5. Sea el espacio vectorial de matrices complejas 2 x 2 hermiticas donde una matriz hermitica (o autoadjunta) sera igual a su adjunta osea una matriz sera igual a su transpuesta conjugada $(A^\dagger)^i_j \rightarrow (A^\dagger)^j_i \equiv A^j_i$

$$A \leftrightarrow \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} = A^\dagger = \begin{pmatrix} z_1^* & z_3^* \\ z_2^* & z_4^* \end{pmatrix} \left\{ \begin{array}{l} z_1^* = z_1 = \text{Real} \\ z_4^* = z_4 = \text{Real} \\ z_3 = z_2^* \ \text{Complejos} \end{array} \right.$$

(a). Muestre que las matrices de Pauli $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ forman base para ese espacio

Sea $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Probemos que cumplen ser hermiticas

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sigma_0^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad // \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_2^\dagger = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \Big| \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Sean entonces base se tiene que cumplir

$$a_0 \sigma_0 + a_1 \sigma_1 + a_2 \sigma_2 + a_3 \sigma_3 = 0$$

no lo usaras

$$a_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + a_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & a_1 \\ a_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -ia_2 \\ ia_2 & 0 \end{pmatrix} + \begin{pmatrix} a_3 & 0 \\ 0 & -a_3 \end{pmatrix} = \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix}$$

$$\begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ igualando } \left\{ \begin{array}{l} a_3 = 0 \\ a_1 - ia_2 = 0 \\ a_1 + ia_2 = 0 \\ -a_3 = 0 \end{array} \right.$$

Entonces $a_3 = 0$ $+ \begin{cases} a_1 - ia_2 = 0 \\ a_1 + ia_2 = 0 \end{cases}$, $a_1 = 0$

y reemplazando $0 - ia_2 = 0$ implica que $a_2 = 0$ $\quad a_1 = 0, \ a_2 = 0, \ a_3 = 0$

(b) Compruebe que la base es ortogonal con el siguente producto interno

$\langle a|b\rangle \doteq T_r(A^{\dagger}B)$

Sea entonces ortogonal si $\langle a|b\rangle = 0$

probemos

$\langle \sigma_1|\sigma_2\rangle = T_r(\sigma_1^{\dagger}\sigma_2)$

$$T_r\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right) = T_r\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i + (-i) = 0 \checkmark$$

$\langle \sigma_1|\sigma_3\rangle = T_r(\sigma_1^{\dagger}\sigma_2)$

$$T_r\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = T_r\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 0 + 0 = 0 \checkmark$$

$\langle \sigma_2|\sigma_3\rangle = T_r(\sigma_2^{\dagger}\sigma_3)$

$$T_r\left(\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = T_r\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = 0 + 0 = 0 \checkmark$$

$\langle \sigma_3|\sigma_1\rangle = T_r(\sigma_3^{\dagger}\sigma_1)$

$$T_r\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = T_r\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 0 + 0 = 0 \checkmark$$

$\langle \sigma_2|\sigma_1\rangle = T_r(\sigma_2^{\dagger}\sigma_1)$

$$T_r\left(\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = T_r\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i + i = 0 \checkmark$$

$\langle \sigma_3|\sigma_2\rangle = T_r(\sigma_3^{\dagger}\sigma_2)$

$$T_r\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = 0 + 0 = 0 \checkmark$$