Chapter 8: The Bootstrap

All of Statistics, Wasserman

4. Let $X_1, \dots, X_n$ be distinct observations. Show that there are $\binom{2n-1}{n}$ distinct boostrap samples.

Proof. Since every (unordered) bootstrap sample is completely determined by the number of times each X_i appears, there is a bijection between the set of bootstrap samples and the following set:

$$\left\{(j_1, \dots, j_n) : j_i \in \mathbb{Z}_{\geq 0} \text{ and } \sum_{i=1}^{n} j_i = n\right\} \tag{1}$$

where j_i is associated to the number of times X_i appears in the bootstrap sample. By the stars-and-bars formula, the number of such n-tuples is $\binom{2n-1}{n}$. □

5. Let $X_1, \dots, X_n$ be iid observations with $\mu = EX_1$ and $\sigma^2 = VX_1$. Let $X_1^*, \dots, X_n^*$ be a bootstrap sample, and let $\overline{X^*}_n$ denote its sample mean. Find $E(\overline{X^*}_n \mid X_1, \dots, X_n)$, $V(\overline{X^*}_n \mid X_1, \dots, X_n)$, $E(\overline{X^*}_n)$, and $V(\overline{X^*}_n)$.

Solution. Let Y be a random variable such that

$$P(Y = X_i \mid X_1, \dots, X_n) = 1/n \text{ for all } i. \tag{2}$$

Then

$$E(Y \mid X_1, \dots, X_n) = \sum_{i=1}^{n} X_i \cdot P(Y = X_i \mid X_1, \dots, X_n) = \sum_{i=1}^{n} X_i \cdot \frac{1}{n} = \overline{X}_n,$$

and

$$E(Y^2 \mid X_1, \dots, X_n) = \sum_{i=1}^{n} X_i^2 \cdot P(Y = X_i \mid X_1, \dots, X_n) = \sum_{i=1}^{n} X_i^2 \cdot \frac{1}{n} = \overline{X^2}_n.$$

In particular,

$$V(Y \mid X_1, \dots, X_n) = \overline{X^2}_n - \left(\overline{X}_n\right)^2.$$

Since X_i^* satisfies (2) for all i,

$$E(\overline{X^*}_n \mid X_1, \dots, X_n) = E(X_1^* \mid X_1, \dots, X_n) = \overline{X}_n,$$

$$V(\overline{X^*}_n \mid X_1, \dots, X_n) = \frac{V(X_1^*)}{n} = \frac{\overline{X^2}_n - \left(\overline{X}_n\right)^2}{n} = \frac{n-1}{n^2} S_n^2.$$

By the iterated expectations property,

$$E(\overline{X^*}_n) = E(\overline{X}_n) = \mu.$$

By the law of total expectation/variance,

$$\begin{aligned} V(\overline{X^*}_n) &= E\left(V(\overline{X^*}_n \mid X_1, \dots, X_n)\right) + V\left(E(\overline{X^*}_n \mid X_1, \dots, X_n)\right) \\ &= E\left(\frac{n-1}{n^2} S_n^2\right) + V\left(\overline{X}_n\right) \\ &= \frac{n-1}{n^2}\sigma^2 + \frac{\sigma^2}{n} \\ &= \frac{2n-1}{n^2}\sigma^2. \end{aligned}$$

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7. Let $X_1, \ldots, X_n \sim U(0, \theta)$. Let $\widehat{\theta} = \max\{X_1, \ldots, X_n\}$. Show that $P(\widehat{\theta} = \theta) = 0$ and yet $P(\widehat{\theta}^* = \widehat{\theta}) \approx .632$.

Proof. Since $\widehat{\theta}$ is continuous, $P(\widehat{\theta} = \theta) = 0$. On the other hand,

$$\begin{aligned}
P(\widehat{\theta}^* \neq \widehat{\theta}) &= P(\max\{X_1^*, \ldots, X_n^*\} \neq \max\{X_1, \ldots, X_n\}) \\
&= P(X_i^* \neq \max\{X_1, \ldots, X_n\} \text{ for all } i) \\
&= \prod_{i=1}^{n} P(X_i^* \neq \max\{X_1, \ldots, X_n\}) \\
&= \prod_{i=1}^{n} (1 - 1/n) \\
&= (1 - 1/n)^n.
\end{aligned}$$

Hence

$$P(\widehat{\theta}^* = \widehat{\theta}) = 1 - (1 - (1/n))^n \xrightarrow{n \to \infty} 1 - e^{-1} \approx 0.632.$$

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