HomeWork 4 - Diego Jara

LÉVY PROCESSES SIMULATIONS

Question 1:

To generate a vector `T`, we use the command `linspace` with the arguments 0, 1 and N, to create N = 53 uniformly spaced points ranging from 0 to 1. Each point in the vector corresponds to a week, representing time expressed in years.

Question 2:

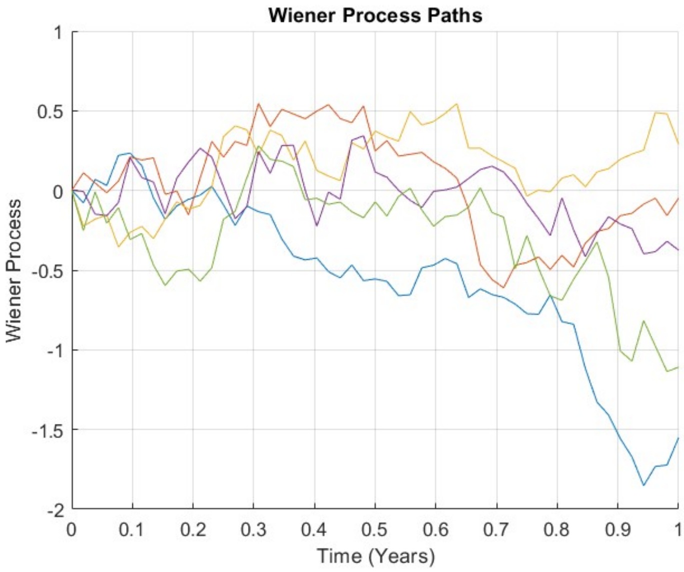
To model a Wiener process, we build a function called `wiener` that takes a vector `T` as an argument and returns a Wiener process `W1` over time. The Wiener process is a fundamental mathematical concept used to describe a continuous-time stochastic process with random increments.

The key idea behind the Wiener process is to generate random increments, denoted as dW_t. The resulting figure shows M paths of the Wiener process that follow a Gaussian distribution with a mean of zero and a variance proportional to the time step ΔT_n. In other words, the change in the Wiener process at each time step is a random variable drawn from a Gaussian distribution with mean zero and standard deviation proportional to the square root of the time step. This randomness in increments gives rise to the characteristic "random walk" behavior of the Wiener process.

The function `wiener(T)` generates the Wiener process by first computing the time step `dt` $= \Delta T_n$ as the differences between consecutive points in the vector `T`. It then generates a vector of Gaussian random numbers `dW` with zero mean and unit variance using `randn` and scales each element by `sqrt(dt)` to obtain the increments ΔW_n.

Next, the function uses the cumulative sum (`cumsum`) of the increments to compute the Wiener process `W1`. Starting from an initial value of zero, each subsequent value is obtained by adding the corresponding increment to the previous value. This process continues for all time steps, generating a continuous path of the Wiener process.

To visualize the Wiener process, we plot M = 5 paths on the same figure, resulting in the following plot:



The resulting figure shows M paths of the Wiener process, where each path represents a realization of the stochastic process, and due to the random nature of the increments, we observe different trajectories for each path. Despite the variations in individual paths, they share common statistical properties, such as zero mean and time-dependent volatility, making the Wiener process a versatile and widely used stochastic model.

In finance, the Wiener process has significant implications, particularly in option pricing models such as the Black-Scholes model. The model assumes that the underlying asset's price follows a Wiener process, representing the asset's random fluctuations over time. The unpredictability and continuous-time nature of the Wiener process make it an essential tool for modeling various financial instruments and risk management scenarios.

Question 3:

In this question, we define the Brownian process B with a drift $\mu = 0.1$ and volatility $\sigma = 0.3$ as follows:

$$dB_t = \mu \cdot dt + \sigma \cdot dW_t$$

$$B_0 = 0$$

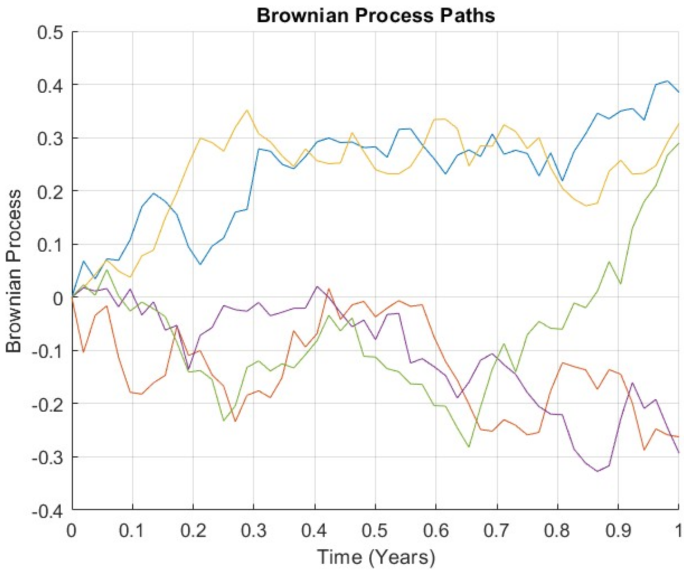
To generate a path of the Brownian process, we build a function called `brownian` that takes a vector `T`, drift μ, and volatility σ as arguments. The Brownian process is a stochastic process that models the continuous random motion of a random variable, with the drift component representing the expected average rate of change and the volatility capturing the dispersion of the random fluctuations.

The key idea behind the Brownian process is to generate random increments dW_t, which follow a Gaussian distribution with zero mean and variance proportional to the time step dt $= \Delta T_n$. This randomness in increments gives rise to the characteristic "random walk" behavior, where the process takes continuous, unpredictable paths over time.

The function `brownian(T, mu, sigma)` computes the time step dt as the differences between consecutive points in the vector `T`. It then generates a vector of Gaussian random numbers `dW` with zero mean and unit variance using `randn` and scales each element by $\sqrt{dt}$ to obtain the increments dW_t. Next, the function computes the differential component dB_t of the Brownian process using the drift μ and volatility σ.

Using the cumulative sum (`cumsum`) of the increments, the function computes the Brownian process `B`. Starting from an initial value $B_0 = 0$, each subsequent value is obtained by adding the corresponding increment to the previous value, resulting in a continuous path of the Brownian process.

To visualize the Brownian process, we plot M = 5 paths on the same figure, resulting in the following plot:



The resulting figure shows M paths of the Brownian process, where each path represents a realization of the stochastic process. Due to the random nature of the increments, we observe different trajectories for each path. However, all paths exhibit the characteristic features of the Brownian motion, such as continuous and unpredictable movements.

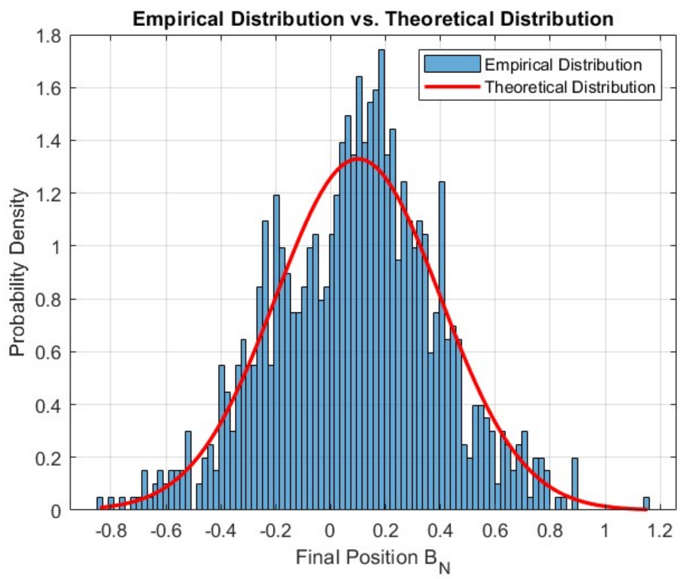
In finance, just like the Wiener process, the Brownian process plays a crucial role in option pricing models like the Black-Scholes model. It assumes that the underlying asset's price follows a Brownian motion, which represents the asset's random fluctuations over time. The volatility component in the Brownian process is of particular significance in pricing options and understanding market risk, as it captures the asset's uncertainty and variability.

Question 4:

In this question, we explore the statistical properties of the previously defined Brownian process by generating `M = 1000` paths and analyzing the empirical distribution of the last position B_N. Additionally, we compare the empirical distribution with the theoretical distribution derived from the properties of the Brownian motion.

To generate the paths, we employ a loop that iteratively computes M realizations of the Brownian process using the `brownian` function. In each iteration, we store the final position B_N in the vector `BN`. The resulting vector `BN` contains the final positions of the Brownian process from all the paths.

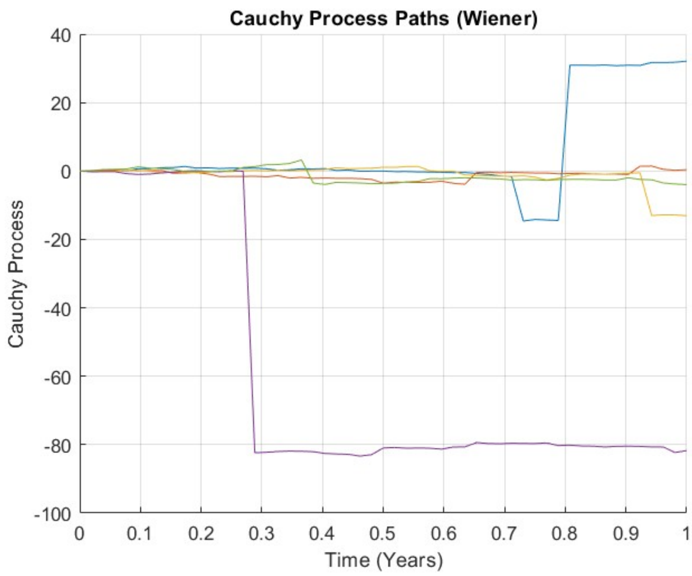
We use the `histogram` function to plot the empirical distribution of the last position B_N, with 100 bins and normalized as the probability density function (PDF). This provides insights into the Brownian process's overall behavior across multiple realizations. To compare with the theoretical distribution, we generate the theoretical PDF using `linspace` and `normpdf`, displaying it as a red curve. The final plot results as follows:

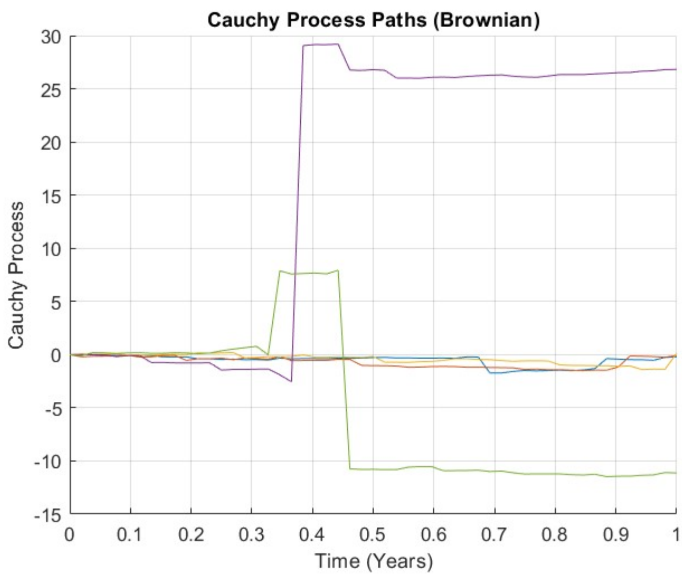


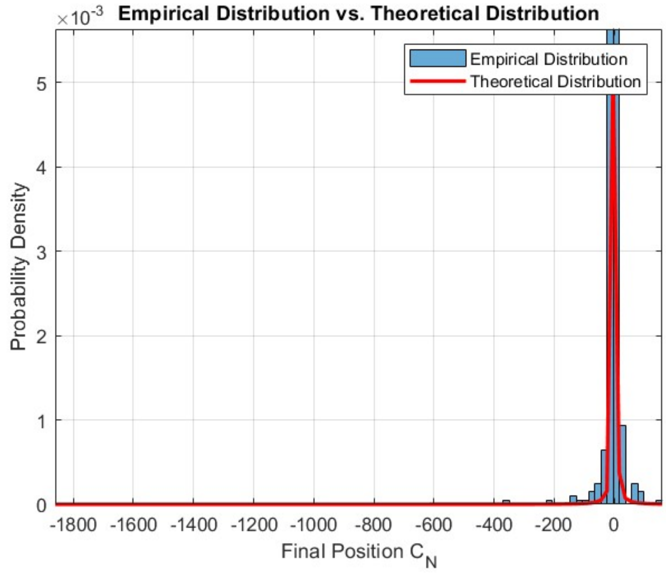
We note that the last values of the Brownian process distribute approximately normally due to the Central Limit Theorem. As the process accumulates independent Gaussian random increments over time, their sum tends to follow a normal distribution. This property is fundamental in finance, especially in option pricing models like Black-Scholes, where it enables efficient calculations and risk assessments.

Question 6:

To study the behavior of the Cauchy process, we create two new functions that generate paths using variables drawn from a Cauchy distribution, similar to the `wiener` and `brownian` functions. The new functions are called `cauchy2` and `cauchy3` for redoing questions 2 and 3, respectively. The plots are as follows:







In the first plot, we observe the paths of the Wiener process generated using variables drawn from the Cauchy distribution. Despite still displaying a "random walk" behavior, the paths show more extreme and erratic movements compared to the Gaussian-distributed Wiener process. Similarly, the second plot shows the paths of the Brownian process with Cauchy-distributed variables, also exhibiting more erratic behavior.

The third plot illustrates the empirical distribution of the last positions (C_N) of the Cauchy process. Unlike the Brownian process, which follows a normal distribution, the last positions of the Cauchy process align with the Cauchy distribution, displaying extreme variance in the sample.

This behavior can be explained by the properties of the Cauchy distribution, which has heavy tails and lacks finite moments (mean and variance). As a result, the Cauchy process exhibits more extreme and unpredictable movements, making it a distinctive example of a Lévy process with unique characteristics compared to the Brownian process.