Chapter 6: Models, Statistical Inference and Learning

All of Statistics, Wasserman

1. Let $X_1, \ldots, X_n \sim \text{Poisson}(\lambda)$ and let $\widehat{\lambda}_n = \overline{X}_n$. Find the bias, se, and MSE of this estimator.

Solution. Recall that in general, the sample mean satisfies $E\overline{X}_n = \mu$ and $V\overline{X}_n = \frac{\sigma^2}{n}$. Furthermore, if $X \sim \text{Poisson}(\lambda)$, then $EX = \lambda$ and $VX = \lambda$. Therefore,

$$\begin{aligned}
\text{bias}(\widehat{\lambda}_n) &= \lambda - \lambda = 0, \\
\text{se}(\widehat{\lambda}_n) &= \sqrt{\frac{\lambda}{n}}, \\
\text{MSE}(\widehat{\lambda}_n) &= 0 + \frac{\lambda}{n} = \frac{\lambda}{n}.
\end{aligned}$$

$\Diamond$

2. Let $X_1, \ldots, X_n \sim U(0, \theta)$ and let $\widehat{\theta} = \max\{X_1, \ldots, X_n\}$. Find the bias, se, and MSE of this estimator.

Solution. By definition,

$$P(\widehat{\theta} \le x) = \prod_{i=1}^{n} P(X_i \le x) = \left(\frac{x}{\theta}\right)^n \text{ for } 0 < x < \theta$$

Then the pdf of $\widehat{\theta}$ is

$$f(x) = \frac{n}{\theta^n} x^{n-1} \text{ for } 0 < x < \theta.$$

Calculating directly,

$$\begin{aligned}
E_\theta(\widehat{\theta}_n) &= \int_0^\theta x \left(\frac{n}{\theta^n} x^{n-1}\right) dx = \frac{n}{\theta^n} \int_0^\theta x^n \, dx = \frac{n}{\theta^n} \cdot \frac{\theta^{n+1}}{n+1} = \frac{n}{n+1}\theta, \\
E_\theta\left((\widehat{\theta}_n)^2\right) &= \int_0^\theta x^2 \left(\frac{n}{\theta^n} x^{n-1}\right) dx = \frac{n}{\theta^n} \int_0^\theta x^{n+1} \, dx = \frac{n}{\theta^n} \frac{\theta^{n+2}}{n+2} = \frac{n}{n+2}\theta^2.
\end{aligned}$$

In particular,

$$V_\theta(\widehat{\theta}_n) = E_\theta\left((\widehat{\theta}_n)^2\right) - \left(E_\theta(\widehat{\theta}_n)\right)^2 = \frac{n}{n+2}\theta^2 - \left(\frac{n}{n+1}\theta\right)^2 = \frac{n}{(n+1)^2(n+2)}\theta^2.$$

Therefore,

$$\begin{aligned}
\text{bias}(\widehat{\theta}_n) &= \frac{n}{n+1}\theta - \theta = -\frac{1}{n+1}\theta, \\
\text{se}(\widehat{\theta}_n) &= \sqrt{\frac{n}{(n+1)^2(n+2)}\theta^2}, \\
\text{MSE}(\widehat{\theta}_n) &= \left(-\frac{1}{n+1}\theta\right)^2 + \frac{n}{(n+1)^2(n+2)}\theta^2 = \frac{2}{(n+1)(n+2)}\theta^2.
\end{aligned}$$

$\Diamond$

3. Let $X_1, \ldots, X_n \sim U(0, \theta)$ and let $\widehat{\theta} = 2\overline{X}_n$. Find the bias, se, and MSE of this estimator.

Solution. Recall that if $X \sim U(0, \theta)$, then $EX = \frac{\theta}{2}$ and $VX = \frac{\theta^2}{12}$. Therefore,

$$\begin{aligned}
\operatorname{bias}(\widehat{\theta}_n) &= E_\theta(2\overline{X}_n) - \theta = 2 \cdot \frac{\theta}{2} - \theta = 0, \\
\operatorname{se}(\widehat{\theta}_n) &= \sqrt{V(2\overline{X}_n)} = \sqrt{4\frac{\theta^2/12}{n}} = \frac{\theta}{\sqrt{3n}}, \\
\operatorname{MSE}(\widehat{\theta}_n) &= 0 + \frac{\theta^2}{3n} = \frac{\theta^2}{3n}.
\end{aligned}$$

$\Diamond$