O3FESA

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version 0.1
December 2, 2023

1 What is O3FESA?

O3FESA (**O**pen-source **O**bject-**O**riented **F**inite **E**lement **S**hell **A**nalysis) is a software written in `C++` using the objected-oriented programming paradigm.

2 Shell theories

2.1 First-order shear deformation theory

2.1.1 Displacement field

$$u_1(x, y, z, t) = u(x, y, t) + z\phi_x(x, y, t) \tag{2.1a}$$

$$u_2(x, y, z, t) = v(x, y, t) + z\phi_y(x, y, t) \tag{2.1b}$$

$$u_3(x, y, z, t) = w(x, y, t) \tag{2.1c}$$

The displacement vector is defined as

$$q = \begin{Bmatrix} u \\ v \\ w \\ \phi_x \\ \phi_y \end{Bmatrix} \tag{2.2}$$

2.1.2 Strain

The Green-Lagrange strain tensor is given by:

$$E_{jk} = \frac{1}{2}\left(\frac{\partial u_j}{\partial X_k} + \frac{\partial u_k}{\partial X_j} + \frac{\partial u_m}{\partial X_j}\frac{\partial u_m}{\partial X_k}\right), \qquad j,k,l = 1,2,3 \tag{2.3}$$

If the rotation of transverse normals are moderate, the strain-displacement relations in Eq. (2.3) simplifies to von Kármán strains:

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} + \phi_x \\ \frac{\partial w}{\partial x} + \phi_y \end{Bmatrix} + z\begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \\ 0 \\ 0 \end{Bmatrix} \tag{2.4}$$

or in a compact form

$$\epsilon = \epsilon^0 + z\epsilon^1 \tag{2.5}$$

2.1.3 Principle of Virtual Work

The Principle of Virtual Work (PVW) states that the sum of internal and external virtual works must be zero:

$$\delta W^{\text{int}} + \delta W^{\text{ext}} = 0 \tag{2.6}$$

The virtual internal work is related to the virtual strain energy, given by

$$\delta W^{\text{int}} = \int_V \delta\epsilon^T \sigma \, \mathrm{d}V \tag{2.7}$$

The stress vector, σ, is defined as

$$\sigma = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = C\left(\epsilon - \epsilon_0\right) \tag{2.8}$$

in which C is the material constitutive matrix and ϵ_0 is initial deformation vector. These relations are defined as

$$C = \begin{bmatrix} \bar{Q}_m & 0 \\ 0 & \bar{Q}_s \end{bmatrix} \tag{2.9}$$

and

$$\epsilon_0 = \begin{Bmatrix} \alpha \\ 0 \end{Bmatrix} \Delta T \tag{2.10}$$

Substituting Eqs. (2.5) in Eq. (2.7) yields

$$\delta W^{\text{int}} = \int_\Omega \int_{-h/2}^{h/2} \left(\delta\epsilon^T \sigma + z\delta\epsilon^T \sigma \right) \mathrm{d}z \, \mathrm{d}\Omega \tag{2.11}$$

Integrating (2.5) through the thickness, one obtains

$$\delta W^{\text{int}} = \int_\Omega \left(\delta\epsilon_m^T N + \delta\epsilon_b^T M + \delta\epsilon_s^T Q \right) \mathrm{d}\Omega \tag{2.12}$$

where the strains components, stress and thermal resultants, respectively

$$\epsilon_m = \begin{Bmatrix} \dfrac{\partial u}{\partial x} + \dfrac{1}{2}\dfrac{\partial w}{\partial x}^2 \\ \dfrac{\partial v}{\partial y} + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2 \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} + \dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} \end{Bmatrix} \tag{2.13a}$$

$$\epsilon_b = \begin{Bmatrix} \dfrac{\partial \phi_x}{\partial x} \\ \dfrac{\partial \phi_y}{\partial y} \\ \dfrac{\partial \phi_x}{\partial y} + \dfrac{\partial \phi_y}{\partial x} \end{Bmatrix} \tag{2.13b}$$

$$\epsilon_s = \begin{Bmatrix} \dfrac{\partial w}{\partial x} + \phi_x \\ \dfrac{\partial w}{\partial x} + \phi_y \end{Bmatrix} \tag{2.13c}$$

$$\begin{Bmatrix} N \\ M \\ Q \end{Bmatrix} = \begin{bmatrix} A & B & 0 \\ B & D & 0 \\ 0 & 0 & A_s \end{bmatrix} \begin{Bmatrix} \epsilon_m \\ \epsilon_b \\ \epsilon_s \end{Bmatrix} - \begin{Bmatrix} N_T \\ M_T \\ 0 \end{Bmatrix} = \hat{C}\hat{\epsilon} - \hat{\Psi} \tag{2.14}$$

The membrane strain, Eq. (2.13a), can be considered as a linear and nonlinear parts

$$\epsilon_m = \epsilon_m^l + \epsilon_m^{nl} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} + \begin{Bmatrix} \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2 \\ \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2 \\ \dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} \end{Bmatrix} \tag{2.15}$$

Furthermore, the strains in Eq. (2.13) can written as a function of a derivative operators, that is,

$$\epsilon_m^l = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 & 0 & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \\ \phi_x \\ \phi_y \end{Bmatrix} = \mathcal{L}_m q \tag{2.16a}$$

$$\epsilon_m^{nl} = \frac{1}{2} \begin{bmatrix} \frac{\partial w}{\partial x} & 0 \\ 0 & \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial y} & \frac{\partial w}{\partial x} \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial y} & 0 & 0 \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \\ \phi_x \\ \phi_y \end{Bmatrix} = \frac{1}{2} \Theta \mathcal{L}_\theta q \tag{2.16b}$$

$$\epsilon_b = \begin{bmatrix} 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial y} \\ 0 & 0 & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \\ \phi_x \\ \phi_y \end{Bmatrix} = \mathcal{L}_b q \tag{2.16c}$$

$$\epsilon_s = \begin{bmatrix} 0 & 0 & \frac{\partial}{\partial x} & 1 & 0 \\ 0 & 0 & \frac{\partial}{\partial y} & 0 & 1 \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \\ \phi_x \\ \phi_y \end{Bmatrix} = \mathcal{L}_s q \tag{2.16d}$$

With exception of Eq. (2.16b), that depends on the displacement, the first variation of Eq. (2.16) has the form:

$$\delta \epsilon_* = \mathcal{L}_* \delta q \tag{2.17}$$

while

$$\delta \epsilon_m^{nl} = \Theta \mathcal{L}_\theta \delta q \tag{2.18}$$

Now, the virtual internal work can now be written as

$$\delta W^{\text{int}} = \int_\Omega \delta \hat{\epsilon}^T \hat{\sigma} \mathrm{d}\Omega \tag{2.19}$$

in which

$$\delta\hat{\epsilon} = \begin{bmatrix} \mathcal{L}_m \\ \mathcal{L}_b \\ \mathcal{L}_s \end{bmatrix} \delta q + \begin{bmatrix} \Theta\mathcal{L}_\theta \\ 0 \\ 0 \end{bmatrix} \delta q \tag{2.20a}$$

$$\hat{\sigma} = \begin{Bmatrix} N \\ M \\ Q \end{Bmatrix} \tag{2.20b}$$

The virtual external work can be defined as

$$\delta W^{\text{ext}} = -\left(\int_\Omega \delta q^T f \; \mathrm{d}\Omega + \delta q^T g \right) \tag{2.21}$$

where the first term represents the external load due to loading acting on and area Ω and second one the work done by concentrated loading, where f and g are the distributed load vector and nodal point load vector, respectively. The PVW may be now defined as

$$\int_\Omega \delta\hat{\epsilon}^T \hat{\sigma} \mathrm{d}\Omega = \int_\Omega \delta q^T f \; \mathrm{d}\Omega + \sum_j \delta q^j g^j \tag{2.22}$$

2.1.4 Finite element discretization

The displacement field within an element, Eq. (2.2), is given by

$$q^{\mathcal{E}} = \hat{N} \hat{q}^{\mathcal{E}} \tag{2.23}$$

where

$$\hat{N} = \begin{bmatrix} \tilde{N} & \tilde{0} & \tilde{0} & \tilde{0} & \tilde{0} \\ \tilde{0} & \tilde{N} & \tilde{0} & \tilde{0} & \tilde{0} \\ \tilde{0} & \tilde{0} & \tilde{N} & \tilde{0} & \tilde{0} \\ \tilde{0} & \tilde{0} & \tilde{0} & \tilde{N} & \tilde{0} \\ \tilde{0} & \tilde{0} & \tilde{0} & \tilde{0} & \tilde{N} \end{bmatrix} \tag{2.24}$$

in which $\tilde{N} = [\tilde{N}_1, \tilde{N}_2, \cdots, \tilde{N}_n]$ is the shape functions vector, $\tilde{0} = [0]_n$ is a null vector with n elements, in which n is the number of nodes. The local element displacement vector $\hat{q}^{\mathcal{E}}$ is given by

$$\hat{q}^{\mathcal{E}} = \begin{Bmatrix} \hat{u}^{\mathcal{E}} \\ \hat{v}^{\mathcal{E}} \\ \hat{w}^{\mathcal{E}} \\ \hat{\phi}_x^{\mathcal{E}} \\ \hat{\phi}_y^{\mathcal{E}} \end{Bmatrix} \tag{2.25}$$

where, for the sake of generality, $\hat{\gamma} = [\gamma_1, \gamma_2, \cdots, \gamma_n]^T$ are nodal displacements. Substituting Eq. (2.23) in Eq. (2.22) yields

$$\delta\hat{q}^{\mathcal{E}T} \left(\int_{\Omega^{\mathcal{E}}} \boldsymbol{B}^{\mathcal{E}T} \hat{\sigma}^{\mathcal{E}} \, \mathrm{d}\Omega^{\mathcal{E}} - \hat{f}^{\mathcal{E}} \right) = 0 \tag{2.26}$$

where $\boldsymbol{B}^{\mathcal{E}}$ is the strain-displacement matrix

$$\boldsymbol{B}^{\mathcal{E}} = \begin{bmatrix} \boldsymbol{B}_m \\ \boldsymbol{B}_b \\ \boldsymbol{B}_s \end{bmatrix} + \begin{bmatrix} \Theta^{\mathcal{E}} \boldsymbol{B}_\theta \\ 0 \\ 0 \end{bmatrix} = \boldsymbol{B}^l + \boldsymbol{B}^{nl\,\mathcal{E}} \tag{2.27}$$

$\hat{\sigma}^{\mathcal{E}}$ is the resultant internal forces

$$\hat{\sigma}^{\mathcal{E}} = \hat{\boldsymbol{C}}^{\mathcal{E}} \left(\boldsymbol{B}^l + \frac{1}{2} \boldsymbol{B}^{nl\,\mathcal{E}} \right) \hat{q}^{\mathcal{E}} - \Psi^{\mathcal{E}} \tag{2.28}$$

and $\hat{f}^{\mathcal{E}}$ is the equivalent nodal load vector

$$\hat{f}^{\mathcal{E}} = \int_{\Omega^{\mathcal{E}}} \hat{N} f^{\mathcal{E}} \, \mathrm{d}\Omega^{\mathcal{E}} \tag{2.29}$$

With regard to Eq. (2.16b), one notices that $\Theta^{\mathcal{E}}$ depends on the displacement within each element, and so the nonlinear part of the strain-displacement matrix.

The strain-displacement matrix $\boldsymbol{B}^{\mathcal{E}}$ is written in terms of its membrane component $\boldsymbol{B}_m$,

$$\boldsymbol{B}_m = \begin{bmatrix} \dfrac{\partial \tilde{N}}{\partial x} & 0 & 0 & 0 & 0 \\ 0 & \dfrac{\partial \tilde{N}}{\partial y} & 0 & 0 & 0 \\ \dfrac{\partial \tilde{N}}{\partial y} & \dfrac{\partial \tilde{N}}{\partial x} & 0 & 0 & 0 \end{bmatrix} \tag{2.30}$$

bending component $\boldsymbol{B}_b$

$$\boldsymbol{B}_b = \begin{bmatrix} 0 & 0 & 0 & \dfrac{\partial \tilde{N}}{\partial x} & 0 \\ 0 & 0 & 0 & 0 & \dfrac{\partial \tilde{N}}{\partial y} \\ 0 & 0 & 0 & \dfrac{\partial \tilde{N}}{\partial y} & \dfrac{\partial \tilde{N}}{\partial x} \end{bmatrix} \tag{2.31}$$

shear component $\boldsymbol{B}_s$

$$\boldsymbol{B}_s = \begin{bmatrix} 0 & 0 & \dfrac{\partial \tilde{N}}{\partial x} & N & 0 \\ 0 & 0 & \dfrac{\partial \tilde{N}}{\partial y} & 0 & N \end{bmatrix} \tag{2.32}$$

and gradient component $\boldsymbol{B}_\theta$

$$\boldsymbol{B}_\theta = \begin{bmatrix} 0 & 0 & \dfrac{\partial \tilde{N}}{\partial x} & 0 & 0 \\ 0 & 0 & \dfrac{\partial \tilde{N}}{\partial y} & 0 & 0 \end{bmatrix} \tag{2.33}$$

Thus, the nonlinear equilibrium equation within the element become

$$\psi^{\mathcal{E}} = \int_{\Omega^{\mathcal{E}}} \boldsymbol{B}^{\mathcal{E}T} \hat{\sigma}^{\mathcal{E}} \, \mathrm{d}\Omega^{\mathcal{E}} - \hat{f}^{\mathcal{E}} = 0 \tag{2.34}$$

which is also know as residual. The first term of Eq. (2.34) can be defined as

$$\int_{\Omega^{\mathcal{E}}} \boldsymbol{B}^{\mathcal{E}T} \hat{\sigma}^{\mathcal{E}} \, \mathrm{d}\Omega^{\mathcal{E}} = \boldsymbol{K}^{\mathcal{E}} \hat{q}^{\mathcal{E}} - \hat{\Psi}^{\mathcal{E}} \tag{2.35}$$

where $\boldsymbol{K}^{\mathcal{E}}$ is the standard stiffness matrix

$$\boldsymbol{K}^{\mathcal{E}} = \int_{\Omega^{\mathcal{E}}} \left(\boldsymbol{B}^{l} + \boldsymbol{B}^{nl\,\mathcal{E}} \right)^{T} \hat{\boldsymbol{C}}^{\mathcal{E}} \left(\boldsymbol{B}^{l} + \frac{1}{2} \boldsymbol{B}^{nl\,\mathcal{E}} \right) \, \mathrm{d}\Omega^{\mathcal{E}} \tag{2.36}$$

and $\hat{\Psi}^{\mathcal{E}}$ is a thermal load vector

$$\hat{\Psi}^{\mathcal{E}} = \int_{\Omega^{\mathcal{E}}} \left(\boldsymbol{B}^{l} + \boldsymbol{B}^{nl\,\mathcal{E}} \right)^{T} \Psi^{\mathcal{E}} \, \mathrm{d}\Omega^{\mathcal{E}} \tag{2.37}$$

The stiffness matrix can be rewritten as a sum of linear and nonlinear stiffness matrices:

$$K^{\mathcal{E}} = K^{l\mathcal{E}} + K^{nl\mathcal{E}} \tag{2.38}$$

where

$$\begin{aligned} K^{l\mathcal{E}} = & \int_{\Omega^{\mathcal{E}}} B_m^T A^{\mathcal{E}} B_m \,\mathrm{d}\Omega^{\mathcal{E}} + \int_{\Omega^{\mathcal{E}}} B_m^T B^{\mathcal{E}} B_b \,\mathrm{d}\Omega^{\mathcal{E}} + \int_{\Omega^{\mathcal{E}}} B_b^T B^{\mathcal{E}} B_m \,\mathrm{d}\Omega^{\mathcal{E}} \\ & + \int_{\Omega^{\mathcal{E}}} B_b^T D^{\mathcal{E}} B_b \,\mathrm{d}\Omega^{\mathcal{E}} + \int_{\Omega^{\mathcal{E}}} B_s^T A_s^{\mathcal{E}} B_s \,\mathrm{d}\Omega^{\mathcal{E}} \end{aligned} \tag{2.39}$$

and

$$\begin{aligned} K^{nl\mathcal{E}} = & \frac{1}{2}\int_{\Omega^{\mathcal{E}}} B_m^T A^{\mathcal{E}} \Theta^{\mathcal{E}} B_\theta \,\mathrm{d}\Omega^{\mathcal{E}} + \frac{1}{2}\int_{\Omega^{\mathcal{E}}} B_b^T B^{\mathcal{E}} \Theta^{\mathcal{E}} B_\theta \,\mathrm{d}\Omega^{\mathcal{E}} \\ & + \int_{\Omega^{\mathcal{E}}} B_\theta^T \Theta^{T\mathcal{E}} A^{\mathcal{E}} B_m \,\mathrm{d}\Omega^{\mathcal{E}} + \int_{\Omega^{\mathcal{E}}} B_\theta^T \Theta^{T\mathcal{E}} B^{\mathcal{E}} B_b \,\mathrm{d}\Omega^{\mathcal{E}} \\ & + \frac{1}{2}\int_{\Omega^{\mathcal{E}}} B_\theta^T \Theta^{T\mathcal{E}} A^{\mathcal{E}} \Theta^{\mathcal{E}} B_\theta \,\mathrm{d}\Omega^{\mathcal{E}} \end{aligned} \tag{2.40}$$

In compact forms, the linear and nonlinear stiffness matrix are given by,

$$K^{l\mathcal{E}} = K_{pp}^{\mathcal{E}} + K_{pb}^{\mathcal{E}} + K_{bp}^{\mathcal{E}} + K_{bb}^{\mathcal{E}} + K_{ss}^{\mathcal{E}} \tag{2.41}$$

and

$$K^{nl\mathcal{E}} = \frac{1}{2}K_{p\theta}^{\mathcal{E}} + \frac{1}{2}K_{b\theta}^{\mathcal{E}} + K_{\theta p}^{\mathcal{E}} + K_{\theta b}^{\mathcal{E}} + \frac{1}{2}K_{\theta\theta}^{\mathcal{E}} \tag{2.42}$$

respectively. The thermal load vector can be written as

$$\hat{\Psi}^{\mathcal{E}} = \int_{\Omega^{\mathcal{E}}} B_m^T N_T^{\mathcal{E}} \,\mathrm{d}\Omega^{\mathcal{E}} + \int_{\Omega^{\mathcal{E}}} B_b^T M_T^{\mathcal{E}} \,\mathrm{d}\Omega^{\mathcal{E}} + \int_{\Omega^{\mathcal{E}}} B_\theta^T \Theta^{T\mathcal{E}} N_T^{\mathcal{E}} \,\mathrm{d}\Omega^{\mathcal{E}} \tag{2.43}$$

The last term of Eq. (2.43) may be rewritten as

$$\int_{\Omega^{\mathcal{E}}} B_\theta^T \Theta^{T\mathcal{E}} N_T^{\mathcal{E}} \,\mathrm{d}\Omega^{\mathcal{E}} = \left(\int_{\Omega^{\mathcal{E}}} B_\theta^T \begin{bmatrix} N_T^1 & N_T^3 \\ N_T^3 & N_T^2 \end{bmatrix} B_\theta \,\mathrm{d}\Omega^{\mathcal{E}} \right) \hat{q}^{\mathcal{E}} = K_T^{\mathcal{E}} \tag{2.44}$$

Thus, one may define the thermal load vector as

$$\hat{\Psi}^{\mathcal{E}} = \hat{f}_{N_T}^{\mathcal{E}} + \hat{f}_{M_T}^{\mathcal{E}} + K_T^{\mathcal{E}} \tag{2.45}$$

The residual can now be rewritten as

$$\psi^{\mathcal{E}} = \bar{K}^{\mathcal{E}} \hat{q}^{\mathcal{E}} - \bar{f}^{\mathcal{E}} \tag{2.46}$$

where $\bar{K}^{\mathcal{E}}$ is given by

$$\bar{K}^{\mathcal{E}} = K^{l^{\mathcal{E}}} + K^{nl^{\mathcal{E}}} - K_T^{\mathcal{E}} \tag{2.47}$$

and

$$\bar{f}^{\mathcal{E}} = \hat{f}^{\mathcal{E}} + \hat{f}_{N_T}^{\mathcal{E}} + \hat{f}_{M_T}^{\mathcal{E}} \tag{2.48}$$

2.1.5 Solution to nonlinear equilibrium equations

An approximation of the residual (unbalanced forces) may be obtained by equating to zero the linearized Taylor's series expansion of $\psi^{\mathcal{E}}$ in the neighborhood of an initial estimate of the total displacement, $\hat{q}_i^{\mathcal{E}}$, as

$$\psi^{\mathcal{E}}(\hat{q}_{i+1}^{\mathcal{E}}) \simeq \psi^{\mathcal{E}}(\hat{q}_i^{\mathcal{E}}) + T^{\mathcal{E}} \Delta \hat{q}_i^{\mathcal{E}} = 0 \tag{2.49}$$

where $T^{\mathcal{E}}$ is known as the tangent stiffness matrix. It is evaluated as

$$T^{\mathcal{E}} = \frac{\partial \psi^{\mathcal{E}}(\hat{q}_i^{\mathcal{E}})}{\partial \hat{q}_i^{\mathcal{E}}} \tag{2.50}$$

The tangent matrix may be written as

$$T^{\mathcal{E}} = \begin{bmatrix} \dfrac{\partial \psi_1^{\mathcal{E}}}{\partial \hat{q}_1^{\mathcal{E}}} & \dfrac{\partial \psi_1^{\mathcal{E}}}{\partial \hat{q}_2^{\mathcal{E}}} & \cdots & \dfrac{\partial \psi_1^{\mathcal{E}}}{\partial \hat{q}_m^{\mathcal{E}}} \\ \dfrac{\partial \psi_2^{\mathcal{E}}}{\partial \hat{q}_1^{\mathcal{E}}} & \dfrac{\partial \psi_2^{\mathcal{E}}}{\partial \hat{q}_2^{\mathcal{E}}} & \cdots & \dfrac{\partial \psi_2^{\mathcal{E}}}{\partial \hat{q}_m^{\mathcal{E}}} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial \psi_m^{\mathcal{E}}}{\partial \hat{q}_1^{\mathcal{E}}} & \dfrac{\partial \psi_m^{\mathcal{E}}}{\partial \hat{q}_2^{\mathcal{E}}} & \cdots & \dfrac{\partial \psi_m^{\mathcal{E}}}{\partial \hat{q}_m^{\mathcal{E}}} \end{bmatrix} \tag{2.51}$$

where m is the element number of degrees of freedom. For convenience, the tangent matrix is rewritten with respect to Eq. (2.34) as

$$T^{\mathcal{E}} = \int_{\Omega^{\mathcal{E}}} \left(B^{\mathcal{E}} \frac{\partial \hat{\sigma}^{\mathcal{E}}}{\partial \hat{q}^{\mathcal{E}}} + \frac{\partial B^{\mathcal{E}}}{\partial \hat{q}^{\mathcal{E}}} \hat{\sigma}^{\mathcal{E}} \right) \, \mathrm{d}\Omega^{\mathcal{E}} \tag{2.52}$$

The first term of Eq. (2.52) may be rewritten as (see Eq. (2.28))

$$\begin{aligned}\int_{\Omega^{\mathcal{E}}} \boldsymbol{B}^{\mathcal{E}T} \frac{\partial \hat{\sigma}^{\mathcal{E}}}{\partial \hat{q}^{\mathcal{E}}} \, \mathrm{d}\Omega^{\mathcal{E}} &= \int_{\Omega^{\mathcal{E}}} \boldsymbol{B}^{\mathcal{E}T} \hat{\boldsymbol{C}}^{\mathcal{E}} \frac{\partial}{\partial \hat{q}^{\mathcal{E}}} \left[\left(\boldsymbol{B}^{l} + \frac{1}{2} \boldsymbol{B}^{nl^{\mathcal{E}}} \right) \hat{q}^{\mathcal{E}} - \Psi^{\mathcal{E}} \right] \, \mathrm{d}\Omega^{\mathcal{E}} \\ &= \int_{\Omega^{\mathcal{E}}} \boldsymbol{B}^{\mathcal{E}T} \hat{\boldsymbol{C}}^{\mathcal{E}} \boldsymbol{B}^{\mathcal{E}} \, \mathrm{d}\Omega^{\mathcal{E}} \\ &= \boldsymbol{K}^{l^{\mathcal{E}}} + \boldsymbol{K}^{nl_2{}^{\mathcal{E}}}\end{aligned} \tag{2.53}$$

where

$$\boldsymbol{K}^{nl_2{}^{\mathcal{E}}} = \boldsymbol{K}^{\mathcal{E}}_{p\theta} + \boldsymbol{K}^{\mathcal{E}}_{b\theta} + \boldsymbol{K}^{\mathcal{E}}_{\theta p} + \boldsymbol{K}^{\mathcal{E}}_{\theta b} + \boldsymbol{K}^{\mathcal{E}}_{\theta\theta} \tag{2.54}$$

The second term of Eq. (2.52) is written as

$$\begin{aligned}\int_{\Omega^{\mathcal{E}}} \frac{\partial \boldsymbol{B}^{\mathcal{E}T}}{\partial \hat{q}^{\mathcal{E}}} \hat{\sigma}^{\mathcal{E}} \, \mathrm{d}\Omega^{\mathcal{E}} &= \int_{\Omega^{\mathcal{E}}} \boldsymbol{B}_{\theta}^{T} \frac{\partial \Theta^{\mathcal{E}T}}{\partial \hat{q}^{\mathcal{E}}} N^{\mathcal{E}} \, \mathrm{d}\Omega^{\mathcal{E}} \\ &= \int_{\Omega^{\mathcal{E}}} \boldsymbol{B}_{\theta}^{T} \begin{bmatrix} \frac{\partial}{\partial \hat{q}^{\mathcal{E}}} \left(\frac{\partial w^{\mathcal{E}}}{\partial x} \right) N_1^{\mathcal{E}} + \frac{\partial}{\partial \hat{q}^{\mathcal{E}}} \left(\frac{\partial w^{\mathcal{E}}}{\partial y} \right) N_3^{\mathcal{E}} \\ \frac{\partial}{\partial \hat{q}^{\mathcal{E}}} \left(\frac{\partial w^{\mathcal{E}}}{\partial x} \right) N_3^{\mathcal{E}} + \frac{\partial}{\partial \hat{q}^{\mathcal{E}}} \left(\frac{\partial w^{\mathcal{E}}}{\partial y} \right) N_2^{\mathcal{E}} \end{bmatrix} \mathrm{d}\Omega^{\mathcal{E}} \\ &= \int_{\Omega^{\mathcal{E}}} \boldsymbol{B}_{\theta}^{T} \begin{bmatrix} N_1^{\mathcal{E}} & N_3^{\mathcal{E}} \\ N_3^{\mathcal{E}} & N_2^{\mathcal{E}} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial \hat{q}^{\mathcal{E}}} \left(\frac{\partial w^{\mathcal{E}}}{\partial x} \right) \\ \frac{\partial}{\partial \hat{q}^{\mathcal{E}}} \left(\frac{\partial w^{\mathcal{E}}}{\partial y} \right) \end{Bmatrix} \mathrm{d}\Omega^{\mathcal{E}}\end{aligned} \tag{2.55}$$

The derivative terms may be evaluated as

$$\frac{\partial}{\partial \hat{q}_j^{\mathcal{E}}} \left[\frac{\partial \left(\tilde{N}^{\alpha} \hat{w}_{\alpha}^{\mathcal{E}} \right)}{\partial \lambda} \right] = \begin{cases} \frac{\partial \tilde{N}}{\partial \lambda}, & \text{if } \hat{q}_j^{\mathcal{E}} = \hat{w}_{\alpha}^{\mathcal{E}} \\ 0, & \text{otherwise} \end{cases} \tag{2.56}$$

which is the definition of the strain-displacement matrix $\boldsymbol{B}_{\theta}$. Thus,

$$\begin{aligned}\int_{\Omega^{\mathcal{E}}} \frac{\partial \boldsymbol{B}^{\mathcal{E}T}}{\partial \hat{q}^{\mathcal{E}}} \hat{\sigma}^{\mathcal{E}} &= \int_{\Omega^{\mathcal{E}}} \boldsymbol{B}_{\theta}^{T} \begin{bmatrix} N_1^{\mathcal{E}} & N_3^{\mathcal{E}} \\ N_3^{\mathcal{E}} & N_2^{\mathcal{E}} \end{bmatrix} \boldsymbol{B}_{\theta} \, \mathrm{d}\Omega^{\mathcal{E}} \\ &= \boldsymbol{K}^{\sigma^{\mathcal{E}}}\end{aligned} \tag{2.57}$$

Finally, the tangent stiffness matrix is given by

$$\boldsymbol{T}^{\mathcal{E}} = \boldsymbol{K}^{l^{\mathcal{E}}} + \boldsymbol{K}^{nl_2{}^{\mathcal{E}}} + \boldsymbol{K}^{\sigma^{\mathcal{E}}} \tag{2.58}$$