and

$$\begin{aligned}
\int_{S_1} \mathbf{F} \cdot \hat{n}_{S_2}\, dS &\approx \mathbf{F}\left(x_0 - \frac{\Delta x}{2}, y_0, z_0\right) \cdot (-\hat{i})\ \Delta y\ \Delta z \\
&= -P\left(x_0 - \frac{\Delta x}{2}, y_0, z_0\right)\ \Delta y\ \Delta z.
\end{aligned}$$

Summing we have

$$\begin{aligned}
\int_{S_1+S_2} \mathbf{F} \cdot \hat{n}\, dS &\approx \left(P\left(x_0 + \frac{\Delta x}{2}, y_0, z_0\right) - P\left(x_0 - \frac{\Delta x}{2}, y_0, z_0\right)\right)\ \Delta y\ \Delta z \\
&= \frac{P\left(x_0 + \frac{\Delta x}{2}, y_0, z_0\right) - P\left(x_0 - \frac{\Delta x}{2}, y_0, z_0\right)}{\Delta x}\ \Delta x\ \Delta y\ \Delta z
\end{aligned}$$

$$\Longrightarrow \frac{1}{\Delta x\ \Delta y\ \Delta z} \int_{S_1+S_2} \mathbf{F} \cdot \hat{n}\, dS \approx \frac{P\left(x_0 + \frac{\Delta x}{2}, y_0, z_0\right) - P\left(x_0 - \frac{\Delta x}{2}, y_0, z_0\right)}{\Delta x}.$$

Recalling that $\Delta V = \Delta x\ \Delta y\ \Delta z$ and then taking the limit as $\Delta V \to 0$ (that is, as $\Delta x \to 0$, $\Delta y \to 0$, and $\Delta z \to 0$) we get

$$\begin{aligned}
\lim_{\Delta V \to 0} \frac{1}{\Delta V} \int_{S_1+S_2} \mathbf{F} \cdot \hat{n}\, dS &= \lim_{\Delta V \to 0} \frac{P\left(x_0 + \frac{\Delta x}{2}, y_0, z_0\right) - P\left(x_0 - \frac{\Delta x}{2}, y_0, z_0\right)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{P\left(x_0 + \frac{\Delta x}{2}, y_0, z_0\right) - P\left(x_0 - \frac{\Delta x}{2}, y_0, z_0\right)}{\Delta x} \\
&= \left.\frac{\partial P}{\partial x}\right|_{(x_0, y_0, z_0)}.
\end{aligned}$$

In the limit the approximation becomes an equality. Also, since there is no $\Delta y$ or $\Delta z$ on the right hand side then $\Delta V \to 0$ becomes simply $\Delta x \to 0$.

*Question 9.1* Let $S_3$ and $S_4$ be the sides of the region $V$ that are perpendicular to the $y$-axis, that is, the right and left sides of the cube. Let $S_5$ and $S_6$ be the sides that are perpendicular to the $z$-axis, that is, the top and bottom of the cube. Repeat these calculations for surfaces $S_3 + S_4$ and $S_5 + S_6$ to get

$$\begin{aligned}
\lim_{\Delta V \to 0} \frac{1}{\Delta V} \int_{S_3+S_4} \mathbf{F} \cdot \hat{n}\, dS &= \left.\frac{\partial Q}{\partial y}\right|_{(x_0, y_0, z_0)}, \\
\lim_{\Delta V \to 0} \frac{1}{\Delta V} \int_{S_5+S_6} \mathbf{F} \cdot \hat{n}\, dS &= \left.\frac{\partial R}{\partial z}\right|_{(x_0, y_0, z_0)}.
\end{aligned}$$

Summing all these terms together we have that

$$\begin{aligned}
\operatorname{div} \mathbf{F} &= \lim_{\Delta V \to 0} \frac{1}{\Delta V} \int_{\partial V} \mathbf{F} \cdot \hat{n}\, dS \\
&= \lim_{\Delta V \to 0} \frac{1}{\Delta V} \int_{S_1+S_2+S_3+S_4+S_5+S_6} \mathbf{F} \cdot \hat{n}\, dS \\
&= \left.\frac{\partial P}{\partial x}\right|_{(x_0, y_0, z_0)} + \left.\frac{\partial Q}{\partial y}\right|_{(x_0, y_0, z_0)} + \left.\frac{\partial R}{\partial z}\right|_{(x_0, y_0, z_0)} \\
&= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.
\end{aligned}$$

In other words, we have just found the formula

| Formula for divergence **F** in Cartesian coordinates | $\text{div}\,\mathbf{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z}.$ |
|---|---|

Since we can take the divergence at any point, the point is usually left off of the notation. We use $\nabla \cdot \mathbf{F}$ as a mnemonic device to help us remember this formula.

Now we turn to looking at the divergence theorem (sometimes called Gauss's theorem) from vector calculus. We will "derive" (in a very non-rigorous fashion) the divergence theorem. This is relatively straight-forward given how we defined div **F**. We will make use of the fact that the flux through a surface $S$ is the sum of the fluxes through the subsurfaces $S_i$ of $S$,

$$\int_S \mathbf{F} \cdot \hat{n}\, dS = \sum_i \int_{S_i} \mathbf{F} \cdot \hat{n}\, dS.$$

In order to understand this consider two adjacent volumes $V_1$ and $V_2$ with a common surface $S_c$ as shown in Fig. 9.6. Surface $S_c$ of $V_1$ has outward pointing normal $\hat{n_1}$ while surface $S_c$ of $V_2$ has outward pointing normal $\hat{n_2}$. Clearly $\hat{n_2} = -\hat{n_1}$. It is easy to see from Fig. 9.6 that

$$\int_{S_c} \mathbf{F} \cdot \hat{n_1} dS + \int_{S_c} \mathbf{F} \cdot \hat{n_2} dS = \int_{S_c} \mathbf{F} \cdot \hat{n_1} dS - \int_{S_c} \mathbf{F} \cdot \hat{n_1} dS = 0.$$

Given a large volume $V$ with surface $\partial V$, as shown in Fig. 9.7, it can be subdivided into $N$ smaller volumes $V_i$. Furthermore, the fluxes out of all the internal surfaces of these smaller volumes cancel with each other,

$$\begin{aligned}\int_{\partial V} \mathbf{F} \cdot \hat{n}\, dS &\approx \sum_{i=1}^{N} \left( \frac{1}{\Delta V_i} \int_{S_i} \mathbf{F} \cdot \hat{n}\, dS \right) \Delta V_i \\ &= \sum_{i=1}^{N} (\text{div}\,\mathbf{F})_i\ \Delta V_i.\end{aligned}$$

As we take the limit as $N \to \infty$ and $|\Delta V_i| \to 0$, then $\sum V_i \to V$ and $\sum \partial V_i \to \partial V$ since the inner surfaces all cancel. So when we take the limit we get the following equality,

$$\begin{aligned}\int_{\partial V} \mathbf{F} \cdot \hat{n}\, dS &= \lim_{\substack{N \to \infty \\ |\Delta V_i| \to 0}} \sum_{i=1}^{N} (\text{div}\,\mathbf{F})_i\ \Delta V_i \\ &= \int_V \text{div}\,\mathbf{F}\, dV,\end{aligned}$$

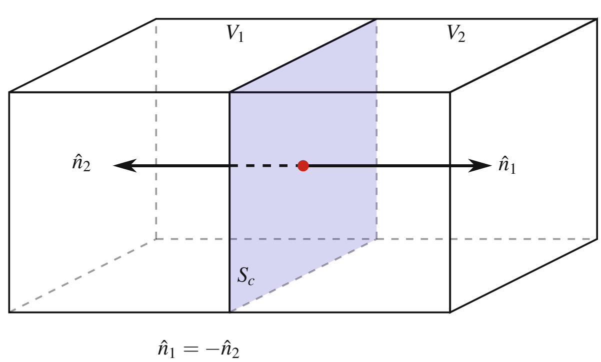


**Fig. 9.6** Two adjacent cubical volumes $V_1$ and $V_2$ that share a common surface $S_c$. Surface $S_c$ of $V_1$ has outward pointing normal $\hat{n_1}$ while surface $S_c$ of $V_2$ has outward pointing normal $\hat{n_2}$ where $\hat{n_2} = -\hat{n_1}$

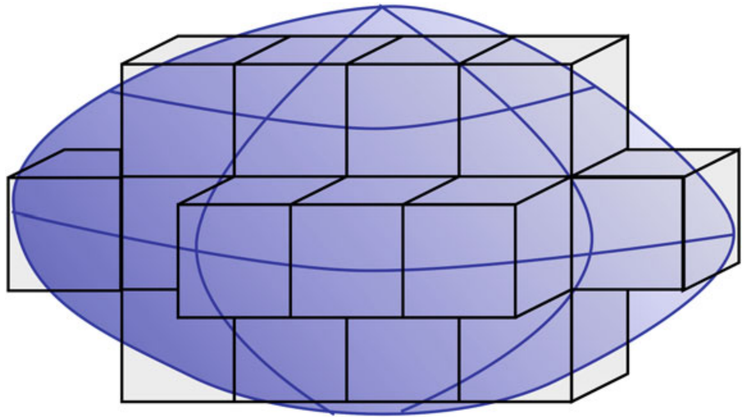

**Fig. 9.7** An irregularly shaped volume $V$ covered by smaller cubical volumes $V_i$. As $\Delta V_i \to 0$ the irregularly shaped volume is approximated better and better

which is exactly the divergence theorem. Recalling that the boundary of $V$ is $\partial V = S$, and writing $\mathbf{F} \cdot d\mathbf{S}$ for $\mathbf{F} \cdot \hat{n}\, dS$, the **divergence theorem** is usually written as

$$\boxed{\text{Divergence Theorem} \qquad \int_{\partial V} \mathbf{F} \cdot d\mathbf{S} = \int_V \text{div}\, \mathbf{F}\, dV.}$$

In this section we have used the definition of the divergence of a vector field $\mathbf{F}$ on $\mathbb{R}^3$ written in Cartesian coordinates to find an expression for div $\mathbf{F}$. But Cartesian coordinates are not the only coordinate system used on $\mathbb{R}^3$. Cylindrical and spherical coordinate systems, introduced in Sect. 6.5, are also very commonly used coordinate systems on $\mathbb{R}^3$.

*Question 9.2* Suppose that $\mathbf{F} = F_r\hat{e}_r + F_\theta\hat{e}_\theta + F_z\hat{e}_z$ is a vector field on $\mathbb{R}^3$ written with respect to cylindrical coordinates. The vector $\hat{e}_r$ is the unit vector in the direction of increasing $r$, $\hat{e}_\theta$ is the unit vector in the direction of increasing $\theta$, and $\hat{e}_z$ is the unit vector in the direction of increasing $z$. Using the cylindrical volume element shown in Fig. 6.14 and a procedure similar to that of this section, show that

$$\text{div}\, \mathbf{F} = \frac{1}{r}\frac{\partial(rF_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}.$$

*Question 9.3* Suppose that $\mathbf{F} = F_r\hat{e}_r + F_\theta\hat{e}_\theta + F_\phi\hat{e}_\phi$ is a vector field on $\mathbb{R}^3$ written with respect to spherical coordinates. The vector $\hat{e}_r$ is the unit vector in the direction of increasing $r$, $\hat{e}_\theta$ is the unit vector in the direction of increasing $\theta$, and $\hat{e}_\phi$ is the unit vector in the direction of increasing $\phi$. Using the spherical volume element shown in Fig. 6.16 and a procedure similar to that of this section, show that

$$\text{div}\, \mathbf{F} = \frac{1}{r^2}\frac{\partial(r^2F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta F_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi}.$$

## 9.2 Curl

Given a vector field $\mathbf{F} = P\hat{i} + Q\hat{j} + R\hat{k}$, in vector calculus classes the curl of the vector field is generally defined as

$$\text{curl}\, \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}.$$

Often you will also see something like this as well

$$\text{curl}\, \mathbf{F} = \nabla \times \mathbf{F},$$

where $\nabla$ was defined earlier to be the operator

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

and the usual mnemonic device for remembering the "cross product" $\times$ of two vectors is employed:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}.$$

It is important to recognize that this formulation of $\nabla \times \mathbf{F}$ is really nothing more than a mnemonic device to help us reconstruct the formula for curl $\mathbf{F}$, assuming we remember how to take the determinant of a $3 \times 3$ matrix. Again, as in the case of divergence, we will take a slightly different approach and define curl in a more geometric way that will allow both the formula for curl $\mathbf{F}$ and Stokes' theorem to just fall out. However, before we actually make the definition we need to understand some necessary background material.

We begin by considering the geometrical meaning of the dot product

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\ |\mathbf{w}|\ \cos\theta$$

as shown in Fig. 9.8. If $|\mathbf{v}| = 1$, that is, $\mathbf{v}$ is a unit vector, then we have $\mathbf{v} \cdot \mathbf{w} = |\mathbf{w}|\ \cos\theta$, which is the amount of $\mathbf{w}$ that is pointing in the $\mathbf{v}$ direction. Now suppose we want to somehow measure the "amount" of vector field $\mathbf{F}$ that is in the direction of some curve $C$. We can use the dot product just defined to find this, see Fig. 9.10. After that we will then want to integrate this "amount" we found.

However, we need to first understand line integrals of vector fields. Again, we assume you have had a vector calculus call and so we will cover this material without going into much detail. However, we need to make one notational comment to help you avoid confusion. We will be integrating around closed curves, which we will denote by $C$. These closed curves will be parameterized by arc length, which we will denote by the lower case $s$. We will denote by capital $S$ a surface whose boundary is curve $C$, that is, $C = \partial S$. If the curve $C$ lies in a plane we will assume the surface $S$ also lies in that plane. If this is the case then the unit normal to the surface $S$ will be denoted $\hat{n}$. To make our lives easier we will assume this is the case in the following.

Assume the curve $C$ is parameterized by arc length. A point $s$ on curve $C$ has coordinates $x = x(s)$, $y = y(s)$, and $z = z(s)$ and a point $s + \Delta s$ has coordinates $x + \Delta x = x(s + \Delta s)$, $y + \Delta y = y(s + \Delta s)$, and $z + \Delta z = z(s + \Delta s)$, which gives

$$\begin{aligned} \Delta x &= x(s + \Delta s) - x(s), \\ \Delta y &= y(s + \Delta s) - y(s), \\ \Delta z &= z(s + \Delta s) - z(s). \end{aligned}$$

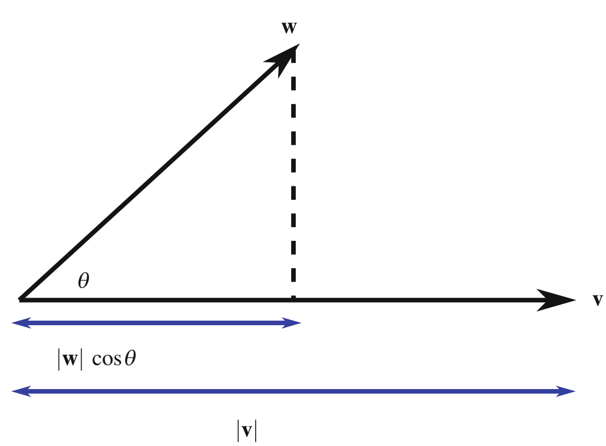


**Fig. 9.8** The geometrical meaning of the dot product as $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\ |\mathbf{w}|\ \cos\theta$

Letting $\Delta r$ be the vector from the point $\big(x(s), y(s), z(s)\big)$ to the point $\big(x(s+\Delta s), y(s+\Delta s), z(s+\Delta s)\big)$ we have

$$\frac{\Delta r}{\Delta s} = \frac{\Delta x}{\Delta s}\hat{i} + \frac{\Delta y}{\Delta s}\hat{j} + \frac{\Delta z}{\Delta s}\hat{k},$$

We define the new vector $\hat{t}$ by taking the limit of the right hand side as $\Delta s \to 0$, which of course means that $\Delta x, \Delta y, \Delta z \to 0$ as well,

$$\begin{aligned}\hat{t} &= \lim_{\Delta s \to 0} \frac{\Delta x}{\Delta s}\hat{i} + \frac{\Delta y}{\Delta s}\hat{j} + \frac{\Delta z}{\Delta s}\hat{k} \\ &= \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k}.\end{aligned}$$

From Fig. 9.9 we can see that $\hat{t}$ is a tangent vector to the curve $C$. Furthermore, as $\Delta s \to 0$ it is easy to see that $|\Delta r| \to \Delta s$. Thus $|\hat{t}| = 1$ and so $\hat{t}$ is the unit tangent vector to the curve $C$. Also, notice that formally we could write $\hat{t}ds = \hat{i}dx + \hat{j}dy + \hat{k}dz$, which we define to be $d\mathbf{s}$.

Now we turn to the line integral of a vector field $\mathbf{F} = P\hat{i} + Q\hat{j} + R\hat{k}$ along a curve $C$, which is parameterized by arc length $s$. The amount of $\mathbf{F}$ going in curve $C$'s direction is given by $\mathbf{F} \cdot \hat{t}$, where $\hat{t}$ is the unit tangent vector to $C$. See Figs. 9.10 and 9.11. Suppose we break curve $C$ into small segments, each of length $\Delta s$. Then the amount of $\mathbf{F}$ going in the direction of $C$ over each segment can be approximated by $(\mathbf{F} \cdot \hat{t})\Delta s$,

$$\begin{aligned}(\mathbf{F} \cdot \hat{t})\Delta s &= \left(P\hat{i} + Q\hat{j} + R\hat{k}\right) \cdot \left(\frac{\Delta x}{\Delta s}\hat{i} + \frac{\Delta y}{\Delta s}\hat{j} + \frac{\Delta z}{\Delta s}\hat{k}\right)\Delta s \\ &= \left(P\frac{\Delta x}{\Delta s} + Q\frac{\Delta y}{\Delta s} + R\frac{\Delta z}{\Delta s}\right)\Delta s\end{aligned}$$

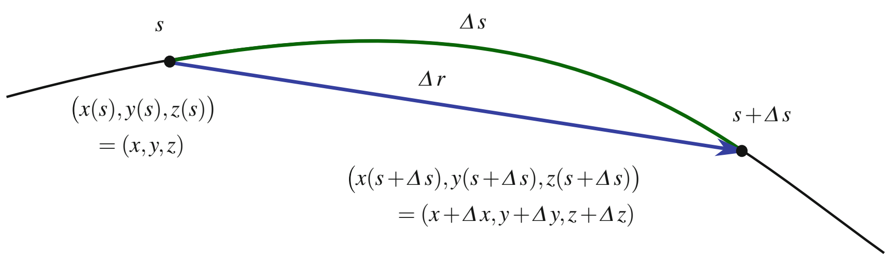


**Fig. 9.9** The unit tangent vector $\hat{t}$ to the curve is the limit of $\Delta r/\Delta s = \big(\Delta x/\Delta s\big)\hat{i} + \big(\Delta y/\Delta s\big)\hat{j} + \big(\Delta z/\Delta s\big)\hat{k}$

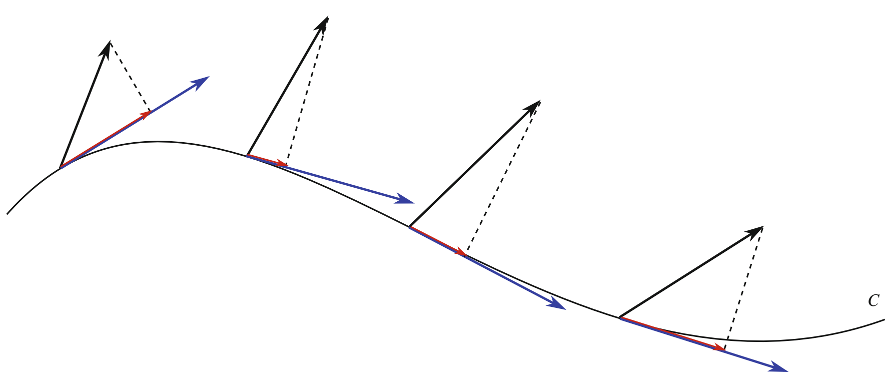


**Fig. 9.10** The vector field $\mathbf{F}$ (black) is defined along a curve $C$. We want to know the "part" of $\mathbf{F}$ that is tangent to $C$. To find this we consider the unit tangent vectors to $\hat{t}$ to $C$ (blue) and find $\mathbf{F} \cdot \hat{t}$ (red) to find the "amount" of $\mathbf{F}$ that is pointing in the $\hat{t}$ direction and is thus tangent to the curve $C$. We do this at every point of the curve $C$. Also see Fig. 9.11

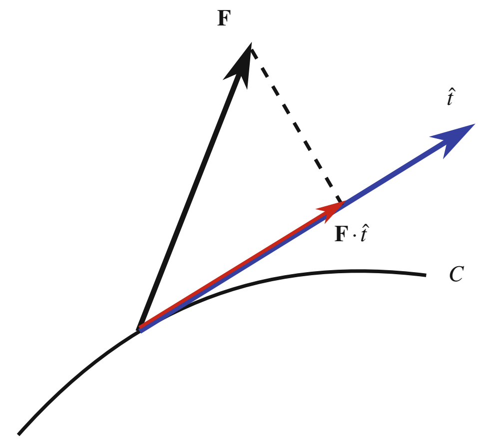


**Fig. 9.11** A close-up of a section of Fig. 9.10. We want to know the "part" of $\mathbf{F}$ that is tangent to $C$. To find this we consider the unit tangent vectors to $\hat{t}$ to $C$ (blue) and find $\mathbf{F}\cdot\hat{t}$ (red) to find the "amount" of $\mathbf{F}$ that is pointing in the $\hat{t}$ direction and is thus tangent to the curve $C$

so we get

$$\int_C \mathbf{F}\cdot\hat{t}\, ds = \lim_{|\Delta s|\to 0}\sum\left(P\frac{\Delta x}{\Delta s} + Q\frac{\Delta y}{\Delta s} + R\frac{\Delta z}{\Delta s}\right)\Delta s.$$

Using the notation from the last paragraph the integral $\int_C \mathbf{F}\cdot\hat{t}\, ds$ can also be written as $\int_C \mathbf{F}\cdot d\mathbf{s}$.

Now that we know what it means to integrate a vector field along a curve we are ready to define **curl F** at a point $(x_0, y_0, z_0)$ as

$$\boxed{\text{Definition of curl}\qquad \hat{n}\cdot \operatorname{curl}\mathbf{F} = \lim_{|\Delta S|\to 0}\frac{1}{\Delta S}\int_C \mathbf{F}\cdot\hat{t}\, ds}$$

where $S$ is the surface bounded by the closed curve $C = \partial S$, $\Delta S$ is the area of that surface, $\hat{n}$ is the unit normal vector to that surface, the $s$ in $ds$ is the infinitesimal arc length element, and the surface area $\Delta S$ shrinks to zero about the point $(x_0, y_0, z_0)$. Recall, a normal vector is orthogonal to, that is at a right angle to, the plane that surface $S$ is in. The curve $C$ along with $\mathbf{F}$ and the component of $\mathbf{F}$ along the curve is shown in Fig. 9.12. Integrating this, dividing by $\Delta S$, and then taking the limit as $\Delta S \to 0$ gives the right hand side of the definition of curl of $\mathbf{F}$. In essence, the curl is the "circulation" per unit area of vector field $\mathbf{F}$ over an infinitesimal path around some point.

*Question 9.4* Using Fig. 9.12 and the vector field $\mathbf{F}$ as shown, estimate $\int_C \mathbf{F}\cdot\hat{t}\, ds$ based on how the integral was defined as the limit of the sum of $(\mathbf{F}\cdot\hat{t})\Delta s$. In other words, is the integral a large or small, positive or negative number, or is it close to zero? Explain your answer in terms of the definition of the integral.

*Question 9.5* Draw a picture like that in Fig. 9.12 with a vector field $\mathbf{F}$ which is circulating around $(x_0, y_0, z_0)$ in a counter-clockwise direction. Assume the tangent vectors to $C$ are also in a counter-clockwise direction. Is the integral $\int_C \mathbf{F}\cdot\hat{t}\, ds$ a large or small, positive or negative number, or is it close to zero? Explain why using the picture and how the integral was defined. Repeat for a vector field $\mathbf{F}$ which is circulating in the clockwise direction.

We want to try to wrap our heads around this odd definition. As hopefully the last two questions illustrated, we are trying to measure how the vector field $\mathbf{F}$ is "circulating" around some point. As we take the limit as $|\Delta S| \to 0$ then we are getting the circulation per unit area at the point $(x_0, y_0, z_0)$. But notice the left hand side of the definition, curl $\mathbf{F}$ is being doted with the unit normal vector $\hat{n}$ to the surface $S$, which tells us that curl $\mathbf{F}$ must be a vector itself.

The problem that makes this definition a little tricky is that in $\mathbb{R}^3$, there are an infinite number of planes that pass through the point $(x_0, y_0, z_0)$, and thus an infinite number of possible surfaces $S$, with corresponding boundaries $\partial S = C$, that can be shrunk to area zero around the point $(x_0, y_0, z_0)$. That is, in essence, the reason for the odd definition of curl that requires the dot product. This definition has to hold true for every possible plane through $(x_0, y_0, z_0)$ and every possible surface $S$ in that plane that shrinks to area zero around $(x_0, y_0, z_0)$. The dot product of $\hat{n}$, the unit normal to surface $S$, with curl $\mathbf{F}$ on the left hand side corresponds with, or compensates for, choosing a particular plane in which to place the surface $S$ and curve $C$ on the right hand side.

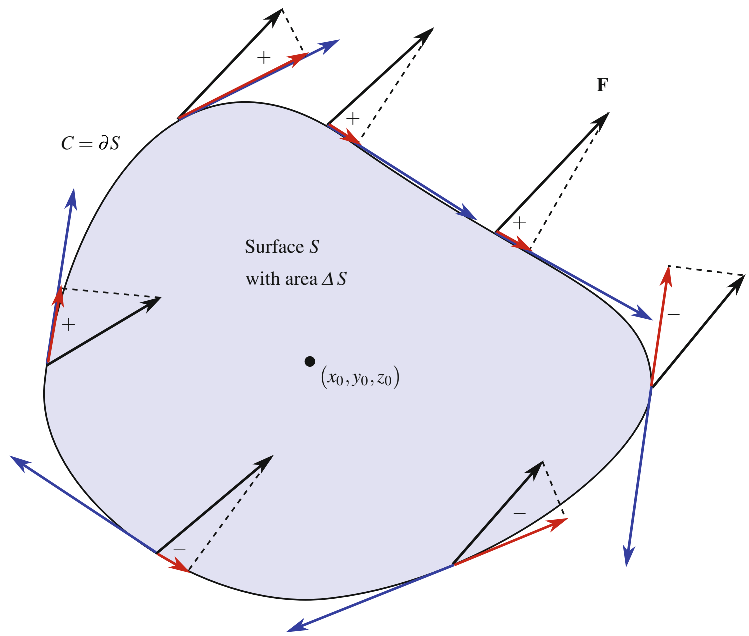


**Fig. 9.12** The vector field **F** (black) along a closed curve $C$ is shown. Unit tangent vectors to $C$ are shown (blue) along with the “part” of **F** along $C$ (red). Notice that sometimes the “part” of **F** along $C$ is positive and sometimes it is negative

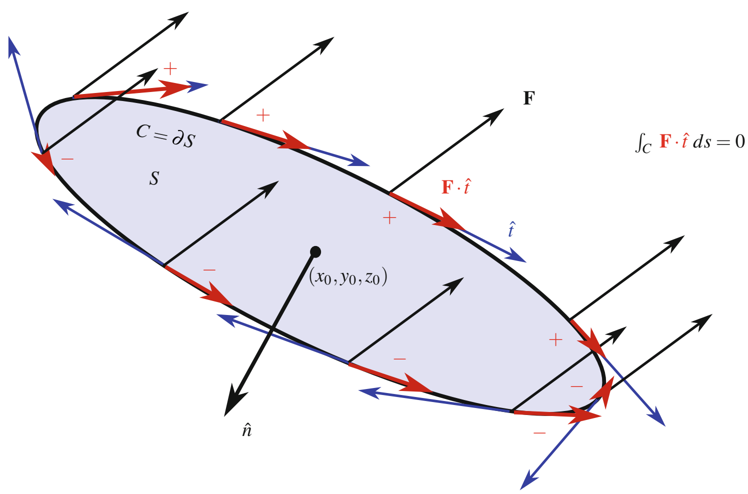


**Fig. 9.13** A picture in which $\hat{n} \cdot \text{curl}\,\mathbf{F} = 0$. We have shown all the components that show up in the definition of curl **F** except the vector curl **F**

Let us take a few moments to explore this a little deeper. Consider Fig. 9.13 where a surface $S$ is shown with boundary $\partial S = C$ around the point $(x_0, y_0, z_0)$. The unit tangent vectors $\hat{t}$ to the curve $C$ are shown in blue at several points along $C$ and the vector field **F** is shown in black at these same points along $C$. The portion of each field vector in the direction of $\hat{t}$ is shown as $\mathbf{F} \cdot \hat{t}$ in red, and its sign is also indicated, positive if $\mathbf{F} \cdot \hat{t}$ points in the direction $\hat{t}$ and negative if $\mathbf{F} \cdot \hat{t}$ points in direction $-\hat{t}$. We have drawn all vectors in the vector field **F** as parallel to each other in an attempt to show no circulation around the point $(x_0, y_0, z_0)$ is happening. This means the “amount” of positive $\mathbf{F} \cdot \hat{t}$ is exactly the same as the “amount” of negative $\mathbf{F} \cdot \hat{t}$ leading to $\int_C \mathbf{F} \cdot \hat{t}\, ds = 0$. The figure also shows the unit normal $\hat{n}$ to surface $S$ drawn at the point $(x_0, y_0, z_0)$. We draw $\hat{n}$ so it is following the right-hand rule with respect to the unit tangent vectors $\hat{t}$ (blue). We do this just to standardize the positive and negative of $\mathbf{F} \cdot \hat{t}$ between different surfaces. Using our definition of curl we have

$$\hat{n} \cdot \text{curl}\,\mathbf{F} = \lim_{|\Delta S| \to 0} \frac{1}{\Delta S} \int_C \mathbf{F} \cdot \hat{t}\, ds = 0.$$

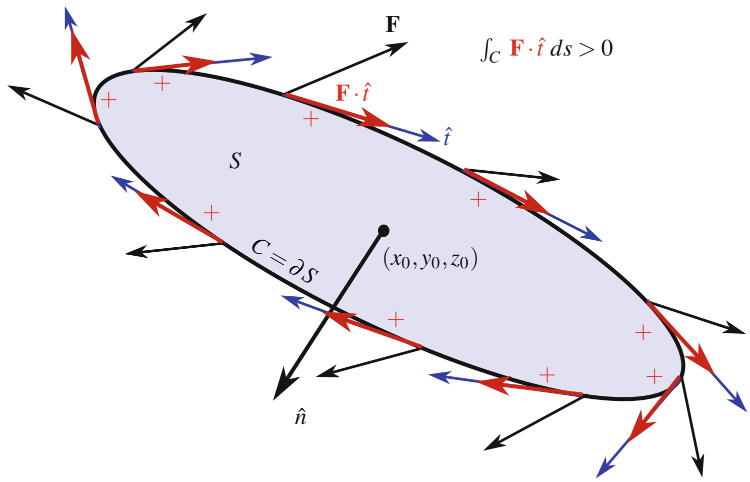


**Fig. 9.14** A picture in which $\hat{n} \cdot \text{curl}\,\mathbf{F} > 0$. Again we have shown all the components that show up in the definition of curl $\mathbf{F}$ except the vector curl $\mathbf{F}$

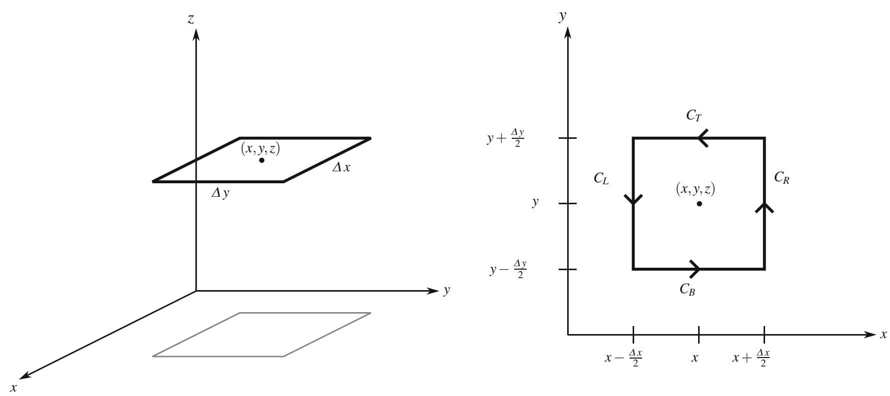


**Fig. 9.15** A square curve around the point $(x, y, z)$. This curve is in the plane parallel to the $xy$-plane (left), which has the unit normal $\hat{k}$. Looking down on this same square curve from the top (right). This square curve is composed of four line segments, $C_T$, $C_B$, $C_L$, and $C_R$

So, while we may not actually yet know what the vector curl $\mathbf{F}$ is, we know it must be perpendicular to $\hat{n}$ since $\hat{n} \cdot \text{curl}\,\mathbf{F} = 0$.

Now consider Fig. 9.14 where almost everything is the same as the last figure, except that this time the vectors from the vector field $\mathbf{F}$ are in some sense circulating around the point $(x_0, y_0, z_0)$. Notice how now all the $\mathbf{F} \cdot \hat{t}$ point in the same direction as $\hat{t}$ and are therefore positive. This means that

$$\hat{n} \cdot \text{curl}\,\mathbf{F} = \lim_{|\Delta S| \to 0} \frac{1}{\Delta S} \int_C \mathbf{F} \cdot \hat{t}\, ds > 0.$$

So, again, while we may not actually yet know what the vector curl $\mathbf{F}$ is, we know that when dotted with $\hat{n}$ it is positive.

So, actually finding what the vector curl $\mathbf{F}$ is requires us to choose surfaces, with their corresponding normals, carefully. Clearly, any vector, including the vector curl $\mathbf{F}$ is a linear combination of the Euclidian unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$. In order to find the component of curl $\mathbf{F}$ that goes in the $\hat{i}$ direction we need to find curl $\mathbf{F} \cdot \hat{i}$. Similarly, to find the component of curl $\mathbf{F}$ that goes in the $\hat{j}$ direction we need to find curl $\mathbf{F} \cdot \hat{j}$, and to find the component of curl $\mathbf{F}$ that goes in the $\hat{k}$ direction we need to find curl $\mathbf{F} \cdot \hat{k}$. So by choosing an $S$ in a plane parallel to the $yz$-plane we get $\hat{n} = \hat{i}$, which allows us to find the $\hat{i}$ term of curl $\mathbf{F}$. Similarly, choosing an $S$ in a plane parallel to the $xz$-plane gives $\hat{n} = \hat{j}$, which allows us to find the $\hat{j}$ term of curl $\mathbf{F}$ and choosing an $S$ in a plane parallel to the $xy$-plane gives $\hat{n} = \hat{k}$, which allows us to find the $\hat{k}$ term of curl $\mathbf{F}$.

We begin with by choosing a path abound the point $(x_0, y_0, z_0)$ in a plane parallel to the $xy$-plane, which has unit normal $\hat{k}$. Any path will do, so for simplicity we will choose a square path as shown in Fig. 9.15. Looking at this path from the top allows us to label the top, bottom, left, and right sides of the curve as $C_T$, $C_B$, $C_L$, and $C_R$ as shown. Along $C_B$ we have the unit tangent $\hat{t} = \hat{i}$, which gives $\mathbf{F} \cdot \hat{t} = \mathbf{F} \cdot \hat{i} = P$. So we have

$$\int_{C_B} \mathbf{F} \cdot \hat{t}\, ds \;=\; \int_{C_B} P\, ds \;\approx P\left(x_0, y_0 - \frac{\Delta y}{2}, z_0\right) \Delta x.$$

Along $C_T$ we have the unit the unit tangent $\hat{t} = -\hat{i}$, which gives $\mathbf{F} \cdot \hat{t} = \mathbf{F} \cdot (-\hat{i}) = -P$. So we have

$$\int_{C_T} \mathbf{F} \cdot \hat{t}\, ds \;=\; \int_{C_B} -P\, ds \;\approx -P\left(x_0, y_0 + \frac{\Delta y}{2}, z_0\right) \Delta x.$$

Combining we have

$$\int_{C_T+C_B} \mathbf{F} \cdot \hat{t}\, ds = \frac{-\left(P\left(x_0, y_0 + \frac{\Delta y}{2}, z_0\right) - P\left(z_0, y_0 - \frac{\Delta y}{2}, z_0\right)\right)}{\Delta y} \Delta x \Delta y$$

$$\Longrightarrow \lim_{|\Delta S| \to 0} \frac{1}{\Delta S} \int_{C_T+C_B} \mathbf{F} \cdot \hat{t}\, ds = \lim_{\Delta y \to 0} \frac{-\left(P\left(x_0, y_0 + \frac{\Delta y}{2}, z_0\right) - P\left(z_0, y_0 - \frac{\Delta y}{2}, z_0\right)\right)}{\Delta y} = -\frac{\partial P}{\partial y}.$$

Along $C_R$ we have the unit tangent $\hat{t} = \hat{j}$, which gives $\mathbf{F} \cdot \hat{t} = \mathbf{F} \cdot \hat{j} = Q$. So we have

$$\int_{C_R} \mathbf{F} \cdot \hat{t}\, ds \;=\; \int_{C_R} Q\, ds \;\approx Q\left(x_0 + \frac{\Delta x}{2}, y_0, z_0\right) \Delta y.$$

Along $C_L$ we have the unit tangent $\hat{t} = \hat{-j}$, which gives $\mathbf{F} \cdot \hat{t} = \mathbf{F} \cdot (-\hat{j}) = -Q$. So we have

$$\int_{C_L} \mathbf{F} \cdot \hat{t}\, ds \;=\; \int_{C_L} -Q\, ds \;\approx -Q\left(x_0 - \frac{\Delta x}{2}, y_0, z_0\right) \Delta y.$$

Combining we have

$$\int_{C_R+C_L} \mathbf{F} \cdot \hat{t}\, dx = \frac{Q\left(x_0 + \frac{\Delta x}{2}, y_0, z_0\right) - Q\left(x_0 - \frac{\Delta x}{2}, y_0, z_0\right)}{\Delta x} \Delta x \Delta y$$

$$\Longrightarrow \lim_{|\Delta S| \to 0} \frac{1}{\Delta S} \int_{C_R+C_L} \mathbf{F} \cdot \hat{t}\, ds = \lim_{\Delta x \to 0} \frac{Q\left(x_0, y_0 + \frac{\Delta y}{2}, z_0\right) - Q\left(x_0, y_0 - \frac{\Delta y}{2}, z_0\right)}{\Delta x} = \frac{\partial Q}{\partial x}.$$

Since the normal to the $xy$-plane is $\hat{k}$ we get

$$\hat{k} \cdot \operatorname{curl} \mathbf{F} = \lim_{|\Delta S| \to 0} \frac{1}{\Delta S} \int_{C_L+C+R+C_B+C_T} \mathbf{F} \cdot \hat{t}\, ds = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$

So this is the component of curl $\mathbf{F}$ in the $\hat{k}$ direction.

*Question 9.6* Repeat the above calculations to find that the component of curl $\mathbf{F}$ in the $\hat{i}$ direction is $\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}$ and the component of curl $\mathbf{F}$ in the $\hat{j}$ direction is $\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}$.

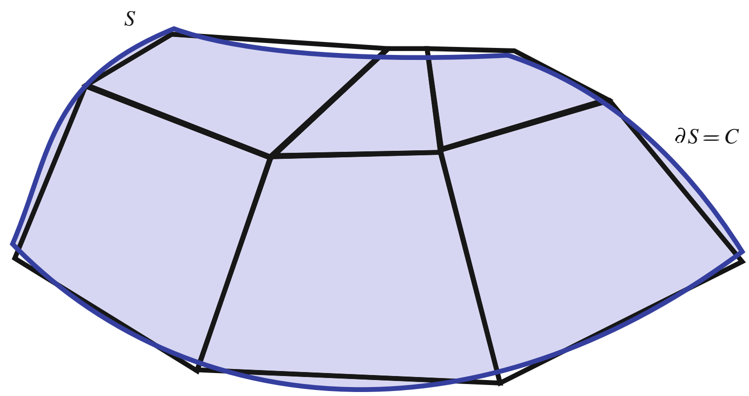


**Fig. 9.16** A surface $S$ with boundary the closed curve $\partial S = C$ covered by smaller surfaces $S_i$. As $\Delta S_i \to 0$ the surface $S$ is approximated better and better

Thus, from our definition of curl we have derived the formula

Formula for curl **F** in Cartesian Coordinates

$$\boxed{\operatorname{curl} \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\imath} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}.}$$

Stokes' theorem basically falls out of this definition. We will "derive" the theorem in a very non-rigorous way. Given any surface $S$, not necessarily in a plane, whose boundary is the closed curve $C$, we can break up at surface into subsurfaces $S_i$ with boundaries $C_i$, as in Fig. 9.16. By a line of reasoning similar to that in the last section, if two $C_i$ share and edge we have

$$\int_A^B \mathbf{F} \cdot \hat{t}\, ds = -\int_B^A \mathbf{F} \cdot \hat{t}\, ds$$

so these terms cancel out, see Fig. 9.17. Thus

$$\begin{aligned}
\int_{\partial S = C} \mathbf{F} \cdot \hat{t}\, ds &= \lim_{\substack{N\to\infty \\ |\Delta S_i|\to 0}} \sum_{i=0}^{N} \int_{C_i} \mathbf{F} \cdot \hat{t}\, ds \\
&= \lim_{\substack{N\to\infty \\ |\Delta S_i|\to 0}} \sum_{i=0}^{N} \left(\frac{1}{\Delta S_i}\int_{C_i} \mathbf{F} \cdot \hat{t}\, ds\right) \Delta S_i \\
&= \lim_{\substack{N\to\infty \\ |\Delta S_i|\to 0}} \sum_{i=0}^{N} \underbrace{\lim_{|\Delta S_i|\to 0}\left(\frac{1}{\Delta S_i}\int_{C_i} \mathbf{F} \cdot \hat{t}\, ds\right)}_{(\hat{n}_i \cdot \operatorname{curl} \mathbf{F})_i} \Delta S_i \\
&= \lim_{\substack{N\to\infty \\ |\Delta S_i|\to 0}} \sum_{i=0}^{N} (\hat{n}_i \cdot \operatorname{curl} \mathbf{F})_i\, \Delta S_i \\
&= \int_S (\hat{n} \cdot \operatorname{curl} \mathbf{F})\, dS,
\end{aligned}$$

which is exactly Stokes' theorem,

Stokes' Theorem

$$\boxed{\int_{\partial S} \mathbf{F} \cdot \hat{t}\, ds = \int_S (\hat{n} \cdot \operatorname{curl} \mathbf{F})\, dS.}$$

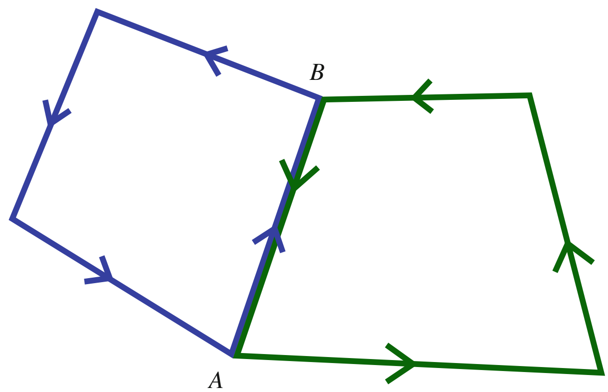


**Fig. 9.17** Two adjacent subsurfaces that share a boundary $C_i$, which is the line segment between points $A$ and $B$

An alternative way of writing Stokes' theorem is

$$\boxed{\text{Stokes' Theorem} \qquad \int_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S}.}$$

In this section we have used the definition of the curl of a vector field $\mathbf{F}$ on $\mathbb{R}^3$ written in Cartesian coordinates to find an expression for curl $\mathbf{F}$. But of course cylindrical and spherical coordinate systems, introduced in Sect. 6.5, are also very commonly used coordinate systems on $\mathbb{R}^3$.

*Question 9.7* Suppose that $\mathbf{F} = F_r \hat{e}_r + F_\theta \hat{e}_\theta + F_z \hat{e}_z$ is a vector field on $\mathbb{R}^3$ written with respect to cylindrical coordinates. The vector $\hat{e}_r$ is the unit vector in the direction of increasing $r$, $\hat{e}_\theta$ is the unit vector in the direction of increasing $\theta$, and $\hat{e}_z$ is the unit vector in the direction of increasing $z$. Using curves that correspond to the edges of each face of the cylindrical volume element shown in Fig. 6.14 and a procedure similar to that of this section, show that

$$\text{curl}\, \mathbf{F} = (\text{curl}\, \mathbf{F})_r \hat{e}_r + (\text{curl}\, \mathbf{F})_\theta \hat{e}_\theta + (\text{curl}\, \mathbf{F})_z \hat{e}_z,$$

where

$$\begin{aligned}
(\text{curl}\, \mathbf{F})_r &= \frac{1}{r} \frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z}, \\
(\text{curl}\, \mathbf{F})_\theta &= \frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}, \\
(\text{curl}\, \mathbf{F})_z &= \frac{1}{r} \frac{\partial (r F_\theta)}{\partial r} - \frac{1}{r} \frac{\partial F_r}{\partial \theta}.
\end{aligned}$$

*Question 9.8* Suppose that $\mathbf{F} = F_r \hat{e}_r + F_\theta \hat{e}_\theta + F_\phi \hat{e}_\phi$ is a vector field on $\mathbb{R}^3$ written with respect to spherical coordinates. The vector $\hat{e}_r$ is the unit vector in the direction of increasing $r$, $\hat{e}_\theta$ is the unit vector in the direction of increasing $\theta$, and $\hat{e}_\phi$ is the unit vector in the direction of increasing $\phi$. Using curves that correspond to the edges of each face of the spherical volume element shown in Fig. 6.16 and a procedure similar to that of this section, show that

$$\text{curl}\, \mathbf{F} = (\text{curl}\, \mathbf{F})_r \hat{e}_r + (\text{curl}\, \mathbf{F})_\theta \hat{e}_\theta + (\text{curl}\, \mathbf{F})_\phi \hat{e}_\phi,$$

where

$$\begin{aligned}
(\text{curl}\, \mathbf{F})_r &= \frac{1}{r \sin\theta} \frac{\partial (\sin\theta F_\phi)}{\partial \theta} - \frac{1}{r \sin\theta} \frac{\partial F_\theta}{\partial \phi}, \\
(\text{curl}\, \mathbf{F})_\theta &= \frac{1}{r \sin\theta} \frac{\partial F_r}{\partial \phi} - \frac{1}{r} \frac{\partial (r F_\phi)}{\partial r}, \\
(\text{curl}\, \mathbf{F})_\phi &= \frac{1}{r} \frac{\partial (r F_\theta)}{\partial r} - \frac{1}{r} \frac{\partial F_r}{\partial \theta}.
\end{aligned}$$

## 9.3 Gradient

The gradient is more straight-forward than either divergence or curl. In vector calculus the gradient of the function $f : \mathbb{R}^3 \to \mathbb{R}$ is often simply defined to be the vector field

$$\text{grad } f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}.$$

Generally grad $f$ is written as $\nabla f$. Additionally, in vector calculus we were told that the directional derivative of a function $f$ in the direction of the unit length vector $u$ can be written as $D_u f = \nabla f \cdot u$ or as $u[f] = \nabla f$.

We essentially go backwards and define the **gradient** of the function $f$ to be the vector field, which when dotted with the unit length vector $u$, gives the directional derivative of $f$ in the direction $u$,

| Definition of gradient | $\text{grad} f \cdot u = u[f].$ |
|---|---|

In Sect. 2.3 we basically showed that one could write

| Formula for grad **F** in Cartesian Coordinates | $\text{grad } f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$ |
|---|---|

when using the Cartesian coordinate system. Thus we see that the usual vector calculus definition agrees with our definition. Of course, at this point you should see the similarities between the gradient of $f$, grad $f$, and the one-form $df$.

We already discussed in the last section what the integral of a vector field along a curve was. The integral of the vector field **F** along a curve $C$ was given by $\int_C \mathbf{F} \cdot \hat{t} ds$, where $\hat{t}$ was the unit tangent vector along the curve $C$, which was parameterized by arc length $s$. Writing the curve $C = c(s) = \big(x(s), y(s), z(s)\big)$ then we have

$$\begin{aligned} c'(s) &= \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right) \\ &= \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k} \end{aligned}$$

which is clearly the tangent to the curve $c(s)$. Also, in a manor similar to above we could formally write $c'(s)ds = \hat{i}dx + \hat{j}dy + \hat{k}dz$ and define this as $d\mathbf{s}$. Suppose we let $\mathbf{F} = \text{grad } f$, then the integral of grad $f$ along the curve $c(s)$ from $c(a)$ to $c(b)$, where $a, b \in \mathbb{R}$ and $a \le b$, would be

$$\begin{aligned} \int_C \text{grad } f \cdot c'(s)\, ds &= \int_a^b \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right) \cdot \left(\frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k}\right)\, ds \\ &= \int_a^b \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds}\, ds \\ &= \int_a^b \frac{d}{ds} f(c(s))\, ds \\ &= f(c(b)) - f(c(a)). \end{aligned}$$

The second to last equality comes from the chain rule and the last equality comes form the fundamental theorem of calculus. Writing $c'(s)\ ds$ as $d\mathbf{s}$ we get the fundamental theorem of line integrals,

| Fundamental theorem of line integrals | $\int_C \text{grad } f \cdot d\mathbf{s} = f(c(b)) - f(c(a)).$ |
|---|---|

In cylindrical coordinates a similar sort of argument would give

$$\text{grad } f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{e}_\theta + \frac{\partial f}{\partial z}\hat{e}_z$$

where $\hat{e}_r$ is the unit vector in the direction of increasing $r$, $\hat{e}_\theta$ is the unit vector in the direction of increasing $\theta$, and $\hat{e}_z$ is the unit vector in the direction of increasing $z$. In spherical coordinates we would have

$$\text{grad } f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{e}_\phi$$

where of course $\hat{e}_r$ is the unit vector in the direction of increasing $r$, $\hat{e}_\theta$ is the unit vector in the direction of increasing $\theta$, and $\hat{e}_\phi$ is the unit vector in the direction of increasing $\phi$.

*Question 9.9* The Laplacian of a function $f$ is defined to be $\text{div}(\text{grad } f) = \nabla \cdot (\nabla f) = \nabla \cdot \nabla f$. Sometimes this is written as $\nabla^2 f$ or as $\Delta f$. Find the Laplacian of $f$ in Cartesian, cylindrical, and spherical coordinates.

## 9.4 Upper and Lower Indices, Sharps, and Flats

Before going any further now is a good time to introduce a particular notational convention. This notational convention will be useful for what follows. So far in this book we have been trying to explain mathematical concepts and so have primarily used mathematical notations. However, as we move toward a number of other examples and topics the notation that is generally used in physics will be useful.

Different notations have different strengths and weaknesses and differential geometry is one of those areas of mathematics that is very notation heavy. Mathematical notation is good for understanding theory, what mathematical objects actually are, the spaces that mathematical objects live in, and what the underlying mathematical operations and actions are. Physics notation is generally much better suited for performing computations and calculating things, even though the notation may obscure the mathematical reality underneath the computations. To be a good mathematician or physicist you really need to be comfortable with both sets of notations.

First we will introduce Einstein summation notation. Simply put, the essence of Einstein summation notation is that when you see repeated upper and lower indices you sum. That is, if you see something like

$$a^i e_i$$

that really means

$$\sum_i a^i e_i.$$

Similarly,

$$a_i e^i \equiv \sum_i a_i e^i.$$

Einstein was once reported to have quipped that the summation notation was his great contribution to mathematics! Summation notation is used extensively in tensor calculus, which we will introduce in Appendix A. Without going into the meaning of the expression at the moment, consider

$$T^{ijk}_{lm}\Lambda^l_{jk}.$$

With Einstein summation notation this actually means

$$\sum_j\sum_k\sum_l T^{ijk}_{lm}\Lambda^l_{jk}.$$

As you can see, every time an upper index and a lower index is repeated we sum over that index. Since there is both an upper and a lower index $j$, $k$, and $l$ then we sum over $j$, $k$, and $l$.

Consider the vector space $V = \text{span}\{e_1, e_2, e_3\}$. The elements of of $V$ can be written in terms of the basis vectors $e_1, e_2, e_3$ with the use of coefficients. That is, $v \in V$ can be written as

$$v = v^1 e_1 + v^2 e_2 + v^3 e_3 = \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = v^i e_i,$$

where the vector coefficients $v^1, v^2, v^3$ are real numbers. Notice that the vector coefficients are written with upper indices. As usual, we continue to think of a vector as a column matrix.

The dual space of V is $V^* = \text{Span}\{e_1^*, e_2^*, e_3^*\}$, where $e_1^*, e_2^*, e_3^*$ are simply the dual elements of $e_1, e_2, e_3$. That is, we have

$$\langle e_i^*, e_j \rangle = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

To keep in line with Einstein summation notation from now on we will write the dual basis elements with upper indices as $e^1, e^2, e^3$. That is, $e_1^* \equiv e^1$, $e_2^* \equiv e^2$, $e_3^* \equiv e^3$, so we have $\langle e^i, e_j \rangle = e^i(e_j) = \delta_{ij}$. Sometimes you will see the Kronecker delta function written as $\delta^i_j$ to keep consistent with Einstein summation notation. The elements of $V^* = \text{Span}\{e^1, e^2, e^3\}$, sometimes called co-vectors, are written as

$$\alpha = \alpha_1 e^1 + \alpha_2 e^2 + \alpha_3 e^3 = [\alpha_1, \alpha_2, \alpha_3] = \alpha_i e^i,$$

where the co-vector coefficients $\alpha_1, \alpha_2, \alpha_3$ are real numbers. Again, notice that the co-vector coefficients are written with lower indices. As usual, we continue to think of a co-vector as a row matrix. So,

$$\begin{aligned}
\langle \alpha, v \rangle &= \alpha(v) \\
&= (\alpha_1 e^1 + \alpha_2 e^2 + \alpha_3 e^3)(v^1 e_1 + v^2 e_2 + v^3 e_3) \\
&= \alpha_1 e^1 (v^1 e_1 + v^2 e_2 + v^3 e_3) \\
&\quad + \alpha_2 e^2 (v^1 e_1 + v^2 e_2 + v^3 e_3) \\
&\quad + \alpha_3 e^3 (v^1 e_1 + v^2 e_2 + v^3 e_3) \\
&= \alpha_1 v^1 e^1(e_1) + \alpha_1 v^2 e^1(e_2) + \alpha_1 v^3 e^1(e_3) \\
&\quad + \alpha_2 v^1 e^2(e_1) + \alpha_2 v^2 e^2(e_2) + \alpha_2 v^3 e^2(e_3) \\
&\quad + \alpha_3 v^1 e^3(e_1) + \alpha_3 v^2 e^3(e_2) + \alpha_3 v^3 e^3(e_3) \\
&= \alpha_1 v^1 + \alpha_2 v^2 + \alpha_3 v^3.
\end{aligned}$$

What a messy calculation, not something you would want to do if you had many more then three basis elements, though of course the pattern is pretty clear. The same calculation using traditional summation notation would be

$$\begin{aligned}
\langle \alpha, v \rangle &= \alpha(v) \\
&= \left( \sum_{i=1}^{3} \alpha_i e^i \right) \left( \sum_{j=1}^{3} v^j e_j \right) \\
&= \sum_{i=1}^{3} \alpha_i e^i \left( \sum_{j=1}^{3} v^j e_j \right)
\end{aligned}$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} \alpha_i v^j e^i \left(e_j\right)$$

$$= \sum_{i=1}^{3} \alpha_i v^i. \qquad \text{(Only } j = i \text{ terms survive.)}$$

Finally we do the same calculation with Einstein summation notation

$$\begin{aligned}
\langle \alpha, v \rangle &= \alpha(v) \\
&= (\alpha_i e^i)(v^j e_j) \\
&= \alpha_i e^i (v^j e_j) \\
&= \alpha_i v^j e^i (e_j) \\
&= \alpha_i v^i. \qquad \text{(Only } j = i \text{ terms survive.)}
\end{aligned}$$

Once you get comfortable with Einstein summation notation your hand does quite a bit less moving.

Now, let's take a look at all of this in the context of differential forms. For the moment we will assume that $M = \mathbb{R}^3$. We will write the Cartesian coordinate functions of $M$ as $x^1, x^2, x^3$, with upper indices instead of lower indices. Mathematicians tend to use lower indices for coordinate functions, which is in fact what we have done up till now, but by writing coordinate functions with upper indices Einstein summation notation works nicely. So, at a point $p \in M$ we have the tangent space at $p$ given by

$$\begin{aligned}
T_p\mathbb{R}^3 &= \text{span} \left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \left.\frac{\partial}{\partial x^2}\right|_p, \left.\frac{\partial}{\partial x^3}\right|_p \right\} \\
&= \text{span} \left\{ \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right\},
\end{aligned}$$

where the second line has the base point suppressed in the notation. The cotangent space at $p \in M$ is given by

$$\begin{aligned}
T_p^*\mathbb{R}^3 &= \text{Span} \left\{ \left.dx^1\right|_p, \left.dx^2\right|_p, \left.dx^3\right|_p \right\} \\
&= \text{Span} \left\{ dx^1, dx^2, dx^3 \right\}.
\end{aligned}$$

Clearly we can consider the index in $dx^i$ as an upper index. By convention we consider the index in $\frac{\partial}{\partial x^i}$ as a lower index because it is in the "denominator" of the expression. Therefore elements $v \in T_p\mathbb{R}^3$ are written as

$$v = v^1 \frac{\partial}{\partial x^1} + v^2 \frac{\partial}{\partial x^2} + v^3 \frac{\partial}{\partial x^3} = v^i \frac{\partial}{\partial x^i},$$

where $v^1, v^2, v^3$ are real numbers. Again vectors have coefficients with upper indices. Elements $\alpha \in T_p^*\mathbb{R}^3$ are written as

$$\alpha = \alpha_1 dx^1 + \alpha_2 dx^2 + \alpha_2 dx^3 = \alpha_i dx^i,$$

where $\alpha^1, \alpha^2, \alpha^3$ are real numbers. As we can see, covectors, that is, differential forms, have coefficients with lower indices.

**In summary, vector basis elements are written with lower indices and vector components are written with upper indices. Covector (differential form) basis elements are written with upper indices and covector (differential form) components are written with lower indices.**

Now we introduce the so called **musical isomorphisms**. In reality, the musical isomorphisms depend on something called a metric. Very roughly you can think of a metric as something that allows you to measure the distance between two points. We will discuss the **Euclidian metric** later on, after we have introduced tensors, but it is, in essence, the usual distance function on $\mathbb{R}^n$ that we are familiar with. However, right now we will not be that mathematically rigorous and give a "definition" of the musical isomorphisms that does not explicitly rely on the metric. You should simply keep in mind that at the moment we are sweeping some of the mathematical theory under the rug, so to speak.

The musical isomorphisms are called musical because of the notation that is used, the **flat** $\flat$ and the **sharp** $\sharp$. If you have ever sung or played an instrument you know that the $\flat$ tells you to lower the pitch of a musical note and the $\sharp$ tells you to raise the pitch of a musical note. Similarly, the $\flat$ isomorphism is said to "lower indices" while the $\sharp$ isomorphism is said to "raise indices."

The $\flat$ isomorphism is given by

$$\begin{aligned} \flat : T_pM &\longrightarrow T_p^*M \\ v^i \frac{\partial}{\partial x^i} &\longmapsto v_i dx^i \qquad \text{where } v_i = v^i. \end{aligned}$$

The way this mapping is written is $v^\flat$, just like the flat is written in music, so we would write

$$v^\flat = \left( v^i \frac{\partial}{\partial x^i} \right)^\flat \equiv v_i dx^i,$$

where $v_i = v^i$. So for example we would have

$$\left( 7\frac{\partial}{\partial x^1} + 3\frac{\partial}{\partial x^2} - 6\frac{\partial}{\partial x^3} \right)^\flat = 7dx^1 + 3dx^2 - 6dx^3.$$

Notice what is happening, we are turning a vector, where the vector components are written with upper indices, into a one-form, where the one-form components are written with lower indices, so we are "lowering" the indices, which explains the use of the $\flat$ symbol.

The $\sharp$ isomorphism is given by

$$\begin{aligned} \sharp : T_p^*M &\longrightarrow T_pM \\ \alpha_i dx^i &\longmapsto \alpha^i \frac{\partial}{\partial x^i} \qquad \text{where } \alpha^i = \alpha_i. \end{aligned}$$

Again, the way this mapping is written is $\alpha^\sharp$, just like the sharp is written in music, so we would write

$$\alpha^\sharp = \left( \alpha_i dx^i \right)^\sharp \equiv \alpha^i \frac{\partial}{\partial x^i},$$

where $\alpha^i = \alpha_i$. So for example, we would have

$$\left( 3dx^1 - 9dx^2 + 8dx^3 \right)^\sharp = 3\frac{\partial}{\partial x^1} - 9\frac{\partial}{\partial x^2} + 8\frac{\partial}{\partial x^3}.$$

We are turning a differential one-form, or co-vector, where components are written with lower indices, into a vector, where components are written with upper indices, so we are "raising" the indices, which explains the use of the $\sharp$ symbol.

**You should note that these "definitions" of the flat and sharp operators only work if we are writing our vectors and one-forms in Cartesian coordinates.** This is the price we pay for not giving the real definitions of the flat and short operators in terms of the metric on a manifold. So, for the moment we can only take the flats of vectors and the sharps of one-forms if they are written in terms of Cartesian coordinates but not if our vectors or one-forms are written with respect to any other coordinate system.

If $\mathbf{F}$ is a vector field on $M$, that is a section of the tangent bundle $TM$, then $\mathbf{F}^\flat$ is a differential one-form on $M$, that is, a section of the cotangent bundle $T^*M$. And if $\alpha$ is a differential one-form on $M$, that is a section of the cotangent bundle $T^*M$, then $\alpha^\sharp$ is a vector field on $M$, that is, a section of the tangent bundle $TM$. Recalling that we always write vectors as column matrices and one-forms as row matrices, we can think of $\flat$ as turning a column matrix into into a row matrix and $\sharp$ as turning a row matrix into a column matrix.

Now, let us remind ourselves very briefly of the Hodge star operator from Sect. 5.6. We had computed the following Hodge star mappings

$$
\begin{aligned}
* &: \bigwedge^0\left(\mathbb{R}^3\right) \longrightarrow \bigwedge^3\left(\mathbb{R}^3\right) \\
* &: \bigwedge^1\left(\mathbb{R}^3\right) \longrightarrow \bigwedge^2\left(\mathbb{R}^3\right) \\
* &: \bigwedge^2\left(\mathbb{R}^3\right) \longrightarrow \bigwedge^1\left(\mathbb{R}^3\right) \\
* &: \bigwedge^3\left(\mathbb{R}^3\right) \longrightarrow \bigwedge^0\left(\mathbb{R}^3\right).
\end{aligned}
$$

Which were given by

$$
\begin{aligned}
&* 1 = dx^1 \wedge dx^2 \wedge dx^3 = dx \wedge dy \wedge dz, \\
&* dx = dy \wedge dz, \quad *dy = dz \wedge dx, \quad *dz = dx \wedge dy, \\
&* dy \wedge dz = dx, \quad *dz \wedge dx = dy, \quad *dx \wedge dy = dz, \\
&* dx \wedge dy \wedge dz = 1.
\end{aligned}
$$

Suppose we have the vector field $\mathbf{F} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}$. We will define the mapping $(* \circ \flat)$ as first flatting the vector field to get a one-form and then Hodge staring that one-form to get a two form;

$$
\begin{aligned}
(* \circ \flat)\mathbf{F} &\equiv *(\mathbf{F}^\flat) \\
&= *\left(\left(P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}\right)^\flat\right) \\
&= *\,(Pdx + Qdy + Rdz) \\
&= Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy.
\end{aligned}
$$

With this final mapping in hand, we are now ready to look at the relationship between standard vector calculus and differential forms.

## 9.5 Relationship to Differential Forms

### *9.5.1 Grad, Curl, Div and Exterior Differentiation*

Understanding the relationship between differential forms and vector calculus really amounts to little more then seeing that the appropriate mappings lead to everything working out nicely. We will essentially look at each piece, one at a time. As we go through this process we will discover the generalized Stokes' theorem, usually just called Stokes' theorem. The generalized Stokes' theorem, often simply called Stokes' theorem, is a simultaneous generalization of the three vector calculus theorems you already know,

- fundamental theorem of line integrals,
- (vector calculus) Stokes' theorem,
- (vector calculus) divergence theorem.

The generalized Stokes' theorem will allow us to use differential forms to rewrite all three of these theorems in a very nice and compact way.

Now we will show that the following diagram commutes.

$$\begin{array}{ccc} C\left(\mathbb{R}^3\right) & \xrightarrow{\text{grad}} & T\mathbb{R}^3 \\ \Big\downarrow \text{id} & & \Big\downarrow \flat \\ \bigwedge^0\left(\mathbb{R}^3\right) & \xrightarrow{d} & \bigwedge^1\left(\mathbb{R}^3\right) \end{array}$$

First of all we make sure we recall what the spaces in the four corners are. In the upper left hand corner we have the space $C\left(\mathbb{R}^3\right)$, which is simply the space of continuous functions on the manifold $\mathbb{R}^3$. In the upper right hand corner the space $T\mathbb{R}^3$ is simply the tangent bundle of $\mathbb{R}^3$, which contains all the vector fields on $\mathbb{R}^3$. In the lower left hand corner is $\bigwedge^0\left(\mathbb{R}^3\right)$, the zero-forms on $\mathbb{R}^3$, which are just the continuous functions on $\mathbb{R}^3$. Notice that the space $\bigwedge^0\left(\mathbb{R}^3\right)$ is exactly the same space as $C\left(\mathbb{R}^3\right)$. In the lower left hand corner is the space $\bigwedge^1\left(\mathbb{R}^3\right) = T^*\mathbb{R}^3$, the one-forms on $\mathbb{R}^3$. The mappings between these spaces are grad, $\flat$, $d$, and id, the identity mapping.

"The diagram commutes" is one of those math phrases that you will hear a lot if you are a math major, and even if you are a physics major you may hear it from time to time. Let us take a moment to explain what it means. If you start with a function $f \in C\left(\mathbb{R}^3\right)$ then regardless of the path you take, either across the top then down the right hand side or down the left hand side and then across the bottom, you will get the same element in $\bigwedge^1\left(\mathbb{R}^3\right)$. That is, we have

$$(\text{grad } f)^\flat = d\,(\text{id}\,(f))\,.$$

So saying the above diagram commutes is exactly the same thing as saying $(\text{grad } f)^\flat = d\,(\text{id}\,(f)) = df$. Now we will show this equality. Given a function on $\mathbb{R}^3$, $f \in C(\mathbb{R}^3)$ we know that grad turns that function into a vector field $\nabla f \in T(\mathbb{R})$ according to

$$\begin{aligned} \nabla f &= \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} \\ &= \frac{\partial f}{\partial x}\frac{\partial}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial}{\partial z}. \end{aligned}$$

Flatting this we have

$$(\nabla f)^\flat = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

Notice, this is exactly equal to the exterior derivative of $f$,

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz,$$

hence we have shown that

$$df = (\nabla f)^\flat\,.$$

Similarly, we want to show the below diagram commutes as well.

$$\begin{array}{ccc} T\mathbb{R}^3 & \xrightarrow{\text{curl}} & T\mathbb{R}^3 \\ \Big\downarrow \flat & & \Big\downarrow {*}\circ\flat \\ \bigwedge^1\left(\mathbb{R}^3\right) & \xrightarrow{d} & \bigwedge^2\left(\mathbb{R}^3\right) \end{array}$$

That is, we want to show that $d(\mathbf{F}^\flat) = *\big((\text{curl }\mathbf{F})^\flat\big)$. Given a vector field $\mathbf{F} \in T\mathbb{R}^3$. Recalling the definition of the mapping we defined in the last section, $(* \circ \flat)\mathbf{F} \equiv *(\mathbf{F}^\flat)$, we want to find $(* \circ \flat)(\text{curl }\mathbf{F})$. First, we know

$$\text{curl }\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\frac{\partial}{\partial x} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\frac{\partial}{\partial y} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\frac{\partial}{\partial z}.$$

So we get

$$\begin{aligned}
(* \circ \flat)(\text{curl }\mathbf{F}) &= *\big((\text{curl }\mathbf{F})^\flat\big) \\
&= *\left(\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)dx + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)dy + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dz\right) \\
&= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx \wedge dy.
\end{aligned}$$

Now we want to compute $d(\mathbf{F}^\flat)$. First we find

$$\begin{aligned}
\mathbf{F}^\flat &= \left(P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}\right)^\flat \\
&= Pdx + Qdy + Rdz.
\end{aligned}$$

Then we take the exterior derivative of this

$$\begin{aligned}
d(\mathbf{F}^\flat) &= d(Pdx + Qdy + Rdz) \\
&= dP \wedge dx + dQ \wedge dy + dR \wedge dz \\
&= \left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy + \frac{\partial P}{\partial z}dz\right) \wedge dx \\
&\quad + \left(\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy + \frac{\partial Q}{\partial z}dz\right) \wedge dy \\
&\quad + \left(\frac{\partial R}{\partial x}dx + \frac{\partial R}{\partial y}dy + \frac{\partial R}{\partial z}dz\right) \wedge dz \\
&= \frac{\partial P}{\partial y}dy \wedge dx + \frac{\partial P}{\partial z}dz \wedge dx + \frac{\partial Q}{\partial x}dx \wedge dy + \frac{\partial Q}{\partial z}dz \wedge dy \\
&\quad + \frac{\partial R}{\partial x}dx \wedge dz + \frac{\partial R}{\partial y}dy \wedge dz \\
&= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx \wedge dy.
\end{aligned}$$

Thus we have just shown that $d(\mathbf{F}^\flat) = (* \circ \flat)(\text{curl }\mathbf{F})$. Finally, we would like to show that this diagram commutes.

$$\begin{array}{ccc}
T\mathbb{R}^3 & \xrightarrow{\text{div}} & C(\mathbb{R}^3) \\
\big\downarrow{\scriptstyle *\circ\flat} & & \big\downarrow{\scriptstyle *} \\
\bigwedge^2(\mathbb{R}^3) & \xrightarrow{d} & \bigwedge^3(\mathbb{R}^3)
\end{array}$$

Given $\mathbf{F}$ we have

$$\text{div}\,\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Taking the Hodge star of this we have

$$\begin{aligned} *(\text{div}\,\mathbf{F}) &= *\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) \\ &= \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx \wedge dy \wedge dz. \end{aligned}$$

Next, we find

$$\begin{aligned} (* \circ \flat)\mathbf{F} &= *(\mathbf{F}^\flat) \\ &= *\left(\left(P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}\right)^\flat\right) \\ &= *\,(Pdx + Qdy + Rdz) \\ &= Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy \end{aligned}$$

and then taking the exterior derivative of this,

$$\begin{aligned} d\,((* \circ \flat)\mathbf{F}) &= d\,(Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy) \\ &= dP \wedge dy \wedge dz + dQ \wedge dz \wedge dx + dR \wedge dx \wedge dy \\ &= \frac{\partial P}{\partial x} dx \wedge dy \wedge dz + \frac{\partial Q}{\partial y} dy \wedge dz \wedge dx + \frac{\partial R}{\partial z} dz \wedge dx \wedge dy \\ &= \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx \wedge dy \wedge dz. \end{aligned}$$

Hence we have found that

$$d\,((* \circ \flat)\mathbf{F}) = *(\text{div}\,\mathbf{F}),$$

which was what we wanted to find. When we connect these three diagrams together what we get is this:

$$\begin{tikzcd} C\left(\mathbb{R}^3\right) \arrow[r, "\text{grad}"] \arrow[d, "\text{id}"] & T\mathbb{R}^3 \arrow[r, "\text{curl}"] \arrow[d, "\flat"] & T\mathbb{R}^3 \arrow[r, "\text{div}"] \arrow[d, "*\circ\flat"] & C\left(\mathbb{R}^3\right) \arrow[d, "*"] \\ \bigwedge^0\left(\mathbb{R}^3\right) \arrow[r, "d"] & \bigwedge^1\left(\mathbb{R}^3\right) \arrow[r, "d"] & \bigwedge^2\left(\mathbb{R}^3\right) \arrow[r, "d"] & \bigwedge^3\left(\mathbb{R}^3\right) \end{tikzcd}$$

So, we have very clear relationship between the three vector calculus operators and the exterior derivative. The gradient, the curl, and the divergence are all nothing other than exterior derivatives in a different guise. Vector calculus turns out to be another way of formulating and presenting exterior derivatives and differential forms on $\mathbb{R}^3$. In vector calculus everything is kept as a vector, yet utilized in the same way that forms are utilized. The problem with vector calculus is that it can not be generalized to $\mathbb{R}^n$ for $n > 3$ or to general manifolds, while our notions of differential forms and exterior derivatives can be generalized to both $\mathbb{R}^n$ for $n > 3$ and to general manifolds.

*Question 9.10* Let $\mathbf{F} = P\hat{i} + Q\hat{j} + R\hat{k}$ be a vector field and $f$ a continuous function. Show that

(a) Show curl (grad $\mathbf{F}$)=0.
(b) Show div (curl $\mathbf{F}$)=0.
(c) Explain why you would expect this given what you know about the exterior derivative $d$.

*Question 9.11* Prove the following identities:

(a) $\text{grad } f = (df)^\sharp$
(b) $\text{curl } \mathbf{F} = [*(df\flat)]^\sharp$
(c) $\text{div } \mathbf{F} = *d(*(\mathbf{F}^\flat))$
(d) $v \times w = [*(v^\flat \wedge w^\flat)]^\sharp$ for vector fields $v$ and $w$.
(e) $(v \cdot w) dx \wedge dy \wedge dz = v^\flat \wedge *(w^\flat)$

### 9.5.2 Fundamental Theorem of Line Integrals

Now, let's take a look at the fundamental theorem of line integrals. Suppose we are given a curve $C$, given by $c(s) = \big(x(s), y(s), z(s)\big)$ with end points $c(a) = \big(x(a), y(a), z(a)\big)$ and $c(b) = \big(x(b), y(b), z(b)\big)$, where $a, b \in \mathbb{R}$ and $a \le b$, then the fundamental theorem of line integrals is given by

$$\int_C (\text{grad } f) \cdot d\mathbf{s} = f(c(b)) - f(c(a)).$$

We will write the boundary of curve $C$ as $\partial C = \big\{c(b)\big\} - \big\{c(a)\big\}$. Boundaries are explained in depth in Chap. 11. By doing that then we can view, or define, $f(c(b)) - f(c(a))$ as the degenerate form of the integral of $f$ on $\partial C$, that is, as

$$\int_{\partial C} f \equiv f(c(b)) - f(c(a)).$$

We admit, this notation for the zero dimensional case may seem little contrived, but by making this definition we are able to have one consistent notation for the generalized Stokes' theorem. This allows us to write the right hand side of the fundamental theorem of line integrals as $\int_{\partial C} f$.

Let the curve $C$ be parameterized by $s$ and given by $c$, so we have $C = c(s) = \big(x(s), y(s), z(s)\big)$ and $c'(s) = \frac{dx}{ds}\hat{\imath} + \frac{dy}{ds}\hat{\jmath} + \frac{dz}{dt}\hat{s}$. If $\mathbf{F}$ is a vector field we have the integral of $\mathbf{F}$ along $C$ as

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{s} \equiv \int_C \mathbf{F} \cdot c'(s)\, ds \\
&= \int_C \left(P\hat{\imath} + Q\hat{\jmath} + R\hat{k}\right) \cdot \left(\frac{dx}{ds}\hat{\imath} + \frac{dy}{ds}\hat{\jmath} + \frac{dz}{ds}\hat{k}\right) ds \\
&= \int_C \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt}\right) ds \\
&= \int_C \mathbf{F}^\flat.
\end{aligned}$$

If $\mathbf{F} = \text{grad } f$ then we have $(\text{grad } f)^\flat = df$. In other words, we can write the right hand side of the fundamental theorem of line integrals as $\int_C df$. Combining this with what we had above, we could write the fundamental theorem of line integrals as

$$\int_C df = \int_{\partial C} f.$$

Recognizing that $f$ is in fact a zero-form, which we could denote at $\alpha$ and $C$ is a one dimensional manifold $M$ we could write the fundamental theorem of line integrals as

$$\int_M d\alpha = \int_{\partial M} \alpha.$$

### 9.5.3 *Vector Calculus Stokes' Theorem*

Now we turn to look at the vector calculus version of Stokes' theorem,

$$\int_S \text{curl}\,\mathbf{F}\cdot\hat{n}\;dS = \int_{\partial S}\mathbf{F}\cdot c'(s)\;ds.$$

From above we know we can rewrite the right hand side of the Stokes' theorem as $\int_{\partial S}\mathbf{F}^\flat$.

For the left hand side of Stokes' theorem we want to do something similar to what we did above, which requires us to show that

$$\int_S \text{curl}\,\mathbf{F}\cdot\hat{n}\;dS = \int_S *\left((\text{curl}\,\mathbf{F})^\flat\right).$$

In order to make the notation simpler we will actually show

$$\int_S \mathbf{F}\cdot\hat{n}\;dS = \int_S *\left(F^\flat\right).$$

We then get the required identity simply by replacing the vector field $\mathbf{F}$ by the vector field curl $\mathbf{F}$.

However, instead of a tidy computation like when we showed that $\int_C \mathbf{F}\cdot dr = \int_C \mathbf{F}^\flat$ in the line integral case, here it is easiest to go back to the basic definition of flux through a surface. This argument will not be rigorous but should suffice to help you understand what is going on. An infinitesimal area $dS$ with normal $\hat{n} = n_1\hat{i} + n_2\hat{j} + n_3\hat{k}$ can be decomposed into infinitesimal pieces in the $xy$-plane, the $yz$-plane, and the $xz$-plane. Here $n_1 dS$ is the piece of $dS$ that is in the $yz$-plane. This piece of $dS$ has the volume form $dy\wedge dz$. Similarly, $n_2 dS$ is the piece of $dS$ that is in the $xz$-plane and has the volume form $dz\wedge dx$. Finally $n_3 dS$ is the piece of $dS$ in the $xy$-plane which has volume form $dx\wedge dy$. See Fig. 9.18. Thus we can write

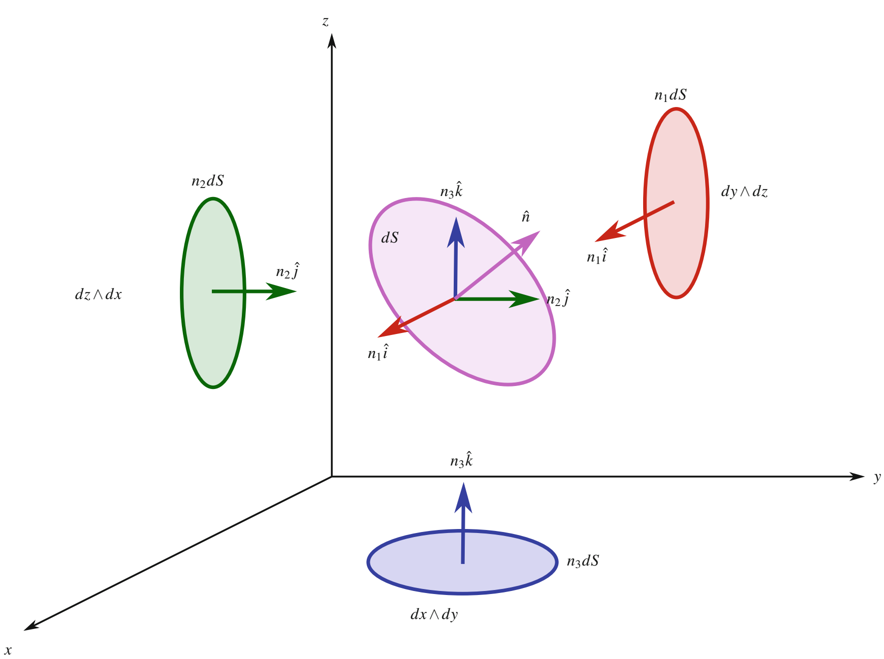


**Fig. 9.18** The infinitesimal area $dS$ with the unit normal vector $\hat{n} = n_1\hat{i}+n_2\hat{j}+n_3\hat{k}$ is decomposed into $n_1dS$, $n_2dS$, and $n_3dS$, with unit normals $n_1\hat{i}$, $n_2\hat{j}$, and $n_3\hat{k}$, respectively

$\mathbf{F} \cdot \hat{n}\ dS$ as $\mathbf{F} \cdot d\mathbf{S}$ and so have

$$\begin{aligned}\int_S \mathbf{F} \cdot d\mathbf{S} &= \int_S \left(P\hat{i} + Q\hat{j} + R\hat{k}\right) \cdot \left(n_1\hat{i} + n_2\hat{j} + n_3\hat{k}\right)\ dS \\ &= \int_S (Pn_1 + Qn_2 + Rn_3)\ dS \\ &= \int_S Pn_1\ dS + Qn_2\ dS + Rn_3\ dS \\ &= \int_S Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy \\ &= \int_S * \left(\mathbf{F}^\flat\right)\end{aligned}$$

which was what we wanted to show. When we replace $\mathbf{F}$ by curl $\mathbf{F}$ we have

$$\begin{aligned}\int_S \text{curl}\ \mathbf{F} \cdot d\mathbf{S} &= \int_S * \left((\text{curl}\ \mathbf{F})^\flat\right) \\ &= \int_S d(\mathbf{F}^\flat).\end{aligned}$$

Thus we have rewritten the left hand side of Stokes' theorem. Putting the right hand and left hand pieces together together we can rewrite the vector calculus version of Stokes' theorem as

$$\int_S d(\mathbf{F}^\flat) = \int_{\partial S} F^\flat.$$

Clearly, $\mathbf{F}^\flat$ is a one-form $\alpha$ and $S$ is a two dimensional manifold $M$, which means we have written Stokes' theorem as

$$\int_M d\alpha = \int_{\partial M} \alpha.$$

### *9.5.4 Divergence Theorem*

Finally, we take a look at the divergence theorem from vector calculus,

$$\int_V \text{div}\ \mathbf{F}\ dV = \int_{\partial V} \mathbf{F} \cdot d\mathbf{S}.$$

We notice that the integrand on the left hand side is nothing more than $*(\text{div}\ \mathbf{F})$, which can be rewritten as $d\big((* \circ \flat)\mathbf{F}\big)$. We have also seen the right hand side $\int_{\partial V} \mathbf{F} \cdot dS$ written as $\int_{\partial V} *(\mathbf{F}^\flat)$. This allows us to write the divergence theorem as

$$\int_V d\left((* \circ \flat)\mathbf{F}\right) = \int_{\partial V} *(\mathbf{F}^\flat).$$

Writing the three dimensional manifold $V$ as $M$ and the three two-form $(* \circ \flat)\mathbf{F}$ as $\alpha$ we have the divergence theorem as

$$\int_M d\alpha = \int_{\partial M} \alpha.$$

Thus we have written all three of our vector calculus theorems in the same form in differential forms notation. This identity,

$$\boxed{\int_M d\alpha = \int_{\partial M} \alpha}$$

is exactly the generalized Stokes' theorem that we will prove in a couple of chapters.

## 9.6 Summary, References, and Problems

### *9.6.1 Summary*

Geometrically the divergence measures how much the vector field "diverges", or "spreads out", at the point $(x_0, y_0, z_0)$. Given a small three dimensional region $V$ about the point $(x_0, y_0, z_0)$ with boundary $\partial V$ and volume $\Delta V$ the divergence of $\mathbf{F}$ at $(x_0, y_0, z_0)$ is defined by

Definition of divergence
$$\operatorname{div} \mathbf{F} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \int_{\partial V} \mathbf{F} \cdot \hat{n} \, dS.$$

In essence, the curl measures the "circulation" per unit area of vector field $\mathbf{F}$ over an infinitesimal path around some point. Suppose $S$ is the surface bounded by the closed curve $C = \partial S$, $\Delta S$ is the area of that surface, $\hat{n}$ is the unit normal vector to that surface, the $s$ in $ds$ is the infinitesimal arc length element, and the surface area $\Delta S$ shrinks to zero about the point $(x_0, y_0, z_0)$. Then the curl $\mathbf{F}$ at a point $(x_0, y_0, z_0)$ is defined as

Definition of curl
$$\hat{n} \cdot \operatorname{curl} \mathbf{F} = \lim_{|\Delta S| \to 0} \frac{1}{\Delta S} \int_C \mathbf{F} \cdot \hat{t} \, ds.$$

We essentially go backwards and define the gradient of the function $f$ to be the vector field, which when dotted with the unit length vector $u$, gives the directional derivative of $f$ in the direction $u$,

Definition of gradient
$$\operatorname{grad} f \cdot u = u[f].$$

Using these geometrical definitions of divergence, curl, and gradient, it is possible to obtain the standard formula definitions from vector calculus. Letting $\mathbf{F} = P\hat{i} + Q\hat{j} + R\hat{k}$ we have

Formula for divergence $\mathbf{F}$ in Cartesian coordinates
$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z},$$

Formula for curl $\mathbf{F}$ in Cartesian Coordinates
$$\operatorname{curl} \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k},$$

Formula for grad $\mathbf{F}$ in Cartesian Coordinates
$$\operatorname{grad} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}.$$

In keeping with Einstein summation notation vector basis elements are written with lower indices and vector components are written with upper indices. Covector (differential form) basis elements are written with upper indices and covector (differential form) components are written with lower indices. This allows us to give the musical isomorphisms in terms of

Cartesian coordinates. The $\flat$ isomorphism is given by

$$\begin{aligned} \flat : T_p M &\longrightarrow T_p^* M \\ v^i \frac{\partial}{\partial x^i} &\longmapsto v_i dx^i \qquad \text{where } v_i = v^i, \end{aligned}$$

while the $\sharp$ isomorphism is given by

$$\begin{aligned} \sharp : T_p^* M &\longrightarrow T_p M \\ \alpha_i dx^i &\longmapsto \alpha^i \frac{\partial}{\partial x^i} \qquad \text{where } \alpha^i = \alpha_i. \end{aligned}$$

Keep in mind that these are not actually the mathematical definitions of the musical isomorphism but instead the formula for them in Cartesian coordinates. But these isomorphisms, along with the Hodge star operator, provide the link between vector calculus and differential forms. The following diagram commutes.

$$\begin{tikzcd} C(\mathbb{R}^3) \arrow[r, "\substack{\nabla \\ \text{grad}}"] \arrow[d, "\text{id}"] & T\mathbb{R}^3 \arrow[r, "\substack{\nabla\times \\ \text{curl}}"] \arrow[d, "\flat"] & T\mathbb{R}^3 \arrow[r, "\substack{\nabla\cdot \\ \text{div}}"] \arrow[d, "*\circ\flat"] & C(\mathbb{R}^3) \arrow[d, "*"] \\ \bigwedge^0(\mathbb{R}^3) \arrow[r, "d"] & \bigwedge^1(\mathbb{R}^3) \arrow[r, "d"] & \bigwedge^2(\mathbb{R}^3) \arrow[r, "d"] & \bigwedge^3(\mathbb{R}^3) \end{tikzcd}$$

This allows the following identities from vector calculus to be written in terms of the exterior derivative,

$$\begin{array}{ccc} \nabla \times (\nabla f) = 0 & \qquad\qquad & \nabla \cdot (\nabla \times \mathbf{F}) = 0 \\ \Updownarrow & & \Updownarrow \\ d(df) = 0 & & d(d\alpha) = 0. \end{array}$$

Notice that $d(df) = 0$ and $d(d\alpha) = 0$ of course just follow from the Poincaré lemma. It also allows the three major theorems from vector calculus to be written in terms of differential forms. All three of these theorems are in fact special cases of the differential forms version of Stokes' theorem,

$$\begin{array}{ccc} \underline{\text{Fund. Thm. Line Integrals}} & \underline{\text{Stokes' Theorem}} & \underline{\text{Divergence Theorem}} \\ f\big(c(b)\big) - f\big(c(a)\big) = \int_C \nabla f \cdot d\mathbf{s} & \int_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} & \int_{\partial V} \mathbf{F} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{F}\, dV \\ \Downarrow & \Downarrow & \Downarrow \\ \int_{\partial C} \alpha = \int_C d\alpha & \int_{\partial S} \alpha = \int_S d\alpha & \int_{\partial V} \alpha = \int_V d\alpha. \end{array}$$

### 9.6.2 *References and Further Reading*

The geometric approach to divergence, curl, and gradient presented in the first half of this chapter is largely inspired by and follows Schey [38] who probably gives the nicest and most intuitively understandable geometrical introductions to these vector calculus operators available. In particular, refer to this book for a more detailed derivation of the divergence, curl, gradient, and Laplacian in both cylindrical and spherical coordinates. The relationship between the divergence, curl, and gradient operators and exterior differentiation is presented almost everywhere, but see in particular Abraham, Marsden, and Ratiu [1], Hubbard and Hubbard [27], or Edwards [18]. The musical isomorphism, sharp and flat, are introduced, for example, in Abraham, Marsden, and Ratiu [1] and in Marsden and Ratiu [32] as well as, very briefly, in Burke [8] and Darling [12].

### 9.6.3 Problems

$f_1(x, y, z) = 5xy^2 - 4x^3y$ $\quad f_5(x, y, z) = x^2y^3z^4$ $\quad f_9(x, y, z) = x^2y^3z - 3xyz^2$

$f_2(x, y, z) = y\ln(z)$ $\quad f_6(x, y, z) = \sqrt{x + yz}$ $\quad f_{10}(x, y, z) = x^2 + y^2 - 5z^2$

$f_3(x, y, z) = 1 + 2x\sqrt{y}$ $\quad f_7(x, y, z) = xe^y + ye^z + ze^x$ $\quad f_{11}(x, y, z) = -x^2y - y^3z^2 + z^4x^5$

$f_4(x, y, z) = x^4 - x^2y^3$ $\quad f_8(x, y, z) = xe^{yz}$ $\quad f_{12}(x, y, z) = 3x - 4y + 5z$

*Question 9.12* For the functions listed above find grad $f_i$. Then find $(\text{grad} f_i)^\flat$.

*Question 9.13* For the functions listed above find $df_i$. Compare with $(\text{grad } f_i)^\flat$ from Question 9.12.

*Question 9.14* For the functions listed above find curl(grad $f_i$). Then find $d(df)$. Compare.

$\mathbf{F}_1 = 2x^2\hat{i} + (x + y)\hat{j}$ $\quad \mathbf{F}_5 = x^2\hat{i} + xy\hat{j} - (x^2 + y^2 + z^2)\hat{k}$ $\quad \mathbf{F}_9 = e^{xy}\hat{i} - e^{yz}\hat{j} + e^{xz}\hat{k}$

$\mathbf{F}_2 = -x\hat{i} + (x - 2y)\hat{j}$ $\quad \mathbf{F}_6 = yz^2\hat{i} + yz\hat{j} + (2x - 2y)\hat{k}$ $\quad \mathbf{F}_{10} = \sqrt{xyz}\hat{i} + \sqrt{xy}\hat{j} + \sqrt{x}\hat{k}$

$\mathbf{F}_3 = 2x^2\hat{i} - 3y^3\hat{j}$ $\quad \mathbf{F}_7 = y\hat{i} + z\hat{j} + x\hat{k}$ $\quad \mathbf{F}_{11} = xy^2z\hat{i} - 2x^2y^3z\hat{j} + 3xy^3z^2\hat{k}$

$\mathbf{F}_4 = 3\hat{j} + 4\hat{k}$ $\quad \mathbf{F}_8 = -x^2\hat{i} + y^3\hat{j} - z^4\hat{k}$ $\quad \mathbf{F}_{12} = y^2\hat{i} + (e^x + e^y)\hat{j} - (x^3 - 2y)\hat{k}$

*Question 9.15* For the vectors listed above, find curl $\mathbf{F}_i$. Then find $(* \circ \flat)(\text{curl } \mathbf{F}_i)$.

*Question 9.16* For the vectors listed above find $\mathbf{F}_i^\flat$. Then find $d(\mathbf{F}_i^\flat)$. Compare with $(* \circ \flat)(\text{curl } F_i)$ from Question 9.15.

*Question 9.17* For the vectors listed above find div(curl $\mathbf{F}_i$). Then find $d(d\mathbf{F}_i^\flat)$. Compare.

$\mathbf{G}_1 = -x^2y\hat{i} - xy^2\hat{j}$ $\quad \mathbf{G}_5 = \sqrt{xy}\hat{i} + xy\hat{j} + (3x + 2y + z)\hat{k}$ $\quad \mathbf{G}_9 = e^{xy}\hat{i} - e^{yz}\hat{j} + e^{xz}\hat{k}$

$\mathbf{G}_2 = (x + 2y)\hat{i} + 2xy\hat{j}$ $\quad \mathbf{G}_6 = yz^2\hat{i} + yz\hat{j} - (x^2 + y^2 + z^2)\hat{k}$ $\quad \mathbf{G}_{10} = x^3yz\hat{i} + x^2y\hat{j} + x\hat{k}$

$\mathbf{G}_3 = x^2y^2\hat{i} - 3x^2y^2\hat{j}$ $\quad \mathbf{G}_7 = \sqrt{yz}\hat{i} + \sqrt{xz}\hat{j} + \sqrt{xy}\hat{k}$ $\quad \mathbf{G}_{11} = 3xyz\hat{i} + \sqrt{xyz}\hat{j} + (4y - z)\hat{k}$

$\mathbf{G}_4 = 3x\hat{j} + 4y\hat{k}$ $\quad \mathbf{G}_8 = e^x\hat{i} + e^y\hat{j} - e^z\hat{k}$ $\quad \mathbf{G}_{12} = (x^2 + y^2 + z^2)\hat{i} + 3x\hat{j} - e^{xyz}\hat{k}$

*Question 9.18* For the vectors listed above find div$(\mathbf{G}_i)$. Then find $*(\text{div } \mathbf{G}_i)$.

*Question 9.19* For the vectors listed above find $(* \circ \flat)\mathbf{G}_i$. Then find $d\big((* \circ \flat)\mathbf{G}_i\big)$. Compare with $*(\text{div } \mathbf{G}_i)$ from Question 9.18.

*Question 9.20* Find a function $f$ such that $\nabla f = xy^2\hat{i} + x^2y\hat{j}$ and then use that function, along with the Fundamental Theorem of Line Integrals, to evaluate $\int_C \nabla f \cdot ds$ along the curve $C$ given by $\gamma(t) = \left(t + \sin\left(\frac{1}{2}\pi t\right), t + \cos\left(\frac{1}{2}\pi t\right)\right)$, $0 \le t \le 1$. Write the integral in terms of differential forms.

*Question 9.21* Find a function $f$ such that $\nabla f = yz\hat{i} + xz\hat{j} + (xy + 2z)\hat{k}$ and then use that function, along with the Fundamental Theorem of Line Integrals, to evaluate $\int_C \nabla f \cdot ds$ along the line segment from $(1, 0, -2)$ to $(4, 6, 3)$.

*Question 9.22* Find a function $f$ such that $\nabla f = (2xz+y^2)\hat{i}+2xy\hat{j}+(x^2+3z^2)\hat{k}$ and then use that function, along with the Fundamental Theorem of Line Integrals, to evaluate $\int_C \nabla f \cdot ds$ along the curve $C$ given by $\gamma(t) = (t^2, t + 1, 2t - 1)$, $0 \le t \le 1$. Write the integral in terms of differential forms.

*Question 9.23* Find a function $f$ such that $\nabla f = y^2\cos(z)\hat{i} + 2xy\cos(z)\hat{j} - xy^2\sin(z)\hat{k}$ and then use that function, along with the Fundamental Theorem of Line Integrals, to evaluate $\int_C \nabla f \cdot ds$ along the curve $C$ given by $\gamma(t) = (t^2, \sin(t), t)$, $0 \le t \le \pi$. Write the integral in terms of differential forms.

*Question 9.24* Use Stokes' theorem to evaluate $\int_S (\nabla \times \mathbf{F}) \cdot dS$ where $\mathbf{F} = 2y\cos(z)\hat{\imath} + e^x \sin(z)\hat{\jmath} + xe^y\hat{k}$ and $S$ is the portion of the sphere $x^2 + y^2 + z^2 = 16$, $z \geq 0$, oriented upwards. Then write the integral in terms of differential forms.

*Question 9.25* Use Stokes' theorem to evaluate $\int_S (\nabla \times \mathbf{F}) \cdot dS$ where $\mathbf{F} = e^{xy}\cos(z)\hat{\imath} + x^2 z\hat{\jmath} + xy\hat{k}$ and $S$ is the hemisphere $x = \sqrt{1 - y^2 - z^2}$, oriented in the direction of the positive $x$-axis. Then write the integral in terms of differential forms.

*Question 9.26* Use Stokes' theorem to evaluate $\int_C \mathbf{F} \cdot ds$ where $\mathbf{F} = yz\hat{\imath} + 2xz\hat{\jmath} + e^{xy}\hat{k}$ and $C$ is the circle $x^2 + y^2 = 9$ in the plane $z = 4$. $C$ is oriented counterclockwise when viewed from above. Then write the integral in terms of differential forms.

*Question 9.27* Use the Divergence theorem to calculate the surface integral $\int_S \mathbf{F} \cdot dS$ where $\mathbf{F} = x^2 z^3\hat{\imath} + 2xyz^3\hat{\jmath} + xz^4\hat{k}$ and $S$ is the surface of the box with vertices $(\pm 3, \pm 2, \pm 2)$. Then write the integral in terms of differential forms.

*Question 9.28* Use the Divergence theorem to calculate the surface integral $\int_S \mathbf{F} \cdot dS$ where $\mathbf{F} = e^x \sin(y)\hat{\imath} + e^x \cos(y)\hat{\jmath} + yz^2\hat{k}$ and $S$ is the surface of the box with bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$. Then write the integral in terms of differential forms.

*Question 9.29* Use the Divergence theorem to calculate the surface integral $\int_S \mathbf{F} \cdot dS$ where $\mathbf{F} = 3xy^2\hat{\imath} + xe^z\hat{\jmath} + z^3\hat{k}$ and $S$ is the surface of the solid bounded by the planes $x = -1$, $x = 1$, and the cylinder $y^2 + z^2 = 4$. Then write the integral in terms of differential forms.

# Chapter 10
# Manifolds and Forms on Manifolds

In this chapter we reintroduce manifolds in a somewhat more mathematically rigorous manor while simultaneously trying not to overwhelm you with details. As always we will still place an emphasis on conceptual understanding and the big picture. Manifold theory is a vast and rich subject and there are numerous books that present manifolds in a completely rigorous manor with all the gory technical details made explicit. When you understand this chapter you will be prepared to tackle these texts.

Section one reintroduces manifolds while section two reintroduces the idea of tangent spaces. Section three gives an abstract presentation of push-forwards of vectors and pull-backs of forms. We will then discuss differential forms and both differentiation and integration on manifolds in section four. Even though general manifolds are a more abstract setting than $\mathbb{R}^n$, you should not be surprised that most of the conceptual ideas you have already become familiar with carry through essentially unchanged. In that sense we are simply repeating what we have already done, only in a more abstract setting.

## 10.1 Definition of a Manifold

In the beginning of this book we made a big point to make a distinction between the manifold $\mathbb{R}^n$ and the vector space $\mathbb{R}^n$. The tangent space at each point $p$ of the manifold $\mathbb{R}^n$, that is the space $T_p(\mathbb{R}^n)$, was equivalent to the vector space $\mathbb{R}^n$. We told you that we made this distinction because in general most manifolds are not vector spaces at all. Most manifolds lack a lot of the important properties that Euclidian spaces, the manifolds we have dealt with throughout calculus, have. We begin this section by trying to explain this in more detail.

As you already know, manifolds are spaces that you can generally think of as being locally Euclidian; meaning that if you zoom in on a single point of the manifold, a very small neighborhood of that point essentially looks like the manifold $\mathbb{R}^n$ for some $n$, see Figs. 2.8 through 2.10. Because of this fact one can actually do calculus on manifolds just as one does calculus on the manifold $\mathbb{R}^n$.

However, there are some distinct differences between manifolds and the Euclidian spaces $\mathbb{R}^n$. First, in the Euclidian space $\mathbb{R}^n$ there is a natural way in which tangent vectors can be **parallel transported** to another point. For example when we transport the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}_p \in T_p\mathbb{R}^2$ to $\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \in T_q\mathbb{R}^2$, where $p \neq q$, these two vectors are, in an intuitively obvious way, parallel to each other. See Fig. 10.1. In fact, this is so obvious that we rarely think about what underlying and unstated assumptions we need to make in order to know that two vectors at different points are parallel to each other.

As we have discussed before, $S^2$ is a manifold since a small neighborhood of every point of $S^2$ looks like $\mathbb{R}^2$. However, consider $S^2$ as shown in Fig. 10.2 with the point $p$ being the "north pole" and the point $q$ being on the "equator." Now consider $v_p \in T_pS^2$. What vector in $T_qS^2$ could we consider as parallel in some sense to $v_p$? That is, what is the parallel transport of $v_p$ to the point $q$? It certainly is not obvious from Fig. 10.2. It turns out that some additional structure on the manifold is needed in order to find the parallel transports of vectors on manifolds. This additional structure is called a **connection**. It is essentially a way to "connect" nearby tangent spaces (whatever "nearby" means!) Connections are beyond the scope of this book, but they play an essential role in manifold theory. On a general manifold we have to specify a connection explicitly, while on a Euclidian space manifold a connection is already implicitly built into our intuitive understanding of Euclidian



J. P. Fortney, *A Visual Introduction to Differential Forms and Calculus on Manifolds*,
https://doi.org/10.1007/978-3-319-96992-3_10

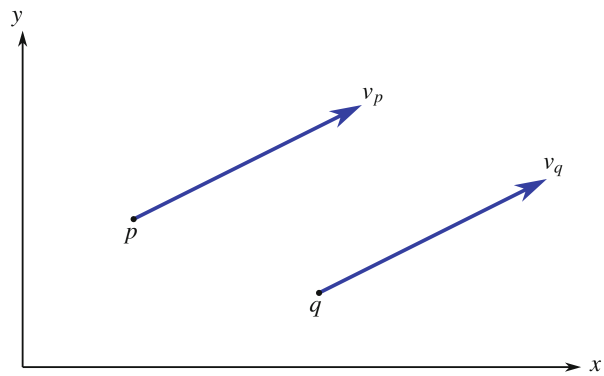


**Fig. 10.1** The vector $2e_1 + e_2$ drawn at two different points $p$ and $q$ in the manifold $\mathbb{R}^2$. These two vectors are parallel to each other in an intuitively obvious way

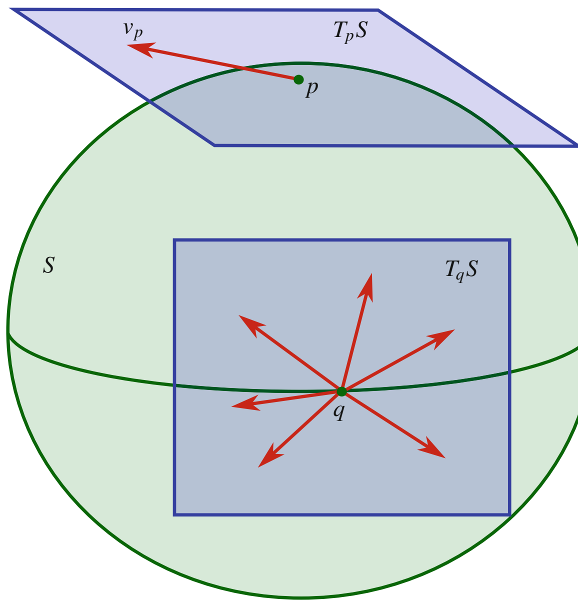


**Fig. 10.2** The vector $v_p$ where $p$ is the "north pole." Which of the vectors at $q$ on the "equator" are parallel to $v_p$?

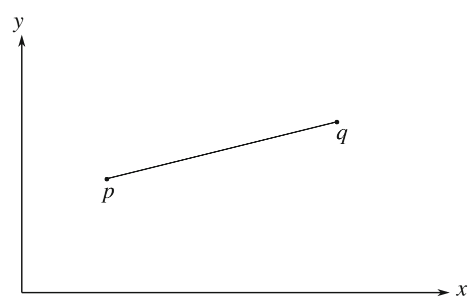


**Fig. 10.3** A straight line between two points $p$ and $q$ in the manifold $\mathbb{R}^2$. What straight means in Euclidian space is naturally intuitive

space. This major difference between general manifolds and Euclidian space manifolds $\mathbb{R}^n$ is one of the reasons we have taken such care with the distinction between the two.

Now consider the points $p$ and $q$ in $\mathbb{R}^2$ as shown in Fig. 10.3. There is a very natural and intuitive idea of what a straight line between the two points would look like. And, given the Cartesian coordinate system we know how to measure the distance between the two points. We also know that the torus $T^2$ is a manifold since a very small neighborhood of every point of $T^2$ looks like $\mathbb{R}^2$. However, what would a "straight line" between the points $p$ and $q$ on a torus look like? What

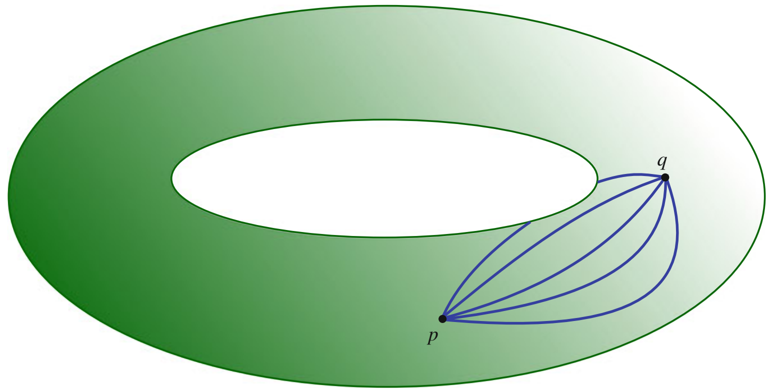


**Fig. 10.4** The torus $T^2$ shown with two points $p$ and $q$ on it. Several lines connecting these points are shown. What would a "straight line" connecting two points on the manifold $T^2$ look like? What does "straight" even mean in this situation?

does "straight" even mean in this situation? It is not obvious from Fig. 10.4. And for that matter, how would one measure the distance between the two points on a torus? To answer this question is that in addition to a connection you would need a structure on the manifold called a **metric**. We will actually learn a little more about metrics later, though we will not go into much depth. In summary, on a general manifold we have to specify a metric explicitly, while on a Euclidian space manifold a metric, called the Euclidian metric, is already implicitly built into our intuitive understanding of Euclidian space. This is another major difference between general manifolds and our familiar Euclidian space manifolds $\mathbb{R}^n$ and is yet another reason we have to be careful with the distinction between the two.

All of this highlights the real difference between general manifolds and the Euclidian space manifolds $\mathbb{R}^n$ that you are used to. However, in everything we have done we have gone to some effort to make a distinction between the "manifold" $\mathbb{R}^n$ and the tangent spaces of the manifold, which are basically the vector spaces $\mathbb{R}^n$. Since you are already used to this moving into the general case with manifolds should hopefully not be too difficult.

Now we are ready to give a rigorous definition for what a differentiable manifold is. Here we will stick with what are called differentiable manifolds. In reality there are a lot of different types of manifolds that have different properties. Since this is not a class on manifold theory we will stick with the simplest kind. General manifolds retain some of the structure of the Euclidian space manifolds, but will not retain either the connection or the Euclidian metric from the Euclidian space manifolds.

An $n$-dimensional **manifold** is a space $M$ that can be completely covered by a collection of **local coordinate neighborhoods** $U_i$ with one-to-one mappings $\phi_i : U_i \to \mathbb{R}^n$, which are called a **coordinate maps**. Together $U_i$ and $\phi_i$ are called **coordinate patch** or a **chart**, which is generally denoted as $(U_i, \phi_i)$. The set of all the charts together, $\big\{(U_i, \phi_i)\big\}$, is called a **coordinate system** or an **atlas** of $M$. Since the $U_i$ cover all of $M$ we write that $M = \bigcup U_i$. Also, since $\phi_i$ is one-to-one it is invertible, so $\phi_i^{-1}$ exists and is well defined. See Fig. 10.5 for a two-dimensional example. The terms atlas and chart are particularly illustrative; what is an atlas of the world but a collection of individual charts? Here the word chart is a nautical word for map. Finally, if two charts have a non-empty intersection, $U_i \cap U_j \neq \varnothing$, then the functions $\phi_j \circ \phi_i^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ are called **transition functions**. There is a lot of important terminology here so we summarize it:

$$
\begin{aligned}
U_i &\quad : \quad \text{coordinate neighborhood,} \\
\phi_i : U_i \to \mathbb{R}^n &\quad : \quad \text{coordinate map,} \\
(U_i, \phi_i) &\quad : \quad \text{coordinate patch/chart,} \\
\big\{(U_i, \phi_i)\big\} &\quad : \quad \text{coordinate system/atlas,} \\
\phi_j \circ \phi_i^{-1} : \mathbb{R}^n \to \mathbb{R}^n &\quad : \quad \text{transition function.}
\end{aligned}
$$

Now we consider what happens to a point $r \in U_i \cap U_i \neq \emptyset$. We have $\phi_i(r) \in \mathbb{R}^n$ and $\phi_j(r) \in \mathbb{R}^n$. Furthermore, we have $\phi_j \circ \phi_i^{-1}$ sending $\phi_i(U_i \cap U_j) \subset \mathbb{R}^n$ to $\phi_j(U_i \cap U_j) \subset \mathbb{R}^n$, that is, $\phi_j \circ \phi_i^{-1}$ is a map of a subset of $\mathbb{R}^n$ to another subset of $\mathbb{R}^n$, and so the mapping $\phi_j \circ \phi_i^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ with domain $\phi_i(U_i \cap U_j)$ and range $\phi_j(U_i \cap U_j)$ is the sort of mapping that we know how to differentiate from multivariable calculus. A **differentiable manifold** is a set $M$, together with a collection of charts $(U_i, \phi_i)$, where $M = \bigcup U_i$, such that every mapping $\phi_j \circ \phi_i^{-1}$, where $U_i \cap U_j \neq \emptyset$, is differentiable.

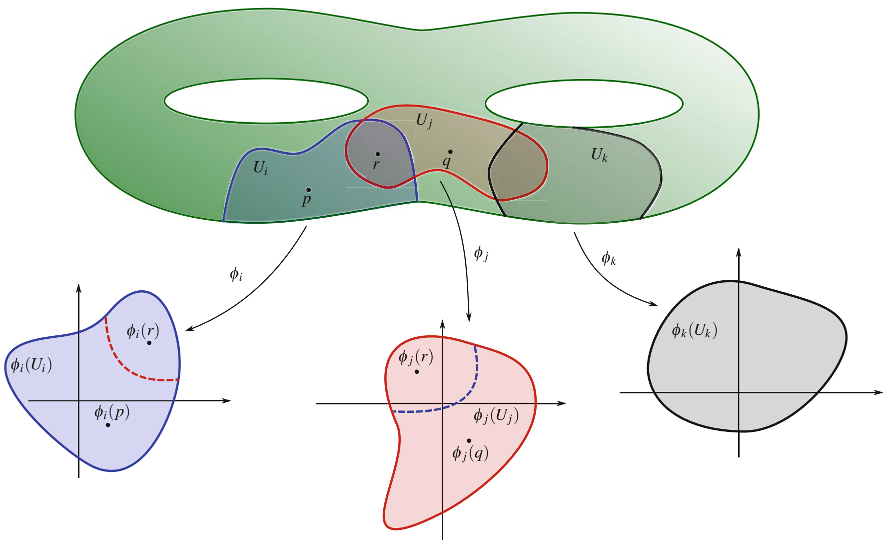


**Fig. 10.5** Here a two-dimensional manifold, the double torus, is shown along with three chats $(U_i, \phi_i)$, $(U_j, \phi_j)$, and $(U_k, \phi_k)$. Notice the point $r \in U_i \cap U_i$. Clearly $\phi_j \circ \phi_i^{-1}$ sends $\phi_i(U_i \cap U_j) \subset \mathbb{R}^2$ to $\phi_j(U_i \cap U_j) \subset \mathbb{R}^2$

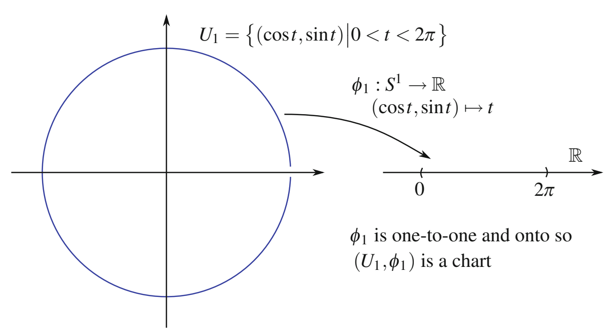


**Fig. 10.6** The chart $(U_1, \phi_1)$ on $S^1$. $U_1$ is the subset of $S^1$ that consists of the points $(\cos t, \sin t)$ for $0 < t < 2\pi$ and $\phi_1 : U_1 \to \mathbb{R}$ is defined by $(\cos t, \sin t) \mapsto t$

Let's see how this works with an simple explicit example. Consider the unit circle $S^1$ in $\mathbb{R}^2$. We will show that $S^1$ is indeed a differentiable manifold. Let $U_1$ be the subset of $S^1$ that consists of the points $(\cos t, \sin t)$ for $0 < t < 2\pi$. Then $\phi_1 : U_1 \to \mathbb{R}$, defined by $(\cos t, \sin t) \mapsto t$, is a one-to-one mapping onto the open interval $(0, 2\pi) \subset \mathbb{R}$, so $(U_1, \phi_1)$ is a chart on $S^1$. See Fig. 10.6.

Similarly, let $U_2$ be the subset of $S^1$ that consists of the points $(\cos\tau, \sin\tau)$ for $-\pi < \tau < \pi$. Then $\phi_2 : U_2 \to \mathbb{R}$, defined by $(\cos\tau, \sin\tau) \mapsto \tau$, is a one-to-one mapping onto a subset of $\mathbb{R}$, so $(U_2, \phi_2)$ is another chart on $S^1$. See Fig. 10.7. The domains $U_1$ and $U_2$ cover all of $S^1$, that is, $S^1 = U_1 \cup U_2$. We say that $(U_1, \phi_1)$ and $(U_2, \phi_2)$ are an atlas for $S^1$.

*Question 10.1* What point of $S^1$ is not included in $U_1$? What point of $S^1$ is not included in $U_2$? Why were these points not included? Could they have been included? Even if they had been included, what problem still arises?

Notice $U_1 \cap U_2$ has two components, the top open half of $S^1$ and the bottom open half of $S^1$. See Fig. 10.8. On the top open half we have $\phi_1^{-1} \circ \phi_2(t) = t$ for $0 < t < \pi$, which is differentiable, and on the bottom half we have $\phi_1^{-1} \circ \phi_2(t) = t - 2\pi$ for $\pi < t < 2\pi$, which is also differentiable. Hence $S^1$ is a differentiable manifold.

At this point in most books on differential geometry one usually encounters the definitions for both orientated manifolds and manifolds with boundary. Based on Sect. 1.2 you already know that volumes come with a sign. The idea of an oriented manifold is related to this, it is essentially when a manifold comes with a way to determine when a volume is positive and when it is negative. We will discuss oriented manifolds in Sect. 10.4 when we discuss integration on manifolds. Manifolds with a boundary are essentially what you would think them to be, manifolds that have some sort of a boundary or "edge" to them. What a boundary or "edge" actually is can be defined very precisely in terms of coordinate charts. In this book we will

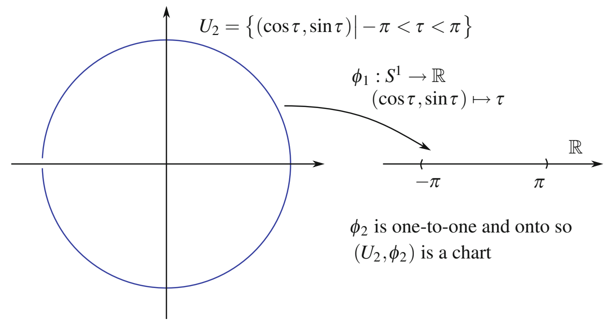


**Fig. 10.7** The chart $(U_2, \phi_2)$ on $S^1$. $U_2$ is the subset of $S^1$ that consists of the points $(\cos\tau, \sin\tau)$ for $-\pi < \tau < \pi$ and $\phi_2 : U_2 \to \mathbb{R}$ is defined by $(\cos\tau, \sin\tau) \mapsto \tau$

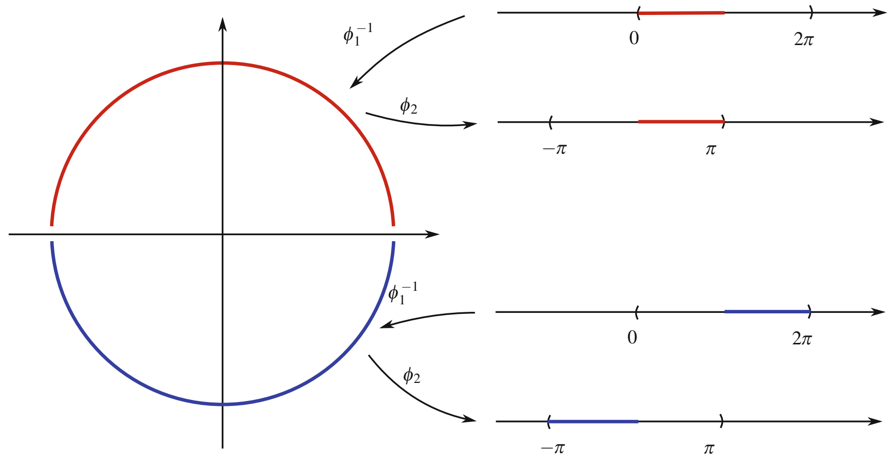


**Fig. 10.8** $U_1 \cap U_2$ has two components, the top open half of $S^1$ (red) and the bottom open half of $S^1$ (blue). On the top open half we have $\phi_1^{-1} \circ \phi_2(t) = t$ for $0 < t < \pi$ and on the bottom half we have $\phi_1^{-1} \circ \phi_2(t) = t - 2\pi$ for $\pi < t < 2\pi$. Both of these are differentiable making $S^1$ a differentiable manifold

not discuss manifolds with boundary explicitly, though we will explore the concepts of boundary more closely in the specific context of Stoke's theorem in Chap. 11.

## 10.2 Tangent Space of a Manifold

Recall that when we introduced the tangent space before we did so by considering the manifold $\mathbb{R}^n$. The examples we considered had $n = 2$ or 3, but the idea generalized to any $n$. See Figs. 2.14 through 2.17. We chose a point $p \in \mathbb{R}^n$ and then considered all the vectors $v_p$ that originated at that point. We called the set of all vectors $v_p$ originating at the point $p$ the tangent space of $\mathbb{R}^n$ at $p$ and denoted it as $T_p\mathbb{R}^n$. We then saw that $T_p\mathbb{R}^n$ was essentially the same space as the vector space $\mathbb{R}^n$. In this section we want to give a somewhat more abstract presentation of tangent spaces.

For example, given a general manifold $M$ you may wonder what we mean by vectors emanating from a point $p \in M$. After all, $\mathbb{R}^n$ is clearly also a vector space in addition to being a differentiable manifold, which allow us to talk about vectors, whereas a general differential manifold need not be a vector space at all meaning that there are not any vectors in the manifold. Consider the sphere $S^2 \subset \mathbb{R}^3$ as shown in Fig. 10.9. We can draw all sorts of vectors at the point $p \in S^2$ which are tangent to $S^2$, but these vectors are actually in the space $\mathbb{R}^3$, which is the space that the manifold $S^2$ is embedded in, and not in the manifold $S^2$ itself.

When we originally introduced tangent spaces we illustrated the general idea of the tangent space to a manifold by drawing a few manifolds and saying the tangent space at a point $p$ was in essence the tangent plane (or hyperplane) to the manifold at that point. For example, the manifold $S^2$ along with a few of its tangent planes is shown in Fig. 10.10.