BLACK-SCHOLES BACKTESTING

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This tutorial verifies that the delta-hedging strategy manages to replicate the payoff of a European vanilla call on some currency like the Chilean observed dollar.

A trader sold a European vanilla call on the observed dollar with maturity T=1 year and strike K=500. He is short the option and long the hedging portfolio. He initially receives the premium from the client and reinvests the proceeds in a domestic MMA with interest rate ρ and a foreign MMA with interest rate q. The trader rebalances his position at a frequency Δt.

This problem uses the canonic notations and a volatility $\sigma = 0.08$.

Market:

Underlying spot: $S_0 = 499.75$; Domestic risk free interest rate: $r = 0.05$; Foreign risk free interest rate: $q = 0.01$.

European Vanilla Call:

Underlying S

Strike: K=500

Maturity: T = 1year.

NB: 1 year = 252 trading days = 12 months = 48 weeks

The trader hedges his position dynamically as described below:

1. At the rebalancing date t_i, the value of the option is V_i while the value of the hedging portfolio is H_i (initially, $H_0 = V_0$).
2. The trader's portfolio net value X_i reads $X_i = H_i - V_i$ (initially, $X_0 = 0$).
3. He computes the delta Δi of the option and goes long (buys) Δ_i USD which he invests in a foreign MMA and invests (or borrows) $B_i = H_i - \Delta_i \cdot S_i CLP$ in a domestic MMA.
4. The trader rebalances his position at a frequency Δt. At the next rebalancing date $t_{i+1} = t_i + \Delta t$, the value of the hedge is: $H_{i+1} = \Delta_i \cdot e^{q\Delta t} S_{i+1} + B_i e^{r\Delta t}$.

The value of the observed dollar is stochastic. Under the Black-Scholes model, one can update the value of the observed dollar index as:

$$S_{i+1} = S_i e^{(\mu - \sigma^2/2)\Delta t + \sigma\sqrt{\Delta t} z_i}$$

The random variable z_i is a standard Gaussian variable.

Question 1:

Using the Black-Scholes formula, what is the premium V_0 paid by the investor?

Question 2:

For a rebalancing frequency Δt of one trading day (= 1/252 years), generate M=1000 paths for the observed dollar using a physical drift $\mu = 0.15$ until maturity T (one year = 252 trading days).

Plot the theoretical and empirical (with Q = 50 buckets) distribution the distribution of $\ln S_T$ on the same graph.

Question 3:

For each rebalancing date of each path, compute:

a. The composition of the hedge portfolio (the quantities Δi and B_i),

b. The value of the hedging portfolio H at the beginning and end of the period,

c. The value of the option V_i, and

d. The $P\&L$ $Y_i = (H_{i+1} - H_i) - (V_{i+1} - V_i)$.

Justify which of these quantities a priori massively depend on the physical drift μ.

Question 4:

Plot the mean and standard deviation for the $P\&L$ Y and the net position X for each rebalancing date as a function of time. Comment the results.

Question 5:

Plot the distribution of delta at the 125^{th} rebalancing day and at maturity and comment the results.

Question 6:

Redo questions 2, 3, and 4 for a rebalancing frequency of 1 week and 1 month. Comment the results.

Question 7:

Redo questions 2, 3, and 4 for a physical drift $\mu = -0.15$ and $\mu = 0$. Does the physical drift appear relevant for the replication of the option payoff?

Question 8:

The true volatility was $\sigma = 0.08$ and the true physical drift was $\mu = 0.15$. The market (including the trader) misestimated the volatility and took a value of 0.10 for the valuation and hedging. A trader rebalances his position every business day. What happens to his $P\&L$ and position?

Question 9:

Redo question 8 for a mistaken market volatility of 0.06. Comment the results.

Question 10:

The trader managed to rightly estimate the volatility of 0.08. However, the market keeps on using a volatility of 0.10. The trader must use the (wrong) market volatility to price the option (he thus charged his client more than what he thought necessary) and his (correct) estimate of the volatility for the hedge. What happens to his $P\&L$ and position? Comment the results.