Quantum Mechanics I - HW2

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February 18, 2016

The linear space of finite dimension (vectors) on which matrices act as linear operators is an example of a mathematical space (Hilbert Space) on which some Quantum Mechanics can be done.

5. Let x and φ be n-dimensional vectors (complex). We define the inner product $\langle x, \varphi\rangle$ as

$$\langle x, \varphi\rangle \equiv x^\dagger \varphi = \sum_i x_i^* \varphi_i \tag{1}$$

Where x_i and φ_i are components of the vectors. Prove the following properties which $\langle x, \varphi\rangle$ shares with the inner product defined for functions $(f \circ g) = \int_{-\infty}^{\infty} \mathrm{d}x f^* g$.

$$\langle x, x\rangle \geq 0 \tag{i}$$
$$\langle x, x\rangle = 0 \iff x = 0 \tag{ii}$$
$$\langle x, \varphi\rangle = \langle \varphi, x\rangle^* \tag{iii}$$
$$\langle x, a\varphi\rangle = a\langle x, \varphi\rangle \tag{iv}$$
$$\langle ax, \varphi\rangle = a^*\langle x, \varphi\rangle \tag{v}$$
$$\langle ax, \varphi\rangle = \langle x, a^*\varphi\rangle \tag{vi}$$
$$\langle x_1 + x_2, \varphi\rangle = \langle x_1, \varphi\rangle + \langle x_2, \varphi\rangle \tag{vii}$$

Proof for *i*.

$$\begin{gathered}\langle x, x\rangle = \sum_i x_i^* x_i = \sum_i |x_i|^2 \\ \text{Since } |x| \geq 0 \to \sum_i |x_i|^2 \geq 0 \\ \text{It follows } \langle x, x\rangle \geq 0\end{gathered} \tag{i proof}$$

Proof for *ii*.

$$\begin{gathered}\langle x, x\rangle = \sum_i |x_i|^2 \text{ but } |x| = 0 \iff x = 0 \\ \text{Then it follows } \langle x, x\rangle = 0 \iff x = 0\end{gathered} \tag{ii proof}$$

Proof for *iii*.

$$\langle x, \varphi\rangle = \sum_i x_i^* \varphi_i = \left(\sum_i x_i \varphi_i^*\right)^* = \left(\sum_i \varphi_i^* x_i\right)^* = \langle \varphi, x\rangle^* \tag{iii proof}$$

Proof for *iv.*

$$\langle x, a\varphi\rangle = \sum_i x_i^* a\varphi_i = a\sum_i x_i^*\varphi_i = a\langle x, \varphi\rangle \tag{\textit{iv proof}}$$

Proof for *v.*

$$\langle ax, \varphi\rangle = \sum_i (ax_i)^*\varphi_i = \sum_i a^* x_i^*\varphi_i = a^*\sum_i x_i^*\varphi_i = a^*\langle x, \varphi\rangle \tag{\textit{v proof}}$$

Proof for *vi.*

$$\langle ax, \varphi\rangle = \sum_i (ax_i)^*\varphi_i = \sum_i a^* x_i^*\varphi_i = \sum_i x_i^*(a^*\varphi_i) = \langle x, a^*\varphi\rangle \tag{\textit{vi proof}}$$

Proof for *vii.*

$$\begin{aligned}\langle x_1 + x_2, \varphi\rangle &= \sum_i (x_1 + x_2)^*\varphi = \sum_i (x_{1i}^* + x_{2i}^*)\varphi = \sum_i \left(x_{1i}^*\varphi_i + x_{2i}^*\varphi_i\right) \\ &= \sum_i x_{1i}^*\varphi + \sum_i x_{2i}^*\varphi = \langle x_1, \varphi\rangle + \langle x_2, \varphi\rangle\end{aligned} \tag{\textit{vii proof}}$$

6. Let us write from now on in two dimensional vector space. Let M be

$$M = \begin{pmatrix} m_1 & a \\ a^* & m_2 \end{pmatrix}, \text{ with } m_{i=1,2} \in \mathbb{R} \tag{2}$$

i. Prove that $M = M^\dagger$ (M is hermitian) and $M^\dagger = (M^T)^*$

$$M^\dagger = \begin{pmatrix} m_1 & a^* \\ a & m_2 \end{pmatrix}^* = \begin{pmatrix} m_1 & a \\ a^* & m_2 \end{pmatrix} = M \tag{3}$$

ii. Find the eigenvalues λ_i of $Mx = \lambda x$, where x is a 2-dimensional vector. For any given square matrix 2×2 of the form

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{4.a}$$

We can define the eigenvalues and eigenvectors, respectively, as follows

$$\lambda_i = \frac{a+d}{2} \pm \sqrt{\frac{(a+d)^2}{4} - (ad - bc)} \tag{4.b}$$

$$x_i = \begin{pmatrix} \lambda_i - d \\ c \end{pmatrix} \tag{4.c}$$

Now using (4.b) we find the eigenvalues for M to be

$$\begin{aligned}\lambda_1 =& \frac{m_1 + m_2}{2} + \sqrt{\frac{(m_1 + m_2)^2}{4} - \left(m_1 m_2 - |a|^2\right)} \\ \lambda_2 =& \frac{m_1 + m_2}{2} - \sqrt{\frac{(m_1 + m_2)^2}{4} - \left(m_1 m_2 - |a|^2\right)}\end{aligned} \tag{5}$$

iii. Prove that λ_i are real

$$
\begin{aligned}
\lambda\langle x, x\rangle =& \langle x, \lambda x\rangle \text{ from relation } (iv)\\
=& \langle x, Mx\rangle \text{ from definition of eigenvalues and eigenvectors}\\
=& \langle M^* x, x\rangle \text{ from relation } (vi)\\
=& \langle Mx, x\rangle \text{ from (3)}\\
=& \langle \lambda x, x\rangle \text{ from definition of eigenvalues and eigenvectors}\\
=& \lambda^* \langle x, x\rangle \text{ from relation } (v)
\end{aligned} \tag{6}
$$

Finally from (6) and taking into account (*i proof*) we get

$$\lambda\langle x, x\rangle = \lambda^* \langle x, x\rangle \to \lambda = \lambda^* \to \lambda \in \mathbb{R} \tag{7}$$

iv. Find the eigenvectors x_i (corresponding to λ_i) such that are normalized to 1 ($x^\dagger x = 1$). Again using (4.c) we find the eigenvectors of M to be

$$
\begin{aligned}
x_1 &= \begin{pmatrix} \frac{m_1+m_2}{2} + \sqrt{\frac{(m_1+m_2)^2}{4} - \left(m_1 m_2 - |a|^2\right)} - m_2 \\ a^* \end{pmatrix}\\
&= \begin{pmatrix} \frac{m_1-m_2}{2} + \sqrt{\frac{(m_1-m_2)^2}{4} + |a|^2} \\ a^* \end{pmatrix}\\
x_2 &= \begin{pmatrix} \frac{m_1+m_2}{2} - \sqrt{\frac{(m_1+m_2)^2}{4} - \left(m_1 m_2 - |a|^2\right)} - m_2 \\ a^* \end{pmatrix}\\
&= \begin{pmatrix} \frac{m_1-m_2}{2} - \sqrt{\frac{(m_1-m_2)^2}{4} + |a|^2} \\ a^* \end{pmatrix}
\end{aligned} \tag{8}
$$

The above vectors are not normalized, so now we need to find each vector's norm and then divide them by the norm. Taking into account they are complex vectors then the norm is equal to

$$|x| = \sqrt{\langle x, x\rangle} \tag{9}$$

So working with equation (9) we find the norm to be

$$
\begin{aligned}
|x_1| &= \sqrt{\frac{(m_1-m_2)^2}{2} + (m_1-m_2)\sqrt{\frac{(m_1-m_2)^2}{4} + |a|^2} + |a|^2}\\
|x_2| &= \sqrt{\frac{(m_1-m_2)^2}{2} - (m_1-m_2)\sqrt{\frac{(m_1-m_2)^2}{4} + |a|^2} + |a|^2}
\end{aligned} \tag{10}
$$

Finally diving the vectors from (8) by their norms (10) we can normalize them

$$
\begin{aligned}
x_{1 \text{ normalized}} &= \frac{\begin{pmatrix} \frac{m_1-m_2}{2} + \sqrt{\frac{(m_1-m_2)^2}{4} + |a|^2} \\ a^* \end{pmatrix}}{\sqrt{\frac{(m_1-m_2)^2}{2} + (m_1-m_2)\sqrt{\frac{(m_1-m_2)^2}{4} + |a|^2} + |a|^2}} \\
x_{2 \text{ normalized}} &= \frac{\begin{pmatrix} \frac{m_1-m_2}{2} - \sqrt{\frac{(m_1-m_2)^2}{4} + |a|^2} \\ a^* \end{pmatrix}}{\sqrt{\frac{(m_1-m_2)^2}{2} - (m_1-m_2)\sqrt{\frac{(m_1-m_2)^2}{4} + |a|^2} + |a|^2}}
\end{aligned}
\tag{11}
$$

v. Prove that $x_i^* x_j$ $(i \neq j)$ is zero, meaning the eigenvectors are orthonormal. For simplicity lets factor out the the norms of the vectors and just calculate $x_i^* x_j$ and define de vectors as follows

$$
\begin{aligned}
A \equiv& \frac{m_1 - m_2}{2} \\
B \equiv& \sqrt{\frac{(m_1 - m_2)^2}{4} + |a|^2}
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
x_{1 \text{ normalized}} &= \frac{1}{\sqrt{\langle x_1, x_1 \rangle}} \begin{pmatrix} A + B \\ a^* \end{pmatrix} \\
x_{2 \text{ normalized}} &= \frac{1}{\sqrt{\langle x_2, x_2 \rangle}} \begin{pmatrix} A - B \\ a^* \end{pmatrix}
\end{aligned}
\tag{13}
$$

So using the above equations (12) and (13) $x_i^* x_j$ is

$$
\begin{aligned}
\langle x_1, x_2 \rangle =& (A + B)(A - B) + aa^* = A^2 - B^2 + |a|^2 \\
=& \frac{(m_1 - m_2)^2}{4} - \left(\frac{(m_1 - m_2)^2}{4} + |a|^2 \right) + |a|^2 \\
=& \frac{(m_1 - m_2)^2}{4} - \frac{(m_1 - m_2)^2}{4} - |a|^2 + |a|^2 \\
=& \, 0
\end{aligned}
\tag{14}
$$

7. Let a be now real. Let U be delivered as follows

$$
U = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}
\tag{15}
$$

i. Prove that U is unitary ($U^\dagger U = UU^\dagger = 1$). First of we need to compute $U^\dagger$

$$U^\dagger = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \tag{16}$$

Now lets compute $U^\dagger U$

$$\begin{aligned} U^\dagger U &= \begin{pmatrix} \cos(\theta)\cos(\theta) + \sin(\theta)\sin(\theta) & -\cos(\theta)\sin(\theta) + \sin(\theta)\cos(\theta) \\ -\sin(\theta)\cos(\theta) + \cos(\theta)\sin(\theta) & \sin(\theta)\sin(\theta) + \cos(\theta)\cos(\theta) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\theta)^2 + \sin(\theta)^2 & \sin(\theta)\cos(\theta) - \sin(\theta)\cos(\theta) \\ \sin(\theta)\cos(\theta) - \sin(\theta)\cos(\theta) & \sin(\theta)^2 + \cos(\theta)^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{17}$$

ii. Make a transformation (M from 6.) and find the angle θ.

$$\text{diag}(\lambda_1, \lambda_2) = UMU^\dagger \tag{18}$$

First of to simplify the process the above equation can be written as

$$\begin{aligned} \text{diag}(\lambda_1, \lambda_2)U &= UMU^\dagger U \\ \text{diag}(\lambda_1, \lambda_2)U &= UM \end{aligned} \tag{19}$$

Now lets compute both sides of (19)

$$\begin{aligned} \text{diag}(\lambda_1, \lambda_2)U &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1 \cos(\theta) & \lambda_1 \sin(\theta) \\ -\lambda_2 \sin(\theta) & \lambda_2 \cos(\theta) \end{pmatrix} \end{aligned} \tag{20}$$

$$\begin{aligned} UM &= \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} m_1 & a \\ a & m_2 \end{pmatrix} \\ &= \begin{pmatrix} m_1 \cos(\theta) + a\sin(\theta) & a\cos(\theta) + m_2 \sin(\theta) \\ -m_1 \sin(\theta) + a\cos(\theta) & -a\sin(\theta) + m_2 \cos(\theta) \end{pmatrix} \end{aligned} \tag{21}$$

We can now say that if two matrices are equal it means that each term of one matrix is equal the the corresponding term in the second matrix. Taking this into account we can say that each $_{1,1}$term of the matrices are equal, so we have

$$
\begin{aligned}
m_1 \cos(\theta) + a \sin(\theta) &= \lambda_1 \cos(\theta) \\
m_1 + a \tan(\theta) &= \lambda_1 \\
\tan(\theta) &= \frac{\lambda_1 - m_1}{a} \\
\theta = \arctan &\left(\frac{\lambda_1 - m_1}{a}\right)
\end{aligned} \tag{22}
$$

8. Let $\hat{A}$ be an arbitrary (not necessarily hermitian) operator acting in the space of functions. Prove $\hat{O} = i\left(\hat{A} - \hat{A}^\dagger\right)$ is hermitian ($\hat{O} = \hat{O}^\dagger$).

$$
\begin{aligned}
\hat{O}^\dagger &= \left(i\left(\hat{A} - \hat{A}^\dagger\right)\right)^\dagger = \; i^\dagger \left(\hat{A} - \hat{A}^\dagger\right)^\dagger \\
&= \; -i\left(\hat{A}^\dagger - \hat{A}^{\dagger\dagger}\right) = \; -i\left(\hat{A}^\dagger - \hat{A}\right) \\
&= i\left(\hat{A} - \hat{A}^\dagger\right) = \hat{O}
\end{aligned} \tag{23}
$$