Quantum Mechanics I - HW2

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The linear space of finite dimension (vectors) on which matrices act as linear operators is an example of a mathematical space (Hilbert Space) o which some Quantum Mechanics can be done.

5. Let x and φ be n-dimensional vectors (complex). We define the inner product $\langle x, \varphi \rangle$ as

$$\langle x, \varphi \rangle \equiv x^\dagger \varphi = \sum_i x_i^* \varphi_i \tag{1}$$

Where x_i and φ_i are components of the vectors. Prove the following properties which $\langle x, \varphi \rangle$ shares with the inner product defined for functions.

$$\langle x, x \rangle \geq 0 \tag{i}$$
$$\langle x, x \rangle = 0 \iff x = 0 \tag{ii}$$
$$\langle x, \varphi \rangle = \langle \varphi, x \rangle^* \tag{iii}$$
$$\langle x, a\varphi \rangle = a \langle x, \varphi \rangle \tag{iv}$$
$$\langle ax, \varphi \rangle = a^* \langle x, \varphi \rangle \tag{v}$$
$$\langle ax, \varphi \rangle = \langle x, a^* \varphi \rangle \tag{vi}$$
$$\langle x_1 + x_2, \varphi \rangle = \langle x_1, \varphi \rangle + \langle x_2, \varphi \rangle \tag{vii}$$

Proof for *i.*

$$\begin{gathered} \langle x, x \rangle = \sum_i x_i^* x_i = \sum_i |x_i|^2 \\ \text{Since } |x| \geq 0 \rightarrow \sum_i |x_i|^2 \geq 0 \\ \text{It follows } \langle x, x \rangle \geq 0 \end{gathered} \tag{i proof}$$

Proof for *ii.*

$$\begin{gathered} \langle x, x \rangle = \sum_i |x_i|^2 \text{ but } |x| = 0 \iff x = 0 \\ \text{Then it follows } \langle x, x \rangle = 0 \iff x = 0 \end{gathered} \tag{ii proof}$$

Proof for *iii.*

$$\langle x, \varphi \rangle = \sum_i x_i^* \varphi_i = \left(\sum_i x_i^{**} \varphi_i^* \right)^* = \left(\sum_i x_i \varphi_i^* \right)^* = \left(\sum_i \varphi_i^* x_i \right)^* = \langle \varphi, x \rangle^* \tag{iii proof}$$

Proof for *iv.*

$$\langle x, a\varphi\rangle = \sum_i x_i^* a\varphi_i = a\sum_i x_i^*\varphi_i = a\langle x,\varphi\rangle \tag{iv proof}$$

Proof for *v.*

$$\langle ax, \varphi\rangle = \sum_i (ax_i)^*\varphi_i = \sum_i a^* x_i^*\varphi_i = a^*\sum_i x_i^*\varphi_i = a^*\langle x,\varphi\rangle \tag{v proof}$$

Proof for *vi.*

$$\langle ax, \varphi\rangle = \sum_i (ax_i)^*\varphi_i = \sum_i a^* x_i^*\varphi_i = \sum_i x_i^*(a^*\varphi_i) = \langle x, a^*\varphi\rangle \tag{vi proof}$$

Proof for *vii.*

$$\begin{aligned}\langle x_1 + x_2, \varphi\rangle &= \sum_i (x_1+x_2)^*\varphi = \sum_i (x_{1i}^* + x_{2i}^*)\varphi = \sum_i \left(x_{1i}^*\varphi_i + x_{2i}^*\varphi_i\right)\\ &= \sum_i x_{1i}^*\varphi + \sum_i x_{2i}^*\varphi = \langle x_1,\varphi\rangle + \langle x_2,\varphi\rangle\end{aligned} \tag{vii proof}$$

6. Let us write from now on in two dimensional vector space. Let M be

$$M = \begin{pmatrix} m_1 & a \\ a^* & m_2\end{pmatrix}, \text{ with } m_{i=1,2}\in\mathbb{R} \tag{2}$$

i. Prove that $M = M^\dagger$ (M is hermitian) and $M^\dagger = (M^T)^*$

$$M^\dagger = \begin{pmatrix} m_1 & a^* \\ a & m_2\end{pmatrix}^* = \begin{pmatrix} m_1 & a \\ a^* & m_2\end{pmatrix} = M \tag{3}$$

ii. Find the eigenvalues λ_i of $Mx = \lambda x$, where x is a 2-dimensional vector. For any given square 2×2 of the form

$$M = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \tag{4.a}$$

We can define the eigenvalues and eigenvectors, respectively, as follows

$$\lambda_i = \frac{a+d}{2} \pm \sqrt{\frac{(a+d)^2}{4} - (ad-bc)} \tag{4.b}$$

$$x_i = \begin{pmatrix}\lambda_i - d \\ c\end{pmatrix} \tag{4.c}$$

Now using (4.b) we find the eigenvalues for M to be

$$\begin{aligned}\lambda_1 =& \frac{m_1+m_2}{2} + \sqrt{\frac{(m_1+m_2)^2}{4} - \left(m_1m_2 - |a|^2\right)}\\ \lambda_2 =& \frac{m_1+m_2}{2} - \sqrt{\frac{(m_1+m_2)^2}{4} - \left(m_1m_2 - |a|^2\right)}\end{aligned} \tag{5}$$

iii. Prove that λ_i are real

$$
\begin{aligned}
\lambda\langle x, x\rangle =&\langle x, \lambda x\rangle \text{ from relation } (iv)\\
=&\langle x, Mx\rangle \text{ from definition of eigenvalues and eigenvectors}\\
=&\langle M^* x, x\rangle \text{ from relation } (vi)\\
=&\langle Mx, x\rangle \text{ from } (3)\\
=&\langle \lambda x, x\rangle \text{ from definition of eigenvalues and eigenvectors}\\
=&\lambda^*\langle x, x\rangle \text{ from relation } (v)
\end{aligned}
\tag{6}
$$

Finally from (6) and taking into account (*i proof*) we get

$$
\lambda\langle x, x\rangle = \lambda^*\langle x, x\rangle \rightarrow \lambda = \lambda^* \rightarrow \lambda \in \mathbb{R} \tag{7}
$$

iv. Find the eigenvectors x_i (each corresponding to λ_i) such that are normalized to 1. Again using (4.c) we find the eigenvectors of M to be

$$
\begin{aligned}
x_1 =& \begin{pmatrix} \frac{m_1+m_2}{2} + \sqrt{\frac{(m_1+m_2)^2}{4} - \left(m_1 m_2 - |a|^2\right)} - m_2 \\ a^* \end{pmatrix}\\
x_2 =& \begin{pmatrix} \frac{m_1+m_2}{2} - \sqrt{\frac{(m_1+m_2)^2}{4} - \left(m_1 m_2 - |a|^2\right)} - m_2 \\ a^* \end{pmatrix}
\end{aligned}
\tag{8}
$$

v. Prove that $x_i^* x_j$ $(i \neq j)$ is zero, meaning the eigenvectors are orthonormal.