Discrete-Time Dynamic Consensus on the Max Value

Diego Deplano*, Mauro Franceschelli*, Alessandro Giua*

*Department of Electrical and Electronic Engineering,
University of Cagliari, Cagliari, Italy

15th European Workshop on
Advanced Control and Diagnosis, ACD 2019, Bologna, Italy

Outline

Outline

Problem set-up

Undirected Network $\to \mathcal{G} = (\mathcal{V}, \mathcal{E})$

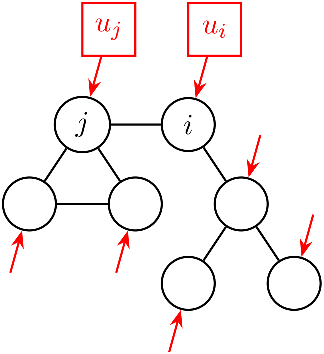
Set of agents $\to \mathcal{V} = \{1, \ldots, n\}$

Set of interactions $\to \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$

State of agent i $\to x_i \in \mathbb{R}$

Input of agent i $\to u_i \in \mathbb{R}$

Neighbors of agent i $\to \mathcal{N}_i = \{j | (j, i) \in \mathcal{E}\}$



$$x_i(k+1) = f_i\left(u_i(k),\ x_i(k),\ x_j(k) : j \in \mathcal{N}_i\right) \tag{1}$$

Static Consensus Problem

CONSENSUS $$\lim_{k\to\infty} \|x_i(k) - g(x_j(0) : j \in \mathcal{V})\| = 0 \quad \forall i \in \mathcal{V}$$

CONSENSUS AVERAGE $$\lim_{k\to\infty} \|x_i(k) - \frac{1}{n}\sum_{j\in\mathcal{V}} x_j(0)\| = 0 \quad \forall i \in \mathcal{V}$$

CONSENSUS MAX VALUE[1-6] $$\lim_{k\to\infty} \|x_i(k) - \max_{j\in\mathcal{V}} x_j(0)\| = 0 \quad \forall i \in \mathcal{V}$$

[1] **Cortés** (2008)
[2-3] **B. M. Nejad et al.** (2009-2010)
[4] **F. Iutzeler et al.** (2012)
[5] **S. Zhang et al.** (2016)
[6] **M. Abdelrahim et al.** (2017)

Static Consensus Problem

CONSENSUS $$\lim_{k\to\infty} \|x_i(k) - g(x_j(0) : j \in \mathcal{V})\| = 0 \quad \forall i \in \mathcal{V}$$

CONSENSUS AVERAGE $$\lim_{k\to\infty} \|x_i(k) - \frac{1}{n}\sum_{j\in\mathcal{V}} x_j(0)\| = 0 \quad \forall i \in \mathcal{V}$$

CONSENSUS MAX VALUE[1-6] $$\lim_{k\to\infty} \|x_i(k) - \max_{j\in\mathcal{V}} x_j(0)\| = 0 \quad \forall i \in \mathcal{V}$$

[1] Cortés (2008)
[2-3] B. M. Nejad et al. (2009-2010)
[4] F. Iutzeler et al. (2012)
[5] S. Zhang et al. (2016)
[6] M. Abdelrahim et al. (2017)

Static Consensus Problem

CONSENSUS $$\lim_{k\to\infty} \|x_i(k) - g(x_j(0) : j \in \mathcal{V})\| = 0 \quad \forall i \in \mathcal{V}$$

CONSENSUS AVERAGE $$\lim_{k\to\infty} \|x_i(k) - \frac{1}{n}\sum_{j\in\mathcal{V}} x_j(0)\| = 0 \quad \forall i \in \mathcal{V}$$

CONSENSUS MAX VALUE[1-6] $$\lim_{k\to\infty} \|x_i(k) - \max_{j\in\mathcal{V}} x_j(0)\| = 0 \quad \forall i \in \mathcal{V}$$

[1] **Cortés** (2008)
[2-3] **B. M. Nejad et al.** (2009-2010)
[4] **F. Iutzeler et al.** (2012)
[5] **S. Zhang et al.** (2016)
[6] **M. Abdelrahim et al.** (2017)

Dynamic Consensus Problem

DYNAMIC CONSENSUS $\lim_{k\to\infty} \|x_i(k) - g(u_j(k) : j \in \mathcal{V})\| \leq e \quad \forall i \in \mathcal{V}$

DYNAMIC CONSENSUS MAX VALUE $\lim_{k\to\infty} \|x_i(k) - \max_{j\in\mathcal{V}} u_j(k)\| \leq e \quad \forall i \in \mathcal{V}$

No solutions to the dynamic consensus on max value problem

Main Contribution

We propose the first protocol to solve the dynamic consensus on max value problem with a bounded relative error $\hat{\varepsilon}$.
Denoting the max value $\bar{u}(k) = \max_{j\in\mathcal{V}} u_j$, for each agent i it holds

$$\lim_{k\to\infty} \|x_i(k) - \bar{u}(k)\| \leq \hat{\varepsilon} \cdot \bar{u}(k).$$

Dynamic Consensus Problem

DYNAMIC CONSENSUS $\lim_{k\to\infty} \|x_i(k) - g(u_j(k) : j \in \mathcal{V})\| \leq e \quad \forall i \in \mathcal{V}$

DYNAMIC CONSENSUS MAX VALUE $\lim_{k\to\infty} \|x_i(k) - \max_{j\in\mathcal{V}} u_j(k)\| \leq e \quad \forall i \in \mathcal{V}$

No solutions to the dynamic consensus on max value problem

Main Contribution

We propose the first protocol to solve the dynamic consensus on max value problem with a bounded relative error $\hat{\varepsilon}$.
Denoting the max value $\bar{u}(k) = \max_{j\in\mathcal{V}} u_j$, for each agent i it holds

$$\lim_{k\to\infty} \|x_i(k) - \bar{u}(k)\| \leq \hat{\varepsilon} \cdot \bar{u}(k).$$

Dynamic Consensus Problem

DYNAMIC CONSENSUS $$\lim_{k\to\infty} \|x_i(k) - g(u_j(k) : j \in \mathcal{V})\| \leq e \quad \forall i \in \mathcal{V}$$

DYNAMIC CONSENSUS MAX VALUE $$\lim_{k\to\infty} \|x_i(k) - \max_{j\in\mathcal{V}} u_j(k)\| \leq e \quad \forall i \in \mathcal{V}$$

No solutions to the dynamic consensus on max value problem

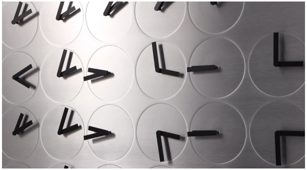
Main Contribution

We propose the first protocol to solve the dynamic consensus on max value problem with a bounded relative error $\hat{\varepsilon}$.
Denoting the max value $\bar{u}(k) = \max_{j\in\mathcal{V}} u_j$, for each agent i it holds

$$\lim_{k\to\infty} \|x_i(k) - \bar{u}(k)\| \leq \hat{\varepsilon} \cdot \bar{u}(k).$$

Applications

DISTRIBUTED SYNCHRONIZATION



Time-synchronization[7]

NETWORK PARAMETER ESTIMATION

Cardinality[8], highest degree[9]

[7] **Z. Dengchang et al.,** *Time Synchronization in Wireless Sensor Networks Using Max and Average Consensus Protocol* (2013).
[8] **R. Lucchese et al.** *Network cardinality estimation using max consensus: The case of Bernoulli trials* (2015).
[9] **T. Borsche and S. A. Attia,** *On leader election in multi-agent control systems* (2010).

Outline

Proposed Protocol

Protocol: Dynamic Max-Consensus (DMC)

Input : Tuning parameter $\alpha \in (0, 1)$;
Time interval $T \in \mathbb{Z}$.

Set : Initial conditions $x_i(0)$ at any arbitrary value in $\mathbb{R}$

for $k = 0, 1, 2, \ldots$ **each node** i **does**

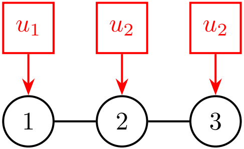
Collect $x_j(k)$ from all neighbors $j \in \mathcal{N}_i$

Update the current state according to

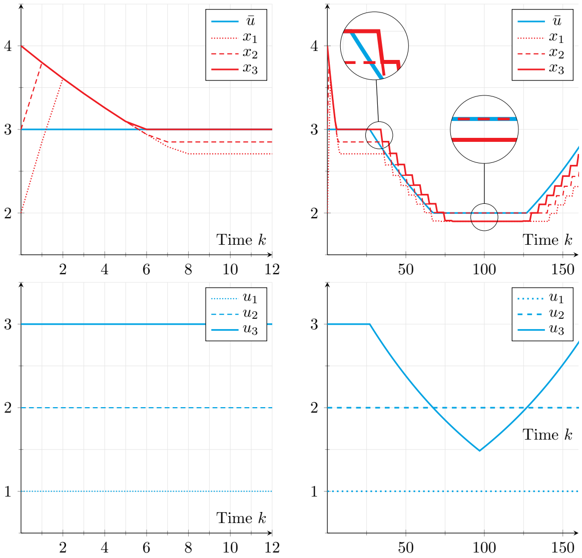
$$x_i(k+1) = \max_{j \in \mathcal{N}_i \bigcup \{i\}} \{\alpha \cdot x_j(k), u_i(k - mod(k, T))\} \tag{2}$$

Function $mod(k, T)$ *denotes the modulo operation, i.e., it outputs the remainder after the division of the integer* k *by* T

A simple example



$$\alpha = 0.95$$
$$\Pi = 0.01$$
$$T = 5$$



Steady state error analysis

Theorem 1

Consider a MAS executing DMC Protocol with tuning parameter $\alpha \in (0,1)$ and time interval $T \in \mathbb{N}$ and consider constant and positive input reference signals u_i. If graph $\mathcal{G}$ is connected, then there exists $\bar{k} \in \mathbb{Z}$ such that for any time $k \geq \bar{k}$ and any initial condition $x(0) \in \mathbb{R}^n$ the relative error is bounded by

$$\varepsilon(k) = \max_{i \in V} \frac{\|x_i(k) - \bar{u}\|}{\bar{u}} \leq 1 - \alpha^{\delta(\mathcal{G})}, \tag{3}$$

where $\delta(\mathcal{G})$ denotes the diameter of graph $\mathcal{G}$ and $\lceil \cdot \rceil$ denotes the ceil function.

Proof Sketch - Steady state error

1) Initial time is $k^\star$

2) Worst case is when

$$\max_{i \in \mathcal{V}} x_i(k^\star) \geq \max_{i \in \mathcal{V}} u_i = \bar{u}$$

3) The maximum state decreases by a factor α, then there exists a time $k_1 \geq k^\star$ such that

$$\alpha^{k_1 - k^\star} \max_{i \in \mathcal{V}} x_i(k^\star) \leq \bar{u}$$

4) For any $k \geq k_1$ it holds

$$\max_{i \in \mathcal{V}} x_i(k_1) = \bar{u}$$

5) For any $k \geq k_1 + \delta(\mathcal{G})$ it holds

$$\min_{i \in \mathcal{V}} x_i(k) = \alpha^{\delta(\mathcal{G})} \bar{u}$$

6) Finally

$$\varepsilon(k) = \max_{i \in V} \frac{\|x_i(k) - \bar{u}\|}{\bar{u}} \leq \frac{-\alpha^{\delta(\mathcal{G})} \bar{u} + \bar{u}}{\bar{u}} = 1 - \alpha^{\delta(\mathcal{G})}$$

Tracking error analysis

Assumption: Bounded relative input's change

Each unknown exogenous reference signal is strictly positive, $u_i(k) > 0$ and their relative change is bounded by a constant $\Pi \in (0, 1)$, i.e.,

$$\frac{|u_i(k+1) - u_i(k)|}{|u_i(k)|} \leq \Pi, \quad \forall i \in V,\ \forall k \geq 0.$$

Any continuous time signal with bounded relative change can be oversampled to reduce its relative changes.

Tracking error analysis

Theorem 2

Consider a MAS executing DMC Protocol with tuning parameter $\alpha \in (0, 1)$ and time interval $T \in \mathbb{N}$ and consider consider time-varying input reference signals $u_i(k)$ satisfying the assumption. If graph $\mathcal{G}$ is connected, and the tuning parameters α and T satisfy

$$(1-\alpha) > \Pi, \quad T \geq \left\lceil \frac{\delta(\mathcal{G})}{1 - \log_\alpha(1-\Pi)} \right\rceil,$$

then there exists $\bar{k} \in \mathbb{Z}$ such that for any time $k \geq \bar{k}$ and any initial condition $x(0) \in \mathbb{R}^n$ the relative error is bounded by

$$e(k) = \max_{i \in V} \frac{|x_i(k) - \bar{u}(k)|}{\bar{u}(k)} \leq \max\left\{ \frac{1}{(1-\Pi)^T} - 1, 1 - \frac{\alpha^{\delta(\mathcal{G})}}{(1+\Pi)^{T+\delta(\mathcal{G})}} \right\} \quad (4)$$

where $\delta(\mathcal{G})$ denotes the diameter of graph $\mathcal{G}$, Π is the maximum relative change of the inputs and $\lceil \cdot \rceil$ denotes the ceil function.

Proof Sketch - Tracking error

1) Consider time steps k multiple of T $\qquad k = \lambda T$ with $\lambda \in \mathbb{N}$

2) Initial time is $\qquad k^\star = \lambda^\star T$

3) Worst case is when $\qquad \max_{i \in \mathcal{V}} x_i(\lambda^\star T) \geq \max_{i \in \mathcal{V}} u_i(\lambda^\star T) = \bar{u}(\lambda^\star T)$

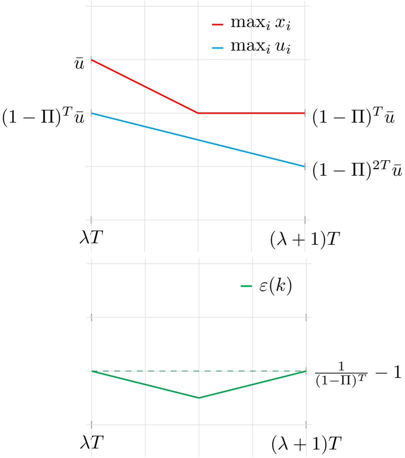
4) Since $\alpha < 1 - \Pi$ there exists $\bar{\lambda}$ such that for $\lambda \geq \bar{\lambda}$ it holds $\qquad \max_{i \in \mathcal{V}} x_i(\lambda T) \leq \bar{u}((\lambda - 1)T)$

5) For $\lambda < \bar{\lambda}$ the error decreases

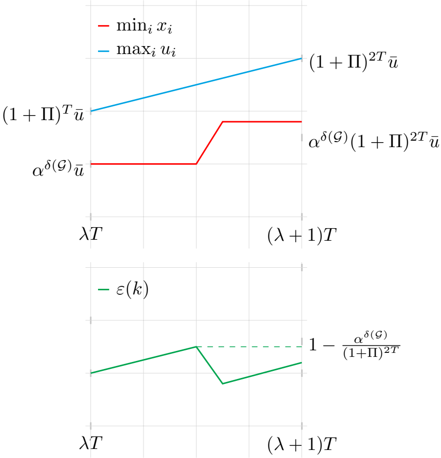
6) What happen for $\lambda \geq \bar{\lambda}$?

Proof Sketch - Tracking error

Max input is decreasing

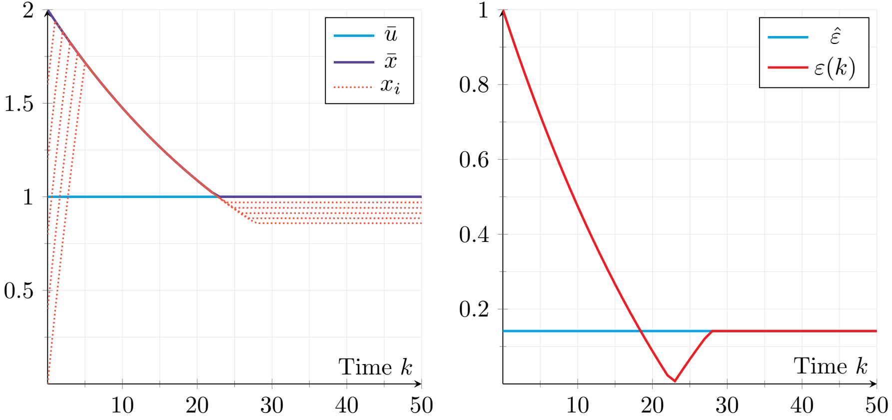


Max input is increasing

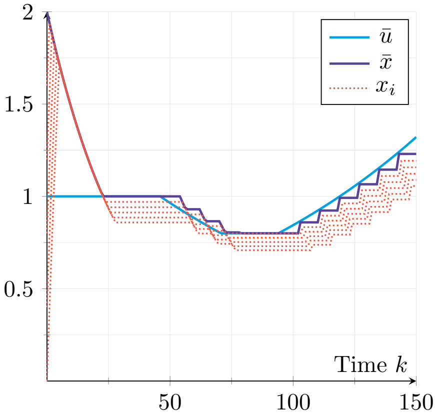


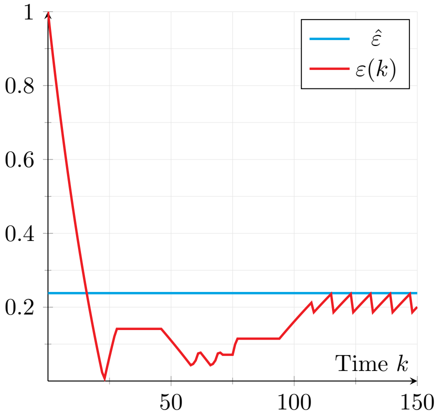
Outline

Constant inputs

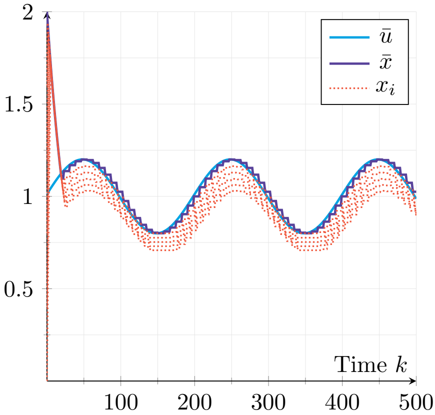


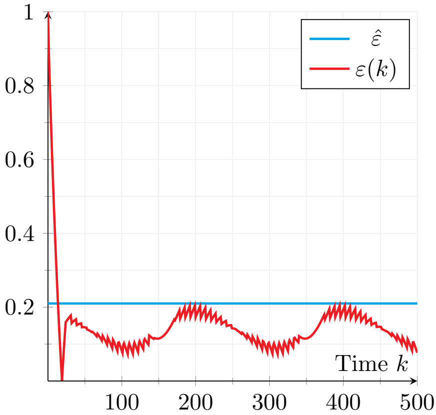
Two steps inputs



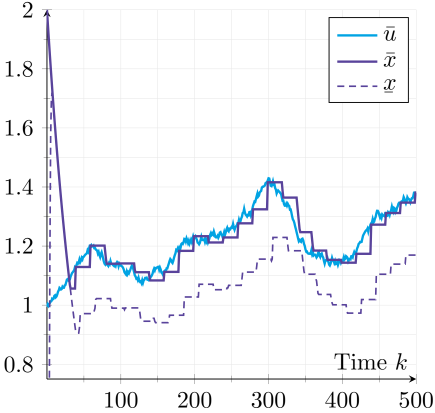


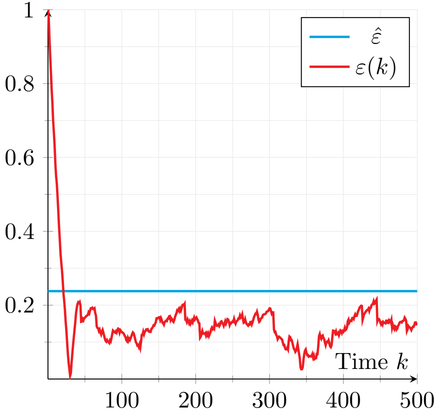
Sinusoidal input





Large Network with random input signals





Outline

Conclusions and Future works

Conclusions

- We proposed the first distributed protocol for multi-agent systems to solve the problem of tracking the maximum of a set of positive time-varying input reference signals.
- The protocol is guaranteed to achieve bounded relative error in both steady state with cosntant inputs and tracking of time-varying inputs..
- Main advantage versus static max-consensus: dealing with time-varying inputs without re-initialization.

Future works

- Novel and improved protocols able to deal with negative inputs and with improved error performance.
- Application of our dynamic consensus on the max value protocols for size-estimation of time-varying anonymous network.

Conclusions and Future works

Conclusions

- We proposed the first distributed protocol for multi-agent systems to solve the problem of tracking the maximum of a set of positive time-varying input reference signals.
- The protocol is guaranteed to achieve bounded relative error in both steady state with cosntant inputs and tracking of time-varying inputs..
- Main advantage versus static max-consensus: dealing with time-varying inputs without re-initialization.

Future works

- Novel and improved protocols able to deal with negative inputs and with improved error performance.
- Application of our dynamic consensus on the max value protocols for size-estimation of time-varying anonymous network.

Thank you for your attention

Questions?

email:
diego.deplano@unica.it