5. Toss a fair coin until two heads appear. What is the probability that exactly k tosses are needed?

Solution. There are exaclty $k-1$ (H,T)-sequences with (i) length k, (ii) exactly two heads, and (iii) heads as its final entry. Indeed, there is one for each possible position of the non-final head. Since the number of arrangements after k tosses is 2^k and each arrangement is equally likely, the probability that exactly k tosses are needed is $\frac{k-1}{2^k}$.

6. Let $\Omega = \mathbb{N}$. Prove that there does not exist a uniform distribution on Ω.

Solution. Suppose $F \subset \mathbb{N}$ is finite. If $P(F) > 0$, then

$$P\left(\bigcup_{n=0}^{\infty}(F + n \max F)\right) = \sum_{n=0}^{\infty} P(F) = \infty.$$

Therefore, every finite set has probability 0. In particular,

$$P(\mathbb{N}) = P\left(\bigcup_{n=0}^{\infty}\{n\}\right) = \sum_{n=0}^{\infty} P(\{n\}) = 0.$$

8. Suppose that $P(A_i) = 1$ for each i. Prove that

$$P\left(\bigcap_{i=1}^{\infty} A_i\right) = 1.$$

Solution. First notice that

$$P(A_1 \cap A_2) = P(A_1) + P(A_2) - P(A_1 \cup A_2) = 1 + 1 - 1 = 1.$$

Then by induction, $P\left(\bigcap_{i=1}^{n} A_i\right) = 1$ for all n. Let $B_n = \bigcap_{i=1}^{n} A_i$. Then (B_n) is monotone decreasing and $B_n \to \bigcap_{i=1}^{\infty} A_i$. Therefore, by continuity,

$$P\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} P(B_n) = 1.$$

10. The Monty Hall problem.

Solution. As suggested, consider

$$\Omega = \{(\omega_1, \omega_2) : \omega_i = 1, 2, 3\}$$

where ω_1 is the prize and ω_2 is the door Monty opens. Suppose we initially choose door 1 and Monty chooses door 2. Calculating directly,

$$P(\omega_1 = 3 \mid \omega_2 = 2) = \frac{P(\omega_2 = 2 \mid \omega_1 = 3)P(\omega_1 = 3)}{P(\omega_2 = 2)} = \frac{1 \cdot 1/3}{1/2} = 2/3$$

where $P(\omega_2 = 2 \mid \omega_1 = 3) = 1$ since Monty always chooses an incorrect door.

11. There are three cards. The first is green on both sides, the second is red on both sides and the third is green on one side and red on the other. We choose a card at random and we see one side (also chosen at random). If the side we see is green, what is the probability that the other side is also green?

Solution. Let $X =$ the color that we are shown and $Y =$ the color of the other side. Calculating directly,

$$P(Y = G \mid X = G) = \frac{P(X = G \text{ and } Y = G)}{P(X = G)} = \frac{1/3}{3/6} = 2/3$$

since $P(X = G) = \frac{\#\text{ of green sides}}{\#\text{ of sides}}$.

13. Suppose that a fair coin is tossed repeatedly until both a head and tail have appeared at least once. What is the probability that three tosses will be required?

Solution. Exactly[1] three tosses will be required only in two cases: HHT and TTH. Since there are $2^3 = 8$ (H,T)-sequences of length 3 (and each of them is equally likely), then the probability that (exactly) three cases will be required is $2/8 = 1/4$.

15. The probability that a child has blue eyes is $1/4$. Assume independence between children. Consider a family with 3 children.

(a) If it is known that at least one child has blue eyes, what is the probability that at least two children have blue eyes?

(b) If it is known that the youngest child has blue eyes, what is the probability that at least two children have blue eyes?

Solution. Let $X = \#$ of children with blue eyes. Let $X_i = 1$ if the i-th child has blue eyes and $X_i = 0$ otherwise.

(a) Calculating directly,

$$\begin{aligned} P(X \geq 2 \mid X \geq 1) &= \frac{P(X \geq 2)}{P(X \geq 1)} = \frac{1 - P(X < 2)}{1 - P(X < 1)} \\ &= \frac{1 - \big((3/4)^3 + 3(1/4)(3/4)^2\big)}{1 - (3/4)^3}. \end{aligned}$$

since, $P(X < 1) = P(X = 0)$, $P(X < 2) = P(X = 0) + P(X = 1)$, and

$$P(X = 1) = \sum_{i=1}^{3} P(\text{only } i\text{-th child has blue eyes}).$$

[1]Contrary to my initial understanding, the phrasing of the problem does not ask the probability that at least three tosses will be required.

(b) Calculating directly,

$$P(X \geq 2 \mid X_3 = 1) = \frac{P(X \geq 2 \text{ and } X_3 = 1)}{P(X_3 = 1)} =$$

$$\frac{P(X_1 + X_2 \geq 1 \text{ and } X_3 = 1)}{P(X_3 = 1)} = P(X_1 + X_2 \geq 1) =$$

$$1 - P(X_1 + X_2 = 0) = 1 - (3/4)^2 = 7/16.$$

Alternatively,

$$P(X_1 + X_2 \geq 1) = P(X_1 = 1 \text{ or } X_2 = 1) =$$

$$P(X_1 = 1) + P(X_2 = 1) - P(X_1 = 1 \text{ and } X_2 = 1) =$$

$$1/4 + 1/4 - (1/4)^2 = 7/16.$$

18. Let $(A_i)_{i=1}^k$ be a partition of Ω and let $B \subset \Omega$ be such that $P(B) > 0$. Prove that if $P(A_1 \mid B) > P(A_1)$, then $P(A_i \mid B) < P(A_i)$ for some $i = 2, \ldots, k$.
Solution. Suppose otherwise. Then

$$\sum_{i=1}^{k} P(A_i \mid B) < \sum_{i=1}^{k} P(A_i) = 1.$$

A contradiction since

$$\sum_{i=1}^{k} P(A_i \mid B) = \sum_{i=1}^{k} \frac{P(A_i \cap B)}{P(B)} = \frac{1}{P(B)} \sum_{i=1}^{k} P(A_i \cap B) = \frac{1}{P(B)} P(B) = 1.$$

19. Suppose that 30 percent of computer owners use a Macintosh, 50 percent use Windows, and 20 percent use Linux. Suppose that 65 percent of the Mac users have succumbed to a computer virus, 82 percent of the Windows users get the virus, and 50 percent of the Linux users get the virus. We select a person at random and learn that her system was infected with the virus. What is the probability that she is a Windows user?
Solution. By Bayes,

$$P(U = W \mid V = 1) = \frac{(.82)(.50)}{(.63)(.30) + (.82)(.50) + (.50)(.20)}.$$

20. A box contains 5 coins and each has a different probability of showing heads:

$$p_1 = 0,\ p_2 = 1/4,\ p_3 = 1/2,\ p_4 = 3/4,\ p_5 = 1.$$

Let H denote heads is obtained and C_i denote the event that coin i is selected.

(a) Select a coin at random and toss it. Suppose a head is obtained. Find $P(C_i \mid H)$ for $i = 1, \ldots, 5$.

(b) Toss the coin again. What is the probability of another head?

(c) Now suppose the experiment was carried out as follows: We select a coin at random and toss it until a head is obtained. Find $P(C_i \mid B_4)$ where $B_4 =$ 'the first head is obtained in toss 4.

Solution.

(a) By Bayes,

$$P(C_i \mid H) = \frac{P(H \mid C_i)P(C_i)}{\sum_{j=1}^{5} P(H \mid C_j)P(C_j)} = \frac{P(H \mid C_i)}{\sum_{j=1}^{5} P(H \mid C_j)}$$

since $P(C_j) = 1/5$ for all j.

(b) Let $H_j =$ head occurs on toss j. Then

$$P(H_2 \mid H_1) = \frac{P(H_1 \cap H_2)}{P(H_1)} =$$

$$\frac{\sum_{i=1}^{5} P(H_1 \cap H_2 \cap C_i)}{\sum_{i=1}^{5} P(H_1 \cap C_i)} = \frac{0^2 + (1/4)^2 + (1/2)^2 + (3/4)^2 + 1^2}{0 + 1/4 + 1/2 + 3/4 + 1}.$$

(c) By Bayes,

$$P(C_i \mid B_4) = \frac{P(B_4 \mid C_i)P(C_i)}{\sum_{j=1}^{5} P(B_4 \mid C_j)P(C_j)} =$$

$$\frac{((1-p_i)^3 p_i)(1/5)}{\sum_{i=1}^{5}((1-p_j)^3 p_j)(1/5)} = \frac{(1-p_i)^3 p_i}{\sum_{i=1}^{5}(1-p_j)^3 p_j}.$$