Quantum Mechanics I HW1

Diego Ramírez Milano (201214691)

Febrero 7 de 2016

Given the the wave function

$$\Psi(x,t) = e^{-iEt/\hbar} N e^{-x^2/2\lambda^2} \tag{1}$$

$$\Psi^*(x,t) = e^{iEt/\hbar} N e^{-x^2/2\lambda^2} \tag{2}$$

Where N is the normalization factor and $\lambda = \sqrt{\frac{\hbar}{m\omega}}$.
The Hamiltonian operator for the harmonic oscillator reads

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}kx^2 = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}m\omega^2 x^2 \tag{3}$$

Help

$$\int_0^\infty e^{-a^2x^2} = \frac{\sqrt{\pi}}{2a},\ a > 0 \tag{4}$$

$$\int_0^\infty x^2 e^{-a^2x^2} = \frac{\sqrt{\pi}}{4a^3},\ a > 0 \tag{5}$$

$$\int_{\mathbb{R}} \Psi\Psi^* \mathrm{d}x = 1 \tag{6}$$

1. Find the normalization factor N.

$$\int_{\mathbb{R}} \Psi\Psi^* \mathrm{d}x = \int_{\mathbb{R}} e^{-iEt/\hbar} N e^{-x^2/2\lambda^2} e^{iEt/\hbar} N e^{-x^2/2\lambda^2} \mathrm{d}x = \int_{\mathbb{R}} N^2 e^{-x^2/\lambda^2} \mathrm{d}x \tag{7}$$

Now we use the substitution

$$\begin{aligned} u &= \frac{x}{\lambda} \\ \lambda \mathrm{d}u &= \mathrm{d}x \end{aligned} \tag{8}$$

$$\begin{aligned} \lambda N^2 \int_{\mathbb{R}} e^{-u^2} \mathrm{d}u &= 1 \\ \lambda N^2 \sqrt{\pi} &= 1 \\ \boxed{N = \frac{1}{\sqrt{\lambda\sqrt{\pi}}} = \sqrt[4]{\frac{m\omega}{\hbar\pi}}} \end{aligned} \tag{9}$$

2. Calculate the probability for the particle to be in the interval $[0, \infty]$

$$\int_0^{\infty} \Psi\Psi^* \mathrm{d}x = N^2 \int_0^{\infty} e^{-x^2/\lambda^2} \mathrm{d}x \tag{10}$$

Using same substitution as (**??**) we get

$$\begin{gathered} \lambda N^2 \int_0^{\infty} e^{-u^2} \mathrm{d}x = N^2 \lambda \frac{\sqrt{\pi}}{2} \\ \boxed{P = \frac{\lambda}{\lambda\sqrt{\pi}} \frac{\sqrt{\pi}}{2} = \frac{1}{2}} \end{gathered} \tag{11}$$

3. Calculate the values $\langle x \rangle$, $\langle x^2 \rangle$, $\langle \hat{P}_x \rangle$ and $\langle \hat{P}_x^2 \rangle$.

For $\langle x \rangle$:

$$\langle x \rangle = \int_{\mathbb{R}} x\Psi\Psi^* \mathrm{d}x = N^2 \int_{\mathbb{R}} x e^{-x^2/\lambda^2} \mathrm{d}x \tag{12}$$

Now we use the substitution

$$\begin{aligned} u &= \frac{x^2}{\lambda^2} \\ \mathrm{d}u &= \frac{2x}{\lambda^2} \mathrm{d}x \end{aligned} \tag{13}$$

$$\begin{aligned} &= N^2 \frac{\lambda^2}{2} \int_{\mathbb{R}} e^{-u} \mathrm{d}u = -N^2 \frac{\lambda^2}{2} \left(e^{-u} \right) \\ &= -\frac{\lambda}{2\sqrt{\pi}} e^{-x^2/\lambda^2} \Big|_{-\infty}^{\infty} \\ &= \boxed{-\frac{\lambda}{2\sqrt{\pi}} (0 - 0) = 0} \end{aligned} \tag{14}$$

For $\langle x^2 \rangle$:

$$\langle x^2 \rangle = \int_{\mathbb{R}} x^2 \Psi\Psi^* \mathrm{d}x = N^2 \int_{\mathbb{R}} x^2 e^{-x^2/\lambda^2} \mathrm{d}x \tag{15}$$

We take advantage now of help number 2 in equation (**??**) to say:

$$\begin{gathered} N^2 \int_{\mathbb{R}} x^2 e^{-x^2/\lambda^2} \mathrm{d}x = N^2 \frac{\sqrt{\pi}\lambda^3}{2} \\ \boxed{\langle x^2 \rangle = \frac{\lambda^2}{2} = \frac{\hbar}{2m\omega}} \end{gathered} \tag{16}$$

For $\langle \hat{P}_x \rangle$:

$$
\begin{aligned}
\langle \hat{P}_x \rangle =& N^2 \int_{\mathbb{R}} e^{-x^2/2\lambda^2} \left(-i\hbar \frac{\mathrm{d}}{\mathrm{d}x} e^{-x^2/2\lambda^2} \right) \mathrm{d}x \\
=& \frac{N^2 i\hbar}{\lambda^2} \int_{\mathbb{R}} x e^{-x^2/\lambda^2} \mathrm{d}x \\
=& \boxed{\frac{N^2 i\hbar}{\lambda^2} \langle x \rangle = 0}
\end{aligned} \tag{17}
$$

For $\langle \hat{P}_x^2 \rangle$:

$$
\langle \hat{P}_x^2 \rangle = -\int_{\mathbb{R}} \hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} \Psi \Psi^* \mathrm{d}x = -N^2 \hbar^2 \int_{\mathbb{R}} e^{-x^2/2\lambda^2} \frac{\mathrm{d}^2}{\mathrm{d}x^2} e^{-x^2/2\lambda^2} \mathrm{d}x \tag{18}
$$

In a separate place we compute the derivatives

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} e^{-x^2/2\lambda^2} &= -\frac{x}{\lambda} e^{-x^2/2\lambda^2} \\
\frac{\mathrm{d}^2}{\mathrm{d}x^2} e^{-x^2/2\lambda^2} &= \left(\frac{x^2 - \lambda^2}{\lambda^4} \right) e^{-x^2/2\lambda^2}
\end{aligned} \tag{19}
$$

Therefore using what we just computed we get:

$$
\begin{aligned}
\langle \hat{P}_x^2 \rangle =& - N^2 \hbar^2 \int_{\mathbb{R}} e^{-x^2/2\lambda^2} \left(\frac{x^2 - \lambda^2}{\lambda^4} \right) e^{-x^2/2\lambda^2} \mathrm{d}x \\
=& - \frac{\hbar^2}{\lambda^4} \left(\int_{\mathbb{R}} N^2 x^2 e^{-x^2/2\lambda^2} \mathrm{d}x - \int_{\mathbb{R}} N^2 \lambda^2 e^{-x^2/2\lambda^2} \mathrm{d}x \right) \\
=& - \frac{\hbar^2}{\lambda^4} \left(\langle x^2 \rangle - \lambda^2 \right) = -\frac{\hbar^2}{\lambda^4} \left(\frac{\lambda^2}{2} - \lambda^2 \right) \\
=& \boxed{\frac{\hbar^2}{2\lambda^2} = \frac{mw}{2}}
\end{aligned} \tag{20}
$$

4. Find the eigenvalue E in $\hat{H}\Psi = E\Psi$

We now implement what is stated in equation (**??**) to the wave function, so we get

$$
\begin{aligned}
\hat{H}\Psi =& - \frac{\hbar^2}{2m} \frac{\mathrm{d}^2}{\mathrm{d}x^2} \Psi + \frac{1}{2} m\omega^2 x^2 \Psi \\
=& - \frac{\hbar^2}{2m} e^{-iEt/\hbar} N e^{-x^2/2\lambda^2} \left(\frac{x^2 - \lambda^2}{\lambda^4} \right) + \frac{1}{2} m\omega^2 x^2 e^{-iEt/\hbar} N e^{-x^2/2\lambda^2} \\
=& - \frac{\hbar^2}{2m} \Psi \left(\frac{x^2 - \lambda^2}{\lambda^4} \right) + \frac{1}{2} m\omega^2 x^2 \Psi \\
=& \Psi \left(-\frac{\hbar^2}{2m} \left(\frac{x^2 - \lambda^2}{\lambda^4} \right) + \frac{1}{2} m\omega^2 x^2 \right)
\end{aligned} \tag{21}
$$

From here we can now solve for E:

$$\begin{aligned} E\Psi =& \Psi\left(-\frac{\hbar^2}{2m}\left(\frac{x^2-\lambda^2}{\lambda^4}\right)+\frac{1}{2}m\omega^2x^2\right) \\ E =& \boxed{-\frac{\hbar^2}{2m}\left(\frac{x^2-\lambda^2}{\lambda^4}\right)+\frac{1}{2}m\omega^2x^2} \end{aligned} \tag{22}$$

Finally replacing λ in equation (**??**) we can simplify the expression to

$$\begin{aligned} E =& -\frac{\hbar^2m^2\omega^2x^2}{2m\hbar^2}+\frac{\hbar^2m\omega}{2m\hbar}+\frac{1}{2}m\omega^2x^2 \\ =& -\frac{1}{2}m\omega^2x^2+\frac{\hbar\omega}{2}+\frac{1}{2}m\omega^2x^2=\boxed{\frac{\hbar\omega}{2}} \end{aligned} \tag{23}$$

5. Calculate the probability current j_x

$$\begin{aligned} j_x =& \frac{i\hbar}{2m}\left(\Psi\frac{\mathrm{d}}{\mathrm{d}x}\Psi^*-\Psi^*\frac{\mathrm{d}}{\mathrm{d}x}\Psi\right) \\ =& \frac{i\hbar}{2m}\left(N^2e^{-x^2/2\lambda^2}\frac{\mathrm{d}}{\mathrm{d}x}e^{-x^2/2\lambda^2}-N^2e^{-x^2/2\lambda^2}\frac{\mathrm{d}}{\mathrm{d}x}e^{-x^2/2\lambda^2}\right) \\ =& \frac{i\hbar N^2}{2m}e^{-x^2/2\lambda^2}\left(\frac{\mathrm{d}}{\mathrm{d}x}e^{-x^2/2\lambda^2}-\frac{\mathrm{d}}{\mathrm{d}x}e^{-x^2/2\lambda^2}\right) \\ =& \boxed{0} \end{aligned} \tag{24}$$