Chapter 8
Double Integrals

Abstract Double integrals arise in a variety of scientific contexts, essentially as a way to calculate the product of quantities that vary. They are introduced in the first multivariable calculus course, together with the iterated (repeated) integrals that are often used to evaluate them. This chapter concentrates on definitions and properties, and begins with a problem in gravitational attraction that leads to double integrals. It then introduces a precise notion of area called *Jordan content*, and uses that to define the integral. The next chapter concentrates on evaluation, using iterated integrals, curvilinear coordinates, and the change of variables formula.

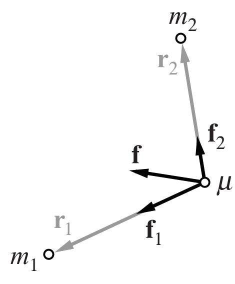
8.1 Example: gravitational attraction

Newton's law of **universal gravitation** says that between any two masses there is an attractive force that is proportional to each of the masses and to the inverse square of the distance between them.

Force is a vector quantity. To describe the force that one mass m exerts on another μ, we begin with the vector $\mathbf{r}$ that gives the position of m with respect to μ. Then the force acts in the direction of the unit vector $\mathbf{u} = \mathbf{r}/\|\mathbf{r}\|$, and its magnitude is proportional to μ and to $m/\|\mathbf{r}\|^2$. If we let G denote the proportionality constant, as customary, then we can write

$$\text{force} = \mu\mathbf{f}, \quad \text{where } \mathbf{f} = \frac{Gm}{\|\mathbf{r}\|^3}\mathbf{r}.$$

The gravitational field



According to Newton's second law of motion, the vector $\mathbf{f}$ is the **acceleration** of the "test mass" μ; $\mathbf{f}$ depends only on the mass m and on the *position* of μ in relation to m. Such a vector function of position is called a *vector field*. This particular vector field is the **gravitational field** due to the mass m. To determine the gravitational force that m exerts on a test mass μ located anywhere in space, just multiply the acceleration field vector $\mathbf{f}$ at that point by the size μ of the test mass. The gravitational field

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defined by a collection of masses m_i, $i = 1, \dots, N$ is just the vector sum of their individual fields:

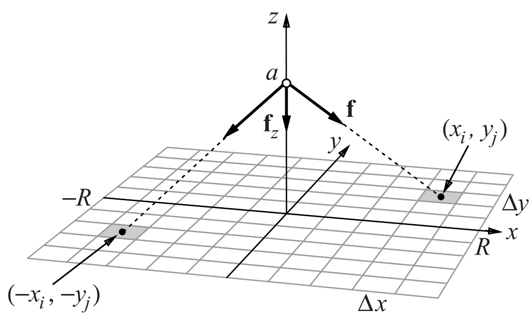
$$\mathbf{f} = \sum_{i=1}^{N} \mathbf{f}_i = \sum_{i=1}^{N} \frac{Gm_i}{\|\mathbf{r}_i\|^3}\mathbf{r}_i.$$

The gravitational field of a large square plate

Our formula appears to define the gravitational field of only a finite number of masses that are concentrated at discrete points. However, by using limit processes (in which sums become integrals), we can extend the formula to continuous distributions of matter. To see how this happens, let us analyze the gravitational field of a large homogeneous plate of uniform thickness. For such a plate, the mass of any piece is simply proportional to its area A:

$$\text{mass} = \rho A.$$

The constant of proportionality ρ gives the mass per unit area, or the *mass density*, of the plate. We determine the gravitational field of the plate only for points directly above the center of the plate. (Eventually, we assume that the plate is so large that it is effectively infinite in extent. In that case, every location on the plate is like every other; we are able to think of any point on the plate as its "center.")



For now, let the plate be the square $-R \le x, y \le R$ in the (x,y)-plane. We want to determine the gravitational field at the point $(0,0,a)$, $a > 0$, on the z-axis. We can use the additivity of the field: divide the plate into a number of small cells, approximate the field due to each cell, and then add the results to get an estimate of the field due to the whole plate.

Approximating the field

We choose cells that form a grid of small congruent squares of dimensions $\Delta x = \Delta y = R/k$ and mass $\rho\,\Delta x\,\Delta y$. If R/k is small enough, we can approximate the field due to a single square by imagining all its mass is concentrated at its center. If the center is at $(x_i, y_j, 0)$, then

$$\mathbf{r} = (x_i, y_j, 0) - (0,0,a) = (x_i, y_j, -a),$$

so the gravitational field due to this one square is approximately

$$\mathbf{f}_{ij} = \frac{G\rho\,\Delta x\,\Delta y}{(x_i^2 + y_j^2 + a^2)^{3/2}}\,(x_i, y_j, -a).$$

As the figure above indicates, the horizontal component of $\mathbf{f}_{ij}$ is canceled by the horizontal component of the field due to the square centered at symmetrically opposite point $(-x_i, -y_j)$ on the plate. Thus, for the whole field $\mathbf{f}$, only its vertical component is nonzero. In fact, the four points (x_i, y_j), $(x_i, -y_j)$, $(-x_i, y_j)$, and $(-x_i, -y_j)$ contribute the same vertical component, so we can restrict ourselves to squares in the first quadrant, writing the four contributions together as the scalar

$$4\,\frac{G\rho\,\Delta x\,\Delta y}{(x_i^2+y_j^2+a^2)^{3/2}}\cdot(-a) = \frac{-4G\rho\,a\,\Delta x\,\Delta y}{(x_i^2+y_j^2+a^2)^{3/2}}.$$

Therefore, the vertical component of the field at $z = a$ that is due to the whole plate is approximately the double sum

$$\text{field} \approx \sum_{i=1}^{k}\sum_{j=1}^{k} \frac{-4G\rho\,a\,\Delta x\,\Delta y}{(x_i^2+y_j^2+a^2)^{3/2}}.$$

The coordinates (x_i, y_j) of the centers of the squares in the first quadrant start with $(x_1, y_1) = (\Delta x/2, \Delta y/2)$ and then increase by steps of Δx and Δy:

$$\begin{aligned} x_1 &= \Delta x/2, & y_1 &= \Delta y/2, \\ x_i &= x_{i-1} + \Delta x,\ i = 2,\ldots,k, & y_j &= y_{j-1} + \Delta y,\ j = 2,\ldots,k. \end{aligned}$$

Computing the double sum

Using a simple program such as the the one below, we can compute the double sum for any given value of a. To simplify the computation, however, we first choose units that make $4G\rho = 1$. Then, keeping in mind that the dimensions of the plate should be large in comparison to the distance to a point in the field (i.e., $R \gg a$), we choose $R = 32$ and then do a sequence of calculations for $a = 0.2$, 0.1, and 0.05.

```
       PROGRAM: The gravitational field of a large plate
a = .2
R = 32
k = 64
dx = R / k
dy = dx
sum = 0
x = dx / 2
FOR i = 1 TO k
  y = dy / 2
  FOR j = 1 TO k
    sum = sum - a * dx * dy / (x ^ 2 + y ^ 2 + a ^ 2) ^ (3 / 2)
    y = y + dy
  NEXT j
  x = x + dx
NEXT i
PRINT k, sum
```

How the sum varies with k

The results appear in the table below. Each column indicates how the estimate of field strength changes as the number of squares increases when a is fixed. (Note that there are k^2 squares, and thus more than a million when $k = 1024$.) It appears that the sums are converging to some fixed value as k increases. For example, when $a = 0.2$, the first seven or eight digits of the sum have "stabilized" when $k = 1024$, suggesting that a limit is emerging:

$$\text{field at } 0.2 = \lim_{k\to\infty} \text{sum} = -1.561\,957\ldots.$$

But for the same value of k, only the first four digits appear to have stabilized when $a = 0.1$, and only the first two when $a = 0.05$. However, when k is doubled, even the sum for 0.05 appears to have stabilized out to four digits. This suggests

$$\text{field at } 0.1 = -1.566\ldots, \qquad \text{field at } 0.05 = -1.568\ldots.$$

$a = 0.2$		$a = 0.1$		$a = 0.05$	
k	sum	k	sum	k	sum
64	-1.233	64	-0.757	128	-0.759
128	-1.526	128	-1.238	256	-1.240
256	$-1.561\,691$	256	$-1.530\,399$	512	-1.533
512	$-1.561\,957\,628$	512	$-1.566\,110$	1024	$-1.568\,320$
1024	$-1.561\,957\,637$	1024	$-1.566\,377$	2048	$-1.568\,587$

The Riemann integral as a limit of sums

We recognize that the double sums are, in fact, *Riemann* sums for the function

$$\varphi_a(x,y) = \frac{-4G\rho a}{(x^2+y^2+a^2)^{3/2}};$$

thus, the limiting value that we are seeking for each a is the Riemann integral of that function for the given a:

$$\text{field at } a = \lim_{k\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{k} \varphi_a(x_i,y_j)\,\Delta x\,\Delta y = \iint\limits_{\substack{0\le x\le R,\\ 0\le y\le R}} \varphi_a(x,y)\,dx\,dy.$$

In this expression, "$dx\,dy$" is sometimes called the **element of area**; it represents the limit of the area $\Delta x\,\Delta y$ of a rectangle in the Cartesian grid. When other coordinates are used, the element of area may have a different form. However, our expression for a double integral always contains an element of area.

"Integral as product"

The integral arises here in a typical way: it is a number that is essentially the product of two quantities. (For example, field strength is the product of a mass and the reciprocal of a distance squared.) But the quantities involved are variable, so their product cannot be found directly as a single number. The remedy is to restrict the quantities being multiplied to small cells on which they become nearly constant. (For example, restrict to small squares on the plate). Now calculate a representative product on each cell, and then add the results over all cells. The sum gives an approximation to the numerical value we seek. To get a better approximation, make the cells even smaller. If the sums tend to a limit as the cells get smaller, that limit is defined to be the integral. In Chapter 8.3, we use these ideas to define the integral and catalogue some of its properties.

How the field varies with position

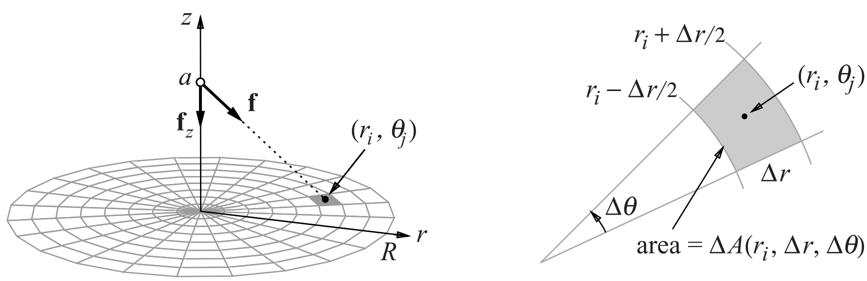
Let us return to the estimates of the gravitational field provided by our calculations. Notice that field strength does not vary with the inverse square of the distance a to the plate. On the contrary, the calculations suggest that field strength is essentially constant for all $a \ll R$. If $R = \infty$, then $a \ll R$ for all finite a, so it seems reasonable to speculate that the gravitational field of an infinite plate is exactly constant. But we cannot check this directly: if $R = \infty$, the Riemann sums we use to estimate the field are not defined. (The BASIC program breaks down on its second line.) Indeed, we define the Riemann integral only for a bounded domain.

Improper integrals

But there is a standard way out of the difficulty: determine the value of the integral as a function of R, and then see if the values tend to a well-defined limit as $R \to \infty$. When the limit exists, it is called the *improper integral*. Improper double integrals arise from unbounded functions as well as from unbounded domains; we define both kinds in Chapter 9. However, we can even now confirm that an infinite homogeneous plate produces a constant gravitational field; we just need to start with a different shape.

The gravitational field of a circular plate

Because we are interested ultimately in the field of the infinite plate, let us allow ourselves to change the shape of its finite approximation. Specifically, we change the plate from a square to a circle, and then exploit the circular symmetry by changing from Cartesian to polar coordinates. This change leads to a one-variable improper integral that determines the field of an infinite plate.



Let the plate have radius R and be centered at the origin; then it is given by the inequalities $0 \le r \le R$, $0 \le \theta < 2\pi$. To divide the plate into small cells, it is natural to use equally spaced concentric circles $r =$ constant and radial lines $\theta =$ constant, with spacings

$$\Delta r = \frac{R}{k}, \quad \Delta\theta = \frac{2\pi}{l}, \quad k, l \text{ positive integers.}$$

Area of a cell

The cells are not uniform in size; their areas grow with r. Choose the representative point (r_i, θ_j) at the center of the cell. Each cell is the portion of a circular ring of thickness Δr that is cut out by a central angle $\Delta\theta$. The area of the whole ring between $r_i - \Delta r/2$ and $r_i + \Delta r/2$ is

$$\pi\left(r_i + \frac{\Delta r}{2}\right)^2 - \pi\left(r_i - \frac{\Delta r}{2}\right)^2 = 2\pi r_i \Delta r.$$

A single cell occupies the fraction $\Delta\theta/2\pi$ of this ring, so its area is

$$\left(\frac{\Delta\theta}{2\pi}\right) 2\pi r_i \Delta r = r_i \,\Delta r\, \Delta\theta = \Delta A(r_i, \Delta r, \Delta\theta) = \Delta A,$$

and we can write its mass as $\rho\,\Delta A$. Assuming the mass is concentrated at (r_i, θ_j), we estimate the cell's contribution to the gravitational field at $z = a$ is approximately

$$\frac{-G\rho a\,\Delta A}{(r_i^2 + a^2)^{3/2}}.$$

Therefore, the field at $z = a$ that is due to the whole plate is now approximated by the double sum

$$\text{field at } a \approx \sum_{i=1}^{k} \sum_{j=1}^{l} \frac{-G\rho a\,\Delta A}{(r_i^2 + a^2)^{3/2}}.$$

Simplifications due to symmetry

Notice that the values θ_j are absent: all cells in a given ring (i.e., with fixed r_i) make the same contribution. This symmetry with respect to θ means we can write the inner sum (where i, and hence r_i, is fixed) as

$$\sum_{j=1}^{l} \frac{-G\rho a\,\Delta A}{(r_i^2 + a^2)^{3/2}} = \frac{-G\rho a}{(r_i^2 + a^2)^{3/2}} \sum_{j=1}^{l} \Delta A = \frac{-G\rho a}{(r_i^2 + a^2)^{3/2}} 2\pi r_i \Delta r,$$

because $\sum \Delta A$ is just the area of an entire circular ring. Thus the field due to the whole plate reduces to a sum over a single index:

$$\text{field at } a \approx -2\pi G\rho a \sum_{i=1}^{k} \frac{r_i\,\Delta r}{(r_i^2 + a^2)^{3/2}}.$$

But this is just a Riemann sum for the one-variable function

$$h(r) = -2\pi G\rho a \,\frac{r}{(r^2 + a^2)^{3/2}}.$$

Therefore, as $k \to \infty$ and $\Delta r \to 0$, the sum becomes the ordinary (i.e., single) integral

$$\begin{aligned}\int_0^R h(r)\,dr &= -2\pi G\rho a \int_0^R \frac{r\,dr}{(r^2 + a^2)^{3/2}} \\ &= 2\pi G\rho a\, \frac{1}{(r^2 + a^2)^{1/2}}\bigg|_0^R = 2\pi G\rho a \left(\frac{1}{(R^2 + a^2)^{1/2}} - \frac{1}{a}\right).\end{aligned}$$

Because the sum becomes a better and better approximation to the field as $\Delta r \to 0$ (when $k \to \infty$), we write

$$\text{field at } a = -2\pi G\rho a \int_0^R \frac{r\,dr}{(r^2 + a^2)^{3/2}} = 2\pi G\rho \left(\frac{a}{\sqrt{R^2 + a^2}} - 1\right).$$

Infinite plates and improper integrals

Let us compare this result with what we computed for the square. Thus we set $4G\rho = 1$ and $R = 32$, and then find

$$\text{field at } 0.2 = -1.56098, \quad \ldots \text{at } 0.1 = -1.56589, \quad \ldots \text{at } 0.05 = -1.56834.$$

These values are virtually identical with the corresponding ones for the square. Now that we have an exact formula for the field, it is easy to see what happens as the plate becomes infinite in extent, that is, as we let $R \to \infty$. Because

$$\lim_{R\to\infty} \frac{a}{\sqrt{R^2+a^2}} = 0,$$

we find that

$$\text{field} = 2\pi G\rho \cdot (-1) = -2\pi G\rho = \text{constant};$$

the field is indeed independent of the distance a from the plate. Furthermore, after the normalization $4G\rho = 1$, the field strength takes on the constant value $-\pi/2 = -1.570793\ldots$. In fact, we express the field of the infinite plate as an improper integral, that is, as the limit of a sequence of "proper" integrals:

$$\int_0^\infty \frac{r\,dr}{(r^2+a^2)^{3/2}} = \lim_{R\to\infty} \int_0^R \frac{r\,dr}{(r^2+a^2)^{3/2}} = \lim_{R\to\infty} \left(\frac{1}{(R^2+a^2)^{1/2}} - \frac{1}{a} \right) = -\frac{1}{a}.$$

We evaluate improper double integrals in a similar way (cf. Chapter 9.2).

Double integral in polar coordinates

Let us return to the finite circular plane and the double sum formula

$$\sum_{i=1}^{k} \sum_{j=1}^{l} \frac{-G\rho a \Delta A}{(r_i^2+a^2)^{3/2}}$$

that expresses the approximate field strength in polar coordinates. As we did with Cartesian coordinates, we can recognize that these are Riemann sums for the function

$$\psi_a(r,\theta) = \frac{-G\rho a}{(r^2+a^2)^{3/2}}.$$

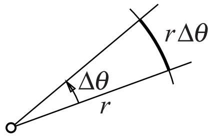
The exact value of the field strength is thus the Riemann integral that represents the limit of these sums:

$$\text{field at } a = \lim_{\substack{k\to\infty \\ l\to\infty}} \sum_{i=1}^{k} \sum_{j=1}^{l} \psi_a(r_i,\theta_j)\,\Delta A = \iint\limits_{\substack{0\le r\le R, \\ 0\le\theta<2\pi}} \psi_a(r,\theta)\,dA.$$

In this expression, dA is the element of area (cf. p. 272); it represents the limit of the area ΔA of a cell in the polar coordinate grid.

$dA = r\,dr\,d\theta$

For a Cartesian grid, $dA = dx\,dy$ is just the product of the "elements of length" dx and dy for the individual coordinates. However, $dA \neq dr\,d\theta$ for a polar grid. On the contrary, we have already noted that $\Delta A = (\Delta r)(r\Delta\theta)$. Although $\Delta\theta$ is dimensionless, $r\Delta\theta$ does have the dimensions of a length: $r\Delta\theta$ is the length of the arc



subtended by the angle $\Delta\theta$ on the circle of radius r. Informally, then, $r\,d\theta$ and dr are the "elements of length" whose product is the element of area $dA = r\,dr\,d\theta$. Thus we write

$$\text{field at } a = \iint_{\substack{0\le r\le R,\\ 0\le\theta<2\pi}} \psi_a(r,\theta)\,r\,dr\,d\theta = -G\rho a \iint_{\substack{0\le r\le R,\\ 0\le\theta<2\pi}} \frac{r\,dr\,d\theta}{(r^2+a^2)^{3/2}}.$$

8.2 Area and Jordan content

Riemann integration: a sketch

The definition of the Riemann integral of a function over a region is simple in outline. First, partition the region into many small pieces; then multiply the "size" of each piece by a value that the function takes on somewhere in that piece, and sum those products; finally, repeat the process with ever smaller pieces and take the limit of the computed sums. To convert this sketch into something useful and precise, one of the questions we must decide is what kind of pieces can be used to make up a partition.

Partitioning 2-dimensional regions

When the function depends on just a single real variable x, the answer is immediate: each small piece is an interval $a \le x \le b$ whose "size" is its length, $b-a$. But if the function depends on two real variables, x and y, the answer is not so clear. Certainly there is a 2-dimensional analogue for an interval; it is a rectangle $a \le x \le b$, $c \le y \le d$ in the (x,y)-plane with sides parallel to the axes, whose "size" is given by its area $A = (b-a)(d-c)$. But now consider what happens under a change of variables. On the line, a small interval is generally transformed into another small interval. On the plane, however, a small rectangle is transformed into a quadrilateral with curved sides (see Chapter 4), so these more general shapes appear in partitions as naturally as rectangles do. We therefore get a better answer to the question by focusing not on the shape of a small piece but on whether we can assign it a "size."

Jordan content

The *size* of a region will be its *area*, of course; we have to worry about admissible shapes because not every region in the plane has a well-defined area. For example, there is no consistent way to assign a nonzero area to the set of points in the unit square that have rational coordinates (the **rational points**); see below, page 279. If this is not immediately evident, however, it may be because our notion of *area* is more intuitive than precise. Thus, we construct a precise notion of *size* (called *Jordan content*) that captures our intuitive ideas about area, extends immediately to higher dimensions, and fits well with the process of integration.

Plane topology

Before we discuss Jordan content, though, we must establish some basic topology concerning the interior and the boundary of a set in the plane.

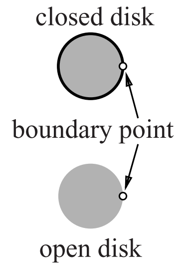
Definition 8.1 *The **open** (respectively, **closed**) **disk** of radius $r > 0$ centered at the point $\mathbf{p}$ in $\mathbb{R}^2$ is the set of all points $\mathbf{x}$ in $\mathbb{R}^2$ for which $\|\mathbf{x}-\mathbf{p}\| < r$ (respectively, $\|\mathbf{x}-\mathbf{p}\| \le r$).*

Definition 8.2 *A point* $\mathbf{p}$ *is an* ***interior point of a set S*** *in* $\mathbb{R}^2$ *if some open disk centered at* $\mathbf{p}$ *is contained entirely in S.*

Definition 8.3 *A point* $\mathbf{q}$ *is an* ***exterior point of S*** *if it is an interior point of* S^c*, the complement of S in* $\mathbb{R}^2$*. A point* $\mathbf{b}$ *is a* ***boundary point of S*** *if it is neither an interior nor an exterior point of S.*

Open and closed sets

Every interior point of S lies inside S, of course; what makes it an *interior* point, however, is the fact that it is surrounded by points that also lie inside S. Likewise, every exterior point lies outside S and is surrounded by points outside S. An individual boundary point may lie either inside or outside S, but every open disk centered at a boundary point contains at least one point in S and one point outside S (see Exercise 8.5). For example, the open and closed disks with a given radius and center have the same boundary points, namely the points on the circle with that radius and center. They represent two extremes: the closed disk contains all its boundary points, but the open disk contains none. These become the models for *closed* and *open* sets in general.



Definition 8.4 *A set is* ***closed*** *if it contains all its boundary points; it is* ***open*** *if it contains none of them.*

Thus every point of an open set is an interior point. It is more common to define a closed set, however, as one whose complement is open. The following theorem shows that our definition is equivalent to the usual one.

Theorem 8.1. *The set S is closed* $\Leftrightarrow$ *The complement* S^c *is open.*

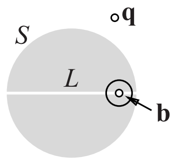
Proof. S is closed $\Leftrightarrow S^c$ contains no boundary points of S
$\Leftrightarrow S^c$ contains only exterior points of S
$\Leftrightarrow$ all points of S^c are interior points of S^c
$\Leftrightarrow S^c$ is open. □

Definition 8.5 *The* ***interior of S****, denoted* $^\circ\mathbf{S}$*, is the set of interior points of S; the* ***boundary of S****, denoted* $\boldsymbol{\partial S}$*, is the set of boundary points of S; the* ***closure of S****, denoted* $\overline{\mathbf{S}}$*, is* $S \cup \partial S$*.*

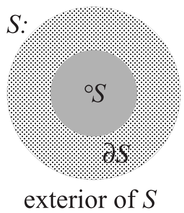
Thus, S is open if $S = {}^\circ S$ and is closed if $S = \overline{S}$. We "open" S (i.e., create its interior) by removing from S all its boundary points; we "close" S (create its closure) by adding to S all its boundary points. The symbol we have introduced for *boundary* may have no good rationale at the moment, but its aptness should become clear when we reconsider Green's theorem in Chapter 10; see especially page 427.

The boundary can be nonintuitive

When S is a familiar shape—such as a polygon—its interior, exterior, and boundary are what we expect. But when S is less familar, intuition may be a poor guide. For example, even though a polygon's boundary separates the interior from the exterior, this is not true for all sets. For one thing, the boundary may not be a finite collection of curves or line segments. Take S to be the plane minus the origin; then ∂S is just the origin and there are no exterior points at all. According to Exercise 8.5, every disk centered at a boundary point must contain a point outside S; in this case, the only such point is the center of the disk itself.



For an even more instructive example, take S to be the open unit disk centered at the origin, minus the portion L of the x-axis that lies within 1 unit of the origin. The exterior points $\mathbf{q}$ of S are those for which $\|\mathbf{q}\| > 1$. The boundary of S is a pair of curves: the unit circle and the line segment L. Any neighborhood of a point on the unit circle does indeed contain both interior and exterior points of S, but that is not true of L. At any point $\mathbf{b}$ of L, a sufficiently small open disk centered at $\mathbf{b}$ will contain no exterior points of S whatsoever. Therefore, we cannot say that L "separates" the interior of S from its exterior.



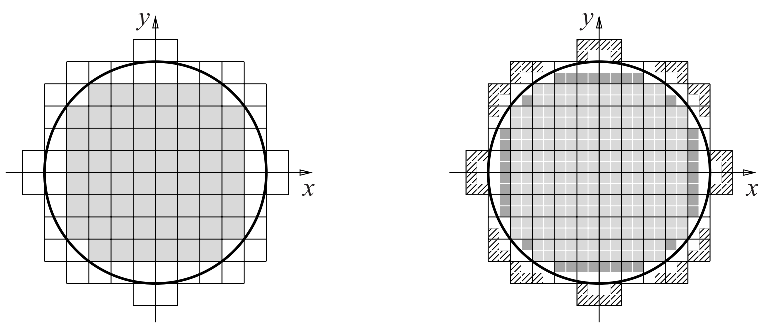
For a set that cannot be sketched easily, its interior and boundary may be even more nonintuitive. For example, take S to be the set of all points in the closed disk of radius 1 together with all *rational* points in the disk of radius 2, both centered at the origin. The interior of S is the open disk of radius 1, and the exterior of S consists of all points $\mathbf{q}$ with $\|\mathbf{q}\| > 2$. The boundary of S is everything else; it is the annulus of points $\mathbf{b}$ with $1 \leq \|\mathbf{b}\| \leq 2$. Here is a set whose boundary is "thick;" although ∂S does separate the interior and the exterior of S, it is not a simple 1-dimensional curve. Another example in the same vein is the set of all rational points in the plane. The interior and the exterior are both empty, so the boundary is the entire plane. As we show, sets such as these that have "thick" boundaries will always fail to have Jordan content.

To define Jordan content, we use a method that we introduce now in an intuitive and ad hoc way to find the areas of two particular sets. In one case, the method succeeds; in the other, it does not, illustrating how a set can fail to have Jordan content.

Areas from squares

The fundamental shape whose area we know is a square. Consider how we might use only squares to approximate the area of the closed unit disk S. Cover the plane with a grid of squares of width w. If L of them lie entirely inside S, then the area of S must be at least Lw^2. If U of them meet S, then the area of S can be no more than Uw^2. For example, if $w = 1/5$ and the origin is one of the grid intersection points (below, left), then we can use 3–4–5 right triangles to show that $L = 60$ gray squares (15 in each quadrant) lie inside S, and $U = 104$ squares altogether meet S, implying

$$2.4 = \frac{60}{25} = Lw^2 < \text{area}\, S < Uw^2 = \frac{104}{25} = 4.16.$$



These are lower and upper bounds for the area, but neither is a good estimate for the correct value, $\pi \approx 3.14$. The smaller fails to count area that it should and the larger counts area that it should not. The difference between the bounds—which indicates the coarseness of the estimates—equals the total area $44/25 = 1.76$ of the white squares in the figure on the left.

Refine the grid to get a better estimate

To get better estimates, use smaller squares; then more of the area inside S will be counted, and less outside. Take $w = 1/10$, giving us four small squares in each large square (above, right). We find nine additional small squares (darker gray) inside S in each quadrant, so $L = 4 \times 60 + 36 = 276$. Furthermore, 14 new small squares (hatched) in each quadrant now lie completely outside S, so $U = 4 \times 104 - 56 = 360$, and the new bounds are

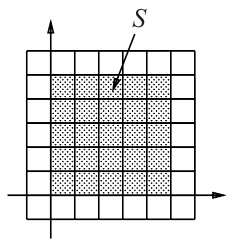
$$2.76 = Lw^2 < \operatorname{area} S < Uw^2 = 3.6.$$

The difference between the new bounds (which is the area of the 84 small white squares in the figure on the right) is less than half what it was in the previous stage; in this sense, the new estimates are twice as good as the old. (Their average, 3.18, is within $1\frac{1}{4}$% of the true area.)

Area of the boundary

Calculations made with further refinements of the grid (see Exercise 8.17) suggest that the difference between the bounds—as measured by the area of the white squares—shrinks to zero, forcing our estimates of the area of S toward a single value. Notice that those white squares also cover the boundary circle, implying that the area of the boundary is zero. The circle is, of course, a pair of graphs; as we show (Theorem 8.10, p. 284), the graph of a continuous function on a bounded interval will always have zero area. In summary, S has an area, and its boundary is "thin" enough to have zero area.

A set with no area



For our second example, take S to be the set of rational points in the unit square that lies in the first quadrant with a corner at the origin. Let us use the same method of counting squares to find the area of S, assuming it has an area. As before, we cover the plane with a grid of squares of width w. But now, because S has no interior points, no grid square lies entirely inside S. In other words, there are no solid gray squares; L is always zero. On the other hand, many grid squares have a point in common with S, so $U > 0$. For example, if $w = 1/5$, then we can count 25 such grid squares inside the unit square itself, plus 5 more meeting the unit square along each of its four sides, plus 1 more at each each corner; thus $U = 25 + 4 \times 5 + 4 = 49$. If S does have an area, then

$$0 = Lw^2 < \operatorname{area} S < Uw^2 = \frac{49}{25} = 1.96.$$

When we set $w = 1/10$ to refine the grid, then $U = 100 + 4 \times 10 + 4 = 144$ and the bounds become

$$0 = Lw^2 < \operatorname{area} S < Uw^2 = \frac{144}{100} = 1.44.$$

More generally, if $w = 1/2^n$, then $U = 2^{2n} + 4 \times 2^n + 4$ and the bounds are

$$0 = Lw^2 < \operatorname{area} S < Uw^2 = 1 + \frac{1}{2^{n-2}} + \frac{1}{2^{2n-2}}.$$

No matter how small we make w, the bounds never get any better than

$$0 < \operatorname{area} S \le 1.$$

Our method of counting squares thus fails to assign a meaningful value to the area of S.

A boundary with positive area

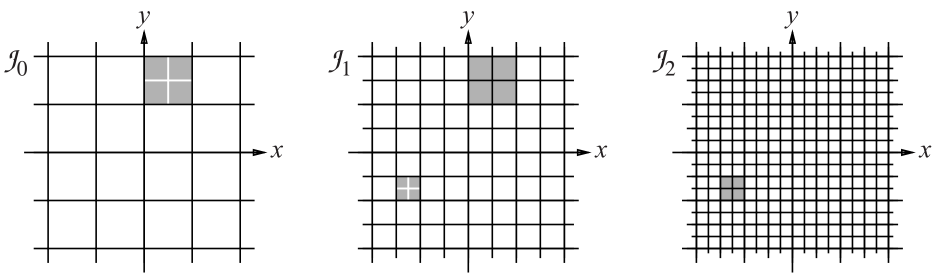
How is this failure connected to the size, or "thickness," of ∂S? For the closed disk, the difference $Uw^2 - Lw^2$ was the area of the white squares that covered the boundary circle; it served as an upper bound on the area of that circle. For the rational points in the square, the difference is

$$Uw^2 - Lw^2 = Uw^2 > 1$$

for all $w > 0$; in particular, the difference does not converge to zero. Indeed, the area of ∂S is 1. To see why, recall from our earlier examples (p. 278) that $\partial S = \overline{S}$ is the closed unit square, so area $\partial S =$ area $\overline{S} = 1$. Thus, in contrast with the disk, the set of rational points in the square does not have an area, but its boundary is "thick" and does have positive area.

Defining Jordan content

We can now formalize the method we use to define the *Jordan content* of a set. There are three steps. First, we count grid squares to get monotonic sequences of "inner" and "outer" areas. Second, we compute the limiting areas as the grids are refined. Third, we see whether the two limits are equal; if they are, the set has Jordan content equal to the common value. Although it might seem reasonable to choose the grids based on how well they are adapted to a given set, it is not a priori evident that the value obtained from one sequence of grids would then equal the value obtained from another. Thus, to eliminate ambiguity, we always use just one collection of grids, $\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2, \ldots$. Later (pp. 287ff.), we in fact introduce other grids and prove that they yield the same value given by the grids $\mathcal{J}_k$.



The grids $\mathcal{J}_k$

The grid $\mathcal{J}_0$ consists of the closed unit squares in the (x,y)-plane that are bounded by the vertical lines $x =$ integer and $y =$ integer. To get the squares of the next grid $\mathcal{J}_1$, divide each unit square into four congruent subsquares. Because every square of $\mathcal{J}_1$ lies entirely in a single square of $\mathcal{J}_0$, we say $\mathcal{J}_1$ is a *refinement* of $\mathcal{J}_0$.

Use the same procedure to get $\mathcal{J}_2$ as a refinement of $\mathcal{J}_1$, and so on. The squares in the grid $\mathcal{J}_k$ at stage k have width $w = 1/2^k$.

Definition 8.6 *Let* $\underline{\boldsymbol{J}}_{\boldsymbol{k}}(\boldsymbol{S})$ *denote the total area of the squares in* $\mathcal{J}_k$ *that are entirely contained in the bounded set S, and let* $\overline{\boldsymbol{J}}_{\boldsymbol{k}}(\boldsymbol{S})$ *denote the total area of the squares in* $\mathcal{J}_k$ *that intersect S.*

$\underline{J}_k(S)$ and $\overline{J}_k(S)$

The "inner" and "outer" area estimates for S ($\underline{J}_k(S)$ and $\overline{J}_k(S)$, respectively) at the various stages are nested together in the following chain:

$$0 \le \underline{J}_0(S) \le \cdots \le \underline{J}_k(S) \le \underline{J}_{k+1}(S) \le \cdots \le \overline{J}_{l+1}(S) \le \overline{J}_l(S) \le \cdots \le \overline{J}_0(S) < \infty.$$

To see this, note first that $0 \le \underline{J}_0(S)$ because our squares all have positive area. Also, $\overline{J}_0(S)$ is finite because the bounded set S can meet only a finite number of unit squares. The remaining inequalities in the chain follow from the next two lemmas.

Lemma 8.1. *For any bounded set S and integer* $k \ge 0$, $\underline{J}_k(S) \le \underline{J}_{k+1}(S)$ *and* $\overline{J}_{k+1}(S) \le \overline{J}_k(S)$.

Proof. If a square is counted in $\underline{J}_k(S)$, then its four subsquares, with the same total area, are counted in $\underline{J}_{k+1}(S)$; hence $\underline{J}_k(S) \le \underline{J}_{k+1}(S)$.

Similarly, if a square is counted in $\overline{J}_{k+1}(S)$, then the square in $\mathcal{J}_k$ that contains it is counted in $\overline{J}_k(S)$; that implies $\overline{J}_{k+1}(S) \le \overline{J}_k(S)$. □

Lemma 8.2. *For any bounded set S and integers* $k,l \ge 0$, $\underline{J}_k(S) \le \overline{J}_l(S)$.

Proof. First note that $\underline{J}_j(S) \le \overline{J}_j(S)$ for every $j \ge 0$, because any square counted in $\underline{J}_j(S)$ is also counted in $\overline{J}_j(S)$. Then (using Lemma 8.1),

$$\begin{aligned} k \ge l \quad &\Rightarrow \quad \underline{J}_k(S) \le \overline{J}_k(S) \le \overline{J}_l(S); \\ k < l \quad &\Rightarrow \quad \underline{J}_k(S) \le \underline{J}_l(S) \le \overline{J}_l(S). \end{aligned}$$ □

The nested inequalities imply that the sequence $\underline{J}_k(S)$ of "inner areas" is monotonic increasing and bounded above; hence it has a finite limit. The sequence $\overline{J}_k(S)$ is monotonic decreasing and bounded below, so it has a finite limit, too.

Definition 8.7 *The* ***inner*** *and* ***outer Jordan content*** *of the bounded set S are the respective limits*

Inner and outer Jordan content

$$\underline{\boldsymbol{J}}(\boldsymbol{S}) = \lim_{k\to\infty} \underline{J}_k(S) \ \textit{ and } \ \overline{\boldsymbol{J}}(\boldsymbol{S}) = \lim_{k\to\infty} \overline{J}_k(S).$$

Lemma 8.2 implies that $\underline{J}(S) \le \overline{J}(S)$, but $\underline{J}(S)$ and $\overline{J}(S)$ need not be equal. For example, when S is the set of rational points in the unit square, our earlier work shows that $\underline{J}(S) = 0$ and $\overline{J}(S) = 1$.

Definition 8.8 *If* $\underline{J}(S) = \overline{J}(S)$*, then we say S* ***is Jordan measurable****, or is a* ***J-set****, and its* ***Jordan content*** *is* $\boldsymbol{J}(\boldsymbol{S}) = \underline{J}(S) = \overline{J}(S)$.

Jordan content and area

The definition of Jordan content is now clear; however, it is not yet evident that the Jordan content of a set equals its usual area, not even for one of the original grid squares! Most of the rest of this section is devoted to establishing the properties of Jordan content. From that emerges its connection with area. (Exercise 8.7, for example, establishes that the Jordan content of a square in $\mathcal{J}_k$ is indeed its area, $1/2^{2k}$.) The first property is the fundamental one we saw in the two illustrative examples: a set has Jordan content precisely when its boundary is "thin" enough to have Jordan content equal to zero.

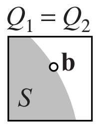
Theorem 8.2. *The set S is Jordan measurable $\Leftrightarrow J(\partial S) = 0$.*

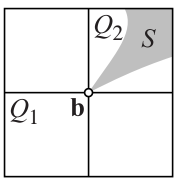
Proof. We make use of the fact that

$$S \text{ is Jordan measurable} \Leftrightarrow \lim_{k\to\infty} \left(\overline{J}_k(S) - \underline{J}_k(S)\right) = 0.$$

The number $\overline{J}_k(S) - \underline{J}_k(S)$ is the total area of the squares that meet S but are not entirely contained in S. Each such square thus contains a point $\mathbf{p}$ in S and a point $\mathbf{q}$ not in S. Also, it is a convex set that contains the entire line segment connecting $\mathbf{p}$ and $\mathbf{q}$. Therefore, by Exercise 8.6, this square contains a point of ∂S, so it is counted in $\overline{J}_k(\partial S)$; hence

$$\overline{J}_k(S) - \underline{J}_k(S) \le \overline{J}_k(\partial S).$$





Conversely, suppose a square Q_1 in $\mathcal{J}_k$ contains a point $\mathbf{b}$ in ∂S. We claim $\mathbf{b}$ must also lie in one of the squares Q_2 that is counted in $\overline{J}_k(S) - \underline{J}_k(S)$. If $\mathbf{b}$ is an interior point of the square Q_1, then every sufficiently small open disk centered at $\mathbf{b}$ lies in Q_1. But by Exercise 8.5, every such disk contains at least one point in S and at least one point not in S. Thus we can take $Q_2 = Q_1$.

If, on the contrary, $\mathbf{b}$ lies on either a side or a corner of Q_1, then it also meets either one or three squares adjacent to Q_1 in $\mathcal{J}_k$. Because $\mathbf{b}$ is in ∂S, at least one of these (two or four) squares contains a point in S and at least one contains a point not in S. For suppose each square contained exclusively one kind of point or the other. Because $\mathbf{b}$ is in each of these squares, it must be both in S and not in S. This is impossible, so at least one square Q_2 contains both kinds of points; Q_2 is counted in $\overline{J}_k(S) - \underline{J}_k(S)$.

Thus, each square Q_1 counted in $\overline{J}_k(\partial S)$ is either equal to a square Q_2 that is counted in $\overline{J}_k(S) - \underline{J}_k(S)$, or it is one of the eight neighbors of Q_2 in the grid $\mathcal{J}_k$. The total area of squares Q_1 is therefore not larger than nine times the total area of squares Q_2:

$$\overline{J}_k(\partial S) \le 9\left(\overline{J}_k(S) - \underline{J}_k(S)\right).$$

The two displayed inequalities now allow us to write

$$\begin{aligned} S \text{ is Jordan measurable} &\Leftrightarrow \lim_{k\to\infty} \left(\overline{J}_k(S) - \underline{J}_k(S)\right) = 0 \\ &\Leftrightarrow \lim_{k\to\infty} \overline{J}_k(\partial S) = 0 \\ &\Leftrightarrow J(\partial S) = 0. \end{aligned}$$

□

The next several results concern primarily outer Jordan content and sets whose Jordan content is zero. They are useful on their own and they culminate in the theorem (Theorem 8.10) that the graph of continuous function on a bounded interval has Jordan content zero.

Outer content and Jordan content zero

Theorem 8.3. *Let S and T be bounded sets with $S \subseteq T$; then $\overline{J}(S) \leq \overline{J}(T)$.*

Proof. Every square in $\mathcal{J}_k$ that meets S also meets T, so $\overline{J}_k(S) \leq \overline{J}_k(T)$. The inequality is preserved in the limit as $k \to \infty$, so $\overline{J}(S) \leq \overline{J}(T)$. □

Corollary 8.4 *If T has Jordan content zero, then so does every subset of T.*

Proof. If $S \subseteq T$ and $\overline{J}(T) = 0$, then $\overline{J}(S) = 0$. □

Theorem 8.5. *If S and T are bounded sets, then $\overline{J}(S \cup T) \leq \overline{J}(S) + \overline{J}(T)$.*

Proof. Every square in $\mathcal{J}_k$ that meets $S \cup T$ meets either S or T (or both); thus

$$\overline{J}_k(S \cup T) \leq \overline{J}_k(S) + \overline{J}_k(T).$$

The inequality is preserved in the limit as $k \to \infty$. □

Corollary 8.6 *If $S_1, \ldots, S_p$ are bounded sets, then*

$$\overline{J}(S_1 \cup \cdots \cup S_p) \leq \overline{J}(S_1) + \cdots + \overline{J}(S_p). \qquad \square$$

Corollary 8.7 *The union of a finite number of sets that have Jordan content zero also has Jordan content zero.* □

In particular, every finite set of points has Jordan content zero.

Corollary 8.8 *Suppose that, for any $\varepsilon > 0$, the set S is contained in a finite number of sets whose total Jordan content is less than ε. Then S has Jordan content zero.*

Proof. Suppose $S \subseteq T_1 \cup \cdots \cup T_p$ and

$$\overline{J}(T_1) + \cdots + \overline{J}(T_p) = J(T_1) + \cdots + J(T_p) < \varepsilon.$$

Then $\overline{J}(S) < \varepsilon$ for every $\varepsilon > 0$, so $\overline{J}(S) = 0$. □

Theorem 8.9. *The Jordan content of a square in $\mathcal{J}_k$ is its ordinary area, $1/2^{2k}$, and the Jordan content of the rectangle $[a,b] \times [c,d]$ is its ordinary area $(b-a)(d-c)$.*

Proof. See the exercises. □

We now introduce the notions of the *floor* and *ceiling* of a real number as convenient tools for our work. They are used immediately in the next proof.

Floor and *ceiling*

Definition 8.9 *The **floor of the real number x**, denoted $\lfloor x \rfloor$, is the largest integer m for which $m \leq x$. The **ceiling of x**, denoted $\lceil x \rceil$, is the smallest integer M for which $x \leq M$.*

The Jordan content of a graph

Theorem 8.10. *The graph of a continuous function on a bounded interval has Jordan content zero.*

Proof. Let $y = f(x)$ be continuous on the interval $A \le x \le B$. We show that the graph of f is contained in a finite number of rectangles whose total area is less than any preassigned $\varepsilon > 0$. Corollary 8.8 then implies that the graph has Jordan content zero.

Because the interval is closed and bounded, f is *uniformly* continuous (see a text on analysis), which means that for any $\varepsilon > 0$, there is a $\delta > 0$ for which

$$|u-v| < \delta \quad \Rightarrow \quad |f(u)-f(v)| < \frac{\varepsilon}{4(B-A)},$$

for any $A \le u, v \le B$. A bound like $\varepsilon/4(B-A)$ is chosen with hindsight, of course; the reason emerges below. Let

$$\alpha = \left\lfloor \frac{A}{\delta} \right\rfloor, \quad \beta = \left\lceil \frac{B}{\delta} \right\rceil,$$

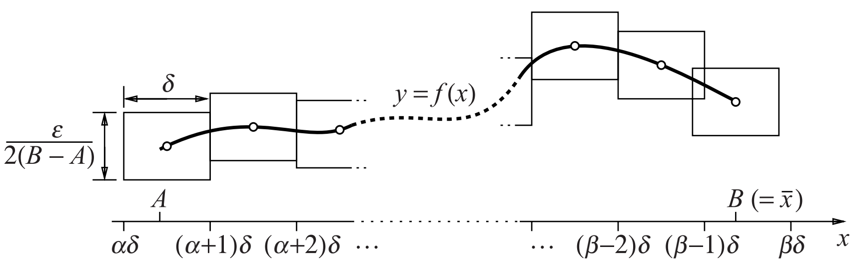
so that $\alpha\delta \le A < (\alpha+1)\delta$ and $(\beta-1)\delta < B \le \beta\delta$. Now partition the x-axis into nonoverlapping closed intervals of length δ, beginning at $\alpha\delta$ and ending at $\beta\delta$. Then A lies in the first interval and B in the last. For clarity, we want these two to be different intervals, so we require $(\alpha+1)\delta < (\beta-1)\delta$; it is sufficient to take $2\delta < B-A$ (Exercise 8.9).

Suppose u and v lie in the same interval. If $\bar{x}$ is the midpoint of that interval, then $|u-\bar{x}| \le \delta/2 < \delta$, $|v-\bar{x}| \le \delta/2 < \delta$, so

$$\begin{aligned} |f(u)-f(v)| &\le |f(u)-f(\bar{x})| + |f(\bar{x})-f(v)| \\ &< \frac{\varepsilon}{4(B-A)} + \frac{\varepsilon}{4(B-A)} = \frac{\varepsilon}{2(B-A)}. \end{aligned}$$

The inequalities imply that the graph of $y = f(x)$ is entirely contained inside $\beta-\alpha$ closed rectangles, each of which is δ units wide and $\varepsilon/2(B-A)$ units tall. (As the figure below demonstrates, it may be necessary to take $\bar{x}$ to be A or B in the first or last interval, so there may be some overlapping at the ends.) The total area of the rectangles is

$$(\beta-\alpha) \times \delta \times \frac{\varepsilon}{2(B-A)}.$$

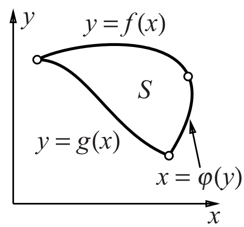


But $(\beta-\alpha)\delta < B-A+2\delta$; because we have also taken $2\delta < B-A$, it follows that $(\beta-\alpha)\delta < 2(B-A)$ and thus

$$\text{total area} = \frac{(\beta-\alpha)\delta}{2(B-A)}\varepsilon < \varepsilon.$$

The graph of f is therefore contained in a finite number of sets whose total Jordan content can be made smaller than any preassigned positive number ε. By Corollary 8.8, the graph of f has Jordan content zero. □

Corollary 8.11 *Suppose S is a bounded set in the (x,y)-plane whose boundary consists of a finite number of curves, each of which is the graph of a continuous function $y = f(x)$ or $x = \varphi(y)$. Then S is Jordan measurable.*



Proof. Each graph has Jordan content zero. There are only finitely many in ∂S, so ∂S likewise has Jordan content zero. By Theorem 8.2 (p. 282), S itself is Jordan measurable. □

Theorem 8.12. *If S and T are Jordan measurable sets, then so are $S\cup T$ and $S\cap T$, and*

$$J(S\cup T) \le J(S)+J(T),$$
$$J(S\cap T) \le J(S), \quad J(S\cap T) \le J(T).$$

Proof. Each boundary point of either $S\cup T$ or $S\cap T$ is a boundary point of S or of T:

$$\partial(S\cup T) \subseteq \partial S\cup\partial T, \quad \partial(S\cap T) \subseteq \partial S\cup\partial T.$$

By hypothesis, ∂S and ∂T have Jordan content zero; hence, so do their subsets $\partial(S\cup T)$ and $\partial(S\cap T)$ (Corollary 8.4). Consequently, $S\cup T$ and $S\cap T$ are both Jordan measurable. Theorem 8.5 then implies

$$J(S\cup T) = \overline{J}(S\cup T) \le \overline{J}(S)+\overline{J}(T) = J(S)+J(T).$$

Because $S\cap T \subseteq S$, Theorem 8.3 implies $J(S\cap T) = \overline{J}(S\cap T) \le \overline{J}(S) = J(S)$. Finally, $S\cap T \subseteq T$, so the same argument gives $J(S\cap T) \le J(T)$. □

Definition 8.10 *Two sets **overlap** if their interiors have a nonempty intersection. They are **nonoverlapping** if their interiors are disjoint.*

Theorem 8.13. *If S and T are bounded Jordan measurable sets that do not overlap, then*

$$J(S\cup T) = J(S)+J(T).$$

Proof. Because S and T have disjoint interiors, a grid square Q_1 that counts in the area $\underline{J}_k(S)$ cannot be entirely contained in T, so it does not count in $\underline{J}_k(T)$. Similarly, a grid square that counts in $\underline{J}_k(T)$ does not count in $\underline{J}_k(S)$. Of course, every grid square that counts in one or the other counts in $\underline{J}_k(S\cup T)$, so

$$\underline{J}_k(S)+\underline{J}_k(T) \le \underline{J}_k(S\cup T).$$

In the limit, $\underline{J}(S)+\underline{J}(T) \le \underline{J}(S\cup T)$, and then $J(S)+J(T) \le J(S\cup T)$ because S, T, and $S\cup T$ are Jordan measurable. Finally, with Theorem 8.12 we have

$$J(S)+J(T) = J(S\cup T). \qquad \square$$

Finite additivity of Jordan content

This leads immediately to the **finite additivity** of Jordan content.

Corollary 8.14 *If $S_1, \dots, S_p$ are Jordan measurable sets, and no two overlap, then $S_1\cup\cdots\cup S_p$ is Jordan measurable and*

$$J(S_1\cup\cdots\cup S_p) = J(S_1)+\cdots+J(S_p). \qquad \square$$

Overlapping sets

When sets overlap, there is still a definite relation between the area of their union and their individual areas. We make use of the **set difference $\boldsymbol{T\setminus S}$**; this is the set of points in T that are not in S (i.e., $T\setminus S = T\cap S^c$). The definition does not assume $S\subseteq T$.

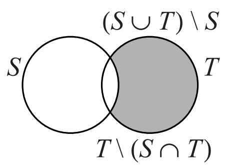
Lemma 8.3. *Suppose T and $S\subseteq T$ are Jordan measurable; then $T\setminus S$ is Jordan measurable and $J(T\setminus S) = J(T)-J(S)$.*

Proof. Becuase $\partial(T\setminus S)\subseteq \partial T\cup\partial S$ (see Exercise 8.14.b), $\partial(T\setminus S)$ has area zero and $T\setminus S$ is Jordan measurable. Because $T=(T\setminus S)\cup S$ and $T\setminus S$ and S do not overlap (they are disjoint), we have

$$J(T) = J(T\setminus S)+J(S). \qquad \square$$

Theorem 8.15. *Suppose S and T are Jordan measurable; then*

$$J(S\cup T)+J(S\cap T) = J(S)+J(T).$$



Proof. Because $S\subseteq (S\cup T)$ and $(S\cap T)\subseteq T$, and all these are Jordan measurable, the lemma applies to both set differences

$$(S\cup T)\setminus S = T\setminus(S\cap T).$$

Because the set differences are equal, the lemma implies

$$J(S\cup T)-J(S) = J(T)-J(S\cap T);$$

to get the theorem, just rearrange the terms. $\square$

Another way to state the theorem that perhaps indicates more explicitly how the overlap $S\cap T$ affects the area of the union is

$$J(S\cup T) = J(S)+J(T)-J(S\cap T).$$

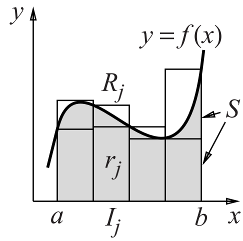
Area under a graph

We can equate the integral of a function with the Jordan content of the region under its graph, making a useful connection between area and Jordan content.

Theorem 8.16. *Let $y = f(x)$ be continuous and nonnegative on $a \le x \le b$, and let S be the region in the (x,y)-plane bounded by the vertical lines $x = a$ and $x = b$, the x-axis, and the graph of f. Then S is Jordan measurable and*

$$J(S) = \int_a^b f(x)\,dx.$$

Proof. The boundary of S has Jordan content zero (Theorem 8.10), so S is Jordan measurable.



To get estimates for the integral of f, subdivide the interval $a \le x \le b$ into K equal pieces $I_1, \ldots, I_K$, each of length $\Delta x = (b-a)/K$. Let m_j and M_j the the minimum and maximum values of $y = f(x)$ on I_j; then

$$m_1\Delta x + \cdots + m_K\Delta x \le \int_a^b f(x)\,dx \le M_1\Delta x + \cdots + M_K\Delta x,$$

and these bounds converge to the value of the integral as $K \to \infty$.

Now let r_j be the rectangle with base I_j and height m_j, and let R_j be the rectangle with the same base but height M_j. Then $r_1, \ldots, r_K$ are nonoverlapping rectangles whose union is contained in S, and $R_1, \ldots, R_K$ are nonoverlapping rectangles whose union contains S:

$$r_1 \cup \cdots \cup r_K \subseteq S \subseteq R_1 \cup \cdots \cup R_K.$$

By the finite additivity of Jordan content (Corollary 8.14),

$$J(r_1 \cup \cdots \cup r_K) = J(r_1) + \cdots + J(r_k),$$
$$J(R_1 \cup \cdots \cup R_K) = J(R_1) + \cdots + J(R_K),$$

and the set inclusions imply

$$J(r_1) + \cdots + J(r_k) \le J(S) \le J(R_1) + \cdots + J(R_K).$$

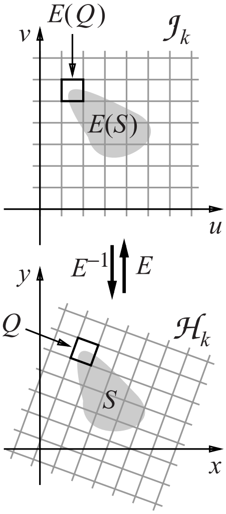
But $J(r_j) = m_j\Delta x$ and $J(R_j) = M_j\Delta x$, so

$$m_1\Delta x + \cdots + m_K\Delta x \le J(S) \le M_1\Delta x + \cdots + M_K\Delta x.$$

Thus, $J(S)$ has the same bounds as the integral; these bounds therefore converge to $J(S)$ as well as to the integral. □

Jordan content via other grids

In elementary geometry, congruent figures have the same area; we now prove they have the same Jordan content, too. We need to show that Jordan content is preserved by the translations, rotations, and reflections that link congruent figures. Such invariance is not immediately obvious, because we have restricted ourselves to a single collection of grids $\mathcal{J}_k$. For example, a translate of S does not, in general, have the same relation to $\mathcal{J}_k$ that S itself does. However, if we translate the grid as well and can show that the translated grid yields the same Jordan content as the original grid, then it follows that Jordan content is invariant under translations. This leads us to the task of showing that the Jordan content of a set can be determined



just as well by many other grids. We concentrate on grids obtained from $\mathcal{J}_k$ by a Euclidean motion or, more generally, by a coordinate change.

For simplicity, we begin with a Euclidean motion $E:\mathbb{R}^2\to\mathbb{R}^2$. The action of E is given by an orthogonal matrix R (either a rotation or a reflection) followed by a translation. We can write formulas for E and its inverse as

$$\mathbf{u}=E(\mathbf{x})=R\mathbf{x}+\mathbf{a},\quad \mathbf{x}=E^{-1}(\mathbf{u})=R^{-1}(\mathbf{u}-\mathbf{a})=R^{\dagger}\mathbf{u}+\mathbf{b},$$

where $\mathbf{b}=-R^{\dagger}\mathbf{a}$ and $R^{\dagger}=R^{-1}$ because R is orthogonal. If S is a bounded set in the plane, then its image $E(S)$ under a Euclidean motion stands in the same relation to the grid $\mathcal{J}_k$ that S itself does to the new grid $\mathcal{H}_k=E^{-1}(\mathcal{J}_k)$. Each time a square Q in $\mathcal{H}_k$ counts in estimating the area of S, its image $E(Q)$ in $\mathcal{J}_k$ counts in estimating the area of $E(S)$.

But there is still a problem, because we do not know that Q has the same Jordan content as $E(Q)$. (This is precisely the question we are trying to settle: the invariance of Jordan content under Euclidean motions!) Nevertheless, Q is certainly Jordan measurable, because ∂Q is just a finite collection of line segments with Jordan content zero. Making no assumption about the value of $J(Q)$, we now construct the analogues of $\underline{J}_k$, $\overline{J}_k$, $\underline{J}$, and $\overline{J}$ for the grids $\mathcal{H}_k$ (cf. Definition 8.6, page 281).

$\underline{H}_k$ and $\overline{H}_k$ for $\mathcal{H}_k$

Definition 8.11 *Let* $\underline{\boldsymbol{H}}_{\boldsymbol{k}}(\boldsymbol{S})$ *denote the total Jordan content of the squares in* $\mathcal{H}_k$ *that are entirely contained in the bounded set S, and let* $\overline{\boldsymbol{H}}_{\boldsymbol{k}}(\boldsymbol{S})$ *denote the total Jordan content of the squares in* $\mathcal{H}_k$ *that intersect S.*

We can express these very compactly using set and summation notation:

$$\underline{H}_k(S)=\sum_{\substack{Q\in\mathcal{H}_k\\ Q\subseteq S}}J(Q),\quad \overline{H}_k(S)=\sum_{\substack{Q\in\mathcal{H}_k\\ Q\cap S\neq\phi}}J(Q)$$

The values of $\underline{H}_k(S)$ and $\overline{H}_l(S)$ are nested in the same way as the values of $\underline{J}_k(S)$ and $\overline{J}_l(S)$; this gives us the limits

$$\underline{H}(S)=\lim_{k\to\infty}\underline{H}_k(S),\quad \overline{H}(S)=\lim_{k\to\infty}\overline{H}_k(S),$$

and the inequality $\underline{H}(S)\le\overline{H}(S)$.

H content

Definition 8.12 *If* $\underline{H}(S)=\overline{H}(S)$*, we say that* ***S is H measurable*** *and we define the* ***H content of S*** *to be* $\boldsymbol{H}(\boldsymbol{S})=\underline{H}(S)=\overline{H}(S)$.

Lemma 8.4. *Suppose S is Jordan measurable, then* $\underline{H}_k(S)\le J(S)\le\overline{H}_k(S)$ *for every* $k=0,1,2,\ldots$.

Proof. Suppose $Q_1,\ldots,Q_p$ are the squares of $\mathcal{H}_k$ that are counted in $\underline{H}_k(S)$. Because they are nonoverlapping and $Q_1\cup\cdots\cup Q_p\subseteq S$, the finite additivity of J implies

$$\underline{H}_k(S)=J(Q_1)+\cdots+J(Q_p)=J(Q_1\cup\cdots\cup Q_p)\le J(S).$$

The inequality $J(S)\le\overline{H}_k(S)$ is proven in a similar way. □

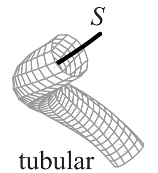
Corollary 8.17 *Suppose a Jordan measurable set S is also H measurable, then $H(S) = J(S)$.* □

But must a J-measurable set also be H measurable? The corollary directs our attention to the difference $\overline{H}_k(S) - \underline{H}_k(S)$, because

$$S \text{ is } H \text{ measurable} \Leftrightarrow \lim_{k\to\infty} \left(\overline{H}_k(S) - \underline{H}_k(S)\right) = 0$$

(cf. the comment at the beginning of the proof of Theorem 8.2, p. 282). Therefore, if we can show that $\overline{H}_k(S) - \underline{H}_k(S)$ is smaller than any preassigned $\varepsilon > 0$ when k is sufficiently large, we shall have shown that $H(S) = J(S)$. To complete our argument, it is useful to introduce the notion of a *tubular neighborhood*.

Definition 8.13 *If S is a bounded set, the* ***tubular neighborhood of S of width $w > 0$*** *is the set of points T that are within distance w of some point of S.*

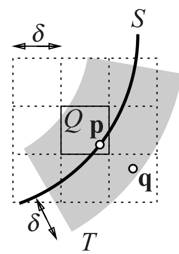


The tubular neighborhood of S of width w is the union of the open disks of radius w centered at all the points of S. If S is a smooth curve in space, and w is small enough, then the tubular neighborhood looks like a tube with S at its core; together, they resemble a coaxial cable.

Lemma 8.5. *Suppose S has Jordan content zero and $\varepsilon > 0$ is given. Then S has a tubular neighborhood T of some width $\delta > 0$ for which $\overline{J}(T) < \varepsilon$.*

Proof. Because $J(S) = 0$, we know $\overline{J}_k(S) \to 0$ as $k \to \infty$. Choose K so large that $\overline{J}_K(S) < \varepsilon/9$. The squares Q in $\mathcal{J}_K$ that are counted in $\overline{J}_K(S)$ cover S and have total Jordan content less than $\varepsilon/9$.

Define T to be the tubular neighborhood of S of width $\delta = 1/2^K$, and let $\mathbf{q}$ be a point in T. By definition, $\mathbf{q}$ is within distance $\delta = 1/2^K$ of some point $\mathbf{p}$ in S. Let Q be a square counted in $\overline{J}_K(S)$ that contains $\mathbf{p}$. Because the squares in $\mathcal{J}_K$ are closed and have width $\delta = 1/2^K$, the point $\mathbf{q}$ is either in Q or one of the eight neighbors of Q. Now $\mathbf{q}$ is an arbitrary point of T, so T is covered by the squares Q and their immediate neighbors, whose total Jordan content is less than $9 \times \varepsilon/9 = \varepsilon$; hence $\overline{J}(T) \leq \overline{J}_K(T) < \varepsilon$. □



Theorem 8.18. *Suppose S is Jordan measurable; then it is H measurable and $J(S) = H(S)$.*

Proof. By Corollary 8.17 and the discussion following it, we need only show that, given any $\varepsilon > 0$, there is a K such that

$$\overline{H}_K(S) - \underline{H}_K(S) < \varepsilon.$$

Note: the sequence $\overline{H}_k(S) - \underline{H}_k(S)$ decreases monotonically as k increases, so we also have $\overline{H}_k(S) - \underline{H}_k(S) < \varepsilon$ for all $k \geq K$.

By hypothesis, S is Jordan measurable, so ∂S has Jordan content zero. Therefore, using the given $\varepsilon > 0$ and Lemma 8.5, we know ∂S has a tubular neighborhood T of some width $\delta > 0$ for which $\overline{J}(T) < \varepsilon$. Now choose K so that the *diameter* (see below, Definition 8.14) of any grid square Q in $\mathcal{H}_K$ is less than δ. The diameter of Q is its diagonal length $\sqrt{2}/2^K < 1/2^{K-1}$, so it is sufficient to choose

$$K > 1 + \log_2(1/\delta).$$

Consider the difference $\overline{H}_K(S) - \underline{H}_K(S)$. Adapting the arguments of the proof of Theorem 8.2 from $\mathcal{J}_k$ to $\mathcal{H}_K$, we draw two conclusions:

- $\overline{H}_K(S) - \underline{H}_K(S)$ is the total Jordan content of the squares Q in $\mathcal{H}_K$ that meet S but are not entirely contained in S.
- Each such square Q contains a point of ∂S.

The diameter of Q is less than δ (by construction); thus the entire square Q lies within the tubular neighborhood T. Hence, the total Jordan content of the squares Q counted in $\overline{H}_K(S) - \underline{H}_K(S)$ is less than the outer Jordan content of T; that is,

$$\overline{H}_K(S) - \underline{H}_K(S) < \varepsilon. \qquad \square$$

Jordan content of congruent figures

One of our main objectives, to show that congruent figures have the same Jordan content, now follows as an immediate corollary.

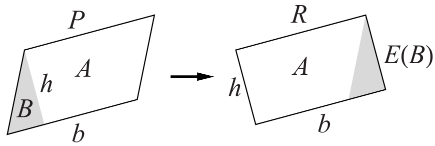
Corollary 8.19 *If S is Jordan measurable and $E : \mathbb{R}^2 \to \mathbb{R}^2$ is a Euclidean motion, then $E(S)$ is Jordan measurable and $J(E(S)) = J(S)$.*

Proof. $J(E(S)) = H(S) = J(S)$. $\square$

Jordan content and ordinary area

Thus, if a rectangle R with sides of length l and w lies anywhere in the (x,y)-plane, a Euclidean motion E will transform it into the rectangle $E(R) : 0 \le x \le l,\ 0 \le y \le w$. By Exercise 8.8, $J(E(R)) = lw$; therefore, $J(R) = lw$. If P is a parallelogram with base b and height h, then P can be decomposed into nonoverlapping sets A and B so that A and a translate $E(B)$ are nonoverlapping and form a rectangle R with the same base and height. Therefore,

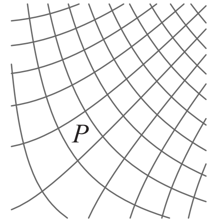
$$J(P) = J(A) + J(B) = J(A) + J(E(B)) = J(R) = bh.$$



If T is a triangle with base b and height h, then similar geometric arguments show that $J(T) = \frac{1}{2}bh$. Because a polygon can be written as a union of nonoverlapping triangles, it follows that the Jordan content of any polygon equals its ordinary area.

General grids $\mathcal{G}_k$ and G content

Under what conditions will a more general collection $\mathcal{G}_k$ $(k = 0, 1, 2, \ldots)$ of grids on the plane determine Jordan content? We assume that each grid is a refinement of its predecessor, and also that the individual cells P of a grid are closed, connected, nonoverlapping sets with positive Jordan content that together cover $\mathbb{R}^2$. The cells need not be congruent or even straight-sided. We define:



- $\underline{G}_k(S)$ is the total Jordan content of the cells P of $\mathcal{G}_k$ that are entirely contained in S;
- $\overline{G}_k(S)$ is the total Jordan content of the cells P of $\mathcal{G}_k$ that meet S.

$$\underline{G}_k(S) = \sum_{\substack{P \in \mathcal{G}_k \\ P \subseteq S}} J(P), \quad \overline{G}_k(S) = \sum_{\substack{P \in \mathcal{G}_k \\ P \cap S \neq \phi}} J(P).$$

Because $\mathcal{G}_k$ refines its predecessor, $\underline{G}_k(S)$ increases monotonically with k to its limiting value $\underline{G}(S)$, the inner G content of S. Likewise, $\overline{G}_k(S)$ decreases monotonically to the outer G content, $\overline{G}(S)$, and

$$\underline{G}_k(S) \leq \underline{G}(S) \leq \overline{G}(S) \leq \overline{G}_k(S).$$

If the inner and outer G content are equal, we say S is ***G* measurable** and has ***G* content** $\boldsymbol{G(S)} = \underline{G}(S) = \overline{G}(S)$. Under what conditions will $G(S) = J(S)$? For H content, the answer was provided by Theorem 8.18, whose proof involved the *linear* dimensions (i.e., diameters) of the grid elements.

Diameter $\delta(S)$ and mesh size $\|\mathcal{G}\|$

Definition 8.14 *The **diameter** of the set S, denoted $\boldsymbol{\delta(S)}$, is the maximum distance between any two points in its closure $\overline{S}$. The **mesh size** $\|\mathcal{G}\|$ of a grid $\mathcal{G}$ is the smallest upper bound on the size of the diameters of the elements P of $\mathcal{G}$.*

Theorem 8.20. *If $\|\mathcal{G}_k\| \to 0$ as $k \to \infty$, then every Jordan-measurable set S is G measurable, and $J(S) = G(S)$.*

Proof. This argument imitates the proof of Theorem 8.18 and the lemmas preceding it. First of all, because S is Jordan measurable,

$$\underline{G}_k(S) \leq J(S) \leq \overline{G}_k(S)$$

(cf. Lemma 8.4). Hence, to prove the theorem it suffices to show

$$\lim_{k\to\infty} \left(\overline{G}_k(S) - \underline{G}_k(S)\right) = 0.$$

Suppose $\varepsilon > 0$ is given; we wish to show that there is an integer $K = K(\varepsilon)$ for which

$$\overline{G}_k(S) - \underline{G}_k(S) < \varepsilon,$$

for all $k > K$. Because S is Jordan measurable, $J(\partial S) = 0$ and ∂S has a tubular neighborhood T of some width δ for which $\overline{J}(T) < \varepsilon$ (Lemma 8.5). Because $\|\mathcal{G}_k\| \to 0$ as $k \to \infty$, we can choose K so that $\|\mathcal{G}_k\| < \delta$ for all $k > K$. Each cell P in the grid $\mathcal{G}_k$ contains a point $\mathbf{p}$ in S and a point $\mathbf{q}$ not in S. Because P is connected, there is a continuous path in P from $\mathbf{p}$ to $\mathbf{q}$, and that path must contain a point of ∂P (Exercise 8.6). If $k > K$, then the diameter of P is less than δ, so P is entirely contained within the tubular neighborhood T. Hence the total Jordan content of the cells P counted in $\overline{G}_k(S) - \underline{G}_k(S)$ is less than $\overline{J}(T) < \varepsilon$; that is,

$$\overline{G}_k(S) - \underline{G}_k(S) < \varepsilon. \qquad \square$$

Area magnification by a linear map

We are now able to extend to Jordan content our earlier observation about the area magnification factor of a linear map (Theorem 2.8, p. 42). We make frequent use of the fact that the Jordan content of a polygon P is its ordinary (absolute) area; thus, if $L(P)$ is its polygonal image under a linear map, then

$$J(L(P)) = |\det L|\, J(P).$$

Lemma 8.6. *If $L : \mathbb{R}^2 \to \mathbb{R}^2$ is linear and $J(S) = 0$, then $J(L(S)) = 0$.*

Proof. If L is not invertible, the proof is immediate, because the whole image $L(\mathbb{R}^2)$ is a line, so $L(S)$ is a finite line segment and automatically has Jordan content zero.

If L is invertible (i.e., $\det L \neq 0$), then we show that $L(S)$ is contained in a union of sets whose total Jordan content is less than any positive number ε. The lemma then follows from Corollary 8.8 (p. 283).

By hypothesis, $J(S) = 0$ so $\overline{J}_K(S) < \varepsilon / |\det L|$ for some integer K. That is, S is covered by squares Q whose total Jordan content is less than $\varepsilon / |\det L|$. Therefore $L(S)$ is covered by the images $L(Q)$ of those squares. Because $J(L(Q)) = |\det L|\, J(Q)$, the total Jordan content of the sets covering $L(S)$ is less than ε. $\square$

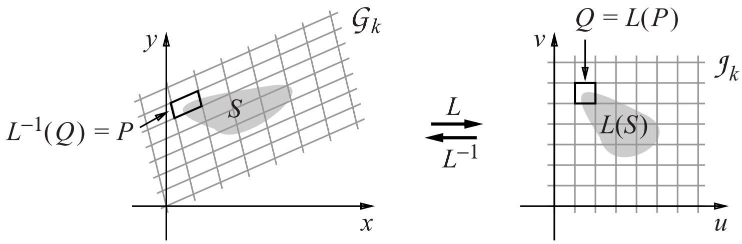
Corollary 8.21 *The image of a Jordan-measurable set under a linear map is Jordan measurable.* $\square$

The multiplier for Jordan content

We can now show that the Jordan content multiplier of a linear map is the absolute value of its determinant.

Theorem 8.22. *Suppose S is Jordan measurable and $L : \mathbb{R}^2 \to \mathbb{R}^2$ is linear; then $J(L(S)) = |\det L|\, J(S)$.*

Proof. If L is not invertible, then $\det L = 0$, and $L(S)$ is a bounded subset of the line $L(\mathbb{R}^2)$. Thus $J(L(S)) = 0 = |\det L|\, J(S)$.



For an invertible map L, we adapt the argument we used to prove that a Euclidean motion preserves Jordan content (pp. 287–290). The key to the argument is to note that $L(S)$ has the same relation to the grid $\mathcal{J}_k$ that S itself does to the new grid $\mathcal{G}_k = L^{-1}(\mathcal{J}_k)$ (see the figure below). Of course, to use the G-content functions associated with $\mathcal{G}_k$ (as defined above, p. 291), we want Theorem 8.20 to hold. We must therefore check that the maximum diameter $\|\mathcal{G}_k\|$ of a cell P of $\mathcal{G}_k$ tends to zero as $k \to \infty$. Exercise 8.16 establishes that the diameters of Q and $P = L^{-1}(Q)$ are linked by a constant σ that depends only on L^{-1} and not on k: $\delta(P)=\sigma\delta(Q)$. Thus all cells P of $\mathcal{G}_k$ have the same diameter: $\|\mathcal{G}_k\|=\sigma\delta(Q)$. Because $\delta(Q)=\sqrt{2}/2^k$, it follows that $\|\mathcal{G}_k\|\to 0$ as $k\to\infty$. By Theorem 8.20, the given Jordan-measurable set S is also G measurable, and $J(S)=G(S)$.

As P is contained in S precisely when $Q=L(P)$ is contained in $L(S)$, we have (using $P=L^{-1}(Q)$ as well)

$$\underline{G}_k(S)=\sum_{\substack{P\in\mathcal{G}_k\\ P\subseteq S}} J(P)=\sum_{\substack{Q\in\mathcal{J}_k\\ Q\subseteq L(S)}} J(L^{-1}(Q)).$$

Because Q is a polygon, $J(L^{-1}(Q))=|\det L^{-1}|J(Q)$, so the last sum becomes

$$|\det L^{-1}|\sum_{\substack{Q\in\mathcal{J}_k\\ Q\subseteq L(S)}} J(Q)=|\det L^{-1}|\,\underline{J}_k(L(S)),$$

or just $\underline{G}_k(S)=|\det L^{-1}|\,\underline{J}_k(L(S))$. Because $|\det L^{-1}|=|\det L|^{-1}$, we have

$$\underline{J}_k(L(S))=|\det L|\,\underline{G}_k(S).$$

In the limit as $k\to\infty$, $J(L(S))=|\det L|\,G(S)=|\det L|\,J(S)$. □

Grids for integration

When we take up integration in the next section, we need an even larger class of grids than sequences $\mathcal{G}_0,\mathcal{G}_1,\mathcal{G}_2,\ldots$ of successive refinements of the sort we have considered so far. The nonnegative integers that we use to index these grids have a natural order that is imparted to the grids themselves: there is a "first" grid, then a "second," and so on. When we say that each grid refines its predecessor, we make implicit use of that order.

So when we enlarge the class of grids, such a larger collection $\{\mathcal{G}\}$ typically has no natural ordering. Thus, even though one grid in the collection may be a refinement of another, the notion of "predecessor" is now missing, and we are no longer able to say that a grid refines its predecessor. Nevertheless, we still assume the cells P in any grid are closed, connected, nonoverlapping sets with positive Jordan content, and together they cover $\mathbb{R}^2$. Then we define (exactly as we did for the more restricted class of grids $\mathcal{G}_k$):

- $\underline{G}_{\mathcal{G}}(S)$ is the total Jordan content of the cells P of $\mathcal{G}$ that are entirely contained in S.
- $\overline{G}_{\mathcal{G}}(S)$ is the total Jordan content of the cells P of $\mathcal{G}$ that meet S.

$$\underline{G}_{\mathcal{G}}(S)=\sum_{\substack{P\in\mathcal{G}\\ P\subseteq S}} J(P),\quad \overline{G}_{\mathcal{G}}(S)=\sum_{\substack{P\in\mathcal{G}\\ P\cap S\neq\phi}} J(P)$$

$\underline{G}_{\mathcal{G}}(S)$ and $\overline{G}_{\mathcal{G}}(S)$ vary with $\|\mathcal{G}\|$

We should next get inner and outer G content (i.e., $\underline{G}(S)$ and $\overline{G}(S)$). When grids were indexed by integers $k=0,1,2,\ldots$, we just took limits of $\underline{G}_k(S)$ and $\overline{G}_k(S)$ as $k\to\infty$ (and monotonicity guaranteed that the limits existed). But for an arbitrary collection of grids $\{\mathcal{G}\}$, there is no index that supplies an ordering. Nevertheless, if we consider how $\underline{G}_{\mathcal{G}}(S)$ and $\overline{G}_{\mathcal{G}}(S)$ vary with mesh size $\|\mathcal{G}\|$, we see that they do have well-defined limits, at least if S has Jordan content. We then have a way, once again, to define G content.

To see how this happens, suppose S is Jordan measurable. Then, for any grid $\mathcal{G}$,

$$\underline{G}_{\mathcal{G}}(S) \leq J(S) \leq \overline{G}_{\mathcal{G}}(S)$$

(cf. Lemma 8.4). Therefore, if we can show that

$$\lim_{\|\mathcal{G}\|\to 0} \left(\overline{G}_{\mathcal{G}}(S) - \underline{G}_{\mathcal{G}}(S)\right) = 0,$$

then we know the following limits exist:

$$\underline{G}(S) = \lim_{\|\mathcal{G}\|\to 0} \underline{G}_{\mathcal{G}}(S) = J(S), \quad \overline{G}(S) = \lim_{\|\mathcal{G}\|\to 0} \overline{G}_{\mathcal{G}}(S) = J(S).$$

We can then say

- S is G measurable.
- $G(S) = \underline{G}(S) = \overline{G}(S)$.
- $J(S) = G(S)$.

Hence, for any given $\varepsilon > 0$, we must show there is a $\delta > 0$ for which

$$\|\mathcal{G}\| < \delta \quad \Rightarrow \quad \overline{G}_{\mathcal{G}}(S) - \underline{G}_{\mathcal{G}}(S) < \varepsilon.$$

By Lemma 8.5 we know that ∂S has a tubular neighborhood T of some positive width δ for which $\overline{J}(T) < \varepsilon$. Now suppose $\|\mathcal{G}\| < \delta$. Then, by essentially the same argument as in the proof of Theorem 8.20, the total Jordan content of the cells P that are counted in $\overline{G}_{\mathcal{G}}(S) - \underline{G}_{\mathcal{G}}(S)$ is less than $\overline{J}(T) < \varepsilon$. In other words,

$$\|\mathcal{G}\| < \delta \quad \Rightarrow \quad \overline{G}_{\mathcal{G}}(S) - \underline{G}_{\mathcal{G}}(S) < \varepsilon,$$

as required. We state the conclusion as a theorem.

Theorem 8.23. *If $\{\mathcal{G}\}$ is an infinite collection of integration grids whose mesh sizes $\|\mathcal{G}\|$ come arbitrarily close to zero, then G content is defined for all sets S that have Jordan content, and $G(S) = J(S)$.* □

Use area to denote Jordan content

As we have seen, the Jordan content of any plane figure of elementary Euclidean geometry is its ordinary area. For that reason, we now go back to the simpler and more familiar term *area*. Thus, we say that S **has area** if it is Jordan measurable; in that case, the **area** of S is denoted $A(S) = J(S)$. For a more general set S, its **inner area** $\underline{A}(S)$ is its inner Jordan content $\underline{J}(S)$, and its **outer area** $\overline{A}(S)$ is its outer Jordan content $\overline{J}(S)$.

Volume in $\mathbb{R}^3$

With grids of cubes instead of squares, it requires virtually no alterations to transfer the theory of Jordan content from $\mathbb{R}^2$ to $\mathbb{R}^3$. Let us assume that has been done. Then, following what we just did in $\mathbb{R}^2$, we call the Jordan content of a set S in $\mathbb{R}^3$ its **volume**, and write $V(S) = J(S)$. In fact, we can use the same method to get the analogue of area or volume in any dimension.

8.3 Riemann and Darbou integrals

We now introduce double integrals and establish their properties; in the next chapter we develop methods for evaluating them. We define the Riemann integral of a function $z = f(x,y)$ on a closed bounded set S that has area (i.e., is Jordan measurable). We assume f is bounded on S and is extended to all of $\mathbb{R}^2$ by setting $f(x,y) = 0$ when (x,y) is not in S.

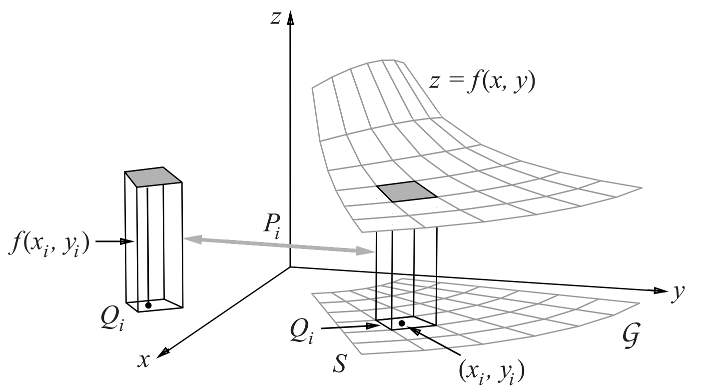
Integration grids

Let $\mathcal{G}$ be a grid of the sort we considered near the end of the previous section. Thus, the cells Q of $\mathcal{G}$ are closed, bounded, nonoverlapping sets that have area. We let $A(Q)$ denote the area of Q and $\delta(Q)$ its diameter (cf. p. 291); the diameters have a finite bound $\|\mathcal{G}\|$, the *mesh size* of $\mathcal{G}$. The cells of $\mathcal{G}$ must cover S, but they need not cover all of $\mathbb{R}^2$. Furthermore, those cells need not be congruent, nor need they have straight sides. We call $\mathcal{G}$ an **integration grid**.

Riemann sums

Let $Q_1,\ldots,Q_N$ be all the cells of $\mathcal{G}$ that meet S; we write the area $A(Q_i)$ as ΔA_i. A **Riemann sum for f over S** is an expression of the form

$$\sum_{i=1}^{N(\mathcal{G})} f(x_i,y_i)\,\Delta A_i,$$



where (x_i,y_i) is a point of Q_i, $i = 1,\ldots,N$. Note that the sum depends upon the grid $\mathcal{G}$ and the points (x_i,y_i), as well as on f and S. By writing $N = N(\mathcal{G})$ as well, we call attention to the fact that the number of cells that meet S depends on $\mathcal{G}$.

Approximating volumes

To interpret such a sum it helps to let f be positive and continuous, as in the figure above. Then $f(x_i,y_i)$ is approximately the height of the prism P_i that has base Q_i, vertical sides, and an irregular top formed by the graph of f; $f(x_i,y_i)\,\Delta A_i$ is approximately its volume. The Riemann sum therefore approximates the total volume of the solid that lies above S and under the graph. To get a better aproximation, make the individual cells smaller; more exactly, use a new grid $\mathcal{G}$ with a smaller mesh size $\|\mathcal{G}\|$. In fact, we expect all Riemann sums will be as close as we wish to the actual volume, as long as the mesh size is sufficiently small, *independently of the way the points are chosen in a grid*. This leads us to the definition of the *Riemann integral*.

The Riemann integral

Definition 8.15 *If the Riemann sums for f over S have a limit that is independent of the points (x_i, y_i) as $\|\mathcal{G}\| \to 0$, then we say **f is integrable over S** and the **integral** is that limit:*

$$\iint_S f(x,y)\,dA = \lim_{\|\mathcal{G}\|\to 0} \sum_{i=1}^{N(\mathcal{G})} f(x_i,y_i)\,\Delta A_i.$$

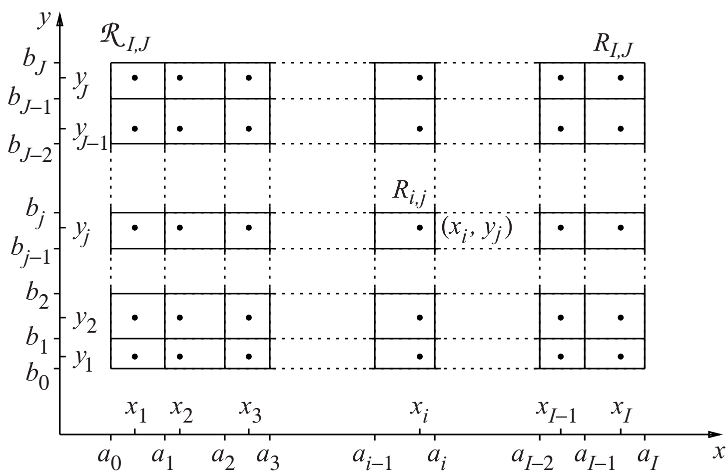
More exactly, this a *Riemann* integral, which is different from the Darboux integral that we introduce presently, as well as from other kinds of integrals that we do not consider. To make it clear that convergence to the limit is uniform with respect to the points chosen in the cells of a grid, we put the definition more formally, as follows. Given an $\varepsilon > 0$, there is a $\delta > 0$ such that, for every grid $\mathcal{G}$ with $\|\mathcal{G}\| < \delta$,

$$\left| \sum_{i=1}^{N(\mathcal{G})} f(x_i,y_i)\,\Delta A_i - \iint_S f(x,y)\,dA \right| < \varepsilon,$$

regardless of the choice of points (x_i, y_i) within the cells of $\mathcal{G}$.

Absolute (unoriented) double integrals

Because the domain is 2-dimensional, we call this a *double* integral. (A rectangular grid, as in the following example, provides an even more compelling reason for the name.) More particularly, this is an *absolute*, or *unoriented*, double integral, because the grid cells Q_i are given no orientation and their areas $\Delta A_i = A(Q_i)$ are always nonnegative.



Here is a special class of grids that are frequently used to construct Riemann sums. First partition the x- and y-axes into nonoverlapping intervals: Riemann double sums

$$[a_{i-1},a_i] : a_{i-1} \le x \le a_i, \quad i = 1,\dots I,$$
$$[b_{j-1},b_j] : b_{j-1} \le y \le b_j, \quad j = 1,\dots J.$$

Let $\Delta x_i = a_i - a_{i-1}$ and $\Delta y_j = b_j - b_{j-1}$ denote the lengths of these intervals. Now form the rectangles $R_{i,j} : [a_{i-1},a_i] \times [b_{j-1},b_j]$; the area of $R_{i,j}$ is the product $A(R_{i,j}) = \Delta x_i \, \Delta y_j$. These rectangles are cells of a grid $\mathcal{R}_{I,J}$ that is natural to index with a *pair* of integers I,J (in contrast, e.g., to the grid $\mathcal{J}_k$).

Let x_i be a point in the x-interval $[a_{i-1},a_i]$, and let y_j be a point in the y-interval $[b_{j-1},b_j]$. Then (x_i,y_j) is a point in $R_{i,j}$, and a Riemann sum for f over S naturally takes the form of a *double* sum:

$$\sum_{i=1}^{I}\sum_{j=1}^{J} f(x_i,y_j)\, \Delta x_i \, \Delta y_j.$$

If f is integrable, then these sums approach the integral of f as the grid mesh size tends to zero. In that case it is natural to replace the "element of area" dA by $dx\,dy$ and to write the limit itself as

$$\iint_S f(x,y)\, dx\, dy = \lim_{\|\mathcal{R}\| \to 0} \sum_{i=1}^{I(\mathcal{R})} \sum_{j=1}^{J(\mathcal{R})} f(x_i,y_j) \Delta x_i \, \Delta y_j.$$

The two summation signs on the right now explain why we use a pair of integral signs to denote the integral, and they also suggest why we call it a *double* integral.

As we have already noted, the terms in a Riemann sum are products of lengths $f(x_i,y_i)$ and areas ΔA_i, so we usually think of a Riemann sum and the resulting integral as *volumes*. We pursue this below, but first we make a connection between double integrals and areas.

Theorem 8.24. *A constant function $f(x,y) = c$ is integrable over every set S that has area, and*
$$\iint_S c\, dA = cA(S) = c \times \text{area}\, S.$$

Proof. Let $\{\mathcal{G}\}$ be an infinite collection of integration grids whose mesh sizes $\|\mathcal{G}\|$ get arbitrarily close to zero. Let $\overline{G}_{\mathcal{G}}$ be the outer content function associated with $\mathcal{G}$ (cf. p. 293). Because $f(x,y) = 0$ outside S, by construction, the grid cells Q_i of $\mathcal{G}$ that make a nonzero contribution to a Riemann sum for f are precisely the ones that meet S; thus

$$\sum_{i=1}^{N} c\, \Delta A_i = c \sum_{Q_i \cap S \ne \phi} A(Q_i) = c\, \overline{G}_{\mathcal{G}}(S).$$

The collection $\{\mathcal{G}\}$ satisfies the hypotheses of Theorem 8.23; therefore, because S has area, it is G measurable and $G(S) = A(S)$. Hence, $\overline{G}_{\mathcal{G}}(S) \to A(S)$ as $\|\mathcal{G}\| \to 0$.

This limit does not depend on the collection $\{\mathcal{G}\}$; therefore we conclude all Riemann sums have the same limit, namely $cA(S)$, and f is thus integrable. $\square$

Theorem 8.25. *Any bounded function f on a set S of zero area is integrable, and*

$$\iint_S f(x,y)\,dA = 0.$$

Proof. Suppose B is a bound on f: $|f(x,y)| \leq B$ for all (x,y) in $\mathbb{R}^2$. Let $\varepsilon > 0$ be given. By Lemma 8.5, we can construct a tubular neighborhood of S of width $\delta > 0$ for which $\overline{J}(T) < \varepsilon/B$. Let $\mathcal{G}$ be any grid for which $\|\mathcal{G}\| < \delta$. In a Riemann sum, the cells Q_i of $\mathcal{G}$ that meet S are contained entirely in T; thus, for any choice of $\mathbf{p}_i$ in Q_i, we have

$$\left|\sum_{i=1}^{N} f(\mathbf{p}_i)\,\Delta A_i - 0\right| \leq \sum_{i=1}^{N} |f(\mathbf{p}_i)|\,\Delta A_i \leq B\sum_{i=1}^{N} \Delta A_i \leq B\,\overline{J}(T) < \varepsilon.$$

This shows that f is integrable and the value of the integral is 0. $\square$

Properties of double integrals

Arguments similar to those in the last proof can be used to establish the following general properties of double integrals; see the exercises. These properties are formally the same as those of integrals of a single-variable function.

Theorem 8.26. *Suppose f and g are integrable over S; then so are cf and $f \pm g$, and*

$$\iint_S cf\,dA = c\iint_S f\,dA, \quad \iint_S (f \pm g)\,dA = \iint_S f\,dA \pm \iint_S g\,dA. \qquad \square$$

Theorem 8.27. *Suppose f is integrable over the nonoverlapping sets R and S; then f is integrable over $R \cup S$ and*

$$\iint\limits_{R\cup S} f\,dA = \iint\limits_{R} f\,dA + \iint\limits_{S} f\,dA. \qquad \square$$

Additivity and linearity of the integral

The second theorem says that the integral is additive over sets in the same sense that area (Jordan content) is; see Corollary 8.14, page 286. The first theorem says that the integral acts as a linear operator on functions.

Theorem 8.28. *Suppose f is integrable over S and $f(x,y) \geq 0$ on S; then*

$$\iint_S f(x,y)\,dA \geq 0. \qquad \square$$

Corollary 8.29 *Suppose f and g are integrable over S, and $f(x,y) \leq g(x,y)$ for every point (x,y) in S; then*

$$\iint_S f(x,y)\,dA \leq \iint_S g(x,y)\,dA. \qquad \square$$

Corollary 8.30 *Suppose f is integrable over S and $m \le f(x,y) \le M$ for every point (x,y) in S; then*

$$mA(S) \le \iint_S f(x,y)\,dA \le MA(S). \qquad \square$$

Proof. By Theorem 8.26, $g(x,y) = f(x,y) - m \ge 0$ is integrable, and

$$0 \le \iint_S g(x,y)\,dA = \iint_S f(x,y)\,dA - \iint_S m\,dA = \iint_S f(x,y)\,dA - mA(S),$$

which leads to the first of the stated inequalities. The second is obtained in a similar way. $\square$

Integrate only bounded functions

We have a standing assumption that integration applies only to bounded functions. That assumption is essential in the last corollary. For example, let S be the unit interval on the x-axis, so $A(S) = 0$ as a region in the plane. Let

$$f(x,y) = \begin{cases} 1/\sqrt{x} & 0 < x \le 1,\ y = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Now consider Riemann sums for f on S constructed with the grids $\mathcal{J}_n$ used to define Jordan content. Let Q_1 be the square $[0,1/2^n] \times [0,1/2^n]$ and let $\mathbf{p}_1 = (1/2^{6n},0)$. Then

$$f(\mathbf{p}_1)\,\Delta A_1 = \frac{1}{\sqrt{1/2^{6n}}} \cdot \frac{1}{2^{2n}} = 2^{3n} \cdot \frac{1}{2^{2n}} = 2^n \to \infty \quad \text{as } n \to \infty,$$

so no Riemann sum that contains this term can converge to a finite value. In other words, f is not integrable, even though its domain has area zero.

Bounds and the Darboux integral

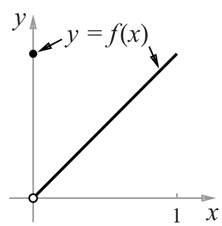
We now turn to the Darboux integral. It is constructed from sums involving upper and lower bounds of a function on each cell of a grid. Although the Darboux and Riemann integrals have similar definitions (and we eventually show they have the same value), the Darboux integral is defined more in the style of Jordan content: there are analogues of inner and outer areas on a grid and inner and outer content as limits.

Least upper bound; greatest lower bound

Suppose f is bounded on S, and $\mathcal{G}$ is an integration grid whose cells $Q_1,\ldots,Q_N$ cover S. We do not assume f is Riemann integrable over S. Let M_i be the smallest of the upper bounds (the "least upper bound") for f on Q_i, and let m_i be the largest of the lower bounds (the "greatest lower bound"). In other words,

$$m_i \le f(\mathbf{p}_i) \le M_i$$

for all points $\mathbf{p}_i$ in Q_i, and these bounds are the best possible. To see what "best possible" means here, consider the following example:



$$f(x) = \begin{cases} 1 & x = 0, \\ x & 0 < x \le 1, \end{cases}$$

where $Q : 0 \le x \le 1$. Then 0 is a lower bound for f on Q, but no larger number $\varepsilon > 0$ is a lower bound, because we can always choose $\bar{x}$ in Q so that $f(\bar{x}) < \varepsilon$ (e.g., take $\bar{x} = \varepsilon/2$). Thus $m = 0$ is the *greatest* lower bound, which is also called an *infimum*. Because there is no "smallest" positive real number, the set of values $f(x)$ has no minimum, but it does have an *infimum*.

inf and sup

Definition 8.16 *Any nonempty set of numbers Z that is bounded below has a* ***greatest lower bound,*** **glb** $\boldsymbol{Z}$*, or* ***infimum,*** **inf** $\boldsymbol{Z}$*; if Z is bounded above, it has a* ***least upper bound,*** **lub** $\boldsymbol{Z}$*, or* ***supremum,*** **sup** $\boldsymbol{Z}$.

Lower and upper Darboux sums

Definition 8.17 *The* ***lower*** *and* ***upper Darboux sums*** *for f over S and the grid $\mathcal{G}$ are, respectively,*

$$\underline{D}_{\mathcal{G}}(f,S) = \sum_{i=1}^{N} m_i \Delta A_i, \quad \overline{D}_{\mathcal{G}}(f,S) = \sum_{i=1}^{N} M_i \Delta A_i.$$

Lower and upper Darboux sums give us lower and upper bounds on all possible Riemann sums that can be constructed with the grid $\mathcal{G}$; that is,

$$\underline{D}_{\mathcal{G}}(f,S) \le \sum_{i=1}^{N} f(\mathbf{p}_i)\Delta A_i \le \overline{D}_{\mathcal{G}}(f,S),$$

no matter how $\mathbf{p}_i$ is chosen in Q_i. The following lemma, which says no lower sum is larger than any upper sum, plays the same role here that Lemma 8.2, page 281, does for the inner and outer area estimates of Jordan content.

Lemma 8.7. *For every pair of integration grids $\mathcal{H}$ and $\mathcal{G}$,*

$$\underline{D}_{\mathcal{H}}(f,S) \le \overline{D}_{\mathcal{G}}(f,S).$$

Proof. We construct the **common refinement,** $\mathcal{K}$, of $\mathcal{H}$ and $\mathcal{G}$. The cells of $\mathcal{K}$ consist of the intersections $P \cap Q$, where P is a cell in $\mathcal{H}$ and Q is a cell in $\mathcal{G}$. Then $\mathcal{K}$ does indeed refine both $\mathcal{H}$ and $\mathcal{G}$, so the usual arguments about refinements imply

$$\underline{D}_{\mathcal{H}}(f,S) \le \underline{D}_{\mathcal{K}}(f,S) \le \overline{D}_{\mathcal{K}}(f,S) \le \overline{D}_{\mathcal{G}}(f,S). \qquad \square$$

Thus each upper sum is an upper bound for all lower sums, and each lower sum is a lower bound for all upper sums. Consequently, the following least upper bound and the greatest lower bound are well-defined.

Lower and upper Darboux integrals

Definition 8.18 *The* ***lower Darboux integral*** $\underline{\mathbf{D}}(\boldsymbol{f},\boldsymbol{S})$ ***of f over S*** *is the least upper bound of the numbers $\underline{D}_{\mathcal{G}}(f,S)$, over all grids $\mathcal{G}$. Similarly, the* ***upper Darboux integral*** $\overline{\mathbf{D}}(\boldsymbol{f},\boldsymbol{S})$ *is the greatest lower bound of the numbers $\overline{D}_{\mathcal{G}}(f,S)$.*

Theorem 8.31. $\underline{D}(f,S) \le \overline{D}(f,S)$.

Proof. Choose $\mathcal{G}$ arbitrarily; by Lemma 8.7, $\overline{D}_{\mathcal{G}}(f,S)$ is an upper bound for all possible lower sums, so it is at least as large as their least upper bound:

$$\underline{D}(f,S) \le \overline{D}_{\mathcal{G}}(f,S).$$

By this inequality, the lower integral $\underline{D}(f,S)$ is a lower bound for all possible upper sums (because $\mathcal{G}$ is arbitrary), so it is at least as small as their greatest lower bound:

$$\underline{D}(f,S) \le \overline{D}(f,S). \qquad \square$$

Definition 8.19 *If $\underline{D}(f,S) = \overline{D}(f,S)$, then **f is Darboux integrable over S**, and its **Darboux integral** is $\boldsymbol{D(f,S)} = \underline{D}(f,S) = \overline{D}(f,S)$.* The Darboux integral

The next two theorems establish that the two notions of integral are equivalent.

Theorem 8.32. *If f is Riemann integrable on S, then it is also Darboux integrable, and the two integrals are equal.*

Proof. Because f is bounded, its upper and lower Darboux sums are defined for all grids. To prove the theorem, it is enough to show that, for any given $\varepsilon > 0$, there is a grid $\mathcal{G}$ for which

$$\overline{D}_{\mathcal{G}}(f,S) - \underline{D}_{\mathcal{G}}(f,S) < \varepsilon.$$

Because $\varepsilon > 0$ is arbitrary, it then follows that $\overline{D}(f,S) - \underline{D}(f,S) = 0$ and that f is Darboux integrable. Moreover, because every Riemann sum is trapped between $\underline{D}_{\mathcal{G}}(f,S)$ and $\overline{D}_{\mathcal{G}}(f,S)$, so is the Riemann integral. The Darboux integral is trapped the same way, so the two integrals must be equal.

Using the given $\varepsilon > 0$ and the hypothesis that f is Riemann integrable, choose $\delta > 0$ so that, for every integration grid $\mathcal{G}$ with $\|\mathcal{G}\| < \delta$,

$$\left| \sum_{i=1}^{N} f(\mathbf{p_i})\,\Delta A_i - \iint_S f(x,y)\,dA \right| < \frac{\varepsilon}{4},$$

regardless of how $\mathbf{p}_i$ is chosen in the cell Q_i of the grid $\mathcal{G}$ (cf. Definition 8.15). What we take from this is the fact that the difference between any two Riemann sums for f with the grid $\mathcal{G}$ is less than $\varepsilon/2$.

Fix a grid $\mathcal{G}$ for which $\|\mathcal{G}\| < \delta$, and let $Q_1, \ldots, Q_N$ be the cells of $\mathcal{G}$ that meet S. Construct the the lower and upper Darboux sums

$$\underline{D}_{\mathcal{G}}(f,S) = \sum_{i=1}^{N} m_i \Delta A_i, \quad \overline{D}_{\mathcal{G}}(f,S) = \sum_{i=i}^{N} M_i \Delta A_i,$$

in the usual way, and set

$$A = \sum_{i=i}^{N} \Delta A_i = \overline{G}_{\mathcal{G}}(S),$$

the outer G content of S with respect to the grid $\mathcal{G}$.

Because m_i is the greatest lower bound of $f(\mathbf{x})$ on Q_i, $m_i + \varepsilon/4A$ is not a lower bound. In other words, there is a point $\mathbf{p}_i$ in each Q_i for which

$$f(\mathbf{p}_i) < m_i + \frac{\varepsilon}{4A}.$$

Therefore,

$$\sum_{i=1}^{N} f(\mathbf{p}_i)\,\Delta A_i < \sum_{i=1}^{N} m_i\,\Delta A_i + \frac{\varepsilon}{4A}\sum_{i=i}^{N}\Delta A_i, = \underline{D}_{\mathcal{G}}(f,S) + \frac{\varepsilon}{4}.$$

In a similar way, there is a point $\mathbf{q}_i$ in each Q_i for which

$$M_i - \frac{\varepsilon}{4A} < f(\mathbf{q}_i),$$

and a similar argument shows that

$$\overline{D}_{\mathcal{G}}(f,S) - \frac{\varepsilon}{4} < \sum_{i=1}^{N} f(\mathbf{q}_i)\,\Delta A_i.$$

Subtracting the first inequality from the second, we find

$$\overline{D}_{\mathcal{G}}(f,S) - \underline{D}_{\mathcal{G}}(f,S) - \frac{\varepsilon}{2} < \sum_{i=1}^{N} f(\mathbf{q}_i)\,\Delta A_i - \sum_{i=1}^{N} f(\mathbf{p}_i)\,\Delta A_i < \frac{\varepsilon}{2}.$$

The last inequality in this sequence is just the fact that any two Riemann sums differ by less than $\varepsilon/2$. Hence $\overline{D}_{\mathcal{G}}(f,S) - \underline{D}_{\mathcal{G}}(f,S) < \varepsilon$; by what was said above, this completes the proof. □

Theorem 8.33. *If f is Darboux integrable on S, then it is also Riemann integrable, and the two integrals are equal.*

Proof. Let D be the value of the Darboux integral of f on S. We must show that, for any given $\varepsilon > 0$, there is a $\delta > 0$ so that, for any integration grid $\mathcal{G}$ with $\|\mathcal{G}\| < \delta$,

$$\left|\sum_{i=1}^{N} f(\mathbf{p}_i)\,\Delta A_i - D\right| < \varepsilon,$$

regardless of how the point $\mathbf{p}_i$ is chosen in the cell Q_i of $\mathcal{G}$.

Every Darboux integrable function is bounded, by definition; choose B so that $|f(x,y)| \le B$ on S. The definition also implies that upper and lower Darboux sums for f get arbitrarily close to D. Thus, for the ε given above, we can select a particular grid $\mathcal{H}$ for which

$$D - \frac{\varepsilon}{2} < \underline{D}_{\mathcal{H}}(f,S) \text{ and } \overline{D}_{\mathcal{H}}(f,S) < D + \frac{\varepsilon}{2}.$$

Suppose the cells of $\mathcal{H}$ that cover S are $P_1,\ldots,P_J$. Let ∂P denote the set of boundary points of all these cells. Because each P_j has area, $A(\partial P_j) = 0$ and thus $A(\partial P) = 0$. By Lemma 8.5, page 289, ∂P has a tubular neighborhood T of some width $\delta > 0$ for which

$$\overline{J}(T) < \frac{\varepsilon}{4B}.$$

Now let $\mathcal{G}$ be any integration grid for which $\|\mathcal{G}\| < \delta$. We claim that

$$D-\varepsilon < \underline{D}_{\mathcal{G}}(f,S) \text{ and } \overline{D}_{\mathcal{G}}(f,S) < D+\varepsilon.$$

Every Riemann sum constructed with the grid $\mathcal{G}$ lies between $\underline{D}_{\mathcal{G}}(f,S)$ and $\overline{D}_{\mathcal{G}}(f,S)$; thus it follows from the claim that

$$D-\varepsilon < \sum_{i=1}^{N} f(\mathbf{p}_i)\,\Delta A_i < D+\varepsilon,$$

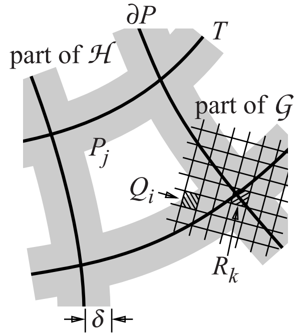
which is equivalent to what we need to prove.

Proving $D-\varepsilon < \underline{D}_{\mathcal{G}}(f,S)$

We now prove the first of the inequalities in the claim; the second can be proven by essentially the same argument. We divide the cells of $\mathcal{G}$ into two classes, as follows.

- $R_1,\dots,R_K$ lie entirely within the tubular neighborhood T.
- $Q_1,\dots,Q_N$ contain points outside T.

Now let $\mathcal{K}$ be the common refinement of $\mathcal{H}$ and $\mathcal{G}$; by definition, the cells of $\mathcal{K}$ are $P_j \cap Q_i$ and $P_j \cap R_l$. But because the diameter of each Q_i is less than δ, Q_i does not meet ∂P, and thus lies entirely in a single cell P_j of $\mathcal{H}$. In other words, $P_j \cap Q_i$ is either empty or it is just Q_i. The cells of $\mathcal{K}$ are therefore



$$Q_1,\dots,Q_N, \quad \text{and} \quad \begin{matrix} P_1\cap R_1, & \dots, & P_1\cap R_K, \\ P_2\cap R_1, & \dots, & P_2\cap R_K, \\ \vdots & \ddots & \vdots \\ P_J\cap R_1, & \dots, & P_J\cap R_K. \end{matrix}$$

We have

$$\sum_{j=1}^{J} A(P_j\cap R_k) = A(R_k), \qquad \sum_{k=1}^{K} A(R_k) < \overline{J}(T) < \frac{\varepsilon}{4B}.$$

We now construct the Darboux lower sums associated with $\mathcal{G}$ and $\mathcal{K}$. For this we need the greatest lower bound of f on each cell of each of these grids:

$$m_i = \inf_{\mathbf{p}\in Q_i} f(\mathbf{p}), \quad \widehat{m}_k = \inf_{\mathbf{p}\in R_k} f(\mathbf{p}), \quad m_{jk} = \inf_{\mathbf{p}\in P_j\cap R_k} f(\mathbf{p}).$$

Then

$$\underline{D}_{\mathcal{K}}(f,S) = \sum_{i=1}^{N} m_i A(Q_i) + \sum_{k=1}^{K}\sum_{j=1}^{J} m_{jk} A(P_j\cap R_k),$$
$$\underline{D}_{\mathcal{G}}(f,S) = \sum_{i=1}^{N} m_i A(Q_i) + \sum_{k=1}^{K} \widehat{m}_k A(R_k) = \sum_{i=1}^{N} m_i A(Q_i) + \sum_{k=1}^{K}\sum_{j=1}^{J} \widehat{m}_k A(P_j\cap R_k).$$

Subtracting, we find

$$\begin{aligned}\underline{D}_{\mathcal{K}}(f,S)-\underline{D}_{\mathcal{G}}(f,S) &= \sum_{k=1}^{K}\sum_{j=1}^{J}(m_{jk}-\widehat{m}_k)A(p_j\cap R_k)\\ &\le 2B\sum_{k=1}^{K}\sum_{j=1}^{J}A(P_j\cap R_k)\le 2B\sum_{k=1}^{K}A(R_k)<2B\cdot\frac{\varepsilon}{4B}=\frac{\varepsilon}{2},\end{aligned}$$

or $\underline{D}_{\mathcal{K}}(f,S)<\underline{D}_{\mathcal{G}}(f,S)+\varepsilon/2$. Because $\mathcal{K}$ is a refinement of $\mathcal{H}$,

$$\underline{D}_{\mathcal{H}}(f,S)\le\underline{D}_{\mathcal{K}}(f,S).$$

We thus have a sequence of inequalities,

$$D-\frac{\varepsilon}{2}<\underline{D}_{\mathcal{H}}(f,S)\le\underline{D}_{\mathcal{K}}(f,S)<\underline{D}_{\mathcal{G}}(f,S)+\frac{\varepsilon}{2},$$

that together establish the first claim, $D-\varepsilon<\underline{D}_{\mathcal{G}}(f,S)$. □

The Riemann-Darboux integral

The common value produced by the two definitions is sometimes called the **Riemann–Darboux integral**. However, we usually just call it the *integral*, and employ the two definitions interchangeably, depending on which is more useful in a particular situation. For example, the proof of the next theorem uses the Darboux characterization of the integral.

Theorem 8.34. *Suppose f is integrable on S; then so is $|f|$ and*

$$\left|\iint_S f(x,y)\,dA\right|\le\iint_S|f(x,y)|\,dA.$$

Proof. First we prove $|f|$ is integrable; it is enough to show that, given any $\varepsilon>0$, there is a grid $\mathcal{H}$ for which

$$\overline{D}_{\mathcal{H}}(|f|,S)-\underline{D}_{\mathcal{H}}(|f|,S)<\varepsilon.$$

To analyze the upper and lower Darboux sums for both $|f|$ and f, let $Q_1,\ldots,Q_N$ be the cells in an arbitrary grid $\mathcal{G}$, and let

$$\begin{aligned}m_i^* &= \inf_{\mathbf{p}\in Q_i}|f(\mathbf{p})|, \qquad & m_i &= \inf_{\mathbf{p}\in Q_i}f(\mathbf{p}),\\ M_i^* &= \sup_{\mathbf{p}\in Q_i}|f(\mathbf{p})|, \qquad & M_i &= \sup_{\mathbf{p}\in Q_i}f(\mathbf{p}).\end{aligned}$$

Now it is always true that $M_i^*-m_i^*\le M_i-m_i$ (see Exercise 8.20); thus, for any grid $\mathcal{G}$,

$$\begin{aligned}\overline{D}_{\mathcal{G}}(|f|,S)-\underline{D}_{\mathcal{G}}(|f|,S) &= \sum_{i=1}^{N}\left(M_i^*-m_i^*\right)\Delta A_i,\\ &\le\sum_{i=1}^{N}\left(M_i-m_i\right)\Delta A_i=\overline{D}_{\mathcal{G}}(f,S)-\underline{D}_{\mathcal{G}}(f,S).\end{aligned}$$

Because f is integrable, by hypothesis, there is always a grid $\mathcal{H}$ for which

$$\overline{D}_{\mathcal{H}}(f,S) - \underline{D}_{\mathcal{H}}(f,S) < \varepsilon;$$

then $\overline{D}_{\mathcal{H}}(|f|,S) - \underline{D}_{\mathcal{H}}(|f|,S) < \varepsilon$ as well, so $|f|$ is integrable.

Finally, because $-|f(x,y)| \leq f(x,y) \leq |f(x,y)|$ and the integral is monotone (cf. Corollary 8.29),

$$-\iint_S |f(x,y)|\, dA \leq \iint_S f(x,y)\, dA \leq \iint_S |f(x,y)|\, dA. \qquad \square$$

The integral of a continuous function

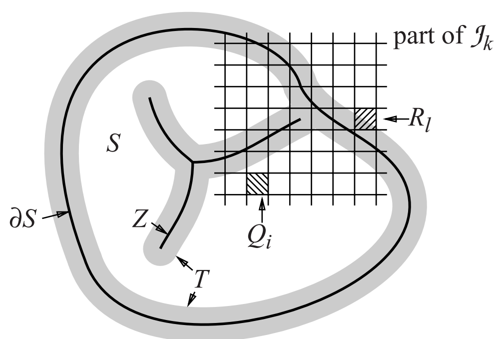
One of the fundamental results of calculus is that a continuous function is integrable. However, because we extend every function to the whole plane by setting it equal to zero outside its given domain, even a continuous function becomes, in general, discontinuous across the boundary of that domain. This causes difficulties in proving integrability, because some cells of a grid straddle the boundary; in those cells, the function can take widely different values, even if the cell is small. In proving that a continuous function is integrable, we, however, take all this into account, and even allow certain other discontinuities in the function.

Theorem 8.35. *Let Z be a subset of S with $A(Z) = 0$. If f is bounded and continuous on $S \setminus Z$, then f is integrable on S.*

Proof. We show that, given any $\varepsilon > 0$, there is a grid $\mathcal{J}_k$ (one of the original grids of congruent squares) for which

$$\overline{D}_{\mathcal{J}_k}(f,S) - \underline{D}_{\mathcal{J}_k}(f,S) < \varepsilon.$$

Let B be a global bound for f; that is, $|f(x,y)| \leq B$ for all points (x,y) in $\mathbb{R}^2$. Let T be a tubular neighborhood of $Z \cup \partial S$ of positive width, chosen so that $\overline{J}(T) < \varepsilon/4B$.



Now T contains all points of discontinuity of f, so f is continuous everywhere on $S \setminus T$. Furthermore, T is open so $S \setminus T$ is closed and bounded, implying f is uniformly continuous there. Thus, for the given ε, we can choose a $\delta > 0$ so that if $\mathbf{p}$ and $\mathbf{q}$ are in $S \setminus T$, then

$$\|\mathbf{p}-\mathbf{q}\| < \delta \;\Rightarrow\; |f(\mathbf{p})-f(\mathbf{q})| < \frac{\varepsilon}{2A(S)}.$$

Here $A(S)$ is the area of S; by Theorem 8.25, we may assume that $A(S) > 0$.

Now consider any grid $\mathcal{J}_k$ for which $\|\mathcal{J}_k\| < \delta$ ($k > 1+\log_2(1/\delta)$ suffices), and divide the squares of $\mathcal{J}_k$ into two classes:

- $Q_1,\ldots,Q_N$ lie entirely within $S\setminus T$.
- $R_1,\ldots,R_L$ meet the tubular neighborhood T.

Let

$$m_i = \inf_{\mathbf{p}\in Q_i} f(\mathbf{p}), \qquad \widehat{m}_l = \inf_{\mathbf{p}\in R_l} f(\mathbf{p}),$$
$$M_i = \sup_{\mathbf{p}\in Q_i} f(\mathbf{p}), \qquad \widehat{M}_l = \sup_{\mathbf{p}\in R_l} f(\mathbf{p}).$$

Then the difference between the upper and lower Darboux sums over $\mathcal{J}_k$ is

$$\overline{D}_{\mathcal{J}_k}(f,S) - \underline{D}_{\mathcal{J}_k}(f,S) = \sum_{i=1}^{N}(M_i - m_i)A(Q_i) + \sum_{l=1}^{L}(\widehat{M}_l - \widehat{m}_l)A(R_l).$$

Consider the first sum on the right. Because f is continuous on each closed bounded set Q_i (because Q_i lies entirely in $S\setminus T$), Q_i contains points $\mathbf{q}_i$ and $\mathbf{p}_i$ at which f attains its supremum and its infimum, respectively:

$$M_i = f(\mathbf{q}_i), \quad m_i = f(\mathbf{p}_i).$$

But because Q_i has diameter less than δ, we have $\|\mathbf{q}_i - \mathbf{p}_i\| < \delta$, so

$$M_i - m_i = f(\mathbf{q}_i) - f(\mathbf{p}_i) < \frac{\varepsilon}{2A(S)}.$$

Therefore,

$$\sum_{i=1}^{N}(M_i - m_i)A(Q_i) = \sum_{i=1}^{N}\big(f(\mathbf{q}_i) - f(\mathbf{p}_i)\big)A(Q_i) < \frac{\varepsilon}{2A(S)}\sum_{i=1}^{N}A(Q_i).$$

All the squares Q_i lie entirely within S; thus their total area is not greater than $A(S)$, implying

$$\sum_{i=1}^{N}A(Q_i) \le A(S) \;\text{ and }\; \sum_{i=1}^{N}(M_i - m_i)A(Q_i) < \frac{\varepsilon}{2A(S)}A(S) = \frac{\varepsilon}{2}.$$

Now consider the second sum, the one involving the cells R_l. Because $\widehat{M}_l$ and $-\widehat{m}_l$ are both bounded by B,

$$\sum_{l=1}^{L}(\widehat{M}_l - \widehat{m}_l)A(R_l) \leq 2B\sum_{l=1}^{L}A(R_l).$$

Because the squares R_l cover T and they all meet T, they are precisely the squares involved in computing the outer area of T with the grid $\mathcal{J}_k$:

$$\sum_{l=1}^{L}A(R_l) = \overline{J}_k(T).$$

By the definition of outer Jordan content, we know $\overline{J}_k(T)$ decreases monotonically to $\overline{J}(T)$ as $k \to \infty$. Because $\overline{J}(T) < \varepsilon/4B$ by construction, we must have $\overline{J}_K(T) < \varepsilon/4B$ as well when K is sufficiently large, implying

$$\sum_{l=1}^{L}(\widehat{M}_l - \widehat{m}_l)A(R_l) < 2B \cdot \frac{\varepsilon}{4B} = \frac{\varepsilon}{2}.$$

Therefore, if $k > K$ and $k > 1 + \log_2(1/\delta)$, then

$$\overline{D}_{\mathcal{J}_k}(f,S) - \underline{D}_{\mathcal{J}_k}(f,S) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad \square$$

For functions of a single variable, there is an analogue of the preceding theorem that is particularly useful because it provides us with a large class of integrable functions. For example, it implies that a function $y = f(x)$ with only a finite number of finite jump discontinuities is integrable. To prove it, just adapt—and simplify—the preceding proof; see Exercise 8.18.

Integrating single-variable functions

Theorem 8.36. *Suppose $f(x)$ is bounded and continuous on a closed interval $[a,b]$ minus a finite set of points; then f is integrable on $[a,b]$.* $\square$

The proof of Theorem 8.35 also implies that **restricted** Riemann sums, using only the cells contained in the interior of the domain of integration, have the same limit as unrestricted sums. The following corollary provides the details.

Restricted Riemann sums

Corollary 8.37 *Suppose f is bounded and integrable on S; then*

$$\iint_S f(x,y)\,dA = \lim_{\|\mathcal{G}\|\to 0} \sum_{Q_j \subset {}^\circ S} f(\mathbf{q}_j)A(Q_j),$$

where the sum is taken over only those cells Q_j of $\mathcal{G}$ that lie within ${}^\circ S$, the interior of S.

Proof. Let $\varepsilon > 0$ be given. Then, because f is integrable over S, there is a $\delta > 0$ such that any grid $\mathcal{G}$ with $\|\mathcal{G}\| < \delta$ has

$$\left| \iint_S f(x,y)\,dA - \sum_{P_i \cap S \neq \phi} f(\mathbf{p}_i)A(P_i) \right| < \frac{\varepsilon}{2}.$$

(The sum on the right is an ordinary, unrestricted Riemann sum over all cells P_i of $\mathcal{G}$ that meet S.) Suppose $|f(x,y)| \le B$ for all (x,y) in S. Let T be a tubular neighborhood of ∂S for which $\overline{J}(T) < \varepsilon/2B$, and suppose that T has width $w > 0$ (cf. Definition 8.13, p. 289). Further restrict $\mathcal{G}$, if necessary, so that $\|\mathcal{G}\| < w$; then all the cells P_i of $\mathcal{G}$ that appear in the unrestricted Riemann sum above are contained in $S \cup T$. Divide these cells into two classes:

- $Q_1, \dots, Q_J$ lie entirely within $S \setminus \partial S = {}^\circ S$.
- $R_1, \dots, R_K$ meet ∂S (and hence lie entirely within T).

Then, for any $\mathbf{r}_k$ in R_k, $k = 1, \dots, K$,

$$\left| \sum_{R_k} f(\mathbf{r}_k) A(R_k) \right| \le \sum_{R_k} |f(\mathbf{r}_k)| A(R_k) \le B \sum_{R_k} A(R_k),$$

but because $R_1 \cup \dots R_K \subseteq T$, we have

$$\sum_{R_k} A(R_k) \le \overline{J}(T) < \frac{\varepsilon}{2B} \quad \text{and} \quad \left| \sum_{R_k} f(\mathbf{r}_k) A(R_k) \right| < \frac{\varepsilon}{2}.$$

Consequently, for any $\mathbf{q}_j$ in Q_j ($j = 1, \dots, J$),

$$\begin{aligned} &\left| \iint_S f(x,y)\, dA - \sum_{Q_j} f(\mathbf{q}_j) A(Q_j) \right| \\ &\qquad \le \left| \iint_S f(x,y)\, dA - \sum_{P_i} f(\mathbf{p}_i) A(P_i) \right| + \left| \sum_{R_k} f(\mathbf{r}_k) A(R_k) \right| < \varepsilon, \end{aligned}$$

where $\mathbf{p}_i = \mathbf{q}_j$ when $P_i = Q_j$ and $\mathbf{p}_i = \mathbf{r}_k$ when $P_i = R_k$. This proves that the integral is the limit of restricted Riemann sums. □

Integrals as volumes

Theorem 8.35 also provides a way to connect double integrals to volumes. In the following theorem, the volume of a solid W is its 3-dimensional Jordan content, denoted $V(W)$. We make use of the analogue of Theorem 8.9, that the volume of the rectangular parallelepiped $[a, a+l] \times [b, b+w] \times [c, c+h]$ is the product lwh.

Theorem 8.38. *Suppose $f \ge 0$ is bounded and integrable on a closed bounded set S that has area, and W is the solid region in $\mathbb{R}^3$ that lies between S and the graph of $z = f(x,y)$. Then W has volume, and*

$$V(W) = \iint_S f(x,y)\, dA.$$

Proof. We show that, given any $\varepsilon > 0$, there is a grid $\mathcal{J}_k$ of squares in the plane for which

$$\underline{D}_{\mathcal{J}_k}(f,S) - \varepsilon < \underline{V}(W) \le \overline{V}(W) \le \overline{D}_{\mathcal{J}_k}(f,S).$$

(Here $\underline{V}(W)$ is the **inner volume** of W—that is, the 3-dimensional inner Jordan content $\underline{J}(W)$; similarly, $\overline{V}(W) = \overline{J}(W)$ is the **outer volume**.) Because f is integrable over S and ε can be any positive number, the inequalities show that $V(W)$ exists and equals the integral of f over S.

Inner and outer volume

We begin by choosing a global bound B for f: $|f(x,y)| \leq B$ on $\mathbb{R}^2$. Then, because $A(\partial S) = 0$, we can choose k so large that the squares $R_1, \ldots, R_L$ of $\mathcal{J}_k$ that meet ∂S have total area less than ε/B. Let $Q_1, \ldots, Q_N$ be the remaining squares of $\mathcal{J}_k$ that meet S; they lie entirely within $S \setminus \partial S$. Let

$$m_i = \inf_{\mathbf{p}\in Q_i} f(\mathbf{p}), \qquad \widehat{m}_l = \inf_{\mathbf{p}\in R_l} f(\mathbf{p}),$$
$$M_i = \sup_{\mathbf{p}\in Q_i} f(\mathbf{p}), \qquad \widehat{M}_l = \sup_{\mathbf{p}\in R_l} f(\mathbf{p}).$$

Taking into account the fact that every $\widehat{m}_l \leq B$ and that the total area of the squares R_l is less than ε/B, we find that the lower Darboux sum for f over S is

$$\begin{aligned}\underline{D}_{\mathcal{J}_k}(f,S) &= \sum_{i=1}^{N} m_i A(Q_i) + \sum_{l=1}^{L} \widehat{m}_l A(R_l) \leq \sum_{i=1}^{N} m_i A(Q_i) + B \sum_{l=1}^{L} A(R_l)\\ &< \sum_{i=1}^{N} m_i A(Q_i) + B \cdot \frac{\varepsilon}{B} = \sum_{i=1}^{N} m_i A(Q_i) + \varepsilon,\end{aligned}$$

or

$$\underline{D}_{\mathcal{J}_k}(f,S) - \varepsilon < \sum_{i=1}^{N} m_i A(Q_i) = \sum_{i=1}^{N} V(P_i).$$

In the last sum, P_i is the parallelepiped with base Q_i and height m_i; its volume is $m_i A(Q_i)$. These parallelepipeds are nonoverlapping sets that are entirely contained in W, so their total volume is not larger than the inner volume of W:

$$\sum_{i=1}^{N} V(P_i) \leq \underline{V}(W).$$

This gives $\underline{D}_{\mathcal{J}_k}(f,S) - \varepsilon < \underline{V}(W)$, the first of the two inequalities we must establish.

The second inequality is more straightforward. In the formula for the upper Darboux sum,

$$\overline{D}_{\mathcal{J}_k}(f,S) = \sum_{i=1}^{N} M_i A(Q_i) + \sum_{l=1}^{L} \widehat{M}_l A(R_l),$$

each term is the volume of a parallelepiped based on one of the squares Q_i or R_l. These parallelepipeds are nonoverlapping and their union entirely contains W. Consequently, their total volume $\overline{D}_{\mathcal{J}_k}(f,S)$ is at least as large as the outer volume of W:

$$\overline{V}(W) \leq \overline{D}_{\mathcal{J}_k}(f,S). \qquad \square$$

Triple integrals over 3-dimensional regions

With cubes replacing squares, we can define and calculate the Jordan content of a region D in $\mathbb{R}^3$ (cf. p. 295). Then, modifying the exposition at the beginning of this section by using a grid $\mathcal{G}$ whose cells are cubes Q_i instead of squares we can define the Riemann triple integral of a function $f(x,y,z)$ over a 3-dimensional region D as

$$\iiint_D f(x,y,z)\,dV = \lim_{\|\mathcal{G}\|\to 0} \sum_{i=1}^{N(\mathcal{G})} f(x_i,y_i,z_i)\,\Delta V_i,$$

where $\Delta V_i = J(Q_i)$, the Jordan content, or volume, of Q_i. Compare this to Definition 8.15 for double integrals. All the theorems and corollaries of this section have natural extensions to triple integrals. In particular (cf. Theorem 8.35), a function that is bounded and continuous on a region $D \setminus Z$, where D has volume and Z has volume zero, is integrable. Having made these observations, we now assume that triple integrals are available for our future work.

Set functions

Jordan content is an example of a **set function**: it assigns a real number to each of the sets in a certain collection. There are numerous other examples, including integrals themselves. In many cases, a set function even has a derivative. We end this section by showing that the derivative of a suitable set function is a point function whose integral equals the original set function. This is, in fact, a version of the fundamental theorem of calculus. To fix ideas, we first explore some examples.

Example: mass and mass density

Imagine a thin flat plate that lies over a portion of the (x,y)-plane, and suppose it has a continuous but nonuniform mass distribution. Let S be a subset of the plane with positive area, and let $M(S)$ be the total mass of the portion of the plate that lies over S. If $A(S)$ is the area of S, then

$$\frac{M(S)}{A(S)} = \text{average mass density over } S.$$

Intuitively, the mass density $\rho(x,y)$ of the plate at the point (x,y) should be the limit of $M(S)/A(S)$ as the set S "shrinks down" to (x,y), in the sense that $\delta(S) \to 0$ for sets S that contain (x,y); $\delta(S)$ is the diameter of S (Definition 8.14, p. 291). Thus, mass *distribution* is a set function, mass *density* is a point function, and the second is the derivative of the first. That is,

$$\text{mass density at } (x,y) = \rho(x,y) = M'(x,y) = \lim_{\delta(S)\to 0} \frac{M(S)}{A(S)},$$

for (x,y) in S, if the limit exists. A related example is a 3-dimensional solid with a continuous but nonuniform mass distribution. Let D be any region of positive volume $V(D)$ in $\mathbb{R}^3$ that contains the point (x,y,z), and let $M(D)$ be the mass of the portion of the solid that lies in D; then

$$\rho(x,y,z) = M'(x,y,z) = \lim_{\delta(D)\to 0} \frac{M(D)}{V(D)}$$

is the mass density of the solid at (x,y,z), if the limit exists.

Another physical example is the hydrostatic force a liquid applies to the walls of its container. The force $F(S)$ on any portion S of the surface of the container is a set function. If S has area, then $F(S)/A(S)$ is the average pressure (force per unit area) on S; as S shrinks down to a point, this ratio approaches the pressure at that point. Electric charge on a plate, and the related charge density, show that a set function can take negative, as well as positive, values.

Example: probability and probability density

For a different kind of example (using subsets of $\mathbb{R}$ instead of $\mathbb{R}^2$, for simplicity), suppose X is a random variable (cf. p. 20) that takes real values. For any subset S of $\mathbb{R}$ that has a length $L(S)$, we define

$$P(S) = \text{probability that } X \text{ lies in } S.$$

Probability is a set function. The corresponding probability density function $p(x)$ should be the limit of $P(S)/L(S)$ as S "shrinks down" to x. A common example is the normal density function

$$P'(x) = p(x) = \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}},$$

which determines the normal probability function

$$P(S) = \frac{1}{\sqrt{2\pi}} \int_S \mathrm{e}^{-x^2/2}\,dx.$$

Here we find a set function that is the integral of its derivative.

Example: set functions of integral type

Integrals provide a very general class of set functions. Define

$$F(S) = \iint_S f(x,y)\,dA,$$

where $f(x,y)$ is a fixed function that is bounded and continuous on some fixed open set Ω in $\mathbb{R}^2$. Then F is a set function; it assigns a real number to each subset S of Ω that has area.

Let (x,y) be a point in Ω, and let S_n be a collection of closed subsets of Ω with positive area that all contain (x,y). Let m_n and M_n be the minimum and maximum values, respectively, of f on S_n. Then, by Corollary 8.29, page 298,

$$m_n \le \frac{F(S_n)}{A(S_n)} \le M_n.$$

Suppose $\delta(S_n) \to 0$ as $n \to \infty$. The continuity of f implies that $m_n \to f(x,y)$ and $M_n \to f(x,y)$ as $n \to \infty$. In other words,

$$\lim_{n\to\infty} \frac{F(S_n)}{A(S_n)} = f(x,y).$$

Because this limit is independent of the choice of the sets S_n used to compute it, we define it to be the **derivative** of F at (x,y), and write

$$F'(x,y) = \lim_{n\to\infty} \frac{F(S_n)}{A(S_n)}.$$

Although F is a set function, its derivative $F' = f$ is a point function. We call F a *set function of integral type*. The following theorem summarizes our observations.

Theorem 8.39. *A set function of integral type has a deriviative, and the set function is equal to the integral of its derivative.* □

A set function may not be of integral type

Note that continuous mass distributions and normal probability are both set functions of integral type:

$$M(S) = \iint_S \rho(x,y)\,dA, \quad M'(x,y) = \rho(x,y);$$

$$P(S) = \int_S \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx, \qquad P'(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}.$$

But not all set functions are. Here is a simple example to the contrary. Let

$$\widetilde{M}(S) = \begin{cases} 1 & \text{if } S \text{ contains the origin,} \\ 0 & \text{otherwise.} \end{cases}$$

You can think of $\widetilde{M}$ as the set function associated with a unit point mass concentrated at the origin on $\mathbb{R}$. To show $\widetilde{M}$ is not of integral type, suppose the contrary. That is, suppose

$$\widetilde{M}(S) = \int_S g(x)\,dx$$

for some integrable function $g(x)$ (that need not even be continuous). Now suppose $Q_1 = [-1,0]$, $Q_2 = [0,1]$, and $S = [-1,1]$; by definition of $\widetilde{M}$,

$$\widetilde{M}(Q_1) = \widetilde{M}(Q_2) = \widetilde{M}(S) = 1.$$

However, by assumption we have

$$\widetilde{M}(S) = \int_{-1}^{1} g(x)\,dx = \int_{-1}^{0} g(x)\,dx + \int_{0}^{1} g(x)\,dx = \widetilde{M}(Q_1) + \widetilde{M}(Q_2) = 2,$$

a contradiction. The contradiction arises because the integral is additive on nonoverlapping sets (Theorem 8.27, p. 298), but $\widetilde{M}$ is not:

$$\widetilde{M}(S_1 \cup S_2) \neq \widetilde{M}(S_1) + \widetilde{M}(S_2).$$

A set function cannot be of integral type unless it possesses, at the outset, all the relevant properties of a Riemann integral.

Exercises

8.1. a. Adapt the program that estimates the gravitational field of a large plate (p. 271) to a version of BASIC or a similar language and use it to reproduce the table of values of the field that are found in the text.

b. In the original computation of the gravitational field, we assumed the plate density was constant and took $4G\rho = 1$. Recompute all the tabular values assuming that $4G\rho = 1/(1+x^2+y^2)$. This provides estimates for the double integral

$$\iint_{\substack{0\le x\le R,\\ 0\le y\le R}} \frac{-a\,dx\,dy}{(1+x^2+y^2)(x^2+y^2+a^2)^{3/2}}.$$

c. In which case is the plate less massive, and in which case is the gravitational field weaker? Does the less massive plate have the weaker field?

8.2. In Chapter 9.3, pages 342–343, the area of the curved region $1 \le x^2 - y^2 \le 2$, $1 \le 2xy \le 3$, in the (x,y)-plane is given by the double integral

$$\iint_{\substack{1\le u\le 2,\\ 1\le v\le 3}} \frac{du\,dv}{4\sqrt{u^2+v^2}}.$$

a. Approximate the integral by a Riemann sum using a 2×4 grid of squares with the integrand evaluated at the center of each square. Use a modification of the BASIC program in the previous question to show the value of the Riemann sum is 0.204 806.

b. Obtain additional approximations using a 20×40 grid and a 200×400 grid. How close are these to the estimate 0.205 213 found anlytically on page 342?

8.3. Adapt the previous BASIC program to estimate the value of the integral

$$\iint_S 4(x^2+y^2)\,dx\,dy$$

on the square $S: 0.2 \le x \le 1, 0.2 \le y \le 1$ (cf. Exercise 9.38.c, p. 384). Evaluate the function at the center of each grid square. Show that, with a 4×4 grid, the value is 2.0992 and with a 20×20 grid the value is 2.1156. How large must the grid be to make the value 2.11626?

8.4. Is the interior of the complement of S equal to the complement of the interior of S? If not, does either of these sets always contain the other?

8.5. Suppose $\mathbf{b}$ is a boundary point of S. Show that every open disk centered at $\mathbf{b}$ contains at least one point $\mathbf{p}$ in S and also at least one point $\mathbf{q}$ that is not in S.

8.6. Suppose $\mathbf{p}$ is a point in S and $\mathbf{q}$ is a point not in S. Show that at least one point on any continuous curve from $\mathbf{p}$ to $\mathbf{q}$ is in ∂S.

8.7. Let Q be a square in the grid $\mathcal{J}_k$, and let $m > k$. Show that

$$\underline{J}_m(Q) = \frac{1}{2^{2k}} \text{ and } \overline{J}_m(Q) = \frac{1}{2^{2k}} + 4\frac{2^{m-k}}{2^{2m}} + 4\frac{1}{2^{2m}}.$$

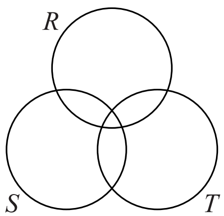
Conclude that Q is Jordan measurable and $J(Q) = 1/2^{2k}$.

8.8. Show that the rectangle $a \le x \le b, c \le y \le d$ has Jordan content $(b-a)(d-c)$.

8.9. Suppose $\delta > 0$, $\alpha\delta \le a < (\alpha+1)\delta$, and $(\beta-1)\delta < b \le \beta\delta$. Show that $2\delta < b-a$ implies $(\alpha+1)\delta < (\beta-1)\delta$.

8.10. Show that $\underline{J}(S) = 0 \Leftrightarrow {}^\circ S = \phi$ (i.e., the interior of S is empty).

8.11. Suppose S is Jordan measurable and ${}^\circ S \subseteq T \subseteq \overline{S}$. Show that T is Jordan measurable and $J(T) = J({}^\circ S) = J(S) = J(\overline{S})$.



8.12. Suppose R, S, and T are Jordan measurable; show that

$$\begin{aligned} J(R\cup S\cup T) &= J(R)+J(S)+J(T) \\ &\quad -J(R\cap S)-J(S\cap T)-J(T\cap R)+J(R\cap S\cap T). \end{aligned}$$

This includes showing that all the sets on the right-hand side of the equation are Jordan measurable.

8.13. Generalize the result in the previous exercise to four sets, and then to p sets $S_1, \ldots, S_p$.

8.14. Let S and T denote Jordan-measurable bounded subsets of the plane.

a. Give an example in which $\partial(T\setminus S)$ has a point in ∂S that is not in ∂T.
b. Prove that $\partial(T\setminus S) \subseteq \partial T \cup \partial S$.

8.15. Modify the proof of Lemma 8.5 so that it works with cubes and sets in $\mathbb{R}^3$.

8.16. Let the linear map $L : \mathbb{R}^2 \to \mathbb{R}^2$ be given by the matrix

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and suppose Q is the square $0 \le x \le w$, $0 \le y \le w$. Show that the ratio $\sigma = \delta(L(Q))/\delta(Q)$ of the diameters of Q and its image is the larger of the two numbers

$$\frac{a^2+b^2+c^2+d^2 \pm 2(ab+cd)}{\sqrt{2}},$$

and thus depends only on L, not on Q. Make a sketch to illustrate how these numbers are connected to $L(Q)$.

8.17. Confirm (e.g., by writing suitable programs) that the inner and outer areas, $\underline{J}_k$ and $\overline{J}_k$, of the unit disk have the values indicated in the following table.

k	Inner Squares	Outer Squares	$\underline{J}_k$	$\overline{J}_k$
0	0	12	0	12
1	4	24	1	6
2	32	68	2	4.25
3	164	232	2.5625	3.625
4	732	864	2.859375	3.375
5	3080	3340	3.007813	3.261719
6	12596	13112	3.075195	3.201172
7	50920	51948	3.107910	3.170654
8	204836	206888	3.125549	3.156860
9	821424	825524	3.133484	3.149124

8.18. Prove Theorem 8.36.

8.19. Let S be the unit square in $\mathbb{R}^2$, and let

$$f(x,y) = \begin{cases} 1 & \text{if } x \text{ and } y \text{ are irrational,} \\ 0 & \text{otherwise.} \end{cases}$$

For an arbitrary grid $\mathcal{G}$ determine the upper and lower Darboux sums $\overline{D}_{\mathcal{G}}(f,S)$ and $\underline{D}_{\mathcal{G}}(f,S)$. What are the values of the upper and lower Darboux integrals of f on S? Is f Darboux integrable on S?

8.20. Suppose f is integrable on a closed cell Q, and

$$m^* = \inf_{\mathbf{p}\in Q} |f(\mathbf{p})|, \qquad m = \inf_{\mathbf{p}\in Q} f(\mathbf{p}),$$
$$M^* = \sup_{\mathbf{p}\in Q} |f(\mathbf{p})|, \qquad M = \sup_{\mathbf{p}\in Q} f(\mathbf{p}).$$

Show that $M^* - m^* \leq M - m$.

8.21. Suppose a thin flat plate is a disk of radius R centered at the origin of $\mathbb{R}^2$. Suppose its mass distribution is circularly symmetric and that the mass of the disk of radius α centered at the origin is $\alpha/(1+\alpha)$, for every $0 \leq \alpha \leq R$.

a. What is the mass of an annulus whose radii are $a - \Delta r/2$ and $a + \Delta r/2$?
b. What is the mass $M(S)$ of the piece S of this annulus cut off by radial lines $\theta = b - \Delta\theta/2$ and $\theta = b + \Delta\theta/2$. What is the area $A(S)$.
c. Determine the mass density at the point (a,b) on the plate as

$$\rho(a,b) = \lim_{\substack{\Delta r\to 0 \\ \Delta\theta\to 0}} \frac{M(S)}{A(S)}.$$

d. Using ρ, verify that the mass of the disk of radius α has the value it should; that is, verify

$$\iint_{x^2+y^2\leq\alpha^2} \rho(x,y)\,dA = \frac{\alpha}{1+\alpha}.$$

e. Repeat all the previous analysis assuming that the mass of the disk of radius α centered at the origin is just α.

8.22. Let S be a closed bounded set with area in the (x,y)-plane. The **moment** of S about the y-axis is the integral

$$\iint_S x\,dA.$$

Estimate the moment of the square $S: 0.2 \leq x \leq 1, 0.2 \leq y \leq 1$ about the y-axis by adapting the BASIC program of Exercise 8.3, above. Use a 4×4 grid and a 20×20 grid.