Summary

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Allocation: Theory

Firms i in an industry each produce output Y_i using a single input X_i according to the production function:

$$Y_i = A_i X_i^{\beta}$$

where A_i is firm productivity and $\beta \in (0,1)$ is a parameter. Say that the industry is efficient if it produces the maximum possible output given the total amount of input used. Let lowercase letters denote natural logarithms, so that $x_i = lnX_i$. Assume that there is a fixed number of firms N.

(a) What is the coefficient in a regression of y_i on a_i in an efficient industry?

First let's define what are the conditions for an efficient industry. An industry is efficient if:

$$\max_{\{X_i\}_{i=1}^N} \sum_{i=1}^N A_i X_i^{\beta}$$
$$s.t. \sum_{i=1}^N X_i = X$$

Where X is the total amount of inputs used. First order conditions imply immediately that:

$$\beta A_i X_i^{\beta-1} = \lambda \ \forall i \rightarrow X_i = \left(\frac{\lambda}{A_i \beta}\right)^{\frac{1}{\beta-1}} \ \forall i$$

Using the constraint we have that:

$$\sum_{i=1}^N X_i = X \rightarrow \sum_{i=1}^N \left(\frac{\lambda}{A_i \beta}\right)^{\frac{1}{\beta-1}} = X$$

So we can solve for λ:

$$\lambda = \beta X^{\beta-1} \left(\sum_{i=1}^N A_i^{\frac{1}{1-\beta}}\right)^{1-\beta}$$

Now, we can replace λ in the equation of X_i to get:

$$X_i = X \frac{A_i^{\frac{1}{1-\beta}}}{\sum_{i=1}^N A_i^{\frac{1}{1-\beta}}}$$

Note that this implies that if $A_i > A_j$ then $X_i > X_j$. Now, we can take logs and get:

$$x_i = x + \frac{1}{1-\beta}a_i - \ln\Big[\sum_{i=1}^{N} A_i^{\frac{1}{1-\beta}}\Big]$$

Now, define the following regression model:

$$y_i = \alpha_0 + \alpha_1 a_i + \epsilon_i$$

Then, the population coefficient would be:

$$\alpha_1 = \frac{Cov(y_i, a_i)}{Var(a_i)} = \frac{Cov(a_i + \beta x_i, a_i)}{Var(a_i)} = 1 + \beta\frac{Cov(x_i, a_i)}{Var(a_i)}$$

Note that:

$$Cov(x_i, a_i) = Cov(x + \frac{1}{1-\beta}a_i - \ln\Big[\sum_{i=1}^{N} A_i^{\frac{1}{1-\beta}}\Big], a_i) = \frac{1}{1-\beta}Var(a_i) - Cov(\ln\Big[\sum_{i=1}^{N} A_i^{\frac{1}{1-\beta}}\Big], a_i)$$

Using this result we obtain that:

$$\alpha_1 = 1 + \beta\frac{Cov(x_i, a_i)}{Var(a_i)} = 1 + \frac{\beta}{1-\beta} - \beta\frac{Cov(\ln\Big[\sum_{i=1}^{N} A_i^{\frac{1}{1-\beta}}\Big], a_i)}{Var(a_i)}$$

For a fixed number of firms N we have that $\ln\Big[\sum_{i=1}^{N} A_i^{\frac{1}{1-\beta}}\Big]$ is a constant. Therefore, we have that:

$$\alpha_1 = 1 + \beta\frac{Cov(x_i, a_i)}{Var(a_i)} = 1 + \frac{\beta}{1-\beta} = \frac{1}{1-\beta}$$

Finally, note that this is positively biased for 1 that is the true coefficient of a_i in the log production function.

(b) What is the coefficient in a regression of y_i on xi in an efficient industry?

Define the following regression model:

$$y_i = \gamma_0 + \gamma_1 x_i + \eta_i$$

Then, the population coefficient would be:

$$\gamma_1 = \frac{Cov(y_i, x_i)}{Var(x_i)} = \frac{Cov(a_i + \beta x_i, x_i)}{Var(x_i)} = \frac{Cov(a_i, x_i)}{Var(x_i)} + \beta$$

Now, note that:

$$Var(x_i) = \Big(\frac{1}{1-\beta}\Big)^2 Var(a_i) + Var\Big(\ln\Big[\sum_{i=1}^{N} A_i^{\frac{1}{1-\beta}}\Big]\Big) - 2\frac{1}{1-\beta}Cov(\ln\Big[\sum_{i=1}^{N} A_i^{\frac{1}{1-\beta}}\Big], a_i)$$

Using also the result from part **(a)** we get that:

$$\gamma_1 = \frac{Cov(a_i, x_i)}{Var(x_i)} + \beta = \frac{\frac{1}{1-\beta}Var(a_i) - Cov(\ln\left[\sum_{i=1}^{N} A_i^{\frac{1}{1-\beta}}\right], a_i)}{\left(\frac{1}{1-\beta}\right)^2 Var(a_i) + Var\Big(\ln\left[\sum_{i=1}^{N} A_i^{\frac{1}{1-\beta}}\right]\Big) - 2\frac{1}{1-\beta}Cov(\ln\left[\sum_{i=1}^{N} A_i^{\frac{1}{1-\beta}}\right], a_i)} + \beta$$

Again using that for a fixed number of firms N we have that $\ln\left[\sum_{i=1}^{N} A_i^{\frac{1}{1-\beta}}\right]$ is a constant, we get that:

$$\gamma_1 = \frac{Cov(a_i, x_i)}{Var(x_i)} + \beta = \frac{1-\beta}{1} + \beta = 1$$

Finally, note that this is positively biased for β that is the true coefficient of x_i in the log production function.

(c) Suppose that output and inputs are each traded in a perfectly competitive market with a price of 1. Is the equilibrium efficient?

The given assumptions imply that the FOC for profit maximization of a given firm is just:

$$\beta A_i {X_i^*}^{\beta-1} = 1 \; \forall i$$

Which essentially means that $\lambda = 1$, and therefore we have that:

$$X_i^* = \left(\frac{1}{\beta A_i}\right)^{\frac{1}{\beta-1}} \; \forall i$$

Therefore, the competitive equilibrium is efficient.

(d) In the competitive equilibrium described in (c), is β identified from the data $\{(x_i, y_i)\}_{i=1}^N$? Is $\{A_i\}_{i=1}^N$ identified?

Given that the competetitive equilibrium is efficient, we can use the result from **(b)**. Then, for the following regression model:

$$y_i = \gamma_0 + \gamma_1 x_i + \eta_i$$

we have that:

$$\gamma_1 = \frac{Cov(a_i, x_i)}{Var(x_i)} + \beta = 1$$

So β is not identified. If we knew β we could compute $\{A_i\}_{i=1}^N$ by doing:

$$A_i = \exp(y_i - \beta x_i)$$

But as we cannot learn it from the data $\{(x_i, y_i)\}_{i=1}^N$, then $\{A_i\}_{i=1}^N$ is not identified.

(e) A central planner assigns inputs to firms to ensure that output is identical across firms, i.e. that $Y_i = Y$ for all i. How much less total output is produced than would be the case under an efficient allocation of the same total amount of inputs?

Start by noting that if $Y_i = Y\ \forall i$ then it must be true that:

$$A_i X_i^{\beta} = Y\ \forall i \rightarrow X_i = \left(\frac{Y}{A_i}\right)^{\frac{1}{\beta}}$$

Using the feasibility constraint for total inputs X we get that:

$$\sum_{i=1}^{N} X_i = X \rightarrow \sum_{i=1}^{N} \left(\frac{Y}{A_i}\right)^{\frac{1}{\beta}} = X$$

Solving for Y we get that:

$$Y = \left(\frac{X}{\sum_{i=1}^{N}\left(\frac{1}{A_i}\right)^{\frac{1}{\beta}}}\right)^{\beta}$$

Therefore, we have that:

$$X_i = X\frac{\left(\frac{1}{A_i}\right)^{\frac{1}{\beta}}}{\sum_{i=1}^{N}\left(\frac{1}{A_i}\right)^{\frac{1}{\beta}}}$$

Now, note that:

$$NY = N\left(\frac{X}{\sum_{i=1}^{N}\left(\frac{1}{A_i}\right)^{\frac{1}{\beta}}}\right)^{\beta}$$

As for the efficient allocation for the same amount of total input X, using the result from (c), we have that:

$$\sum_{i=1}^{N} Y_i^* = \sum_{i=1}^{N} A_i X_i^{*\beta} = \sum_{i=1}^{N} A_i \left(\frac{1}{\beta A_i}\right)^{\frac{\beta}{\beta-1}}$$

After some algebraic operations we get that:

$$\sum_{i=1}^{N} Y_i^* = \beta^{\frac{\beta}{1-\beta}} \sum_{i=1}^{N} A_i^{\frac{1}{1-\beta}}$$

Now, note that we can use the fact that $\sum_{i=1}^{N} X_i^* = \sum_{i=1}^{N}\left(\frac{1}{\beta A_i}\right)^{\frac{1}{\beta-1}} X$ to get:

$$\sum_{i=1}^{N} Y_i^* = \frac{X}{\beta}$$

Putting all together we have that total output that is sacrificed by a central planner that allocates inputs such that $Y_i = Y$ relative to an efficient allocation is then:

$$\sum_{i=1}^{N} Y_i^* - NY = \frac{X}{\beta} - N\left(\frac{X}{\sum_{i=1}^{N}\left(\frac{1}{A_i}\right)^{\frac{1}{\beta}}}\right)^{\beta}$$

(f) In the competitive equilibrium described in (c), are $\{\beta_i, A_i\}_{i=1}^{N}$ identified if we now posit that $Y_i = A_i X_i^{\beta_i}$? Provide some economic intuition.

In this case for an equilibrium as in **(c)** we know that:

$$\beta_i A_i X_i^{\beta_i - 1} = 1$$

Multiplying both sides by X_i we have that:

$$\beta_i A_I X_i^{\beta_i - 1} = X_i \rightarrow \beta_i Y_i = X_i$$

So that:

$$\beta_i = \frac{X_i}{Y_i}$$

Therefore, we have that:

$$Y_i = A_i X_i^{\frac{X_i}{Y_i}} \rightarrow A_i = \frac{Y_i}{X_i^{\frac{X_i}{Y_i}}}$$

Intuitively, if we know the functional form of the production function we can use its implications - i.e. that the expenditure on input is a constant share of output value - to find the parameters.

Allocation: Evidence

(a) Estimate a regression of y_i on x_i separately for each country-year and report the results. Comment on whether they are consistent with efficiency in the sense of question 1.

The results are displayed in the table below:

	(1)	(2)	(3)	(4)	(5)
	Brazil 2003	India 2002	Indonesia 2003	SriLanka 2004	Thailand 2004
x	0.698***	0.815***	0.774***	0.906***	0.852***
	(0.0555)	(0.0614)	(0.0573)	(0.0743)	(0.0243)
_cons	1.937***	0.852	3.040**	-0.166	0.803**
	(0.479)	(0.648)	(0.961)	(0.691)	(0.282)
p-value test beta_hat = 1	0.000	0.004	0.000	0.208	0.000
R-squared	.63262	.6902463	.5859608	.7159755	.8722668
Observations	94	81	131	61	182

Standard errors in parentheses

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Note that I report the p-value of an F-test of the estimated coefficient of x_i, $\hat{\beta}$, being equal to 1. That is because, from part **1(b)** we know that in an efficient industry we should expect this coefficient to be equal to 1. The only country-year in which we cannot reject $\hat{\beta} = 1$ is Sri Lanka 2004.

(b) Suppose that each firm has its own coefficient β_i. Use the assumptions from 1(f) to estimate β_i for each firm. Compute, for each country-year, the percent increase in textile output from moving to an efficient allocation of capital.

(c) Now suppose that all textile firms in the world have the same coefficient β. Using your answers to 1(b) and 2(a), identify the country-year c^* that is closest to an efficient allocation of capital. Estimate β under the assumption that the economy in c^* is competitive in the sense of 1(c) and that the rental rate on capital is 2.5 percent of its value.

(d) Using your estimate of β from 2(c), compute, for each country-year, the percent increase in textile output from moving to an efficient allocation of capital.

(e) Which country shows the smallest gain from reallocation? Why?