Quantum Mechanics I - HW2

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The linear space of finite dimension (vectors) on which matrices act as linear operators is an example of a mathematical space (Hilbert Space) on which some Quantum Mechanics can be done.

5. Let x and φ be n-dimensional vectors (complex). We define the inner product $\langle x, \varphi \rangle$ as

$$\langle x, \varphi \rangle \equiv x^\dagger \varphi = \sum_i x_i^* \varphi_i \tag{1}$$

Where x_i and φ_i are components of the vectors. Prove the following properties which $\langle x, \varphi \rangle$ shares with the inner product defined for functions $(f \circ g) = \int_{-\infty}^{\infty} \mathrm{d}x f^* g$.

$$\langle x, x \rangle \geq 0 \tag{i}$$
$$\langle x, x \rangle = 0 \iff x = 0 \tag{ii}$$
$$\langle x, \varphi \rangle = \langle \varphi, x \rangle^* \tag{iii}$$
$$\langle x, a\varphi \rangle = a\langle x, \varphi \rangle \tag{iv}$$
$$\langle ax, \varphi \rangle = a^*\langle x, \varphi \rangle \tag{v}$$
$$\langle ax, \varphi \rangle = \langle x, a^*\varphi \rangle \tag{vi}$$
$$\langle x_1 + x_2, \varphi \rangle = \langle x_1, \varphi \rangle + \langle x_2, \varphi \rangle \tag{vii}$$

6. Let us write from now on in two dimensional vector space. Let M be

$$M = \begin{pmatrix} m_1 & a \\ a^* & m_2 \end{pmatrix}, \text{ with } m_{i=1,2} \in \mathbb{R} \tag{2}$$

i. Prove that $M = M^\dagger$ (M is hermitian).

ii. Find the eigenvalues λ_i of $Mx = \lambda x$, where x is a 2-dimensional vector.

iii. Prove that λ_i are real.

iv. Find the eigenvectors x_i (corresponding to λ_i) such that are normalized to 1 ($x^\dagger x = 1$).

v. Prove that $x_i^* x_j$ ($i \neq j$) is zero, meaning the eigenvectors are orthonormal.

7. Let a be now real. Let U be delivered as follows

$$U = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \tag{3}$$

i. Prove that U is unitary ($U^\dagger U = UU^\dagger = 1$).

ii. Make a transformation (M from 6.)

$$\mathrm{diag}(\lambda_1, \lambda_2) = UMU^\dagger \tag{4}$$

and find the angle θ. What transformation on x from $Mx = \lambda x$ is necessary to make it an eigenvector?

8. Let $\hat{A}$ be an arbitrary (not necessarily hermitian) operator acting in the space of functions. Prove $\hat{O} = i(\hat{A} - \hat{A}^\dagger)$ is hermitian ($\hat{O} = \hat{O}^\dagger$).