Lyapunov-Free Analysis for Consensus of Nonlinear Discrete-Time Multi-Agent Systems

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Outline

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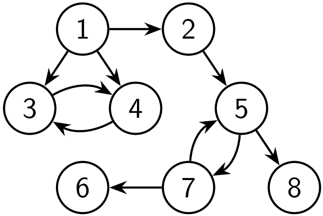
Problem set-up

$$\begin{aligned}
\text{Network} &\rightarrow \mathcal{G} = (\mathcal{V}, \mathcal{E}) \\
\text{Set of agents} &\rightarrow \mathcal{V} = \{1, \dots, n\} \\
\text{Set of interactions} &\rightarrow \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \\
\text{State of the system} &\rightarrow x \in \mathbb{R}^n \\
\text{State of agent } i &\rightarrow x_i \in \mathbb{R} \\
\text{Neighbors of agent } i &\rightarrow \mathcal{N}_i = \{j | (j, i) \in \mathcal{E}\}
\end{aligned}$$



$$x_i^+ = f_i\left(x_i, x_j : j \in \mathcal{N}_i\right) \tag{1}$$

$$x^+ = f(x) = \begin{bmatrix} f_1(\cdot) \\ \vdots \\ f_n(\cdot) \end{bmatrix} \tag{2}$$

Nonlinear maps

Linear Multi-Agent Systems

Classical Perron-Frobenius Theory deals with nonnegative matrices.

Nonlinear Multi-Agent Systems

Nonlinear Perron-Frobenius Theory[LemmensNussbaum2012] deals with order-preserving maps.

For consensus ... more properties

Row-stochasticity
+
Irreducibility

?

Previous works

- A. Jadbabaie et al., 2003.[jadbabaie2003coordination]
- Y. G. Sun et al., 2009.[Sun2009]
- A. Olshevsky et al., 2008.[Olshevsky2008]
- L. Moreau, 2005.[Moreau2005]
- M. Franceschelli et al., 2017.[Fran2017]

Motivations

- Not general results on nonlinear Discrete-Time (DT) Multi-Agent Systems (MAS).
  - Common approach: ad-hoc Lyapunov functions for specific applications
- Extend convergence results for DT MAS derived by non-negative matrix theory (Perron-Frobenius Theory) to a general class of nonlinear systems (Nonlinear Perron Frobenious Theory).
- Address heterogeneity in MAS for a class of nonlinear local interactions rules with unknown network topology.

Main contribution

Identify criteria for the local interaction rule $f_i(\cdot)$ of agent i, to establish convergence to consensus of the MAS, without identifying Lyapunov functions.

Outline

Definition: Order Preservation

A continuous map $f : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ is said to be order-preserving if for all $x, y \in \mathbb{R}^n_{\geq 0}$ it holds the following (elementwise)

$$x \leq y \Leftrightarrow f(x) \leq f(y)$$

A nonnegative

$$\Updownarrow$$

$A : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$

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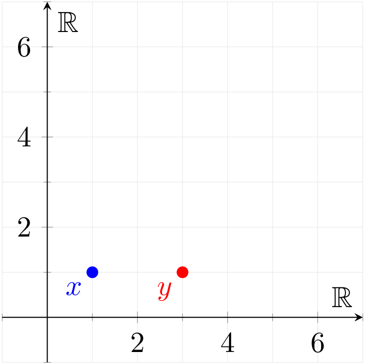

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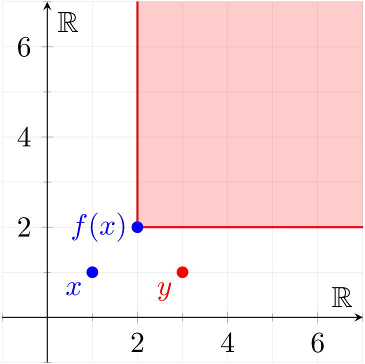

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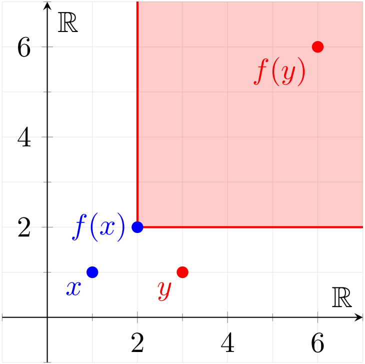

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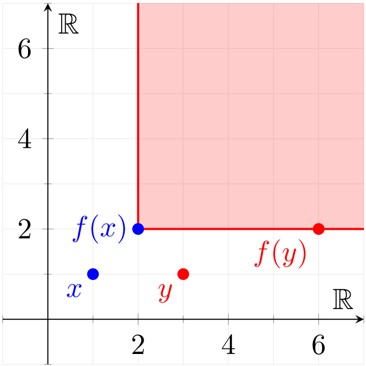

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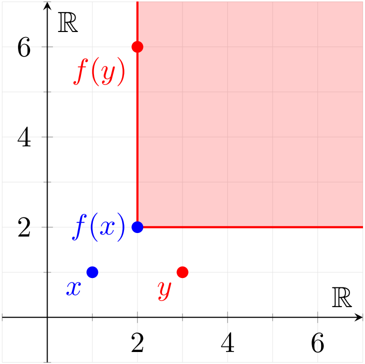

Definition: Sub-homogeneity

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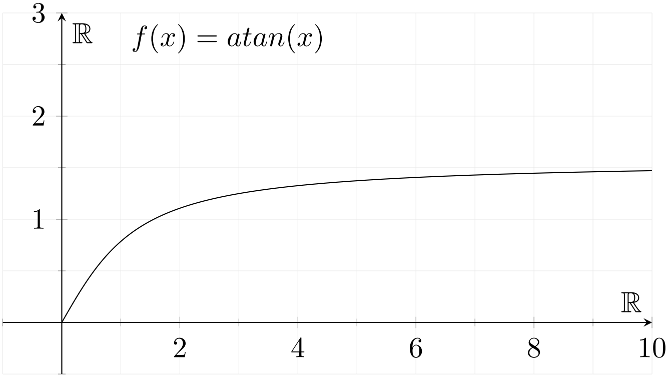

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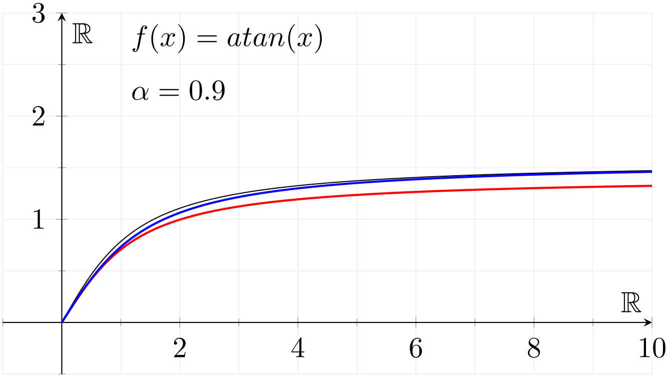

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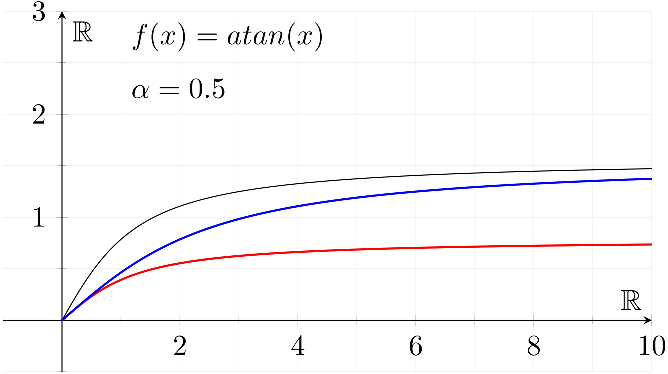

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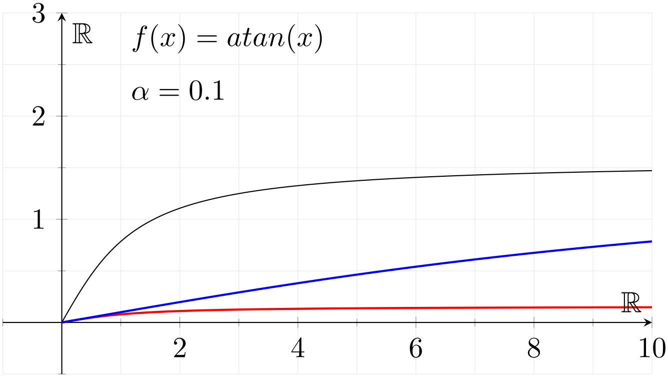
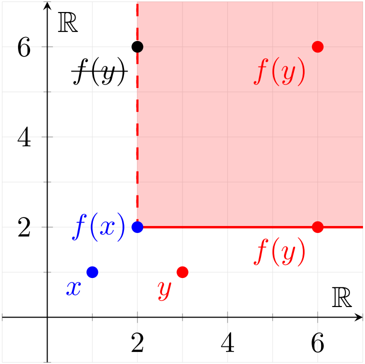


Definition: Local Strong Order-Preservation

A continuous map $f : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ is said to be locally strongly order-preserving if for all $x \leq y, x \neq y$, each f_i satisfies:

$$x_i = y_i \Rightarrow f_i(x) \leq f_i(y)$$
$$x_i < y_i \Rightarrow f_i(x) < f_i(y)$$



Outline

Convergence to Fixed Points

Theorem: Convergence to Fixed Points

If the map $f : \mathbb{R}^n_{\geq 0} \rightarrow \mathbb{R}^n_{\geq 0}$ is locally strongly order-preserving and sub-homogeneous and if f has a Lyapunov stable fixed point, then all trajectories converge to a fixed point.

Proof sketch

- Local Strong Order-Preservation (LSOP) $\Rightarrow$ Order-Preservation (OP)
- OP & sub-homogeneity (SH) $\Rightarrow$[GaubertAkian2006, Nussbaum1988] sup-norm non-expansiveness (NE);
- Lyapunov stable point & NE $\Rightarrow$[weller87, LemmensNussbaum2012] all trajectories are bounded;
- LSOP & SH $\Rightarrow$[Jiang1996] all trajectories converge to a fixed point.

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Theorem[GaubertAkian2006, Nussbaum1988]

If $f : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ is an order-preserving map, then it is sub-homogeneous if and only if is sup-norm non-expansive.

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Theorem[weller87, LemmensNussbaum2012]

If $f : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ is a sup-norm non-expansive map, then either all trajectories $f^k(x)$ with $x \in \mathbb{R}^n_{\geq 0}$ are unbounded or they converge to a periodic point $\bar{x}$ with period p, i.e., $\lim_{k\to\infty} f^{kp}(x) = \bar{x}$.

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Theorem[Jiang1996]

Let $f : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ be sub-homogeneous and locally strongly order-preserving. If all trajectories are bounded, then each trajectory converges to a periodic point with period equal to 1, i.e., a fixed point.

Convergence to Consensus Theorem

Theorem: Consensus for nonlinear DT MAS

If a set of (possibly heterogeneous) local interaction rules $f_i(x_i, x_j : j \in \mathcal{N}_i)$ with $i \in \mathcal{V}$ satisfies the next properties:

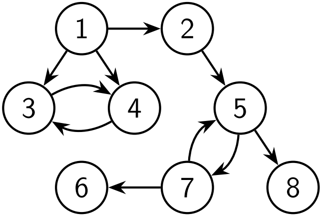
1. $f_i(x) \geq 0$ for all $x \in \mathbb{R}^n_{\geq 0}$;
2. $\partial f_i / \partial x_i > 0$ and $\partial f_i / \partial x_j \geq 0$ for $i \neq j$;
3. $\alpha f_i(x) \leq f_i(\alpha x)$ for all $\alpha \in [0, 1]$ and $x \in \mathbb{R}^n_{\geq 0}$;
4. $x_i^+ = f_i(x_i, x_j : j \in \mathcal{N}_i) = x_i$ if and only if $x_i = x_j$ for all $j \in \mathcal{N}_i$;
5. Graph $\mathcal{G}$ has a rooted directed spanning tree;

then the MAS in (**??**) converges asymptotically to the consensus state.

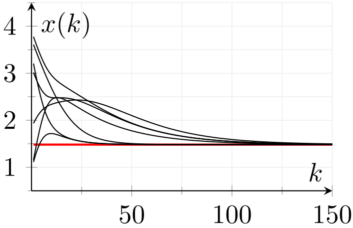
Proof sketch:

- $(1) \Rightarrow f$ leaves the cone $\mathbb{R}^n_{\geq 0}$ invariant, i.e., $f : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$;
- $(2) \Rightarrow f : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ is locally strongly order-preserving;
- $(3) \Rightarrow f : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ is sub-homogeneous;
- (4) & $(5) \Rightarrow$ the fixed point set is $\{x \in \mathbb{R}^n_{\geq 0} : x = c\mathbf{1}, \;\; c \in \mathbb{R}^+\}$.

Outline



$$x_i^+ = x_i + 0.1 \sum_{j \in \mathcal{N}_i^{in}} \text{atan}(x_j - x_i)$$



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   - Worst case: all $x_j = 0$
   - $N = \max_i \mathcal{N}_i^{in} \leq 7$
   - $f_i \geq x_i - 0.7 * \text{atan}(x_i) \geq 0$
2. f locally strongly order preserving?
   - $j \in \mathcal{N}_i^{in} \Rightarrow \frac{\partial f_i}{\partial x_j} \geq \frac{0.1}{1+(x_j-x_i)^2} > 0$
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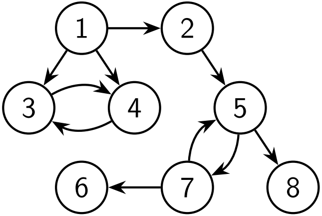
4. Is the consensus state a fixed point?
   - If $x = c\mathbb{1}$ then $x_i^+ = x_i$
5. Has $\mathcal{G}$ a rooted directed spanning tree?
   - Yes



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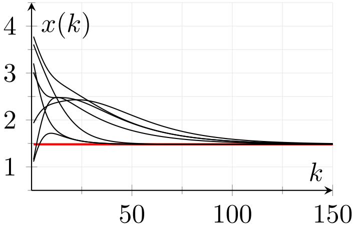

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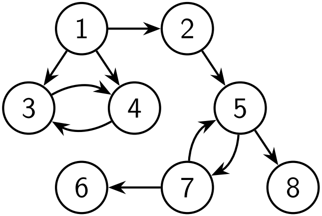

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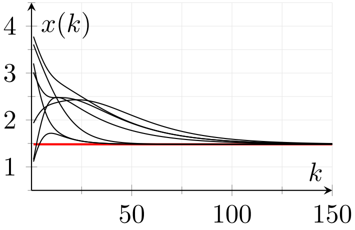

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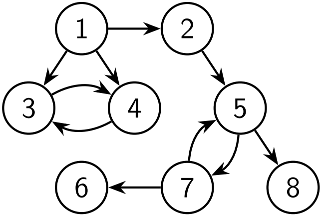

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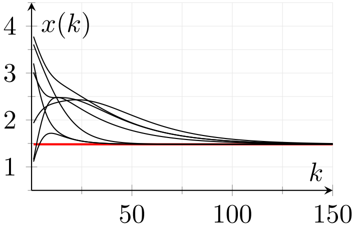

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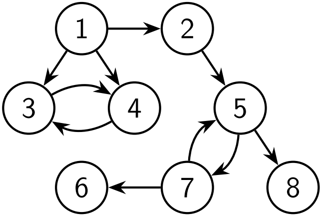

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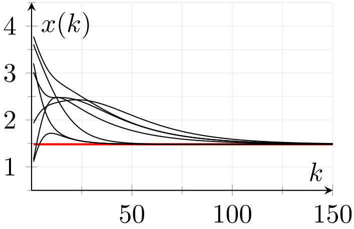

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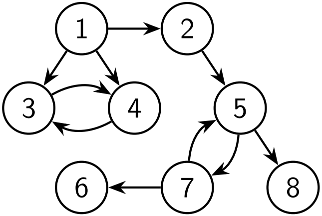

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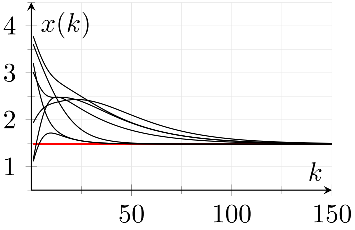

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   - $j \in \mathcal{N}_i^{in} \Rightarrow \frac{\partial f_i}{\partial x_j} \geq \frac{0.1}{1+(x_j - x_i)^2} > 0$
     $j \notin \mathcal{N}_i^{in} \Rightarrow \frac{\partial f_i}{\partial x_j} = 0$
   - $\frac{\partial f_i}{\partial x_i} \geq 1 - \frac{0.7}{1+(x_j - x_i)^2} > 0$
3. f sub-homogeneous?
   - $\alpha \sum_{j \in \mathcal{N}_i^{in}} \text{atan}(x_j - x_i) \leq \sum_{j \in \mathcal{N}_i^{in}} \text{atan}(\alpha x_j - \alpha x_i)$
4. Is the consensus state a fixed point?
   - If $x = c\mathbf{1}$ then $x_i^+ = x_i$
5. Has $\mathcal{G}$ a rooted directed spanning tree?
   - Yes

Outline

Conclusions

- We addressed the problem of consensus for nonlinear DT MAS.
- We extended convergence results for linear DT MAS to a general class of nonlinear DT MAS, allowing heterogeneous local interaction rules.
- The proposed method is inspired by nonlinear Perron-Frobenius theory and is not based on Lyapunov arguments.
- We gave an instructive example to prove the convergence of a MAS.

Future works

- Relax conditions of our main results.
- Address time-varying network topology.

Thank you for your attention

Questions?

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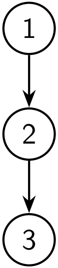
Sup-norm Paracontractive

A continuous map $f : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ is said to be sup-norm paracontractive if for all $x \in \mathbb{R}^n_{\geq 0} \setminus F_f$, where F_f is the set of the fixed points, and $y \in F_f$ it holds that

$$\|f(x), f(y)\|_\infty < \|x, y\|_\infty.$$



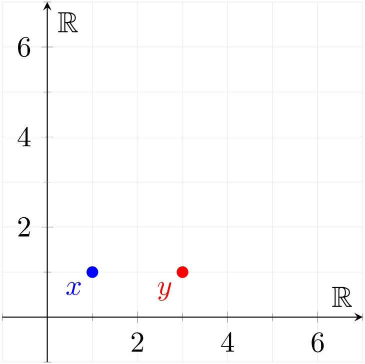
- $f(x) = \begin{bmatrix} x_1 \\ \sqrt{x_1 x_2} \\ \frac{1}{2}(x_2 + x_3) \end{bmatrix}$
- $F_f = \{c\mathbf{1} | c \in \mathbb{R}_{\geq 0}\}$
- $\bar{x} = x(0) = [\alpha, \beta, \beta]^T$
- $f(\bar{x}) = [\alpha, \sqrt{\alpha\beta}, \beta]^T$
- $\bar{y} = f(\bar{y} = [\gamma, \gamma, \gamma]^T$
- $0 < \alpha < \beta < \gamma$

- $\|f(\bar{x}), f(\bar{y})\|_\infty = \gamma - \alpha$
- $\|\bar{x}, \bar{y}\|_\infty = \gamma - \alpha$
- $\|f(\bar{x}), f(\bar{y})\|_\infty = \|\bar{x}, \bar{y}\|_\infty$
- f is sup-norm nonexpansive
- f is not sup-norm paracontractive

Definition: Sup-norm Non-Expansiveness

Denote the sup-norm metric as follow $\|x, y\|_\infty = \max_i x_i - y_i$. A continuous map $f : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ is said to be sup-norm non-expansive if for all $x, y \in \mathbb{R}^n_{\geq 0}$ it holds

$$\|f(x), f(y)\|_\infty \leq \|x, y\|_\infty.$$



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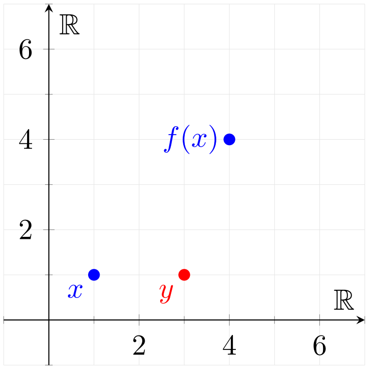

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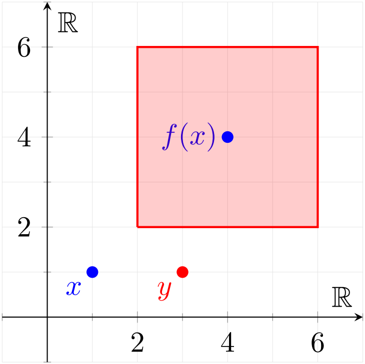

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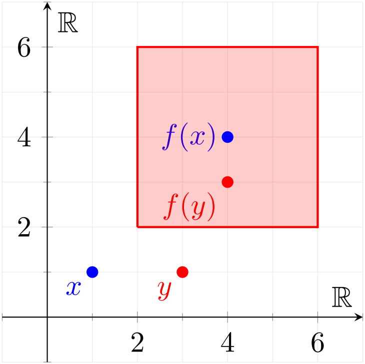

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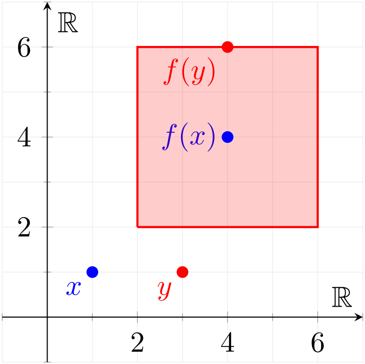

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