Chapter 4: Inequalities

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1. Let $X \sim \text{Exponential}(\beta)$. Find $P(|X - \mu_X| \geq k\sigma_X)$ for $k > 1$. Compare this to the bound you get from Chebyshev's inequality.

Solution. Since $X \sim \text{Exponential}(\beta)$, then $\mu_X = \beta$ and $\sigma_X = \beta$. Calculating directly,

$$\begin{aligned}
P(|X - \mu_X| \geq k\sigma_X) &= P(X - \beta \geq k\beta) + P(X - \beta \leq -k\beta) \\
&= P(X \geq (k+1)\beta) + P(X \leq (1-k)\beta) \\
&= P(X \geq (k+1)\beta) + 0 \\
&= \int_{(k+1)\beta}^{\infty} \frac{1}{\beta} \exp\left(-\frac{x}{\beta}\right) dx \\
&= \left[-\exp\left(-\frac{x}{\beta}\right)\right]_{(k+1)\beta}^{\infty} \\
&= \exp\left(-(k+1)\right)
\end{aligned}$$

where $P(X \leq (1-k)\beta) = 0$ since $k > 1$ and $X \geq 0$. In contrast, the Chebyshev bound is $\frac{1}{k^2}$. ◊

2. Let $X \sim \text{Poisson}(\lambda)$. Use Chebyshev's inequality to show that $P(X \geq 2\lambda) \geq 1/\lambda$.

Proof. Recall that $\mu_X = \lambda$ and $\sigma_X^2 = \lambda$. Setting $t = \lambda$ in $P(|X - \mu| \geq t) \leq \frac{\sigma^2}{t^2}$ yields the desired result. (Note that $P(X - \lambda \leq \lambda) = 0$ since $X \geq 0$.) □

4. Let $X_1, \ldots, X_n \sim \text{Bernoulli}(p)$. Let $\alpha > 0$ be fixed and define $\epsilon_n = \sqrt{\frac{\log(2/\alpha)}{2n}}$. Let $\hat{p}_n = \overline{X}_n$ and define $C_n = (\hat{p}_n - \epsilon_n, \hat{p}_n + \epsilon_n)$. Use Hoeffding's inequality to show that $P(p \in C_n) \geq 1 - \alpha$.

Proof. Since $2\exp(-2n\epsilon_n^2) = \alpha$, the desired result is an immediate consequence of Theorem 4.5. For completeness, we will prove Hoeffding's inequality implies Theorem 4.5: Let $X_1, \ldots, X_n \sim \text{Bernoulli}(p)$ and set $Y_i = X_i - p$. Then $EY_i = 0$ and $a_i := -p \leq Y_i \leq 1 - p =: b_i$. Therefore, by Hoeffding, for any $t > 0$,

$$P\left(\sum_{i=1}^{n} Y_i \geq \epsilon\right) \leq \exp\left(-t\epsilon + \frac{t^2}{8}\sum_{i=1}^{n}(b_i - a_i)^2\right) = \exp\left(-t\epsilon + \frac{nt^2}{8}\right).$$

Optimizing the RHS over $t > 0$ (i.e. solving $\left(-t\epsilon + \frac{nt^2}{8}\right)' = 0$) yields $t = \frac{4\epsilon}{n}$. Thus

$$P\left(\sum_{i=1}^{n} Y_i \geq \epsilon\right) \leq \exp\left(-\frac{2\epsilon^2}{n}\right).$$

In particular,

$$P(\overline{X}_n - p \geq \eta) = P\left(\sum_{i=1}^{n} Y_i \geq n\eta\right) \leq \exp\left(-2n\eta^2\right).$$

Applying the same argument to $-Y_i$ yields

$$P(\overline{X}_n - p \leq -\eta) = P\left(\sum_{i=1}^{n} (-Y_i) \geq n\eta\right) \leq \exp\left(-2n\eta^2\right).$$

Therefore,

$$P(|\overline{X}_n - p| > \eta) \leq P(\overline{X}_n - p \geq \eta) + P(\overline{X}_n - p \leq -\eta) \leq 2\exp\left(-2n\eta^2\right).$$

$\square$

5. Prove Mill's inequality.

Proof. Following the hints,

$$\begin{aligned}
P(|Z| \geq t) &= 2P(Z \geq t) && \text{(by symmetry)} \\
&= 2\int_t^\infty \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)\, dx \\
&= \sqrt{\frac{2}{\pi}} \int_t^\infty \exp(-x^2/2)\, dx \\
&\leq \sqrt{\frac{2}{\pi}} \int_t^\infty \frac{x}{t} \exp(-x^2/2)\, dx && (x/t > 1 \text{ if } x > t) \\
&= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{t} \left[-\exp(-x^2/2)\right]_t^\infty \\
&= \sqrt{\frac{2}{\pi}} \frac{\exp(-t^2/2)}{t}.
\end{aligned}$$

$\square$

7. Let $X_1, \dots, X_n \sim N(0,1)$. Bound $P(|\overline{X}_n| \geq t)$ using Mill's inequality. Compare to the Chebyshev bound.

Solution. Note that $\overline{X}_n \sim N(0, 1/n)$. In particular, $\sqrt{n} \cdot \overline{X}_n \sim N(0,1)$. Therefore, by Mill

$$P(|\overline{X}_n| \geq t) = P(|Z| \geq t\sqrt{n}) \leq \sqrt{\frac{2}{\pi}} \frac{\exp(-(t\sqrt{n})^2/2)}{t\sqrt{n}}$$

On the other hand, Chebyshev's bound is $\frac{1/n}{t^2}$ which is larger than Mill's bound. $\diamondsuit$