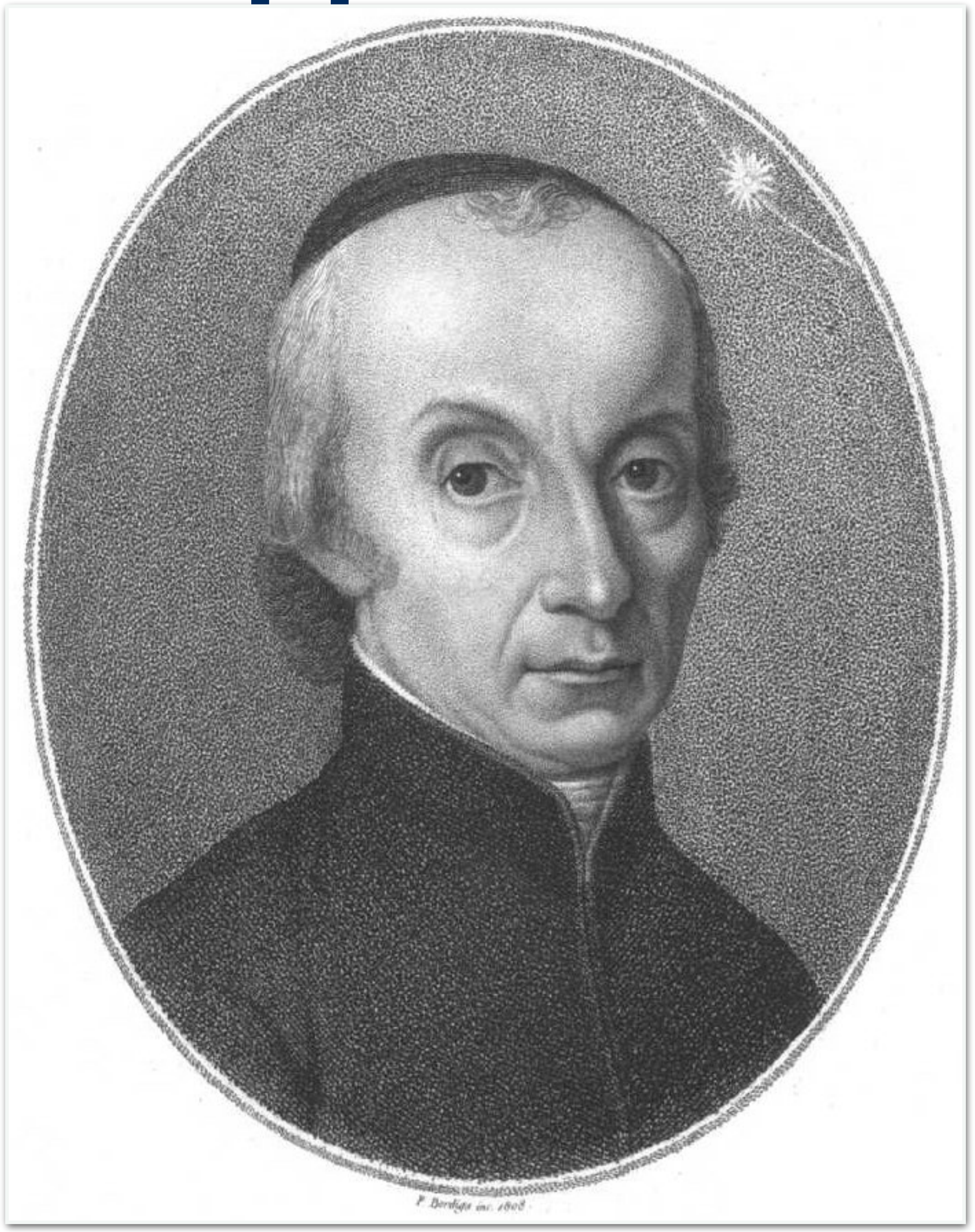
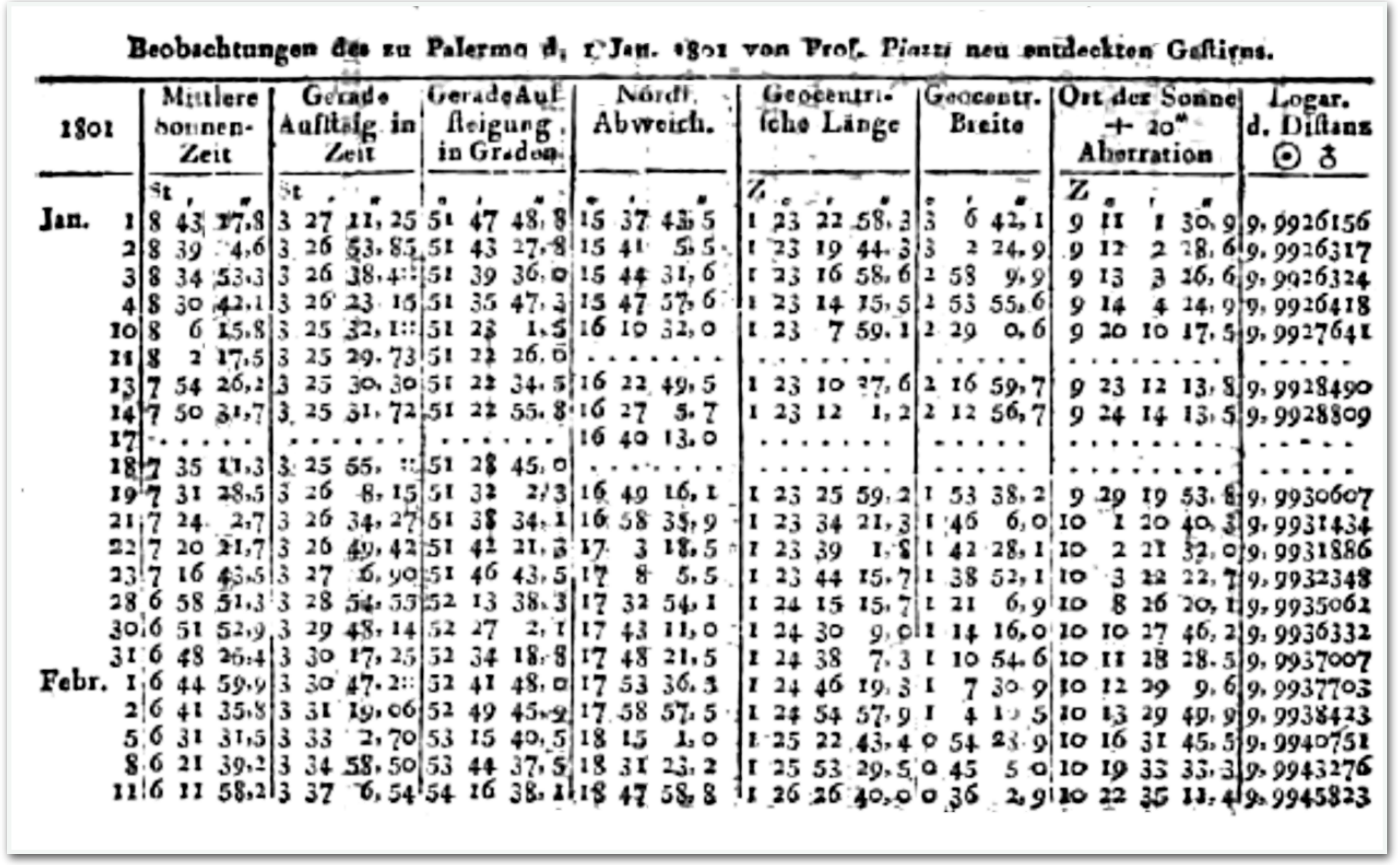
Giuseppe Piazzi and



Giuseppe Piazzi



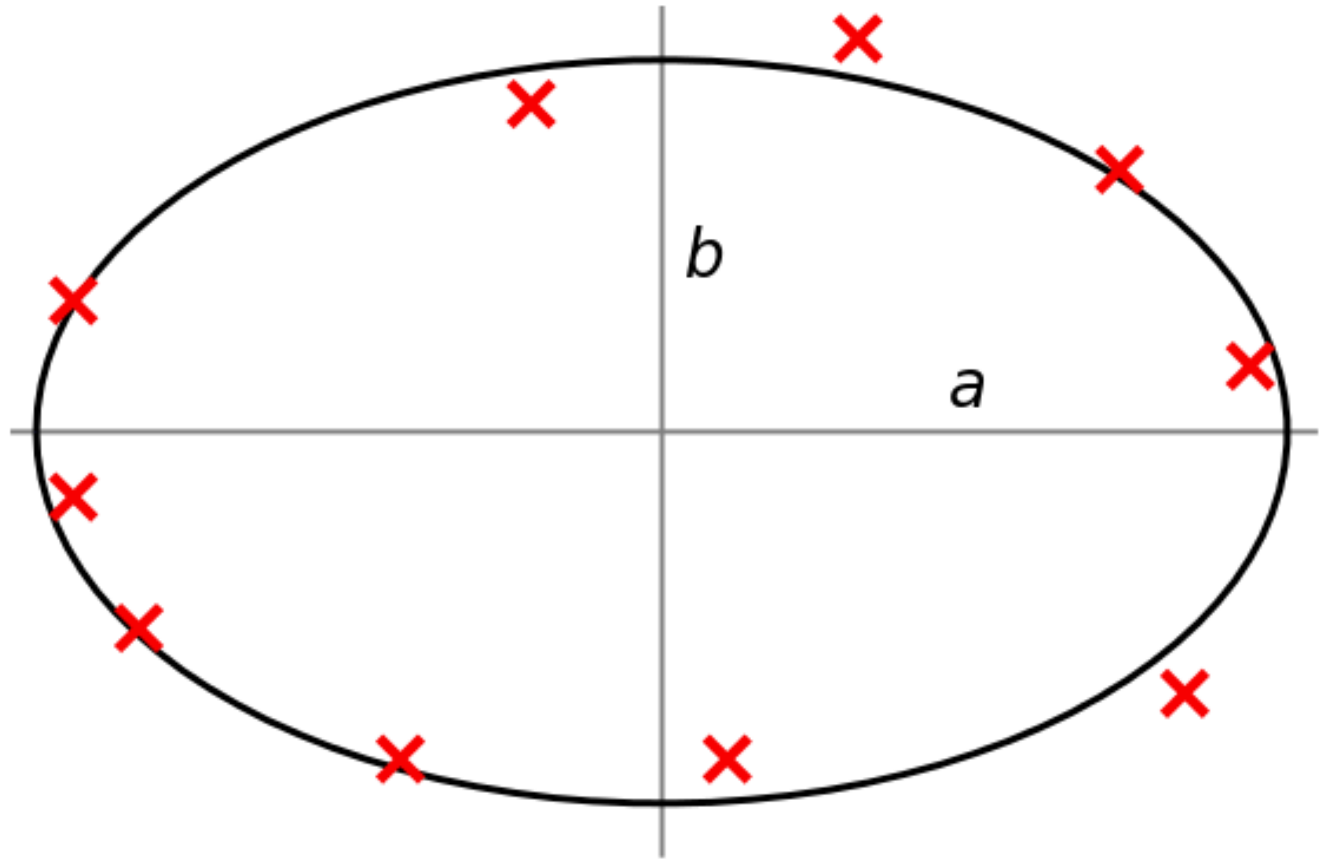
Beobachtungen des zu Palermo d. 1. Jan. 1801 von Prof. Piazzi neu entdeckten Gestirns.

1801	Mittlere Sonnen-Zeit	Gerade Aufstg. in Zeit	Gerade Aufsteigung in Graden	Nördl. Abweich.	Geocentrische Länge	Geocentr. Breite	Ort der Sonne + 20" Aberration	Logar. d. Distanz ⊙ ♂
	St ′ ″	St ′ ″	° ′ ″	° ′ ″	Z ° ′ ″	° ′ ″	Z ° ′ ″	
Jan. 1	8 43 27,8	3 27 11,25	51 47 48,8	15 37 43,5	1 23 22 58,3	3 6 42,1	9 11 1 30,9	9,9926156
2	8 39 4,6	3 26 53,85	51 43 27,8	15 41 5,5	1 23 19 44,3	3 2 24,9	9 12 2 28,6	9,9926317
3	8 34 53,3	3 26 38,4::	51 39 36,0	15 44 31,6	1 23 16 58,6	2 58 9,9	9 13 3 26,6	9,9926324
4	8 30 42,1	3 26 23,15	51 35 47,3	15 47 57,6	1 23 14 15,5	2 53 55,6	9 14 4 24,9	9,9926418
10	8 6 15,8	3 25 32,1::	51 23 1,5	16 10 32,0	1 23 7 59,1	2 29 0,6	9 20 10 17,5	9,9927641
11	8 2 17,5	3 25 29,73	51 22 26,0	....	....	.....	....	.....
13	7 54 26,2	3 25 30,30	51 22 34,5	16 22 49,5	1 23 10 27,6	2 16 59,7	9 23 12 13,8	9,9928490
14	7 50 31,7	3 25 31,72	51 22 55,8	16 27 5,7	1 23 12 1,2	2 12 56,7	9 24 14 13,5	9,9928809
17	......	......	......	16 40 13,0	....	.....	....	.....
18	7 35 11,3	3 25 55,::	51 28 45,0	....	....	.....	....	.....
19	7 31 28,5	3 26 8,15	51 32 2,3	16 49 16,1	1 23 25 59,2	1 53 38,2	9 29 19 53,8	9,9930607
21	7 24 2,7	3 26 34,27	51 38 34,1	16 58 35,9	1 23 34 21,3	1 46 6,0	10 1 20 40,3	9,9931434
22	7 20 21,7	3 26 49,42	51 42 21,3	17 3 18,5	1 23 39 1,8	1 42 28,1	10 2 21 32,0	9,9931886
23	7 16 43,5	3 27 6,90	51 46 43,5	17 8 5,5	1 23 44 15,7	1 38 52,1	10 3 22 22,7	9,9932348
28	6 58 51,3	3 28 54,55	52 13 38,3	17 32 54,1	1 24 15 15,7	1 21 6,9	10 8 26 20,1	9,9935062
30	6 51 52,9	3 29 48,14	52 27 2,1	17 43 11,0	1 24 30 9,0	1 14 16,0	10 10 27 46,2	9,9936332
31	6 48 26,4	3 30 17,25	52 34 18,8	17 48 21,5	1 24 38 7,3	1 10 54,6	10 11 28 28,5	9,9937007
Febr. 1	6 44 59,9	3 30 47,2::	52 41 48,0	17 53 36,3	1 24 46 19,3	1 7 30,9	10 12 29 9,6	9,9937703
2	6 41 35,8	3 31 19,06	52 49 45,9	17 58 57,5	1 24 54 57,9	1 4 15,5	10 13 29 49,9	9,9938423
5	6 31 31,5	3 33 2,70	53 15 40,5	18 15 1,0	1 25 22 43,4	0 54 23,9	10 16 31 45,5	9,9940751
8	6 21 39,2	3 34 58,50	53 44 37,5	18 31 23,2	1 25 53 29,5	0 45 5,0	10 19 33 33,3	9,9943276
11	6 11 58,2	3 37 6,54	54 16 38,1	18 47 58,8	1 26 26 40,0	0 36 2,9	10 22 35 11,4	9,9945823

Piazzi's 24 observations

Carl Friedrich Gauss

Carl Friedrich Gauss
‘Princeps mathematicorum’

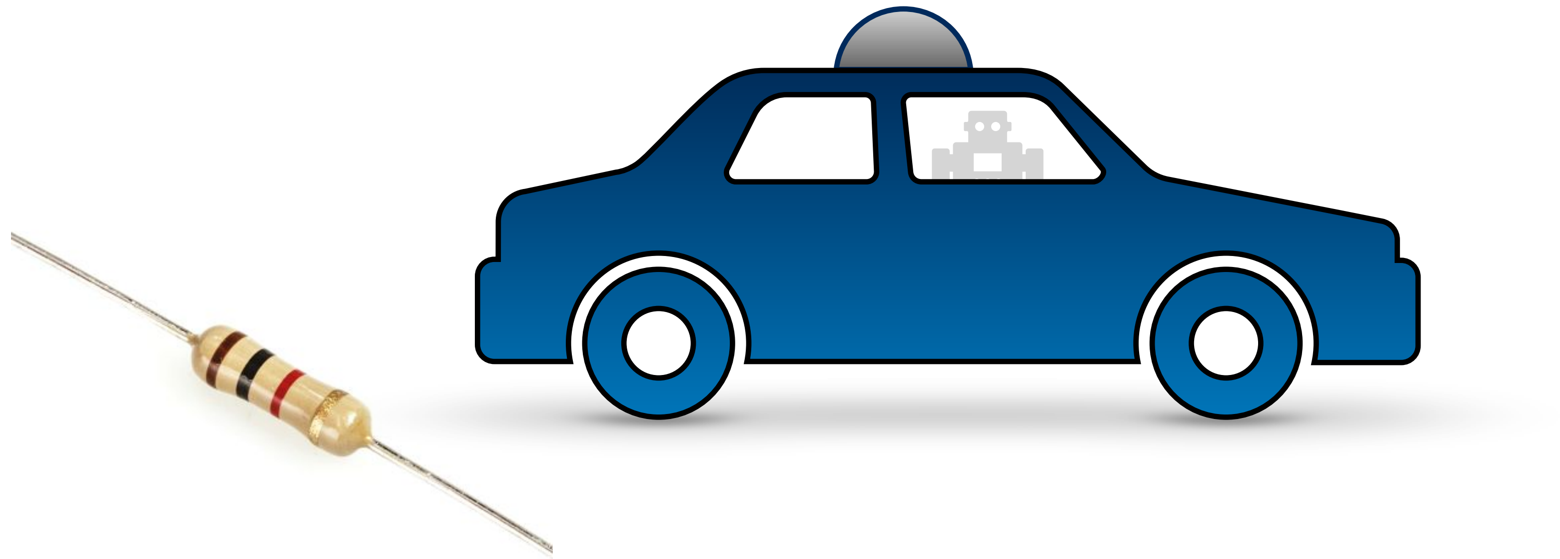


*Gauss used the **method of least squares** to determine the orbital parameters of Ceres.*

Least Squares

*The **most probable value** of the unknown quantities will be that in which the **sum of the squares** of the differences between the actually observed and the computed values multiplied by numbers that measure the degree of precision is a **minimum**.*

- Carl Friedrich Gauss



Resistor in the drive-system of a car

Multimeter

Estimating Resistance

Measurement	Resistance (Ohms)
1	1068
2	988
3	1002
4	996

***R**?*

Estimating Resistance

Measurement	Resistance (Ohms)
1	1068
2	988
3	1002
4	996

Let x be the resistance. Assume it is a **constant**, but **unknown**.

We make measurements, y , of the resistance.

We model our measurements as corrupted by noise v.

$$\boxed{y = x + v}$$

Estimating Resistance

Measurement	Resistance (Ohms)
1	1068
2	988
3	1002
4	996

Measurement Model

'Actual' resistance *Measurement noise*

$$y_1 = x + v_1$$

$$y_2 = x + v_2$$

$$y_3 = x + v_3$$

$$y_4 = x + v_4$$

Estimating Resistance

#	Resistance (Ohms)
1	1068
2	988
3	1002
4	996

Measurement Model

$$y_1 = x + v_1$$

$$y_2 = x + v_2$$

$$y_3 = x + v_3$$

$$y_4 = x + v_4$$

Squared Error

$$e_1^2 = (y_1 - x)^2$$

$$e_2^2 = (y_2 - x)^2$$

$$e_3^2 = (y_3 - x)^2$$

$$e_4^2 = (y_4 - x)^2$$

The squared error *criterion*: $\hat{x}_{\mathrm{LS}} = \operatorname{argmin}_x \left(e_1^2 + e_2^2 + e_3^2 + e_4^2\right) = \mathscr{L}_{\mathrm{LS}}(x)$

The 'best' estimate of resistance is the one that minimizes the sum of squared errors

Minimizing the Squared Error

$$\boxed{\hat{x}_{\mathrm{LS}} = \operatorname{argmin}_x \left(e_1^2 + e_2^2 + e_3^2 + e_4^2\right) = \mathscr{L}_{\mathrm{LS}}(x)}$$

Let's re-write our criterion using vectors:

This matrix is called the ***'Jacobian'***

$$\mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \mathbf{y} - \mathbf{H}x$$

$$= \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} x$$

Minimizing the Squared Error

$$\boxed{\hat{x}_{\mathrm{LS}} = \operatorname{argmin}_{x} \left(e_1^2 + e_2^2 + e_3^2 + e_4^2\right) = \mathscr{L}_{\mathrm{LS}}(x)}$$

Now, we can express our criterion as follows,

$$\begin{aligned}
\mathscr{L}_{\mathrm{LS}}(x) = e_1^2 + e_2^2 + e_3^2 + e_4^2 &= \mathbf{e}^T\mathbf{e} \\
&= (\mathbf{y} - \mathbf{H}x)^T(\mathbf{y} - \mathbf{H}x) \\
&= \mathbf{y}^T\mathbf{y} - x^T\mathbf{H}^T\mathbf{y} - \mathbf{y}^T\mathbf{H}x + x^T\mathbf{H}^T\mathbf{H}x
\end{aligned}$$

Minimizing the Squared Error

$$\boxed{\mathscr{L}(x) = \mathbf{e}^T\mathbf{e} = \mathbf{y}^T\mathbf{y} - x^T\mathbf{H}^T\mathbf{y} - \mathbf{y}^T\mathbf{H}x + x^T\mathbf{H}^T\mathbf{H}x}$$

To minimize this, we can compute the partial derivative with respect to our parameter, set to 0, and solve for an extremum:

$$\left.\frac{\partial \mathscr{L}}{\partial x}\right|_{x=\hat{x}} = -\mathbf{y}^T\mathbf{H} - \mathbf{y}^T\mathbf{H} + 2\hat{x}^T\mathbf{H}^T\mathbf{H} = 0$$

$$-2\mathbf{y}^T\mathbf{H} + 2\hat{x}^T\mathbf{H}^T\mathbf{H} = 0$$

Re-arranging, we arrive at:

$$\hat{x}_{\mathrm{LS}} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{y}$$

x-hat minimizes our squared error criterion!

Minimizing the Squared Error

Careful! We will only be able to solve for $\hat{x}$ if $(\mathbf{H}^T\mathbf{H})^{-1}$ exists

If we have m measurements, and n unknown parameters, then:

$$\mathbf{H} \in \mathbb{R}^{m \times n} \qquad \mathbf{H}^T\mathbf{H} \in \mathbb{R}^{n \times n}$$

This means that $(\mathbf{H}^T\mathbf{H})^{-1}$ exists only if there are at least as many measurements as there are unknown parameters:

$$m \geq n$$

Minimizing the Squared Error

Returning to our problem, we see that:

$$\mathbf{y} = \begin{bmatrix} 1068 \\ 988 \\ 1002 \\ 996 \end{bmatrix} \qquad \mathbf{H} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

#	Resistance (Ohms)
1	1068
2	988
3	1002
4	996

$$\hat{x}_{\mathrm{LS}} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{y}$$

$$= \left([1111] \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right)^{-1} [1111] \begin{bmatrix} 1068 \\ 988 \\ 1002 \\ 996 \end{bmatrix} = \frac{1}{4}(1068 + 988 + 1002 + 996) = 1013.5 \quad \text{Ohms}$$

The least squares solution is just the mean of our measurements!

Method of Least Squares I

- Our measurement model, $y = x + v$, is **linear**
- Measurements are **equally weighted**
  (we do not suspect that some have more noise than others)

Batch Least Squares

In our previous formulation, we assumed we had all of our measurements available when we computed our estimate:

	Resistance Measurements (Ohms)	
#	Multimeter A ($\sigma = 20 \text{ Ohms}$)	Multimeter B ($\sigma = 2 \text{ Ohms}$)
1	1068	
2	988	
3		1002
4		996

'Batch Solution' $\hat{x}_{\mathrm{WLS}} = (\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H})^{-1}\mathbf{H}^T\mathbf{R}^{-1}\mathbf{y}$

Recursive Estimation

- What happens if we have a *stream* of data? Do we need to re-solve for our solution every time? Can we do something smarter?

$$\hat{x}_1 = (\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H})^{-1}\mathbf{H}^T\mathbf{R}^{-1}\mathbf{y}_1$$

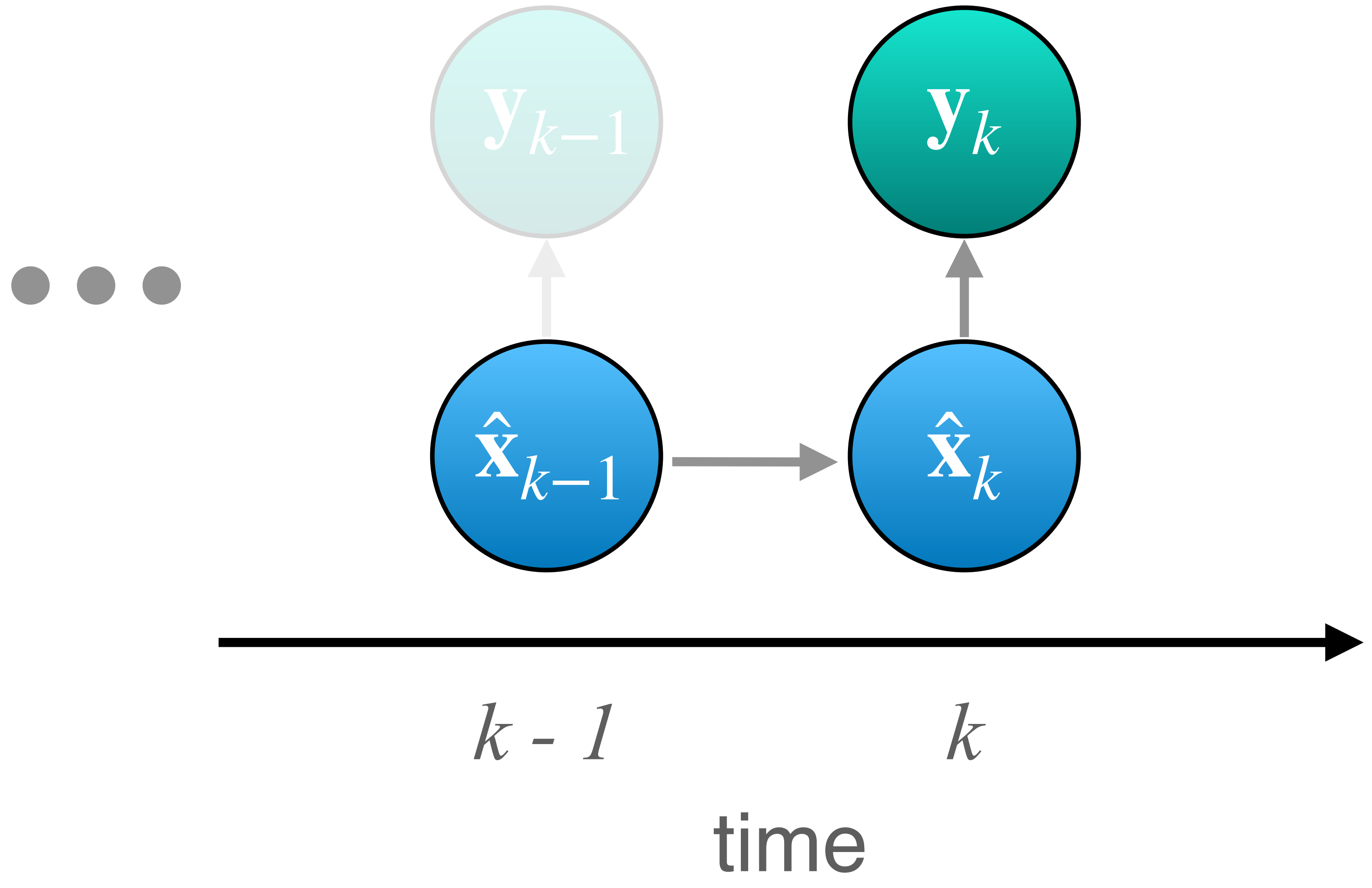
$$\hat{x}_2 = (\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H})^{-1}\mathbf{H}^T\mathbf{R}^{-1}\mathbf{y}_{1:2}$$

$$\vdots$$

	Resistance (Ohms)	
Time	Multimeter A	Multimeter B
t = 1 sec	1068	
t = 2 sec	988	
t = 3 sec		1002
t = 4 sec		996

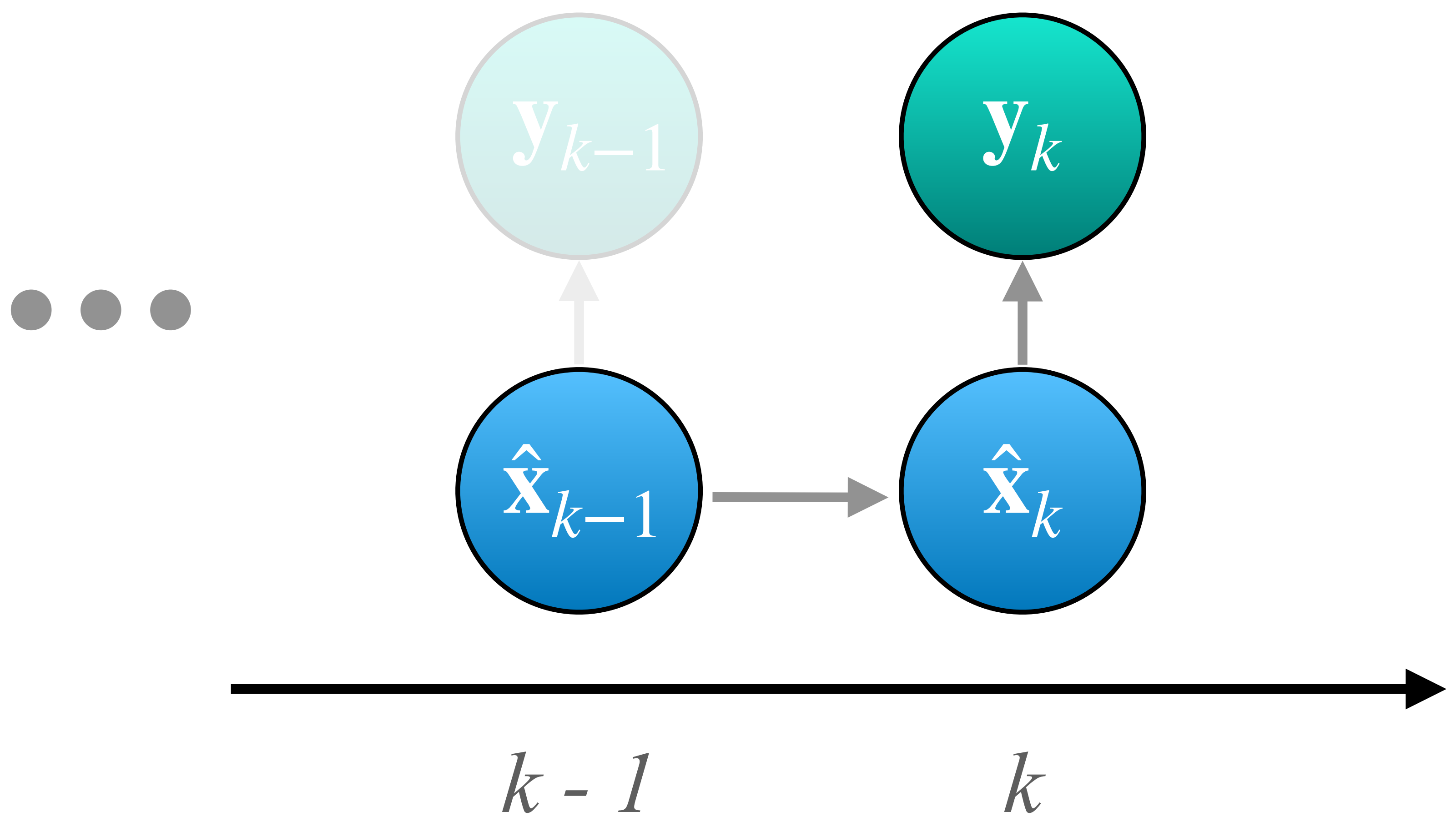
Linear Recursive Estimator

- We can use a *linear recursive estimator*
- Suppose we have an optimal estimate, $\hat{\mathbf{x}}_{k-1}$, of our unknown parameters at time interval $k-1$
- Then we obtain a new measurement at time k: $\mathbf{y}_k = \mathbf{H}_k\mathbf{x} + \mathbf{v}_k$



Goal: compute $\hat{\mathbf{x}}_k$ as a function of $\mathbf{y}_k$ and $\hat{\mathbf{x}}_{k-1}$!

Linear Recursive Estimator



- We can use a *linear recursive update:*

$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_{k-1} + \mathbf{K}_k \left(\mathbf{y}_k - \mathbf{H}_k \hat{\mathbf{x}}_{k-1}\right)$$

- We update our new state as a linear combination of the previous best guess and the current measurement *residual (or error)*, weighted by a gain matrix $\mathbf{K}_k$

Recursive Least Squares

- But what is the gain matrix $\mathbf{K}_k$?
- We can compute it by minimizing a similar least squares criterion, but this time we'll use a probabilistic formulation.
- We wish to minimize the **expected value of the sum of squared errors** of our current estimate at time step k:

$$\begin{aligned}\mathscr{L}_{\mathrm{RLS}} &= \mathbb{E}[(x_k - \hat{x}_k)^2] \\ &= \sigma_k^2\end{aligned}$$

- If we have n unknown parameters at time step k, we generalize this to

$$\begin{aligned}\mathscr{L}_{\mathrm{RLS}} &= \mathbb{E}[(x_{1k} - \hat{x}_{1k})^2 + \ldots + (x_{nk} - \hat{x}_{nk})^2] \\ &= \mathrm{Trace}(\mathbf{P}_k)\end{aligned}$$

Estimator ***covariance***

Recursive Least Squares

- Using our linear recursive formulation, we can express covariance as a function of $\mathbf{K}_k$

$$\mathbf{P}_k = (\mathbf{1} - \mathbf{K}_k\mathbf{H}_k)\mathbf{P}_{k-1}(\mathbf{1} - \mathbf{K}_k\mathbf{H}_k)^T + \mathbf{K}_k\mathbf{R}_k\mathbf{K}_k^T$$

- We can show (through matrix calculus) that this is minimized when

$$\mathbf{K}_k = \mathbf{P}_{k-1}\mathbf{H}_k^T(\mathbf{H}_k\mathbf{P}_{k-1}\mathbf{H}_k^T + \mathbf{R}_k)^{-1}$$

- With this expression, we can also simplify our expression for $\mathbf{P}_k$:

$$\begin{aligned}\mathbf{P}_k &= \mathbf{P}_{k-1} - \mathbf{K}_k\mathbf{H}_k\mathbf{P}_{k-1} \\ &= (\mathbf{1} - \mathbf{K}_k\mathbf{H}_k)\mathbf{P}_{k-1}\end{aligned}$$

Our covariance ‘shrinks’ with each measurement

Recursive Least Squares | Algorithm

1. Initialize the estimator

$$\hat{\mathbf{x}}_0 = \mathbb{E}[\mathbf{x}]$$

$$\mathbf{P}_0 = \mathbb{E}[(\mathbf{x} - \hat{\mathbf{x}}_0)(\mathbf{x} - \hat{\mathbf{x}}_0)^T]$$

2. Set up the measurement model, defining the Jacobian and the measurement covariance matrix:

$$\mathbf{y}_k = \mathbf{H}_k\mathbf{x} + \mathbf{v}_k$$

3. Update the estimate of $\hat{\mathbf{x}}_k$ and the covariance $\mathbf{P}_k$ using:

$$\mathbf{K}_k = \mathbf{P}_{k-1}\mathbf{H}_k^T(\mathbf{H}_k\mathbf{P}_{k-1}\mathbf{H}_k^T + \mathbf{R}_k)^{-1}$$

$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_{k-1} + \mathbf{K}_k\left(\mathbf{y}_k - \mathbf{H}_k\hat{\mathbf{x}}_{k-1}\right)$$

$$\mathbf{P}_k = (\mathbf{1} - \mathbf{K}_k\mathbf{H}_k)\mathbf{P}_{k-1}$$

Important! Our parameter covariance ‘shrinks’ with each measurement

Summary | Recursive Least Squares

- RLS produces a ‘running estimate’ of parameter(s) for *a stream of measurements*

- RLS is a linear recursive estimator that minimizes the (co)variance of the parameter(s) at the current time

Summary | The Method of Least

- Pioneered by Gauss to determine the orbit of the planetoid *Ceres*
- Least squares finds the parameters which minimize the *Least Squares Criterion*
- Ordinary least squares assumes that measurements are weighted equally, measurement model is linear