A FIRST COURSE IN LINEAR ALGEBRA

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Spectral Theory: Eigenvalues and Eigenvectors

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Adapted by

LYRYX SERVICE COURSE SOLUTION

Definition of Eigenvalues and Eigenvectors

Eigenvalues and Eigenvectors

Definition

Let A be an $n \times n$ matrix, λ a real number, and $X \neq 0$ an n-vector. If $AX = \lambda X$, then λ is an eigenvalue of A, and X is an eigenvector of A corresponding to λ, or a λ-eigenvector.

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Example

Let $A = \left[\begin{array}{cc} 1 & 2 \\ 1 & 2 \end{array} \right]$ and $X = \left[\begin{array}{c} 1 \\ 1 \end{array} \right]$. Then

$$AX = \left[\begin{array}{cc} 1 & 2 \\ 1 & 2 \end{array} \right] \left[\begin{array}{c} 1 \\ 1 \end{array} \right] = \left[\begin{array}{c} 3 \\ 3 \end{array} \right] = 3 \left[\begin{array}{c} 1 \\ 1 \end{array} \right] = 3X.$$

This means that 3 is an eigenvalue of A, and $\left[\begin{array}{c} 1 \\ 1 \end{array} \right]$ is an eigenvector of A corresponding to 3 (or a 3-eigenvector of A).

The Characteristic Polynomial

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Then

$$\begin{aligned} \lambda X - AX &= 0 \\ \lambda IX - AX &= 0 \\ (\lambda I - A)X &= 0 \end{aligned}$$

Suppose $\lambda I - A$ has an inverse. Then,

$$\begin{aligned} X &= IX \\ &= ((\lambda I - A)^{-1}(\lambda I - A))X \\ &= (\lambda I - A)^{-1}((\lambda I - A)X) \\ &= (\lambda I - A)^{-1}0 \\ &= 0 \end{aligned}$$

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$$\det(\lambda I - A) = 0.$$

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Theorem

Let A be an $n \times n$ matrix.

1. *The eigenvalues of A are the roots of $c_A(x)$.*
2. *The λ-eigenvectors X are the nontrivial solutions to $(\lambda I - A)X = 0$.*

Characteristic Polynomial

Example

The characteristic polynomial of $A = \left[\begin{array}{rr} 4 & -2 \\ -1 & 3 \end{array}\right]$ is

$$\begin{aligned}
c_A(x) &= \det\left(\left[\begin{array}{cc} x & 0 \\ 0 & x \end{array}\right] - \left[\begin{array}{rr} 4 & -2 \\ -1 & 3 \end{array}\right]\right) \\
&= \det\left[\begin{array}{cc} x-4 & 2 \\ 1 & x-3 \end{array}\right] \\
&= (x-4)(x-3) - 2 \\
&= x^2 - 7x + 10
\end{aligned}$$

Basic Eigenvectors

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A basic eigenvector of an $n \times n$ matrix A is any nonzero multiple of a basic solution to $(\lambda I - A)X = 0$, where λ is an eigenvalue of A.

Note: Any (nonzero) linear combination of basic eigenvectors of A is again an eigenvector of A.

Finding Eigenvalues and Eigenvectors

Multiplicity of an Eigenvalue

Recall that the eigenvalues of A are the roots of the characteristic polynomial given by $c_A(x) = \det(xI - A)$.

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The multiplicity of an eigenvalue λ of A is the number of times λ occurs as a root of $c_A(x)$.

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- **Eigenvalues:** Find λ by solving the equation

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- **Check:** For each pair of λ, X check that $AX = \lambda X$.

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$$c_A(x) = x^2 - 7x + 10 = (x-2)(x-5)$$

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We want to find the eigenvalues (λ) and eigenvectors (X) of A.

Eigenvalues: To find λ, solve $c_A(x) = 0$

$$c_A(x) = x^2 - 7x + 10 = (x-2)(x-5) = 0$$

Example (continued)

Therefore the roots are found by

$$x - 2 = 0 \rightarrow x = 2$$

$$x - 5 = 0 \rightarrow x = 5$$

A has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 5$. They each have multiplicity equal to 1.

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Eigenvectors: To find the basic eigenvector of A corresponding to $\lambda_1 = 2$, solve $(2I - A)X = 0$. The matrix $2I - A$ is given by

$$\left[\begin{array}{rr} 2 & 0 \\ 0 & 2 \end{array}\right] - \left[\begin{array}{rr} 4 & -2 \\ -1 & 3 \end{array}\right] = \left[\begin{array}{rr} -2 & -2 \\ 1 & -2 \end{array}\right]$$

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Solving the system $(2I - A)X = 0$ is done as follows.

$$\left[\begin{array}{rr|r} -2 & 2 & 0 \\ 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rr|r} 1 & -1 & 0 \\ -2 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rr|r} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Example (continued)

The general solution, in parametric form, is

$$X = \left[\begin{array}{c} t \\ t \end{array} \right] = t \left[\begin{array}{c} 1 \\ 1 \end{array} \right] \text{ where } t \in \mathbb{R}.$$

Example (continued)

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Therefore, the basic eigenvector of A associated with $\lambda_1 = 2$ is

$$X_1 = \left[\begin{array}{c} 1 \\ 1 \end{array} \right]$$

Example (continued)

The general solution, in parametric form, is

$$X = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ where } t \in \mathbb{R}.$$

Therefore, the basic eigenvector of A associated with $\lambda_1 = 2$ is

$$X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

To find the basic eigenvector of A corresponding to $\lambda_2 = 5$, solve $(5I - A)X = 0$:

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Example (continued)

The general solution, in parametric form, is

$$X = \left[\begin{array}{r} -2s \\ s \end{array} \right] = s \left[\begin{array}{r} -2 \\ 1 \end{array} \right] \text{ where } s \in \mathbb{R}.$$

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Therefore the basic eigenvector corresponding to $\lambda_2 = 5$ is

$$X_2 = \left[\begin{array}{r} -2 \\ 1 \end{array} \right]$$

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Therefore the basic eigenvector corresponding to $\lambda_2 = 5$ is

$$X_2 = \left[\begin{array}{r} -2 \\ 1 \end{array} \right]$$

Check: Check that $AX_1 = \lambda_1 X_1$.

$$\left[\begin{array}{rr} 4 & -2 \\ -1 & 3 \end{array} \right] \left[\begin{array}{r} 1 \\ 1 \end{array} \right] = \left[\begin{array}{r} 2 \\ 2 \end{array} \right] = 2 \left[\begin{array}{r} 1 \\ 1 \end{array} \right]$$

You can also check that $AX_2 = \lambda_2 X_2$.

Another Example

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For $A = \begin{bmatrix} 3 & -4 & 2 \\ 1 & -2 & 2 \\ 1 & -5 & 5 \end{bmatrix}$, find $c_A(x)$ and the eigenvalues and eigenvectors of A.

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Solution: Eigenvalues:

$$\det(xI - A) = \left|\begin{array}{ccc} x-3 & 4 & -2 \\ -1 & x+2 & -2 \\ -1 & 5 & x-5 \end{array}\right| = \left|\begin{array}{ccc} x-3 & 4 & -2 \\ 0 & x-3 & -x+3 \\ -1 & 5 & x-5 \end{array}\right|$$

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$$\det(xI - A) = \left| \begin{array}{ccc} x-3 & 4 & -2 \\ -1 & x+2 & -2 \\ -1 & 5 & x-5 \end{array} \right| = \left| \begin{array}{ccc} x-3 & 4 & -2 \\ 0 & x-3 & -x+3 \\ -1 & 5 & x-5 \end{array} \right|$$

$$= \left| \begin{array}{ccc} x-3 & 4 & 2 \\ 0 & x-3 & 0 \\ -1 & 5 & x \end{array} \right| = (x-3) \left| \begin{array}{cc} x-3 & 2 \\ -1 & x \end{array} \right|$$

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$$\det(xI - A) = \begin{vmatrix} x-3 & 4 & -2 \\ -1 & x+2 & -2 \\ -1 & 5 & x-5 \end{vmatrix} = \begin{vmatrix} x-3 & 4 & -2 \\ 0 & x-3 & -x+3 \\ -1 & 5 & x-5 \end{vmatrix}$$

$$= \begin{vmatrix} x-3 & 4 & 2 \\ 0 & x-3 & 0 \\ -1 & 5 & x \end{vmatrix} = (x-3)\begin{vmatrix} x-3 & 2 \\ -1 & x \end{vmatrix}$$

$$c_A(x) = (x-3)(x^2 - 3x + 2) = (x-3)(x-2)(x-1)$$

Example (continued)

Solving for the roots of $c_A(x)$, we find that the eigenvalues of A are $\lambda_1 = 3, \lambda_2 = 2$, and $\lambda_3 = 1$. All these eigenvalues have multiplicity of 1.

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Eigenvectors: Find the eigenvector corresponding to $\lambda_1 = 3$: solve $(3I - A)X = 0$.

$$\left[\begin{array}{rrr|r} 0 & 4 & -2 & 0 \\ -1 & 5 & -2 & 0 \\ -1 & 5 & -2 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

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Thus $X = \left[\begin{array}{r} \frac{1}{2}t \\ \frac{1}{2}t \\ t \end{array}\right] = t\left[\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{array}\right]$, $t \in \mathbb{R}$. The basic eigenvector is $\left[\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{array}\right]$.

Recall that any nonzero multiple of a basic eigenvector is again an eigenvector. Choosing $t = 2$ gives us $X_1 = \left[\begin{array}{c} 1 \\ 1 \\ 2 \end{array}\right]$ as an eigenvector corresponding to $\lambda_1 = 3$.

Example (continued)

Find the eigenvector corresponding to $\lambda_2 = 2$: solve $(2I - A)X = 0$.

$$\left[\begin{array}{rrr|r} -1 & 4 & -2 & 0 \\ -1 & 4 & -2 & 0 \\ -1 & 5 & -3 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Example (continued)

Find the eigenvector corresponding to $\lambda_2 = 2$: solve $(2I - A)X = 0$.

$$\left[\begin{array}{rrr|r} -1 & 4 & -2 & 0 \\ -1 & 4 & -2 & 0 \\ -1 & 5 & -3 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $X = \left[\begin{array}{r} 2s \\ s \\ s \end{array}\right] = s\left[\begin{array}{r} 2 \\ 1 \\ 1 \end{array}\right]$, $s \in \mathbb{R}$.

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Thus $X = \left[\begin{array}{r} 2s \\ s \\ s \end{array}\right] = s \left[\begin{array}{r} 2 \\ 1 \\ 1 \end{array}\right]$, $s \in \mathbb{R}$.

The basic eigenvector is $\left[\begin{array}{r} 2 \\ 1 \\ 1 \end{array}\right]$. Again, we can take a multiple of this.

Choosing $s = 3$ gives us $X_2 = \left[\begin{array}{r} 6 \\ 3 \\ 3 \end{array}\right]$ as an eigenvector corresponding to $\lambda_2 = 2$.

Example (continued)

Basic eigenvectors corresponding to $\lambda_3 = 1$: solve $(I - A)X = 0$.

$$\left[\begin{array}{rrr|r} -2 & 4 & -2 & 0 \\ -1 & 3 & -2 & 0 \\ -1 & 5 & -4 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Example (continued)

Basic eigenvectors corresponding to $\lambda_3 = 1$: solve $(I - A)X = 0$.

$$\left[\begin{array}{rrr|r} -2 & 4 & -2 & 0 \\ -1 & 3 & -2 & 0 \\ -1 & 5 & -4 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $X = \left[\begin{array}{c} r \\ r \\ r \end{array}\right] = r \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$, $r \in \mathbb{R}$.

Example (continued)

Basic eigenvectors corresponding to $\lambda_3 = 1$: solve $(I - A)X = 0$.

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Thus $X = \left[\begin{array}{c} r \\ r \\ r \end{array}\right] = r \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$, $r \in \mathbb{R}$.

The basic eigenvector is $\left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$. Choosing $r = 4$, we have that

$X_3 = \left[\begin{array}{c} 4 \\ 4 \\ 4 \end{array}\right]$ is an eigenvector corresponding to $\lambda_3 = 1$.

Example (continued)

Basic eigenvectors corresponding to $\lambda_3 = 1$: solve $(I - A)X = 0$.

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Thus $X = \left[\begin{array}{c} r \\ r \\ r \end{array}\right] = r \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$, $r \in \mathbb{R}$.

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$X_3 = \left[\begin{array}{c} 4 \\ 4 \\ 4 \end{array}\right]$ is an eigenvector corresponding to $\lambda_3 = 1$.

Remember that you can (and should!) check your work!

Eigenvalues and Eigenvectors for Special Types of Matrices

Similar Matrices

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How do similar matrices help us in spectral theory?

Suppose A and B are similar matrices. Then A and B have the same eigenvalues.

Using Similar and Elementary Matrices

Example

Find the eigenvalues for the matrix

$$A = \left[\begin{array}{rrr} 33 & 105 & 105 \\ 10 & 28 & 30 \\ -20 & -60 & -62 \end{array} \right]$$

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$$A = \begin{bmatrix} 33 & 105 & 105 \\ 10 & 28 & 30 \\ -20 & -60 & -62 \end{bmatrix}$$

Solution: We will use elementary matrices to simplify A before finding the eigenvalues. Left multiply A by $E(2,2)$, and right multiply by the inverse of $E(2,2)$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 33 & 105 & 105 \\ 10 & 28 & 30 \\ -20 & -60 & -62 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 33 & -105 & 105 \\ 10 & -32 & 30 \\ 0 & 0 & -2 \end{bmatrix}$$

Notice that the resulting matrix and A are similar matrices (with $E(2,2)$ playing the role of P) so they have the same eigenvalues.

Example (continued)

We do this step again, on the resulting matrix above.

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 33 & -105 & 105 \\ 10 & -32 & 30 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 15 \\ 10 & -2 & 30 \\ 0 & 0 & -2 \end{bmatrix} = B$$

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We do this step again, on the resulting matrix above.

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Again by properties of similar matrices, the resulting matrix here (labeled B) has the same eigenvalues as our original matrix A. The advantage is that it is much simpler to find the eigenvalues of B than A.

Finding these eigenvalues follows the usual procedure and is left as an exercise.

Eigenvalues for a Triangular Matrix

Recall that the determinant of a triangular matrix is found by multiplying the entries on the main diagonal. Similarly, there is a “shortcut” to finding eigenvalues of a triangular matrix.

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Let

$$A = \begin{bmatrix} 4 & 5 & -44 \\ 0 & 3 & 176 \\ 0 & 0 & 2 \end{bmatrix}$$

Find the eigenvalues of A.

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Recall that the determinant of a triangular matrix is found by multiplying the entries on the main diagonal. Similarly, there is a "shortcut" to finding eigenvalues of a triangular matrix.

Example

Let

$$A = \begin{bmatrix} 4 & 5 & -44 \\ 0 & 3 & 176 \\ 0 & 0 & 2 \end{bmatrix}$$

Find the eigenvalues of A.

Solution: As usual, we need to solve $\det(xI - A) = 0$.

$$\det(xI - A) = \det \begin{bmatrix} x-4 & -5 & 44 \\ 0 & x-3 & -176 \\ 0 & 0 & x-2 \end{bmatrix} = (x-4)(x-3)(x-2) = 0$$

Example (continued)

$$(x-4)(x-3)(x-2)=0$$

Therefore the eigenvalues of A are $\lambda_1 = 4, \lambda_2 = 3$, and $\lambda_3 = 2$.

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Notice that these are the entries of the main diagonal of A!

Example (continued)

$$(x - 4)(x - 3)(x - 2) = 0$$

Therefore the eigenvalues of A are $\lambda_1 = 4, \lambda_2 = 3$, and $\lambda_3 = 2$.

Notice that these are the entries of the main diagonal of A!

This is true in general! The eigenvalues of a triangular matrix A are exactly the entries of the main diagonal of A.

Geometric Intepretation of Eigenvalues and Eigenvectors

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How does the linear transformation affect the eigenvectors of the matrix?

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Let V be a nonzero vector in $\mathbb{R}^2$. We denote by L_V the unique line in $\mathbb{R}^2$ that contains V and the origin.

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Let V be a nonzero vector in $\mathbb{R}^2$. We denote by L_V the unique line in $\mathbb{R}^2$ that contains V and the origin.

Lemma

Let $V = \begin{bmatrix} a \\ b \end{bmatrix}$ be a nonzero vector in $\mathbb{R}^2$. Then L_V is the set of all scalar multiples of V, i.e.,

$$L_V = \mathbb{R}V = \{tV \mid t \in \mathbb{R}\}.$$

Definition

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Let A be a 2×2 matrix and let $V \neq 0$ be a vector in $\mathbb{R}^2$. Then L_V is A-invariant if and only if V is an eigenvector of A.

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Theorem

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This theorem provides a geometrical method for finding the eigenvectors of a 2×2 matrix.

Geometric Example

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Let $m \in \mathbb{R}$ and consider the linear transformation $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$, i.e., reflection in the line $y = mx$.

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Claim. $X_1 = \left[\begin{array}{c} 1 \\ m \end{array} \right]$ is an eigenvector of A corresponding to eigenvalue $\lambda = 1$.
The reason for this: $X_1 = \left[\begin{array}{c} 1 \\ m \end{array} \right]$ lies in the line $y = mx$, and hence

$$Q_m \left[\begin{array}{c} 1 \\ m \end{array} \right] = \left[\begin{array}{c} 1 \\ m \end{array} \right], \text{ implying that } A \left[\begin{array}{c} 1 \\ m \end{array} \right] = 1 \left[\begin{array}{c} 1 \\ m \end{array} \right].$$

Example (continued)

More generally, any vector $\begin{bmatrix} k \\ km \end{bmatrix}$, $k \neq 0$, lies in the line $y = mx$ and is an eigenvector of A.

Another way of saying this is that the line $y = mx$ is A-invariant for the matrix

$$A = \frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}.$$

Another Geometric Example

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Let θ be a real number, and $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ rotation through an angle of θ, induced by the matrix

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The reason for this: a line L in $\mathbb{R}^2$ is A invariant if and only if θ is an integer multiple of π.