon
- While the TF-Kan Zhou allocation underperforms the tangency allocation in the analysis based on the main data set, it represents one of the best allocations, in the control data set. This indicates that the results of the out-of-sample analysis are not robust, and depend on the data set used

The better performance might be caused by more robust parameter estimates for allocations accounting for uncertainty. In particular:

- for the LW-tangency allocation, the variance-covariance estimate is not that biased by single values in the return series. This seems to improve the estimation over all simulations
- for the GUW allocations, a higher ϵ shrinks the portfolio more from the mean-variance portfolio towards the minimum variance portfolio. As the minimum variance portfolio only depends on the estimate of $\mathbf{\Sigma}$, the parameter uncertainty fades out for higher values of ϵ

- for TF-Kan Zhou allocations, the shrinkage works differently than the GUW shrinkage, as it shrinks the tangency allocation towards the minimum-variance allocation. Therefore, it relies less on the mean-return estimate and more on the variance-covariance estimate. This, according to Kan and Zhou is also a consequence of the non-perfect correlation of estimation errors

Hypothesis 2: *Utility-optimized portfolios perform better relative to benchmark allocations*

This hypothesis does not find support. Firstly, we find contradicting results among models that take investors' preferences into account. Secondly, benchmark portfolios show comparably solid performance.

This holds especially for the case of the minimum variance allocation, which showed promising results in each analysis. Hence, the hypothesis can not be supported.

A reason for that is, that even though utility-optimized allocations aim to provide an optimal solution, the uncertainty induced by the estimation of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ is reflected in a bad performance.

The good performance of the minimum-variance allocation under uncertainty is probably caused by the lower number of parameters to be estimated. As it represents the least volatile allocation, the CER is positively affected. However, it should be noted, that the performance of the minimum-variance allocation highly depends on the returns yielded by the least correlated and volatile assets. Thus, the performance is dependent on the return data set.

Hypothesis 3: *Larger estimation windows lead to more accurate parameter estimates (less uncertainty) and therefore better performance*

On one hand, the hypothesis holds for all utility-optimized allocations and the minimum variance allocation in the expected CER analysis. This seems reasonable, since the estimators of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, by construction become less uncertain in the Monte Carlo analysis. As already stated above, the minimum-variance allocation performs well, since it is the least volatile allocation. Hence, for longer estimation windows, the more precise estimate of $\boldsymbol{\Sigma}$ exploits this advantage even further.

On the other hand, we can not find the same behaviour among the risk parity allocation and the LPM allocation. This probably stems from the fact that both allocations do not

optimize the CER, as they are neither aiming to increase the expected return, nor to decrease the portfolio variance.

In the out-of-sample application however, we can not see such a clear pattern anymore. Especially, for long estimation windows, the CER decreases. While this might seem surprising on first sight, it illustrates the issue raised in sec. 4.5.3, that a too long estimation period might imply using irrelevant data.

Hypothesis 4: *The performance of the out-of-sample analysis follows the expected performance (CER)*

As already outlined above, the out-of-sample performance does not always follow the expected performance. It deviates for longer estimation windows, and also shows opposing results with respect to the performance of the tangency allocation. The out-of-sample performance of the tangency allocation is substantially better than the one of the mean-variance allocation, whereas in the expected CER analysis, the opposite was predicted. This trend, however, reverses in the short sale constrained case, and also in the out-of-sample analysis based on the control data. Thus, especially these portfolios show a low degree of predictability in the analysis.

These contradicting results probably exist, as the results of the Monte Carlo analysis are less stable in the case of tangency and the mean-variance allocation. Where in turn, this lower stability stems from the higher sensitivity towards uncertainty. Another possible reason might be that returns do not follow a normal distribution, as was assumed in the analysis of expected CER. Also, for the expected CER analysis, 10,000 simulations were drawn. This is considerably higher than the 457 months of out-of-sample returns.

It should also be noted, that the GUW allocations, as well as the benchmark portfolios overall show coherent results, among both analyses.

Thus, we observe a lower predictability of the out-of-sample performance, for the (LW-)tangency and the mean-variance allocation.

Hypothesis 5: *Short-sale constrained portfolios perform better under uncertainty than non-constrained asset allocations*

Both, the expected CER analysis and the out-of-sample performance analysis show a similar behaviour, with respect to introducing a short sale constraint. Even though we can see a poor performance of the tangency, LW-tangency and the mean-variance allocation in the unconstrained case, this sharply improves, when introducing the constraint.

A reason for this might be the following. Whilst, in the unrestricted case single positions can theoretically have any weight within $-\infty$ and $+\infty$, as could be seen in fig. 5.4 and 5.5, in the short sale constrained case weights are restricted to much less extreme positions. In a one-fund setting there is no opportunity to leverage. Hence, the maximum possible weight of a single asset is restricted to 1. In a two-fund setting, this weight is restricted to ω_p. Thus, uncertainty impacts allocations less considerably.

Contradicting the pattern of the upper discussed portfolios, we observe a minor performance of allocations that showed a good performance without a constraint. A possible reason is, that the full potential of diversification can not be exploited.

Additional Findings:

Worth to mention is that the extension to the GUW-allocation, proposed in sec. 3.3.3 yields interesting results. We see that even though it shrinks on average less than the GUW-allocation with $\epsilon = 3$, it can show a superior performance.[54] This indicates, that it allocates a stronger weight towards the mean-variance portfolio, if parameters are less uncertain. Hence, in cases when the allocation is expected to perform well.

Moreover, the TF-Kan Zhou allocation performs better for the control data set in the expected CER analysis, as well as in the out-of-sample analysis. The reason for that might be in the nature of the data, however, we also can not exclude coincidence.

6.2 Comparison of Empirical Findings with Respect to Further Research

For the comparison, we focus on three different articles conducting a study of performance, namely the ones from Kan and Zhou (2007), Garlappi et al. (2007) and Demiguel et al. (2009). While Kan and Zhou perform the expected CER analysis, Garlappi et al. and Demiguel et al. focus on the out-of-sample performance.

[54] see fig. 5.3 and tab. 5.8

6.2.1 Comparison Expected CER

Kan and Zhou examine the following strategies relevant to us

- Three-fund allocation of Kan and Zhou (2007)
- Ambiguity aversion allocation of Garlappi et al. (2007)
- Minimum-variance allocation
- Tangency allocation

Their analysis comprises estimation lengths from 60 months up to 480 months. For the longest estimation window, they find that their proposed three-fund strategy actually performs the best. However, for shorter estimation windows, the authors observe the same pattern seen in our results. The ambiguity aversion portfolio (GUW) performs best among all portfolios in their, as well as, in our analysis. On the other hand, considering the results that our control dataset yields, we see that the three-fund allocation of Kan Zhou already performs best from 96 months onward. This suggests, that using a different data set, can have an impact on the ranking of the strategies in the Monte Carlo analysis.

Nonetheless, we should keep in mind that increasing the estimation window length does not necessarily lead to improved parameter estimation. As explained in sec. 4.5.3 companies, industries and even economies change over time, which impels an upper bound to the estimation period. This problem can be seen in the out-of-sample analysis, where especially, for long estimation windows, we observe a decreasing performance.

Another common finding is, that the minimum variance portfolio generally shows a better performance than the tangency allocation, and that the tangency allocation represents the worst performing allocation.

Worth to mention is also, that they can not find positive expected CER for the tangency allocation with the classic plug-in estimators until an estimation length of 360 months. This underlines the significant impact of parameter uncertainty on the allocation.

Considering the expected CER under a short-sale constraint, there is unfortunately no results to be compared, as Kan and Zhou only performed their analysis in a non-restricted setting.

6.2.2 Comparison Out-of-Sample Performance

Demiguel et al. (2009) examine the certainty equivalent return and the Sharpe ratio based on various data sets. Comparing their research to our findings, we find results, depicted in the following.

1/N allocation:

The authors find that the portfolios tested, do not show a CER that is "statistically superior" to the $1/N$ strategy.[55] Comparing their conclusion with our findings, we see that the 1/N-strategy generally yields confirming results.

Considering the Sharpe ratios, the authors also conclude that the $1/N$ strategy outperforms the Mean Variance portfolio under uncertainty. This tendency is also found in our analysis (see tab. 5.9).

TF-Kan Zhou (three-fund allocation):

The performance of three fund allocation is poor, for all the data sets considered in their work. This seems in line with what we found, with a slight improvement in T.

In terms of Sharpe ratio, the authors find that the three-fund performance is highly dependent on the data set used, therefore not showing any consistent pattern. These contradicting findings are coherent with our analysis, where very different Sharpe ratios were found for the main data set and the control data set.

Minimum-variance allocation:

While overall, the authors find promising results CERs for the minimum-variance allocation, we can also see that it depends on the data set used, as suggested before.

According to the Sharpe ratios, the minimum-variance allocation is found to have attractive risk-adjusted performance across different data sets. The same has been found in our the out-of-sample analysis. It appears that focusing only on the variance-covariance input yields positive effects both in terms of risk adjusted returns and CER, however depending on the data set.

[55] With "statistically superior" they refer to p-values, obtained when testing for a difference in Sharpe ratio

Mean-variance allocation:

The authors find poor results with respect to the CER. This is confirmed for almost all the data sets considered. This is consistent with our analysis.

In terms of risk adjusted returns, they find that the mean-variance allocation performs significantly worse than the 1/N allocation in 5 out of 6 cases (in 2/6 cases significantly). This trend is confirmed, in our analysis, as we consistently find Sharpe ratios below the ones of 1/N-allocation.

Garlappi et al. (2007) perform an out-of-sample performance analysis of the following allocations, relevant to us:

- Mean-variance allocation
- Minimum-variance allocation
- Ambiguity aversion allocation (GUW)

As a performance indicator, the authors consider the ratio between the mean return and the standard deviation.

	Short Sale Allowed		*Short Sale Not Allowed*	
Portfolio	Our results	Garlappi et al.	Our results	Garlappi et al.
Mean-Variance	0.0413	0.0190	0.1221	0.1774
GUW $\epsilon = 0.5$	0.0550	0.0841	0.1338	0.2391
GUW $\epsilon = 1$	0.0855	0.1491	0.1372	0.2533
GUW $\epsilon = 1.5$	0.1106	0.1909	0.1385	0.2607
GUW $\epsilon = 2$	0.1256	0.2144	0.1399	0.2647
GUW $\epsilon = 3$	0.1398	0.2369	0.1425	0.2695
Min-Variance	0.1594	0.2827	0.1491	0.2831

Table 6.1: Comparison of our findings for $\frac{\bar{r}}{\sigma}$ with Garlappi et al. (2007)

Same as Garlappi et al., we observe that the return-risk ratio increases in ϵ. This pattern reflects the increasing shrinkage towards the minimum-variance allocation, as can be seen in tab. 6.1.

Moreover, the performance of the mean-variance allocation increases considerably, when imposing a short-sale constraint. This result is coherent with our findings.

However, we can see that the performance in resulting from our analysis is notably lower. This difference is probably caused by the different length of the period examined. In our case the time frame encloses the global financial crisis, hence, a period of high volatility and low returns.

6.3 Limitations and Suggestion for Further Research

The limitations to the analyses performed indicate possible areas of research.

The assumption of CARA utility characteristics is not realistic as a wealthy investor is willing to give up the same absolute amount to avoid a risky gamble, as a poor one does. In addition to that, the quadratic utility curve, implied by only considering the first and second moment of a return distribution does not seem to be a good proxy. Return distributions are usually not normally distributed. This deviation from normality can also be seen in our data set and hence show higher moments (Skew, Kurtosis). Thus, even though it is easier to derive the optimal allocation in such a setting, research with different utility characteristics has to be conducted.

Moreover, the assumption of normally distributed returns is not fulfilled for a majority of asset classes, as was seen with our data. However, it is unclear to which extent this non-normality impacts out-of-sample performance.

In our analysis we see that the allocation expected to perform best, is not necessarily excelling in an out-of-sample application. The reasons for this deviation might be both the high sensitivity towards inputs and the non-normal nature of returns with varying parameters over time. Nonetheless, these are just suppositions. Thus, research in this area could yield further evidence.

While Demiguel et al. (2009), test for differences in Sharpe ratio, the test applied assumes IID normally distributed return series. However, if the assumptions are not kept, the results of the test are invalid (Ledoit and Wolf, 2008). In our case, the resulting portfolio returns do not comply with the assumptions. Thus, they were not tested. In addition to that, we did not perform a test of significance for the CER in the out-of-sample application, as there is no adequate test available.

7 Conclusion

The Markowitz framework assumes, that input parameters are certain. However, this assumption is unrealistic, as $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ can not be estimated without imprecision.

In this thesis, we therefore studied different techniques accounting for parameter uncertainty in the Markowitz setting, and their performance in comparison with alternative strategies. The options examined, comprise shrinkage techniques in a Bayesian estimation setting, developed by Ledoit and Wolf (2004) and Kan and Zhou (2007), as well as the ambiguity aversion model of Garlappi et al. (2007). Moreover, the effect of changing the estimation window length, and of imposing constraints on portfolio weights, has been assessed.

For the tangency and the one-fund mean-variance allocation, we found highly volatile results in the out-of-sample application. These deviate considerably from the expectation. The discrepancy, however, is not observed in the short-sale constrained case.

Moreover, we found evidence, that accounting for uncertainty increases the expected performance of the Markowitz model. The Ledoit-Wolf shrinkage applied in a tangency setting shows consistent improvement in both, the expected, and the real-life out-of-sample performance. However, as for the tangency and the mean-variance allocation, the expected performance is not a good predictor. The out-of-sample performance of the Garlappi et al. model, and the three-fund allocation is congruent with what was expected. Especially, the Garlappi et al. (2007) model performs consistently well.

Following the ambiguity aversion framework developed by Garlappi et al., we proposed an extension, that considers the reliability of the data, considered to estimate the input parameters. We see evidence, that yet on average applying a lower shrinkage towards the minimum-variance allocation than in a constant ambiguity aversion setting, it can show superior results. In addition to that, we did not find support, that increasing the estimation window necessarily improves the out-of-sample performance.

Lastly, it should be pointed out that benchmark portfolios do show a competitive performance. This raises legitimate doubt, whether an application of the Markowitz framework, or its extensions represent a viable direction to pursue.

"A little uncertainty is good for everyone." - Henry Kissinger

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Appendices

A Derivations in the MV-Setting

A.1 Deriving the Efficient Frontier

The following problem has to be solved

$$\begin{aligned} \min_{\boldsymbol{\pi}} \quad & \boldsymbol{\pi}^\top \boldsymbol{\Sigma} \boldsymbol{\pi} \\ s.t \quad & \boldsymbol{\pi}^\top \boldsymbol{\mu} = \bar{\mu} \\ & \boldsymbol{\pi}^\top \mathbf{1} = 1 \end{aligned} \tag{1.1}$$

where $\mathbf{1}$ is a unit-vector of dimension $N \times 1$.

Using the method of Lagrange multipliers, the Lagrange function is given by

$$\mathcal{L}(\boldsymbol{\pi}, \alpha, \beta) = \boldsymbol{\pi}^\top \boldsymbol{\Sigma} \boldsymbol{\pi} + \alpha(\bar{\mu} - \boldsymbol{\pi}^\top \boldsymbol{\mu}) + \beta(1 - \boldsymbol{\pi}^\top \mathbf{1}) \tag{1.2}$$

and to determine the optimal investment strategy, we need to solve for $\boldsymbol{\pi}^*$ fulfilling the following first order conditions

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\pi}} = 0, \quad \frac{\partial \mathcal{L}}{\partial \alpha} = 0 \quad \frac{\partial \mathcal{L}}{\partial \beta} = 0 \tag{1.3}$$

where the latter two just gives the two constraints.

Given that the variance-covariance matrix is symmetric, hence $\boldsymbol{\Sigma}^\top = \boldsymbol{\Sigma}$ and following the standard rules for differentiation, it follows that:

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\pi}} = 2\boldsymbol{\Sigma}\boldsymbol{\pi} - \alpha\boldsymbol{\mu} - \beta\mathbf{1} = 0 \tag{1.4}$$

and solving for $\boldsymbol{\pi}$, we have that:

$$\boldsymbol{\pi} = \frac{1}{2}\alpha\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \frac{1}{2}\beta\Sigma^{-1}\mathbf{1} \tag{1.5}$$

Being the transposed vector of weights:

$$\boldsymbol{\pi}^\top = \frac{1}{2}\alpha\boldsymbol{\mu}^\top\boldsymbol{\Sigma}^{-1} + \frac{1}{2}\beta\mathbf{1}^\top\boldsymbol{\Sigma}^{-1} \tag{1.6}$$

We substitute this into the two constraints of (2.10), thus obtaining:

$$\boldsymbol{\pi}^\top\boldsymbol{\mu} = \frac{1}{2}\alpha\boldsymbol{\mu}^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \frac{1}{2}\beta\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} = \bar{\mu} \tag{1.7}$$

$$\boldsymbol{\pi}^\top\mathbf{1} = \frac{1}{2}\alpha\boldsymbol{\mu}^\top\boldsymbol{\Sigma}^{-1}\mathbf{1} + \frac{1}{2}\beta\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\mathbf{1} = 1 \tag{1.8}$$

Defining the following constants

$$A = \boldsymbol{\mu}^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}, \quad B = \boldsymbol{\mu}^\top\boldsymbol{\Sigma}^{-1}\mathbf{1} = \mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}, \quad C = \mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\mathbf{1} \tag{1.9}$$

and

$$D = AC - B^2$$

it follows, that we can write Equation (1.7) and (1.8) as

$$\frac{1}{2}\alpha A + \frac{1}{2}\beta B = \bar{\mu} \tag{1.10}$$

and

$$\frac{1}{2}\alpha B + \frac{1}{2}\beta C = 1 \tag{1.11}$$

Isolating α in equation (1.7), the first Lagrange multiplier, in terms of the second one β, the following is obtained:

$$\alpha = 2\frac{\left(\bar{\mu} - 0.5\beta\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\right)}{\boldsymbol{\mu}^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}} \tag{1.12}$$

now substituting the previously defined constants A,B and C as well as (1.12) into eq. (1.8) we get

$$\beta = -\frac{2}{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1}\mathbf{1}} \left(\frac{\bar{\mu} - 0.5\beta \mathbf{1}^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{\boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}} \left(\boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1}\mathbf{1}\right) - 1 \right)$$

$$\beta = -\frac{2}{C} \left(\frac{(\bar{\mu} - 0.5\beta B)B}{A} - 1 \right)$$

$$\beta = \frac{-2\bar{\mu}B + \beta B^2}{AC} + \frac{2}{C}$$

$$\beta AC = -2\bar{\mu}B + \beta B^2 + 2A$$

$$\beta = \frac{2A - 2\bar{\mu}B}{(AC - B^2)}$$

$$\beta = 2\frac{A - B\bar{\mu}}{D}$$

and

$$\alpha = 2\frac{C\bar{\mu} - B}{D}$$

We now substitute these into (1.5), and get that the mean-variance efficient portfolio, for the return $\bar{\mu}$ is defined as:

$$\boldsymbol{\pi}(\bar{\mu}) = \frac{C\bar{\mu} - B}{D}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \frac{A - B\bar{\mu}}{D}\boldsymbol{\Sigma}^{-1}\mathbf{1} \tag{1.13}$$

Now we can derive the portfolio variance by substituting the weight vector $\boldsymbol{\pi}(\bar{\mu})$ into the portfolio variance equation (2.5). Thus obtaining,

$$\sigma^2(\bar{\mu}) = \boldsymbol{\pi}(\bar{\mu})^\top \boldsymbol{\Sigma} \boldsymbol{\pi}(\bar{\mu})$$

$$\sigma^2 = \left(\frac{C\bar{\mu} - B}{D} \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} + \frac{A - B\bar{\mu}}{D} \boldsymbol{\Sigma}^{-1} \mathbf{1} \right)^\top \boldsymbol{\Sigma} \left(\frac{C\bar{\mu} - B}{D} \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} + \frac{A - B\bar{\mu}}{D} \boldsymbol{\Sigma}^{-1} \mathbf{1} \right)$$

finally we obtain the following expression

(2.15),

$$\sigma^2 = \frac{(-2B\bar{\mu} + C\bar{\mu}^2 + A)}{D} \tag{1.14}$$

A.2 Portfolio Weights for CARA Investor in One-Fund Setting

The maximization problem

$$\begin{aligned} \max_{\boldsymbol{\pi}} \quad & \boldsymbol{\pi}^\top \boldsymbol{\mu} - \frac{\gamma}{2} \boldsymbol{\pi}^\top \boldsymbol{\Sigma} \boldsymbol{\pi} \\ s.t. \quad & \boldsymbol{\pi}^\top \mathbf{1} = 1 \end{aligned} \tag{1.15}$$

yields the following Lagrangian function

$$\mathcal{L} = \boldsymbol{\pi}^\top \boldsymbol{\mu} - \frac{\gamma}{2} \boldsymbol{\pi}^\top \boldsymbol{\Sigma} \boldsymbol{\pi} + \alpha \left(1 - \boldsymbol{\pi}^\top \mathbf{1} \right) \tag{1.16}$$

inducing the first order conditions, we have

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\pi}} = \boldsymbol{\mu} - \gamma \boldsymbol{\pi} \boldsymbol{\Sigma} - \alpha \mathbf{1} = 0 \tag{1.17}$$

and

$$\frac{\partial \mathcal{L}}{\partial \alpha} = 1 - \boldsymbol{\pi}^\top \mathbf{1} = 0 \tag{1.18}$$

which can be rewritten as the following expression

$$\boldsymbol{\pi} = \frac{1}{\gamma}\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu} - \alpha\mathbf{1}\right) \tag{1.19}$$

substituting into $\frac{\partial \mathcal{L}}{\partial \alpha}$ yields

$$1 - \frac{1}{\gamma}\boldsymbol{\mu}^\top\boldsymbol{\Sigma}^{-1}\mathbf{1} + \frac{1}{\gamma}\alpha\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\mathbf{1} = 0$$

with B and C as defined in eq. (1.9) we have,

$$1 - \frac{1}{\gamma}B + \frac{1}{\gamma}\alpha C = 0$$

$$\alpha = \frac{B-\gamma}{C}$$

substituting alpha back into eq. (1.19) then yields

$$\boldsymbol{\pi} = \frac{1}{\gamma}\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu} - \frac{B-\gamma}{C}1\right) \tag{1.20}$$

A.3 Global Minimum Variance Portfolio Weights

The following function is to be minimized

$$\sigma^2(\bar{\mu}) = \frac{C\bar{\mu}^2 - B\bar{\mu} + A}{D} \tag{1.21}$$

Thus, we have

$$\frac{\partial\sigma^2(\bar{\mu})}{\partial\bar{\mu}} = \frac{2C\bar{\mu} - B}{D} \quad \Leftrightarrow \quad \bar{\mu} = \frac{B}{C} \tag{1.22}$$

and as $C = \mathbf{1}^\top\boldsymbol{\Sigma}\mathbf{1} > 0$, for a positive definit matrix $\boldsymbol{\Sigma}$, a minimum is obtained for $\frac{\partial\sigma^2(\bar{\mu})}{\partial\bar{\mu}} = 0$. The result can be substituted into eq. (1.21), in order to obtain the minimum variance:

$$\sigma^2(\frac{B}{C}) = \frac{C(\frac{B}{C})^2 - B(\frac{B}{C}) + A}{D} = \frac{AC - B^2}{CD} = \frac{1}{C} \tag{1.23}$$

And when substituting $\bar{\mu} = \frac{B}{C}$ into the previously defined eq. (1.13), we obtain the portfolio weights of the global minimum variance portfolio as:

$$\begin{aligned} \boldsymbol{\pi}\left(\frac{B}{C}\right) &= \frac{C\frac{B}{C} - B}{D}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \frac{A - B\frac{B}{C}}{D}\boldsymbol{\Sigma}^{-1}\mathbf{1} \\ &= \frac{AC - B^2}{CD}\boldsymbol{\Sigma}^{-1}\mathbf{1} \\ &= \frac{1}{C}\boldsymbol{\Sigma}^{-1}\mathbf{1} \end{aligned} \tag{1.24}$$

A.4 Tangency Portfolio Weights

Given the tangency portfolio weights as:

$$\boldsymbol{\pi}_{\tan} = \frac{1}{B - Cr_f}\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu} - r_f\mathbf{1}\right) \tag{1.25}$$

where the sum of the fractions invested in each asset is then 1, hence $\boldsymbol{\pi}_{\tan} = \sum_{i=1}^{n} \pi_i = 1$.

Given CARA preferences and the optimal amount invested in the risky asset defined in (2.26), it follows that:

$$\omega^* = \frac{\mu_{tan} r_f}{\gamma\sigma_{tan}^2} \tag{1.26}$$

The portfolio weights in the risky assets then become:

$$\boldsymbol{\pi}_{tan}^* = \omega^*\boldsymbol{\pi}_{\tan} = \frac{\mu_{\tan} - r_f}{\gamma\sigma^2}\frac{\boldsymbol{\Sigma}^{-1}\left(\mu - r_f\mathbf{1}\right)}{B - Cr_f} = \frac{\lambda}{\gamma\sigma}\frac{\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu} - r_f\mathbf{1}\right)}{B - Cr_f} \tag{1.27}$$

with λ being the Sharpe Ratio. Given that:

$$\lambda = \frac{\mu_{\tan} - r_f}{\sigma} = \left(B - Cr_f\right)\sigma = \left(\mathbf{1}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \mathbf{1}^T\boldsymbol{\Sigma}^{-1}\mathbf{1}r_f\right)\sigma \tag{1.28}$$

The weights on each single assets are then:

$$\boldsymbol{\pi}^*_{tan} = \frac{1}{\gamma}\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu} - r_f\mathbf{1}\right) \tag{1.29}$$

B Derivations in the Ambiguity Aversion Setting as in Garlappi, Uppal, Wang (2007)

B.1 Solving the Joint Representation of Uncertainty

While the optimization problem is represented as in the separate representation as:

$$\max_{\boldsymbol{\pi}} \min_{\boldsymbol{\mu}} \quad \boldsymbol{\pi}^\top\boldsymbol{\mu} - \frac{\gamma}{2}\boldsymbol{\pi}^\top\boldsymbol{\Sigma}\boldsymbol{\pi} \tag{2.1}$$

The constraint written in joint representation is defined as follows

$$\frac{T\left(T-N\right)}{\left(T-1\right)N}(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu})^\top\boldsymbol{\Sigma}^{-1}\left(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu}\right) \leq \epsilon \tag{2.2}$$

Hence, the resulting probabilistic statement:

$$P\left[\frac{T\left(T-N\right)}{\left(T-1\right)N}(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu})^\top\boldsymbol{\Sigma}^{-1}\left(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu}\right) \leq \epsilon\right] = 1 - p \tag{2.3}$$

B.1.1 Inner Minimization over $\boldsymbol{\mu}$

Solving the optimization defined above requires to first solve the minimization

$$\min_{\boldsymbol{\mu}} \boldsymbol{\pi}^\top\boldsymbol{\mu} - \frac{\gamma}{2}\boldsymbol{\pi}^\top\boldsymbol{\Sigma}\boldsymbol{\pi} \tag{2.4}$$

s.t.

$$(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu})^\top\boldsymbol{\Sigma}^{-1}\left(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu}\right) \leqslant \varepsilon \tag{2.5}$$

$$\boldsymbol{\pi}^\top \mathbf{1} = 1 \tag{2.6}$$

Hence, the Lagrangian function is expressed as:

$$\mathcal{L}(\boldsymbol{\mu}) = \boldsymbol{\pi}^\top \boldsymbol{\mu} - \frac{\gamma}{2}\boldsymbol{\pi}^\top \boldsymbol{\Sigma}\boldsymbol{\pi} - \lambda\left(\varepsilon - (\hat{\boldsymbol{\mu}} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu})\right) \tag{2.7}$$

Imposing the $FOC \overset{!}{=} 0$, leads to

$$\frac{\delta\mathcal{L}(\boldsymbol{\mu})}{\delta\boldsymbol{\mu}} = 0 \quad \Leftrightarrow \quad \boldsymbol{\pi}^\top + 2\lambda\boldsymbol{\Sigma}^{-1}(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu})(-1) = 0 \tag{2.8}$$

which, solving for $\boldsymbol{\mu}$ results the optimal value $\boldsymbol{\mu}^*$:

$$\boldsymbol{\mu}^* = \hat{\boldsymbol{\mu}} - \frac{1}{2\lambda}\boldsymbol{\Sigma}\boldsymbol{\pi} \tag{2.9}$$

When substituting (2.9) in the original Lagrangian function (2.7) and simplifying we obtain the following

$$\Leftrightarrow \mathcal{L}(\boldsymbol{\mu}^*, \lambda) = \boldsymbol{\pi}^\top \hat{\boldsymbol{\mu}} - \left(\frac{1}{4\lambda} - \frac{\gamma}{2}\right)\boldsymbol{\pi}^\top \boldsymbol{\Sigma}\boldsymbol{\pi} - \lambda\varepsilon \tag{2.10}$$

B.1.2 Inner Maximization over λ

The upper derived Lagrangian subject to the following FOC:

$$0 \overset{!}{=} \frac{\partial\mathcal{L}(\boldsymbol{\pi}, \lambda)}{\partial\lambda} = -\frac{1}{4\lambda^2}\boldsymbol{\pi}^\top \boldsymbol{\Sigma}\boldsymbol{\pi} - \varepsilon \tag{2.11}$$

$$\Leftrightarrow \lambda = \frac{1}{2}\sqrt{\frac{\boldsymbol{\pi}^\top \boldsymbol{\Sigma}\boldsymbol{\pi}}{\varepsilon}} \tag{2.12}$$

so that the maximization problem dependent on the weight vector $\boldsymbol{\pi}$ becomes

$$\mathcal{L}(\boldsymbol{\pi}, \lambda) = \boldsymbol{\pi}^\top \hat{\boldsymbol{\mu}} - \left(\frac{1}{2\sqrt{\frac{\boldsymbol{\pi}^\top \boldsymbol{\Sigma} \boldsymbol{\pi}}{\varepsilon}}} + \frac{\gamma}{2} \right) \boldsymbol{\pi}^\top \boldsymbol{\Sigma} \boldsymbol{\pi} - \frac{1}{2} \sqrt{\frac{\boldsymbol{\pi}^\top \boldsymbol{\Sigma} \boldsymbol{\pi}}{\varepsilon}} \varepsilon$$

$$\Leftrightarrow \ \mathcal{L}(\boldsymbol{\pi}, \lambda) = \boldsymbol{\pi}^\top \hat{\boldsymbol{\mu}} - \frac{\gamma}{2} \boldsymbol{\pi}^\top \boldsymbol{\Sigma} \boldsymbol{\pi} \left(1 + \frac{2\sqrt{\varepsilon}}{\gamma \sqrt{\boldsymbol{\pi}^\top \boldsymbol{\Sigma} \boldsymbol{\pi}}} \right) \tag{2.13}$$

B.1.3 Outer Maximization over $\boldsymbol{\mu}$

In order to solve the outer maximization

$$\max_{\boldsymbol{\pi}} \ \boldsymbol{\pi}^\top \hat{\boldsymbol{\mu}} - \frac{\gamma}{2} \boldsymbol{\pi}^\top \boldsymbol{\Sigma} \boldsymbol{\pi} \left(1 + \frac{2\sqrt{\varepsilon}}{\gamma \sqrt{\boldsymbol{\pi}^\top \boldsymbol{\Sigma} \boldsymbol{\pi}}} \right) \tag{2.14}$$

First, $\Omega(\boldsymbol{\pi})$ is defined as

$$\Omega(\boldsymbol{\pi}) \equiv \left(1 + \frac{2\sqrt{\varepsilon}}{\gamma \sqrt{\boldsymbol{\pi}^\top \boldsymbol{\Sigma} \boldsymbol{\pi}}} \right)$$

Including the "sum-to-one"-constraint as defined in equation (3.30) the Lagrangian function then becomes the following

$$\mathcal{L}(\boldsymbol{\pi}, \lambda) = \boldsymbol{\pi}^\top \boldsymbol{\Sigma} \boldsymbol{\pi} - \frac{\gamma}{2} \boldsymbol{\pi}^\top \boldsymbol{\Sigma} \boldsymbol{\pi} \Omega(\boldsymbol{\pi}) + \lambda \left(1 - \boldsymbol{\pi}^\top 1 \right) \tag{2.15}$$

Introducing the $FOC : \frac{\partial \mathcal{L}(\boldsymbol{\pi}, \lambda)}{\partial \boldsymbol{\pi}} \stackrel{!}{=} 0$ yields:

$$0 = \hat{\boldsymbol{\mu}}^\top - \frac{\gamma}{2} \left(2 \boldsymbol{\Sigma} \boldsymbol{\pi} \left(\mathbf{1} + \frac{2\sqrt{\varepsilon}}{\gamma \sqrt{\boldsymbol{\pi}^\top \boldsymbol{\Sigma} \boldsymbol{\pi}}} \right) + \boldsymbol{\pi}^\top \boldsymbol{\Sigma} \boldsymbol{\pi} \left(\frac{-2\sqrt{\varepsilon}}{\gamma} \left(\boldsymbol{\pi}^\top \boldsymbol{\Sigma} \boldsymbol{\pi} \right)^{-1.5} \boldsymbol{\Sigma} \boldsymbol{\pi} \right) \right) - \lambda \mathbf{1}^\top \tag{2.16}$$

which can be rewritten as follows

$$\Leftrightarrow 0 = \hat{\boldsymbol{\mu}}^\top - \boldsymbol{\Sigma\pi}\left(\frac{\sqrt{\varepsilon} + \gamma\sqrt{\boldsymbol{\pi}^\top\boldsymbol{\Sigma\pi}}}{\sqrt{\boldsymbol{\pi}^\top\boldsymbol{\Sigma\pi}}}\right) - \lambda\mathbf{1}^\top \tag{2.17}$$

In the following, the portfolio standard deviation is expressed as $\sigma_p = \sqrt{\boldsymbol{\pi}^T\boldsymbol{\Sigma\pi}}$, yielding the following FOC:

$$\hat{\boldsymbol{\mu}}^\top - \boldsymbol{\Sigma\pi}\left(\frac{\sqrt{\varepsilon} + \gamma\sigma_p}{\sigma_p}\right) - \lambda\mathbf{1} = 0 \tag{2.18}$$

Solving for $\boldsymbol{\pi}$ results in:

$$\Leftrightarrow \boldsymbol{\pi} = \frac{\sigma_p}{\sqrt{\varepsilon} + \gamma\sigma_p}\boldsymbol{\Sigma}^{-1}\left(\hat{\boldsymbol{\mu}}^\top - \lambda\mathbf{1}\right) \tag{2.19}$$

and knowing that $\boldsymbol{\pi}^\top\mathbf{1} = 1$, it follows that:

$$\begin{aligned} \boldsymbol{\pi}^\top\mathbf{1} = 1 &= \left(\boldsymbol{\Sigma}^{-\mathbf{1}}\left(\hat{\boldsymbol{\mu}} - \lambda\mathbf{1}\right)\right)^\top\mathbf{1}\left(\frac{\sigma_p}{\sqrt{\varepsilon} + \gamma\sigma_p}\right) \\ \Leftrightarrow 1 &= \left(\hat{\boldsymbol{\mu}}^\top - \lambda\mathbf{1}^\top\right)\boldsymbol{\Sigma}^{-1}\mathbf{1}\left(\frac{\sigma_p}{\sqrt{\varepsilon} + \gamma\sigma_p}\right) \end{aligned} \tag{2.20}$$

With $C = \mathbf{1}^\top\boldsymbol{\Sigma}\mathbf{1}$ and $B = \boldsymbol{\mu}^\top\boldsymbol{\Sigma}\mathbf{1}$, as defined in section (2) (this deviates from the depiction of Garlappi, Uppal, and Wang (2007)) the following solution for λ results:

$$\lambda = \frac{B}{C} - \frac{\sqrt{\varepsilon} + \gamma\sigma_p}{\sigma_p} \tag{2.21}$$

defining $E \equiv \dfrac{\sigma_p}{\sqrt{\varepsilon} + \gamma\sigma_p}$, and $F \equiv -\frac{1}{C}\left(B - \dfrac{1}{E}\right)$ and substituting the upper expression of λ into the portfolio variance expression $\sigma_p^2 = \boldsymbol{\pi}^\top\boldsymbol{\Sigma\pi}$, it can be shown that:

$$
\begin{aligned}
&\boldsymbol{\pi}^\top \boldsymbol{\Sigma} \boldsymbol{\pi} \\
&= \left(\frac{\sigma_p}{\sqrt{\varepsilon} + \gamma \sigma_p} \boldsymbol{\Sigma}^{-1} \left[\hat{\boldsymbol{\mu}} - \frac{1}{C} \left(B - \frac{\sqrt{\varepsilon} + \gamma \sigma_p}{\sigma_p} \right) \mathbf{1}_n \right] \right)^\top \\
&\qquad \times \boldsymbol{\Sigma} \frac{\sigma_p}{\sqrt{\varepsilon} + \gamma \sigma_p} \boldsymbol{\Sigma}^{-1} \left[\hat{\boldsymbol{\mu}} - \frac{1}{C} \left(B - \frac{\sqrt{\varepsilon} + \gamma \sigma_p}{\sigma_p} \right) \mathbf{1}_n \right] \\
&= E^2 \left[\hat{\boldsymbol{\mu}}^\top + \mathbf{1}_n^\top F \right] \boldsymbol{\Sigma}^{-1} \left[\hat{\boldsymbol{\mu}} + F \mathbf{1}_n \right] = E^2 \left(\hat{\boldsymbol{\mu}}^\top + \mathbf{1}_n^\top F \right) \left(\boldsymbol{\Sigma}^{-1} \hat{\boldsymbol{\mu}} + \boldsymbol{\Sigma}^{-1} F \mathbf{1}_n \right) \\
&= E^2 A - \frac{E^2 B}{C} + \frac{1}{C}
\end{aligned}
\tag{2.22}
$$

Simplifying the upper result and substituting E, the following polynomial expression depending on σ_p^2 can be derived

$$
\begin{aligned}
\sigma_p^2 &= \frac{1}{C} \left(C \left(\frac{\sigma_p}{\sqrt{\varepsilon} + \gamma \sigma_p} \right)^2 A - B^2 \left(\frac{\sigma_p}{\sqrt{\varepsilon} + \gamma \sigma_p} \right)^2 + 1 \right) \\
0 &= C \gamma^2 \sigma_p^4 + 2 C \gamma \sqrt{\varepsilon} \sigma_p^3 \\
&\quad + \left(C\varepsilon - AC + B^2 - \gamma^2 \right) \sigma_p^2 - 2\gamma \sqrt{\varepsilon} \sigma_p - \varepsilon
\end{aligned}
\tag{2.23}
$$

C Three Fund allocation

In a Three Fund Asset Allocation, the weights of the resulting portfolio are a function of 2 new variables defined as c and d, which are constants to be chosen optimally, the expected Certainty Equivalent Return (CER onward) that is maximized is

$$
E\left[U\left(\hat{\boldsymbol{\pi}}\left(c,d\right)\right)\right] = E[\hat{\boldsymbol{\pi}}\left(c,d\right)]^\top \mu - \frac{\gamma}{2} E\left[\hat{\boldsymbol{\pi}}(c,d)^\top \Sigma \hat{\boldsymbol{\pi}}\left(c,d\right)\right] \tag{3.1}
$$

The three fund asset allocation strategy invest the amount

$$
\hat{\boldsymbol{\pi}} = \hat{\boldsymbol{\pi}}\left(c,d\right) = \frac{1}{\gamma} \left(c \hat{\boldsymbol{\Sigma}}^{-1} \left(\hat{\boldsymbol{\mu}} - r_f \mathbf{1}^\top \right) + d \hat{\boldsymbol{\Sigma}}^{-1} \mathbf{1} \right) \tag{3.2}
$$

in the risky assets and the amount $1 - \sum_{i=1}^{N} \hat{\pi}_i$ in the risk free asset.

Kan and Zhou (2007) shows that the maximization of eq. (3.5) leads to defining the optimal c and d that maximize the expected out-of-sample performance, given N assets, T observations, as

$$\begin{aligned} c^{**} &= c_3 \left(\frac{\psi^2}{\psi^2 + \frac{N}{T}} \right) \\ d^{**} &= c_3 \left(\frac{\frac{N}{T}}{\psi^2 + \frac{N}{T}} \right) \mu_g \end{aligned} \tag{3.3}$$

with

$$\psi^2 = \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} - \frac{\left(\boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}\right)^2}{\mathbf{1}_N^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}} = \left(\boldsymbol{\mu} - \mu_g \mathbf{1}\right)^\top \boldsymbol{\Sigma}^{-1} \left(\boldsymbol{\mu} - \mu_g \mathbf{1}\right) \tag{3.4}$$

being the squared slope of the segment connecting the portfolio to the minimum variance portfolio, and $\mu_g = \frac{\left(1_N^\top \boldsymbol{\Sigma}^{-1} \mu\right)}{\left(1_N^\top \boldsymbol{\Sigma}^{-1} 1_N\right)}$ the expected excess return of the minimum variance portfolio. Their sample counterpart are as follows:

$$\begin{aligned} \hat{\mu}_g &= \frac{\left(\mathbf{1}^\top \hat{\boldsymbol{\Sigma}}^{-1} \hat{\boldsymbol{\mu}}\right)}{\left(\mathbf{1}^\top \hat{\boldsymbol{\Sigma}}^{-1} \mathbf{1}\right)} \\ \hat{\psi}^2 &= \left(\hat{\boldsymbol{\mu}} - \hat{\mu}_g \mathbf{1}\right)^\top \hat{\boldsymbol{\Sigma}}^{-1} \left(\hat{\boldsymbol{\mu}} - \hat{\mu}_g \mathbf{1}\right) \end{aligned} \tag{3.5}$$

and the unbiased version $\hat{\psi}_a^2$ of ψ^2 being

$$\hat{\psi}_a^2 = \frac{\left(T - N - 1\right) \hat{\psi}^2 - \left(N - 1\right)}{T} + \frac{2\left(\hat{\psi}^2\right)^{\frac{N-1}{2}} \left(1 + \hat{\psi}^2\right)^{-\frac{T-2}{2}}}{T B_{\hat{\psi}^2/\left(1+\hat{\psi}^2\right)} \left(\left(N-1\right)/2, \left(T - N + 1\right)/2\right)} \tag{3.6}$$

where

$$B_x(a,b) = \int_0^x y^{a-1}(1-y)^{b-1}dy \tag{3.7}$$

and the scalar $c_3 = [(T-N-1)(T-N-4)] / [T(T-2)]$. They show that the resulting portfolio in a three fund setting, can finally be written as

$$w^{III} = \frac{c_3}{\gamma}\left[\left(\frac{\hat{\psi}_a^2}{\hat{\psi}_a^2 + \frac{N}{T}}\right)\hat{\boldsymbol{\Sigma}}^{-1}\left(\hat{\boldsymbol{\mu}} - r_f\mathbf{1}^\top\right) + \left(\frac{\frac{N}{T}}{\hat{\psi}_a^2 + \frac{N}{T}}\right)\hat{\mu}_g\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}\right] \tag{3.8}$$

D Additional Results of Analyses

D.1 Summary Statistics Control Data

In the following, we see the data of the control dataset and the results of the data analysis of expected CER and out-of-sample performance

D.1.1 Summary Statistics

Asset	$\hat{\boldsymbol{\mu}}$	$\hat{\boldsymbol{\sigma}}$	*min*	*max*	Skew	Kurtosis
NoDur	1.10%	4.30%	-21.03%	18.88%	-0.296	2.238
Durbl	0.90%	6.41%	-32.63%	42.63%	0.097	4.922
Manuf	1.02%	4.97%	-27.33%	17.51%	-0.517	2.782
Enrgy	1.06%	5.59%	-18.33%	24.56%	0.065	1.179
HiTec	0.98%	6.58%	-25.96%	20.76%	-0.230	1.372
Telcm	0.96%	4.76%	-16.36%	21.36%	-0.247	1.206
Shops	1.06%	5.23%	-28.23%	25.86%	-0.287	2.589
Hlth	1.05%	4.90%	-20.46%	29.52%	0.041	2.667
Utils	0.91%	4.08%	-12.65%	18.84%	-0.139	1.103
Other	0.96%	5.29%	-23.59%	20.28%	-0.494	2.086
S&PCOMP	0.59%	4.46%	-24.68%	15.31%	-0.766	3.093

Table 4.1: Summary statistics of return series data used

Assets	W	p-value	Reject
NoDur	0.98	9.3E-08	Yes
Durbl	0.96	5.9E-12	Yes
Manuf	0.97	3.8E-09	Yes
Enrgy	0.99	3.7E-04	Yes
HiTec	0.98	3.4E-06	Yes
Telcm	0.99	2.7E-05	Yes
Shops	0.98	1.0E-07	Yes
Hlth	0.98	1.1E-07	Yes
Utils	0.99	2.8E-04	Yes
Other	0.98	2.4E-08	Yes
S&PCOMP	0.96	1.5E-11	Yes
All	0.97	2.7E-32	Yes

Table 4.2: Shapiro-Wilk test statistics and p-values; control data set

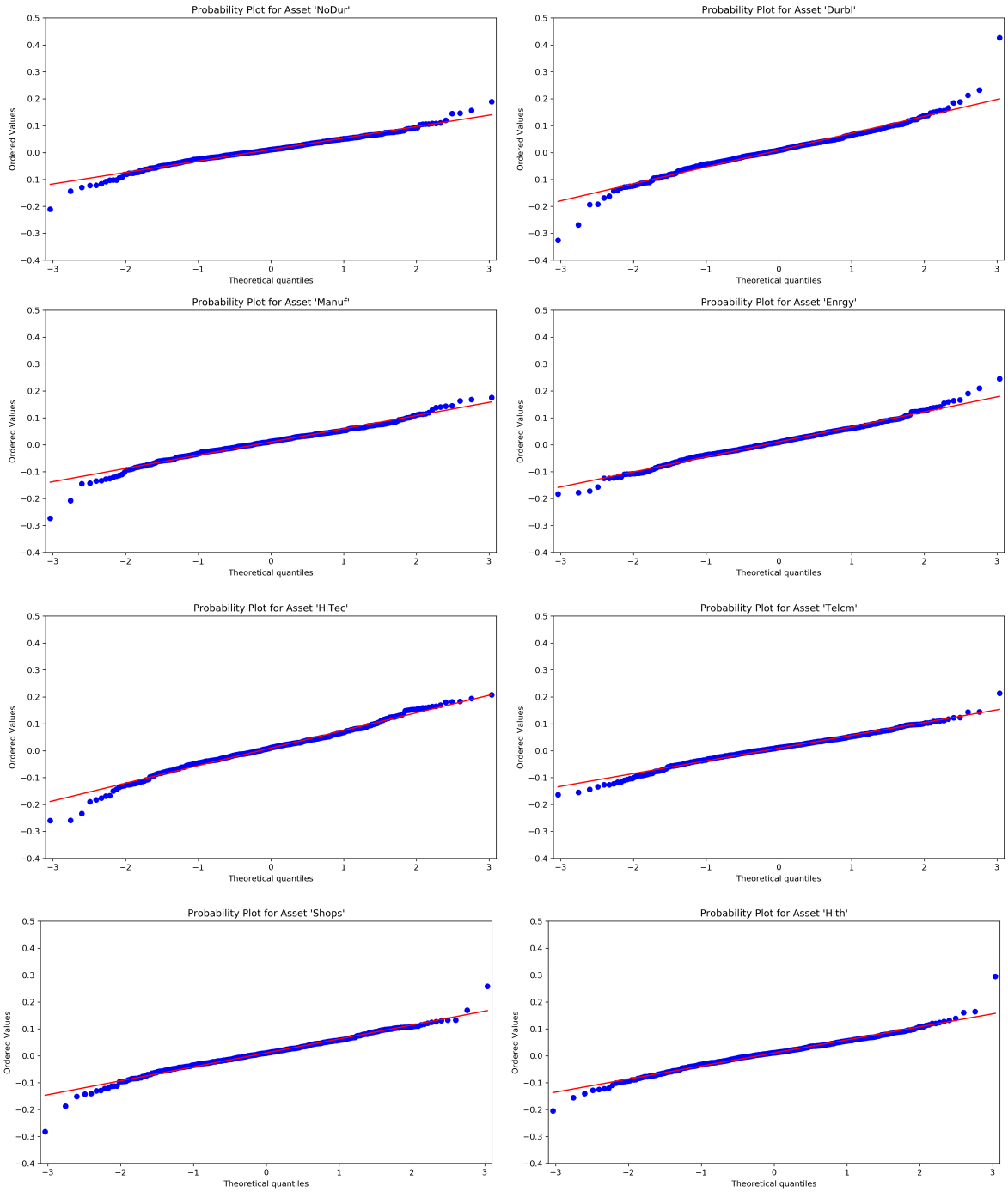

Figure 4.1: QQ-plots of asset returns 1/2

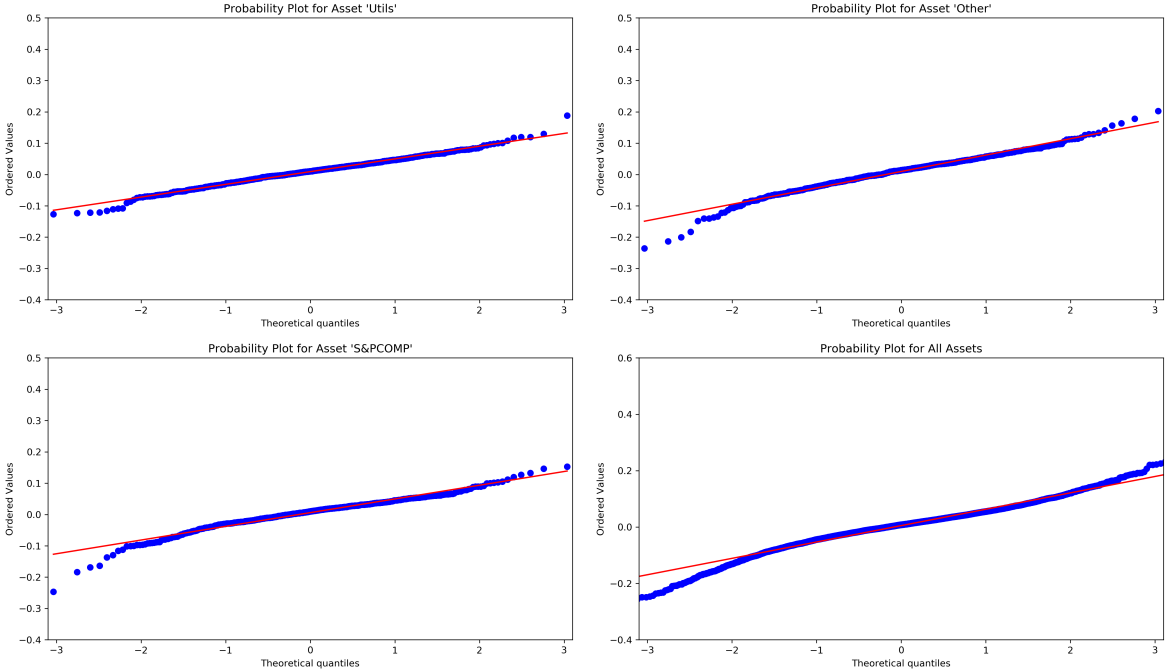

Figure 4.2: QQ-plots of asset returns 2/2

D.2 Expected CER Analysis

D.2.1 T-test for CER=0

	p-value for $H_0 : CER = 0$ and $H_1 : CER \neq 0$				
Utility-Optimized Portolios	48	60	72	96	120
Tangency	0.00	0.00	0.00	0.00	0.00
LW-Tangency	0.00	0.00	0.00	0.00	0.00
Mean-Variance	0.00	0.00	0.00	0.00	0.00
GUW $\epsilon = 0.5$	0.00	0.00	0.00	0.00	0.00
GUW $\epsilon = 1$	0.00	0.00	0.00	0.00	0.00
GUW $\epsilon = 1.5$	0.00	0.00	0.00	0.00	0.00
GUW $\epsilon = 2$	0.00	0.00	0.00	0.00	0.00
GUW $\epsilon = 3$	0.00	0.00	0.00	0.00	0.00
GUW ext.	0.00	0.00	0.00	0.00	0.00
TF - Kan Zhou	0.00	0.00	0.00	0.00	0.00
Benchmark Portfolios					
Min-Variance	0.00	0.00	0.00	0.00	0.00
1 / N	-	-	-	-	-
RP	0.00	0.00	0.00	0.00	0.00
LPM (6.2%)	0.00	0.00	0.00	0.00	0.00

Table 4.3: Two-sided t-test for monthly CER $\gamma = 1$

No results for 1/N as the expected CER is actually certain as it is not estimated based on data.

D.2.2 Different Values of γ

The following tables depict expected monthly CER for different values of γ, both short sale constrained and not constrained.

Short sale allowed

	Estimation window length T				
Utility-Optimized Portolios	48	60	72	96	120
Certain Tangency	1.320%	1.320%	1.320%	1.320%	1.320%
Tangency	-29.595%	-20.102%	-14.937%	-9.770%	-7.058%
LW-Tangency	-25.942%	-18.501%	-14.059%	-9.468%	-6.931%
Mean-Variance	-23.515%	-16.552%	-12.268%	-8.178%	-5.836%
GUW $\epsilon = 0.5$	-2.170%	-1.339%	-0.766%	-0.294%	-0.019%
GUW $\epsilon = 1$	0.014%	0.203%	0.342%	0.446%	0.504%
GUW $\epsilon = 1.5$	0.474%	0.518%	0.551%	0.577%	0.594%
GUW $\epsilon = 2$	0.548%	0.564%	0.580%	0.593%	0.604%
GUW $\epsilon = 3$	0.564%	0.573%	0.584%	0.592%	0.599%
GUW ext.	0.531%	0.533%	0.547%	0.550%	0.555%
TF - Kan Zhou	-2.421%	-1.798%	-1.340%	-0.844%	-0.517%
Benchmark Portfolios					
Min-Variance	0.541%	0.543%	0.546%	0.546%	0.547%
1 / N	0.448%	0.448%	0.448%	0.448%	0.448%
RP	0.461%	0.461%	0.461%	0.461%	0.461%
LPM (6.2%)	0.505%	0.500%	0.496%	0.482%	0.471%

Table 4.4: Expected monthly CER $\gamma = 0.5$; short sale allowed

	Estimation window length T				
Utility-Optimized Portolios	48	60	72	96	120
Certain Tangency	0.700%	0.700%	0.700%	0.700%	0.700%
Tangency	-9.605%	-6.441%	-4.719%	-2.997%	-2.093%
LW-Tangency	-8.388%	-5.907%	-4.427%	-2.897%	-2.051%
Mean-Variance	-7.578%	-5.247%	-3.821%	-2.453%	-1.672%
GUW $\epsilon = 0.5$	-0.621%	-0.307%	-0.100%	0.079%	0.188%
GUW $\epsilon = 1$	0.127%	0.219%	0.289%	0.346%	0.383%
GUW $\epsilon = 1.5$	0.324%	0.361%	0.394%	0.420%	0.439%
GUW $\epsilon = 2$	0.387%	0.409%	0.430%	0.447%	0.460%
GUW $\epsilon = 3$	0.426%	0.440%	0.454%	0.465%	0.475%
GUW ext.	0.351%	0.360%	0.376%	0.385%	0.397%
TF - Kan Zhou	-0.547%	-0.340%	-0.187%	-0.022%	0.087%
Benchmark Portfolios					
Min-Variance	0.443%	0.449%	0.453%	0.456%	0.458%
1 / N	0.339%	0.339%	0.339%	0.339%	0.339%
RP	0.356%	0.356%	0.356%	0.356%	0.356%
LPM (6.2%)	0.367%	0.367%	0.364%	0.349%	0.333%

Table 4.5: Expected monthly CER $\gamma = 1.5$; short sale allowed

	Estimation window length T				
Utility-Optimized Portolios	48	60	72	96	120
Certain Tangency	0.622%	0.622%	0.622%	0.622%	0.622%
Tangency	-7.107%	-4.733%	-3.442%	-2.151%	-1.473%
LW-Tangency	-6.194%	-4.333%	-3.223%	-2.075%	-1.441%
Mean-Variance	-5.628%	-3.871%	-2.800%	-1.770%	-1.184%
GUW $\epsilon = 0.5$	-0.477%	-0.230%	-0.063%	0.085%	0.175%
GUW $\epsilon = 1$	0.090%	0.174%	0.238%	0.292%	0.327%
GUW $\epsilon = 1.5$	0.255%	0.295%	0.330%	0.358%	0.378%
GUW $\epsilon = 2$	0.316%	0.343%	0.366%	0.385%	0.400%
GUW $\epsilon = 3$	0.361%	0.378%	0.395%	0.407%	0.418%
GUW ext.	0.278%	0.291%	0.310%	0.323%	0.337%
TF - Kan Zhou	-0.313%	-0.158%	-0.043%	0.081%	0.163%
Benchmark Portfolios					
Min-Variance	0.394%	0.402%	0.407%	0.411%	0.414%
1 / N	0.284%	0.284%	0.284%	0.284%	0.284%
RP	0.304%	0.304%	0.304%	0.304%	0.304%
LPM (6.2%)	0.298%	0.300%	0.298%	0.283%	0.264%

Table 4.6: Expected monthly CER $\gamma = 2$; short sale allowed

	Estimation window length T				
Utility-Optimized Portolios	48	60	72	96	120
Certain Tangency	0.545%	0.545%	0.545%	0.545%	0.545%
Tangency	-4.608%	-3.026%	-2.165%	-1.304%	-0.852%
LW-Tangency	-3.999%	-2.759%	-2.019%	-1.254%	-0.831%
Mean-Variance	-3.723%	-2.543%	-1.823%	-1.134%	-0.740%
GUW $\epsilon = 0.5$	-0.392%	-0.204%	-0.074%	0.043%	0.116%
GUW $\epsilon = 1$	-0.002%	0.076%	0.136%	0.189%	0.225%
GUW $\epsilon = 1.5$	0.130%	0.175%	0.213%	0.245%	0.268%
GUW $\epsilon = 2$	0.188%	0.221%	0.249%	0.273%	0.290%
GUW $\epsilon = 3$	0.238%	0.261%	0.281%	0.298%	0.311%
GUW ext.	0.147%	0.168%	0.192%	0.212%	0.230%
TF - Kan Zhou	-0.079%	0.025%	0.101%	0.184%	0.238%
Benchmark Portfolios					
Min-Variance	0.296%	0.307%	0.315%	0.321%	0.325%
1 / N	0.175%	0.175%	0.175%	0.175%	0.175%
RP	0.199%	0.199%	0.199%	0.199%	0.199%
LPM (6.2%)	0.159%	0.167%	0.166%	0.150%	0.127%

Table 4.7: Expected monthly CER $\gamma = 3$; short sale allowed

Short sale not allowed

	Estimation window length T				
Utility-Optimized Portolios	48	60	72	96	120
Certain Tangency	1.320%	1.320%	1.320%	1.320%	1.320%
Tangency	-3.729%	-2.653%	-2.119%	-1.321%	-0.930%
LW-Tangency	-3.694%	-2.642%	-2.117%	-1.323%	-0.932%
Mean-Variance	0.437%	0.444%	0.449%	0.458%	0.477%
GUW $\epsilon = 0.5$	0.501%	0.504%	0.507%	0.512%	0.516%
GUW $\epsilon = 1$	0.510%	0.512%	0.515%	0.520%	0.524%
GUW $\epsilon = 1.5$	0.514%	0.517%	0.520%	0.524%	0.528%
GUW $\epsilon = 2$	0.517%	0.519%	0.522%	0.526%	0.530%
GUW $\epsilon = 3$	0.520%	0.522%	0.525%	0.529%	0.532%
GUW ext.	0.514%	0.515%	0.517%	0.520%	0.523%
TF - Kan Zhou	-	-	-	-	-
Benchmark Portfolios					
Min-Variance	0.520%	0.523%	0.526%	0.529%	0.531%
1 / N	0.448%	0.448%	0.448%	0.448%	0.448%
RP	0.461%	0.461%	0.461%	0.461%	0.461%
LPM (6.2%)	0.465%	0.461%	0.451%	0.436%	0.427%

Table 4.8: Expected monthly CER $\gamma = 0.5$; no short sale

	Estimation window length T				
Utility-Optimized Portolios	48	60	72	96	120
Certain Tangency	0.500%	0.500%	0.500%	0.500%	0.500%
Tangency	-0.983%	-0.625%	-0.447%	-0.181%	-0.051%
LW-Tangency	-0.972%	-0.621%	-0.446%	-0.181%	-0.051%
Mean-Variance	0.298%	0.311%	0.320%	0.335%	0.346%
GUW $\epsilon = 0.5$	0.389%	0.394%	0.398%	0.405%	0.411%
GUW $\epsilon = 1$	0.405%	0.409%	0.413%	0.419%	0.424%
GUW $\epsilon = 1.5$	0.412%	0.416%	0.420%	0.426%	0.430%
GUW $\epsilon = 2$	0.417%	0.421%	0.425%	0.430%	0.434%
GUW $\epsilon = 3$	0.422%	0.427%	0.430%	0.435%	0.439%
GUW ext.	0.412%	0.414%	0.416%	0.419%	0.422%
TF - Kan Zhou	-	-	-	-	-
Benchmark Portfolios					
Min-Variance	0.429%	0.433%	0.437%	0.441%	0.444%
1 / N	0.339%	0.339%	0.339%	0.339%	0.339%
RP	0.356%	0.356%	0.356%	0.356%	0.356%
LPM (6.2%)	0.281%	0.264%	0.245%	0.212%	0.190%

Table 4.9: Expected monthly CER $\gamma = 1.5$; no short sale

	Estimation window length T				
Utility-Optimized Portolios	48	60	72	96	120
Certain Tangency	0.472%	0.472%	0.472%	0.472%	0.472%
Tangency	-0.640%	-0.371%	-0.238%	-0.038%	0.059%
LW-Tangency	-0.631%	-0.368%	-0.237%	-0.039%	0.059%
Mean-Variance	0.236%	0.250%	0.260%	0.276%	0.288%
GUW $\epsilon = 0.5$	0.335%	0.341%	0.346%	0.355%	0.361%
GUW $\epsilon = 1$	0.353%	0.359%	0.363%	0.370%	0.375%
GUW $\epsilon = 1.5$	0.362%	0.367%	0.372%	0.378%	0.383%
GUW $\epsilon = 2$	0.367%	0.373%	0.377%	0.383%	0.388%
GUW $\epsilon = 3$	0.374%	0.379%	0.383%	0.389%	0.393%
GUW ext.	0.362%	0.364%	0.367%	0.371%	0.374%
TF - Kan Zhou	-	-	-	-	-
Benchmark Portfolios					
Min-Variance	0.383%	0.388%	0.393%	0.397%	0.400%
1 / N	0.284%	0.284%	0.284%	0.284%	0.284%
RP	0.304%	0.304%	0.304%	0.304%	0.304%
LPM (6.2%)	0.187%	0.164%	0.140%	0.097%	0.068%

Table 4.10: Expected monthly CER $\gamma = 2$; no short sale

	Estimation window length T				
Utility-Optimized Portolios	48	60	72	96	120
Certain Tangency	0.445%	0.445%	0.445%	0.445%	0.445%
Tangency	-0.297%	-0.118%	-0.029%	0.104%	0.169%
LW-Tangency	-0.291%	-0.116%	-0.028%	0.104%	0.169%
Mean-Variance	0.120%	0.135%	0.148%	0.169%	0.186%
GUW $\epsilon = 0.5$	0.232%	0.240%	0.247%	0.258%	0.266%
GUW $\epsilon = 1$	0.253%	0.260%	0.266%	0.275%	0.282%
GUW $\epsilon = 1.5$	0.263%	0.271%	0.276%	0.284%	0.291%
GUW $\epsilon = 2$	0.270%	0.277%	0.283%	0.290%	0.296%
GUW $\epsilon = 3$	0.278%	0.285%	0.290%	0.297%	0.302%
GUW ext.	0.263%	0.268%	0.271%	0.276%	0.281%
TF - Kan Zhou	-	-	-	-	-
Benchmark Portfolios					
Min-Variance	0.292%	0.298%	0.304%	0.309%	0.313%
1 / N	0.175%	0.175%	0.175%	0.175%	0.175%
RP	0.199%	0.199%	0.199%	0.199%	0.199%
LPM (6.2%)	-0.002%	-0.040%	-0.075%	-0.134%	-0.176%

Table 4.11: Expected monthly CER $\gamma = 3$; no short sale

D.2.3 Control Data: No Short Sale

	Estimation window length T				
Utility-Optimized Portolios	48	60	72	96	120
Certain Tangency	1.931%	1.931%	1.931%	1.931%	1.931%
Tangency	-1.146%	-0.442%	-0.075%	0.426%	0.696%
LW-Tangency	-1.093%	-0.413%	-0.059%	0.429%	0.696%
Mean-Variance	0.873%	0.875%	0.879%	0.885%	0.890%
GUW $\epsilon = 0.5$	0.916%	0.918%	0.920%	0.923%	0.926%
GUW $\epsilon = 1$	0.919%	0.920%	0.922%	0.926%	0.928%
GUW $\epsilon = 1.5$	0.919%	0.921%	0.923%	0.926%	0.928%
GUW $\epsilon = 2$	0.919%	0.921%	0.922%	0.925%	0.927%
GUW $\epsilon = 3$	0.918%	0.920%	0.922%	0.924%	0.926%
GUW ext.	0.918%	0.920%	0.922%	0.925%	0.928%
TF - Kan Zhou	-	-	-	-	-
Benchmark Portfolios					
Min-Variance	0.909%	0.909%	0.910%	0.910%	0.911%
1 / N	0.911%	0.911%	0.911%	0.911%	0.911%
RP	0.917%	0.917%	0.917%	0.917%	0.917%
LPM (6.2%)	1.081%	1.101%	1.128%	1.174%	1.204%

Table 4.12: Expected monthly CER $\gamma = 1$; no short sale

D.3 Out-of-Sample Performance Analysis

The tables below depict the results of the out-of-sample CER analysis and the performance measures for different values of γ, in absence of a short sale constraint and when short sales are constrained, respectively.

D.3.1 Portfolio Variance for $\gamma = 1$

For CER and performance measures when $\gamma = 1$ and short sales are allowed, see tables 5.8 and 5.9 respectively.

	Estimation window length T				
Utility-Optimized Portolios	48	60	72	96	120
Tangency	10.31%	9.29%	9.28%	10.55%	11.79%
LW-Tangency	11.04%	11.29%	11.20%	11.91%	13.97%
Mean-Variance	57.18%	47.20%	41.52%	33.33%	28.61%
GUW $\epsilon = 0.5$	21.89%	18.32%	16.02%	13.22%	11.65%
GUW $\epsilon = 1$	11.84%	10.45%	9.32%	8.19%	7.65%
GUW $\epsilon = 1.5$	7.51%	7.05%	6.68%	6.25%	6.07%
GUW $\epsilon = 2$	5.94%	5.78%	5.70%	5.48%	5.41%
GUW $\epsilon = 3$	5.04%	5.01%	5.02%	4.90%	4.89%
GUW ext.	7.08%	7.74%	7.71%	7.59%	7.12%
TF-Kan Zhou	27.08%	24.84%	21.60%	16.75%	13.58%
Benchmark Portfolios					
Min-Variance	4.17%	4.14%	4.09%	4.06%	4.12%
1 / N	4.75%	4.75%	4.75%	4.75%	4.75%
RP	4.64%	4.66%	4.61%	4.65%	4.60%
LPM (6.2%)	7.30%	5.33%	5.17%	5.58%	5.59%

Table 4.13: Monthly standard deviation of portfolios; $\bar{\sigma}_P$, $\gamma = 1$; short sale allowed

D.3.2 Different Values of γ

Short sale allowed

	Estimation window length T				
Utility-Optimized Portolios	48	60	72	96	120
Tangency	-2.260%	-1.288%	-0.941%	-1.229%	-1.330%
LW-Tangency	-1.999%	-0.510%	-0.695%	-0.611%	-0.799%
Mean-Variance	-24.992%	-13.804%	-12.187%	-7.539%	-6.321%
GUW $\epsilon = 0.5$	-2.355%	-0.700%	-0.229%	0.012%	-0.602%
GUW $\epsilon = 1$	-0.214%	0.349%	0.543%	0.579%	0.249%
GUW $\epsilon = 1.5$	0.383%	0.568%	0.643%	0.665%	0.504%
GUW $\epsilon = 2$	0.572%	0.713%	0.684%	0.690%	0.580%
GUW $\epsilon = 3$	0.647%	0.740%	0.681%	0.686%	0.618%
GUW ext.	0.552%	0.809%	0.664%	0.553%	0.398%
TF - Kan Zhou	-2.682%	-3.160%	-3.106%	-0.951%	-1.730%
Benchmark Portfolios					
Min-Variance	0.569%	0.588%	0.606%	0.610%	0.614%
1 / N	0.509%	0.509%	0.509%	0.509%	0.509%
RP	0.558%	0.579%	0.545%	0.522%	0.504%
LPM (0%)	0.983%	0.380%	0.515%	0.545%	0.329%

Table 4.14: Monthly CER out-of-sample; $\gamma = 0.5$; short sale allowed

	Estimation window length T				
Utility-Optimized Portolios	48	60	72	96	120
Tangency	-2.260%	-1.288%	-0.941%	-1.229%	-1.330%
LW-Tangency	-1.999%	-0.510%	-0.695%	-0.611%	-0.799%
Mean-Variance	-24.992%	-13.804%	-12.187%	-7.539%	-6.321%
GUW $\epsilon = 0.5$	-2.355%	-0.700%	-0.229%	0.012%	-0.602%
GUW $\epsilon = 1$	-0.214%	0.349%	0.543%	0.579%	0.249%
GUW $\epsilon = 1.5$	0.383%	0.568%	0.643%	0.665%	0.504%
GUW $\epsilon = 2$	0.572%	0.713%	0.684%	0.690%	0.580%
GUW $\epsilon = 3$	0.647%	0.740%	0.681%	0.686%	0.618%
GUW ext.	0.552%	0.809%	0.664%	0.553%	0.398%
TF - Kan Zhou	-2.682%	-3.160%	-3.106%	-0.951%	-1.730%
Benchmark Portfolios					
Min-Variance	0.482%	0.503%	0.523%	0.527%	0.529%
1 / N	0.396%	0.396%	0.396%	0.396%	0.396%
RP	0.450%	0.471%	0.439%	0.414%	0.399%
LPM (6.2%)	0.826%	0.238%	0.381%	0.389%	0.172%

Table 4.15: Monthly CER out-of-sample; $\gamma = 1.5$; short sale allowed

	Estimation window length T				
Utility-Optimized Portolios	48	60	72	96	120
Tangency	-0.296%	-0.052%	0.033%	-0.038%	-0.063%
LW-Tangency	-0.230%	0.140%	0.094%	0.115%	0.066%
Mean-Variance	-5.925%	-3.166%	-2.785%	-1.589%	-1.267%
GUW $\epsilon = 0.5$	-0.366%	0.115%	0.115%	0.243%	0.136%
GUW $\epsilon = 1$	0.216%	0.431%	0.381%	0.424%	0.328%
GUW $\epsilon = 1.5$	0.383%	0.518%	0.456%	0.477%	0.399%
GUW $\epsilon = 2$	0.441%	0.547%	0.483%	0.498%	0.431%
GUW $\epsilon = 3$	0.475%	0.557%	0.501%	0.512%	0.460%
GUW ext.	0.442%	0.571%	0.468%	0.451%	0.378%
TF - Kan Zhou	-0.399%	-0.519%	-0.507%	0.032%	-0.165%
Benchmark Portfolios					
Min-Variance	0.438%	0.460%	0.481%	0.486%	0.487%
1 / N	0.339%	0.339%	0.339%	0.339%	0.339%
RP	0.396%	0.416%	0.386%	0.360%	0.346%
LPM (0%)	0.747%	0.167%	0.314%	0.311%	0.094%

Table 4.16: Monthly CER out-of-sample; $\gamma = 2$; short sale allowed

	Estimation window length T				
Utility-Optimized Portolios	48	60	72	96	120
Tangency	-0.078%	0.085%	0.141%	0.094%	0.077%
LW-Tangency	-0.034%	0.212%	0.181%	0.195%	0.162%
Mean-Variance	-3.879%	-2.056%	-1.810%	-0.996%	-0.777%
GUW $\epsilon = 0.5$	-0.248%	0.103%	0.071%	0.184%	0.123%
GUW $\epsilon = 1$	0.153%	0.330%	0.270%	0.318%	0.252%
GUW $\epsilon = 1.5$	0.279%	0.401%	0.337%	0.365%	0.306%
GUW $\epsilon = 2$	0.329%	0.429%	0.367%	0.388%	0.334%
GUW $\epsilon = 3$	0.365%	0.445%	0.392%	0.407%	0.362%
GUW ext.	0.322%	0.436%	0.347%	0.347%	0.292%
TF - Kan Zhou	-0.146%	-0.226%	-0.219%	0.140%	0.009%
Benchmark Portfolios					
Min-Variance	0.351%	0.375%	0.397%	0.404%	0.402%
1 / N	0.226%	0.226%	0.226%	0.226%	0.226%
RP	0.288%	0.308%	0.279%	0.252%	0.240%
LPM (0%)	0.590%	0.025%	0.181%	0.155%	-0.062%

Table 4.17: Monthly CER out-of-sample; $\gamma = 3$; short sale allowed

Short sale not allowed

	Estimation window length T				
Utility-Optimized Portolios	48	60	72	96	120
Tangency	-1.766%	-0.137%	-1.004%	-0.033%	0.818%
LW-Tangency	-1.639%	-0.124%	-0.938%	0.055%	0.781%
Mean-Variance	0.544%	0.577%	0.648%	0.466%	0.668%
GUW $\epsilon = 0.5$	0.552%	0.610%	0.572%	0.565%	0.616%
GUW $\epsilon = 1$	0.530%	0.593%	0.571%	0.576%	0.602%
GUW $\epsilon = 1.5$	0.518%	0.589%	0.581%	0.584%	0.593%
GUW $\epsilon = 2$	0.520%	0.584%	0.585%	0.592%	0.588%
GUW $\epsilon = 3$	0.520%	0.565%	0.587%	0.602%	0.587%
GUW ext.	0.514%	0.592%	0.590%	0.612%	0.592%
TF - Kan Zhou	-	-	-	-	-
Benchmark Portfolios					
Min-Variance	0.462%	0.488%	0.521%	0.530%	0.539%
1 / N	0.509%	0.509%	0.509%	0.509%	0.509%
RP	0.457%	0.460%	0.466%	0.463%	0.462%
LPM (0%)	0.456%	0.240%	0.225%	0.547%	0.278%

Table 4.18: Monthly CER out-of-sample; $\gamma = 0.5$; no short sale

	Estimation window length T				
Utility-Optimized Portolios	48	60	72	96	120
Tangency	-0.350%	0.193%	-0.097%	0.227%	0.510%
LW-Tangency	-0.308%	0.197%	-0.075%	0.256%	0.498%
Mean-Variance	0.384%	0.463%	0.457%	0.415%	0.473%
GUW $\epsilon = 0.5$	0.439%	0.492%	0.453%	0.459%	0.490%
GUW $\epsilon = 1$	0.422%	0.488%	0.468%	0.480%	0.485%
GUW $\epsilon = 1.5$	0.421%	0.488%	0.482%	0.491%	0.482%
GUW $\epsilon = 2$	0.428%	0.483%	0.486%	0.501%	0.484%
GUW $\epsilon = 3$	0.425%	0.465%	0.488%	0.509%	0.489%
GUW ext.	0.417%	0.491%	0.489%	0.514%	0.479%
TF - Kan Zhou	-	-	-	-	-
Benchmark Portfolios					
Min-Variance	0.462%	0.488%	0.521%	0.530%	0.539%
1 / N	0.509%	0.509%	0.509%	0.509%	0.509%
RP	0.457%	0.460%	0.466%	0.463%	0.462%
LPM (0%)	0.456%	0.240%	0.225%	0.547%	0.278%

Table 4.19: Monthly CER out-of-sample; $\gamma = 1.5$; no short sale

	Estimation window length T				
Utility-Optimized Portolios	48	60	72	96	120
Tangency	-0.174%	0.234%	0.016%	0.259%	0.471%
LW-Tangency	-0.142%	0.237%	0.033%	0.280%	0.462%
Mean-Variance	0.327%	0.390%	0.405%	0.364%	0.394%
GUW $\epsilon = 0.5$	0.381%	0.431%	0.397%	0.412%	0.430%
GUW $\epsilon = 1$	0.370%	0.438%	0.420%	0.434%	0.429%
GUW $\epsilon = 1.5$	0.374%	0.438%	0.432%	0.446%	0.430%
GUW $\epsilon = 2$	0.382%	0.432%	0.438%	0.455%	0.435%
GUW $\epsilon = 3$	0.378%	0.416%	0.440%	0.463%	0.440%
GUW ext.	0.369%	0.440%	0.439%	0.461%	0.429%
TF - Kan Zhou	-	-	-	-	-
Benchmark Portfolios					
Min-Variance	0.462%	0.488%	0.521%	0.530%	0.539%
1 / N	0.339%	0.339%	0.339%	0.339%	0.339%
RP	0.457%	0.460%	0.466%	0.463%	0.462%
LPM (0%)	0.456%	0.240%	0.225%	0.547%	0.278%

Table 4.20: Monthly CER out-of-sample; $\gamma = 2$; no short sale

	Estimation window length T				
Utility-Optimized Portolios	48	60	72	96	120
Tangency	0.003%	0.275%	0.129%	0.291%	0.432%
LW-Tangency	0.024%	0.277%	0.141%	0.305%	0.426%
Mean-Variance	0.225%	0.258%	0.260%	0.235%	0.264%
GUW $\epsilon = 0.5$	0.270%	0.326%	0.295%	0.321%	0.318%
GUW $\epsilon = 1$	0.274%	0.338%	0.323%	0.343%	0.325%
GUW $\epsilon = 1.5$	0.284%	0.339%	0.336%	0.358%	0.334%
GUW $\epsilon = 2$	0.289%	0.332%	0.341%	0.365%	0.339%
GUW $\epsilon = 3$	0.285%	0.321%	0.343%	0.372%	0.344%
GUW ext.	0.271%	0.341%	0.343%	0.373%	0.328%
TF - Kan Zhou	-	-	-	-	-
Benchmark Portfolios					
Min-Variance	0.462%	0.488%	0.521%	0.530%	0.539%
1 / N	0.226%	0.226%	0.226%	0.226%	0.226%
RP	0.457%	0.460%	0.466%	0.463%	0.462%
LPM (0%)	0.456%	0.240%	0.225%	0.547%	0.278%

Table 4.21: Monthly CER out-of-sample; $\gamma = 3$; no short sale

D.3.3 Control Data: No Short Sale

	Estimation window length T				
Utility-Optimized Portolios	48	60	72	96	120
Tangency	0.232%	0.589%	1.311%	0.856%	0.905%
LW-Tangency	0.136%	0.498%	1.318%	0.866%	0.950%
Mean-Variance	0.859%	0.962%	0.949%	0.909%	0.785%
GUW $\epsilon = 0.5$	0.890%	0.924%	1.004%	0.963%	0.918%
GUW $\epsilon = 1$	0.892%	0.931%	0.972%	0.965%	0.918%
GUW $\epsilon = 1.5$	0.902%	0.937%	0.967%	0.969%	0.929%
GUW $\epsilon = 2$	0.909%	0.944%	0.967%	0.971%	0.936%
GUW $\epsilon = 3$	0.915%	0.945%	0.968%	0.972%	0.944%
GUW ext.	0.928%	0.943%	0.962%	0.982%	0.982%
TF - Kan Zhou	-	-	-	-	-
Benchmark Portfolios					
Min-Variance	0.949%	0.982%	0.984%	0.977%	0.993%
1 / N	0.991%	0.991%	0.991%	0.991%	0.991%
RP	0.998%	1.003%	1.000%	1.001%	1.001%
LPM (6.2%)	1.337%	1.093%	1.023%	1.408%	1.890%

Table 4.22: Monthly CER out-of-sample; $\gamma = 1$; control data set; no short sale

	Estimation Length T				
	48	60	72	96	120
Utility-Optimized Portfolios					
Tangency	0.109	0.139	0.141	0.103	0.105
	(1.54)	*(1.72)*	*(0.62)*	*(0.41)*	*(0.43)*
LW-Tangency	0.107	0.138	0.141	0.103	0.110
	(1.48)	*(1.54)*	*(0.58)*	*(0.39)*	*(0.41)*
Mean-Variance	0.115	0.131	0.128	0.122	0.102
	(0.36)	*(0.30)*	*(0.28)*	*(0.25)*	*(0.20)*
GUW ϵ =0.5	0.148	0.156	0.171	0.165	0.152
	(0.28)	*(0.27)*	*(0.24)*	*(0.20)*	*(0.18)*
GUW ϵ =1	0.160	0.170	0.177	0.178	0.164
	(0.23)	*(0.22)*	*(0.20)*	*(0.17)*	*(0.15)*
GUW ϵ =1.5	0.167	0.176	0.183	0.186	0.171
	(0.21)	*(0.20)*	*(0.18)*	*(0.15)*	*(0.13)*
GUW ϵ =2	0.172	0.181	0.187	0.190	0.176
	(0.19)	*(0.18)*	*(0.16)*	*(0.13)*	*(0.12)*
GUW ϵ =3	0.176	0.184	0.191	0.194	0.181
	(0.18)	*(0.16)*	*(0.14)*	*(0.12)*	*(0.11)*
GUW ext.	0.181	0.186	0.192	0.197	-0.047
	(0.17)	*(0.16)*	*(0.14)*	*(0.15)*	*(0.14)*
TF - Kan Zhou	0.135	0.147	0.168	0.212	0.218
	(54.89)	*(36.93)*	*(27.35)*	*(16.77)*	*(12.23)*
Benchmark Portfolios					
Min-Variance	0.193	0.204	0.206	0.203	0.205
	(0.16)	*(0.14)*	*(0.12)*	*(0.10)*	*(0.09)*
1 / N	0.177	0.177	0.177	0.177	0.177
	(0.04)	*(0.04)*	*(0.04)*	*(0.04)*	*(0.04)*
RP	0.188	0.189	0.188	0.188	0.188
	(0.05)	*(0.04)*	*(0.04)*	*(0.04)*	*(0.04)*
LPM (6.2%)	0.163	0.127	0.120	0.158	0.202
	(1.78)	*(1.67)*	*(1.56)*	*(1.49)*	*(1.73)*

Table 4.23: SR and TO (in parentheses); $\gamma = 1$; control dataset; no short sale; monthly figures

E Discussion

E.1 Out-of-Sample: Comparison with Garlappi et al. (2007)

Portfolio	$\bar{r}$	σ	$\frac{\bar{r}}{\sigma}$	Rank
Panel A: Short sales allowed				
Mean-Variance	0.0118	0.2861	0.0413	7
Min-Variance	0.0066	0.0412	0.1594	1
GUW $\epsilon = 0.5$	0.0064	0.1165	0.0550	6
GUW $\epsilon = 1$	0.0065	0.0765	0.0855	5
GUW $\epsilon = 1.5$	0.0067	0.0607	0.1106	4
GUW $\epsilon = 2$	0.0068	0.0541	0.1256	3
GUW $\epsilon = 3$	0.0068	0.0489	0.1398	2
Panel B: Short sales not allowed				
Mean-Variance	0.0074	0.0609	0.1221	7
Min-Variance	0.0063	0.0421	0.1491	1
GUW $\epsilon = 0.5$	0.0068	0.0507	0.1338	6
GUW $\epsilon = 1$	0.0066	0.0479	0.1372	5
GUW $\epsilon = 1.5$	0.0064	0.0466	0.1385	4
GUW $\epsilon = 2$	0.0064	0.0458	0.1399	3
GUW $\epsilon = 3$	0.0064	0.0448	0.1425	2

Table 5.1: Average return, standard deviation; monthly figures; $T = 120$

Portfolio	$\bar{r}$	σ	$\frac{\bar{r}}{\sigma}$	Rank
Panel A: Short sales allowed				
Mean-Variance	0.0049	0.2557	0.0190	7
Min-Variance	0.0118	0.0419	0.2827	1
GUW $\epsilon = 0.5$	0.0080	0.0953	0.0841	6
GUW $\epsilon = 1$	0.0098	0.0654	0.1491	5
GUW $\epsilon = 1.5$	0.0104	0.0545	0.1909	4
GUW $\epsilon = 2$	0.0107	0.0501	0.2144	3
GUW $\epsilon = 3$	0.0111	0.0419	0.2827	1
Panel B: Short sales not allowed				
Mean-Variance	0.0104	0.0587	0.1774	7
Min-Variance	0.0117	0.0412	0.2831	1
GUW $\epsilon = 0.5$	0.0115	0.0482	0.2391	6
GUW $\epsilon = 1$	0.0116	0.0457	0.2533	5
GUW $\epsilon = 1.5$	0.0116	0.0446	0.2607	4
GUW $\epsilon = 2$	0.0117	0.0440	0.2647	3
GUW $\epsilon = 3$	0.0117	0.0433	0.2695	2

Table 5.2: Results of Garlappi et al. (2007); monthly figures; $T = 120$

F Code

The scripts of the different analyses, as well as the return series data have been uploaded to https://goo.gl/Pw11ye.