Chapter 3: Expectation

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1. Suppose we play a game where we start with c dollars. On each play of the game you either double or half your money, with equal probability. What is your expected fortune after n trials.

Solution. Let X_n be the fortune after n trials. Notice that

$$X_n = c\prod_{i=1}^{n} Y_i, \text{ with } Y_i \in \{2, 1/2\} \text{ iid}$$

By definition,

$$EY_i = 2 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{5}{4}.$$

Then by independence

$$EX_n = c\prod_{i=1}^{n} EY_i = c\left(\frac{5}{4}\right)^n.$$

Alternatively, $X_0 = c$ and

$$X_n \mid X_{n-1} = \begin{cases} 2X_{n-1} & \text{with prob } 1/2, \\ \frac{1}{2}X_{n-1} & \text{with prob } 1/2. \end{cases}$$

In particular,

$$E\left(X_n \mid X_{n-1}\right) = \left(2X_{n-1}\right)\frac{1}{2} + \left(\frac{1}{2}X_{n-1}\right)\frac{1}{2} = \frac{5}{4}X_{n-1}.$$

Furthermore,

$$EX_n = E\left(E\left(X_n \mid X_{n-1}\right)\right) = \frac{5}{4}E\left(X_{n-1}\right).$$

So by induction,

$$EX_n = \frac{5}{4}\left(c\left(\frac{5}{4}\right)^{n-1}\right) = c\left(\frac{5}{4}\right)^n.$$

Alternatively,

$$X_n = c \cdot 2^k \left(\frac{1}{2}\right)^{n-k}$$

where k is the number of times the money was doubled. Furthermore,

$$P\left(X_n = c \cdot 2^k \left(\frac{1}{2}\right)^{n-k}\right) = \binom{n}{k}\left(\frac{1}{2}\right)^n$$

since there are $\binom{n}{k}$ cases, each with probability $\left(\frac{1}{2}\right)^n$, in which $X_n = c \cdot 2^k \left(\frac{1}{2}\right)^{n-k}$. Substituting these values in the definition $EX = \sum xf(x)$ (and simplifying) we obtain the same result as in the previous two proofs. $\diamond$

2. Show that $VX = 0$ iff there is a constant c such that $P(X = c) = 1$.

Proof. Suppose $VX = 0$. Since

$$VX = \int (x - EX)^2 dF(x)$$

and $(x - EX)^2 \geq 0$, then $X - EX = 0$ a.e. with respect to F. In other words, $P(X = EX) = 1$. Conversely, suppose $P(X = c) = 1$. Then $EX = c$ and therefore

$$VX = \int (x - c)^2 dF(x) = 0$$

where the last equality is because $P(X = c) = 1$. □

3. Let $X_1, \ldots, X_n \sim U(0, 1)$ and let $Y = \max\{X_1, \ldots, X_n\}$. Find EY.

Solution. By definition (of Y and iid),

$$P(Y \leq y) = P(X_1 \leq y, \ldots, X_n \leq y) = \prod_{i=1}^{n} P(X_i \leq y) = y^n.$$

Therefore,

$$EY = \int_0^1 y \cdot ny^{n-1}\, dy = \frac{n}{n+1}.$$

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4. A particle starts at the origin of the real line and moves in jumps of one unit. For each jump, the probability that the particle will jump one unit to the left is p and the probability that the particle will jump one unit to the right is $1 - p$. Let X_n be the position of the particle ater n units. Find EX_n and VX_n.

Solution. Notice that $X_n = \sum_{i=1}^n J_i$ where the J_i are iid with

$$J_i = \begin{cases} -1 & \text{with prob } p, \\ 1 & \text{with prob } 1 - p. \end{cases}$$

Furthermore,

$$EJ_i = -1 \cdot p + 1 \cdot (1 - p) = 1 - 2p.$$

Therefore,

$$EX_n = \sum_{i=1}^{n} EJ_i = n - 2np.$$

Similarly,

$$E(J_i^2) = (-1)^2 \cdot p + (1)^2 \cdot (1 - p) = 1.$$

So,

$$VJ_i = E(J_i^2) - (EJ_i)^2 = 1 - (1 - 2p)^2 = 4p(1 - p).$$

Therefore,

$$VX_n = \sum_{i=1}^{n} VJ_i = 4np(1 - p).$$

Alternatively, let L and R be the number of left and right moves. Then $X_n = R - L$ and $n = R + L$. Therefore, $X_n = n - 2L$. But $L \sim \text{Binom}(n, p)$, which allows us to calculate the desired quantities. ◊

5. A fair coin is tossed until a head is obtained. What is the expected number of tosses that will be required?

Solution. Let N be the trial number in which the first head is obtained. Then

$$f_N(n) = P(N = n) = \left(\frac{1}{2}\right)^{n-1} \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^n.$$

Note that the $\left(\frac{1}{2}\right)^{n-1}$ factor corresponds to the first $n-1$ tails. Recall that

$$\sum_{n=1}^{\infty} nr^n = \frac{r}{(1-r)^2}, \qquad |r| < 1.$$

Therefore,

$$EN = \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n = 2.$$

Alternatively, consider $M = N + 1$. Then

$$f_M(k) = f_N(k-1) = \left(\frac{1}{2}\right)^{k-1} = 2\left(\frac{1}{2}\right)^k = 2f_N(k), \qquad k = 2, 3, \ldots$$

Therefore,

$$\begin{aligned} EN &= \sum_{k=1}^{\infty} k f_N(k) \\ &= 1 \cdot f_N(1) + \sum_{k=2}^{\infty} k f_N(k) \\ &= 1 \cdot \frac{1}{2} + \sum_{k=2}^{\infty} k \left(\frac{1}{2} f_M(k)\right) \\ &= \frac{1}{2} + \frac{1}{2} EM \\ &= \frac{1}{2} + \frac{1}{2}(1 + EN). \end{aligned}$$

Solving for EN we obtain the desired result. Alternatively, by the law of total expectation,

$$EN = E(N \mid \text{first toss} = H)P(H) + E(N \mid \text{first toss} = T)P(T) = 1 \cdot \frac{1}{2} + (1 + EN)\frac{1}{2},$$

where $E(N \mid \text{first toss} = T) = 1 + EN$ is due to the memoryless property of the geometric distribution. $\Diamond$

7. Let X be a continuous variable with CDF F. Suppose that $P(X > 0) = 1$ and that EX exists. Show that $EX = \int_0^\infty P(X > x)\,dx$.

Proof. Calculating directly,

$$\begin{aligned} \int_0^\infty P(X > x)\,dx &= \int_0^\infty (1 - F(x))\,dx \\ &= (x(1 - F(x)))_0^\infty - \int_0^\infty x(-f(x))\,dx \\ &= 0 + EX. \end{aligned}$$

Where $\lim_{x\to\infty} x(1 - F(x)) = 0$ since EX exists. $\square$

13. Suppose we generate a random variable X in the following way. First we flip a fair coin. If the coin is heads, take X to have a $U(0,1)$ distribution. If the coin is tails, take X to have a $U(3,4)$ distribution. Find the mean and standard deviation of X.

Solution. By definition,

$$f(x) = \begin{cases} 1/2 & \text{if } x \in (0,1) \cup (3,4), \\ 0 & \text{otherwise} \end{cases}$$

Then

$$EX = \int_0^1 x \cdot \frac{1}{2}\,dx + \int_3^4 x \cdot \frac{1}{2}\,dx = 2.$$

Alternatively, let C denote the value of the coin. Then

$$X \mid C \sim \begin{cases} U(0,1) & \text{if } C = H, \\ U(3,4) & \text{if } C = T. \end{cases} \tag{1}$$

By the law of total expectation

$$EX = E(X \mid C = H)P(C = H) + E(X \mid C = T)P(C = T) = \frac{0+1}{2} \cdot \frac{1}{2} + \frac{3+4}{2}\frac{1}{2} = 2.$$

On the other hand,

$$E(X^2) = \int_0^1 x^2 \cdot \frac{1}{2}\,dx + \int_3^4 x^2 \cdot \frac{1}{2}\,dx = \frac{19}{3}.$$

So

$$VX = E(X^2) - (EX)^2 = \frac{19}{3} - 4 = \frac{7}{3}.$$

Alternatively, by the law of total variance

$$VX = EV(X \mid C) + VE(X \mid C).$$

Equation (1) implies

$$V(X \mid C) = \frac{1}{12}$$

and

$$E(X \mid C) = \begin{cases} 1/2 & \text{if } C = H, \\ 7/2 & \text{if } C = T. \end{cases}$$

Then

$$E\left((E(X \mid C))^2\right) = \left(\frac{1}{2}\right)^2 \frac{1}{2} + \left(\frac{7}{2}\right)^2 \frac{1}{2} = \frac{25}{4}$$

and

$$E\left(E(X \mid C)\right) = \left(\frac{1}{2}\right)\frac{1}{2} + \left(\frac{7}{2}\right)\frac{1}{2} = 2.$$

(Note that we could have simply used the tower property.) Therefore,

$$VX = \frac{1}{12} + \left(\frac{25}{4} - 2^2\right) = \frac{7}{3}.$$

$\Diamond$

14. Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be r.v. and let $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ be constants. Show that

$$\operatorname{Cov}\left(\sum_{i=1}^m a_i X_i, \sum_{j=1}^n b_j Y_j\right) = \sum_{i=1}^m \sum_{j=1}^n a_i b_j \operatorname{Cov}(X_i, Y_j).$$

Proof. Calculating directly,

$$E\left(\left(\sum_{i=1}^m a_i X_i\right)\left(\sum_{j=1}^n b_j Y_j\right)\right) = E\left(\sum_{i=1}^m \sum_{j=1}^n a_i b_j X_i Y_j\right) = \sum_{i=1}^m \sum_{j=1}^n a_i b_j E(X_i Y_j)$$

and

$$E\left(\sum_{i=1}^m a_i X_i\right) E\left(\sum_{j=1}^n b_j Y_j\right) = \left(\sum_{i=1}^m a_i E X_i\right)\left(\sum_{j=1}^n b_j E Y_j\right) = \sum_{i=1}^m \sum_{j=1}^n a_i b_j E X_i E Y_j.$$

Then the identity $\operatorname{Cov}(X, Y) = E(XY) - EXEY$ implies

$$\begin{aligned}
\operatorname{Cov}\left(\sum_{i=1}^m a_i X_i, \sum_{j=1}^n b_j Y_j\right) &= \sum_{i=1}^m \sum_{j=1}^n a_i b_j E(X_i Y_j) - \sum_{i=1}^m \sum_{j=1}^n a_i b_j E X_i E Y_j \\
&= \sum_{i=1}^m \sum_{j=1}^n a_i b_j \left(E(X_i Y_j) - E X_i E Y_j\right) \\
&= \sum_{i=1}^m \sum_{j=1}^n a_i b_j \operatorname{Cov}(X_i, Y_j).
\end{aligned}$$

□

15. Let $f(x, y) = \frac{1}{3}(x + y)$ for $0 \le x \le 1$, $0 \le y \le 2$. Find $V(2X - 3Y + 8)$.

Solution. The identity $V(X - Y) = V(X) + V(Y) - 2\operatorname{Cov}(X, Y)$ implies

$$\begin{aligned}
V(2X - 3Y + 8) &= V(2X) + V(3Y - 8) - 2\operatorname{Cov}(2X, 3Y - 8) \\
&= 4VX + 9VY - 2(2)(3)\operatorname{Cov}(X, Y).
\end{aligned}$$

To compute this variance proceed as follows:

1. Find the marginals $f(x) = \int_0^2 f(x, y)\,dy$ and $f(y) = \int_0^1 f(x, y)\,dx$.
2. Find EX, $E(X^2)$, EY, $E(Y^2)$ using the marginals.
3. Find $VX = E(X^2) - (EX)^2$ and VY (analogously).
4. Find $E(XY) = \int_0^2 \int_0^1 xyf(x, y)\,dxdy$.
5. Find $\operatorname{Cov}(X, Y) = E(XY) - EXEY$.

◊

16. Let $r(X)$ be a function of X and let $s(Y)$ be a function of Y. Show that

$$E\left(r(X)s(Y) \mid X\right) = r(X)E\left(s(Y) \mid X\right)$$

In particular, show that $E(r(X) \mid X) = r(X)$.

Proof. Intuitively, if X is given, $r(X)$ is a constant. Formally,

$$E\left(r(X)s(Y) \mid X\right) = \int r(X)s(y)f(y \mid X)\,dy = r(X)\int s(y)f(y \mid X)\,dy = r(X)E(s(Y) \mid X).$$

For the second equality, let $s(Y) = 1$. □

17. Prove that

$$VY = EV(Y \mid X) + VE(Y \mid X).$$

Proof. Note that

$$\begin{aligned} VY = E\left((Y - EY)^2\right) = E\left((Y - E(Y \mid X) + E(Y \mid X) - EY)^2\right) = \\ E\left((Y - E(Y \mid X))^2 + 2(Y - E(Y \mid X))(E(Y \mid X) - EY) + (E(Y \mid X) - EY)^2\right) = \\ E\left((Y - E(Y \mid X))^2\right) + 2E\left((Y - E(Y \mid X))(E(Y \mid X) - EY)\right) + E\left((E(Y \mid X) - EY)^2\right). \end{aligned}$$

Consider each term separately. The first term:

$$E\left((Y - E(Y \mid X))^2\right) = E\left(E\left((Y - E(Y \mid X))^2 \mid X\right)\right) = EV(Y \mid X)$$

The second term:

$$\begin{aligned} E\left((Y - E(Y \mid X))(E(Y \mid X) - EY)\right) = \\ E\left(E\left((Y - E(Y \mid X))(E(Y \mid X) - EY) \mid X\right)\right) = \\ (E(Y \mid X) - EY)E\left((Y - E(Y \mid X)) \mid X\right) = \\ (E(Y \mid X) - EY)(0) = 0. \end{aligned}$$

Where the second equality is due to exercise 16 ($E(Y \mid X)$ is a function of X). The third term: Since the mean of $E(Y \mid X)$ is precisely EY, by definition

$$E\left((E(Y \mid X) - EY)^2\right) = VE(Y \mid X).$$

□

18. Suppose that $E(X \mid Y) = c$ for some constant c. Show that X and Y are uncorrelated.

Proof. On one hand,

$$E(XY) = E(E(XY \mid Y)) = E(YE(X \mid Y)) = E(Yc) = cEY$$

where the second equality is due to exercise 16. On the other hand,

$$EX = E(E(X \mid Y)) = E(c) = c.$$

Therefore,

$$\operatorname{Cov}(X, Y) = E(XY) - EXEY = cEY - cEY = 0.$$

(Note that $E(X \mid Y) = c$ constant implies $EX = c$.) □

21. Suppose that $E(Y \mid X) = X$. Show that $\operatorname{Cov}(X, Y) = VX$.

Proof. On one hand,

$$E(XY) = E(E(XY \mid X)) = E(XE(Y \mid X)) = E(X^2).$$

where the second equality is due to exercise 16. On the other hand,

$$E(Y) = E(E(Y|X)) = EX.$$

Therefore,

$$\operatorname{Cov}(X, Y) = E(XY) - EXEY = E(X^2) - (EX)^2 = VX.$$

□

22. Let $X \sim U(0,1)$. Let $0 < a < b < 1$. Let

$$Y = \begin{cases} 1 & \text{if } 0 < x < b, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$Z = \begin{cases} 1 & \text{if } a < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Are Y and Z independent?

(b) Find $E(Y \mid Z)$.

Proof. (a) No: On one hand,

$$P(Y = 1, Z = 1) = P(a < X < b) = b - a.$$

On the other hand,

$$P(Y = 1)P(Z = 1) = P(X < b)P(X > a) = b(1 - a).$$

(b) Calculating directly,

$$\begin{aligned} E(Y \mid Z) &= \sum_{y=0,1} y f(y \mid Z) \\ &= f_{Y|Z}(1 \mid Z) \\ &= P(X < b \mid Z = z) \\ &= \begin{cases} P(X < b \mid a < X) & \text{if } Z = 1, \\ P(X < b \mid X \leq a) & \text{if } Z = 0 \end{cases} \\ &= \begin{cases} \frac{b-a}{1-a} & \text{if } Z = 1, \\ 1 & \text{if } Z = 0. \end{cases} \end{aligned}$$

$\square$