



## ABSTRACT

Modern artificial intelligence relies on networks of agents that collect data, process information, and exchange it with neighbors to collaboratively solve optimization and learning problems. We present a novel distributed algorithm to address a broad class of these problems in open networks, where the number of participating agents may vary due to several factors, such as autonomous decisions, heterogeneous resource availability, or DoS attacks. Extending the current literature, the convergence analysis of the proposed algorithm is based on the newly developed Theory of Open Operators, which characterizes an operator as open when the set of components to be updated changes over time, yielding time-varying operators acting on sequences of points of different dimensions and compositions. The mathematical tools and convergence results developed here provide a general framework for evaluating distributed algorithms in open networks. As an illustrative example, the proposed algorithm is used to solve classification problems with logistic loss functions.

## STATE-OF-THE-ART OF OPTIMIZATION/LEARNING IN OPEN MULTI-AGENT SYSTEMS

[Ref.]	Algorithm	Assumptions on the problem	Assumptions on the network	Time independent parameters	Convergence metric	Convergence rate
Hendrickx et al. (2020)	DGD	Static + Smooth + Strongly convex + Minimizers in a ball	✗ Only replacement	✓	Distance from minimizers	Linear (inexact)
Hsieh et al. (2021)	Dual averaging	Static + Lipschitz + Convex + Shared convex constraint set	≈ Vertex-connected <sup>†</sup> (jointly)	✗	Regret	Sublinear (if the network's size is known)
Hayashi (2023)	Sub-gradient	Time-varying + Lipschitz + Convex + Shared compact constraint set	≈ Vertex-connected <sup>†</sup> (jointly)	✗	Regret	Sublinear (if the network's size is bounded)
Deplano et al. (2025)	ADMM	Time-varying + Semicontinuous + Convex + Unconstrained	✓ Connected and sparse <sup>‡</sup>	✓	Distance from minimizers	Linear (exact)

<sup>†</sup>  $\mathcal{G}_k = (\mathcal{V}_k, \mathcal{E}_k)$  is jointly vertex connected if  $\exists B, \kappa \geq 1$  such that at least  $\kappa$  nodes need to be removed to disrupt the connectivity of the union graph  $(\bigcup_{t=k-B}^k \mathcal{V}_t, \bigcup_{t=k-B}^k \mathcal{E}_t)$  for all  $k \in \mathbb{N}$ .  
<sup>‡</sup>  $\mathcal{G}_k = (\mathcal{V}_k, \mathcal{E}_k)$  is sparse if  $|\mathcal{E}_k| \leq \mu |\mathcal{V}_k|$  for some  $\mu > 0$ .

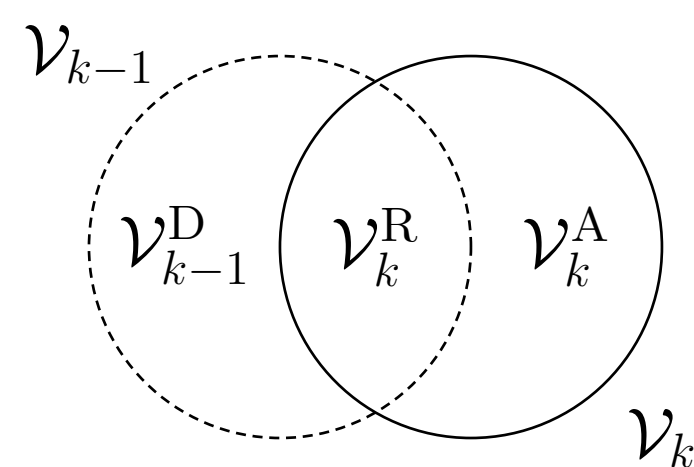
## PROBLEM SET-UP

We consider the following optimization problem

$$\min_{y \in \mathbb{R}^p} \sum_{i \in \mathcal{V}_k} f_k^i(y), \quad (1)$$

where  $p \in \mathbb{N}$  denotes the number of variables,  $f_k^i : \mathbb{R}^p \mapsto \mathbb{R}$  denotes the local objective function of an agent  $i \in \mathcal{V}_k$  in the network at time  $k$ , where  $\mathcal{V}_k$  represents the time-varying set of agents, yielding an **Open Multi-Agent Systems (OMAS)**. The agents are linked according to a graph  $\mathcal{G}_k = (\mathcal{V}_k, \mathcal{E}_k)$ , which is assumed to be undirected and connected at all times. The set of agents linked to the  $i$ -th agent at time  $k$  is denoted by  $\mathcal{N}_k^i = \{j \in \mathcal{V}_k : (i, j) \in \mathcal{E}_k\}$ . We define:

- The set of **remaining agents** is  $\mathcal{V}_k^R = \mathcal{V}_k \cap \mathcal{V}_{k-1}$ ;
- The set of **arriving agents** is  $\mathcal{V}_k^A = \mathcal{V}_k \setminus \mathcal{V}_{k-1}$ ;
- The set of **departing agents** is  $\mathcal{V}_k^D = \mathcal{V}_k \setminus \mathcal{V}_{k+1}$ .



To solve the problem in (1) in a distributed way over an open network of agents, we propose the open version of ADMM, which we call **Open ADMM** and whose implementation is detailed in Algorithm 1. Open ADMM requires each agent  $i \in \mathcal{V}_k$  to update/initialize a state variable  $x_k^{ij} \in \mathbb{R}^p$  for every agent  $j \in \mathcal{N}_k^i$  with which it has an open communication channel. Thus, one needs to define:

- The set of **remaining neighbors** is  $\mathcal{R}_k^i = \mathcal{N}_k^i \cap \mathcal{N}_{k-1}^i$ ;
- The set of **arriving neighbors** is  $\mathcal{A}_k^i = \mathcal{N}_k^i \setminus \mathcal{N}_{k-1}^i$ ;
- The set of **departing neighbors** is  $\mathcal{D}_k^i = \mathcal{N}_k^i \setminus \mathcal{N}_{k+1}^i$ .

We formalize next our set of assumptions.

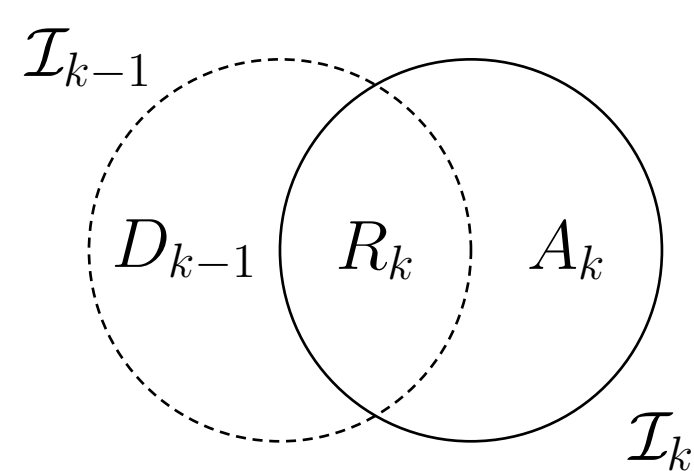
**Assumption 1.** The problem in (1) is such that,  $\forall k \in \mathbb{N}$ :

- (i) the local cost functions  $f_k^i$  are proper, lower semi-continuous, and convex for all  $i \in \mathcal{V}_k$ ;
- (ii) the set of minimizers  $\mathcal{Y}_k^{i,*} \subseteq \mathbb{R}^p$  for each local cost function  $f_k^i$  is not empty;
- (iii) the distance between two consecutive global solutions  $y_k^* \in \mathcal{Y}_k^*$  and  $y_{k-1}^* \in \mathcal{Y}_{k-1}^*$  is upper bounded by a constant  $\sigma \geq 0$ ;
- (iv) the distance between any local solution  $y_k^{i,*} \in \mathcal{Y}_k^{i,*}$  and any global solution  $y_k^* \in \mathcal{Y}_k^*$  is upper bounded by  $\omega \geq 0$ .

## OPEN OPERATOR THEORY [R1]

**Theorem 1:** Consider the iteration of a time-varying open operator  $\mathcal{T}_k : \mathbb{R}^{\mathcal{I}_{k-1}} \rightarrow \mathbb{R}^{\mathcal{I}_k}$  given component-wise for  $i \in \mathcal{I}_k$  by

$$x_k^i = \mathcal{T}_k^i(x_{k-1}) = \begin{cases} F_k^i(x_{k-1}) & \text{if } i \in \mathcal{R}_k = \mathcal{I}_{k-1} \setminus \mathcal{D}_{k-1}, \\ x_k^{A,i} & \text{if } i \in \mathcal{A}_k = \mathcal{I}_k \setminus \mathcal{I}_{k-1}. \end{cases}$$



and let  $\hat{\mathcal{X}}_k := \{x \in \mathbb{R}^{\mathcal{I}_k} \mid x = F_{k+1}(x)\}$  be the trajectory of points of interest (TSI). If:

- (a)  $F_k$  is paracontractive with  $\gamma \in (0, 1)$ , i.e.,  $d(F_{k+1}(x), \hat{\mathcal{X}}_k) \leq \gamma \cdot d(x, \hat{\mathcal{X}}_k)$ ;
- (b) the TSI has bounded variation  $B \geq 0$ , i.e.,  $d_{\text{SH}}(\hat{\mathcal{X}}_k, \hat{\mathcal{X}}_{k-1}) \leq B \sqrt{|\mathcal{R}_k|}$ ;
- (c) the arrival process is bounded with  $H \geq 0$ , i.e.,  $d(x_k^A, \hat{\mathcal{X}}_k) \leq H \sqrt{|\mathcal{A}_k|}$ ;
- (d) the departure process is bounded with  $\beta \in (\gamma, 1)$ , i.e.,  $\sqrt{|\mathcal{I}_k|} \geq \beta \sqrt{|\mathcal{I}_{k-1}|}$ ;

then, the open sequence  $\{x_k \in \mathbb{R}^{\mathcal{I}_k} : k \in \mathbb{N}\}$  converges linearly with rate  $\theta = \gamma/\beta \in (0, 1)$  to the TSI within a radius

$$R = \frac{B + H}{1 - \theta}.$$

## OPEN ADMM [R1]

**Algorithm 1:** Open and distributed ADMM

**Input:** The relaxation  $\alpha \in (0, 1)$  and the penalty  $\rho > 0$

**Output:** The agent return an approximate solution  $y_k^i$  to the optimization problem in (1)

**for**  $k = 0, 1, 2, \dots$  **each agent**  $i \in \mathcal{V}_k$ :

**if**  $i \in \mathcal{V}_k^A$  **is an arriving agent:**

initializes the state variables to a local optimum

$$x_k^{ij} = \rho y_k^{i,*}, \quad \forall j \in \mathcal{N}_k^i$$

**else if**  $i \in \mathcal{V}_k^R$  **is a remaining agent:**

receives  $y_{k-1}^j, x_{k-1}^{ji}$  from each neighbor  $j \in \mathcal{R}_k^i$   
updates the remaining state variable according to

$$x_k^{ij} = (1 - \alpha)x_{k-1}^{ij} - \alpha x_{k-1}^{ji} + 2\rho\alpha y_{k-1}^j, \quad \forall j \in \mathcal{R}_k^i$$

initializes the new state variables to a local optimum

$$x_k^{ij} = \rho y_k^{i,*}, \quad \forall j \in \mathcal{A}_k^i$$

**end if**

updates the output variable

$$y_k^i = \text{prox}_{f_k^i}^{1/\rho n_k^i} \left( \frac{1}{\rho n_k^i} \sum_{j \in \mathcal{N}_k^i} x_k^{ij} \right)$$

transmits  $y_k^i, x_k^{ij}$  to each neighbor  $j \in \mathcal{N}_k^i$

**end for**

**Theorems 2-3:** Consider an OMAS executing Open ADMM to distributedly solve an optimization problem as in (1) under Assumption 1. If the standard iteration is paracontractive with  $\gamma \in (0, 1)$  and the departure process is bounded with  $\beta \in (\gamma, 1)$ , then the open sequence  $\{x_k : k \in \mathbb{N}\}$  generated by the open operator of Open ADMM converges with linear rate  $\theta = \gamma/\beta \in (0, 1)$  to the TSI within a radius  $R$ ,

$$\limsup_{k \rightarrow \infty} \frac{d(x_k, \hat{\mathcal{X}}_k)}{\sqrt{pn_k}} \leq \rho \frac{(\sigma + \omega)}{(1 - \theta)} =: R.$$

Moreover, if  $\exists \mu > 0$  such that  $|\mathcal{E}_k| \leq \mu |\mathcal{V}_k|$ , then, the open sequence of agents' estimates converges linearly to the consensus state on the optimal solutions within a radius  $\mu R$ , namely

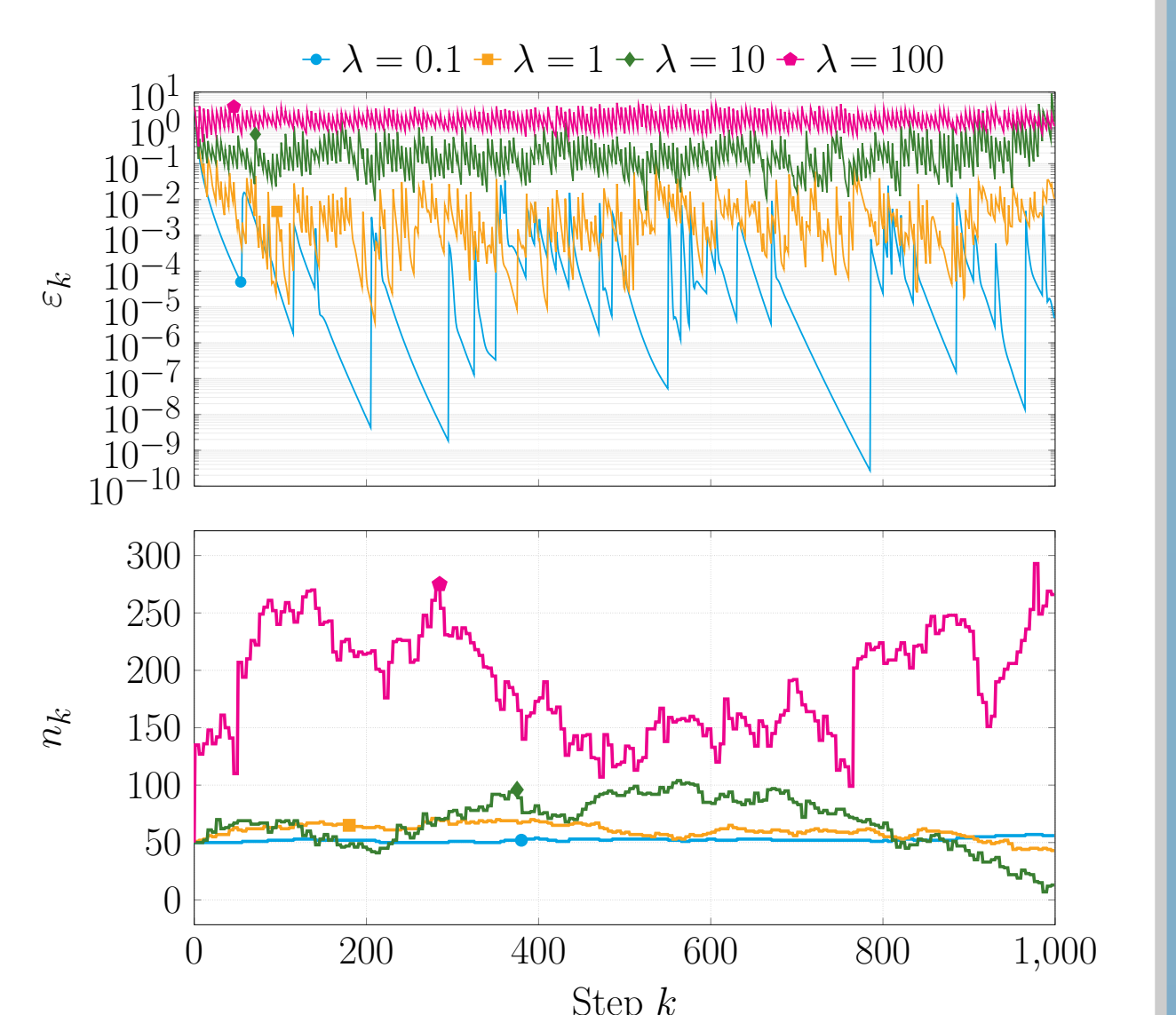
$$\limsup_{k \rightarrow \infty} \frac{d(y_k, \mathcal{C}_k^*)}{\sqrt{pn_k}} = \mu R =: \Delta.$$

## NUMERICAL SIMULATIONS

We apply the Open ADMM to a classification problem, characterized by the (static) local costs

$f_k^i := \frac{1}{m_i} \sum_{j=1}^{m_i} \log(1 + \exp(-b_{i,j} a_{i,j}^T x)) + \frac{\epsilon}{2} \|x\|^2$   
where  $a_{i,j} \in \mathbb{R}^p$  and  $b_{i,j} \in \{-1, 1\}$  are randomly generated pairs of feature vector and label, with  $p = 5$ ,  $m_i = 150$ , and  $\epsilon = 0.05$ .

On the right we show the results for open networks in which the arrival and departure events occur according to the Poisson distribution  $\text{Pois}(\lambda)$  for different values of  $\lambda$ . Arriving agents are connected to a number of remaining agents equal to the average degree in the network, and the network starts with  $n_0 = 50$  agents.



## BIBLIOGRAPHY AND AFFILIATIONS

[R1] Deplano, D., Bastianello, N., Franceschelli, M., and Johansson, K.H., "Optimization and Learning in Open Multi-Agent Systems". Submitted to the Transaction on Automatic Control (April 2025).

D. Deplano and M. Franceschelli are with DIEE, University of Cagliari, 09123 Cagliari, Italy. Emails: {diego.deplano,mauro.franceschelli}@unica.it

N. Bastianello and K. H. Johansson are with the School of Electrical Engineering and Computer Science, and Digital Futures, KTH Royal Institute of Technology, Stockholm, Sweden. Emails: {nicolba,kallej}@kth.se