The whole purpose of this book is to develop a deeper intuitive geometric understanding of what differential forms are, and since exterior differentiation is such a central concept we will attempt to look at it from all of these perspectives, trying to see how they relate to each other, as well as to actually spend some time looking at what is going on geometrically.

4.2 The Local Formula

We are taking things slowly and have simplified our lives considerably by only dealing with manifolds $\mathbb{R}^n$, and even then primarily with the cases of $n = 2$ or 3. In fact, we have even gone a step further in our simplification and only considered $\mathbb{R}^n$ with the standard Cartesian coordinates. These simplifications have allowed us to sidestep a large number of technical issues and to genuinely concentrate on developing an intuitive understanding of what differential forms are. We shall continue in this vein a bit longer, but will caution that by doing so we are also missing some of the richness and subtlety of the subject.

First we will recall what we did in chapter two. Given a function $f : \mathbb{R}^n \to \mathbb{R}$ on the manifold $\mathbb{R}^n$ we want to write the directional derivative of f in the direction of

$$v_p = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}_p$$

at the point $p = (x_{1_0}, x_{2_0}, \ldots, x_{n_0})$. Writing $p + tv_p$ as $p + tv_p = (x_{1_0} + tv_1, x_{2_0} + tv_2, \ldots, x_{n_0} + tv_n)$, and then writing the intermediate functions $x_1(t) = x_{1_0} + tv_1$ through $x_n(t) = x_{n_0} + tv_n$ we have

$$\begin{aligned}
df(v_p) &\equiv v_p[f] \\
&= \lim_{t\to 0} \frac{f\big(x_{1_0} + tv_1, x_{2_0} + tv_2, \cdots, x_{n_0} + tv_n\big) - f\big(x_{1_0}, x_{2_0}, \ldots, x_{n_0}\big)}{t} \\
&= \frac{d}{dt}\Big(f\big(p + tv_p\big)\Big)\Big|_{t=0} \\
&= \frac{d}{dt} f\big(\underbrace{x_{1_0} + tv_1}_{x_1(t)}, \underbrace{x_{2_0} + tv_2}_{x_2(t)}, \ldots, \underbrace{x_{n_0} + tv_n}_{x_n(t)}\big)\Big|_{t=0} \\
&= \frac{\partial f(x_1, \ldots, x_n)}{\partial x_1}\bigg|_p \cdot \frac{dx_1(t)}{dt}\bigg|_{t=0} + \frac{\partial f(x_1, \ldots, x_n)}{\partial x_2}\bigg|_p \cdot \frac{dx_2(t)}{dt}\bigg|_{t=0} + \cdots + \frac{\partial f(x_1, \ldots, x_n)}{\partial x_n}\bigg|_p \cdot \frac{dx_n(t)}{dt}\bigg|_{t=0} \\
&= \frac{\partial f}{\partial x_1}\bigg|_p \cdot v_1 + \frac{\partial f}{\partial x_2}\bigg|_p \cdot v_2 + \cdots + \frac{\partial f}{\partial x_n}\bigg|_p \cdot v_n \\
&= \sum_{i=1}^n v_i \cdot \frac{\partial f}{\partial x_i}\bigg|_p \\
&= \sum_{i=1}^n \frac{\partial f}{\partial x_i}\bigg|_p \, dx_i(v_p) \\
&= \left(\sum_{i=1}^n \frac{\partial f}{\partial x_i}\bigg|_p \, dx_i \right)(v_p)
\end{aligned}$$

and so, leaving off the base point, we have

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\, dx_i.$$

Note, often you see $df(v_p)$ written as $df \cdot v_p$.

As an example, for $\mathbb{R}^2$, if $v_p = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_p$ we found, recalling the use of the chain rule, that

$$\begin{aligned} df(v_p) &\equiv v_p[f] \\ &= \frac{\partial f}{\partial x} v_1 + \frac{\partial f}{\partial y} v_2 \\ &= \frac{\partial f}{\partial x} dx(v_p) + \frac{\partial f}{\partial y} dy(v_p) \\ &= \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right) (v_p) \end{aligned}$$

so that we end up with

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

Similarly, for $\mathbb{R}^3$ we have that

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz.$$

Now we will introduce a little more terminology. If α is a one-form on the manifold M then $\alpha : T_pM \to \mathbb{R}$. That is, α "eats" a vector at each point p of the manifold M and "spits out" a real number for each point p of the manifold M. The space of one-forms is written as T_p^*M or $\bigwedge^1(M)$. If α is a two-form on the manifold M then $\alpha : T_pM \times T_pM \to \mathbb{R}$; it eats two vectors at each point p of the manifold M and spits out a real number for each point p of the manifold M. The space of two-forms is denoted by $\bigwedge^2(M)$. Similarly, if α is an n-form

$$\alpha : \underbrace{T_pM \times T_pM \times \cdots \times T_pM}_{n \text{ times}} \longrightarrow \mathbb{R}$$

eats n vectors at each point p of the manifold M. The space of n-forms is denoted by $\bigwedge^n(M)$.

In other words, we can think of a one-form as eating a point and a vector, a two-form as eating a point and two vectors, and an n-form at eating a point and n vectors. So, what would a zero-form eat? What would it spit out? It stands to reason that a zero-form would eat a point and zero vectors and spit out a real number for each point. And what sorts of things do that? Functions of course. Functions $f : M \to \mathbb{R}$ are things that we are very familiar with, and functions are nothing more than zero-forms. Functions are generally classified by how many times they can be differentiated on a domain, in this case the manifold M. Without getting into the details, we will generally assume our functions can be differentiated as many times as we like, an infinite number of times in fact. Thus we can denote our space of functions as $\mathscr{C}^\infty(M)$ or, if we are thinking of our functions as zero-forms, as $\bigwedge^0(M)$.

In the expression

$$df = \sum \frac{\partial f}{\partial x_i} dx_i$$

d is an **operator** that takes a function, a zero-form, and produces as output a one-form denoted df. The one-form df evaluated at a point p in some sense encodes the information about the tangent plane to f at the point p. In other words, df is the closest linear approximation to f at p. As a linear functional we should recognize that df is also a one-form. But notice, when the operator d is applied to a function and then paired with a vector that is exactly the old-fashioned directional derivative.

We want to extend this idea, we want to see how the operator d can take a one-form and produce as output a two-form, or take in an n-form and produce as output an $(n+1)$-form. In other words, we want an operator

$$d : \bigwedge^n(M) \longrightarrow \bigwedge^{n+1}(M).$$

This operator d will be called exterior differentiation. Since d when applied to a zero-form f is essentially a rewriting of the directional derivative then we can think of exterior differentiation as being a generalization of the directional derivatives from vector calculus. We will begin by simply giving the local (or "in-coordinates") formula for the exterior derivative.

Definition 4.2.1 Suppose f is a zero-form. Then the **exterior derivative of** f is defined by

$$df = \sum \frac{\partial f}{\partial x_i} dx_i.$$

Suppose $\alpha = \sum f_i dx_i$ is a one-form. Then the **exterior derivative of** α is defined by

$$d\alpha = \sum df_i \wedge dx_i.$$

Suppose $\omega = \sum f_{i_1 \cdots i_n} dx_{i_1} \wedge \cdots \wedge dx_{i_n}$ is an n-form. Then the **exterior derivative of** ω is defined by

$$d\omega = \sum df_{i_1 \cdots i_n} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_n}.$$

These three formulas are all very intuitive and nicely cover all the possible cases, but it is of course possible to combine them all into a single formula,

Exterior derivative of an n-form

$$\boxed{d\Big(\sum \alpha_{i_1 \cdots i_n} \, dx_{i_1} \wedge \cdots \wedge dx_{i_n}\Big) = \sum \sum_{j=1}^{n} \frac{\partial \, \alpha_{i_1 \cdots i_n}}{\partial x_{i_j}} \, dx_{i_j} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_n}.}$$

Each f_i in the definition of α and ω are functions, or zero-forms, and hence the df_i in the definition of the exterior derivatives $d\alpha$ and $d\omega$ are given by the definition of the exterior derivatives of a zero-form. Let us see how this works in a simple example. We will consider a one-form on the manifold $\mathbb{R}^2$. Suppose $\alpha = f_1 dx + f_2 dy$ is a one-form on the manifold $\mathbb{R}^2$ for some functions $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$. Then

$$\begin{aligned}
d\alpha &= df_1 \wedge dx + df_2 \wedge dy \\
&= \left(\frac{\partial f_1}{\partial x} dx + \frac{\partial f_1}{\partial y} dy\right) \wedge dx + \left(\frac{\partial f_2}{\partial x} dx + \frac{\partial f_2}{\partial y} dy\right) \wedge dy \\
&= \frac{df_1}{dx} \underbrace{dx \wedge dx}_{=0} + \frac{df_1}{dy} \underbrace{dy \wedge dx}_{=-dx \wedge dy} + \frac{df_2}{dx} dx \wedge dy + \frac{df_2}{dy} \underbrace{dy \wedge dy}_{=0} \\
&= \left(\frac{df_2}{dx} - \frac{df_1}{dy}\right) dx \wedge dy.
\end{aligned}$$

Since α was a one-form then $d\alpha$ is a two-form. We go one step further to see how this two-form acts on the vectors

$$v = v_1 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \text{and} \quad w = w_1 \frac{\partial}{\partial x} + w_2 \frac{\partial}{\partial y} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

We have

$$\begin{aligned}
d\alpha(v, w) &= \left(\frac{df_2}{dx} - \frac{df_1}{dy}\right) dx \wedge dy(v, w) \\
&= \left(\frac{df_2}{dx} - \frac{df_1}{dy}\right) \begin{vmatrix} dx(v) & dx(w) \\ dy(v) & dy(w) \end{vmatrix} \\
&= \left(\frac{df_2}{dx} - \frac{df_1}{dy}\right) \begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix}
\end{aligned}$$

$$= \left(\frac{df_2}{dx} - \frac{df_1}{dy} \right) (v_1 w_2 - w_1 v_2)$$
$$= v_1 w_2 \frac{df_2}{dx} - v_1 w_2 \frac{df_1}{dy} - w_1 v_2 \frac{df_2}{dx} + w_1 v_2 \frac{df_1}{dy}.$$

We will return to this example later when we try to motivate the global formula for exterior differentiation.

Let us consider one more example. Let $\alpha = xy^3z^2\ dx + 5x^2y\ dz$ be a one-form on the manifold $\mathbb{R}^3$. We will find the exterior derivative of α,

$$\begin{aligned}
d\alpha &= d\left(xy^3z^2\ dx + 5x^2y\ dz\right) \\
&= d\big(xy^3z^2\big) \wedge dx + \big(5x^2y\big) \wedge dz \\
&= \left(\frac{\partial\ xy^3z^2}{\partial x} dx + \frac{\partial\ xy^3z^2}{\partial y} dy + \frac{\partial\ xy^3z^2}{\partial z} dz \right) \wedge dx \\
&\quad + \left(\frac{\partial\ 5x^2y}{\partial x} dx + \frac{\partial\ 5x^2y}{\partial y} dy + \frac{\partial\ 5x^2y}{\partial z} dz \right) \wedge dz \\
&= \left(y^3z^2\ dx + 3xy^2z^2\ dy + 2xy^3z\ dz \right) \wedge dx \\
&\quad + \left(10xy\ dx + 5x^2\ dy + 0\ dz \right) \wedge dz \\
&= y^3z^2\ \underbrace{dx \wedge dx}_{=0} + 3xy^2z^2\ \underbrace{dy \wedge dx}_{-dx\wedge dy} + 2xy^3z\ dz \wedge dx \\
&\quad + 10xy\ \underbrace{dx \wedge dz}_{-dz\wedge dx} + 5x^2\ dy \wedge dz + 0\ \underbrace{dz \wedge dz}_{=0} \\
&= -3xy^2z^2\ dx \wedge dy + 5x^2\ dy \wedge dz + \big(2xy^3z - 10xy\big)\ dz \wedge dx.
\end{aligned}$$

We see that $d\alpha$ is a two-form.

Question 4.1 Find $dd\alpha$ for this example.

Question 4.2 Let $\alpha = 2x^2yz^3\ dx + 3yz^2\ dy + 7x^3y^2z^2\ dz$ be a one-form on the manifold $\mathbb{R}^3$. Find $d\alpha$ and $dd\alpha$.

When the exterior derivative is defined by just giving the formulas, as in this section, one generally proceeds to show that the exterior derivative operator d has a number of algebraic properties. One then needs to prove that the formulas remain unchanged under a change of coordinates. This is not at all difficult to do, but since we have not discussed changes of coordinates yet we will not do that here. Now suppose α, β are n-forms and ω is an m-form. Exterior differentiation satisfies the following three algebraic properties:

1. $d(\alpha + \beta) = d\alpha + d\beta$,
2. $d(\alpha \wedge \omega) = d\alpha \wedge \omega + (-1)^n \alpha \wedge d\omega$,
3. for each n-form α, $d(d\alpha) = 0$.

Showing these properties is not difficult, but the general case becomes somewhat notationally tedious.

Question 4.3 Using the notation from the last chapter, let $\alpha = \sum \alpha_I dx^I$, $\beta = \sum \beta_J dx^J$ both be n-forms and $\omega = \sum \omega_K dx^K$ be an m-form. Show the three algebraic properties above.

4.3 The Axioms of Exterior Differentiation

The second approach to introducing exterior differentiation one often encounters is to list the algebraic properties one wants exterior differentiation to have and then to show that such an operation exists and is unique.

Theorem 4.1 *There exists a unique operator*

$$d : \bigwedge^n(M) \longrightarrow \bigwedge^{n+1}(M).$$

called the ***exterior derivative*** *that satisfies the following four properties. Suppose α, β are n-forms and ω is an m-form on M, the operator d satisfies*

1. $d(\alpha + \beta) = d\alpha + d\beta$,
2. $d(\alpha \wedge \omega) = d\alpha \wedge \omega + (-1)^n \alpha \wedge d\omega$,
3. *for each n-form α, $d(d\alpha) = 0$,*
4. *in local coordinates, for each function f, $df = \sum \frac{\partial f}{\partial x_i} dx_i$.*

The first three of these properties are of course the properties that were listed at the end of the last section. Books that take this approach in introducing exterior differentiation essentially give these four properties and then use them to show that an operator d that satisfies them both exists and is unique. In the process of doing this they find the local formula for d. In essence they are simply going in the opposite direction as the last section. The last property is included because we want the operator d, when applied to a zero-form, to be exactly the same as the directional derivative of the zero-form function. By doing this we are essentially forcing our idea of exterior differentiation to be a generalization of the directional derivative. By making this specification, along with listing the properties we want d to have, it is fairly straight forward to derive the actual formula.

First of all we show existence of the operator d. To show existence we use the properties to find a formula for d. Once we know the formula for d we know it exists. And since we find only one formula for it then it must be unique as well. Let us now make explicit a notational convention that we have implicitly been using. If f is a zero-form (function) and α is an n-form then we generally write $f \wedge \alpha$ simply as $f\alpha$. This convention used in the third equality below. Suppose that we have the n-from

$$\alpha = \sum \alpha_{i_1 \cdots i_n} \, dx_{i_1} \wedge \cdots \wedge dx_{i_n}.$$

We apply d to α to get

$$\begin{aligned}
d\alpha &= d\Big(\sum \alpha_{i_1 \cdots i_n} \, dx_{i_1} \wedge \cdots \wedge dx_{i_n}\Big) \\
&= \sum d\Big(\alpha_{i_1 \cdots i_n} \, dx_{i_1} \wedge \cdots \wedge dx_{i_n}\Big) \\
&= \sum \Big(d\alpha_{i_1 \cdots i_n} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_n} + (-1)^0 \alpha_{i_1 \cdots i_n} \, d\big(dx_{i_1} \wedge \cdots \wedge dx_{i_n}\big)\Big) \\
&= \sum \Big(d\alpha_{i_1 \cdots i_n} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_n} + (-1)^0 \alpha_{i_1 \cdots i_n} \, \big(ddx_{i_1} \wedge \cdots \wedge dx_{i_n} + (-1)^1 dx_{i_1} \wedge d\big(dx_{i_2} \wedge \cdots \wedge dx_{i_n}\big)\big)\Big) \\
&\ \ \vdots \\
&= \sum \Big(d\alpha_{i_1 \cdots i_n} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_n} \\
&\qquad + (-1)^0 \alpha_{i_1 \cdots i_n} \, \big(\underbrace{ddx_{i_1}}_{=0} \wedge \cdots \wedge dx_{i_n} + (-1)^1 dx_{i_1} \wedge \underbrace{ddx_{i_2}}_{=0} \wedge \cdots \wedge dx_{i_n} + \cdots + (-1)^{n-1} dx_{i_1} \wedge \cdots \wedge dx_{i_{n-1}} \wedge \underbrace{ddx_{i_n}}_{=0}\big)\Big) \\
&= \sum d\alpha_{i_1 \cdots i_n} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_n} \\
&= \sum \left(\sum_{j=1}^{n} \frac{\partial \, \alpha_{i_1 \cdots i_n}}{\partial x_{i_j}} \, dx_{i_j}\right) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_n} \\
&= \sum \sum_{j=1}^{n} \frac{\partial \, \alpha_{i_1 \cdots i_n}}{\partial x_{i_j}} \, dx_{i_j} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_n}.
\end{aligned}$$

Thus our formula for the exterior differentiation operator d is thus given by

$$d\Big(\sum \alpha_{i_1\cdots i_n}\, dx_{i_1} \wedge \cdots \wedge dx_{i_n}\Big) = \sum \sum_{j=1}^{n} \frac{\partial\, \alpha_{i_1\cdots i_n}}{\partial x_{i_j}}\, dx_{i_j} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_n}.$$

This is exactly the formula for exterior differentiation d that was given in the last section. Notice that in the above string of equalities we used all four properties. Even though we used properties $i - iv$ to find this formula, to be completely rigorous we ought to turn around and show that this formula satisfies the properties. This is essentially what Question 4.3 asked you to do.

Question 4.4 Show where each of the properties $1 - 4$ were used in the above computation.

There is one final comment that needs to be made. What we have done is shown that the exterior differential operator exists and is unique, locally. What we mean by locally is within a single *coordinate patch* where the coordinates we used to find the formula apply. We discuss coordinate patches in Chap. 10 when we discuss general manifolds. However, for a Euclidian manifold $\mathbb{R}^n$ there is only one coordinate patch, so for the Euclidian manifold $\mathbb{R}^n$ we now know that the exterior differential operator exists and is unique on the whole manifold. For a general manifold we need to show existence and uniqueness is global and does not depend on the coordinate patch being used. We will address this issue in Sect. 10.4 once we have the necessary concepts and tools at our disposal.

4.4 The Global Formula

In the last two sections we did not try to address or think about what the geometric meaning behind exterior differentiation was; the last two sections were all about the formula and properties of the exterior derivative. In this section we will start to think about the underlying geometry in an attempt to both justify and find the global formula for the exterior derivative. As far as we are aware you will not see a presentation along these lines in any other introductions to exterior differentiation. However we believe it provides both a geometrical justification for exterior differentiation as well as allowing for a deeper intuitive understanding of both the exterior derivative and the global formula for it. A different geometric approach is presented in the next section.

4.4.1 Exterior Differentiation with Constant Vector Fields

Recall that given a one-form ϕ on the manifold $\mathbb{R}^n$, at each point $p \in \mathbb{R}^n$ we have a mapping $\phi : T_p(\mathbb{R}^n) \to \mathbb{R}$. If we were given a vector field v on manifold $\mathbb{R}^n$, then for each $p \in \mathbb{R}^n$ we have $\phi_p(v_p) \in \mathbb{R}$, that is, a real number. Hence we could view $\phi(v)$ as a function on the manifold $\mathbb{R}^n$. That is, its inputs are points p on the manifold and its outputs are real numbers, that is defined as so:

$$\begin{aligned} \phi(v) : \mathbb{R}^n &\longrightarrow \mathbb{R} \\ p &\longmapsto \phi_p(v_p). \end{aligned}$$

Also, we recall the notation $\langle \cdot, \cdot \rangle$ to mean the canonical pairing between a one-form and a vector, that is, $\phi(v) = \langle \phi, v \rangle$. If ϕ were an n-form then we would write the canonical pairing as $\phi(v_1, \ldots, v_n) = \langle \phi, (v_1, \ldots, v_n) \rangle$.

Now let us go back to the idea that differentiation is some sort of a measure of how a mathematical object, in this case a differential form, varies. For the moment we will continue to consider the special case of a one-form $\alpha = f_1 dx + f_2 dy$ on the manifold $\mathbb{R}^2$ since we can draw the associated pictures easily. Suppose we are given a *constant vector field* v on the manifold $\mathbb{R}^2$, where

$$v_p = v_1 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_p$$

at every point $p \in \mathbb{R}^2$ and $v_1, v_2 \in \mathbb{R}$. The fact that we assume the vector field to be constant simplifies the computations considerably thereby making the underlying ideas clearer. Of course, vector fields need not be constant, and in fact generally they are not. The case of the non-constant vector field is handled next.

One of the ways that we could try to measure how the differential one-form $\alpha : T(\mathbb{R}^2) \to \mathbb{R}$ varies is to instead consider how the function $\langle \alpha, v \rangle$ varies. True, doing this does not just measure how just α varies, instead it measure how $\langle \alpha, v \rangle$ taken together vary. But be that as it may, at least it is an approach that we know how to handle, meaning that we already know how to measure how functions vary in different directions. This is just our usual directional derivative.

But to do this we need to vary $\langle \alpha, v \rangle$ in some direction, say in the direction w. Again, to keep things simple we will choose w to be a constant vector field on the manifold $\mathbb{R}^2$,

$$w_p = w_1 \frac{\partial}{\partial x} + w_2 \frac{\partial}{\partial y} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}_p$$

at every point $p \in \mathbb{R}^2$. Thus, seeing how the function $\langle \alpha, v \rangle$ varies in the direction w means finding the directional derivative of the function $\langle \alpha, v \rangle$ in the direction of w, that is, finding $d\langle \alpha, v \rangle\big(w\big) = w[\langle \alpha, v \rangle]$. In order to do this we first find the function $\langle \alpha, v \rangle$,

$$\begin{aligned}
\langle \alpha, v \rangle &= (f_1 dx + f_2 dy)\left(v_1 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y}\right) \\
&= [f_1, f_2] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\
&= v_1 f_1 + v_2 f_2.
\end{aligned}$$

Notice, this is just some sort of a "weighted sum" of the functions f_1 and f_2 with weights given by v_1 and v_2, which come from our constant vector v. We already know how to take the differential of a function, so we have

$$\begin{aligned}
d\langle \alpha, v \rangle &= \frac{\partial \langle \alpha, v \rangle}{\partial x} dx + \frac{\partial \langle \alpha, v \rangle}{\partial y} dy \\
&= \frac{\partial (v_1 f_1 + v_2 f_2)}{\partial x} dx + \frac{\partial (v_1 f_1 + v_2 f_2)}{\partial y} dy \\
&= \left(v_1 \frac{\partial f_1}{\partial x} + v_2 \frac{\partial f_2}{\partial x}\right) dx + \left(v_1 \frac{\partial f_1}{\partial y} + v_2 \frac{\partial f_2}{\partial y}\right) dy.
\end{aligned}$$

Using this we can finally find the directional derivative of $\langle \alpha, v \rangle$ in the direction of w as follows,

$$\begin{aligned}
d\langle \alpha, v \rangle\big(w\big) &= \big\langle d\langle \alpha, v \rangle, w \big\rangle \\
&= \left[v_1 \frac{\partial f_1}{\partial x} + v_2 \frac{\partial f_2}{\partial x},\ v_1 \frac{\partial f_1}{\partial y} + v_2 \frac{\partial f_2}{\partial y} \right] \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \\
&= \left(v_1 \frac{\partial f_1}{\partial x} + v_2 \frac{\partial f_2}{\partial x}\right) w_1 + \left(v_1 \frac{\partial f_1}{\partial y} + v_2 \frac{\partial f_2}{\partial y}\right) w_2 \\
&= v_1 w_1 \frac{\partial f_1}{\partial x} + v_2 w_1 \frac{\partial f_2}{\partial x} + v_1 w_2 \frac{\partial f_1}{\partial y} + v_2 w_2 \frac{\partial f_2}{\partial y}.
\end{aligned}$$

We try to show this pictorially in Fig. 4.1.

In summary, what we have here,

$$d\langle \alpha, v \rangle\big(w\big) = v_1 w_1 \frac{\partial f_1}{\partial x} + v_2 w_1 \frac{\partial f_2}{\partial x} + v_1 w_2 \frac{\partial f_1}{\partial y} + v_2 w_2 \frac{\partial f_2}{\partial y}$$

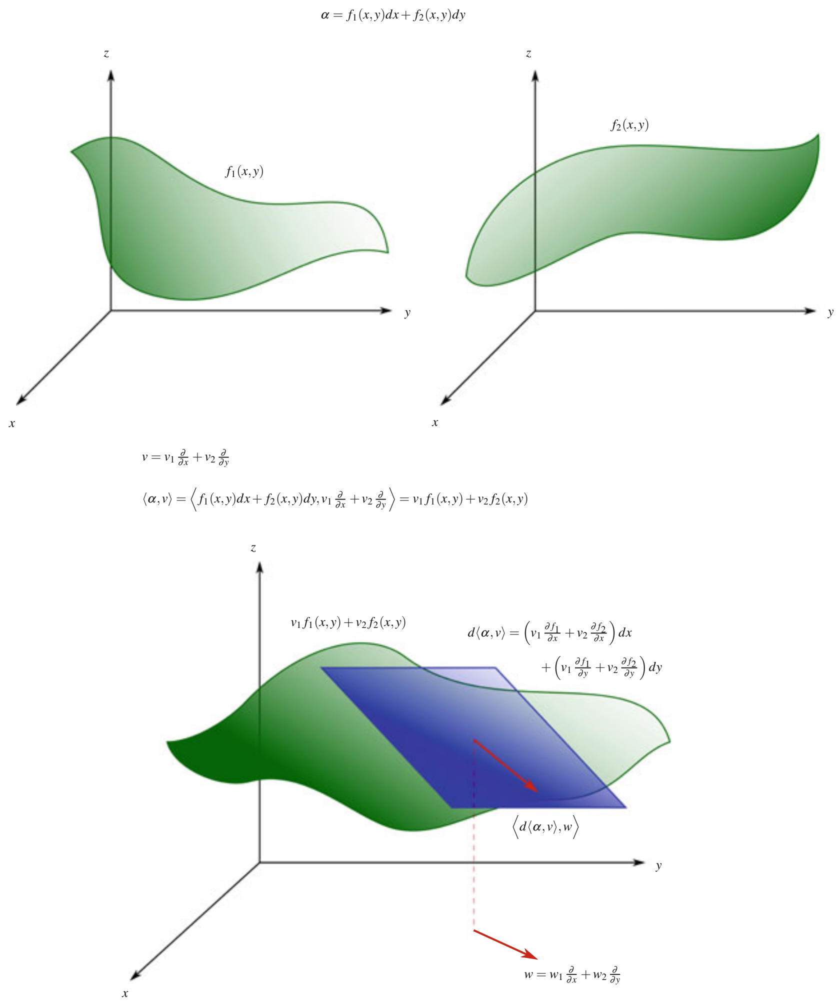

Fig. 4.1 The one-form $\alpha_{(x,y)} = f_1(x,y)dx + f_2(x,y)dy$ on the manifold $\mathbb{R}^2$ is made up of two functions $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$, shown above. Once we are given a vector field v on manifold $\mathbb{R}^3$, here the constant vector field $v = v_1\partial_x + v_2\partial_y$, then this can be used to find the real-valued function $\langle \alpha, v\rangle = v_1 f_1 + v_2 f_2$, which can be viewed as a linear combination of the two functions f_1, f_2. The directional derivative of this function can then be found in the direction of another given vector $w = w_1\partial_x + w_2\partial_y$. Here the differential $d\langle \alpha, v\rangle$ in essence encodes the information about the tangent plane to the function $\langle \alpha, v\rangle$, shown in green

is a function from the manifold $\mathbb{R}^2$ to $\mathbb{R}$. When we input some point $p = (x_0, y_0)$ what we get is

$$\begin{aligned} d\langle \alpha_{(x_0,y_0)}, v\rangle\big(w\big) =& v_1 w_1 \left.\frac{\partial f_1(x,y)}{\partial x}\right|_{(x_0,y_0)} + v_2 w_1 \left.\frac{\partial f_2(x,y)}{\partial x}\right|_{(x_0,y_0)} \\ &+ v_1 w_2 \left.\frac{\partial f_1(x,y)}{\partial y}\right|_{(x_0,y_0)} + v_2 w_2 \left.\frac{\partial f_2(x,y)}{\partial y}\right|_{(x_0,y_0)}, \end{aligned}$$

which is a number that measures how much the function $\langle \alpha, v\rangle = v_1 f_1 + v_2 f_2$ varies as we move along the vector w at the given point $p = (x_0, y_0)$.

This clearly is not simply a measure of how the one-form α varies. Instead, it requires two additional vectors, v and w. The first vector v is used along with the one-form to make the real-valued function $\langle \alpha, v\rangle$ and then the second vector is used to find the directional derivative of this function. While this seems like a somewhat odd way to measure how α varies it is perhaps about as straight-forward as we can expect to get given how complicated a mathematical object a one-form actually is.

But before we try to somehow define the differential of a one-form α with this formula we decide to be cautious and see what happens when we switch the order of the vectors v and w. That is, we want to know what $d\langle \alpha, w\rangle\big(v\big)$ is. After all, if we simply change the order of the vectors v and w we would hope that the results are "the same" in some sense. Proceeding to do just this we have $\langle \alpha, w \rangle = w_1 f_1 + w_2 f_2$, which gives

$$d\langle \alpha, w \rangle = \left(w_1 \frac{\partial f_1}{\partial x} + w_2 \frac{\partial f_2}{\partial x} \right) dx + \left(w_1 \frac{\partial f_1}{\partial y} + w_2 \frac{\partial f_2}{\partial y} \right) dy,$$

resulting in

$$\begin{aligned} d\langle \alpha, w \rangle \big(v \big) &= \big\langle d\langle \alpha, w \rangle, v \big\rangle \\ &= v_1 w_1 \frac{\partial f_1}{\partial x} + v_1 w_2 \frac{\partial f_2}{\partial x} + v_2 w_1 \frac{\partial f_1}{\partial y} + v_2 w_2 \frac{\partial f_2}{\partial y}. \end{aligned}$$

Let us put these two formulas side by side to see how they match up.

$$\begin{aligned} \Big\langle d\langle \alpha, v \rangle, w \Big\rangle &= v_1 w_1 \frac{\partial f_1}{\partial x} + v_2 w_1 \frac{\partial f_2}{\partial x} + v_1 w_2 \frac{\partial f_1}{\partial y} + v_2 w_2 \frac{\partial f_2}{\partial y} \\ \Big\langle d\langle \alpha, w \rangle, v \Big\rangle &= v_1 w_1 \frac{\partial f_1}{\partial x} + v_1 w_2 \frac{\partial f_2}{\partial x} + v_2 w_1 \frac{\partial f_1}{\partial y} + v_2 w_2 \frac{\partial f_2}{\partial y}. \end{aligned}$$

So, these two equalities are not the same as we may have hoped. Nor do they simply differ from each other by a sign. But clearly there is a relation between $d\langle \alpha, v \rangle \big(w \big)$ and $d\langle \alpha, w \rangle \big(v \big)$. The relation between them is a little more complicated. The first and last terms of each equality are identical but the middle two terms are not. We can not use either of these formulas as a definition for the derivative of a one-form. However, all is not lost. A bit of clever thinking may yet salvage the situation.

In the past we have discovered that the "volume" of a parallelepiped has a sign attached to it that depends on the orientation, or order, of the vectors. This sign "ambiguity" has shown up in the definition of determinant, and then in the definition of wedgeproduct. So perhaps we should not be too surprised, or concerned, if it shows up in whatever definition of derivative of a form we end up settling on. So we will use a little "trick" that pops up in mathematics from time to time. What would happen if we take the difference between these two terms as the definition of exterior differentiation? That is, suppose we define the exterior derivative of α, $d\alpha$, by the following formula

$$d\alpha(v, w) = \Big\langle d\langle \alpha, w \rangle, v \Big\rangle - \Big\langle d\langle \alpha, v \rangle, w \Big\rangle.$$

With this formula we then notice that by changing the order of v and w the sign is changed,

$$\begin{aligned} d\alpha(w, v) &= \big\langle d\langle \alpha, v \rangle, w \big\rangle - \big\langle d\langle \alpha, w \rangle, v \big\rangle \\ &= -\Big(\big\langle d\langle \alpha, w \rangle, v \big\rangle - \big\langle d\langle \alpha, v \rangle, w \big\rangle \Big) \\ &= -d\alpha(v, w). \end{aligned}$$

By doing this we have that $d\alpha(v, w)$ and $d\alpha(w, v)$ give us the same answer up to sign, something that we are already perfectly used to and comfortable with. In fact, given the way signs seem to keep behaving in this subject it would be surprising if switching the order of two vectors didn't change the sign of something. So, we will settle on

$$d\alpha(v, w) = \big\langle d\langle \alpha, w \rangle, v \big\rangle - \big\langle d\langle \alpha, v \rangle, w \big\rangle.$$

as our working definition for the derivative of α. (In reality this working definition only works if v and w happen to be constant vector fields, but more on that later.) Geometrically, $d\alpha(v, w)$ measures the change in the function $\langle \alpha, w \rangle$ in the v direction minus the change in the function $\langle \alpha, v \rangle$ in the w direction. We attempt to show this in Fig. 4.2.

Spend a little time thinking about what we have done. Since the one-form $\alpha = f_1 dx + f_2 dy$ is a rather complicated mathematical object, no obvious direct way of measuring how it changes is apparent. Thus we decided to try to measure how α changes indirectly using techniques that we were already familiar with, namely directional derivatives. In order to do this we needed two (constant) vector fields v and w. Thus, once we were given a particular v and w we used these to indirectly measure how α changes.

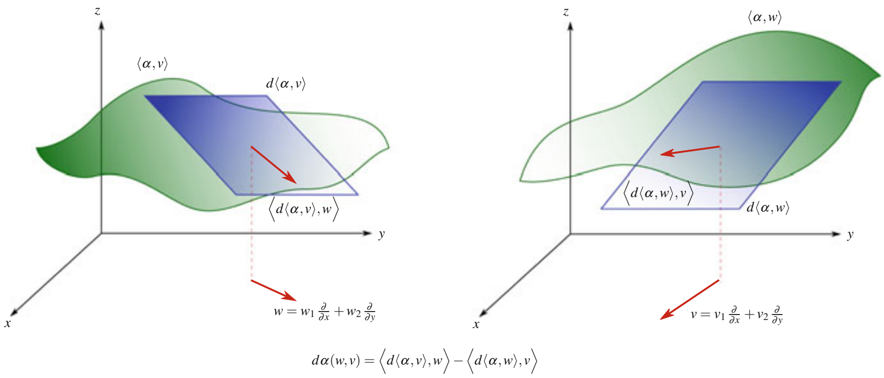

Fig. 4.2 The change in $\langle\alpha, v\rangle$ in the direction w is shown on the left and the change in $\langle\alpha, w\rangle$ in the direction v is shown on the left. The difference in these two values is then used to define $d\alpha(w, v)$. This is an indirect way to measure how α changes

Let us use this to get an actual expression for $d\alpha(v, w)$,

$$
\begin{aligned}
d\alpha(v, w) =& \Big\langle d\langle\alpha, w\rangle, v\Big\rangle - \Big\langle d\langle\alpha, v\rangle, w\Big\rangle \\
=& \left(v_1 w_1 \frac{\partial f_1}{\partial x} + v_1 w_2 \frac{\partial f_2}{\partial x} + v_2 w_1 \frac{\partial f_1}{\partial y} + v_2 w_2 \frac{\partial f_2}{\partial y}\right) \\
& - \left(v_1 w_1 \frac{\partial f_1}{\partial x} + v_2 w_1 \frac{\partial f_2}{\partial x} + v_1 w_2 \frac{\partial f_1}{\partial y} + v_2 w_2 \frac{\partial f_2}{\partial y}\right) \\
=& v_1 w_2 \frac{\partial f_2}{\partial x} - v_1 w_2 \frac{\partial f_1}{\partial y} - v_2 w_1 \frac{\partial f_2}{\partial x} + v_2 w_1 \frac{\partial f_1}{\partial y} \\
=& \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)(v_1 w_2 - w_1 v_2) \\
=& \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) \begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix} \\
=& \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) \begin{vmatrix} dx(v) & dx(w) \\ dy(v) & dy(w) \end{vmatrix} \\
=& \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx \wedge dy(v, w) \\
=& \left(\frac{\partial f_1}{\partial x} \underbrace{dx \wedge dx}_{=0} + \frac{\partial f_1}{\partial y} \underbrace{dy \wedge dx}_{=-dx\wedge dy} + \frac{\partial f_2}{\partial x} dx \wedge dy + \frac{\partial f_2}{\partial y} \underbrace{dy \wedge dy}_{=0}\right)(v, w) \\
=& \left(\underbrace{\left(\frac{\partial f_1}{\partial x} dx + \frac{\partial f_1}{\partial y} dy\right)}_{df_1} \wedge dx + \underbrace{\left(\frac{\partial f_2}{\partial x} dx + \frac{\partial f_2}{\partial y} dy\right)}_{df_2} \wedge dy\right)(v, w) \\
=& \Big(df_1 \wedge dx + df_2 \wedge dy\Big)(v, w).
\end{aligned}
$$

Thus our "geometric" definition of $d\alpha$ as the change in the function $\langle\alpha, w\rangle$ in the v direction minus the change in the function $\langle\alpha, v\rangle$ in the w direction leads us all the way back to our original formula for $d\alpha$, $\sum df_i \wedge dx_i$, when $\alpha = \sum f_i dx_i$. Furthermore, we have a wonderful bonus, $d\alpha$ turns out to be a two-form. Okay, in reality this is actually more than just a "bonus." The fact that this definition results in $d\alpha$ being a two-form is actually a prime reason we settle on this admittedly slightly awkward geometrical meaning for the exterior derivative of a one-form.

Actually, in order to keep the computations and pictures simple we looked at a one-form α on the manifold $\mathbb{R}^2$, but we could have done the whole analysis for any manifold and arrived at the same formula; the computations would have simply been more cumbersome.

We recognize that we tend to bounce between different notations, and we will continue to do so. We are actually doing this on purpose, it is important that you become comfortable with all of these different notations. Different books, papers, and authors all tend to use different notations. Recall that $df(v) = v[f]$ and $d\langle\alpha, w\rangle\big(v\big) = v[\langle\alpha, w\rangle]$. Also, we have $\langle\alpha, w\rangle = \alpha(w)$, so we could write $\big\langle d\langle\alpha, w\rangle, v\big\rangle$ as $v[\alpha(w)]$. Similarly, we have $\big\langle d\langle\alpha, v\rangle, w\big\rangle = w[\alpha(v)]$, so we can also write the definition of the exterior derivative of α, where v and w are constant vector fields, as

Global formula for exterior derivative of a one-form, constant vector fields	$d\alpha(v, w) = \big\langle d\langle\alpha, w\rangle, v\big\rangle - \big\langle d\langle\alpha, v\rangle, w\big\rangle$

or as

Global formula for exterior derivative of a one-form, constant vector fields	$d\alpha(v, w) = v[\alpha(w)] - w[\alpha(v)].$

Notice, these are exactly the first two terms in our global formula given in the overview section,

$$d\alpha(v, w) = v[\alpha(w)] - w[\alpha(v)] - \alpha([v, w]).$$

Thus we see that our "geometric" definition of $d\alpha$ is tied to the global formula for $d\alpha$. However, we are missing the final term $\alpha([v, w])$ from the global formula. This term is zero when both v and w are constant vector fields. Avoiding the mess of this term was the reason we chose constant vector fields v and w. Again, we point out that in this global formula nowhere do the actual coordinates that we are using show up. Sometimes this is called a coordinate-free formula.

Now we will repeat this quickly for a general one-form on a general n-dimensional manifold. The computations apply to what is called a coordinate patch of a general manifold. That is a concept we will introduce in a latter chapter, so for now just imagine that we are on the manifold $\mathbb{R}^n$. Consider the one-form and vectors

$$\begin{aligned} \alpha &= \sum_i \alpha_i\, dx_i, \\ v &= \sum_i v_i\, \partial_{x_i}, \\ w &= \sum_i w_i\, \partial_{x_i}. \end{aligned}$$

Then we have

$$\begin{aligned} \alpha(v) &= \langle\alpha, v\rangle \\ &= \left(\sum_i \alpha_i\, dx_i\right)\left(\sum_i v_i\, \partial_{x_i}\right) \\ &= \sum_i v_i\alpha_i. \end{aligned}$$

As before, $\langle\alpha, v\rangle = \sum_i v_i\alpha_i$ can be viewed as a function on the manifold. Basically this function it is the sum of the functions α_i weighted by v_i. Using d as defined in the directional derivative of a function case, we have

$$\begin{aligned} d\langle\alpha, v\rangle &= \sum_j \frac{\partial\left(\sum_i v_i\alpha_i\right)}{\partial x_j} dx_j \\ &= \sum_j \sum_i v_i \frac{\partial \alpha_i}{\partial x_j} dx_j. \end{aligned}$$

So the directional derivative of the function $\langle \alpha, v\rangle = \sum_i v_i \alpha_i$ in the direction w is given by

$$\begin{aligned}
d\langle \alpha, v\rangle(w) &= \left(\sum_j \sum_i v_i \frac{\partial \alpha_i}{\partial x_j} dx_j\right)\left(\sum_k w_k \partial_{x_k}\right) \\
&= \sum_j \sum_i v_i w_j \frac{\partial \alpha_i}{\partial x_j}.
\end{aligned}$$

Recall, basically $d\langle \alpha, v\rangle$ encodes the tangent space of the function $\langle \alpha, v\rangle$ and $d\langle \alpha, v\rangle(w)$ tell us the change that occurs in as we move by w. A completely identical computation gives us

$$d\langle \alpha, w\rangle(v) = \sum_j \sum_i v_j w_i \frac{\partial \alpha_i}{\partial x_j}.$$

Comparing $d\langle \alpha, v\rangle(w)$ and $d\langle \alpha, w\rangle(v)$ we can see that when $i \neq j$ the terms differ, so like before we will define

$$\begin{aligned}
d\alpha(v, w) &= \big\langle d\langle \alpha, w\rangle, v\big\rangle - \big\langle d\langle \alpha, v\rangle, w\big\rangle \\
&= \sum_j \sum_i v_j w_i \frac{\partial \alpha_i}{\partial x_j} - \sum_j \sum_i v_i w_j \frac{\partial \alpha_i}{\partial x_j} \\
&= \sum_j \sum_i \big(v_j w_i - v_i w_j\big) \frac{\partial \alpha_i}{\partial x_j} \\
&= \sum_j \sum_i \frac{\partial \alpha_i}{\partial x_j} \begin{vmatrix} v_j & w_j \\ v_i & w_i \end{vmatrix} \\
&= \sum_j \sum_i \frac{\partial \alpha_i}{\partial x_j} \begin{vmatrix} dx_j(v) & dx_j(w) \\ dx_i(v) & dx_i(w) \end{vmatrix} \\
&= \sum_j \sum_i \frac{\partial \alpha_i}{\partial x_j}\, dx_j \wedge dx_i(v, w) \\
&= \sum_i \underbrace{\left(\sum_j \frac{\partial \alpha_i}{\partial x_j} dx_j\right)}_{=d\alpha_i} \wedge dx_i(v, w).
\end{aligned}$$

Hence we have the in-coordinates formula

$$d\alpha = \sum_i d\alpha_i \wedge dx_i,$$

which is of course exactly what we had before.

As in the case of the one-forms we could attempt to make an geometrical argument for the global formula of a k-form ω. The first step would be completely analogous. In order to use the concept of the directional derivative of a function we would have to turn a k-from into a function by putting in not one, but k vectors $v_0, \ldots, v_{k-1}$ and then we would measure how that function changes in the direction of yet another vector v_k. However, the next step, where we actually use the math "trick" to define $d\omega$ would be difficult to rationalize. We would end up with

Global formula for exterior derivative of a k-form, constant vector fields	$d\omega(v_0, \ldots, v_k) = \sum_i (-1)^i \Big\langle d\big\langle \omega, (v_0, \ldots, \widehat{v_i}, \ldots, v_k)\big\rangle, v_i \Big\rangle,$

which could also be written as

Global formula for exterior derivative of a k-form, constant vector fields	$d\omega(v_0, \ldots, v_k) = \sum_i (-1)^i v_i\big[\omega(v_0, \ldots, \widehat{v_i}, \ldots, v_k)\big].$

Using this global formula to arrive at the in-coordinates formula would also be a very computationally intense process. Therefore we will forgo it. However, notice that the above formulas are different from the global formula in the overview. Like before, this is because we are using constant vector fields.

4.4.2 Exterior Differentiation with Non-Constant Vector Fields

What we have done for constant vector fields v and w is to make some geometrical augments that $d\alpha(v, w) = v[\alpha(w)] - w[\alpha(v)]$ would be a good definition for the global formula of the exterior derivative of α. We then used this formula to find the in-coordinates formula for the exterior derivative of the one-form $\alpha = \sum_i \alpha_i dx_i$ and found it to be $d\alpha = \sum_i d\alpha_i \wedge dx_i$. We will now assume that this is the form we want the in-coordinates version of the general global formula to take.

Since we are not trying to present an actual proof of the global formula for exterior differentiation but instead are trying to give you a feel for how and why the formula is what it is; we will not attempt a logically rigorous argument. What we will be doing is actually logically backwards. We know what answer we want to get, that is, we know what the in-coordinates formula for the exterior derivative should be, so we will actually use this knowledge of where we want to end up to help us find and understand the general global formula for exterior differentiation. The general global formula for exterior differentiation will actually be proved in Sect. A.7 in a completely different way using a number of concepts and identities that will be introduced in the appendix. The proof in that section is very computational and the formula seems to miraculously appear out of a stew of different ingredients. This section is meant to be more down-to-earth, giving motivation and addressing the underlying geometry to a greater extent.

We begin with the one-form α and two *non-constant vector fields* v and w on $\mathbb{R}^2$,

$$\alpha = f_1 dx + f_2 dy,$$

$$v = g_1 \partial_x + g_2 \partial_y = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix},$$

$$w = h_1 \partial_x + h_2 \partial_y = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix},$$

where $f_1, f_2, g_1, g_1, h_1, h_2$ are real-valued functions on the manifold $\mathbb{R}^2$.

As we said, we will start with the answer we know we want to arrive at. We want the in-coordinates formula for the exterior derivative of α to be $d\alpha = \sum_i d\alpha_i \wedge dx_i$. So first we compute that,

$$\begin{aligned}
d\alpha &= df_1 \wedge dx + df_2 \wedge dy \\
&= \left(\frac{\partial f_1}{\partial x}dx + \frac{\partial f_1}{\partial y}dy\right) \wedge dx + \left(\frac{\partial f_2}{\partial x}dx + \frac{\partial f_2}{\partial y}dy\right) \wedge dy \\
&= \frac{\partial f_1}{\partial x}\underbrace{dx \wedge dx}_{=0} + \frac{\partial f_1}{\partial y}\underbrace{dy \wedge dx}_{=-dx\wedge dy} + \frac{\partial f_2}{\partial x}dx \wedge dy + \frac{\partial f_2}{\partial y}\underbrace{dy \wedge dy}_{=0} \\
&= \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx \wedge dy.
\end{aligned}$$

Plugging in our vector fields v and w we get

$$d\alpha(v, w) = \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial x}\right) dx \wedge dy(v, w)$$

$$
\begin{aligned}
&= \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial x}\right) \begin{vmatrix} dx(v) & dx(w) \\ dy(v) & dy(w) \end{vmatrix} \\
&= \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial x}\right) \begin{vmatrix} g_1 & h_1 \\ g_2 & h_2 \end{vmatrix} \\
&= \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial x}\right) (g_1 h_2 - g_2 h_1) \\
&= g_1 h_2 \frac{\partial f_2}{\partial x} - g_1 h_2 \frac{\partial f_1}{\partial y} - g_2 h_1 \frac{\partial f_2}{\partial x} + g_2 h_1 \frac{\partial f_1}{\partial y}.
\end{aligned}
$$

By now we should be comfortable with this calculation. The only difference is that the constants v_1, v_1, w_1, w_2 have been replaced with the functions g_1, g_2, h_1, h_2. This is the answer we want to get to. Later on we will refer back to this equation.

Our strategy is the same as in the constant vector field case for the same geometric reasons. In fact, based on the constant vector field case we will first hypothesize that the global formula is $d\alpha(v, w) = v[\alpha(w)] - w[\alpha(v)]$. We will do this (admittedly messy) calculation and discover there are a lot more terms than the local formula for $d\alpha(v, w)$ has, which we found just above. We will then group those terms together and call them $-\alpha([v, w])$ thereby giving the full global formula $d\alpha(v, w) = v[\alpha(w)] - w[\alpha(v)] - \alpha([v, w])$ for the one-form case. Of course, at that point you will have no idea what $\alpha([v, w])$ is. We will then spend some time explaining what this final term actually represents. The brackets $[\cdot, \cdot]$, with the dots replaced by vectors, is call the Lie bracket, so $[v, w]$ is the Lie bracket of v and w, and turns out to be another vector field, which is good since it is being eaten by the one-form α.

Now we are ready to begin. The same arguments apply. We want to somehow measure how α varies. Since we are not sure how to do this we first change α to something that we know how to study, namely a function, which we can do by pairing it with the vector field v, which no longer needs to be a constant vector field,

$$
\langle \alpha, v \rangle = [f_1, f_2] \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = f_1 g_1 + f_2 g_2.
$$

Compare this to the case with the constant vector field, where we had obtained $v_1 f_1 + v_2 f_2$ with v_1 and v_2 being constants. The situation is immediately more complicated since it involves products of functions and not just a weighted sum of functions.

In order to find the directional derivative of the function $f_1 g_1 + f_2 g_2$ we first find the differential of the function $\langle \alpha, v \rangle$,

$$
\begin{aligned}
d\langle \alpha, v \rangle =& \frac{\partial (f_1 g_1 + f_2 g_2)}{\partial x} dx + \frac{\partial (f_1 g_1 + f_2 g_2)}{\partial y} dy \\
=& \left(\frac{\partial (f_1 g_1)}{\partial x} + \frac{\partial (f_2 g_2)}{\partial x} \right) dx + \left(\frac{\partial (f_1 g_1)}{\partial y} + \frac{\partial (f_2 g_2)}{\partial y} \right) dy \\
=& \left(\frac{\partial f_1}{\partial x} \cdot g_1 + f_1 \cdot \frac{\partial g_1}{\partial x} + \frac{\partial f_2}{\partial x} \cdot g_2 + f_2 \cdot \frac{\partial g_2}{\partial x} \right) dx \\
&+ \left(\frac{\partial f_1}{\partial y} \cdot g_1 + f_1 \cdot \frac{\partial g_1}{\partial y} + \frac{\partial f_2}{\partial y} \cdot g_2 + f_2 \cdot \frac{\partial g_2}{\partial y} \right) dy.
\end{aligned}
$$

Notice that the required use of the product rule here results in twice the number of terms as we had previously when we were dealing with constant vector fields. And these terms are themselves products of functions. This is what makes dealing with non-constant vector fields so much more complicated. We now find the directional derivative of $\langle \alpha, v \rangle$ in the direction of w by

$$
\begin{aligned}
\Big\langle d\langle \alpha, v \rangle, w \Big\rangle =& \left(\frac{\partial f_1}{\partial x} \cdot g_1 + f_1 \cdot \frac{\partial g_1}{\partial x} + \frac{\partial f_2}{\partial x} \cdot g_2 + f_2 \cdot \frac{\partial g_2}{\partial x} \right) dx \left(\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \right) \\
&+ \left(\frac{\partial f_1}{\partial y} \cdot g_1 + f_1 \cdot \frac{\partial g_1}{\partial y} + \frac{\partial f_2}{\partial y} \cdot g_2 + f_2 \cdot \frac{\partial g_2}{\partial y} \right) dy \left(\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{\partial f_1}{\partial x} \cdot g_1 + f_1 \cdot \frac{\partial g_1}{\partial x} + \frac{\partial f_2}{\partial x} \cdot g_2 + f_2 \cdot \frac{\partial g_2}{\partial x} \right) h_1 \\
&\quad + \left(\frac{\partial f_1}{\partial y} \cdot g_1 + f_1 \cdot \frac{\partial g_1}{\partial y} + \frac{\partial f_2}{\partial y} \cdot g_2 + f_2 \cdot \frac{\partial g_2}{\partial y} \right) h_2 \\
&= g_1 h_1 \frac{\partial f_1}{\partial x} + f_1 h_1 \frac{\partial g_1}{\partial x} + g_2 h_1 \frac{\partial f_2}{\partial x} + f_2 h_1 \frac{\partial g_2}{\partial x} \\
&\quad + g_1 h_2 \frac{\partial f_1}{\partial y} + f_1 h_2 \frac{\partial g_1}{\partial y} + g_2 h_2 \frac{\partial f_2}{\partial y} + f_2 h_2 \frac{\partial g_2}{\partial y}.
\end{aligned}
$$

Next we repeat this procedure exchanging the vector fields v and w,

$$
\langle \alpha, w \rangle = [f_1, f_2] \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = f_1 h_1 + f_2 h_2.
$$

In order to find the directional derivative we need

$$
\begin{aligned}
d\langle \alpha, w \rangle &= \frac{\partial (f_1 h_1 + f_2 h_2)}{\partial x} dx + \frac{\partial (f_1 h_1 + f_2 h_2)}{\partial y} dy \\
&= \left(\frac{\partial (f_1 h_1)}{\partial x} + \frac{\partial (f_2 h_2)}{\partial x} \right) dx + \left(\frac{\partial (f_1 h_1)}{\partial y} + \frac{\partial (f_2 h_2)}{\partial y} \right) dy \\
&= \left(\frac{\partial f_1}{\partial x} \cdot h_1 + f_1 \cdot \frac{\partial h_1}{\partial x} + \frac{\partial f_2}{\partial x} \cdot h_2 + f_2 \cdot \frac{\partial h_2}{\partial x} \right) dx \\
&\quad + \left(\frac{\partial f_1}{\partial y} \cdot h_1 + f_1 \cdot \frac{\partial h_1}{\partial y} + \frac{\partial f_2}{\partial y} \cdot h_2 + f_2 \cdot \frac{\partial h_2}{\partial y} \right) dy.
\end{aligned}
$$

Thus the directional derivative of $\langle \alpha, w \rangle$ in the direction of v is given by

$$
\begin{aligned}
\Big\langle d\langle \alpha, w \rangle, v \Big\rangle &= \left(\frac{\partial f_1}{\partial x} \cdot h_1 + f_1 \cdot \frac{\partial h_1}{\partial x} + \frac{\partial f_2}{\partial x} \cdot h_2 + f_2 \cdot \frac{\partial h_2}{\partial x} \right) dx \left(\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \right) \\
&\quad + \left(\frac{\partial f_1}{\partial y} \cdot h_1 + f_1 \cdot \frac{\partial h_1}{\partial y} + \frac{\partial f_2}{\partial y} \cdot h_2 + f_2 \cdot \frac{\partial h_2}{\partial y} \right) dy \left(\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \right) \\
&= \left(\frac{\partial f_1}{\partial x} \cdot h_1 + f_1 \cdot \frac{\partial h_1}{\partial x} + \frac{\partial f_2}{\partial x} \cdot h_2 + f_2 \cdot \frac{\partial h_2}{\partial x} \right) g_1 \\
&\quad + \left(\frac{\partial f_1}{\partial y} \cdot h_1 + f_1 \cdot \frac{\partial h_1}{\partial y} + \frac{\partial f_2}{\partial y} \cdot h_2 + f_2 \cdot \frac{\partial h_2}{\partial y} \right) g_2 \\
&= h_1 g_1 \frac{\partial f_1}{\partial x} + f_1 g_1 \frac{\partial h_1}{\partial x} + h_2 g_1 \frac{\partial f_2}{\partial x} + f_2 g_1 \frac{\partial h_2}{\partial x} \\
&\quad + h_1 g_2 \frac{\partial f_1}{\partial y} + f_1 g_2 \frac{\partial h_1}{\partial y} + h_2 g_2 \frac{\partial f_2}{\partial y} + f_2 g_2 \frac{\partial h_2}{\partial y}.
\end{aligned}
$$

Recall that in our strategy we were first going to hypothesis that $d\alpha(v, w) = v[\alpha(w)] - w[\alpha(v)]$. We chose this in the constant vector case simply because it was something that seemed to work, and it also resulted in the fact that switching the vectors v and w caused the sign to change. Proceeding with our hypothesis we have

$$
\begin{aligned}
d\alpha(v, w) &\underset{\text{hyp.}}{=} v[\alpha(w)] - w[\alpha(v)] \\
&= \Big\langle d\langle \alpha, w \rangle, v \Big\rangle - \Big\langle d\langle \alpha, v \rangle, w \Big\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \cancel{h_1 g_1 \frac{\partial f_1}{\partial x}} + f_1 g_1 \frac{\partial h_1}{\partial x} + h_2 g_1 \frac{\partial f_2}{\partial x} + f_2 g_1 \frac{\partial h_2}{\partial x} \\
&\quad + h_1 g_2 \frac{\partial f_1}{\partial y} + f_1 g_2 \frac{\partial h_1}{\partial y} + \cancel{h_2 g_2 \frac{\partial f_2}{\partial y}} + f_2 g_2 \frac{\partial h_2}{\partial y} \\
&\quad - \cancel{g_1 h_1 \frac{\partial f_1}{\partial x}} - f_1 h_1 \frac{\partial g_1}{\partial x} - g_2 h_1 \frac{\partial f_2}{\partial x} - f_2 h_1 \frac{\partial g_2}{\partial x} \\
&\quad - g_1 h_2 \frac{\partial f_1}{\partial y} - f_1 h_2 \frac{\partial g_1}{\partial y} - \cancel{g_2 h_2 \frac{\partial f_2}{\partial y}} - f_2 h_2 \frac{\partial g_2}{\partial y}.
\end{aligned}
$$

Notice that four terms cancel out right away. Since we already know what we want our local in-coordinates formula to look like we now turn to that. Based on the computation above, what we are trying to arrive at is

$$
d\alpha(v, w) = g_1 h_2 \frac{\partial f_2}{\partial x} - g_1 h_2 \frac{\partial f_1}{\partial y} - g_2 h_1 \frac{\partial f_2}{\partial x} + g_2 h_1 \frac{\partial f_1}{\partial y}.
$$

These terms are shown in green above. Thus we have eight extra terms

$$
\begin{aligned}
& f_1 g_1 \frac{\partial h_1}{\partial x} + f_2 g_1 \frac{\partial h_2}{\partial x} + f_1 g_2 \frac{\partial h_1}{\partial y} + f_2 g_2 \frac{\partial h_2}{\partial y} \\
& - f_1 h_1 \frac{\partial g_1}{\partial x} - f_2 h_1 \frac{\partial g_2}{\partial x} - f_1 h_2 \frac{\partial g_1}{\partial y} - f_2 h_2 \frac{\partial g_2}{\partial y}.
\end{aligned}
$$

As we said we would do in our strategy, we will group these terms together and call them $-\alpha([v, w])$ resulting in

$$
\begin{aligned}
\alpha([v, w]) = \; & f_1 h_1 \frac{\partial g_1}{\partial x} + f_2 h_1 \frac{\partial g_2}{\partial x} + f_1 h_2 \frac{\partial g_1}{\partial y} + f_2 h_2 \frac{\partial g_2}{\partial y} \\
& - f_1 g_1 \frac{\partial h_1}{\partial x} - f_2 g_1 \frac{\partial h_2}{\partial x} - f_1 g_2 \frac{\partial h_1}{\partial y} - f_2 g_2 \frac{\partial h_2}{\partial y}.
\end{aligned}
$$

which in turn gives us the global formula

$$
d\alpha(v, w) = v[\alpha(w)] - w[\alpha(v)] - \alpha([v, w]).
$$

Of course, at this point the last term in this global formula seems like simply a matter of notational convenience; we have no idea what it actually represents. The fact of the matter is that the third term genuinely does represent something that is geometrically interesting and important, which we now turn to explaining.

The definition of the Lie bracket of two vector fields v and w, which is denoted by $[v, w]$, is defined in terms of how those vector fields act on a function $F : \mathbb{R}^2 \to \mathbb{R}$. Given a vector field w defined on the manifold $\mathbb{R}^2$ we can find the directional derivative of F in the direction of w. At a particular point $p \in \mathbb{R}^2$ we have that $w_p[F]$ is a value telling how quickly F is changing in the w direction at that point p. But if we do not specify a point p the $w[F]$ is another function on the manifold $\mathbb{R}^2$. That is,

$$
\begin{aligned}
w[F] : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\
p &\longmapsto w_p[F].
\end{aligned}
$$

We can again take the directional derivative of this function $w[F]$ in the direction of v. Again, at the point $p \in \mathbb{R}^2$ we have that $v_p\big[w[F]\big]$ is a value that tells how quickly $w[F]$ changes in the v direction at that point p. However, if we do not specify the point then $v\big[w[F]\big]$ is again another function on the manifold $\mathbb{R}^2$,

$$
\begin{aligned}
v\big[w[F]\big] : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\
p &\longmapsto v_p\big[w[F]\big].
\end{aligned}
$$

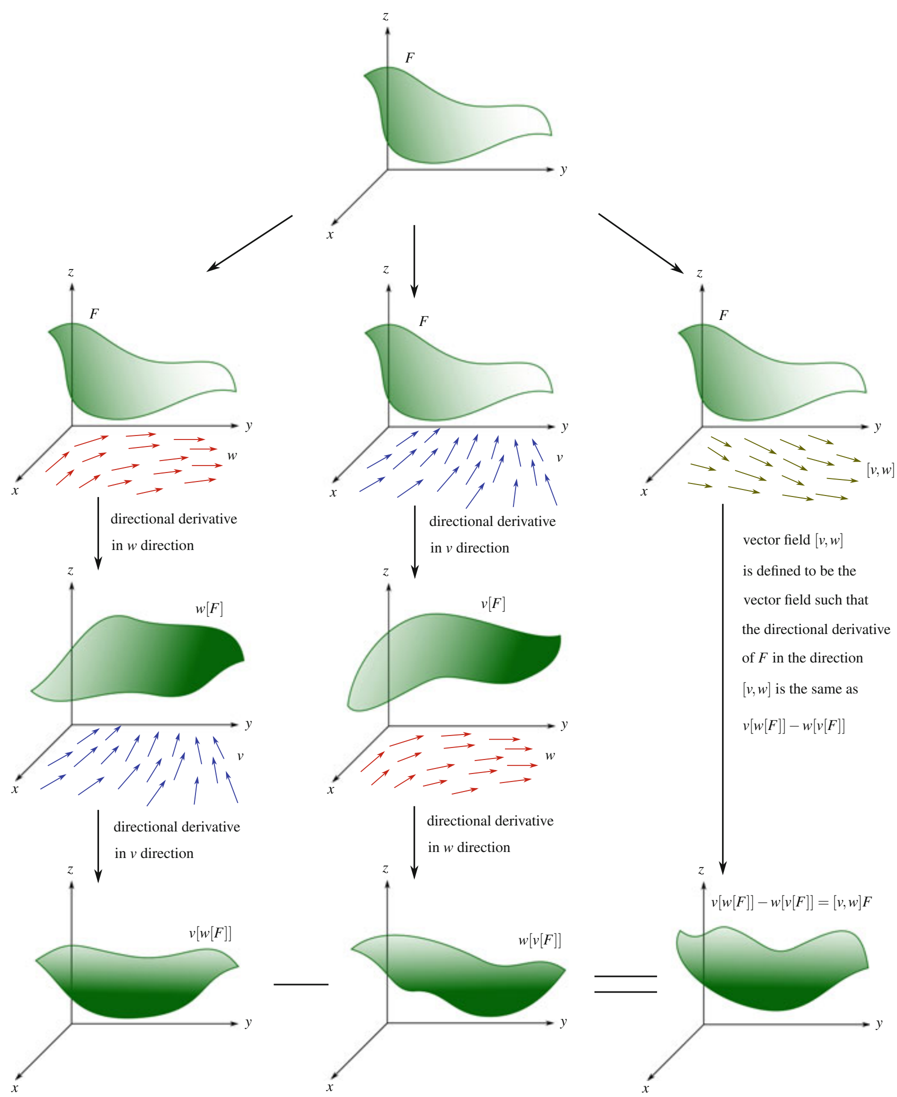

Fig. 4.3 On the left we show the directional derivative of $F : \mathbb{R}^2 \to \mathbb{R}$ taken in the direction of w to give the function $w[F]$, after which the directional derivative of $w[F]$ is taken in the direction of v to give the function $w\big[w[F]\big]$. Down the middle we show the directional derivative of F taken in the direction of v to give the function $v[F]$, after which the directional derivative of $v[F]$ is taken in the direction of w to give $w\big[v[F]\big]$. The difference $v\big[w[F]\big]-w\big[v[F]\big]$ of these two functions is taken to give a new function, shown along the bottom. The Lie bracket of v and w, denoted $[v, w]$, is the vector field which, when applied to the original function F, gives this new function. In other words, $[v, w]\big[F\big] = v\big[w[F]\big]-w\big[v[F]\big]$, as shown on the right

This process is all illustrated on the left side of Fig. 4.3. If we reverse the order of the vector fields and use v first we get the function $v[F] : \mathbb{R}^2 \to \mathbb{R}$. Applying w next we get the function $w\big[v[F]\big] : \mathbb{R}^2 \to \mathbb{R}$. This process is illustrated down the center of Fig. 4.3. Suppose we then took the difference of these two functions, as along the bottom of Fig. 4.3, then we would get a new function

$$v\big[w[F]\big] - w\big[v[F]\big] : \mathbb{R}^2 \longrightarrow \mathbb{R}.$$

It is in fact possible to find a single vector field that does this all in one step. In other words, we can find a single vector field which, when applied to F gives exactly this function $v\big[w[F]\big] - w\big[v[F]\big]$. That single vector field is called the Lie bracket of v and w and is denoted by $[v, w]$. Thus we have

$$[v, w]\big[F\big] = v\big[w[F]\big] - w\big[v[F]\big].$$

This is shown on the right side of Fig. 4.3. Despite the fact that the notation $[v, w]$ has both vector fields v and w in it, $[v, w]$ is simply a single vector field. Also notice that the square brackets are used in two different ways. However, by paying attention to what is inside the square brackets it should be easy to distinguish between the two uses. We will find $[v, w]$ computationally below, but we should note there is also another somewhat geometric way of viewing $[v, w]$. In fact, it turns out that $[v, w]$ is the Lie derivative of w in the direction v. The explanation for this is left until Sect. A.7 and Figs. A.5 and A.6.

Suppose we get rid of the function F in the notation to get

$$[v, w]\big[\,\cdot\,\big] = v\big[w[\cdot]\big] - w\big[v[\cdot]\big].$$

This explains the reason you will almost always see the Lie bracket of v and w somewhat lazily defined simply as

$$[v, w] = vw - wv.$$

Our goal now turns to finding out exactly what the vector field $[v, w]$ is,

$$\begin{aligned}
[v, w]F &= (vw - wv)F \\
&= v[w[F]] - w[v[F]] \\
&= v\left[h_1\frac{\partial F}{\partial x} + h_2\frac{\partial F}{\partial y}\right] - w\left[g_1\frac{\partial F}{\partial x} + g_2\frac{\partial F}{\partial y}\right] \\
&= (g_1\partial_x + g_2\partial_y)\left[h_1\frac{\partial F}{\partial x} + h_2\frac{\partial F}{\partial y}\right] - (h_1\partial_x + h_2\partial_y)\left[g_1\frac{\partial F}{\partial x} + g_2\frac{\partial F}{\partial y}\right] \\
&= g_1\frac{\partial}{\partial x}\left(h_1\frac{\partial F}{\partial x}\right) + g_1\frac{\partial}{\partial x}\left(h_2\frac{\partial F}{\partial y}\right) + g_2\frac{\partial}{\partial y}\left(h_1\frac{\partial F}{\partial x}\right) + g_2\frac{\partial}{\partial y}\left(h_2\frac{\partial F}{\partial y}\right) \\
&\quad - h_1\frac{\partial}{\partial x}\left(g_1\frac{\partial F}{\partial x}\right) - h_1\frac{\partial}{\partial x}\left(g_2\frac{\partial F}{\partial y}\right) - h_2\frac{\partial}{\partial y}\left(g_1\frac{\partial F}{\partial x}\right) - h_2\frac{\partial}{\partial y}\left(g_2\frac{\partial F}{\partial y}\right) \\
&\overset{\substack{\text{prod.}\\\text{rule}}}{=} g_1\frac{\partial h_1}{\partial x}\frac{\partial F}{\partial x} + \cancel{g_1h_1\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\right)} + g_1\frac{\partial h_2}{\partial x}\frac{\partial F}{\partial y} + \cancel{g_1h_2\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)} \\
&\quad + g_2\frac{\partial h_1}{\partial y}\frac{\partial F}{\partial x} + \xcancel{g_2h_1\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)} + g_2\frac{\partial h_2}{\partial y}\frac{\partial F}{\partial y} + \xcancel{g_2h_2\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y}\right)} \\
&\quad - h_1\frac{\partial g_1}{\partial x}\frac{\partial F}{\partial x} - \cancel{h_1g_1\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\right)} - h_1\frac{\partial g_2}{\partial x}\frac{\partial F}{\partial y} - \xcancel{h_1g_2\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)} \\
&\quad - h_2\frac{\partial g_1}{\partial y}\frac{\partial F}{\partial x} - \cancel{h_2g_1\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)} - h_2\frac{\partial g_2}{\partial y}\frac{\partial F}{\partial y} - \xcancel{h_2g_2\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y}\right)} \\
&= g_1\frac{\partial h_1}{\partial x}\frac{\partial F}{\partial x} + g_1\frac{\partial h_2}{\partial x}\frac{\partial F}{\partial y} + g_2\frac{\partial h_1}{\partial y}\frac{\partial F}{\partial x} + g_2\frac{\partial h_2}{\partial y}\frac{\partial F}{\partial y} \\
&\quad - h_1\frac{\partial g_1}{\partial x}\frac{\partial F}{\partial x} - h_1\frac{\partial g_2}{\partial x}\frac{\partial F}{\partial y} - h_2\frac{\partial g_1}{\partial y}\frac{\partial F}{\partial x} - h_2\frac{\partial g_2}{\partial y}\frac{\partial F}{\partial y} \\
&= \left(\left(g_1\frac{\partial h_1}{\partial x} + g_2\frac{\partial h_1}{\partial y} - h_1\frac{\partial g_1}{\partial x} - h_2\frac{\partial g_1}{\partial y}\right)\frac{\partial}{\partial x}\right. \\
&\quad \left. + \left(g_1\frac{\partial h_2}{\partial x} + g_2\frac{\partial h_2}{\partial y} - h_1\frac{\partial g_2}{\partial x} - h_2\frac{\partial g_2}{\partial y}\right)\frac{\partial}{\partial y}\right)[F]
\end{aligned}$$

where the various terms cancel due to the equality of mixed patrials. Thus we have found the form the vector field $[v, w]$ takes, it is given by

$$[v, w] = \left(g_1\frac{\partial h_1}{\partial x} + g_2\frac{\partial h_1}{\partial y} - h_1\frac{\partial g_1}{\partial x} - h_2\frac{\partial g_1}{\partial y}\right)\partial_x$$

$$+\left(g_1\frac{\partial h_2}{\partial x}+g_2\frac{\partial h_2}{\partial y}-h_1\frac{\partial g_2}{\partial x}-h_2\frac{\partial g_2}{\partial y}\right)\partial_y$$
$$=\begin{bmatrix} g_1\frac{\partial h_1}{\partial x}+g_2\frac{\partial h_1}{\partial y}-h_1\frac{\partial g_1}{\partial x}-h_2\frac{\partial g_1}{\partial y} \\ g_1\frac{\partial h_2}{\partial x}+g_2\frac{\partial h_2}{\partial y}-h_1\frac{\partial g_2}{\partial x}-h_2\frac{\partial g_2}{\partial y}\end{bmatrix}.$$

Let us now see what α of the vector field $[v, w]$ is. We have

$$\begin{aligned}\alpha([v,w]) &= \langle \alpha, [v,w]\rangle \\ &= [f_1, f_2]\begin{bmatrix} g_1\frac{\partial h_1}{\partial x}+g_2\frac{\partial h_1}{\partial y}-h_1\frac{\partial g_1}{\partial x}-h_2\frac{\partial g_1}{\partial y} \\ g_1\frac{\partial h_2}{\partial x}+g_2\frac{\partial h_2}{\partial y}-h_1\frac{\partial g_2}{\partial x}-h_2\frac{\partial g_2}{\partial y}\end{bmatrix} \\ &= f_1g_1\frac{\partial h_1}{\partial x}+f_1g_2\frac{\partial h_1}{\partial y}-f_1h_1\frac{\partial g_1}{\partial x}-f_1h_2\frac{\partial g_1}{\partial y} \\ &\quad + f_2g_1\frac{\partial h_2}{\partial x}+f_2g_2\frac{\partial h_2}{\partial y}-f_2h_1\frac{\partial g_2}{\partial x}-f_2h_2\frac{\partial g_2}{\partial y}.\end{aligned}$$

Here notice the notation, $[v, w]$ is the lie bracket of two vector fields, which is itself a vector field while $[f_1, f_2]$ is the co-vector, or row vector, associated with the differential one-form $\alpha = f_1dx + f_2dy$. The way you tell these two mathematically distinct objects apart is by paying close attention to what is inside the brackets, vector fields or functions. But notice what we have gotten, these eight terms are exactly the negative of the eight terms left over from $v[\alpha(w)] - w[\alpha(v)]$. Thus, we define the global formula for the exterior derivative of a one-form α on the manifold $\mathbb{R}^2$ to be $d\alpha(v, w) = v[\alpha(w)] - w[\alpha(v)] - \alpha([v, w])$.

Of course, so far we have only done this for a one-form on the manifold $\mathbb{R}^2$. What about general one-form on the manifold $\mathbb{R}^n$? We will walk through the admittedly complicated calculations in this case as well, in order to see everything works out as expected. We start with

$$\begin{aligned}\alpha &= \sum \alpha_i dx_i, \\ v &= \sum g_i \partial_{x_i}, \\ w &= \sum h_i \partial_{x_i}.\end{aligned}$$

We will begin by computing $[v, w]$. Using a real-valued function F on the manifold, we first find

$$\begin{aligned}v\big[w[F]\big] &= v\left[\sum_i h_i\frac{\partial F}{\partial x_i}\right] \\ &= \left(\sum_j g_j\partial_{x_j}\right)\left[\sum_i h_i\frac{\partial F}{\partial x_i}\right] \\ &= \sum_j g_j\partial_{x_j}\left(\sum_i h_i\frac{\partial F}{\partial x_i}\right) \\ &= \sum_j g_j\sum_i\left(\frac{\partial h_i}{\partial x_j}\cdot\frac{\partial F}{\partial x_i}+h_i\frac{\partial}{\partial x_j}\left(\frac{\partial F}{\partial x_i}\right)\right) \\ &= \sum_j\sum_i\left(g_j\frac{\partial h_i}{\partial x_j}\cdot\frac{\partial F}{\partial x_i}+g_jh_i\frac{\partial}{\partial x_j}\left(\frac{\partial F}{\partial x_i}\right)\right).\end{aligned}$$

Finding the next term is similar and we get

$$w\big[v[F]\big] = \sum_j\sum_i\left(h_j\frac{\partial g_i}{\partial x_j}\cdot\frac{\partial F}{\partial x_i}+h_jg_j\frac{\partial}{\partial x_j}\left(\frac{\partial F}{\partial x_i}\right)\right).$$

Putting these together we get

$$
\begin{aligned}
v\big[w[F]\big] - w\big[v[F]\big] &= \sum_j \sum_i \left(g_j \frac{\partial h_i}{\partial x_j} \cdot \frac{\partial F}{\partial x_i} + g_j h_i \frac{\partial}{\partial x_j}\left(\frac{\partial F}{\partial x_i}\right)\right) - \sum_j \sum_i \left(h_j \frac{\partial g_i}{\partial x_j} \cdot \frac{\partial F}{\partial x_i} + h_j g_j \frac{\partial}{\partial x_j}\left(\frac{\partial F}{\partial x_i}\right)\right) \\
&= \sum_j \sum_i \left(g_j \frac{\partial h_i}{\partial x_j} \cdot \frac{\partial F}{\partial x_i} - h_j \frac{\partial g_i}{\partial x_j} \cdot \frac{\partial F}{\partial x_i}\right) + \underbrace{\sum_i \sum_j g_j h_i \frac{\partial}{\partial x_j}\left(\frac{\partial F}{\partial x_i}\right) - \underbrace{\sum_j \sum_i g_i h_j \frac{\partial}{\partial x_j}\left(\frac{\partial F}{\partial x_i}\right)}_{\text{switch dummy variables}}}_{=0 \text{ by equality of mixed partials}} \\
&= \left(\sum_i \left(\sum_j \left(g_j \frac{\partial h_i}{\partial x_j} - h_j \frac{\partial g_i}{\partial x_j}\right)\right) \frac{\partial}{\partial x_i}\right) F.
\end{aligned}
$$

Taking away the function F we get the actual vector field

$$
[v, w] = \sum_i \left(\sum_j \left(g_j \frac{\partial h_i}{\partial x_j} - h_j \frac{\partial g_i}{\partial x_j}\right)\right) \partial_{x_i}.
$$

Finally, we find

$$
\begin{aligned}
\alpha([v, w]) &= \left(\sum_k \alpha_k dx_k \right) \left(\sum_i \left(\sum_j \left(g_j \frac{\partial h_i}{\partial x_j} - h_j \frac{\partial g_i}{\partial x_j}\right)\right) \partial_{x_i}\right) \\
&= \sum_i \alpha_i \left(\sum_j \left(g_j \frac{\partial h_i}{\partial x_j} - h_j \frac{\partial g_i}{\partial x_j}\right)\right).
\end{aligned}
$$

Now we turn our attention to the expression $v[\alpha(w)] - w[\alpha(v)] - \alpha([v, w])$. Since $v[\alpha(w)] = d(\alpha(v))(w) = \big\langle d(\alpha(v)), w\big\rangle = \big\langle d\langle \alpha, v\rangle, w\big\rangle$ we start off with

$$
\begin{aligned}
\langle \alpha, v\rangle &= \left(\sum_i \alpha_i dx_i \right) \left(\sum_j g_j \partial_{x_j} \right) \\
&= \sum_i \alpha_i g_i.
\end{aligned}
$$

Taking the directional derivative we get

$$
\begin{aligned}
d\langle \alpha, v\rangle &= d\left(\sum_i \alpha_i g_i \right) \\
&= \sum_j \frac{\partial \left(\sum_i \alpha_i g_i \right)}{\partial x_j} dx_j \\
&= \sum_j \left(\sum_i \left(\frac{\partial \alpha_i}{\partial x_j} g_i + \alpha_i \frac{\partial g_i}{\partial x_j}\right)\right) dx_j,
\end{aligned}
$$

which we can then use to find

$$
w[\alpha(v)] = \sum_j h_j \left(\sum_i \left(\frac{\partial \alpha_i}{\partial x_j} g_i + \alpha_i \frac{\partial g_i}{\partial x_j}\right)\right).
$$

Similarly we have

$$v[\alpha(w)] = \sum_j g_j \left(\sum_i \left(\frac{\partial \alpha_i}{\partial x_j} h_i + \alpha_i \frac{\partial h_i}{\partial x_j} \right) \right).$$

Next, we combine everything

$$\begin{aligned}
d\alpha(v, w) &= v[\alpha(w)] - w[\alpha(v)] - \alpha([v, w]) \\
&= \sum_j g_j \left(\sum_i \left(\frac{\partial \alpha_i}{\partial x_j} h_i + \alpha_i \frac{\partial h_i}{\partial x_j} \right) \right) - \sum_j h_j \left(\sum_i \left(\frac{\partial \alpha_i}{\partial x_j} g_i + \alpha_i \frac{\partial g_i}{\partial x_j} \right) \right) - \sum_i \alpha_i \left(\sum_j \left(g_j \frac{\partial h_i}{\partial x_j} - h_j \frac{\partial g_i}{\partial x_j} \right) \right) \\
&= \sum_{i,j} g_j h_i \frac{\partial \alpha_i}{\partial x_j} + \sum_{i,j} g_j \alpha_i \frac{\partial h_i}{\partial x_j} - \sum_{i,j} h_j g_i \frac{\partial \alpha_i}{\partial x_j} - \sum_{i,j} h_j \alpha_i \frac{\partial g_i}{\partial x_j} - \sum_{i,j} \alpha_i g_j \frac{\partial h_i}{\partial x_j} + \sum_{i,j} \alpha_i h_j \frac{\partial g_i}{\partial x_j} \\
&= \sum_{i,j} g_j h_i \frac{\partial \alpha_i}{\partial x_j} - \sum_{i,j} h_j g_i \frac{\partial \alpha_i}{\partial x_j} + \underbrace{\sum_{i,j} g_j \alpha_i \frac{\partial h_i}{\partial x_j} - \sum_{i,j} \alpha_i g_j \frac{\partial h_i}{\partial x_j}}_{=0} \underbrace{- \sum_{i,j} h_j \alpha_i \frac{\partial g_i}{\partial x_j} + \sum_{i,j} \alpha_i h_j \frac{\partial g_i}{\partial x_j}}_{=0} \\
&= \sum_{i,j} (g_j h_i - h_j g_i) \frac{\partial \alpha_i}{\partial x_j} \\
&= \sum_j \sum_i \frac{\partial \alpha_i}{\partial x_j} \begin{vmatrix} g_j & h_j \\ g_i & h_i \end{vmatrix} \\
&= \sum_j \sum_i \frac{\partial \alpha_i}{\partial x_j} \begin{vmatrix} dx_j(v) & dx_j(w) \\ dx_i(v) & dx_i(w) \end{vmatrix} \\
&= \sum_j \sum_i \frac{\partial \alpha_i}{\partial x_j} \, dx_j \wedge dx_i(v, w) \\
&= \sum_i \underbrace{\left(\sum_j \frac{\partial \alpha_i}{\partial x_j} dx_j \right)}_{=d\alpha_i} \wedge dx_i(v, w),
\end{aligned}$$

which is exactly the in-coordinates formula $d\alpha = \sum_i d\alpha_i \wedge dx_i$ that we want. Thus our global formula for a general one-form on the manifold $\mathbb{R}^n$ is actually what we want,

$$\boxed{\text{Global formula for exterior derivative of a one-form,} \quad d\alpha(v, w) = v[\alpha(w)] - w[\alpha(v)] - \alpha([v, w]).}$$

As in the case with constant vector fields, we will not make an attempt to provide a geometric argument for the global formula for the exterior derivative of a general k-form on the manifold $\mathbb{R}^n$. The process would be very computationally intensive and the geometry would not be at all clear. However, we now have all the components we need to have a general intuitive feel for why the formula is what it is. The exterior derivative of a k-form ω will be a $(k+1)$-form, which necessitates $k+1$ input vector fields $v_0, \ldots, v_k$. We start the labeling at 0 simply in order to have a nice clean $(-1)^{i+j}$ term,

$$\boxed{\text{Global formula for exterior derivative of } k\text{-form} \quad d\alpha(v_0, \ldots, v_k) = \sum_i (-1)^i v_i \big[\alpha(v_0, \ldots, \widehat{v_i}, \ldots, v_k)\big] + \sum_{i<j} (-1)^{i+j} \alpha\big([v_i, v_j], v_0, \ldots, \widehat{v_i}, \ldots, \widehat{v_j}, \ldots, v_k\big).}$$

As before, the hat in the notation means that those elements are omitted. This formula will actually be proved in Sect. A.7. The proof there will be entirely computational and require concepts and identities that will not be presented until that chapter.

Question 4.5 Show that in the case of a one-form the global formula for the exterior derivative of a k-form reduces to the global formula for the exterior derivative of a one-form.

4.5 Another Geometric Viewpoint

We begin this section by giving yet another definition for the exterior derivative of a k-form ω. We will then proceed to show that using this definition we arrive at exactly the same formula for $d\omega$ that we had in previous sections. Following that we will look at the geometric meaning of this definition. In many respects, this definition gives the most coherent and comprehensible geometric representation of the exterior derivative and it is a pity that more books do not emphasise this geometry. As mentioned in the introductory section, genuinely understanding this definition of the exterior derivative requires understanding integration of forms first. And generally integration is not introduced before differentiation. However, from a certain perspective differential forms can be viewed as "things one integrates." If differential forms are presented from this perspective integration would be discussed first.

Based on your understanding of integration from calculus you should be able to read this section to gain a general feel for the ideas and the underlying geometry. However, the various calculations rely on ideas, notations, and formulas developed and given in Chaps. 6–11 so you should not expect to understand the details. On a first read simply try to understand the big picture. We also highly recommend revisiting this section after Chap. 11.

We will motivate our new definition of the exterior derivative of a k-form ω by looking at the classical definition of the derivative of a function f. Recall that a function is nothing other than a zero-form. We have

$$\frac{df(x)}{dx} = \lim_{h \to 0} \frac{1}{h}\Big(f(x+h) - f(x) \Big).$$

We are evaluating f at the end points of the interval of length h, denoted by $hP = [x, x+h]$. Here the square brackets represent a closed interval on $\mathbb{R}$. A closed interval is an interval on $\mathbb{R}$ that includes the endpoints. For reasons explained in great detail in chapter 10 the boundary of the interval hP, which is denoted by $\partial(hP)$, is the point $\{(x+h)\}$ minus the point $\{(x)\}$. That is, $\partial(hP) = \partial[x, x+h] = \{(x+h)\} - \{(x)\}$. While this is something that we generally are not interested in calculus, and therefore generally is not done, we can define the integral of a function at a single point to be the function evaluated at that point. Thus

$$\int_{\{(x+h)\}} f \equiv f(x+h) \qquad \text{and} \qquad \int_{-\{(x)\}} f \equiv -f(x).$$

Notice what happens to the negative sign from $-\{(x)\}$. This allows us to rewrite our definition of the derivative of f as

$$\begin{aligned}
\frac{df(x)}{dx} &= \lim_{h \to 0} \frac{1}{h}\Big(f(x+h) - f(x) \Big) \\
&= \lim_{h \to 0} \frac{1}{h}\left(\int_{\{(x+h)\}} f + \int_{-\{(x)\}} f \right) \\
&= \lim_{h \to 0} \frac{1}{h} \int_{\{(x+h)\} - \{(x)\}} f \\
&= \lim_{h \to 0} \frac{1}{h} \int_{\partial(hP)} f.
\end{aligned}$$

Thus we have written the derivative of the zero-form f in terms of the integral of f evaluated at the boundary of the one-dimensional parallelepiped $[x, x+h]$. In calculus it would be absurd to use the formula that follows the final equality as a definition for the derivative of f, but for differential forms this definition leads to a clear understanding of what the exterior derivative means geometrically thus it makes sense to take this formula, applied to k-forms, as a definition. Thus we define for ω a k-form on the manifold $\mathbb{R}^n$, the exterior derivative of ω at the base point of the vectors $v_1, \dots, v_{k+1}$ to be

Exterior derivative of a k-form
$$\boxed{d\omega\big(v_1, \dots, v_{k+1}\big) = \lim_{h \to 0} \frac{1}{h^{k+1}} \int_{\partial(hP)} \omega}$$

where P is the parallelepiped spanned by the vectors $v_1, \dots, v_{k+1}$, hP is that parallelepiped scaled by h, and $\partial(hP)$ is the boundary of the scaled parallelepiped. Again, the meaning of the boundary of a parallelepiped is explained in great detail in chapter 10.

Before exploring the geometric meaning of this definition we will proceed to show that this definition leads to exactly the same formula for $d\omega$ that we would expect from earlier sections. In showing this we are forced to use a number of concepts, notations, and formulas that are actually derived in later chapters. We also have to make use of a few ideas and techniques that are not in the purview of the book and which we therefore present without explanation or a great deal of rigor. The Taylor series of a function is one of these ideas. In the below we will simply assume that the Taylor series converges to the given function f in some neighborhood of the point a.

Suppose $f : \mathbb{R}^n \to \mathbb{R}$. The Taylor series of f about the point $a = (a_1, \dots, a_n)$ is given by

$$\begin{aligned}
f(x_1, \dots, x_n) &= \sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} \frac{(x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n}}{i_1! \cdots i_n!} \left(\frac{\partial^{i_1+\cdots+i_n} f}{\partial^{i_1} x_1 \cdots \partial^{i_n} x_n} \right) \Bigg|_{(a_1,\dots,a_n)} \\
&= f(a_1, \dots, a_n) + \sum_{j=1}^{n} (x_j - a_j) \left. \frac{\partial f}{\partial x_j} \right|_{(a_1,\dots,a_n)} \\
&\quad + \frac{1}{2!} \sum_{j=0}^{n} \sum_{k=0}^{n} (x_j - a_j)(x_k - a_k) \left. \frac{\partial^2 f}{\partial x_j \partial x_k} \right|_{(a_1,\dots,a_n)} \\
&\quad + \frac{1}{3!} \sum_{j=0}^{n} \sum_{k=0}^{n} \sum_{\ell=0}^{n} (x_j - a_j)(x_k - a_k)(x_\ell - a_\ell) \left. \frac{\partial^3 f}{\partial x_j \partial x_k \partial x_\ell} \right|_{(a_1,\dots,a_n)} \\
&\quad + \cdots
\end{aligned}$$

In essence, for points $(x_1, \dots, x_n)$ that are sufficiently close to the point $(a_1, \dots, a_n)$ we can rewrite f as the above sum.

Question 4.6 Show that these two representations of the Taylor series of f are identical.

In finding the formula for $d\omega$ we will make two simplifying adjustments. First, we will write the Taylor series for f at the origin, hence we have $(a_1, \dots, a_n) = (0, \dots, 0)$. Second, we will group all of the second order terms and higher into a remainder function which we will simply write as R. Hence our Taylor series can be rewritten as

$$f(x_1, \dots, x_n) = f(0, \dots, 0) + \sum_{j=1}^{n} x_j \left. \frac{\partial f}{\partial x_j} \right|_{(0,\dots,0)} + R(x_1, \dots, x_n).$$

Next, we will identify the manifold $\mathbb{R}^n$ with the vector space $\mathbb{R}^n$. This will mean that the following argument will only apply to manifolds that are also vector spaces, though it is possible, with the appropriate technical details that we will not concern ourselves with here, to extend the results to general n-dimensional manifolds. Supplying these details will take us too far away from the big picture. Using our identification we have

$$\text{Manifold } \mathbb{R}^n \longleftrightarrow \text{Vector Space } \mathbb{R}^n$$

$$p = (x_1, \dots, x_n) \Longleftrightarrow v = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Notice that the second term of the Taylor series can be written as

$$\sum_{j=1}^{n} x_j \left.\frac{\partial f}{\partial x_j}\right|_{(0,\ldots,0)} = \underbrace{\left[\left.\frac{\partial f}{\partial x_1}\right|_{(0,\ldots,0)}, \cdots, \left.\frac{\partial f}{\partial x_n}\right|_{(0,\ldots,0)} \right]}_{\text{co-vector } df_0} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = df_0(v),$$

where df is the usual differential of f and df_0 is the differential of f taken at the origin $(0, \ldots, 0) = 0$. With this we can write the Taylor series at the "point" $(x_1, \ldots, x_n) = v$ as

$$f(v) = f(0) + df_0(v) + R(v).$$

Now, suppose we are given the vectors $v_1, \ldots, v_{k+1}$. We will label the parallelepiped spanned by these vectors as P. Here the word span is being used differently than in linear algebra. Here we have

$$P = \operatorname{span}\Big\{v_1, \ldots, v_{k+1}\Big\} = \Big\{ t_1 v_1 + \cdots + t_{k+1} v_{k+1} \ \Big|\ 0 \le t_i \le 1, i = 1, \ldots, k+1 \Big\}.$$

We are still identifying the manifold $\mathbb{R}^n$ with the vector space $\mathbb{R}^n$, thus the vector $t_1 v_1 + \cdots + t_{k+1} v_{k+1}$ really means the point given by the vector's coordinates as per the identity above. We can scale this parallelepiped by the factor h to get a new parallelepiped

$$hP = \operatorname{span}\Big\{hv_1, \ldots, hv_{k+1}\Big\} = \Big\{ t_1 v_1 + \cdots + t_{k+1} v_{k+1} \ \Big|\ 0 \le t_i \le h, i = 1, \ldots, k+1 \Big\}.$$

The various faces of hP are given by

$$hP_{(\ell,0)} = \Big\{ t_1 v_1 + \cdots + t_{\ell-1} v_{\ell-1} + 0 v_\ell + t_\ell v_{\ell+1} + \cdots + t_k v_{k+1} \ \Big|\ 0 \le t_i \le h, i = 1, \ldots, k \Big\}$$

or

$$hP_{(\ell,1)} = \Big\{ t_1 v_1 + \cdots + t_{\ell-1} v_{\ell-1} + h v_\ell + t_\ell v_{\ell+1} + \cdots + t_k v_{k+1} \ \Big|\ 0 \le t_i \le h, i = 1, \ldots, k \Big\}$$

where $\ell = 1, \ldots, k+1$. Notice the labeling of the ts. In the faces of hP we now have t_1 through t_k. Continuing to write points as vectors, points of the faces $hP_{(\ell,1)}$ and $hP_{(\ell,0)}$ can be written as

$$\begin{aligned} v_{(\ell,1)} &= hv_\ell + t_1 v_1 + \cdots + t_{\ell-1} v_{\ell-1} + v_\ell v_{\ell+1} + \cdots + t_k v_{k+1} \\ \text{and} \quad v_{(\ell,0)} &= \qquad t_1 v_1 + \cdots + t_{\ell-1} v_{\ell-1} + t_\ell v_{\ell+1} + \cdots + t_k v_{k+1} \end{aligned}$$

respectively. Defining the scaled k-cube hI^k to be

$$hI^k = \Big\{ (t_1, \ldots, t_k) \ \Big|\ 0 \le t_i \le h, i = 1, \ldots, k \Big\},$$

as long as we know what the vectors $v_1, \ldots, v_{k+1}$ are, for each ℓ and any value of h we have a natural identification between each of the faces $hP_{(\ell,1)}$ and $hP_{(\ell,0)}$ to hI^k,

$$\begin{aligned} hP_{(\ell,1)} &\longleftrightarrow hI^k \\ v_{(\ell,1)} &\longleftrightarrow (t_1, \ldots, t_k) \end{aligned}$$

and

$$\begin{aligned} hP_{(\ell,0)} &\longleftrightarrow hI^k \\ v_{(\ell,0)} &\longleftrightarrow (t_1, \ldots, t_k). \end{aligned}$$

We are now going to focus in on the $hP_{(\ell,1)} \to hI^k$ and examine it closely. What we do in this particular case also holds for $hP_{(\ell,0)}$ and for all ℓ. We begin by naming the mapping Φ, thus we have

$$\begin{aligned} hP_{(\ell,1)} &\xrightarrow{\Phi} hI^k \\ v_{(\ell,1)} &\xmapsto{\Phi} (t_1, \ldots, t_k). \end{aligned}$$

Obviously Φ is invertible so we also have

$$\begin{aligned} hP_{(\ell,1)} &\xleftarrow{\Phi^{-1}} hI^k \\ v_{(\ell,1)} &\xleftarrow{\Phi^{-1}} (t_1, \ldots, t_k). \end{aligned}$$

If we are given a point $t = (t_1, \ldots, t_k) \in hI^k$ then we can write $\Phi^{-1}(t) = v_{(\ell,1)}(t)$. In other words, the point $v_{(\ell,1)}$ is written as a function of the point t.

Given the mapping Φ we also have the tangent mapping, also called the push-forward, $T\Phi$. When we are considering the tangent mapping then we are actually viewing the elements $v_{(\ell,1)}$ as vectors. The associated tangent mapping of Φ is

$$\begin{aligned} T(hP_{(\ell,1)}) &\xrightarrow{T\Phi} T(hI^k) \\ v_{(\ell,1)} &\longmapsto T\Phi \cdot v_{(\ell,1)} = \begin{bmatrix} t_1 \\ \vdots \\ t_k \end{bmatrix}. \end{aligned}$$

In particular we are interested in what happens to the original vectors $v_1, \ldots, v_{k+1}$ that were used to define the parallelepiped P,

$$\underbrace{v_m}_{m \neq \ell} \longmapsto T\Phi \cdot v_m = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{bmatrix}.$$

For $T\Phi \cdot v_m$ we have the 1 in the mth slot if $m < \ell$ and the $(m-1)$th slot if $m > \ell$.

Question 4.7 Find $T\Phi^{-1} \cdot e_i$, $i = 1, \ldots, k$, where e_i are the standard Euclidian vectors.

We also have the cotangent map, also call the pull-back, of Φ^{-1},

$$\begin{aligned} \bigwedge^k \left(hP_{(\ell,1)}\right) &\xrightarrow{T^*\Phi^{-1}} \bigwedge^k \left(hI^k\right) \\ \alpha &\longmapsto T^*\Phi^{-1} \cdot \alpha. \end{aligned}$$

In particular, we are interested in the k-forms on $hP_{(\ell,1)}$. Note, $hP_{(\ell,1)}$ is a k-dimensional submanifold of $\mathbb{R}^{k+1}$. Since $\mathbb{R}^{k+1}$ already has the Cartesian coordinate functions $x_1, \ldots, x_{k+1}$ we will simply use these as the coordinate functions of the embedded submanifold $hP_{(\ell,1)}$. This may seem slightly odd since the manifold $hP_{(\ell,1)}$ is k-dimensional yet we are using $k+1$ coordinate functions on it. But this is the simplest approach since the manifold in question is simply a parallelepiped embedded in Euclidian space, which we understand quite well. Also, we will use the $t_1, \ldots, t_k$ as the coordinate functions on hI^k. In essence the t_is are the Cartesian coordinate functions on $\mathbb{R}^k$, which hI^k is embedded in. We begin by looking at k-form basis elements on $hP_{(\ell,1)}$,

$$dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Since both $hP_{(\ell,1)}$ and hI^k are k-dimensional then we know that $T^*\Phi^{-1} \cdot \big(dx_{i_1} \wedge \cdots \wedge dx_{i_k}\big)$ must be of the form $F dt_1 \wedge \cdots \wedge dt_k$. Notice that the dimension of $\bigwedge^k \big(hI^k\big)$ is 1 so $dt_1 \wedge \cdots \wedge dt_k$ is the basis for this space and is hence the volume form on hI^k. We want to find F.

$$
\begin{aligned}
F &= F dt_1 \wedge \cdots \wedge dt_k(e_1, \ldots, e_k) \\
&= T^*\Phi^{-1} \cdot \big(dx_{i_1} \wedge \cdots \wedge dx_{i_k}\big)(e_1, \ldots, e_k) \\
&= dx_{i_1} \wedge \cdots \wedge dx_{i_k} \left(\underbrace{T\Phi^{-1} \cdot e_1}_{v_1}, \ldots, \underbrace{T\Phi^{-1} \cdot e_{\ell-1}}_{v_{\ell-1}}, \underbrace{T\Phi^{-1} \cdot e_\ell}_{v_{\ell+1}}, \ldots, \underbrace{T\Phi^{-1} \cdot e_1}_{v_{k+1}} \right) \\
&= dx_{i_1} \wedge \cdots \wedge dx_{i_k} \left(v_1, \ldots, \widehat{v_\ell}, \ldots, v_{k+1}\right).
\end{aligned}
$$

In other words, we have that $F = dx_{i_1} \wedge \cdots \wedge dx_{i_k} \left(v_1, \ldots, \widehat{v_\ell}, \ldots, v_{k+1}\right)$ and hence that

$$
T^*\Phi^{-1} \cdot \big(dx_{i_1} \wedge \cdots \wedge dx_{i_k}\big) = dx_{i_1} \wedge \cdots \wedge dx_{i_k} \left(v_1, \ldots, \widehat{v_\ell}, \ldots, v_{k+1}\right) dt_1 \wedge \cdots \wedge dt_k.
$$

In Chap. 7 we develop the following formula for integration that involves a change of variables,

$$
\int_R f(x_1, \ldots, x_n)\, dx_1 \wedge \cdots \wedge dx_n = \int_{\Phi(R)} f \circ \Phi^{-1}(\Phi_1, \ldots, \Phi_n)\, T^*\Phi^{-1} \cdot \big(dx_1 \wedge \cdots \wedge dx_n\big).
$$

We will now consider how this formula applies to the case where the region being integrated over is $hP_{(\ell,1)}$ and the change of variable mapping is $\Phi : hP_{(\ell,1)} \to hI^k$. Writing the point $(x_1, \ldots, x_k)$ as $v_{(\ell,1)}$ and the point $(t_1, \ldots, t_k)$ as t, recalling that $\Phi^{-1}(t) = v_{(\ell,1)}(t)$, and writing dx^I for $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, we have

$$
\begin{aligned}
&\int_{hP_{(\ell,1)}} f(v_{(\ell,1)})\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
&= \int_{\Phi(hP_{(\ell,1)})} f \circ \Phi^{-1}(t)\, T^*\Phi^{-1} \cdot \big(dx_{i_1} \wedge \cdots \wedge dx_{i_k}\big) \\
&= \int_{hI^k} f\big(v_{(\ell,1)}(t)\big)\, dx^I(v_1, \ldots, \widehat{v_\ell}, \ldots, v_{k+1})\, dt_1 \wedge \cdots \wedge dt_k \\
&= \underbrace{\int_0^1 \cdots \int_0^1}_{k \text{ times}} f\big(v_{(\ell,1)}(t)\big)\, dx^I(v_1, \ldots, \widehat{v_\ell}, \ldots, v_{k+1})\, \underbrace{dt \cdots dt}_{k \text{ times}}.
\end{aligned}
$$

The computation for the region $hP_{(\ell,0)}$ and for all other values of ℓ are exactly analogous.

Now we have all the ingredients in place to consider the definition of the exterior derivative of ω that was given at the beginning of this section. For ω a k-form on the manifold $\mathbb{R}^n$, the exterior derivative of ω at the base point of the vectors $v_1, \ldots, v_{k+1}$ is defined to be

$$
d\omega\big(v_1, \ldots, v_{k+1}\big) = \lim_{h \to 0} \frac{1}{h^{k+1}} \int_{\partial(hP)} \omega.
$$

To simplify our computations we have already assumed this base point is the origin of the manifold $\mathbb{R}^n$ when we discussed the Taylor series expansion. If the point we were interested in were not the origin we would have to translate the k-form so the point we were interested in coincided with the origin, do the necessary computation, and then translate back. This would add a layer of complexity to the argument we are about to present, but would not fundamentally change anything. In essence we are simplifying our argument by only looking at a single special case which can, with a little extra work, be extended to the general case. The phrase that one often uses in mathematics in situations like this is *without loss of generality*. Thus we will say without loss of generality assume the base point is the origin.

Consider the unit $(k+1)$-cube I^{k+1}. In the Chap. 11 we find the boundary I^{k+1} to be

$$\partial I^{k+1} = \sum_{i=1}^{k+1}\sum_{a=0}^{1}(-1)^{i+a} I^{k+1}_{(i,a)}$$

where $I^{k+1}_{(i,a)}$ is the face of I^{k+1} obtained by holding the ith variable fixed at a, which is either 0 or 1. The $(-1)^{i+a}$ indicate the orientation of that face. This way the orientations of faces opposite each other are opposite. The boundary of any parallelepiped can be found similarly, thus we have

$$\begin{aligned}\partial(hP) &= \sum_{\ell=1}^{k+1}\sum_{a=0}^{1}(-1)^{\ell+a} hP_{(\ell,a)} \\ &= \sum_{\ell=1}^{k+1}\Big((-1)^{\ell} hP_{(\ell,0)} + (-1)^{\ell+1} hP_{(\ell,1)}\Big).\end{aligned}$$

Finally, recall the Taylor series for a function f about the origin. For points v close to the origin we have $f(v) = f(0) + df_0(v) + R(v)$. Now suppose we were given the k-form $\omega = f dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ then the integral on the right hand side of our definition of the exterior derivative of ω can be written as

$$\begin{aligned}\int_{\partial(hP)} \omega &= \int_{\partial(hP)} f\ dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &= \int_{\sum_{\ell=1}^{k+1}((-1)^{\ell} hP_{(\ell,0)} + (-1)^{\ell+1} hP_{(\ell,1)})} f\ \underbrace{dx_{i_1} \wedge \cdots \wedge dx_{i_k}}_{dx^I} \\ &= \sum_{\ell=1}^{k+1}\left((-1)^{\ell}\int_{hP_{(\ell,0)}} f\ dx^I + (-1)^{\ell+1}\int_{hP_{(\ell,1)}} f\ dx^I\right) \\ &= \sum_{\ell=1}^{k+1}(-1)^{\ell-1}\left(\int_{hP_{(\ell,1)}} f\ dx^I - \int_{hP_{(\ell,0)}} f\ dx^I\right) \\ &= \sum_{\ell=1}^{k+1}(-1)^{\ell-1}\left(\int_{hP_{(\ell,1)}} \Big(f(0) + df_0(v) + R(v)\Big)\, dx^I - \int_{hP_{(\ell,0)}} \Big(f(0) + df_0(v) + R(v)\Big)\, dx^I\right) \\ &= \sum_{\ell=1}^{k+1}(-1)^{\ell-1}\left(\int_{hP_{(\ell,1)}} f(0)\ dx^I - \int_{hP_{(\ell,0)}} f(0)\ dx^I \right. \qquad (1) \\ &\qquad + \int_{hP_{(\ell,1)}} df_0(v)\ dx^I - \int_{hP_{(\ell,0)}} df_0(v)\ dx^I \qquad (2) \\ &\qquad \left. + \int_{hP_{(\ell,1)}} R(v)\ dx^I - \int_{hP_{(\ell,0)}} R(v)\ dx^I \right). \qquad (3)\end{aligned}$$

Consider the first and second terms that appear in (1). For each $\ell = 1, \ldots, k+1$ we have

$$\begin{aligned}&\int_{hP_{(\ell,1)}} f(0)\ dx^I - \int_{hP_{(\ell,0)}} f(0)\ dx^I \\ &\quad = \int_{hI^k} f(0) dx^I(v_1, \ldots, \widehat{v_\ell}, \ldots, v_{k+1})\ dt_1 \wedge \cdots \wedge dt_k \\ &\qquad - \int_{hI^k} f(0) dx^I(v_1, \ldots, \widehat{v_\ell}, \ldots, v_{k+1})\ dt_1 \wedge \cdots \wedge dt_k\end{aligned}$$

$$= \int_{hI^k} \underbrace{\Big(\underbrace{f(0)}_{\text{a constant}} - \underbrace{f(0)}_{\text{the same constant}} \Big)}_{=0} dx^I(v_1, \ldots, \widehat{v_\ell}, \ldots, v_{k+1})\, dt_1 \wedge \cdots \wedge dt_k$$

$$= 0.$$

Now consider the third and fourth terms that appear in (2). For each $\ell = 1, \ldots, k+1$ we have

$$\begin{aligned}
&\int_{hP_{(\ell,1)}} df_0(v)\, dx^I - \int_{hP_{(\ell,0)}} df_0(v)\, dx^I \\
&\quad = \int_{hI^k} df_0\big(v_{(\ell,1)}(t)\big) dx^I(v_1, \ldots, \widehat{v_\ell}, \ldots, v_{k+1})\, dt_1 \wedge \cdots \wedge dt_k \\
&\qquad - \int_{hI^k} df_0\big(v_{(\ell,0)}(t)\big) dx^I(v_1, \ldots, \widehat{v_\ell}, \ldots, v_{k+1})\, dt_1 \wedge \cdots \wedge dt_k \\
&\quad = \int_{hI^k} \Big(df_0\big(v_{(\ell,1)}(t)\big) - df_0\big(v_{(\ell,0)}(t)\big) \Big) dx^I(v_1, \ldots, \widehat{v_\ell}, \ldots, v_{k+1})\, dt_1 \wedge \cdots \wedge dt_k \\
&\quad = \int_{hI^k} \Big(df_0\big(hv_\ell + v_{(\ell,0)}(t)\big) - df_0\big(v_{(\ell,0)}(t)\big) \Big) dx^I(v_1, \ldots, \widehat{v_\ell}, \ldots, v_{k+1})\, dt_1 \wedge \cdots \wedge dt_k \\
&\quad = \int_{hI^k} \Big(df_0\big(hv_\ell\big) + \underbrace{df_0\big(v_{(\ell,0)}(t)\big) - df_0\big(v_{(\ell,0)}(t)\big)}_{=0} \Big) dx^I(v_1, \ldots, \widehat{v_\ell}, \ldots, v_{k+1})\, dt_1 \wedge \cdots \wedge dt_k \\
&\quad = \int_{hI^k} \underbrace{df_0\big(hv_\ell\big)\, dx^I(v_1, \ldots, \widehat{v_\ell}, \ldots, v_{k+1})}_{\text{does not depend on } t \in hI^k} dt_1 \wedge \cdots \wedge dt_k \\
&\quad = df_0\big(hv_\ell\big)\, dx^I(v_1, \ldots, \widehat{v_\ell}, \ldots, v_{k+1}) \int_{hI^k} dt_1 \wedge \cdots \wedge dt_k \\
&\quad = df_0\big(hv_\ell\big)\, dx^I(v_1, \ldots, \widehat{v_\ell}, \ldots, v_{k+1}) \underbrace{\int_0^h \cdots \int_0^h}_{k \text{ times}} \underbrace{dt \cdots dt}_{k \text{ times}} \\
&\quad = h\, df_0\big(v_\ell\big)\, dx^I(v_1, \ldots, \widehat{v_\ell}, \ldots, v_{k+1}) \cdot h^k \\
&\quad = h^{k+1} df_0\big(v_\ell\big)\, dx^I(v_1, \ldots, \widehat{v_\ell}, \ldots, v_{k+1}).
\end{aligned}$$

And now we consider the fifth and six terms that appear in (3),

$$\begin{aligned}
&\int_{hP_{(\ell,1)}} R(v)\, dx^I - \int_{hP_{(\ell,0)}} R(v)\, dx^I \\
&\qquad = \int_{hI^k} R\big(v_{(\ell,1)}(t)\big)\, dx^I(v_1, \ldots, \widehat{v_\ell}, \ldots, v_{k+1})\, dt_1 \wedge \cdots \wedge dt_k \\
&\qquad\quad - \int_{hI^k} R\big(v_{(\ell,0)}(t)\big)\, dx^I(v_1, \ldots, \widehat{v_\ell}, \ldots, v_{k+1})\, dt_1 \wedge \cdots \wedge dt_k \\
&\quad = \int_{hI^k} \Big(R\big(v_{(\ell,1)}(t)\big) - R\big(v_{(\ell,0)}(t)\big) \Big)\, dx^I(v_1, \ldots, \widehat{v_\ell}, \ldots, v_{k+1})\, dt_1 \wedge \cdots \wedge dt_k.
\end{aligned}$$

We will only sketch the next part of the argument in enough detail to give you an intuitive understanding. To fill in the details of the argument would require some background in analysis, something we are not assuming in this book. Consider what the remainder term actually is in this case,

$$\begin{aligned}
R(x_1, \ldots, x_n) &= \frac{1}{2!} \sum_{j=0}^{n} \sum_{k=0}^{n} (x_j)(x_k) \left. \frac{\partial^2 f}{\partial x_j \partial x_k} \right|_{(0,\ldots,0)} \\
&\quad + \frac{1}{3!} \sum_{j=0}^{n} \sum_{k=0}^{n} \sum_{m=0}^{n} (x_j)(x_k)(x_m) \left. \frac{\partial^3 f}{\partial x_j \partial x_k \partial x_m} \right|_{(0,\ldots,0)} \\
&\quad + \cdots .
\end{aligned}$$

We are writing the point $(x_1, \ldots, x_n)$ as a vector, which is either

$$\begin{aligned}
& v_{(\ell,1)} = h v_\ell + t_1 v_1 + \cdots + t_{\ell-1} v_{\ell-1} + v_\ell v_{\ell+1} + \cdots + t_k v_{k+1} \\
\text{or} \quad & v_{(\ell,0)} = \phantom{h v_\ell +{}} t_1 v_1 + \cdots + t_{\ell-1} v_{\ell-1} + t_\ell v_{\ell+1} + \cdots + t_k v_{k+1}.
\end{aligned}$$

There are $k+1$ vectors each consisting of n components. Take the absolute value of these $n(k+1)$ components and find the maximum value and call it M. The absolute value of each component in the vectors $v_{(\ell,1)}$ or $v_{(\ell,0)}$ is less than $h(k+1)M$, so we have

$$\begin{aligned}
\left| R(x_1, \ldots, x_n) \right| &\le \frac{1}{2!} \sum_{j=0}^{n} \sum_{k=0}^{n} \left| \big(h(k+1)M\big)\big(h(k+1)M\big) \left. \frac{\partial^2 f}{\partial x_j \partial x_k} \right|_{(0,\ldots,0)} \right| \\
&\quad + \frac{1}{3!} \sum_{j=0}^{n} \sum_{k=0}^{n} \sum_{m=0}^{n} \left| \big(h(k+1)M\big)\big(h(k+1)M\big)\big(h(k+1)M\big) \left. \frac{\partial^3 f}{\partial x_j \partial x_k \partial x_m} \right|_{(0,\ldots,0)} \right| \\
&\quad + \cdots .
\end{aligned}$$

Notice that we can pull out at least an h^2 from every term in this sum. It requires some effort to prove, which we will not go to, but there exists some number K such that

$$\left| R(x_1, \ldots, x_n) \right| \le h^2 K.$$

Using the triangle inequality we have

$$\begin{aligned}
\left| R\big(v_{(\ell,1)}(t)\big) - R\big(v_{(\ell,0)}(t)\big) \right| &\le \left| R\big(v_{(\ell,1)}(t)\big) \right| + \left| R\big(v_{(\ell,0)}(t)\big) \right| \\
&\le h^2 K + h^2 K \\
&= 2h^2 K.
\end{aligned}$$

Similarly, for some value of C we have

$$\begin{aligned}
\left| dx^I(v_1, \ldots, \widehat{v_\ell}, \ldots, v_{k+1}) \right| &= \left| \sum_{\sigma \in S_k} sgn(\sigma) \prod_{\substack{j=1 \\ j \ne \ell}}^{k+1} dx_{\sigma(i_j)}(v_j) \right| \\
&\le \sum_{\sigma \in S_k} \prod_{\substack{j=1 \\ j \ne \ell}}^{k+1} \left| dx_{\sigma(i_j)}(v_j) \right| \\
&\le C.
\end{aligned}$$

Putting this together

$$
\begin{aligned}
&\left| \int_{hI^k} \Big(R\big(v_{(\ell,1)}(t)\big) - R\big(v_{(\ell,0)}(t)\big) \Big)\, dx^I(v_1, \ldots, \widehat{v_\ell}, \ldots, v_{k+1})\, dt_1 \wedge \cdots \wedge dt_k \right| \\
&\quad \le \int_{hI^k} \big(2h^2 K\big) C \; dt_1 \wedge \cdots \wedge dt_k \\
&\quad = h^2\big(2KC\big) \underbrace{\int_0^h \cdots \int_0^h}_{k \text{ times}} \underbrace{dt \cdots dt}_{k \text{ times}} \\
&\quad = h^{k+2}\big(2KC\big).
\end{aligned}
$$

Returning to our original calculation where we had left off,

$$
\begin{aligned}
\int_{\partial(hP)} \omega = \sum_{\ell=1}^{k+1} (-1)^{\ell-1} \bigg(&\int_{hP_{(\ell,1)}} f(0)\, dx^I - \int_{hP_{(\ell,0)}} f(0)\, dx^I \\
&+ \int_{hP_{(\ell,1)}} df_0(v)\, dx^I - \int_{hP_{(\ell,0)}} df_0(v)\, dx^I \\
&+ \int_{hP_{(\ell,1)}} R(v)\, dx^I - \int_{hP_{(\ell,0)}} R(v)\, dx^I \bigg) \\
\text{“}=\text{''} \sum_{\ell=1}^{k+1} (-1)^{\ell-1} \bigg(& \; 0 \\
&+ h^{k+1} df_0\big(v_\ell\big)\, dx^I(v_1, \ldots, \widehat{v_\ell}, \ldots, v_{k+1}) \\
&+ h^{k+2}\big(2KC\big) \bigg).
\end{aligned}
$$

Technically, the last equality is not an equality because of how we obtained the $h^{k+2}\big(2KC\big)$ term, but as we will see in a moment this does not matter. Using this in our definition of $d\omega$ we have

$$
\begin{aligned}
&d\omega\big(v_1, \ldots, v_{k+1}\big) \\
&\quad = \lim_{h \to 0} \frac{1}{h^{k+1}} \int_{\partial(hP)} \omega \\
&\quad = \lim_{h \to 0} \frac{1}{h^{k+1}} \left(\sum_{\ell=1}^{k+1} (-1)^{\ell-1} \Big(0 + h^{k+1} df_0\big(v_\ell\big)\, dx^I(v_1, \ldots, \widehat{v_\ell}, \ldots, v_{k+1}) + h^{k+2}\big(2KC\big) \Big) \right) \\
&\quad = \underbrace{\lim_{h \to 0} \left(\sum_{\ell=1}^{k+1} (-1)^{\ell-1} \frac{0}{h^{k+1}} \right)}_{=0} + \underbrace{\lim_{h \to 0} \left(\sum_{\ell=1}^{k+1} (-1)^{\ell-1} \cancelto{1}{\frac{h^{k+1}}{h^{k+1}}} df_0\big(v_\ell\big)\, dx^I(v_1, \ldots, \widehat{v_\ell}, \ldots, v_{k+1}) \right)}_{\text{no } h} \\
&\qquad + \underbrace{\lim_{h \to 0} \left(\sum_{\ell=1}^{k+1} (-1)^{\ell-1} \frac{h^{k+2}}{h^{k+1}} \big(2KC\big) \right)}_{\longrightarrow 0}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\ell=1}^{k+1} (-1)^{\ell-1} df_0\big(v_\ell\big)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}(v_1, \ldots, \widehat{v_\ell}, \ldots, v_{k+1}) \\
&= df \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}(v_1, \ldots, v_{k+1}),
\end{aligned}
$$

where the base point of 0 was left off the final line. You are asked to show the final equality in the next question.

Question 4.8 Using the definition of the wedgeproduct of a k-form α and an ℓ-form β in terms of a $(k+\ell)$-shuffle,

$$
(\alpha \wedge \beta)\big(v_1, \ldots, v_{k+\ell}\big) = \sum_{\substack{\sigma \text{ is a} \\ (k+\ell)-\text{shuffle}}} \operatorname{sgn}(\sigma)\alpha\big(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\big)\beta\big(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}\big),
$$

show the final equality in the proceeding calculation.

Thus we have shown that this definition of the exterior derivative of $\omega = f dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ leads to the same computational formula we had in previous sections. It is then of course easy to show that given a general k-form $\alpha = \sum \alpha_{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ that

$$
d\alpha = \sum d\alpha_{i_1 \cdots i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.
$$

We now turn our attention to exploring the geometry of this definition. We will look at a two-form on the manifold $\mathbb{R}^3$ in order to allow us to draw pictures. We can write

$$
\begin{aligned}
d\omega(v_1, v_2, v_3) &= \lim_{h \to 0} \frac{1}{h^3} \int_{\partial(hP)} \omega \\
&= \lim_{h \to 0} \frac{1}{h^3} \left(\int_{hP_{(1,1)}} \omega - \int_{hP_{(1,0)}} \omega - \int_{hP_{(2,1)}} \omega + \int_{hP_{(2,0)}} \omega + \int_{hP_{(3,1)}} \omega - \int_{hP_{(3,0)}} \omega \right) \\
&= \lim_{h \to 0} \frac{1}{h^3} \underbrace{\left(\int_{hP_{(1,1)}} \omega - \int_{hP_{(1,0)}} \omega \right)}_{\text{see Fig. 4.4}} - \lim_{h \to 0} \frac{1}{h^3} \underbrace{\left(\int_{hP_{(2,1)}} \omega - \int_{hP_{(2,0)}} \omega \right)}_{\text{see Fig. 4.5}} \\
&\quad + \lim_{h \to 0} \frac{1}{h^3} \underbrace{\left(\int_{hP_{(3,1)}} \omega - \int_{hP_{(3,0)}} \omega \right)}_{\text{see Fig. 4.5}}.
\end{aligned}
$$

Figure 4.4 attempts to show what is going on with the first two terms above. The two faces $hP_{(1,0)}$ and $hP_{(1,1)}$ of the parallelepiped hP spanned by hv_1, hv_2, hv_3 are shown. The faces are defined as

$$
\begin{aligned}
& hP_{(1,0)} = \left\{ 0v_1 + t_2 v_2 + t_3 v_2 \,\middle|\, 0 \le t_i \le h, i = 2, 3 \right\} \\
\text{and} \qquad & hP_{(1,1)} = \left\{ hv_1 + t_2 v_2 + t_3 v_2 \,\middle|\, 0 \le t_i \le h, i = 2, 3 \right\}.
\end{aligned}
$$

The two-form ω, which is defined on $\mathbb{R}^3$, is integrated over each of these two-dimensional spaces. Using our intuition of what integration does, we can very roughly think of this as the "amount" of ω there is on each of these two spaces. When we take the difference of these integrals, $\int_{hP_{(1,1)}} \omega - \int_{hP_{(1,0)}} \omega$, we are trying to measure how much the "amount" of ω changes as we move in the v_1 direction. Of course, the "amount" of ω that we have we have obtained depends on the spaces that we are integrating over, which are faces of the parallelepiped hP, which is itself just a scaled down version of P. Thus, both the direction v_1 and the faces we are integrating over depend on the original parallelepiped P. We will say that the difference of these two integrals gives a measure of *how much the "amount" of ω is changing in the v_1 direction with respect to the parallelepiped P*.

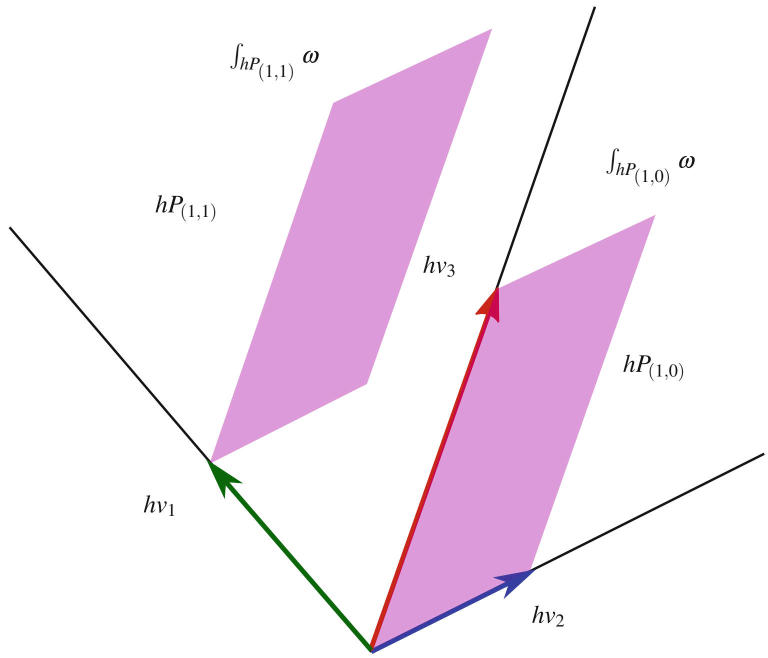

Fig. 4.4 Here the two faces $hP_{(1,0)}$ and $hP_{(1,1)}$ of the parallelepiped hP spanned by hv_1, hv_2, hv_3 are shown. The two-form ω is integrated over each of these faces. The difference of these two integrals, $\int_{hP_{(1,1)}} \omega - \int_{hP_{(1,0)}} \omega$, gives a measure of the "amount" ω is changing in the v_1 direction. Since clearly both the faces of hP we are integrating over and the direction v_1 in which the change is being measured depend on P, we say that we are measuring the change in "amount" of ω in the direction v_1 with respect to P

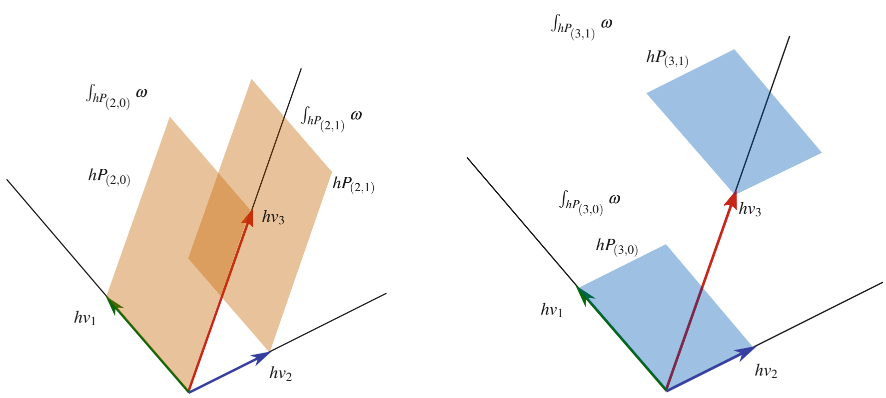

Fig. 4.5 On the left we are finding the difference between two integrals, $\int_{hP_{(2,1)}} \omega - \int_{hP_{(2,0)}} \omega$, which gives a measure on how much the "amount" of ω is changing in the v_2 direction with respect to the P. Similarly, the right we are finding the difference between two integrals, $\int_{hP_{(3,1)}} \omega - \int_{hP_{(3,0)}} \omega$, which gives a measure on how much the "amount" of ω is changing in the v_3 direction with respect to the P

In a similar manner the difference $\int_{hP_{(2,1)}} \omega - \int_{hP_{(2,0)}} \omega$ gives a measure of how much the "amount" of ω is changing in the v_2 direction with respect to the parallelepiped P and the difference $\int_{hP_{(3,1)}} \omega - \int_{hP_{(3,0)}} \omega$ gives a measure how much the "amount" of ω is changing in the v_3 direction with respect to the parallelepiped P. See Fig. 4.5.

By taking the limit as $h \to 0$ of $\frac{1}{h^3}\left(\int_{hP_{(1,1)}} \omega - \int_{hP_{(1,0)}} \omega\right)$ we are in effect finding the rate of change of the "amount" of ω in the direction v_1 with respect to P at the base point. (Recall, in the computation above we had assumed without loss of generality that the base point of the vectors $v_1, \ldots, v_k$ was the origin, but in general the base point could be any point on the manifold.) The limits as $h \to 0$ of $\frac{1}{h^3}\left(\int_{hP_{(2,1)}} \omega - \int_{hP_{(2,0)}} \omega\right)$ and $\frac{1}{h^3}\left(\int_{hP_{(3,1)}} \omega - \int_{hP_{(3,0)}} \omega\right)$ also find the rates of change on the "amount" of ω in the directions v_2 and v_3, respectively, with respect to the parallelepiped P which is spanned by the vectors v_1, v_2, v_3 at the base point. And then on top of that we add/subract these various rates of change. Whether we add or subtract is done in accordance the orientations of the various faces of P. This is explained in detail in Chap. 11.

The number we actually get for $d\omega(v_1, v_2, v_3)$ at a particular point is a somewhat strange combination of the various rates of change in the "amount" of ω in the directions v_1, v_2, and v_3, where those "amounts" are calculated using faces of the parallelepiped spanned by the vectors v_1, v_2, v_3. Thus the number $d\omega(v_1, v_2, v_3)$ is an overall rate of change of the "amount" of ω at the base point that depends on v_1, v_2, v_3. The final number may seem a little odd to you, but all the components that go into finding it have very clear and reasonable geometric meanings. Of course $d\omega$ itself is a three-form that allows us to compute this overall rate of change at a point by plugging in three vectors v_1, v_2, v_3 at that point.

Other than having a feeling for what we mean when we say "amount" of ω, using this definition for the exterior derivative gives us the most concrete geometric meaning for the exterior derivative of a k-form. We have not attempted to explain here what we mean when we say the "amount" of ω, other than simply saying it is the integral of ω over a face of P. The integrals of forms is explained in Chaps. 7–11. In particular, we look at the integration of forms from the perspective of Riemann sums in Sect. 7.2, which gives a very nice idea of what the integration of a form is achieving. However, in three dimensions the integration of forms have some very nice geometrical pictures that can be utilized to get an intuitive idea of what is going on. These pictures rely on how one-, two-, and three-forms can be visualized, which is covered in Chap. 5, and on Stokes' theorem, covered in Chap. 11. We will return to exploring the geometry of the exterior derivatives of one-, two-, and three-forms at the end of Sect. 11.6. Though there are no longer nice pictures to help us visualize differential forms for $k > 3$, the geometric meaning for the exterior derivative of a k-form on an n-dimensional manifold is analogous; there are just more directions and faces of P to be concerned with and all the faces of P are k-dimensional.

This particular definition for the exterior derivative of an n-form ω also presents the clearest intuitive idea of why the generalized Stokes' theorem is true. Using this definition,

$$d\omega\big(v_1, \dots, v_{k+1}\big) = \lim_{h \to 0} \frac{1}{h^{k+1}} \int_{\partial(hP)} \omega$$

$$\Updownarrow \text{ When } h \text{ is very small.}$$

$$d\omega\big(v_1, \dots, v_{k+1}\big) \approx \frac{1}{h^{k+1}} \int_{\partial(hP)} \omega$$

$$\Updownarrow \text{ Multiply both sides by } h^{k+1}.$$

$$h^{k+1} \underbrace{d\omega\big(v_1, \dots, v_{k+1}\big)}_{\text{a number}} \approx \int_{\partial(hP)} \omega$$

$$\Updownarrow \text{ Volume of } hP \text{ is approximately } h^{k+1}.$$

$$\underbrace{d\omega\big(v_1, \dots, v_{k+1}\big)}_{\text{a number}} \int_{hP} \underbrace{dx_1 \wedge \cdots \wedge dx_{k+1}}_{\text{volume form over } hP} \approx \int_{\partial(hP)} \omega$$

$$\Updownarrow \text{ Pull number under integral sign.}$$

$$\int_{hP} \underbrace{d\omega\big(v_1, \dots, v_{k+1}\big)}_{\text{a number}} \underbrace{dx_1 \wedge \cdots \wedge dx_{k+1}}_{\text{volume form over } hP} \approx \int_{\partial(hP)} \omega$$

$$\Updownarrow \text{ Write LHS in abstract notation.}$$

$$\int_{hP} d\omega \approx \int_{\partial(hP)} \omega$$

$$\Updownarrow \text{ Letting } h \to 0.$$

$$\int_{hP} d\omega = \int_{\partial(hP)} \omega.$$

The final implication gives us the generalized Stokes' theorem for the region hP. This is of course not a rigorous proof, that is done in Chap. 11, but it does provide a very nice big-picture idea of what the generalized Stokes' theorem is and a reasonable geometric argument for why it is true.

Question 4.9 Analogous to above, explain the geometric meaning of $d\omega(v_1, v_2, v_3, v_4)$ for a three-form ω on an n-dimensional manifold.

Question 4.10 Analogous to above, explain the geometric meaning of $d\omega(v_1, \ldots, v_k)$ for a k-form ω on an n-dimensional manifold.

4.6 Exterior Differentiation Examples

We now turn our attention to utilizing the various formulas that we have obtained in the previous sections. Probably the simplest way to get used to doing computations involving exterior differentiation is simply to see and work through a series of examples. When doing these computations we will be going into quite a bit of detail just to get you used to and familiar with everything. Once you are familiar with computations of this nature you will be able to skip a number of the steps without any problem.

Suppose we have the two functions $f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}$. We want to find $df \wedge dg$. First we note that we have

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \quad \text{and} \quad dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy.$$

We then take the wedgeproduct of these two one-forms and use the algebraic properties of the wedgeproduct presented in Sect. 3.3.1,

$$\begin{aligned}
df \wedge dg &= \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right) \wedge \left(\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy\right) \\
&= \frac{\partial f}{\partial x}dx \wedge \left(\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy\right) + \frac{\partial f}{\partial y}dy \wedge \left(\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy\right) \\
&= \frac{\partial f}{\partial x}\frac{\partial g}{\partial x}\underbrace{dx \wedge dx}_{=0} + \frac{\partial f}{\partial x}\frac{\partial g}{\partial y}dx \wedge dy + \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}dy \wedge dx + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y}\underbrace{dy \wedge dy}_{=0} \\
&= \frac{\partial f}{\partial x}\frac{\partial g}{\partial y}dx \wedge dy + \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}dy \wedge dx \\
&= \frac{\partial f}{\partial x}\frac{\partial g}{\partial y}dx \wedge dy - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}dx \wedge dy \\
&= \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right)dx \wedge dy \\
&= \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix} dx \wedge dy.
\end{aligned}$$

So, the wedgeproduct of df and dg involves the Jacobian matrix from vector calculus in front of the two-dimensional volume form (area form) $dx \wedge dy$. In other words, we have found the identity

$$\boxed{df \wedge dg = \underbrace{\begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix}}_{Jacobian} \underbrace{dx \wedge dy.}_{area form}}$$

It is perhaps a little tedious, but not difficult to show that for $f, g, h : \mathbb{R}^3 \to \mathbb{R}$ we have

$$\boxed{df \wedge dg \wedge dh = \underbrace{\begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{vmatrix}}_{Jacobian} \underbrace{dx \wedge dy \wedge dz}_{volume form}.}$$

The obvious generalization holds as well. However, the proof of the general formula using the properties of the wedgeproduct is rather tedious. Instead, we will defer the proof of this formula until later when more mathematical machinery has been introduced and we are able to give a much slicker proof.

Question 4.11 Given functions $f, g, h : \mathbb{R}^3 \to \mathbb{R}$ find the above identity for $df \wedge dg \wedge dh$.

Now we look at an example where we have two explicitly given functions. For example, suppose we had a change of coordinates given by $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = x + y$ and $g : \mathbb{R}^2 \to \mathbb{R}$ by $g(x, y) = x - y$. To find $df \wedge dg$ we could take one of two approaches. For example, we could first find df and dg and then use the algebraic properties of differential forms to simplify it or we could simply skip to the identity we found above. We will show both approaches. Following the first approach and first finding df and dg we have

$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{\partial (x+y)}{\partial x}dx + \frac{\partial (x+y)}{\partial y}dy = dx + dy \\ dg &= \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = \frac{\partial (x-y)}{\partial x}dx + \frac{\partial (x-y)}{\partial y}dy = dx - dy. \end{aligned}$$

Then using our algebraic properties of differential forms to simplify we find

$$\begin{aligned} df \wedge dg &= (dx + dy) \wedge (dx - dy) \\ &= dx \wedge (dx - dy) + dy \wedge (dx - dy) \\ &= \underbrace{dx \wedge dx}_{=0} - dx \wedge dy + dy \wedge dx - \underbrace{dy \wedge dy}_{=0} \\ &= -dx \wedge dy - dx \wedge dy \\ &= -2dx \wedge dy. \end{aligned}$$

In the second approach we just use the identity we found earlier to get

$$\begin{aligned} df \wedge dy &= \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix} dx \wedge dy \\ &= \begin{vmatrix} \frac{\partial (x+y)}{\partial x} & \frac{\partial (x+y)}{\partial y} \\ \frac{\partial (x-y)}{\partial x} & \frac{\partial (x-y)}{\partial y} \end{vmatrix} dx \wedge dy \\ &= \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} dx \wedge dy \\ &= -2dx \wedge dy. \end{aligned}$$

It is not at all surprising that our two answers agree.

Let's go into more depth with another example involving more complicated functions. Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^2y + x$ and $g : \mathbb{R}^2 \to \mathbb{R}$ given by $g(x, y) = x^2y^3 + 2xy$. We use our identity,

$$df \wedge dg = \begin{vmatrix} (2xy + 1) & x^2 \\ (2xy^3 + 2y) & (3x^2y^2 + 2x) \end{vmatrix} dx \wedge dy = \Big[(2xy + 1)(3x^2y^2 + 2x) - x^2(2xy^3 + 2y)\Big] dx \wedge dy.$$

This is $df \wedge dg$ on all of our manifold $\mathbb{R}$. Suppose we want to know what $df \wedge dg$ were at a specific point of the manifold? We would have to substitute in that point's values. For example, suppose we wanted to find $df \wedge dg$ at the point $p = (1, 1)$ of manifold $\mathbb{R}^2$. We would substitute to get

$$\begin{aligned}
\Big(df \wedge dg\Big)_{p=(1,1)} &= \Big[\big(2 \cdot 1 \cdot 1 + 1\big)\big(3 \cdot 1^2 \cdot 1^2 + 2 \cdot 1\big) - 1^2\big(2 \cdot 1 \cdot 1^3 + 2 \cdot 1\big)\Big] dx \wedge dy \\
&= \big[(3)(5) - (4)\big] dx \wedge dy \\
&= 11 dx \wedge dy.
\end{aligned}$$

Similarly, if we wanted to find $df \wedge dg$ at the point $p = (-1, 2)$ of manifold $\mathbb{R}^2$ we would substitute to get

$$\begin{aligned}
\Big(df \wedge dg\Big)_{p=(-1,2)} &= \Big[\Big(2 \cdot (-1) \cdot 2 + 1\Big)\Big(3 \cdot (-1)^2 \cdot 2^2 + 2 \cdot (-1)\Big) \\
&\qquad -(-1)^2\Big(2 \cdot (-1) \cdot 2^3 + 2 \cdot (-1)\Big)\Big] dx \wedge dy \\
&= \big[(-3)(10) - (-18)\big] dx \wedge dy \\
&= -12 dx \wedge dy.
\end{aligned}$$

Now suppose we had the two vectors $v_p = \begin{bmatrix} -2 \\ 3 \end{bmatrix}_{(1,1)}$ and $w_p = \begin{bmatrix} -4 \\ -2 \end{bmatrix}_{(1,1)}$ and wanted to find $(df \wedge dg)(v_p, w_p)$. Omitting writing the base point $(1, 1)$ after the first equality we get

$$\begin{aligned}
\big(df \wedge dg\big)_{(1,1)}\left(\begin{bmatrix} -2 \\ 3 \end{bmatrix}_{(1,1)}, \begin{bmatrix} -4 \\ -2 \end{bmatrix}_{(1,1)}\right) &= 11 dx \wedge dy \left(\begin{bmatrix} -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ -2 \end{bmatrix}\right) \\
&= 11 \cdot \begin{vmatrix} dx\left(\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right) & dx\left(\begin{bmatrix} -4 \\ -2 \end{bmatrix}\right) \\ dy\left(\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right) & dy\left(\begin{bmatrix} -4 \\ -2 \end{bmatrix}\right) \end{vmatrix} \\
&= 11 \cdot \begin{vmatrix} -2 & 4 \\ 3 & -2 \end{vmatrix} = 11(4 - 12) \\
&= 88.
\end{aligned}$$

And if we had the vectors $v_p = \begin{bmatrix} 3 \\ 1 \end{bmatrix}_{(-1,2)}$ and $w_p = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{(-1,2)}$ and wanted to find $(df \wedge dg)(v_p, w_p)$ we would have

$$\big(df \wedge dg\big)_{(-1,2)}\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}_{(-1,2)}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{(-1,2)}\right) = -12 \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = -60.$$

Now we do one more example that may look vaguely familiar to you. Write $dx \wedge dy$ in terms of $d\theta \wedge dr$ if $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Doing this gives us

$$\begin{aligned}
dx \wedge dy &= \begin{vmatrix} \frac{dx}{d\theta} & \frac{dx}{dr} \\ \frac{dy}{d\theta} & \frac{dy}{dr} \end{vmatrix} d\theta \wedge dr \\
&= \begin{vmatrix} \frac{d\left(r\cos(\theta)\right)}{d\theta} & \frac{d\left(r\cos(\theta)\right)}{dr} \\ \frac{d\left(r\sin(\theta)\right)}{d\theta} & \frac{d\left(r\sin(\theta)\right)}{dr} \end{vmatrix} d\theta \wedge dr
\end{aligned}$$

$$
\begin{aligned}
&= \begin{vmatrix} -r\sin(\theta) & \cos(\theta) \\ r\cos(\theta) & \sin(\theta) \end{vmatrix} d\theta \wedge dr \\
&= \big(-r\sin^2(\theta) - r\cos^2(\theta)\big) d\theta \wedge dr \\
&= -r d\theta \wedge dr.
\end{aligned}
$$

If you think this looks a little like a polar change of coordinates formula then you would be absolutely correct. We will discuss this in much greater detail soon.

Question 4.12 Suppose $f, g, h : \mathbb{R}^3 \to \mathbb{R}$ are defined by $f(x, y, z) = x^2 y$, $g(x, y, z) = y^2 z^2$, and $h(x, y, z) = xyz$.

(a) Find $df \wedge dg \wedge dh$.
(b) Find $df \wedge dg \wedge dh$ at the point $p = (-1, 1, -1)$.
(c) Find $df \wedge dg \wedge dh_{(-1,1,-1)}\big(u_p, v_p, w_p\big)$ where

$$
u_p = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{(-1,1,-1)}, \quad v_p = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}_{(-1,1,-1)}, \quad w_p = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}_{(-1,1,-1)}.
$$

Over the last few examples and questions we have dealt with a very specific sort of problem. Notice that the number of functions we had (two or three) was the same as the dimension of the manifold the functions were defined on. For example, on the manifold $\mathbb{R}^2$ we had two functions, f and g. On the manifold $\mathbb{R}^3$ we had three functions, f, g, h. Thus, in the first case $df \wedge dg$ gave us a function multiplied by the two dimensional volume form $dx \wedge dy$ and in the second case $df \wedge dg \wedge dh$ gave us a functions multiplied by the three dimensional volume form. Now we will look at a few more general examples.

Question 4.13 Suppose $f, g : \mathbb{R}^3 \to \mathbb{R}$ are given by $f(x, y, z) = x^2 y^3 z^2$ and $g(x, y, z) = xyz^2$.

(a) Find df and dg.
(b) Find $df \wedge dg$. (Notice, you can no longer use the Jacobian matrix, you must use df and dg that you found above and the algebraic properties of the wedgeproduct.)
(c) Find $df \wedge dg$ at the point $(3, 2, 1)$.
(d) Suppose $v_p = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $w_p = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$. Find $df \wedge dg(v_p, w_p)$.

Now we turn to some examples where we find the exterior derivative of a one-form. Suppose that $\phi = x^2 yz dx$ is a one-form on $\mathbb{R}^3$. We will find $d\phi$ using our formula from the last section.

$$
\begin{aligned}
d\phi &= d(x^2 yz) \wedge dx \\
&= \Big(\frac{\partial(x^2yz)}{\partial x} dx + \frac{\partial(x^2yz)}{\partial y} dy + \frac{\partial(x^2yz)}{\partial z} dz \Big) \wedge dx \\
&= \big(2xyz dx + x^2 z dy + x^2 y dz \big) \wedge dx \\
&= 2xyz \underbrace{dx \wedge dx}_{=0} + x^2 z dy \wedge dx + x^2 y dz \wedge dx \\
&= -x^2 z dx \wedge dy - x^2 y dx \wedge dz.
\end{aligned}
$$

For our next example suppose $\phi = \sin(x)dx + e^x \cos(y)dy + 3xydz$. We will find $d\phi$ in three steps, one for each term of ϕ,

$$
d\phi = \underbrace{d\big(\sin(x)\big) \wedge dx}_{(1)} + \underbrace{d\big(e^x \cos(y)\big) \wedge dy}_{(2)} + \underbrace{d\big(3xy\big) \wedge dz}_{(3)}.
$$

The first term gives us

$$\begin{aligned}
d\big(\sin(x)\big) \wedge dx &= \Big(\frac{\partial \sin(x)}{\partial x}dx + \frac{\partial \sin(x)}{\partial y}dy + \frac{\partial \sin(x)}{\partial z}dz\Big) \wedge dx \\
&= \big(\cos(x)dx + 0dy + 0dz\big) \wedge dx \\
&= \cos(x)\underbrace{dx \wedge dx}_{=0} \\
&= 0.
\end{aligned}$$

The second term gives us

$$\begin{aligned}
d\big(e^x \cos(y)\big) \wedge dy &= \Big(\frac{\partial (e^x \cos(y))}{\partial x}dx + \frac{\partial (e^x \cos(y))}{\partial y}dy + \frac{\partial (e^x \cos(y))}{\partial z}dz\Big) \wedge dy \\
&= e^x \cos(y)dx \wedge dy - e^x \sin(y)\underbrace{dy \wedge dy}_{=0} + 0dz \wedge dy \\
&= e^x \cos(y)dx \wedge dy.
\end{aligned}$$

The third term gives us

$$\begin{aligned}
d\big(3xy\big) \wedge dz &= \Big(\frac{\partial 3xy}{\partial x}dx + \frac{\partial 3xy}{\partial y}dy + \frac{\partial 3xy}{\partial z}dz\Big) \wedge dz \\
&= 3ydx \wedge dz + 3xdy \wedge dz + 0dz \wedge dz \\
&= 3xdy \wedge dz - 3ydz \wedge dx.
\end{aligned}$$

Combining everything we have

$$d\phi = e^x \cos(y)dx \wedge dy + 3xdy \wedge dz - 3ydz \wedge dx.$$

Question 4.14 Find a formula for $d\phi$ if $\phi = f_1dx_1 + f_2dx_2 + f_3dx_3$.

Your answer for the last question should have been

$$d\phi = \Big(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\Big)dx_1 \wedge dx_2 + \Big(\frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3}\Big)dx_1 \wedge dx_3 + \Big(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\Big)dx_2 \wedge dx_3.$$

Next we show that if f, g are functions on $\mathbb{R}^n$ that $d(f \cdot g) = df \cdot g + f \cdot dg$.

$$\begin{aligned}
d(f \cdot g) &= \sum_i \frac{\partial f \cdot g}{\partial x_i}dx_i \\
&\underset{\text{prod. rule}}{=} \sum_i \Big(\frac{\partial f}{\partial x_i} \cdot g + f \cdot \frac{\partial g}{\partial x_i}\Big)dx_i \\
&= f \cdot \sum_i \frac{\partial f}{\partial x_i}dx_i + f \sum_i \frac{\partial g}{\partial x_i}dx_i \\
&= g \cdot df + f \cdot dg.
\end{aligned}$$

This is really nothing more than the product rule in somewhat different notation.

Question 4.15 Show that the exterior derivative d has the linearity property, that is, if $\phi = \sum f_i dx_i$ and $\psi = \sum g_i dx_i$ then for $a, b \in \mathbb{R}$,

$$d(a\phi + b\psi) = ad\phi + bd\psi.$$

Question 4.16 If f is a function and ϕ a one-form, show

$$d(f\phi) = df \wedge \phi + f d\phi.$$

Question 4.17 If ϕ, ψ are one-forms, show

$$d(\phi \wedge \psi) = d\phi \wedge \psi + \phi \wedge d\psi.$$

Note, in order of operations multiplication and wedgeproducts take precedence over addition and subtraction.

Next we show that if f is a function on $\mathbb{R}^n$ that $ddf = 0$. This means that no matter what f is that taking the exterior derivative twice gives 0. First, we know $df = \sum_i \frac{\partial f}{\partial x_i} dx_i$, so

$$\begin{aligned}
ddf &= d\Big(\sum_i \frac{\partial f}{\partial x_i} dx_i\Big) \\
&= \sum_i d\Big(\frac{\partial f}{\partial x_i}\Big) \wedge dx_i \\
&= \sum_i \sum_j \frac{\partial}{\partial x_j}\Big(\frac{\partial f}{\partial x_i}\Big) dx_j \wedge dx_i \\
&= \sum_{i,j} \frac{\partial^2 f}{\partial x_j \partial x_i} \qquad \underbrace{dx_j \wedge dx_i}_{\substack{\text{If } i<j \text{ then} \\ dx_j \wedge dx_i = -dx_i \wedge dx_j}} \\
&= \sum_{i<j} \underbrace{\Big(\frac{\partial^2 f}{\partial x_j \partial x_i} - \frac{\partial^2 f}{\partial x_i \partial x_j}\Big)}_{\substack{=0 \\ \text{by equality of mixed partials}}} dx_i \wedge dx_j \\
&= 0.
\end{aligned}$$

Question 4.18 For the functions f, g, show $d(f dg) = df \wedge dg$.

Question 4.19 Simplify

(a) $d(f dg + f df)$,
(b) $d\big((f - g)(df + dg)\big)$,
(c) $d\big((f dg) \wedge (g df)\big)$,
(d) $d(g f df) + d(f dg)$.

As we mentioned earlier, one of our simplifications so far has been to stick with the standard Cartesian coordinate system. Given how familiar most students are with the Cartesian coordinate system it makes sense to introduce a complex idea like differential forms, wedgeproducts, and exterior differentiation in an arena that most students are completely comfortable with. We don't want you to lose sight of the big ideas because you were getting mired down in the details of other coordinate systems. However, other coordinate systems will soon play an increasingly important role for us.

4.7 Summary, References, and Problems

4.7.1 Summary

In this chapter four different approaches to exterior differentiation were explored. In the first approach the local formula, that is a formula that uses coordinates, for the exterior derivative was given, Exterior derivative of an n-form

$$d\Big(\sum \alpha_{i_1\cdots i_n}\, dx_{i_1} \wedge \cdots \wedge dx_{i_n}\Big) = \sum \sum_{j=1}^{n} \frac{\partial\, \alpha_{i_1\cdots i_n}}{\partial x_{i_j}}\, dx_{i_j} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_n}.$$

Using this local formula one can then show the following algebraic properties hold,

(1) $d(\alpha+\beta) = d\alpha + d\beta$,
(2) $d(\alpha \wedge \omega) = d\alpha \wedge \omega + (-1)^n \alpha \wedge d\omega$,
(3) for each n-form α, $d(d\alpha) = 0$.

The second approach to defining the exterior derivative basically goes backwards from this. The three properties are given as defining axioms and then a unique formula in coordinates is derived from these properties.

The third approach to exterior differentiation is an attempt to find a global formula. A global formula differs from a local formula in that it does not rely on coordinates. That is, the coordinates do not show up in the formula. First a global formula for the exterior derivative of a k-form is found using constant vector fields,

Global formula for exterior derivative of a k-form, constant vector fields

$$d\omega(v_0, \ldots, v_k) = \sum_i (-1)^i v_i\big[\omega(v_0, \ldots, \widehat{v_i}, \ldots, v_k)\big].$$

Using the definition of the lie bracket of two vector fields,

$$[v, w]\big[F\big] = v\big[w[F]\big] - w\big[v[F]\big],$$

a global formula for the exterior derivative of a two-form is found for vector fields that are not constant,

Global formula for exterior derivative of a one-form,

$$d\alpha(v, w) = v[\alpha(w)] - w[\alpha(v)] - \alpha([v, w]).$$

The general global formula for the exterior derivative of a k-form with non-constant vector fields is then given,

Global formula for exterior derivative of k-form

$$d\alpha(v_0, \ldots, v_k) = \sum_i (-1)^i v_i\big[\alpha(v_0, \ldots, \widehat{v_i}, \ldots, v_k)\big] + \sum_{i<j} (-1)^{i+j} \alpha\big([v_i, v_j], v_0, \ldots, \widehat{v_i}, \ldots, \widehat{v_j}, \ldots, v_k\big).$$

This formula is not actually proved until Sect. A.7.

The fourth approach to exterior differentiation is a very geometrical approach where the exterior derivative of k-form at a point is defined by

Exterior derivative of a k-form

$$d\omega\big(v_1, \ldots, v_{k+1}\big) = \lim_{h\to 0} \frac{1}{h^{k+1}} \int_{\partial(hP)} \omega.$$

This geometrical definition is revisited in Sect. 11.6 in order to obtain a deeper understanding of the geometry in the three-dimensional case. This geometrical definition of the exterior derivative also gives the most geometrical argument (but not proof) for why Stokes' theorem is true.

4.7.2 References and Further Reading

As the overview explains, in most books exterior differentiation is typically approached in one of four ways. As with the wedgeproduct, this is understandable since exterior differentiation is simply one of many different topics needed to be covered. However, we have tried, as much as is possible, to bring everything together and explore all of these approaches to exterior differentiation so the reader will be completely comfortable with any approach to exterior differentiation that they may encounter in the future. For example, O'Neill [36] uses a local coordinates approach while Tu [46] covers exterior differentiation from a local coordinates perspective early on in the book and then from a global perspective toward the middle of the book. Martin [33], Darling [12], and Flanders [19] all use an axiom-based approach. The geometric viewpoint of the exterior derivative is quite rare, the only place we are aware of where it is actually used as an approach to introducing the exterior derivative is in Hubbard and Hubbard [27] though this geometric meaning of the exterior derivative is also alluded to briefly in Darling [12] and Arnold [3]. Our exposition largely follows that of Hubbard and Hubbard, though it is not as complete or detailed as theirs is.

4.7.3 Problems

Question 4.20 Let $f_1(x,y,z) = x^3y^2z - 2xyz^3$, $f_2(x,y,z) = x^2 - 3y^2 + 6z^4$, $f_3(x,y,z) = xy^2 + yz^2 + zx^2$, and $f_4(x,y,z) = 3x - 2y + 5z - 4$. Find df_1, df_2, df_3, and df_4.

Question 4.21 Let $g_1(x,y,z) = \frac{x^2+y^2}{z}$, $g_2(x,y,z) = \sin(x^2+y^2) + \tan(2z)$, $g_3(x,y,z) = xz + ye^y + 3x - 2$, and $g(x,y,z) = \cos(xy^2z) + \frac{x^2}{yz}$. Find dg_1, dg_2, and dg_3.

Question 4.22 For the functions in Question 4.20 and 4.21 find $d(2f_1+3g_1)$, $d(5f_2+3g_2)$, $d(-2f_3-5g_3)$, and $d(7f_4-3g_3)$.

Question 4.23 For the functions in Question 4.20 find $d(f_1 \cdot f_2)$, $d(f_1 \cdot f_3)$, $d(f_1 \cdot f_4)$, $d(f_2 \cdot f_3)$, $d(f_2 \cdot f_4)$, and $d(f_3 \cdot f_4)$.

Question 4.24 Let $\alpha_1 = 3x^3\,dx + \frac{x+y}{z}\,dy$, $\alpha_2 = -x\,dx + (x-3y)\,dy$, $\alpha_3 = \frac{x^3}{y}\,dx + xyz\,dx + (x^2+y^2+z^2)\,dz$, and $\alpha_4 = y^2z\,dx - xz\,dy + (3x+2)\,dz$. Find $d\alpha_1$, $d\alpha_2$, $d\alpha_3$, and $d\alpha_4$.

Question 4.25 Let $\beta_1 = \sin(x)\,dx + \sin(y)\,dy - \cos(z)\,dz$, $\beta_2 = \sin(xyz)\,dx - \cos(x^2)\,dy + e^y\,dz$, $\beta_3 = \sin^2(xz)\,dx + \cos^3(yz)\,dy - e^{xy}\,dz$, and $\beta_4 = (x^2+y^2+z^2)\,dx - x^z\,dy - xze^y\,dz$. Find $d\beta_1$, $d\beta_2$, $d\beta_3$, and $d\beta_4$.

Question 4.26 For the one-forms in Questions 4.24 and 4.25 find $d(2\alpha_1+3\beta_1)$, $d(5\alpha_2+3\beta_2)$, $d(-2\alpha_3-5\beta_3)$, and $d(7\alpha_4 - 3\beta_3)$.

Question 4.27 For the functions in Question 4.20 and the one-forms in Question 4.24 find $d(f_1\alpha_1)$, $d(f_2\alpha_2)$, $d(f_3\alpha_3)$, and $d(f_4\alpha_4)$.

Question 4.28 For the one-forms in Questions 4.24 and 4.25 find $d(\alpha_1 \wedge \beta_1)$, $d(\alpha_2 \wedge \beta_2)$, $d(\alpha_3 \wedge \beta_3)$, and $d(\alpha_4 \wedge \beta_4)$.

Question 4.29 For the functions in Question 4.20 find d^2f_1, d^2f_2, d^2f_3, and d^2f_4.

Question 4.30 For the one-forms in Questions 4.24 find $d^2\alpha_1$, $d^2\alpha_2$, $d^2\alpha_3$, and $d^2\alpha_4$.

Question 4.31 Let $\gamma_1 = (x^2+y^4)\,dx \wedge dy + s^3y^2z^5\,dy \wedge dz - (3x+2y-4z+7)\,dz \wedge dx$ and $\gamma_2 = xy\sin(z)\,dx \wedge dy - \cos(xyz)\,dy \wedge dz + \sin(xz)\cos(yz)dy \wedge dx$. Find $d\gamma_1$, $d\gamma_2$, and $d(3\gamma_1 - 5\gamma_2)$. Then find $d^2\gamma_1$ and $d^2\gamma_2$.

Question 4.32 If $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, show that $v_p\Big[g(f)\Big] = g'(f)\ v_p\big[f\big]$. Use this to deduce that $d\big(g(f)\big) = g'(f)\,df$.

Question 4.33 If f, g, and h are real-valued functions on $\mathbb{R}^2$ and α is a one-form on $\mathbb{R}^2$, show $d(fgh) = gh\,df + fh\,dg + fg\,dh$ and $\big(df \wedge dg\big)(v,w) = v[f]w[g] - v[g]w[f]$.

Question 4.34 Given the two-forms $\alpha = x_1x_4x_6\,dx_1 \wedge dx_5$, $\beta = x_2^3\sin(x_4)\,dx_3 \wedge dx_5$, and $\gamma = (x_1^2+x_3^2+x_5^2)\,dx_2 \wedge dx_6$ on $\mathbb{R}^6$ find $d\alpha$, $d\beta$, $d\gamma$, $d^2\alpha$, $d^2\beta$, and $d^2\gamma$. Now assume α, β, and γ are two-forms on $\mathbb{R}^8$ and find the same exterior derivatives. Are they the same or different? Explain why or why not.

Use the definition for the integral of a zero-form at a single point given in Sect. 4.5 to answer the following questions.

Question 4.35 Let $P = \big\{(3,2,1)\big\} \subset \mathbb{R}^3$ and let $f = x^3y^2 - z^3$ be a zero-form on $\mathbb{R}^3$. Evaluate $\int_P f$.

Question 4.36 Let $P = -\big\{(-1,2,-1)\big\} \subset \mathbb{R}^3$ and $f = x(y-z+2)$. Evaluate $\int_P f$.

Question 4.37 Let $P = \big\{(5,-5,3)\big\} \subset \mathbb{R}^3$ and $f = x^3(xy^2+yz^2+3)$. Evaluate $\int_P f$.

Question 4.38 Let $P = -\big\{(-4,2,7)\big\} \subset \mathbb{R}^3$ and $f = 2^xy^z$. Evaluate $\int_P f$.

Chapter 5
Visualizing One-, Two-, and Three-Forms

In this chapter we will introduce and discuss at length one of the ways that physicists sometimes visualize “nice” differential forms. In essence, we will be considering ways of visualizing one-forms, two-forms, and three-forms on a vector space. That is, we will find a “cartoon picture” of $\alpha_p \in T_p^*\mathbb{R}^2$ and $\alpha_p \in T_p^*\mathbb{R}^3$. Our picture of α_p will be superimposed on the vector space $T_p\mathbb{R}^3$. This perspective is developed extensively in Misner, Thorne, and Wheeler’s giant book *Gravitation*. In fact, they make some efforts to develop the four-dimensional picture (for space-time) as well, however we will primarily stick to the two and three-dimensional cases here. Section one focuses on the two-dimensional case and sections two through four focuses on the three-dimensional case. Then in section five we expand our cartoon picture to general two and three-dimensional manifolds. Again, this cartoon picture really only applies to “nice” differential forms, of the kind physicists are more likely to encounter, but it is still useful for forms and manifolds that are not overly complicated. Finally, in section six we introduce the Hodge star operator. Our visualizations of forms in three dimensions provide a nice way to visualize what the Hodge star operator does, which makes this a nice place to introduce it. Despite the power and usefulness of the way of visualizing differential forms in physics developed in this chapter, it is rarely encountered in mathematics. One of the reasons for this is that in reality it is not a completely general way of considering forms; when dealing with dimensions greater than four or with more abstract manifolds or with forms that are not “nice” in some sense it breaks down.

5.1 One- and Two-Forms in $\mathbb{R}^2$

We will start out by considering the one-forms $\alpha_p \in T_p^*\mathbb{R}^2$. Recall that a one-form α_p “eats” vectors v_p at the point p on the manifold $\mathbb{R}^2$. We will take the particular one-form dx_p at some point p. For the moment we will suppress the p in the notation and assume everything happens at one particular point p. Recall, the one-form dx is the exterior derivative of the zero-form x, which is itself nothing more than the coordinate function x. But in the case of zero-forms exterior derivatives are simply differentials of functions. Recall, the differential of a function f is defined by $df(v) = v[f]$, the directional derivative of f in the direction v.

But what is the rate of change of the coordinate function x in the x direction? It is one of course. In other words, the slope of the graph of coordinate function x in the x direction is one. We think of the slope as being the “rise” over “run”, the vertical change divided by the horizontal change. But when we defined directional derivatives we made a slight alteration in our definition to accommodate vectors that were not unit vectors. Instead of directional derivatives giving us slopes they give us the “rise” portion of the slope. So, for the coordinate function x the differential dx measures the “rise” that occurs in the x direction. For example, consider a vector

$$v = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

As one moves along the vector v from the point $(0, 0)$ to the point $(3, 2)$ the graph of the coordinate function x “rises” three units, hence

$$dx\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = 3.$$

J. P. Fortney, *A Visual Introduction to Differential Forms and Calculus on Manifolds*,
https://doi.org/10.1007/978-3-319-96992-3_5

Question 5.1 Sketch the coordinate function x on $\mathbb{R}^2$ and find the "rise" of the coordinate function x along the vectors

(a) $\begin{bmatrix}1\\0\end{bmatrix}$, that is, from the point $(0, 0)$ to the point $(1, 0)$,

(b) $\begin{bmatrix}-1\\0\end{bmatrix}$, that is, from the point $(0, 0)$ to the point $(-1, 0)$,

(c) $\begin{bmatrix}2\\2\end{bmatrix}$, that is, from the point $(0, 0)$ to the point $(2, 2)$,

(d) $\begin{bmatrix}-3\\-1\end{bmatrix}$, that is, from the point $(0, 0)$ to the point $(-3, -1)$.

Question 5.2 Sketch the coordinate function y on $\mathbb{R}^2$ and find the "rise" of the coordinate function y along the vectors

(a) $\begin{bmatrix}0\\1\end{bmatrix}$, that is, from the point $(0, 0)$ to the point $(0, 1)$,

(b) $\begin{bmatrix}1\\1\end{bmatrix}$, that is, from the point $(0, 0)$ to the point $(1, 1)$,

(c) $\begin{bmatrix}-1\\2\end{bmatrix}$, that is, from the point $(0, 0)$ to the point $(-1, 2)$,

(d) $\begin{bmatrix}2\\-3\end{bmatrix}$, that is, from the point $(0, 0)$ to the point $(2, -3)$.

This explains the way we originally introduced the one-forms. The way we introduced the one-form dx was as a projection of $v \in T_p\mathbb{R}^2$ onto the ∂_x-axis of $T_p\mathbb{R}^2$. This is exactly the amount of "rise" of the coordinate function x from the point p to the endpoint of v_p. Thus we had

$$\begin{array}{llll} dx\left(\begin{bmatrix}1\\4\end{bmatrix}\right) = 1, & dx\left(\begin{bmatrix}-1\\4\end{bmatrix}\right) = -1, & dx\left(\begin{bmatrix}2\\5\end{bmatrix}\right) = 2, & dx\left(\begin{bmatrix}3\\6\end{bmatrix}\right) = 3, \\ dx\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = 1, & dx\left(\begin{bmatrix}-1\\3\end{bmatrix}\right) = -1, & dx\left(\begin{bmatrix}2\\2\end{bmatrix}\right) = 2, & dx\left(\begin{bmatrix}3\\1\end{bmatrix}\right) = 3, \\ dx\left(\begin{bmatrix}1\\-2\end{bmatrix}\right) = 1, & dx\left(\begin{bmatrix}-1\\-2\end{bmatrix}\right) = -1, & dx\left(\begin{bmatrix}2\\-2\end{bmatrix}\right) = 2, & dx\left(\begin{bmatrix}3\\-1\end{bmatrix}\right) = 3. \end{array}$$

Consider how this is illustrated in the Fig. 5.1. The vertical lines given by the equations $x = -1, x = 0, x = 1, x = 2, x = 3$, etc. act as illustrations of the projections. For example, the vectors $\begin{bmatrix}2\\5\end{bmatrix}$, $\begin{bmatrix}2\\2\end{bmatrix}$, and $\begin{bmatrix}2\\-2\end{bmatrix}$ all terminate on the line $x = 2$. Thus each of these vectors "pierces" the $x = 1$ and $x = 2$ lines. If the vector terminates on a line we consider that the vector 'pierces" the line. Instead of thinking of dx as a projection onto the ∂_x-axis we instead picture dx as the infinite series of lines $x = \pm n$, $n = 0, 1, 2, 3, \ldots$. Using this image we interpret $dx(v)$ as the number of lines that the vector v pierces.

Along with thinking about dx as the lines $x = \pm n$ we also have to attach an orientation. That is shown in the picture as the arrow pointing to the right indicating a positive orientation. Thus the vectors $\begin{bmatrix}-1\\4\end{bmatrix}$, $\begin{bmatrix}-1\\3\end{bmatrix}$, and $\begin{bmatrix}-1\\-2\end{bmatrix}$ all pierce one line, the $x = -1$ line. But since the vectors go "backwards" (in the negative direction) we view it as "piercing" negative one times. If we wanted to consider the differential one-form $-dx$ we would have the same picture, the lines given by $x = \pm n$, but we would draw the orientation in the opposite direction.

Now consider the following,

$$\begin{array}{lll} dx\left(\begin{bmatrix}1.42\\4\end{bmatrix}\right) = 1.42, & dx\left(\begin{bmatrix}1.97\\2\end{bmatrix}\right) = 1.97, & dx\left(\begin{bmatrix}1.1\\-2\end{bmatrix}\right) = 1.1, \\ dx\left(\begin{bmatrix}-1.21\\4\end{bmatrix}\right) = -1.21, & dx\left(\begin{bmatrix}-1.5\\3\end{bmatrix}\right) = -1.5, & dx\left(\begin{bmatrix}-1.92\\-2\end{bmatrix}\right) = -1.92, \end{array}$$

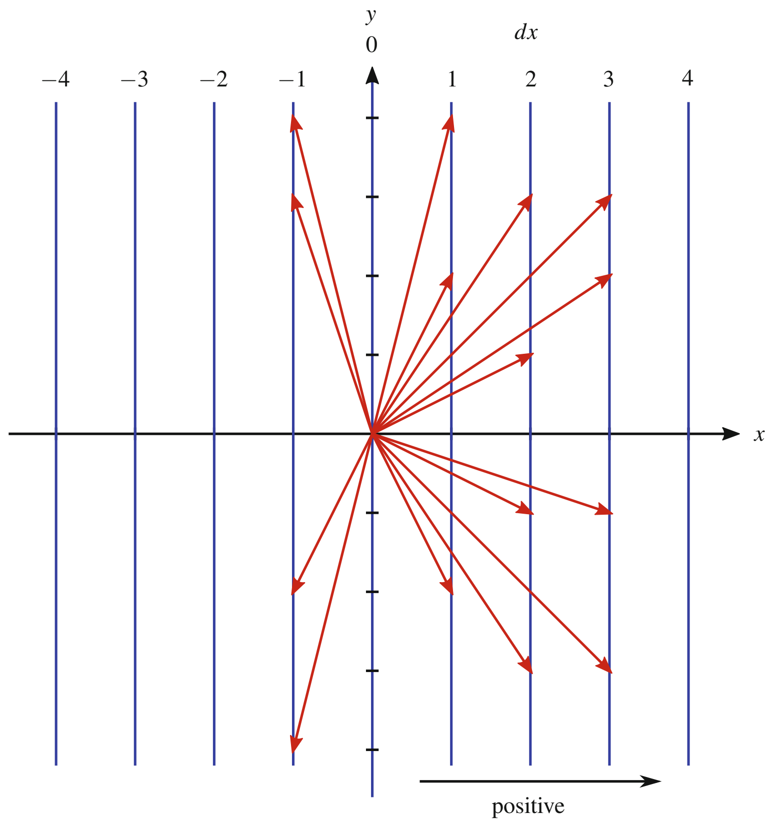

Fig. 5.1 Vectors for which the one-form dx gives outputs of -1, 1, 2, and 3. A positive orientation is shown

$$dx\left(\begin{bmatrix} 2.23 \\ 5 \end{bmatrix}\right) = 2.23, \qquad dx\left(\begin{bmatrix} 2.74 \\ 2 \end{bmatrix}\right) = 2.74, \qquad dx\left(\begin{bmatrix} 2.01 \\ -2 \end{bmatrix}\right) = 2.01,$$

$$dx\left(\begin{bmatrix} 3.69 \\ 6 \end{bmatrix}\right) = 3.69, \qquad dx\left(\begin{bmatrix} 3.91 \\ 1 \end{bmatrix}\right) = 3.91, \qquad dx\left(\begin{bmatrix} 3.34 \\ -1 \end{bmatrix}\right) = 3.34.$$

Clearly $dx\left(\begin{bmatrix} 1.42 \\ 4 \end{bmatrix}\right) = 1.42$ is correct, yet using our picture of dx as the infinite series of lines $x = \pm n$, $n = 0, 1, 2, 3, \ldots$ along with an orientation, the vector $\begin{bmatrix} 1.42 \\ 4.84 \end{bmatrix}$ only pierces one line $x = 1$. So, using the "piercing" picture for dx we have $dx\left(\begin{bmatrix} 1.42 \\ 4 \end{bmatrix}\right) = 1$. Clearly this is not exact, but it does give an approximation. Using this idea of piercing we would have

$$dx\left(\begin{bmatrix} 1.42 \\ 4 \end{bmatrix}\right) = 1, \qquad dx\left(\begin{bmatrix} 1.97 \\ 2 \end{bmatrix}\right) = 1, \qquad dx\left(\begin{bmatrix} 1.1 \\ -2 \end{bmatrix}\right) = 1,$$

$$dx\left(\begin{bmatrix} -1.21 \\ 4 \end{bmatrix}\right) = -1, \qquad dx\left(\begin{bmatrix} -1.5 \\ 3 \end{bmatrix}\right) = -1, \qquad dx\left(\begin{bmatrix} -1.92 \\ -2 \end{bmatrix}\right) = -1.$$

In an identical manor we can represent the differential form $dy_p \in T_p^*\mathbb{R}^2$ as an infinite series of lines $y = \pm n$, $n = 0, 1, 2, 3, \ldots$, along with an orientation, in the vector space $T_p\mathbb{R}^2$. The value of $dy_p(v_p)$ is the number of lines the vector v_p pierces. Again, if the tip of the vector touches a line, even if it does not go through the line, we consider it to pierce the line.