HomeWork 5 - Diego Jara

BLACK-SCHOLES BACKTESTING

Question 1:

To calculate the premium V_0 paid by the investor, we created three functions that replicate the cost for a vanilla call using the Black-Scholes formula, which is the fundamental equation in option pricing.

The first function, `getd1(S0, K, r, q, sigma, T)`, computes the intermediary variable $d1$, encapsulating the interplay of underlying spot price, strike price, domestic risk free interest rate, foreign risk free interest rate, volatility, and time to maturity. This function calculates the variable in accordance with the model:

$$d_1 = \frac{\ln \frac{S_0}{K} + (r - q)\,T}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}$$

The second function, `getd2(d1, sigma, T);`, computes the variable $d2$, building upon $d1$ by incorporating volatility like so:

$$d_2 = d_1 - \sigma\sqrt{T}$$

The third function, `V0 = BS(S0, K, r, q, T, d1, d2, epsilon)`, takes the now computed $d1$ and $d2$ plus the other inputs, and replicates the cost of the vanilla call by evaluating the cumulative probability of normal distributions for $d1$ and $d2$:

$$V_0 = \epsilon \cdot S_0 \cdot e^{-qT} \cdot N\left(\epsilon \cdot d_1\right) - \epsilon \cdot K \cdot e^{-rT} \cdot N\left(\epsilon \cdot d_2\right)$$

Since we are dealing with a call option, $\epsilon = 1$. Considering this, the function with the relevant inputs returns the premium value $V_0 = 26.9108$.

Question 2:

To generate `M = 1000` paths for the observed dollar using the given parameters, we first begin by creating a function called `getRebalancingFrequency(rebalancing_frequency)` that allows us to calculate the time step $\Delta t = dt$ given a specific measurement of time (Trading Days, Weeks or Months). This is very useful for Question 6, where we can evaluate for weekly and monthly rebalancing frequencies by simply changing the argument of the function. But for this question we use $N = 252$ Trading Days, so the time step is calculated as $\Delta t = T/N = 1/252 \approx 0.004$, with maturity $T = 1$ year.

After loading the relevant parameter, we estimate the value of the Chilean observed dollar stochastically using the given formula, for each of the `M = 1000` realizations. Also in the same loop we save the log of each realization's last value $\ln S_T$. We then plot the distribution of these values along with the theoretical distribution, resulting in the following figure:

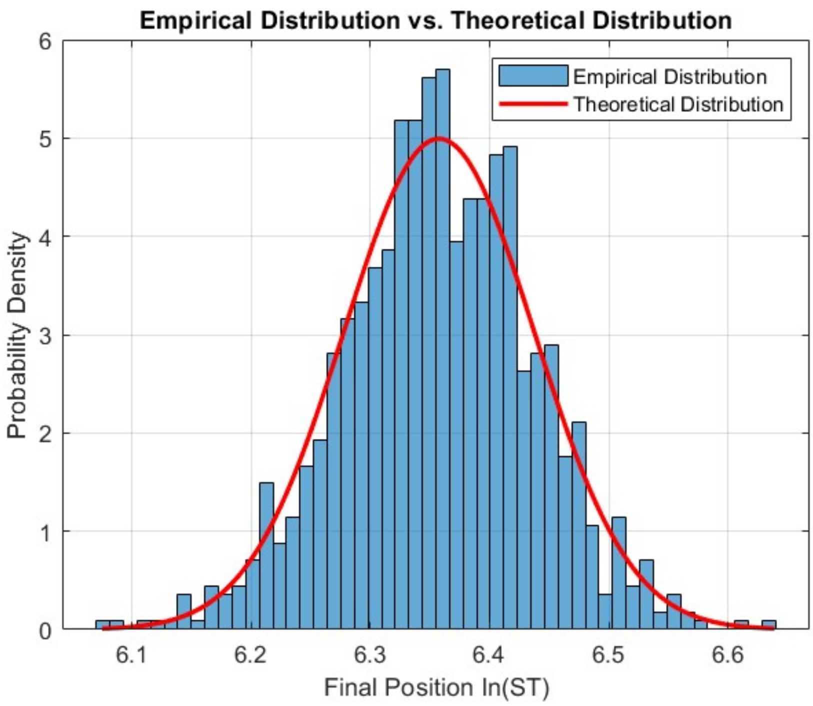

Figure 1: Distribution of last spot values

The plot's Gaussian resemblance emerges due to a blend of factors from the stochastic process and logarithmic transformation of final values. This reflects the Central Limit Theorem and financial modeling's norm. The process's accumulation of multiple random influences, combined with logarithms, aligns the distribution with Gaussian patterns, highlighting their applicability in capturing market uncertainty. The observed distribution underscores the geometric Brownian motion model's suitability for approximating market dynamics.

Question 3:

To compute the desired quantities for each rebalancing date of each of the `M = 1000` paths, we first need to address the time-dependent dynamics of the delta-hedging strategy. For this we create the function `getT_remaining(T, i, dt)`, which computes the remaining time to maturity (T) of the option at each rebalancing step. This is done by subtracting the amount of time steps elapsed based on the index `i` ($i \cdot \Delta t$) from the total maturity time of $T = 1$ year. This dynamic temporal assessment ensures that the evolving portfolio adjustments align accurately with the progressing option lifespan.

From here, we begin first by calculating the classic option greek Δ_i. This metric indicates the quantity of Chilean observed dollar required to hedge the short call position, and is computed with the function `getBSdelta(S_i, K, r, q, sigma, T, epsilon)`. This function draws spot price values from the previously calculated matrix of spot price paths `S_paths` and takes for `T` the remaining time to maturity `T_remaining`, calculating Δ_i for each index i of each path $m \in M$ through the Black-Scholes equation:

$$\Delta_{i,m} = \epsilon \cdot e^{-qT\_remaining} \cdot N\left(\epsilon \cdot d_1\right)$$

After this, we move on to calculate the second component of the hedging portfolio, the domestic MMA for each index i of each path $m \in M$, $B_{i,m}$. For this we input the hedging portfolio, Δ_i, and spot price paths matrices onto the given formula, with $H_0 = V_0 = 26.9108$:

$$B_{i,m} = H_{i,m} - \Delta_{i,m} \cdot S_{i,m} CLP$$

With these quantities calculated, we can now compute the value of the hedging portfolio for each (coming) rebalancing date of frequency $\Delta_t = dt$, for each realization of `M`:

$$H_{i+1,m} = \Delta_{i,m} \cdot e^{q\Delta_t} S_{i+1,m} + B_{i,m} e^{r\Delta_t}$$

Next we calculate the value of the option for each index i of each path $m \in M$, $V_{i,m}$, applying the same functions used in Question 1, but now taking into consideration the remaining time to maturity `T_remaining` and the previously calculated V_0:

$$V_{i,m} = \epsilon \cdot S_{i,m} \cdot e^{-qT\_remaining} \cdot N\left(\epsilon \cdot d_1\right) - \epsilon \cdot K \cdot e^{-rT\_remaining} \cdot N\left(\epsilon \cdot d_2\right)$$

Finally, we compute the P&L for each index i of each path $m \in M$, $Y_{i,m}$, as the difference of the variation in the hedging portfolio and the option's value:

$$Y_{i,m} = \left(H_{i+1,m} - H_{i,m}\right) - \left(V_{i+1,m} - V_{i,m}\right)$$

Since we know that the Black-Scholes model operates under the risk-neutral measure Q, none of the quantities computed using this formula depend massively on the physical drift μ. However, among the quantities calculated, $B_{i,m}$ and, consequently, $H_{i,m}$ are the most dependent on the physical drift. This is because the spot price $S_{i,m}$, which is used in the calculation of $B_{i,m}$, is modeled based on the physical drift μ. Therefore, any change in the value of μ will directly impact the spot price, leading to subsequent changes in the calculation of $B_{i,m}$ and ultimately influencing the dynamics of the hedging portfolio $H_{i,m}$.

Question 4:

To answer this question we begin first by calculating the net value for each index i of each path $m \in M$, $X_{i,m}$, as the difference between the hedging portfolio and the option value:

$$X_{i,m} = H_{i,m} - V_{i,m}$$

We then proceed to plot the mean and standard deviation for both the P&L $Y_{i,m}$ and the net value $X_{i,m}$ for each of the rebalancing date, resulting in the following figure:

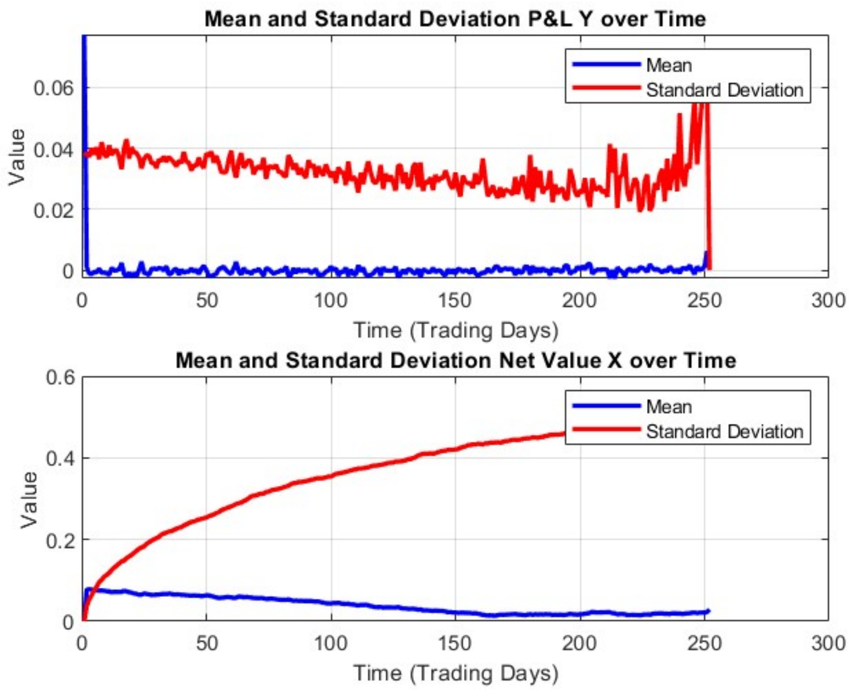

Figure 2: Daily rebalancing moments

The plotted results depict the effectiveness of the delta-hedging strategy for the trader's portfolio. The convergence of the P&L's mean towards zero signifies successful risk mitigation through delta adjustments. The decreasing standard deviation of P&L supports reduced portfolio volatility, while the subsequent increase in volatility towards maturity aligns with options' sensitivity to market shifts ($\Delta_i = 1$). In terms of net value $X_{i,m}$, the decreasing mean and eventual stabilization near zero emphasize the strategy's goal of maintaining a balanced portfolio while the concave-down, increasing trajectory of the standard deviation curve reflects ongoing risk reduction efforts. The increasing standard deviation towards maturity acknowledges the challenges of maintaining a stable net value in dynamic market conditions. These outcomes affirm delta hedging's risk management principles and its adaptability in response to evolving market dynamics.

Question 5

The plot of $\Delta_{i,m}$ at the 125^{th} rebalancing day and at maturity results as follows:

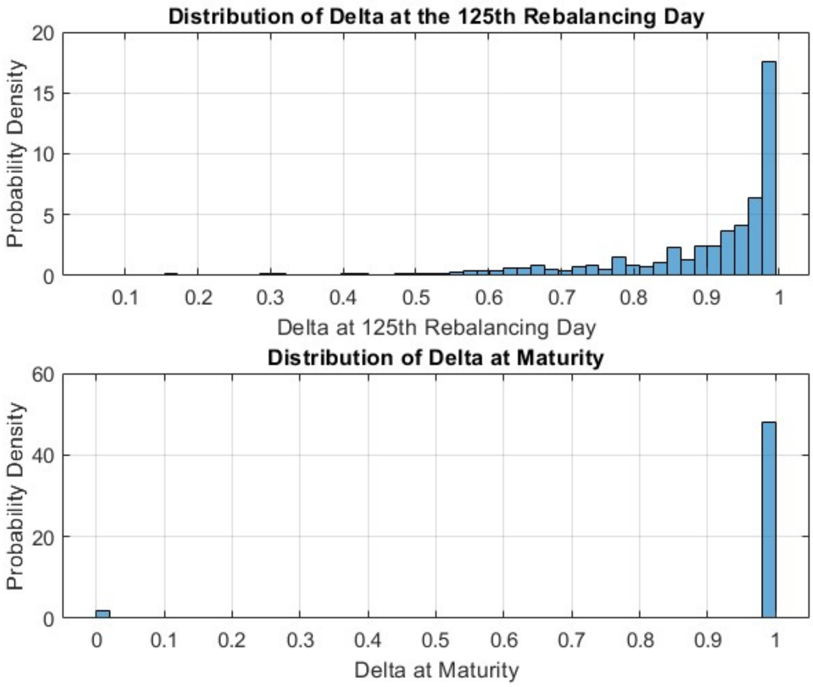

Figure 3: Distributions of $\Delta_{i,m}$

The histogram of $\Delta_{i,m}$ at the 125th rebalancing day illustrates a significant concentration of cases around values close to 1, demonstrating a pronounced increase in delta as the option approaches maturity. While a handful of instances appear below 0.5, the exponential rise in probabilities between 0.5 and 1 aligns with the option's tendency to become more sensitive to market movements as expiration nears. Given the minimal difference between the spot price $S_0 = 499.75$ and the strike price $K = 500$, the probability of exercising the option is high. However, being a European option, $\Delta_{i,m}$ may be below 1 before maturity (unlike American options).

At maturity, the histogram shows a clear dichotomy with most deltas clustering at 1, indicating deep in-the-money positions. Sparse presence of deltas near 0 signifies few out-of-the-money positions. This pattern reaffirms the option's behavior of closely tracking the underlying asset ($\Delta_{i,m} \approx 1$) or having minimal impact ($\Delta_{i,m} \approx 0$) as it reaches maturity.

These distributions align with the expected evolution of delta in relation to the option's lifecycle, with high deltas at maturity showcasing the strategy's adaptation to asset dynamics. The correlation between histogram patterns and anticipated delta behavior underscores the effectiveness of the delta-hedging strategy in managing risk and aligning with options pricing and market dynamics principles.

Question 6

As mention in Question 1, using the `getRebalancingFrequency(rebalancing_frequency)` function, we can change the calculation of the time step $\Delta_t = dt$ by inputting the desired rebalancing frequency. In this case, we changed it to 'weekly' ($N = 48$) and 'monthly' ($N = 12$) resulting in the following plots:

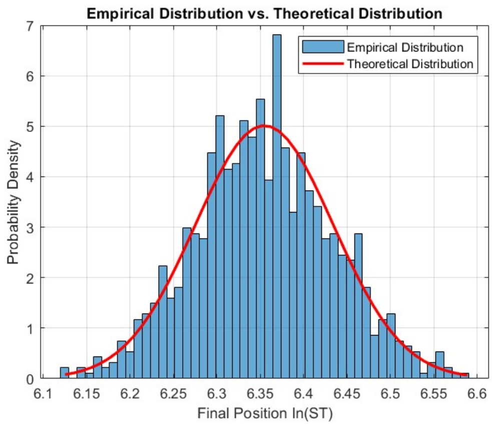

Figure 4: Weekly rebalancing distribution

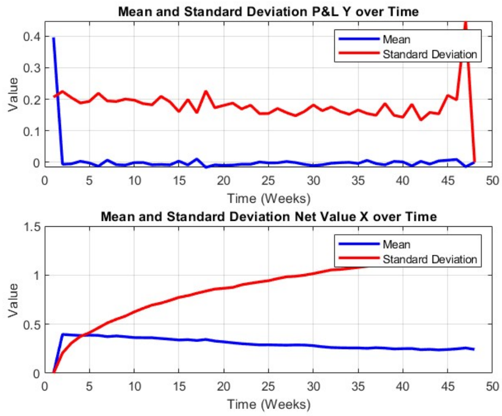

Figure 5: Weekly rebalancing moments

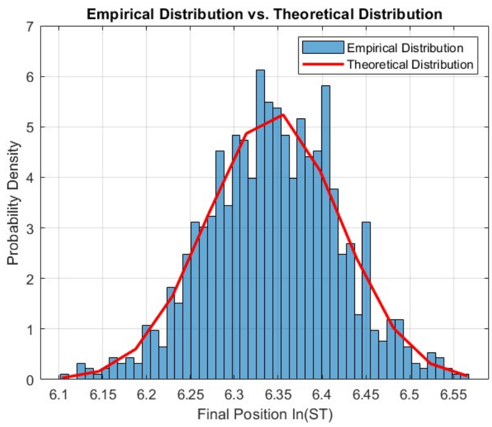

Figure 6: Monthly rebalancing distribution

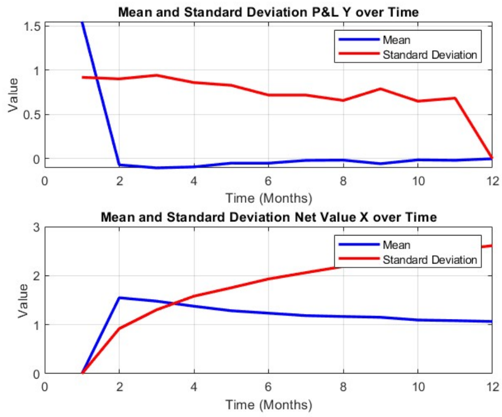

Figure 7: Monthly rebalancing moments

The observed patterns in these plots offer valuable insights into the impact of rebalancing frequency on P&L $Y_{i,m}$ and net value $X_{i,m}$. Transitioning from daily to weekly or monthly rebalancing leads to wider log last position distributions ($\ln S_T$), indicating greater potential for substantial price fluctuations over longer periods. This relationship between frequency and volatility aligns with the notion that less frequent rebalancing allows for larger market movements.

The mean and standard deviation plots for P&L $Y_{i,m}$ show that as rebalancing frequency decreases, mean values and standard deviations rise. This relationship underscores how infrequent rebalancing leads to heightened price swings, contributing to larger potential gains or losses. The same holds true for net value $X_{i,m}$, where both mean and standard deviation increase with decreasing rebalancing frequency.

Question 7

To analyze the effects of negative drift $\mu = -0.15$ and no drift $\mu = 0$ we simply change the value of the μ parameter set for Question 2. The results are as follows:

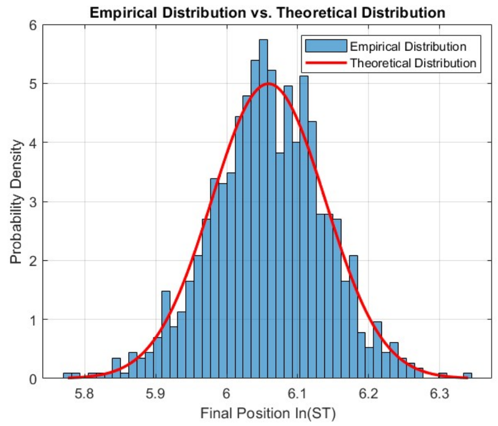

Figure 8: Distribution for $\mu = -0.15$

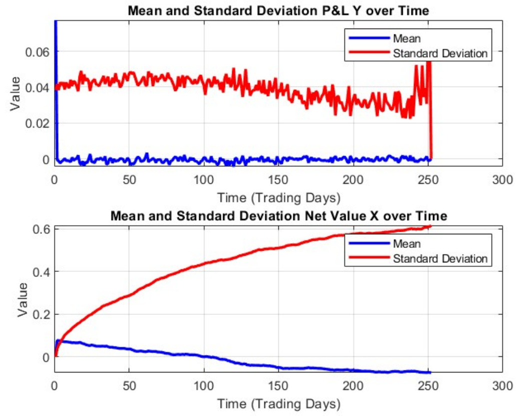

Figure 9: Moments for $\mu = -0.15$

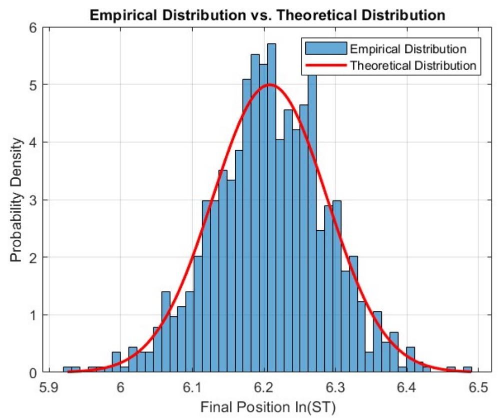

Figure 10: Distribution for $\mu = 0$

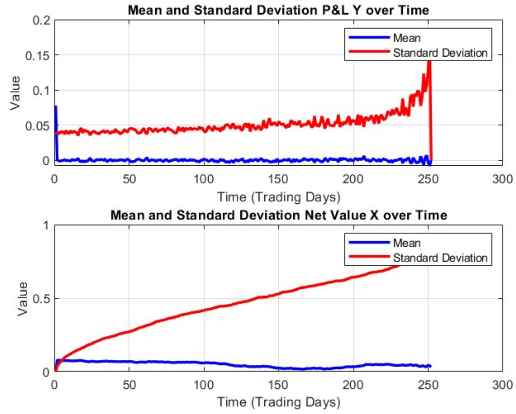

Figure 11: Moments for $\mu = 0$

We note that the distribution of the log of each realization's last value $\ln S_T$ with $\mu = -0.15$ is skewed towards lower values, indicating a higher likelihood of the observed dollar depreciating over time. Conversely, with $\mu = 0$, the distribution shows a more symmetric pattern around the initial value, reflecting a balance between upward and downward movements.

When considering a negative physical drift of $\mu = -0.15$, the mean of P&L $Y_{i,m}$ retains its proximity to zero, but exhibits a slightly increased presence of negative values, reflecting the strategy's effort to counter downward movements. The standard deviation, while following a similar trend to that of $\mu = 0.15$, maintains higher levels for longer periods before diminishing and then surging toward maturity. This behavior underscores the strategy's sensitivity to market downturns, leading to heightened risk exposure.

For net value $X_{i,m}$ with $\mu = -0.15$, the mean strives to approach zero but occasionally dips below it, indicating the strategy's challenge in fully offsetting losses. Meanwhile, the standard deviation showcases a growth trajectory similar to that of $\mu = 0.15$, albeit with more pronounced expansion, reflecting the amplified volatility arising from negative drift.

Conversely, with $\mu = 0$, the mean of P&L $Y_{i,m}$ remains in proximity to zero, albeit with a reduction in noise compared to $\mu = 0.15$. The standard deviation maintains relatively consistent behavior, steadily increasing toward maturity, highlighting the inherent risk and fluctuations associated with the strategy's dynamics.

In terms of net value $X_{i,m}$ under $\mu = 0$, the mean effectively converges to zero, indicative of a strategy adept at minimizing net exposure. However, the standard deviation experiences linear growth below unity, underlining the gradual accrual of risk.

As we mentioned in Question 3, we know that in the Black-Scholes model the physical measure is replaced by an artificial equivalent martingale measure or risk-neutral measure Q. This means that μ has no effect on the option pricing and that is shown here, where in both cases the option value remains at $V_0 = 26.9108$.

Question 8

To analyze the effects of increased volatility, we simply change the value of the σ parameter set for Question 1. In this case, by overestimating the volatility at $\sigma = 0.10$ the call option becomes more expensive due to the increased uncertainty and potential for larger price movements, with the Black-Scholes calculation being at $V_0 = 30.4184$.

Additionally, the plot of the moments for P&L $Y_{i,m}$ and net value $X_{i,m}$ results as follows:

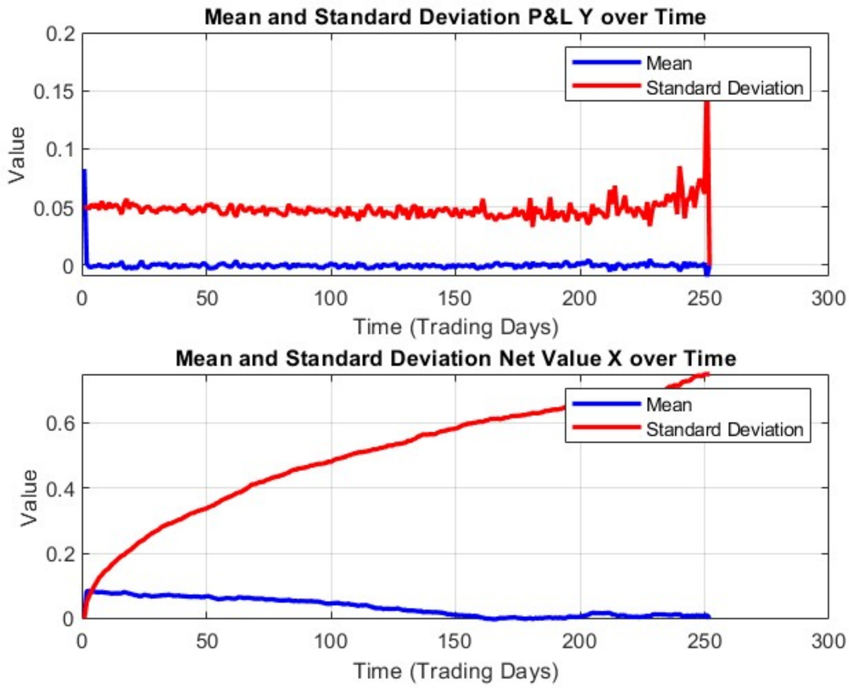

Figure 12: Moments for $\sigma = 0.10$

We note that, despite the similarity in the means of both P&L $Y_{i,m}$ and net value $X_{i,m}$ to those obtained with $\sigma = 0.08$, the standard deviation, particularly for P&L $Y_{i,m}$, exhibits a notable increase. This heightened standard deviation signifies a greater degree of uncertainty and a wider potential range for gains and losses in the portfolio.

Question 9

To analyze the effects of decreased volatility, we simply change the value of the σ parameter set for Question 1. In this case, by underestimating the volatility at $\sigma = 0.06$ the call option becomes more cheap due to reduced uncertainty and potential for smaller price movements, with the Black-Scholes calculation being at $V_0 = 23.6217$.

Additionally, the plot of the moments for P&L $Y_{i,m}$ and net value $X_{i,m}$ results as follows:

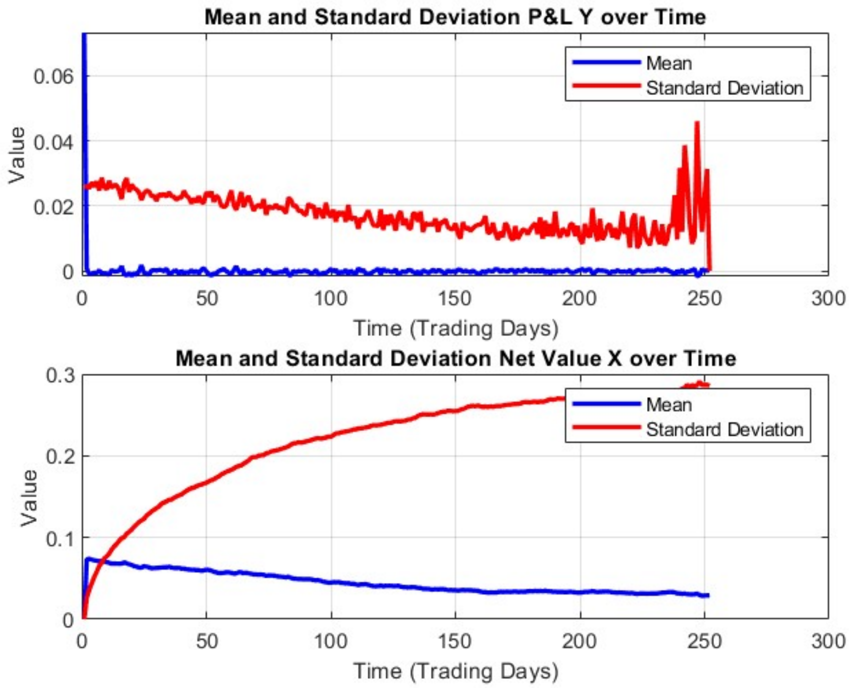

Figure 13: Moments for $\sigma = 0.06$

We note that, similar to the outcomes observed with $\sigma = 0.08$, the means of both P&L $Y_{i,m}$ and net value $X_{i,m}$ remain relatively stable. Notably, the mean of net value $X_{i,m}$ consistently hovers above zero, indicating a more persistent positive bias. Meanwhile, the standard deviation for both quantities experiences a substantial reduction, reflecting decreased levels of volatility and overall portfolio risk.

Question 10

For this question, we use the (wrong) volatility $\sigma = 0.10$ for the pricing of the option, and the simulation of the observed dollar spot price paths. Then, we proceed to calculate the rest of the needed quantities with the (correct) estimate of $\sigma = 0.08$. This results in an overpriced call option of premium $V_0 = 30.4184$ and a plot of the moments for P&L $Y_{i,m}$ and net value $X_{i,m}$ as follows:

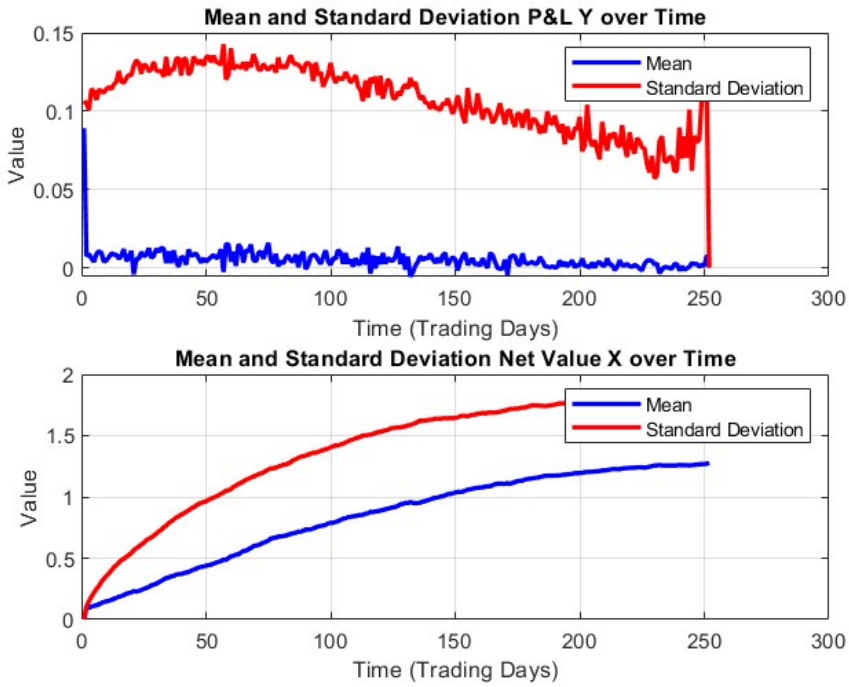

Figure 14: Moments with overpriced option

We note that, in the context of a correct 0.08 volatility estimate versus the market's 0.10 pricing volatility, the P&L $Y_{i,m}$ mean remains steady, while higher volatility emerges due to mismatched pricing. Net value $X_{i,m}$ mean rises progressively, reflecting ongoing gains from pricing mismatch. Both P&L $Y_{i,m}$ and net value $X_{i,m}$ show amplified fluctuations, showcasing the influence of precise risk management within pricing disparities.