Chapter 6
Implicit Functions

Abstract Given a relation between two variables expressed by an equation of the form $f(x,y) = k$, we often want to "solve for y." That is, for each given x in some interval, we expect to find one and only one value $y = \varphi(x)$ that satisfies the relation. The function φ is thus implicit in the relation; geometrically, the locus of the equation $f(x,y) = k$ is a curve in the (x,y)-plane that serves as the graph of the function $y = \varphi(x)$. The problem of implicit functions—and the aim of this chapter—is to determine the function φ from the relation f, or at least to determine that φ exists when its exact form cannot be found. There are analogues of this problem in all dimensions; that is, x and y can be vectors, and the relation $f(x,y) = k$ can expand into a set of equations. However, we begin our analysis with a single equation, because the various impediments to finding the implicit function already occur there.

6.1 A single equation

The circle as a graph

Perhaps the most familiar example of an implicitly defined function is provided by the equation $f(x,y) = x^2 + y^2$. The locus $f(x,y) = k$ is a circle of radius $\sqrt{k}$ if $k > 0$; we can view it as the graph of two different functions,

$$y = \varphi_{\pm}(x) = \pm\sqrt{k - x^2}.$$

But if $k < 0$, the locus is the empty set; there is no implicit function at all. This is the first impediment: there may be no pairs (x,y) whatsoever that satisfy the relation $f(x,y) = k$. We need to know, somehow, that the relation is nonempty; that is, there is at least one point (a,b) for which $f(a,b) = k$, so $\varphi(a) = b$. Think of this point as a kind of "seed" from which the function $y = \varphi(x)$ can "grow."

"Growing φ from a seed" on the locus

For example, when $x^2 + y^2 = k > 0$, we can take either $(0, +\sqrt{k})$ or $(0, -\sqrt{k})$ as a seed; then φ_+ grows out of $(0, \sqrt{k})$, and φ_- grows out of $(0, -\sqrt{k})$. This example calls attention to the fact that we must expect the implicit function to be local, that is, to be defined only on part of the locus $f(x,y) = k$. Different parts of the locus may

J.J. Callahan, *Advanced Calculus: A Geometric View*, Undergraduate Texts in Mathematics,
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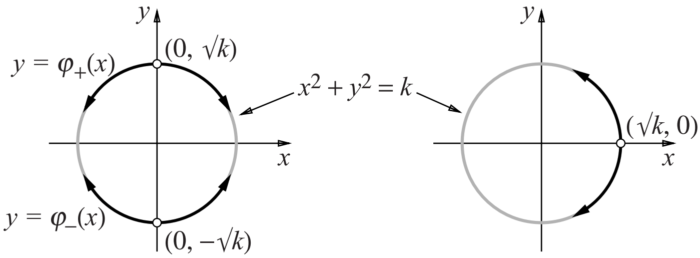
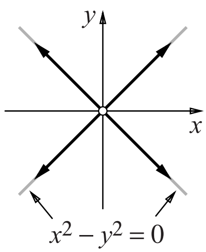
therefore be graphs of different implicit functions. Any point of the form $(a,b) = (a, +\sqrt{k-a^2})$, with $-\sqrt{k} < a < \sqrt{k}$ would serve equally well as a seed for φ_+; likewise, any point $(a, -\sqrt{k-a^2})$ would serve for φ_-.



Points on the locus that are not seeds

This leaves only the points $(\pm\sqrt{k}, 0)$. As the figure on the right shows, $(\sqrt{k}, 0)$ cannot be a seed for a function of x, because the circle has no y-value at all when $x > \sqrt{k}$ and gives two y-values near $y = 0$ when $x < \sqrt{k}$ and x is arbitrarily close to $\sqrt{k}$. There is a similar problem for $(-\sqrt{k}, 0)$. (Of course, $(\sqrt{k}, 0)$ serves perfectly well as a seed for a function $x = \psi(y)$, but we concentrate on x as the independent variable for the moment.) In a different way, there is no seed when $k = 0$. Certainly there is a point on the locus—namely $(a,b) = (0,0)$—but nothing can grow out of it, because the entire locus $x^2 + y^2 = 0$ is just this single point.



Although there is nothing wrong with having two different parts of the locus be the graph of two different implicit functions, we do require that only one implicit function φ should be able to grow out of a given seed on that locus. This is a significant restriction, and places yet another impediment in the way of obtaining φ. We can illustrate the problem with the quadratic equation $f(x,y) = y^2 - x^2 = 0$. The locus is a pair of lines that cross at the origin. Hence, we find that four different implicit functions grow out of a seed at the origin:

$$\varphi_1(x) = x, \quad \varphi_2(x) = -x, \quad \varphi_3(x) = |x|, \quad \varphi_4(x) = -|x|.$$

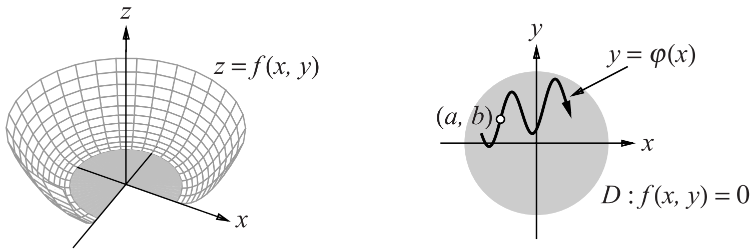
Nothing in either the locus or the seed indicates which of these we should choose; therefore we have failed to determine the implicit function we seek.

A "flat" function has a 2-dimensional locus

The same problem appears in an even more exaggerated form when the locus is not a curve but is a full 2-dimensional region, such as the unit disk $D : x^2 + y^2 \leq 1$. This is what happens for the "flat" function that is defined by the formula

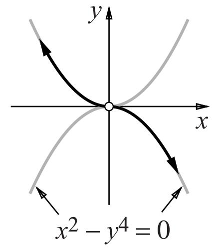
$$\begin{aligned} f(x,y) &= \text{the square of the distance from } (x,y) \text{ to } D \\ &= \begin{cases} 0 & \text{if } x^2 + y^2 \leq 1, \\ (\sqrt{x^2+y^2} - 1)^2 & \text{otherwise.} \end{cases} \end{aligned}$$

The graph of f is flat on D, so the locus $f(x,y) = 0$ is D itself; see below. The points at zero distance from D are precisely the points of D. (Because f measures the square of the distance to D, it is differentiable everywhere, including on the boundary of D.) It is clear that the graph of any continuous function $y = \varphi(x)$ that grows out of a seed (a,b) in the interior of D will lie in D, at least for x sufficiently near a.



A conjecture

Notice what is true about the tangent to the locus $f(x,y) = k$ at a putative seed point (a,b) in each of our examples thus far. When there was no uniquely defined $\varphi(x)$, either there was no tangent ($x^2 + y^2 = 0$), there was more than one tangent (both $x^2 - y^2 = 0$ and the function vanishing on the unit disk), or there was a vertical tangent ($x^2 + y^2 = k$ at $(\pm\sqrt{k}, 0)$). We were successful only when the locus had a single nonvertical tangent at the seed point. It seems reasonable, then, to conjecture that this is a sufficient condition for the existence of a unique implicit function of x.



Unfortunately, a single nonvertical tangent is not enough. Consider the locus $y^2 - x^4 = (y - x^2)(y + x^2) = 0$. It is the union of the two parabolas $y = x^2$ and $y = -x^2$, and it has a single nonvertical tangent at every point, including the origin. Nevertheless, there are still four different implicit functions that grow out of the seed at the origin; the figure shows one of them. (Of course, every other point $(a, \pm a^2)$, $a \neq 0$, on the locus is the seed for a unique implicit function.)

Linearizing the locus

To revise the conjecture, let us make use of the fact that an implicit function is a local object. Then we can search for it with basic tools of local analysis, in particular, with Taylor's theorem. Thus, if we suppose that $f(x,y)$ has continuous second derivatives in a neighborhood of (a,b), its first-order Taylor expansion is

$$f(x,y) = f(a,b) + f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + O(2)$$

in a window centered at (a,b); $\Delta x = x - a$ and $\Delta y = y - b$ are the usual window coordinates. Because $f(a,b) = k$, the equation of the locus $f(x,y) = k$ reduces to

$$f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + O(2) = 0$$

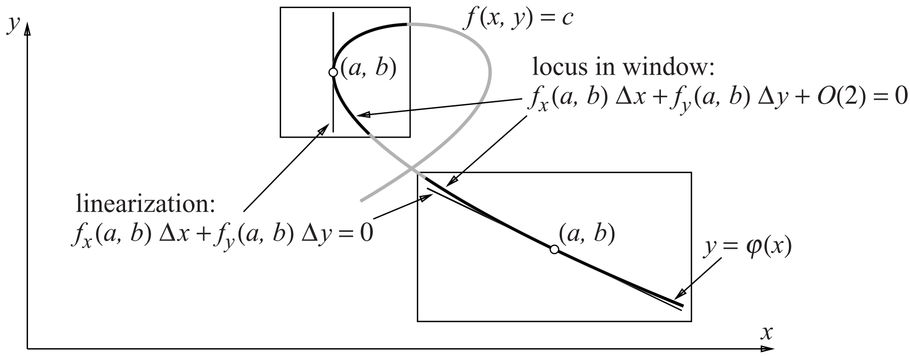
in the window. That is, the terms after $f(a,b)$ in the Taylor expansion must sum to zero. Within the window, this is the equation of the locus. If we delete the higher-order term $O(2)$, the remaining equation is called the **linearization of the locus at** (a,b):

$$f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y = 0.$$

This is a linear equation; if at least one coefficient $f_x(a,b)$, $f_y(a,b)$ is nonzero, it is the equation of a straight line, the tangent line to the locus at (a,b). Furthermore, if $f_y(a,b) \neq 0$, we can solve the linearized equation for Δy,

$$\Delta y = -\frac{f_x(a,b)}{f_y(a,b)}\Delta x,$$

implying that the tangent has finite slope $m = -f_x(a,b)/f_y(a,b)$.



The nature of the linearization

In the figure above, the locus $f(x,y) = k$ has been linearized at two different points (a,b), with fundamentally different results. At the lower right, the linearization has finite slope and implicitly defines the linear function $\Delta y = m\,\Delta x$. In particular, $f_y(a,b) \neq 0$. Furthermore, it is evident that the locus itself determines a unique implicit function $y = \varphi(x)$ when x is near a and $\varphi(a) = b$. We can even see that φ is differentiable at $x = a$ and

$$\varphi'(a) = m = -\frac{f_x(a,b)}{f_y(a,b)} = -\frac{f_x(a,\varphi(a))}{f_y(a,\varphi(a))}.$$

Now compare this to what happens at the second point, in the upper left. The linearization there is the vertical line $\Delta x = 0$. This means that $f_y(a,b) = 0$, and the linearization is not the graph of any implicit (linear) function of the form $\Delta y = m\,\Delta x$. Likewise, no implicit function of x can grow out of the seed point (a,b) on the original locus.

The linearization determines the implicit function

According to our evidence, the condition $f_y(a,b) \neq 0$ guarantees that the linearized locus determines a unique implicit function of x, and that is enough to ensure that the original locus does too. The evidence also suggests that we can connect the derivative of φ (where it exists) to the partial derivatives of f, extending the formula for $\varphi'(a)$ we found above. Just differentiate the identity $k = f(x,\varphi(x))$ using the chain rule to get

$$0 = \frac{d}{dx}f(x,\varphi(x)) = f_x(x,\varphi(x)) + f_y(x,\varphi(x))\,\varphi'(x).$$

Because f_y is continuous by hypothesis, the condition $f_y(a,b) \neq 0$ implies $f_y(x,y) \neq 0$ for all (x,y) sufficiently close to (a,b). This allows us to solve for $\varphi'(x)$:

$$\varphi'(x) = -\frac{f_x(x,\varphi(x))}{f_y(x,\varphi(x))}.$$

Theorem 6.1 (Implicit function theorem). *If $f(x,y)$ has continuous first derivatives in a neighborhood of the point (a,b), and $f(a,b)=k$, $f_y(a,b)\neq 0$, then there is a unique function $y=\varphi(x)$ defined and continuously differentiable on an open interval I containing a for which*

- $f(x,\varphi(x)) = k$ *for all x in I.*
- $\varphi(a) = b$.
- $\varphi'(x) = -\dfrac{f_x(x,\varphi(x))}{f_y(x,\varphi(x))}$ *for all x in I.*

No seed at a critical point

Before we prove the implicit function theorem, let us take a closer look at the condition $f_y(a,b)\neq 0$. It expresses our informal conjecture that the locus $f(x,y)=k$ should have a single nonvertical tangent at the seed point (a,b), but it is both more precise and more restrictive. For example, although the locus $f(x,y)=y^2-x^4=0$ (p. 187) appeared to have a single horizontal tangent at the origin, we find $f_x(0,0)=f_y(0,0)=0$, so the linearized locus is not a horizontal line; it is not a line at all. We call the origin a *critical point* of f.

Definition 6.1 *We say (a,b) is a **critical point** of the differentible function $f(x,y)$ if $f_x(a,b)=f_y(a,b)=0$. If either partial derivative is nonzero, we say (a,b) is a **regular point** of f.*

The implicit function theorem rules out any critical point of f as a seed. Indeed, in most of the problematic examples that led to our original conjecture, we were attempting to make a critical point be a seed.

Near a regular point, the locus is a curve

So suppose (a,b) is a regular point of the function $z=f(x,y)$. Either $f_y(a,b)\neq 0$ and the locus $f(x,y)=f(a,b)$ is the graph of a differentiable function of x near (a,b) (by the implicit function theorem), or else $f_x(a,b)\neq 0$ and, switching the roles of x and y, we see the locus is the graph of a differentiable function of y. In either case, the locus is the graph of some differentiable function and is thus a differentiable curve near (a,b).

Definition 6.2 *If (a,b) is a regular point of the continuously differentiable function $f(x,y)$, and $f(a,b)=k$, then we say (a,b) is a **regular point** of the curve $f(x,y)=k$. If all points on the locus are regular, we say the curve itself is **regular**.*

The locus $f(x,y)=k$ is one of the *level sets*, or *contours*, of f. At a regular point of a contour, at least one of the partial derivatives of f is different from zero. By continuity, that derivative remains nonzero at all sufficiently nearby points. Therefore, near the given regular point, all contours are regular. The following theorem says even more: it says that a suitably chosen coordinate change will "straighten out" those contours. This implies that there is essentially only one way to arrange the contours near a regular point. It also leads to a quick proof of the implicit function theorem.

Theorem 6.2. *Suppose (a,b) is a regular point of a function $z = f(x,y)$ that has continuous first derivatives. Then there is a coordinate change $(u,v) = \mathbf{h}(x,y)$ defined on a window centered at (a,b) that transforms the level curves of f into the coordinate lines $v =$ constant.*

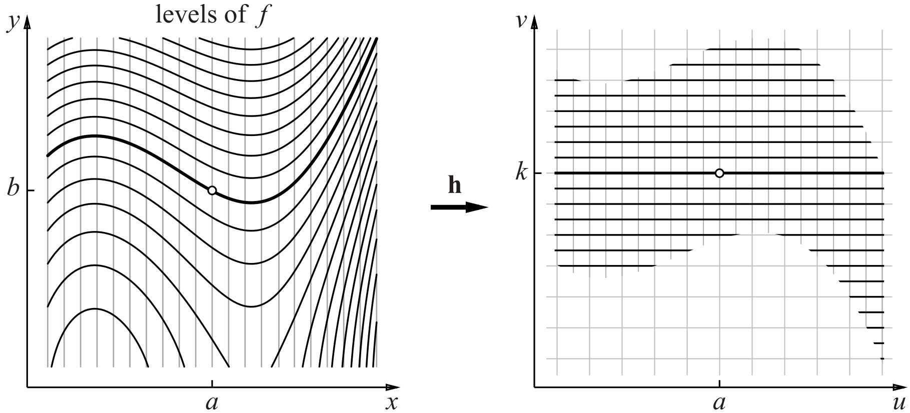
Proof. At least one of $f_x(a,b)$, $f_y(a,b)$ is nonzero; suppose $f_y(a,b) \neq 0$. Define $\mathbf{h}$ by the formulas

$$\mathbf{h} : \begin{cases} u = x, \\ v = f(x,y). \end{cases}$$

(If $f_y(a,b) = 0$, then $f_x(a,b) \neq 0$; set $u = y$ instead; see Exercise 6.10.) Because f has continuous first derivatives, so does $\mathbf{h}$. Moreover,

$$d\mathbf{h}_{(x,y)} = \begin{pmatrix} 1 & 0 \\ f_x(x,y) & f_y(x,y) \end{pmatrix},$$

so $\det d\mathbf{h}_{(a,b)} = f_y(a,b) \neq 0$, implying that $d\mathbf{h}_{(a,b)}$ is invertible. By the inverse function theorem (Theorem 5.2, p. 169), $\mathbf{h}$ has a continuously differentiable inverse defined on a neighborhood of $\mathbf{h}(a,b) = (a, f(a,b)) = (a,k)$. Thus $\mathbf{h}$ is a valid coordinate change near (a,b). Because $v = f(x,y)$, $\mathbf{h}$ transforms each level curve $f(x,y) = \lambda$ into the coordinate line $v = \lambda$. □



Straightening level curves with a nonlinear shear

Thus, near a regular point, the level curves of a real-valued function are part of a curvilinear coordinate system; near that point, those curves are always roughly parallel and evenly spaced. We can see this in the figure above, which shows a coordinate change that straightens the levels of $f(x,y) = y^2 - 4x^2(2x-1)$ near the point $(a,b) = (0.18, 0.417)$; $k = 0.35$. (Note: The origin is not at the intersection of the coordinate axes in either figure.) The coordinate change $\mathbf{h}$ is a nonlinear shear. It maps each vertical line to itself; this is the geometric meaning of the equation $u = x$ in the definition of $\mathbf{h}$. Each vertical line just slides up or down, stretched by different amounts at different points. At the point (α, β) on $x = \alpha$, the vertical stretch factor is $f_y(\alpha, \beta)$. For example, at points in the lower half of the figure above, we can see that the stretch factor is less than 1 (though still positive), because $\mathbf{h}$ shrinks vertical distances there. You can analyze another example, with a simpler function $f(x,y)$, in Exercise 6.8.

Proof of the implicit function theorem

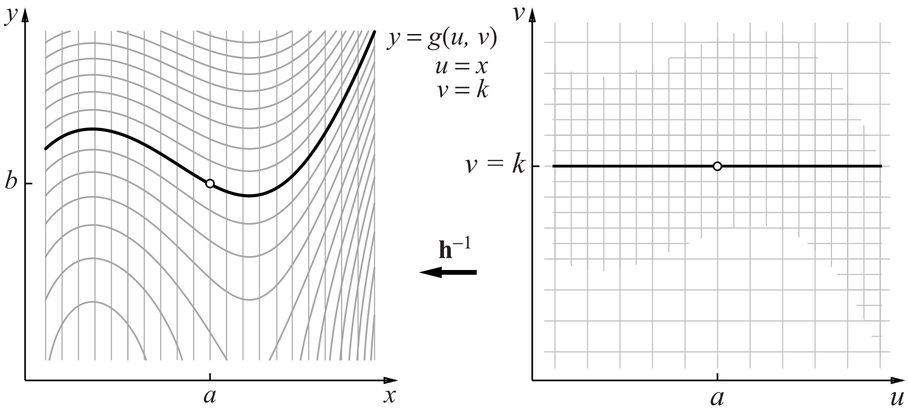
We take up now the proof of the implicit function theorem, Theorem 6.1. Let us write the inverse of $\mathbf{h}$ in terms of components:

$$\mathbf{h}^{-1}: \begin{cases} x = u, \\ y = g(u,v). \end{cases}$$

The first component is just the first component of $\mathbf{h}$ itself. The second component, g, is a continuously differentiable function of u and v. The inverse relation between $\mathbf{h}^{-1}$ and $\mathbf{h}$ implies

$$(x,y) = \mathbf{h}^{-1}(\mathbf{h}(x,y)) = \mathbf{h}^{-1}(x, f(x,y)) = (x, g(x, f(x,y))),$$

and, in particular, $y = g(x, f(x,y))$. Therefore, if $v = f(x,y) = k$, then $y = g(x,k)$. This is the implicit function we seek: $\varphi(x) = g(x,k)$. □



Making coordinate choices

The idea behind the proof of Theorem 6.1 is that it becomes easy to find the implicit function if we use the right coordinate system. We choose coordinates $(x,y) \mapsto (u,v)$ in which the level curves of $f(x,y)$ become coordinate lines $v = \lambda$. Then the implicit function defined by $f(x,y) = k$ is just the constant function $v = k$ in the new coordinates, and the inverse coordinate change $(u,v) \mapsto (x,y)$ converts this line $v = k$ into the graph of a (generally nonlinear) function $y = \varphi(x)$.

Geometry underlies analysis

Underlying the proof is a basic principle we have used several times: coordinates are just labels for points, and we should choose labels that make the geometry most intelligible. In this setting there are two fundamental objects, both geometric: the first is the plane $\mathcal{P}$ and its constituent points $\mathbf{p}$; the second is a real-valued map $\mathcal{F} : \mathcal{P} \to \mathbb{R} : \mathbf{p} \mapsto \mathcal{F}(\mathbf{p})$ defined on $\mathcal{P}$. To describe that map, we introduce analytic tools: coordinates (x,y) to label the points, and the appropriate expression $f(x,y)$ to represent $\mathcal{F}$:

$$
\begin{array}{rcccccc}
\text{analytic level:} & \mathbb{R}^2 & \xrightarrow{f} & \mathbb{R} & \qquad & (x,y) & \xrightarrow{f} & z \\
 & \uparrow & & \| & & \uparrow & & \| \\
\text{geometric level:} & \mathcal{P} & \xrightarrow{\mathcal{F}} & \mathbb{R} & & \mathbf{p} & \xrightarrow{\mathcal{F}} & z
\end{array}
$$

In practice, we start at the analytic level. Conceptually, though, it helps to let the geometry come first; the analysis is then overlaid as a language with which to describe it. For example, equations such as $f(x,y) = k$ are a way to describe level curves of the more fundamental geometric map $\mathbf{p} \mapsto \mathcal{F}(\mathbf{p})$.

Coordinates as languages

We can take the language analogy further. Just as the world of objects and ideas is described by a variety of human languages, the geometry of points and maps can be described by a variety of coordinate systems and analytic expressions. Two human languages are connected by a pair of translation dictionaries; the geometric analogue is a coordinate change. The following diagram shows how the two coordinate systems we used to analyze the level curves of $\mathcal{F}$ are related by the coordinate change $\mathbf{h}$. In the diagram, "v" stands for a *coordinate function* as well as a *coordinate*; as a function, it assigns to the ordered pair (u,v) the number v.

$$
\begin{tikzcd}
\mathbb{R}^2 \arrow[rr, "\mathbf{h}"] \arrow[ddr, "f"'] & & \mathbb{R}^2 \arrow[ddl, "v"] \\
 & \mathcal{P} \arrow[ul] \arrow[ur] \arrow[d, "\mathcal{F}"] & \\
 & \mathbb{R} &
\end{tikzcd}
\qquad
\begin{tikzcd}
(x,y) \arrow[rr, "\mathbf{h}"] \arrow[ddr, "f"'] & & (u,v) \arrow[ddl, "v"] \\
 & \mathbf{p} \arrow[ul] \arrow[ur] \arrow[d, "\mathcal{F}"] & \\
 & k &
\end{tikzcd}
$$

The geometric view

Before we resume our work on the implicit function theorem, let us pause to recall a couple of places where the geometric view has already come to the fore. One was in the study of 2×2 matrices. We viewed them as certain maps of the plane that are characterized geometrically (Theorem 2.6, p. 40) by their eigenvalues and eigenvectors. Like f and v in the diagram above, different matrices can represent the same geometric map. We defined two matrices to be *equivalent* if a linear coordinate change would convert one into the other, and we saw that equivalent matrices had the same geometric action. Another place where we ended up with a geometric view was with the inverse function theorem. According to Corollary 5.4 (p. 176), which was suggested by our work in the example on pages 161–165, a nonlinear map $\mathbf{f}: U^n \to \mathbb{R}^n$ looks like its linear approximation $d\mathbf{f}_\mathbf{a}$ near any point $\mathbf{a}$ where the linear approximation is invertible.

Geometry simplifies

So geometry helps us simplify, and it does so by "lumping together" things (such as equivalent matrices) that we would otherwise treat as distinct. In the diagram above, the coordinate change $\mathbf{h}$ that converts f into v allows us to "lump together" those two functions, and therefore to say that f is essentially a coordinate function. The simplification is this: near a regular point, any real-valued function is essentially just a coordinate function. From this observation we then get, first, the structure of the level curves, and second, the implicit function theorem.

The move from two variables to three is straightforward. The meaning of a regular point of a function or of a locus carries over in a natural way, as does the geometric viewpoint generally. The following theorem and corollary are the main results. Their proofs follow the proofs of the two-variable versions (see the exercises), and also follow from the n-variable versions, below.

Regular points with three variables

Theorem 6.3. *Suppose (a,b,c) is a regular point of a function $s = f(x,y,z)$ that has continuous first derivatives. Then there is a coordinate change $(u,v,w) = \mathbf{h}(x,y,z)$ defined on a window centered at (a,b,c) that transforms the level sets of f into the coordinate planes $w = $ constant.* □

Corollary 6.4 (Implicit function theorem) *Suppose the function $s = f(x,y,z)$ has continuous first derivatives in some open neighborhood of a point (a,b,c), and $f(a,b,c) = k$. If $f_z(a,b,c) \neq 0$, then there is a unique function $z = \varphi(x,y)$ defined on an open neighborhood N of (a,b) for which*

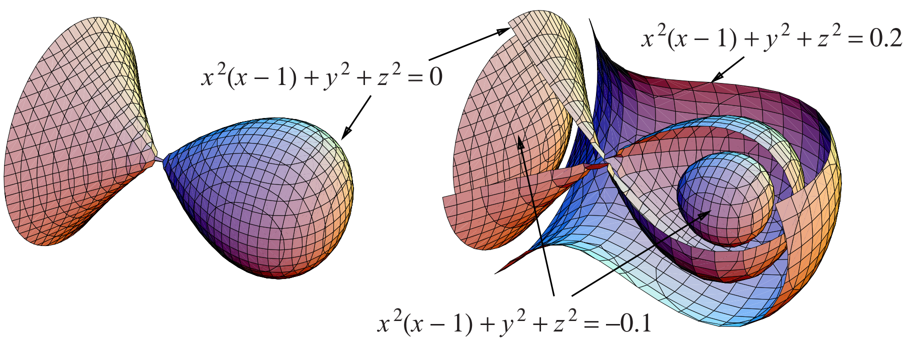
- *$f(x,y,\varphi(x,y)) = k$ for all (x,y) in N.*
- *$\varphi(a,b) = c$.*
- *φ has continuous first derivatives on N, and*

$$\varphi_x(x,y) = -\frac{f_x(x,y,\varphi(x,y))}{f_z(x,y,\varphi(x,y))}, \quad \varphi_y(x,y) = -\frac{f_y(x,y,\varphi(x,y))}{f_z(x,y,\varphi(x,y))}. \qquad \square$$

Different implications of the corollary

As stated, the corollary connects the condition that the partial derivative of f with respect to z is nonzero to the conclusion that z depends on x and y near (a,b,c). However, if instead it is the partial derivative with respect to either y or x that is nonzero, we get the same conclusion *mutatis mutandis* ("the necessary changes being made"). Corollary 6.4 thus stands for three different statements; for example, if $f_y(a,b,c) \neq 0$, then $y = \psi(x,z)$ for some ψ and

$$\psi_x(x,z) = -\frac{f_x(x,\psi(x,z),z)}{f_y(x,\psi(x,z),z)}, \quad \psi_z(x,z) = -\frac{f_z(x,\psi(x,z),z)}{f_y(x,\psi(x,z),z)}.$$



An example

Even though the theorem and corollary above settle our questions about a function of three variables, it is still valuable to look at an individual level set and its linearization at a point, as we did above with a function of two variables. We see above, left, the zero level of

$$f(x,y,z) = x^2(x-1) + y^2 + z^2;$$

on the right, we get some idea how the zero level is nested within the collection of nearby level sets. (Parts of some levels have been "peeled away" to help see inside.)

We expect the locus $f(x,y,z) = k$ to be 2-dimensional, although it may fail to be a proper surface at one or more of its points. That is, a point may fail to be a regular point of the locus. This is what happens to $f(x,y,z) = 0$ at the origin, where it has the shape of the vertex of a cone: no coordinate change can convert the vertex into a simple plane. (In this example, however, every nearby level set contains only regular points of f, at which Theorem 6.3 applies.) In general, a locus $f(x,y,z) = k$ can exhibit all the irregularities that $f(x,y) = k$ did, and many more besides (see Exercise 6.13).

Linearization and the tangent plane

By considering the linearization of $s = f(x,y,z)$, we can see once again what prompts the partial derivative condition that leads us to an implicit function. We begin by assuming, as before, that f has continuous partial derivatives. Then the first-order Taylor expansion of f at a seed point (i.e., $f(a,b,c) = k$) is

$$f(x,y,z) = f(a,b,c) + f_x(a,b,c)\,\Delta x + f_y(a,b,c)\,\Delta y + f_z(a,b,c)\,\Delta z + O(2)$$

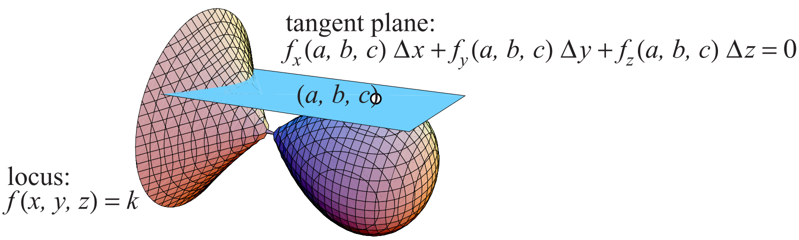
in a window centered at (a,b,c). We take

$$f_x(a,b,c)\,\Delta x + f_y(a,b,c)\,\Delta y + f_z(a,b,c)\,\Delta z + O(2) = 0$$

to be the equation of the locus $f(x,y,z) = k$ in that window, and

$$f_x(a,b,c)\,\Delta x + f_y(a,b,c)\,\Delta y + f_z(a,b,c)\,\Delta z = 0$$

to be the linearization of that locus. If (a,b,c) is a regular point of f, then at least one of the coefficients is nonzero, and the equation describes a plane. Because the difference between the locus and its linearization at (a,b,c) vanishes at least to second order in $(\Delta x, \Delta y, \Delta z)$, the plane is tangent to the original locus at (a,b,c).



More particularly, if $f_z(a,b,c) \neq 0$, then the linearization implicitly defines the linear function

$$\Delta z = -\frac{f_x(a,b,c)}{f_z(a,b,c)}\,\Delta x - \frac{f_y(a,b,c)}{f_z(a,b,c)}\,\Delta y,$$

and this is another version of the equation of the tangent plane at (a,b,c). The analogy with the two-variable case means that the implicit function $z = \varphi(x,y)$ has this equation as its linear approximation near $(x,y) = (a,b)$, and

$$\varphi_x(a,b) = -\frac{f_x(a,b,c)}{f_z(a,b,c)}, \quad \varphi_y(a,b) = -\frac{f_y(a,b,c)}{f_z(a,b,c)}.$$

Regular points with n variables

Finally, let us suppose the number of variables $x_1, x_2, \ldots, x_n$ is arbitrary, with a single relation $f(x_1,\ldots,x_n) = k$ holding between them. Then we expect one of the variables, say x_n, to depend on the others, implying there is a function $x_n = \varphi(x_1,\ldots,x_{n-1})$. The graph of φ is an $(n-1)$-dimensional hypersurface in $\mathbb{R}^n$. Nearby level sets $f(x_1,\ldots,x_n) = \lambda$, for λ near k, should be nested hypersurfaces that together fill a portion of $\mathbb{R}^n$. These expectations are borne out at a regular point of $f(x)$, that is, at a point $(a_1,\ldots,a_n)$ where at least one partial derivative $\partial f/\partial x_i(a_1,\ldots,a_n)$ is nonzero.

Theorem 6.5. *Suppose the function $f : X^n \to \mathbb{R} : \mathbf{x} \mapsto f(\mathbf{x})$ has continuous first derivatives on X^n, and $\mathbf{x} = \mathbf{a}$ is a regular point of $s = f(\mathbf{x})$. Then there is a coordinate change $\mathbf{u} = \mathbf{h}(\mathbf{x})$ defined on a window W^n centered at $\mathbf{x} = \mathbf{a}$ that transforms the level sets of f into the coordinate hyperplanes $u_n =$ constant.*

Proof. Because $\mathbf{a}$ is a regular point of f, at least one of the partial derivatives $f_i(\mathbf{a})$ is nonzero. (We define f_i to be the partial derivative of f with respect to the i-th variable, x_i.) By permuting the variables x_i, if necessary, we may suppose that $f_n(\mathbf{a}) \neq 0$. Define $\mathbf{h} : X^n \to \mathbb{R}^n : \mathbf{x} \mapsto \mathbf{u}$ by

$$\mathbf{h} : \begin{cases} u_1 = x_1, \\ \quad\vdots \\ u_{n-1} = x_{n-1}, \\ u_n = f(x_1,\ldots,x_n). \end{cases}$$

Then $\mathbf{h}$ is continuously differentiable on X^n because f is, and

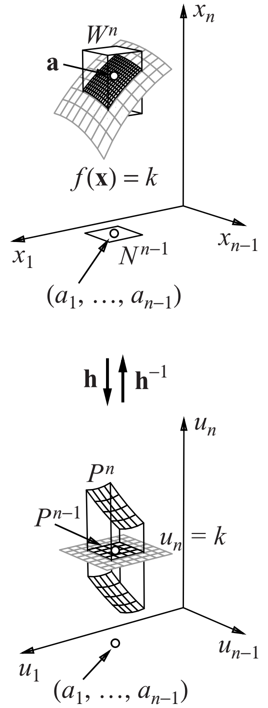
$$d\mathbf{h}_{\mathbf{x}} = \begin{pmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ f_1(\mathbf{x}) & \cdots & f_{n-1}(\mathbf{x}) & f_n(\mathbf{x}) \end{pmatrix}.$$

Therefore $\det d\mathbf{h}_{\mathbf{a}} = f_n(\mathbf{a}) \neq 0$, and the inverse function theorem implies $\mathbf{h}$ is invertible on some neighborhood W^n of $\mathbf{a}$. The coordinate change $\mathbf{h}$ transforms the level set $f(\mathbf{x}) = \lambda$ into the coordinate hyperplane $u_n = \lambda$. $\square$

Corollary 6.6 (Implicit function theorem) *Suppose the function $s = f(x_1,\ldots,x_n)$ has continuous first derivatives on some neighborhood of a point $(a_1,\ldots,a_n)$, and $f(a_1,\ldots,a_n) = k$. If $f_n(a_1,\ldots,a_n) \neq 0$, then there is a unique function $x_n = \varphi(x_1,\ldots,x_{n-1})$ defined on an open neighborhood N^{n-1} of $(a_1,\ldots,a_{n-1})$ for which*

- $f(x_1,\ldots,x_{n-1},\varphi(x_1,\ldots,x_{n-1})) = k$ *for all* $(x_1,\ldots,x_{n-1})$ *in* N^{n-1}.
- $\varphi(a_1,\ldots,a_{n-1}) = a_n$.
- φ *has continuous first derivatives on* N^{n-1}*, and for* $i = 1,\ldots,n-1$,

$$\varphi_i(x_1,\ldots,x_{n-1}) = -\frac{f_i(x_1,\ldots,x_{n-1},\varphi(x_1,\ldots,x_{n-1}))}{f_n(x_1,\ldots,x_{n-1},\varphi(x_1,\ldots,x_{n-1}))}.$$



Proof. Let $\mathbf{h}$ be the coordinate change in Theorem 6.5; because it is the identity on the first $n-1$ coordinates, it is a nonlinear shear that maps each vertical line $(x_1,\ldots,x_{n-1}) = (c_1,\ldots,c_{n-1})$ to itself. Its inverse must do the same, and thus has the form

$$\mathbf{h}^{-1}: \begin{cases} x_1 = u_1, \\ \quad\vdots \\ x_{n-1} = u_{n-1}, \\ x_n = g(u_1,\ldots,u_n). \end{cases}$$

Here g is a real-valued function with continuous derivatives on a neighborhood P^n of the image point $\mathbf{h}(\mathbf{a})$. Let P^{n-1} be the intersection of P^n and the horizontal plane $u_n = k$. Because $\mathbf{h}$ and $\mathbf{h}^{-1}$ preserve vertical lines, it is convenient to put the target space $(u_1,\ldots,u_n)$ directly below the source. Also, for visual clarity, P^n is shown as the sheared image of a box W^n centered at $\mathbf{a}$.

Because $\mathbf{h}^{-1} \circ \mathbf{h}$ is the identity, we can write

$$\begin{aligned}(x_1,\ldots,x_{n-1},x_n) &= \mathbf{h}^{-1}(\mathbf{h}(x_1,\ldots,x_{n-1},x_n)) \\ &= \mathbf{h}^{-1}(u_1,\ldots,u_{n-1},f(x_1,\ldots,x_{n-1},x_n)).\end{aligned}$$

Now assume that the point $(u_1,\ldots,u_{n-1},f(x_1,\ldots,x_{n-1},x_n))$ is in P^{n-1}. In particular, this means $f(x_1,\ldots,x_n) = k$, and we can write

$$\begin{aligned}(x_1,\ldots,x_{n-1},x_n) &= \mathbf{h}^{-1}(u_1,\ldots,u_{n-1},k) \\ &= (x_1,\ldots,x_{n-1},g(x_1,\ldots,x_{n-1},k)).\end{aligned}$$

The figure makes it clear that N^{n-1} (which we must still define) and P^{n-1} are in the same vertical column. Thus we make N^{n-1} the projection of P^{n-1} to the coordinate plane $x_n = 0$; that is, $(x_1,\ldots,x_{n-1})$ is in N^{n-1} if and only if $(x_1,\ldots,x_{n-1},k)$ is in P^{n-1}. Finally, if we set $\varphi(x_1,\ldots,x_{n-1}) = g(x_1,\ldots,x_{n-1},k)$ when $(x_1,\ldots,x_{n-1})$ is in N^{n-1}, then

$$f(x_1,\ldots,x_n) = k \quad\Longleftrightarrow\quad x_n = \varphi(x_1,\ldots,x_{n-1})$$

and $\varphi(a_1,\ldots,a_{n-1}) = g(a_1,\ldots,a_{n-1},k) = a_n$. The expressions for the partial derivatives of φ follow from the chain rule applied to

$$k = f(x_1,\ldots,x_{n-1},\varphi(x_1,\ldots,x_{n-1}));$$

we find $0 = f_i + f_n \cdot \varphi_i$, from which it follows that $\varphi_i = -f_i/f_n$. $\square$

"Symmetrizing" the implicit function theorem

As it is written, the implicit function theorem assumes that $f_n(\mathbf{a}) \neq 0$ at the seed point $\mathbf{a}$. But suppose $f_n(\mathbf{a}) = 0$; the theorem still holds, *mutatis mutandis*, if some other partial derivative is nonzero there. For example, if $f_j(\mathbf{a}) \neq 0$, then we can solve for x_j in terms of the other variables to get a function

$$x_j = \psi(x_1, \dots, \widehat{x}_j, \dots, x_n).$$

Here the circumflex is used to indicate that the variable x_j is missing from the list. The theorem implies $\psi(a_1, \dots, \widehat{a}_j, \dots, a_n) = a_j$, and

$$\psi_i(x_1, \dots, \widehat{x}_j, \dots, x_n) = -\frac{f_i(x_1, \dots, \psi(x_1, \dots, \widehat{x}_j, \dots, x_n), \dots, x_n)}{f_j(x_1, \dots, \psi(x_1, \dots, \widehat{x}_j, \dots, x_n), \dots, x_n)}$$

for every $i = 1, \dots, \widehat{j}, \dots, n$.

Near a regular point $\mathbf{a}$, f looks like $\mathrm{d}f_{\mathbf{a}}$

Suppose $z = f(\mathbf{x})$ has continuous second partial derivatives at $\mathbf{a}$, allowing us to write the first-order Taylor expansion of f (in terms of window coordinates) at $\mathbf{a}$:

$$\begin{aligned}\Delta z = f(\mathbf{a} + \Delta\mathbf{x}) - f(\mathbf{a}) &= f_1(\mathbf{a})\,\Delta x_1 + \cdots + f_n(\mathbf{a})\,\Delta x_n + O(2) \\ &= \mathrm{d}f_{\mathbf{a}}(\Delta\mathbf{x}) + O(2).\end{aligned}$$

If $\mathbf{a}$ is a regular point of f, then it follows from Theorem 6.5 that there are new curvilinear coordinates in which the higher-order terms $O(2)$ disappear: f is transformed into precisely its linear approximation $\mathrm{d}f_{\mathbf{a}}$ near $\mathbf{a}$. The details are in the following corollary, which incidentally is stronger than Taylor's theorem because it requires only continuous first partial derivatives for f.

Corollary 6.7 *Suppose $\mathbf{x} = \mathbf{a}$ is a regular point of the continuously differentiable function $z = f(\mathbf{x})$. Then there is a coordinate change $\Delta\mathbf{v} = \mathbf{g}(\Delta\mathbf{x})$ in a window centered at $\mathbf{a}$ for which*

$$\Delta z = f(\mathbf{a} + \mathbf{g}^{-1}(\Delta\mathbf{v})) - \mathbf{f}(\mathbf{a}) = \mathrm{d}f_{\mathbf{a}}(\Delta\mathbf{v}) = f_1(\mathbf{a})\,\Delta v_1 + \cdots + f_n(\mathbf{a})\,\Delta v_n.$$

Proof. Express the coordinate change $\mathbf{h}$ that is provided by Theorem 6.5 in window coordinates $\Delta\mathbf{x}$ centered at $\mathbf{a}$ and $\Delta\mathbf{u}$ centered at $(a_1, \dots, a_{n-1}, f(\mathbf{a}))$ (so that $\Delta\mathbf{u} = \mathbf{h}(\Delta\mathbf{x})$ and $\mathbf{h}(\mathbf{0}) = \mathbf{0}$):

$$\mathbf{h} : \begin{cases} \Delta u_1 = \Delta x_1, \\ \quad\vdots \\ \Delta u_{n-1} = \Delta x_{n-1}, \\ \Delta u_n = f(\mathbf{a} + \Delta\mathbf{x}) - f(\mathbf{a}); \end{cases} \qquad \mathrm{d}\mathbf{h}_{\mathbf{0}} = \begin{pmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ f_1(\mathbf{a}) & \cdots & f_{n-1}(\mathbf{a}) & f_n(\mathbf{a}) \end{pmatrix}.$$

In terms of these coordinates, f is already transformed into the simple linear function $\Delta z = \Delta u_n$. That is,

$$\Delta z = f(\mathbf{a} + \mathbf{h}^{-1}(\Delta\mathbf{u})) - f(\mathbf{a}) = \Delta u_n.$$

Now consider the linear map $\Delta\mathbf{u} = d\mathbf{h_0}(\Delta\mathbf{v})$, whose nth component function is

$$\Delta u_n = f_1(\mathbf{a})\,\Delta v_1 + \cdots + f_n(\mathbf{a})\,\Delta v_n.$$

Thus, if we set $\mathbf{g} = d\mathbf{h_0}^{-1} \circ \mathbf{h}$ (so that $\mathbf{g}^{-1} = \mathbf{h}^{-1} \circ d\mathbf{h_0}$), then $\mathbf{g}$ is a valid change of window coordinates in a window centered at $\mathbf{a}$, and we get

$$\begin{aligned}\Delta z &= f(\mathbf{a} + \mathbf{g}^{-1}(\Delta\mathbf{v})) - f(\mathbf{a})\\ &= f(\mathbf{a} + \mathbf{h}^{-1}(\Delta\mathbf{u})) - f(\mathbf{a})\\ &= f_1(\mathbf{a})\,\Delta v_1 + \cdots + f_n(\mathbf{a})\,\Delta v_n.\end{aligned}$$ □

Near a regular point, f becomes a coordinate function

Although there is a certain "fitness" to showing that a function can be transformed exactly into its derivative at a regular point, it is useful to know that it can also be transformed into a simple coordinate function. In other words, there is a curvilinear coordinate system in which one of the coordinates is just the value of f there. This result, stated in the next corollary, has already been demonstrated in the last proof.

Corollary 6.8 *Suppose* $\mathbf{x} = \mathbf{a}$ *is a regular point of the continuously differentiable function* $z = f(\mathbf{x})$*. Then there is a coordinate change* $\Delta\mathbf{u} = \mathbf{h}(\Delta\mathbf{x})$ *in a window centered at* $\mathbf{a}$ *for which*

$$\Delta z = f(\mathbf{a} + \mathbf{h}^{-1}(\Delta\mathbf{u})) - \mathbf{f}(\mathbf{a}) = \Delta u_n.$$ □

6.2 A pair of equations

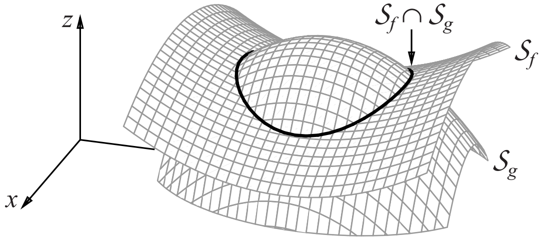
We now suppose that two separate conditions have been imposed on our variables:

$$f(x_1,\ldots,x_n) = k, \quad g(x_1,\ldots,x_n) = l.$$

Because we expect these conditions will allow us to solve for two of the variables in terms of the remaining ones, it is natural to assume that $n > 2$, that is, that there are more variables than conditions. However, if we set aside the matter of implicit functions for the moment, and just consider the geometric implications of the two conditions, there is no reason to exclude $n = 1$ or 2. We take up this possibility in the last section (cf. pp. 214ff.). To begin, however, we assume $n = 3$; this is the most complicated case that we can visualize fully.

In general, the locus is a space curve

Thus, we are dealing with two conditions $f(x,y,z) = k$ and $g(x,y,z) = l$ on three variables. We expect the locus $f = k$ to be a 2-dimensional surface $\mathcal{S}_f$ in $\mathbb{R}^3$, and $g = l$ to be another such surface, $\mathcal{S}_g$. Of course, either of these could fail to look like an ordinary surface at one or more points; for the moment, though, let us assume they are completely regular. The locus determined by two conditions together is the intersection $\mathcal{S}_f \cap \mathcal{S}_g$. We expect the intersection of two surfaces to be a curve in space, but it may not be, even when the surfaces themselves are regular; see the counterexamples below.



Solving for two of the variables

For simplicity, let us assume the intersection locus is, indeed, an ordinary space curve. We should then be able to describe it parametrically:

$$\mathcal{S}_f \cap \mathcal{S}_g : (x(t), y(t), z(t)), \quad t_1 \le t \le t_2.$$

However, we should not be introducing a new variable t. In the spirit of implicit functions, we should, instead, try to express two of x, y, z in terms of the remaining one, and involve no new variable at all. To recover these implicit functions from the parametrization, let us suppose that $x = x(t)$ is invertible, perhaps for t in some smaller interval. If $t = \tau(x)$ is the inverse (on $x_1 \le x \le x_2$), then

$$(x(t), y(t), z(t)) = (x, y(\tau(x)), z(\tau(x))) = (x, \varphi(x), \psi(x)).$$

If $\mathcal{S}_f \cap \mathcal{S}_g$ has more than one point with a given x-value (as in the figure above), we cannot parametrize all of it by x. Restricting the values of x as necessary, we can solve for y and z in terms of x:

$$\begin{aligned} f(x,y,z) &= k, \\ g(x,y,z) &= l, \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} y &= \varphi(x), \\ z &= \psi(x), \end{aligned} \quad x_1 \le x \le x_2.$$

Although we can expect $x(t)$ to be invertible on only a part of the locus $\mathcal{S}_f \cap \mathcal{S}_g$, on a different part we may find that $y(t)$ is invertible. On that subset we can solve for x and z in terms of y; on a subset where $z(t)$ is invertible, we can solve for x and y in terms of z. To summarize: two conditions on three variables implicitly define two of the variables in terms of the third.

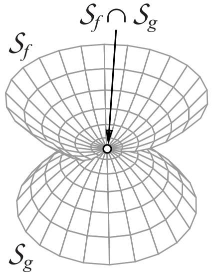
Faulty intersections

Before we discuss precise conditions that allow us to determine those implicit functions, let us see how two ordinary surfaces can fail to intersect in a curve. Thus, we suppose (a, b, c) is a regular point on each of surfaces,

$$\mathcal{S}_f : f(a,b,c) = k, \quad \mathcal{S}_g : g(a,b,c) = l.$$

This means each locus has a well-defined tangent plane at (a, b, c); moreover (Theorem 6.3, p. 193), each locus can be locally transformed into a flat plane by a suitable coordinate change. For these reasons we can regard the locus as a regular surface near (a, b, c).

In each of the following three examples, (a,b,c) is the origin. Moreover, each surface is the graph of a continuously differentiable function, so every point is regular. The first example is



$$\mathcal{S}_f : x^2 + y^2 - z = 0, \quad \mathcal{S}_g : -x^2 - y^2 - z = 0.$$

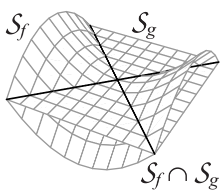
The surfaces are parabolic bowls that meet only at the origin; $\mathcal{S}_f \cap \mathcal{S}_g$ is a point, not a curve. The second example is

$$\mathcal{S}_f : x^2 + y^2 - z = 0, \quad \mathcal{S}_g : z = 0.$$

This time the intersection is a pair of crossed lines, so, once again, it is not a curve. For the third example, let $\mathcal{S}_f$ be the graph of the "flat" function defined on page 186, and let $\mathcal{S}_g$ again be the horizontal plane $z = 0$. The intersection is the whole unit disk in the (x,y)-plane.

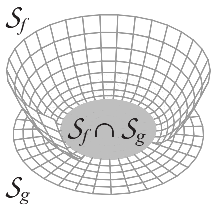
Transversality and general position

The problem with each example is that, even though the two surfaces meet at the origin, they do not cut cleanly across each other at that point. We say that the surfaces fail to be *transverse* there. Surfaces whose intersections are all transverse are also said to be *in general position* with respect to each other. We prove that the intersection of two regular surfaces in general position is a regular curve. To do this, we need to make our informal definition of transversality precise. The key is to note that whenever two surfaces are transverse in the informal sense, so are their tangent planes. But it is much easier to check transversality for the planes than for the surfaces: two planes passing through the same point are either different or identical.



Definition 6.3 *We say that two surfaces in $\mathbb{R}^3$ are* ***transverse*** *at a regular point of intersection if they have different tangent planes at that point.*

For surfaces that are given as the loci of equations, the following theorem gives us a simple and convenient analytic criterion for transversality.



Theorem 6.9. *Suppose* $\mathbf{a} = (a,b,c)$ *lies on both surfaces* $\mathcal{S}_f : f(x,y,z) = k$ *and* $\mathcal{S}_g : g(x,y,z) = l$, *and is a regular point of both* f *and* g. *Then* $\mathcal{S}_f$ *and* $\mathcal{S}_g$ *are transverse at* $\mathbf{a}$ *if and only if the matrix*

$$M = \begin{pmatrix} f_x(\mathbf{a}) & f_y(\mathbf{a}) & f_z(\mathbf{a}) \\ g_x(\mathbf{a}) & g_y(\mathbf{a}) & g_z(\mathbf{a}) \end{pmatrix}$$

has rank 2.

Proof. The equations of the tangent planes of $\mathcal{S}_f$ and $\mathcal{S}_g$ at $\mathbf{a}$ are, respectively,

$$f_x(\mathbf{a})\,\Delta x + f_y(\mathbf{a})\,\Delta y + f_z(\mathbf{a})\,\Delta z = 0,$$
$$g_x(\mathbf{a})\,\Delta x + g_y(\mathbf{a})\,\Delta y + g_z(\mathbf{a})\,\Delta z = 0.$$

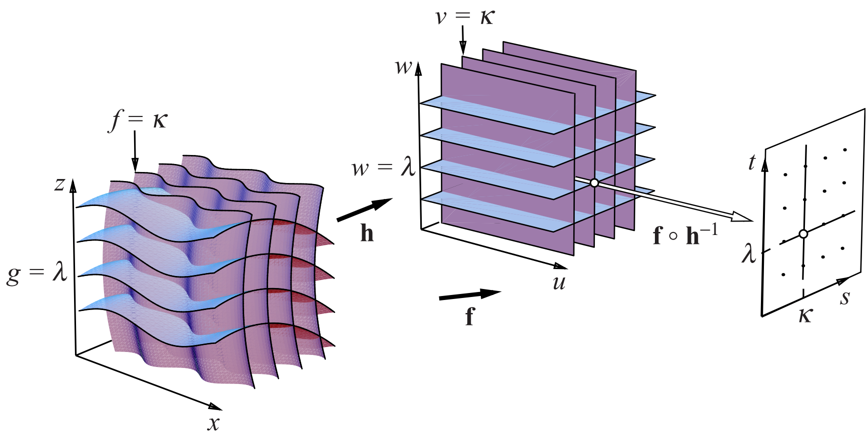
Because $\mathbf{a}$ is a regular point of both functions, each of these equations has at least one nonzero coefficient, and thus determines a well-defined plane. The two planes are different if and only if their coefficent vectors

$$(f_x(\mathbf{a}), f_y(\mathbf{a}), f_z(\mathbf{a})) \text{ and } (g_x(\mathbf{a}), g_y(\mathbf{a}), g_z(\mathbf{a}))$$

are not scalar multiples of each other, and this is true if and only if the matrix M has rank 2. □

Straightening the surfaces

We now return to the question of implicit functions. As in the past, the answer is a consequence of a coordinate change that makes the loci straight. Thus, instead of just $\mathcal{S}_f : f(x,y,z) = k$, we look at the whole family of nearby surfaces $f(x,y,z) = \kappa$, for $\kappa \approx k$. These are nested surfaces that fill a region containing the seed point (a,b,c). Likewise, we look at the family of surfaces $g(x,y,z) = \lambda$ ($\lambda \approx l$) that are nested around $\mathcal{S}_g$; they too fill a region around the seed point. If $\mathcal{S}_f$ and $\mathcal{S}_g$ are transverse at (a,b,c), then (as we prove in a moment), all surfaces in the first family are in general position with respect to all those in the second. After they are straightened by the coordinate change, members of the two families look like the spacers in a case of wine bottles; they intersect in parallel straight lines.



The map defined by f and g

The figure above suggests we should consolidate the functions f and g into a map $\mathbf{f} : X^3 \to \mathbb{R}^2$,

$$\mathbf{f} : \begin{cases} s = f(x,y,z), \\ t = g(x,y,z). \end{cases}$$

Then the surface S_f is the pullback of the (vertical) coordinate line $s = k$ by $\mathbf{f}$:

$$\mathbf{f}^{-1}(k,t) = \{(x,y,z) : f(x,y,z) = k\}.$$

The other surfaces in the same family are the pullbacks $\mathbf{f}^{-1}(\kappa,t)$ of the other vertical coordinate lines. The pullbacks $\mathbf{f}^{-1}(s,\lambda)$ of the horizontal coordinate lines are the second family of surfaces, that is, the ones nested with $\mathcal{S}_g$. The intersection curve $\mathcal{S}_f \cap \mathcal{S}_g$ is the pullback of the single point (k,l) in the (s,t)-plane. (The terms *locus* and *pullback* are roughly equivalent. The first is older and commonly used with real-valued functions; the second is used more generally with maps to an arbitrary target.)

The map $\mathbf{f}$ also gives us a convenient way to indicate when $\mathcal{S}_f$ and $\mathcal{S}_g$ are transverse at the seed point $\mathbf{a} = (a,b,c)$, because the matrix M of the previous theorem is the derivative of $\mathbf{f}$ at $\mathbf{a}$. Here is the theorem that "straightens" the surfaces $f = \kappa$ and $g = \lambda$ simultaneously.

Theorem 6.10. *Let $\mathbf{f} : X^3 \to \mathbb{R}^2$ be continuously differentiable in a neighborhood X^3 of a point $\mathbf{a} = (a,b,c)$, let $\mathbf{f}(a,b,c) = (k,l)$, and assume the derivative $d\mathbf{f}_{\mathbf{a}} : \mathbb{R}^3 \to \mathbb{R}^2$ has maximal rank. Then, on a smaller neighborhood N^3 of (a,b,c), there is a coordinate change $\mathbf{h} : N^3 \to \mathbb{R}^3$ that maps each pullback $\mathbf{f}^{-1}(\kappa,\lambda)$ to the coordinate line $v = \kappa$, $w = \lambda$ in $\mathbf{h}(N^3)$.*

Proof. The 2×3 matrix $d\mathbf{f}_{\mathbf{a}}$ has rank 2; therefore it has two linearly independent columns. By permuting the variables x, y, z, if necessary, we may assume the second and third columns are linearly independent. Thus,

$$D(\mathbf{x}) = \det \begin{pmatrix} f_y(\mathbf{x}) & f_z(\mathbf{x}) \\ g_y(\mathbf{x}) & g_z(\mathbf{x}) \end{pmatrix} \neq 0$$

when $\mathbf{x} = \mathbf{a}$. Because $\mathbf{f}$ is continuously differentiable, $D(\mathbf{x})$ is a continuous function of $\mathbf{x}$ and therefore remains nonzero in some neighbborhood Y^3 of $\mathbf{a}$. Define $\mathbf{h} : Y^3 \to \mathbb{R}^3$ by

$$\mathbf{h} : \begin{cases} u = x, \\ v = f(x,y,z), \\ w = g(x,y,z). \end{cases}$$

Then $\mathbf{h}$ is continuously differentiable, and

$$d\mathbf{h}_{\mathbf{x}} = \begin{pmatrix} 1 & 0 & 0 \\ f_x(\mathbf{x}) & f_y(\mathbf{x}) & f_z(\mathbf{x}) \\ g_x(\mathbf{x}) & g_y(\mathbf{x}) & g_z(\mathbf{x}) \end{pmatrix}.$$

By construction, $\det d\mathbf{h}_{\mathbf{x}} = D(\mathbf{x}) \neq 0$ for all $\mathbf{x}$ in Y^3. According to the inverse function theorem, $\mathbf{h}$ (on a possibly small neighborhood N^3) has a continuously differentiable inverse $\mathbf{h}^{-1}$. By the definition of $\mathbf{f}$,

$$\begin{aligned} f(x,y,z) = \kappa \quad &\Longleftrightarrow \quad v = \kappa, \\ g(x,y,z) = \lambda \quad &\Longleftrightarrow \quad w = \lambda, \end{aligned}$$

whenever (x,y,z) is in N^3. □

Extensions of the theorem

Note that the proof goes beyond what the theorem states: it shows that the coordinate change transforms the individual surfaces $f(x,y,z) = \kappa$ into the coordinate planes $v = \kappa$, and the surfaces $g(x,y,z) = \lambda$ into the coordinate planes $w = \lambda$ of a second family. Moreover, because the two families of coordinate planes are obviously in general position with respect to each other, the same must be true of the original curved families.

The figure also makes it clear that the map

$$\mathbf{f} \circ \mathbf{h}^{-1} : (u, v, w) \to (s, t)$$

is just projection along the first component. That is, each coordinate line $(u, v, w) = (u, \kappa, \lambda)$ parallel to the u-axis projects to the single point (κ, λ). Putting it another way: the pullback (by $\mathbf{f} \circ \mathbf{h}^{-1}$) of any point in the (s, t)-plane is the line parallel to the u-axis that projects to that point.

Corollary 6.11 (Implicit function theorem) *Let $f(x,y,z)$ and $g(x,y,z)$ have continuous first derivatives in some neighborhood of a point $\mathbf{a} = (a, b, c)$, and let $f(a, b, c) = k$, $g(a, b, c) = l$. If the determinant*

$$\begin{vmatrix} f_y(\mathbf{a}) & f_z(\mathbf{a}) \\ g_y(\mathbf{a}) & g_z(\mathbf{a}) \end{vmatrix}$$

is nonzero, then there are unique functions $y = \varphi(x)$, $z = \psi(x)$ defined on an open interval I containing $x = a$ for which

- *$f(x, \varphi(x), \psi(x)) = k$ and $g(x, \varphi(x), \psi(x)) = l$ for all x in I.*
- *$\varphi(a) = b$, $\psi(a) = c$.*
- *φ and ψ are continuously differentiable on I, and*

$$\varphi'(x) = -\frac{\begin{vmatrix} f_x(x, \varphi(x), \psi(x)) & f_z(x, \varphi(x), \psi(x)) \\ g_x(x, \varphi(x), \psi(x)) & g_z(x, \varphi(x), \psi(x)) \end{vmatrix}}{\begin{vmatrix} f_y(x, \varphi(x), \psi(x)) & f_z(x, \varphi(x), \psi(x)) \\ g_y(x, \varphi(x), \psi(x)) & g_z(x, \varphi(x), \psi(x)) \end{vmatrix}},$$

$$\psi'(x) = -\frac{\begin{vmatrix} f_y(x, \varphi(x), \psi(x)) & f_x(x, \varphi(x), \psi(x)) \\ g_y(x, \varphi(x), \psi(x)) & g_x(x, \varphi(x), \psi(x)) \end{vmatrix}}{\begin{vmatrix} f_y(x, \varphi(x), \psi(x)) & f_z(x, \varphi(x), \psi(x)) \\ g_y(x, \varphi(x), \psi(x)) & g_z(x, \varphi(x), \psi(x)) \end{vmatrix}}.$$

Proof. Let $\mathbf{h}$ be the coordinate change in Theorem 6.10. Because $\mathbf{h}$ is the identity on the first coordinate, the same must be true of its inverse:

$$\mathbf{h}^{-1} : \begin{cases} x = u, \\ y = p(u, v, w), \\ z = q(u, v, w). \end{cases}$$

Because $\mathbf{h}^{-1} \circ \mathbf{h}$ is the identity where it is defined,

$$\begin{aligned} (x,y,z) &= \mathbf{h}^{-1} \circ \mathbf{h}(x,y,z) \\ &= \mathbf{h}^{-1}(x, f(x,y,z), g(x,y,z)) \\ &= (x, p(x, f(x,y,z), g(x,y,z)), q(x, f(x,y,z), g(x,y,z))), \end{aligned}$$

implying

$$y = p(x, f(x,y,z), g(x,y,z)), \quad z = q(x, f(x,y,z), g(x,y,z)).$$

These equations reduce to

$$y = p(x,k,l), \quad z = q(x,k,l),$$

when $f(x,y,z) = k$ and $g(x,y,z) = l$, that is, when (x,y,z) lies in $\mathcal{S}_f \cap \mathcal{S}_g$. Let $p(x,k,l) = \varphi(x)$ and $q(x,k,l) = \psi(x)$; as components of the coordinate change $\mathbf{h}^{-1}$, these functions are continuously differentiable in an open neighborhood of $x = a$. By construction,

$$\begin{aligned} k &= f(x,y,z) = f(x, \varphi(x), \psi(x)), \\ l &= g(x,y,z) = g(x, \varphi(x), \psi(x)), \end{aligned}$$

verifying the first condition on φ and ψ.

Because $\mathbf{h}(a,b,c) = (a,k,l)$, it follows that $\mathbf{h}^{-1}(a,k,l) = (a,b,c)$. In terms of components,

$$(a,b,c) = \mathbf{h}^{-1}(a,k,l) = (a, p(a,k,l), q(a,k,l)) = (a, \varphi(a), \psi(a)),$$

so $b = \varphi(a)$ and $c = \psi(a)$, thus verifying the second condition.

We obtain the derivatives of φ and ψ by applying the chain rule to the equations

$$k = f(x, \varphi(x), \psi(x)), \quad l = g(x, \varphi(x), \psi(x)).$$

Suppressing the arguments of the functions for clarity, we find

$$\begin{aligned} 0 &= \frac{dk}{dx} = \frac{d}{dx} f(x, \varphi(x), \psi(x)) = f_x + f_y \cdot \varphi' + f_z \cdot \psi', \\ 0 &= \frac{dl}{dx} = \frac{d}{dx} g(x, \varphi(x), \psi(x)) = g_x + g_y \cdot \varphi' + g_z \cdot \psi'. \end{aligned}$$

If we write these equations in the matrix form

$$\begin{pmatrix} f_y & f_z \\ g_y & g_z \end{pmatrix} \begin{pmatrix} \varphi' \\ \psi' \end{pmatrix} = \begin{pmatrix} -f_x \\ -g_x \end{pmatrix},$$

we can solve them using Cramer's rule to get

$$\varphi' = \frac{\begin{vmatrix} -f_x & f_z \\ -g_x & g_z \end{vmatrix}}{\begin{vmatrix} f_y & f_z \\ g_y & g_z \end{vmatrix}} = -\frac{\begin{vmatrix} f_x & f_z \\ g_x & g_z \end{vmatrix}}{\begin{vmatrix} f_y & f_z \\ g_y & g_z \end{vmatrix}}, \quad \psi' = \frac{\begin{vmatrix} f_y & -f_x \\ g_y & -g_x \end{vmatrix}}{\begin{vmatrix} f_y & f_z \\ g_y & g_z \end{vmatrix}} = -\frac{\begin{vmatrix} f_y & f_x \\ g_y & g_x \end{vmatrix}}{\begin{vmatrix} f_y & f_z \\ g_y & g_z \end{vmatrix}}. \qquad \square$$

Jacobians

We can also express the hypothesis and conclusion of the implicit function theorem in terms of Jacobians (cf. p. 137). The hypothesis is

$$\left.\frac{\partial(f,g)}{\partial(y,z)}\right|_{\mathbf{x}=\mathbf{a}} \neq 0,$$

and the implicit functions $y=\varphi(x)$, $z=\psi(x)$ have derivatives given by

$$\frac{dy}{dx} = -\frac{\partial(f,g)}{\partial(x,z)} \Big/ \frac{\partial(f,g)}{\partial(y,z)}, \qquad \frac{dz}{dx} = -\frac{\partial(f,g)}{\partial(y,x)} \Big/ \frac{\partial(f,g)}{\partial(y,z)}$$

Expressed this way, the derivatives are strikingly similar in form to the derivative of the function $y=\varphi(x)$ that is implicitly defined by the single equation $f(x,y)=k$ (Theorem 6.1, p. 189):

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y}.$$

When does $\mathbf{f}$ "look like" $d\mathbf{f_a}$?

Let us return to the question we have already addressed several times before (Chapters 4.2, 4.3, 5.2, 5.3, especially pp. 119–121, 128–128, 163–165, 175–176): to what extent—and in what way—does a map look like its linear approximation near a given point? Theorem 6.10 deals with a map $\mathbf{f}: U^3 \to \mathbb{R}^2$. Under that assumption that $d\mathbf{f_a}$ has maximal rank (namely 2), it shows that a suitable coordinate change will make $\mathbf{f}$ look like the linear projection $\Pi : \mathbb{R}^{1+2} \to \mathbb{R}^2 : (x,y,z) \mapsto (y,z)$. But according to Theorem 2.19 (p. 50), a coordinate change will likewise make $d\mathbf{f_a}$ into the same projection Π. Coordinate changes thus make $\mathbf{f}$ look like $d\mathbf{f_a}$ near $\mathbf{a}$.

Maximal rank is essential

Maximal rank is essential. To see this, consider the map $\mathbf{f}: \mathbb{R}^3 \to \mathbb{R}^2 : (x,y,z) \to (s,t)$:

$$\mathbf{f}: \begin{cases} s = x, \\ t = (y-z)^3. \end{cases}$$

We have

$$d\mathbf{f_x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3(y-z)^2 & -3(y-z)^2 \end{pmatrix},$$

so the rank of $d\mathbf{f_x}$ is only 1, not 2, at all points in the plane $z=y$ (thus including the origin). Near the origin, $\mathbf{f}$ is geometrically different from $d\mathbf{f_0}$; see Exercise 6.16.

6.3 The general case

In the general case, p equations constrain the values of $k+p$ variables. We expect to find that p of the variables are implicitly determined by the remaining k. Under what conditions can we guarantee that happens, and which variables will be functions of which? Here is the same question, in geometric terms: given a map from (an open set in) $\mathbb{R}^{k+p}$ to $\mathbb{R}^p$, what does the pullback of a point look like? As we have already seen in the low-dimension cases, an answer to the first will follow readily from an answer to the second.

Partial derivatives

Because the source of the map is split into two factors, with k real variables in one and p in the other, it is useful to split the derivative of the map into the two parts—its *"partial" derivatives*—that act separately on these two factors. To define them, we assume $\mathbf{f}: X^{k+p} \to \mathbb{R}^n : (\mathbf{x},\mathbf{y}) \to \mathbf{z}$ is differentiable and $\mathbf{x} = (x_1,\ldots,x_k)$, $\mathbf{y} = (y_1,\ldots,y_p)$. If

$$\mathbf{f}: \begin{cases} z_1 = f_1(x_1,\ldots,x_k,y_1,\ldots,y_p), \\ \quad\vdots \\ z_n = f_n(x_1,\ldots,x_k,y_1,\ldots,y_p), \end{cases}$$

then the derivative of $\mathbf{f}$ is given by the $n \times (k+p)$ matrix

$$d\mathbf{f}_{(\mathbf{x},\mathbf{y})} = \begin{pmatrix} f_{11} & \cdots & f_{1k} & f_{1,k+1} & \cdots & f_{1,k+p} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nk} & f_{n,k+1} & \cdots & f_{n,k+p} \end{pmatrix},$$

where

$$f_{ij} = \begin{cases} \dfrac{\partial f_i}{\partial x_j}(\mathbf{x},\mathbf{y}) & \text{if } j = 1,\ldots,k, \\[2ex] \dfrac{\partial f_i}{\partial y_q}(\mathbf{x},\mathbf{y}) & \text{if } j = k+q \text{ and } q = 1,\ldots,p, \end{cases}$$

and $i = 1,\ldots,n$.

Definition 6.4 *The* ***partial derivatives*** *of* $\mathbf{f}: X^{k+p} \to \mathbb{R}^n$ *are the linear maps* $\partial_1 \mathbf{f}_{(\mathbf{x},\mathbf{y})} = \partial_{\mathbf{x}} \mathbf{f}_{(\mathbf{x},\mathbf{y})} : \mathbb{R}^k \to \mathbb{R}^n$ *and* $\partial_2 \mathbf{f}_{(\mathbf{x},\mathbf{y})} = \partial_{\mathbf{y}} \mathbf{f}_{(\mathbf{x},\mathbf{y})} : \mathbb{R}^p \to \mathbb{R}^n$ *given by the matrices*

$$\partial_1 \mathbf{f}_{(\mathbf{x},\mathbf{y})} = \begin{pmatrix} f_{11} & \cdots & f_{1k} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nk} \end{pmatrix}, \quad \partial_2 \mathbf{f}_{(\mathbf{x},\mathbf{y})} = \begin{pmatrix} f_{1,k+1} & \cdots & f_{1,k+p} \\ \vdots & \ddots & \vdots \\ f_{n,k+1} & \cdots & f_{n,k+p} \end{pmatrix}.$$

Notation: $\partial_1 = \partial_{\mathbf{x}}$

If the derivative of $\mathbf{f}$ is continuous, then so are its partial derivatives. The notation "∂_1" signifies *the partial derivative with respect to the first factor*, and the alternate notation "$\partial_{\mathbf{x}}$" signifies *the partial derivative with respect to the* $\mathbf{x}$ *factor*. As we have done for functions of two real variables (e.g., as with $f_1(x,y) = f_x(x,y)$), we use these notations interchangeably.

Theorem 6.12. *Suppose the map* $\mathbf{f}: X^{k+p} \to \mathbb{R}^p$ *is continuously differentiable, and the derivative* $d\mathbf{f}_{(\mathbf{a},\mathbf{b})} : \mathbb{R}^{k+p} \to \mathbb{R}^p$ *has maximal rank p. Then there is a coordinate change* $\mathbf{h} : (\mathbf{x},\mathbf{y}) \to (\mathbf{u},\mathbf{v})$ *defined in a neighborhood* N^{k+p} *of* $(\mathbf{a},\mathbf{b})$ *that transforms* $\mathbf{f}$ *into the projection* $\boldsymbol{\Pi} : (\mathbf{u},\mathbf{v}) \mapsto \mathbf{v}$*; that is,* $\mathbf{f} \circ \mathbf{h}^{-1} = \boldsymbol{\Pi}$.

Proof. We know p columns of $d\mathbf{f}_{\mathbf{a}}$ are linearly independent. By permuting the variables, if necessary, we may assume that the final p columns are, so the partial derivative $\partial_{\mathbf{y}}\mathbf{f}_{(\mathbf{a},\mathbf{b})} : \mathbb{R}^p \to \mathbb{R}^p$ is invertible. Now use the component functions of $\mathbf{f}$ to define

$$\mathbf{h} : \begin{cases} u_1 = x_1, \\ \quad \vdots \\ u_k = x_k, \\ v_1 = f_1(x_1,\dots,x_k,y_1,\dots,y_p), \\ \quad \vdots \\ v_p = f_p(x_1,\dots,x_k,y_1,\dots,y_p); \end{cases} \qquad \text{schematically,} \quad \mathbf{h} : \begin{cases} \mathbf{u} = \mathbf{x}, \\ \mathbf{v} = \mathbf{f}(\mathbf{x},\mathbf{y}). \end{cases}$$

Then $\mathbf{h}$ is continuously differentiable on X^{k+p} (because $\mathbf{f}$ is), and

$$\mathrm{d}\mathbf{h}_{(\mathbf{x},\mathbf{y})} = \begin{pmatrix} I & O \\ \partial_{\mathbf{x}}\mathbf{f}_{(\mathbf{x},\mathbf{y})} & \partial_{\mathbf{y}}\mathbf{f}_{(\mathbf{x},\mathbf{y})} \end{pmatrix}, \quad \det \mathrm{d}\mathbf{h}_{(\mathbf{x},\mathbf{y})} = \det \partial_{\mathbf{y}}\mathbf{f}_{(\mathbf{x},\mathbf{y})},$$

implying that $\det \mathrm{d}\mathbf{h}_{(\mathbf{a},\mathbf{b})} \neq 0$. By the inverse function theorem (Theorem 5.2, p. 169), $\mathbf{h}$ is invertible on some smaller neighborhood N^{k+p} of $(\mathbf{a},\mathbf{b})$.

To show that $\mathbf{f} \circ \mathbf{h}^{-1} = \boldsymbol{\Pi}$, first write $\mathbf{h}^{-1}$ schematically as

$$\mathbf{h}^{-1} : \begin{cases} \mathbf{x} = \mathbf{u}, \\ \mathbf{y} = \mathbf{g}(\mathbf{u},\mathbf{v}), \end{cases}$$

for a suitable map $\mathbf{g} : \mathbf{h}(N^{k+p}) \to \mathbb{R}^p$. By the definition of an inverse,

$$(\mathbf{u},\mathbf{v}) = \mathbf{h} \circ \mathbf{h}^{-1}(\mathbf{u},\mathbf{v}) = \mathbf{h}(\mathbf{u},\mathbf{g}(\mathbf{u},\mathbf{v})) = (\mathbf{u},\mathbf{f}(\mathbf{u},\mathbf{g}(\mathbf{u},\mathbf{v})))$$

implying

$$\mathbf{v} = \mathbf{f}(\mathbf{u},\mathbf{g}(\mathbf{u},\mathbf{v})) = \mathbf{f}(\mathbf{h}^{-1}(\mathbf{u},\mathbf{v})),$$

as desired. More simply, we know $\mathbf{f} \circ \mathbf{h}^{-1}(\mathbf{u},\mathbf{v}) = \mathbf{v}$ because $\mathbf{f}$ is the second component of $\mathbf{h}$, and $\mathbf{h} \circ \mathbf{h}^{-1}(\mathbf{u},\mathbf{v}) = (\mathbf{u},\mathbf{v})$. □

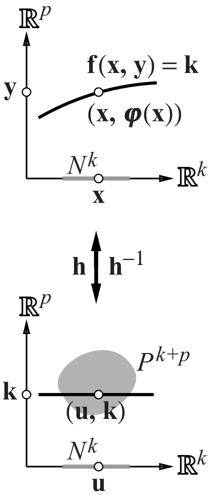
Corollary 6.13 (Implicit function theorem) *Suppose* $\mathbf{f} : X^{k+p} \to \mathbb{R}^p : (\mathbf{x},\mathbf{y}) \to \mathbf{z}$ *is continuously differentable and* $\mathbf{f}(\mathbf{a},\mathbf{b}) = \mathbf{k}$. *If the partial derivative map* $\partial_{\mathbf{y}}\mathbf{f}_{(\mathbf{a},\mathbf{b})} : \mathbb{R}^p \to \mathbb{R}^p$ *is invertible, then there is a unique map* $\mathbf{y} = \boldsymbol{\varphi}(\mathbf{x})$ *defined on a neighborhood* N^k *of* $\mathbf{x} = \mathbf{a}$ *in* $\mathbb{R}^k$ *for which*

- $\mathbf{f}(\mathbf{x},\boldsymbol{\varphi}(\mathbf{x})) = \mathbf{k}$ *for all* $\mathbf{x}$ *in* N^k.
- $\boldsymbol{\varphi}(\mathbf{a}) = \mathbf{b}$.
- $\boldsymbol{\varphi}$ *is continuously differentiable on* N^k, *and*

$$\mathrm{d}\boldsymbol{\varphi}_{\mathbf{x}} = -\left(\partial_{\mathbf{y}}\mathbf{f}_{(\mathbf{x},\boldsymbol{\varphi}(\mathbf{x}))}\right)^{-1} \circ \partial_{\mathbf{x}}\mathbf{f}_{(\mathbf{x},\boldsymbol{\varphi}(\mathbf{x}))} : \mathbb{R}^k \to \mathbb{R}^p.$$

Proof. Let $\mathbf{h}$ be the coordinate change defined on the neighborhood N^{k+p} of $(\mathbf{a},\mathbf{b})$ in $\mathbb{R}^{k+p}$, as provided by Theorem 6.12; let $\mathbf{h}^{-1}$ be its inverse. We wrote

$$\mathbf{h}^{-1}(\mathbf{u},\mathbf{v}) = (\mathbf{u},\mathbf{g}(\mathbf{u},\mathbf{v}))$$



for a suitably defined continuously differentiable map $\mathbf{g}$ on $P^{k+p} = \mathbf{h}(N^{k+p})$, and we saw that

$$\mathbf{v} = \mathbf{f}(\mathbf{u},\mathbf{g}(\mathbf{u},\mathbf{v}))$$

for every $(\mathbf{u},\mathbf{v})$ in P^{k+p}. In particular, $\mathbf{k} = \mathbf{f}(\mathbf{u},\mathbf{g}(\mathbf{u},\mathbf{k}))$.

Now define N^k by the condition

$$\mathbf{u} \text{ is in } N^k \quad \Longleftrightarrow \quad (\mathbf{u},\mathbf{k}) \text{ is in } P^{k+p},$$

and set $\boldsymbol{\varphi}(\mathbf{x}) = \mathbf{g}(\mathbf{x},\mathbf{k})$. Then $\boldsymbol{\varphi}$ is defined on all of N^k, and the equation $\mathbf{k} = \mathbf{f}(\mathbf{u},\mathbf{g}(\mathbf{u},\mathbf{k}))$ translates into

$$\mathbf{f}(\mathbf{x},\boldsymbol{\varphi}(\mathbf{x})) = \mathbf{k} \ \text{ for every } \mathbf{x} \text{ in } N^k.$$

This verifies the first condition. Also, because $\mathbf{h}(\mathbf{a},\mathbf{b}) = (\mathbf{a},\mathbf{f}(\mathbf{a},\mathbf{b})) = (\mathbf{a},\mathbf{k})$,

$$(\mathbf{a},\mathbf{b}) = \mathbf{h}^{-1}(\mathbf{a},\mathbf{k}) = (\mathbf{a},\mathbf{g}(\mathbf{a},\mathbf{k})) = (\mathbf{a},\boldsymbol{\varphi}(\mathbf{a})),$$

it follows that $\boldsymbol{\varphi}(\mathbf{a}) = \mathbf{b}$, verifying the second condition.

The third condition follows from the chain rule applied to the equation $\mathbf{k} = \mathbf{f}(\mathbf{x},\boldsymbol{\varphi}(\mathbf{x}))$. To carry out the differentiation, it will be helpful to define the map $\boldsymbol{\Phi} : N^k \to \mathbb{R}^{k+p} : \mathbf{x} \mapsto (\mathbf{x},\boldsymbol{\varphi}(\mathbf{x}))$. Then $\mathbf{k} = \mathbf{f} \circ \boldsymbol{\Phi}(\mathbf{x})$, so

$$\begin{aligned}\underset{p\times k}{O} = \underset{p\times(k+p)}{d\mathbf{f}_{\boldsymbol{\Phi}(\mathbf{x})}} \circ \underset{(k+p)\times k}{d\boldsymbol{\Phi}_{\mathbf{x}}} &= \left(\partial_{\mathbf{x}}\mathbf{f}_{(\mathbf{x},\boldsymbol{\varphi}(\mathbf{x}))} \ \partial_{\mathbf{y}}\mathbf{f}_{(\mathbf{x},\boldsymbol{\varphi}(\mathbf{x}))}\right)\begin{pmatrix} I \\ d\boldsymbol{\varphi}_{\mathbf{x}} \end{pmatrix} \\ &= \underset{p\times k}{\partial_{\mathbf{x}}\mathbf{f}_{(\mathbf{x},\boldsymbol{\varphi}(\mathbf{x}))}} \circ \underset{k\times k}{I} + \underset{p\times p}{\partial_{\mathbf{y}}\mathbf{f}_{(\mathbf{x},\boldsymbol{\varphi}(\mathbf{x}))}} \circ \underset{p\times k}{d\boldsymbol{\varphi}_{\mathbf{x}}}.\end{aligned}$$

Using the invertibility of $\partial_{\mathbf{y}}\mathbf{f}_{(\mathbf{x},\boldsymbol{\varphi}(\mathbf{x}))}$, we can solve for $d\boldsymbol{\varphi}_{\mathbf{x}}$ to get

$$d\boldsymbol{\varphi}_{\mathbf{x}} = -\left(\partial_{\mathbf{y}}\mathbf{f}_{(\mathbf{x},\boldsymbol{\varphi}(\mathbf{x}))}\right)^{-1}\partial_{\mathbf{x}}\mathbf{f}_{(\mathbf{x},\boldsymbol{\varphi}(\mathbf{x}))},$$

verifying the third condition. □

Summary

This final version of the implicit function theorem echoes the first one (Theorem 6.1, p. 189). In broad outline, it tells us that the locus $\mathbf{f}(\mathbf{x},\mathbf{y}) = \mathbf{k}$ is the graph of a map $\mathbf{y} = \boldsymbol{\varphi}(\mathbf{x})$ for which $d\boldsymbol{\varphi}_{\mathbf{x}} = (\partial_{\mathbf{y}}\mathbf{f}_{(\mathbf{x},\boldsymbol{\varphi}(\mathbf{x}))})^{-1} \circ \partial_{\mathbf{x}}\mathbf{f}_{(\mathbf{x},\boldsymbol{\varphi}(\mathbf{x}))}$, assuming only that $\partial_{\mathbf{y}}\mathbf{f}_{(\mathbf{x},\mathbf{y})}$ is invertible at a seed point $(\mathbf{x},\mathbf{y}) = (\mathbf{a},\mathbf{b})$, where $\mathbf{f}(\mathbf{a},\mathbf{b}) = \mathbf{k}$. The key to the proof is that $\mathbf{f}$ is equivalent to a projection near $(\mathbf{a},\mathbf{b})$; in turn, this follows (Theorem 2.19) from the fact that $d\mathbf{f}_{(\mathbf{a},\mathbf{b})}$ is onto.

Submersions

A map whose derivative is onto is called a *submersion*. Ultimately, the proof of the implicit function theorem can be traced back to the simple fact that $\mathbf{f}$ is a submersion. Submersions have useful behavior with important consequences (beyond the implicit function theorem) that we now pause to explore.

Definition 6.5 *A continuously differentiable map* $\mathbf{f} : X^n \to \mathbb{R}^p$ *is a* ***submersion at*** $\mathbf{c}$ *if* $d\mathbf{f}_\mathbf{c} : \mathbb{R}^n \to \mathbb{R}^p$ *is onto.*

Theorem 6.14. *A map* $\mathbf{f} : X^n \to \mathbb{R}^p$ *is a submersion at* $\mathbf{c}$ *if and only if there is a coordinate change* $\mathbf{h} : N^n \to \mathbb{R}^n$ *defined on a neighborhood* N^n *of* $\mathbf{c}$ *for which* $\mathbf{f} \circ \mathbf{h}^{-1}$ *is a projection.*

Proof. Notice that the "only if" part of the theorem is just a restatement of Theorem 6.12. To prove the converse (the "if" part), let $\mathbf{f} \circ \mathbf{h}^{-1} = \boldsymbol{\Pi}$, a projection. Then

$$d\mathbf{f}_\mathbf{x} = d\boldsymbol{\Pi}_{\mathbf{h}(\mathbf{x})} \circ d\mathbf{h}_\mathbf{x} = \boldsymbol{\Pi} \circ d\mathbf{h}_\mathbf{x},$$

so $\mathbf{f}$ is continuously differentiable on N^n and $d\mathbf{f}_\mathbf{x}$ is onto for every $\mathbf{x}$ in N^n because $\boldsymbol{\Pi}$ and $d\mathbf{h}_\mathbf{x}$ are both onto. □

$\mathbf{f}$ "looks like" $d\mathbf{f}$

Thus, submersions are precisely the maps that are locally equivalent to projections. Moreover, because $\mathbf{f} \circ \mathbf{h}^{-1} = \boldsymbol{\Pi} = d\mathbf{f}_\mathbf{x} \circ d\mathbf{h}_\mathbf{x}^{-1}$, the local coordinate change $\mathbf{h}^{-1} \circ d\mathbf{h}_\mathbf{x}$ transforms $\mathbf{f}$ into its linear approximation $d\mathbf{f}_\mathbf{x}$. This is the generalization of Corollary 6.7, page 197. The next result is a generalization of Corollary 6.8. The result following that is a consequence of the fact that a submersion is equivalent to a local projection.

Corollary 6.15 *If* $\mathbf{f} : X^n \to \mathbb{R}^p$ *is a submersion at* $\mathbf{c}$*, then there are curvilinear coordinates defined near* $\mathbf{c}$ *in which* p *of the* n *coordinate functions are the component functions of* $\mathbf{f}$.

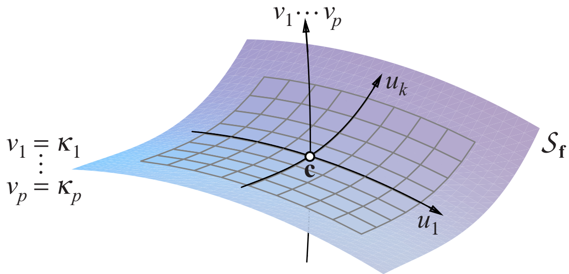
Proof. This follows immediately from the definition of the coordinate change $\mathbf{h}$ in the proof of Theorem 6.12. □

Corollary 6.16 *If* $\mathbf{f} : X^n \to \mathbb{R}^p$ *is a submersion at* $\mathbf{c}$*, then* $\mathbf{f}$ *maps* X^n *onto a neighborhood of* $\mathbf{f}(\mathbf{c})$. □

Embedded surface patches

Submersions give us a valuable way to describe and deal with curved surfaces. To see how this happens, consider first the locus $\mathcal{S}_\mathbf{f} : \mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{k}$ defined by the submersion $\mathbf{f}$. In general, $\mathcal{S}_\mathbf{f}$ is a curved subset of $\mathbb{R}^{k+p}$, but of a special kind. For suppose $\mathbf{c} = (\mathbf{a}, \mathbf{b})$ is a seed point of $\mathbf{f}$; that is, $\mathbf{f}(\mathbf{a}, \mathbf{b}) = \mathbf{k}$. Then the proof of the implicit function theorem provides a coordinate change $\mathbf{h} : (\mathbf{x}, \mathbf{y}) \to (\mathbf{u}, \mathbf{v})$ that "straightens" $\mathcal{S}_\mathbf{f}$ locally and makes it a flat k-dimensional plane near $\mathbf{c}$. In effect, $(u_1, \dots, u_k, v_1, \dots, v_p)$ provides new curvilinear coordinates in $(\mathbf{x}, \mathbf{y})$-space in which equations of the form $v_1 = \kappa_1, \dots, v_p = \kappa_p$ specify $\mathcal{S}_\mathbf{f}$ and the variables $(u_1, \dots, u_k)$ provide a system of curvilinear coordinates on $\mathcal{S}_\mathbf{f}$ itself. The k coordinates $u_1, \dots, u_k$, imply $\mathcal{S}_\mathbf{f}$ is k-dimensional. We now use this characterization of $\mathcal{S}_\mathbf{f}$ as the basis of the definition of an embedded surface patch.

Definition 6.6 *A set* $\mathcal{S}$ *in* $\mathbb{R}^n$ *is an* ***embedded surface patch of dimension k at the point*** $\mathbf{c}$ *if there are coordinates* $(u_1, \dots, u_k, v_1, \dots, v_{n-k})$ *in a window* W^n *centered at* $\mathbf{c}$ *so that* $\mathcal{S}$ *is given by the conditions* $v_1 = \kappa_1, \dots, v_{n-k} = \kappa_{n-k}$ *there. The variables* $\mathbf{u} = (u_1, \dots, u_k)$ *provide coordinates on* $\mathcal{S}$ *in* W^n.



Surface patches of dimension 0 or n

We may abbreviate this term to *surface patch*, *embedding* or just *patch*. We can extend the definition to allow $k = 0$ and $k = n$. An embedded surface patch of dimension 0 at $\mathbf{c}$ is just the point $\mathbf{c}$ itself; it is specified by a full set of n equalities $v_1 = \kappa_1, \ldots, v_n = \kappa_n$. An embedded surface patch of dimension n at $\mathbf{c}$ is just an open set containing $\mathbf{c}$. It is specified by an *empty* set of equalities.

Theorem 6.17. *Suppose $\mathbf{f} : X^n \to \mathbb{R}^p$ is a submersion at a point $\mathbf{c}$ in X^n and $\mathbf{f}(\mathbf{c}) = \mathbf{k}$. Then the pullback $\mathbf{f}^{-1}(\mathbf{k})$ is an embedded surface patch of dimension $n - p$ at the point $\mathbf{c}$.* $\square$

Surface patches and pullbacks

This is just Theorem 6.12 restated using surface patches. The following theorem is its converse; the two taken together imply surface patches are precisely the pullbacks of points by submersions.

Theorem 6.18. *Suppose $\mathcal{S}$ is an embedded surface patch of dimension k at a point $\mathbf{c}$ in $\mathbb{R}^n$. Then there is a submersion $\mathbf{g} : X^n \to \mathbb{R}^{n-k}$ at $\mathbf{c}$ for which $\mathcal{S} = \mathbf{g}^{-1}(\mathbf{g}(\mathbf{c}))$.*

Proof. By hypothesis, there is a window X^n centered at $\mathbf{c}$ and a coordinate change $\mathbf{h} : X^n \to \mathbb{R}^{k+(n-k)} : \mathbf{x} \to (\mathbf{u}, \mathbf{v})$ in terms of which $\mathcal{S}$ is given by the equations $v_1 = v_1(\mathbf{c}), \ldots, v_{n-k} = v_{n-k}(\mathbf{c})$, where the constants $v_j(\mathbf{c})$ are the $\mathbf{v}$-coordinates of the point $\mathbf{c}$. Let us write the components of $\mathbf{h}$ as

$$\mathbf{h} : \begin{cases} u_1 = h_1(x_1, \ldots, x_n), \\ \quad\vdots \\ u_k = h_k(x_1, \ldots, x_n), \\ v_1 = g_1(x_1, \ldots, x_n), \\ \quad\vdots \\ v_{n-k} = g_{n-k}(x_1, \ldots, x_n), \end{cases}$$

and let $\mathbf{g} : X^n \to \mathbb{R}^{n-k}$ be defined by

$$\mathbf{g} : \begin{cases} v_1 = g_1(x_1, \ldots, x_n), \\ \quad\vdots \\ v_{n-k} = g_{n-k}(x_1, \ldots, x_n). \end{cases}$$

Then $\mathbf{g}$ is continuously differentiable because $\mathbf{h}$ is. Moreover, for every $\mathbf{x}$ in W^n, the matrix $d\mathbf{g}_\mathbf{x}$ has maximal rank $n-k$ because it makes up the last $n-k$ rows of the invertible matrix $d\mathbf{h}_\mathbf{x}$. In particular, $d\mathbf{g}_\mathbf{c}$ is onto, and

$$\mathcal{S} = \mathbf{g}^{-1}(v_1(\mathbf{c}), \ldots, v_{n-k}(\mathbf{c})) = \mathbf{g}^{-1}(\mathbf{g}(\mathbf{c})). \qquad \square$$

Dimension and codimension

Although the point $\mathbf{g}(\mathbf{c})$ has dimension 0, its pullback $\mathbf{g}^{-1}(\mathbf{g}(\mathbf{c}))$ has dimension k. The pullback does not preserve dimension. However, the difference in the dimensions of the point and its containing space, namely $n-k$, is the same as the difference in the dimensions of the pullback and its containing space. This suggests that, in discussing pullbacks, we focus on this difference, called the *codimension*. We have already done this for vector spaces and pullbacks of onto linear maps. According to Definition 2.7 (p. 51), the *codimension* of a vector subspace W in a vector space $\mathcal{V}$ is

$$\operatorname{codim} W = \dim \mathcal{V} - \dim W.$$

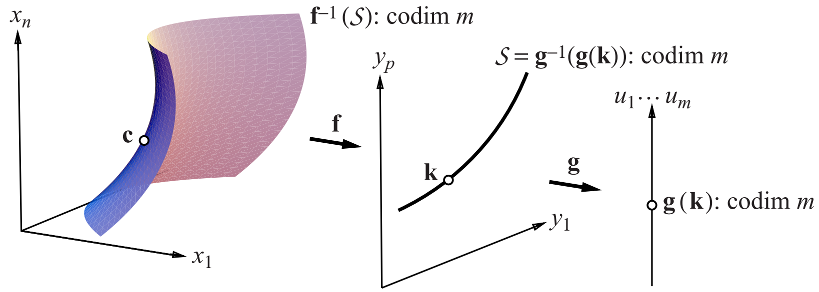
By Corollary 2.21, page 51, any *onto* linear map $L : \mathbb{R}^n \to \mathbb{R}^p$ preserves codimension under pullback; that is, if W is a subspace of codimension m in the target $\mathbb{R}^p$, then the subspace $L^{-1}(W)$ has the same codimension m in the source $\mathbb{R}^n$.

Definition 6.7 *We say an embedded surface patch $\mathcal{S}$ of dimension k in $\mathbb{R}^n$ has* ***codimension*** $\boldsymbol{m = n-k}$.

Note that the codimension of a surface patch is the number of equations (including $m=0$ for an open set) that define the patch in Definition 6.6. Furthermore, the codimension of the surface patch in Theorem 6.18 equals the dimension of the target of the map $\mathbf{g}$ that defines the patch.

Submersions preserve embeddings under pullback

Theorem 6.19. *Suppose $\mathbf{f} : X^n \to \mathbb{R}^p$ is a submersion at $\mathbf{c}$, and $\mathcal{S}$ is an embedded surface patch of codimension m at the point $\mathbf{k} = \mathbf{f}(\mathbf{c})$ in $\mathbb{R}^p$. Then $\mathbf{f}^{-1}(\mathcal{S})$ is an embedded surface patch of codimension m at $\mathbf{c}$ in X^n.*



Proof. According to Theorem 6.18, there is a submersion $\mathbf{g} : W^p \to \mathbb{R}^m$ at $\mathbf{k}$ for which $\mathcal{S} = \mathbf{g}^{-1}(\mathbf{g}(\mathbf{k}))$. Because

$$\mathbf{f}^{-1}(\mathcal{S}) = \mathbf{f}^{-1}(\mathbf{g}^{-1}(\mathbf{g}(\mathbf{k})) = \big(\mathbf{g}\circ\mathbf{f}\big)^{-1}(\mathbf{g}(\mathbf{k})) = \big(\mathbf{g}\circ\mathbf{f}\big)^{-1}\big((\mathbf{g}\circ\mathbf{f})(\mathbf{c})\big),$$

it is sufficient to show that $\mathbf{g}\circ\mathbf{f}: X^n \to \mathbb{R}^m$ is a submersion at $\mathbf{c}$. But by the chain rule,

$$d\big(\mathbf{g}\circ\mathbf{f}\big)_{\mathbf{c}} = d\mathbf{g}_{\mathbf{k}} \circ d\mathbf{f}_{\mathbf{c}},$$

and the composite is onto because the individual maps are. □

Immersions and injections

Submersions handle pullbacks properly; however, they do not behave well with push-forwards. That is, if $\mathcal{S}$ is a surface patch at $\mathbf{c}$, and $\mathbf{f}$ is a submersion at $\mathbf{c}$, the image $\mathbf{f}(\mathcal{S})$ is not, in general, a surface patch; see the exercises. We have faced this dilemma already with linear maps in Chapter 2.3, and we resolved it there (Corollary 2.28, p. 56) by switching from onto to 1–1 linear maps. To handle push-forwards properly, we use an immersion, that is, a map whose derivative is 1–1.

Definition 6.8 *A continuously differentiable map* $\mathbf{f}: X^n \to \mathbb{R}^p$ *is an* ***immersion at*** $\mathbf{c}$ *if* $d\mathbf{f}_{\mathbf{c}}: \mathbb{R}^n \to \mathbb{R}^p$ *is* 1–1 *(implying* $n \le p$*).*

Recall (Theorem 2.27, p. 56) that every 1–1 linear map can be transformed in a simple form, called an *injection* $J: \mathbb{R}^n \to \mathbb{R}^{n+q}$, that is analogous to a projection Π : $\mathbb{R}^{p+k} \to \mathbb{R}^p$. The analogy is easily seen by looking at their matrix representatives:

$$J = \begin{pmatrix} I_{n\times n} \\ O_{q\times n} \end{pmatrix}, \quad \Pi = \begin{pmatrix} O_{p\times k} \; I_{p\times p} \end{pmatrix}.$$

Theorem 6.20. *A map* $\mathbf{f}: X^n \to \mathbb{R}^{n+q}$ *is an immersion at* $\mathbf{c}$ *if and only if there is a coordinate change* $\mathbf{h}: N^{n+q} \to \mathbb{R}^{n+q}$ *defined on a neighborhood* N^{n+q} *of* $\mathbf{f}(\mathbf{c})$ *for which* $\overline{\mathbf{f}} = \mathbf{h}\circ\mathbf{f}$ *is an injection.*

Proof. To prove the "only if" part, we assume $\mathbf{f}$ is an immersion at $\mathbf{c}$. Hence the $(n+q)\times n$-matrix $d\mathbf{f}_{\mathbf{c}}: \mathbb{R}^n \to \mathbb{R}^{n+q}$ is 1–1 and consequently has n linearly independent rows. By rearranging the rows (and the corresponding target variables), if necessary, we assume that the first n rows are linearly independent. Write the target variables as $(\mathbf{y},\mathbf{z})$, where $\mathbf{y} = (y_1,\dots,y_n)$, $\mathbf{z} = (z_1,\dots,z_q)$, and write $\mathbf{f}$ in terms of (vector) components as

$$\mathbf{f}: \begin{cases} \mathbf{y} = \mathbf{f}_1(\mathbf{x}), \\ \mathbf{z} = \mathbf{f}_2(\mathbf{x}). \end{cases}$$

In particular, $\mathbf{f}_1: X^n \to \mathbb{R}^n$ is continuously differentiable and the condition on $d\mathbf{f}_{\mathbf{c}}$ makes $d(\mathbf{f}_1)_{\mathbf{c}}: \mathbb{R}^n \to \mathbb{R}^n$ invertible. By the inverse function theorem (Theorem 5.2, p. 169) $\mathbf{f}_1$ is invertible on some neighborhood N_1^n of $\mathbf{f}_1(\mathbf{c})$ in $\mathbb{R}^n$. Let $N^{n+q} = N_1^n \times \mathbb{R}^q$, and define $\mathbf{h}: N^{n+q} \to \mathbb{R}^{n+q}: (\mathbf{y},\mathbf{z}) \to (\overline{\mathbf{y}},\overline{\mathbf{z}})$ by the vector components

$$\mathbf{h}: \begin{cases} \overline{\mathbf{y}} = \mathbf{f}_1^{-1}(\mathbf{y}), \\ \overline{\mathbf{z}} = -\mathbf{f}_2(\mathbf{f}_1^{-1}(\mathbf{y})) + \mathbf{z}. \end{cases}$$

Because its components are continuously differentiable on N^{n+q}, $\mathbf{h}$ is a valid coordinate change, and it transforms $\mathbf{f}$ into

$$
\begin{aligned}
\bar{\mathbf{f}}(\mathbf{x}) = \mathbf{h} \circ \mathbf{f}(\mathbf{x}) &= \mathbf{h}(\mathbf{f}_1(\mathbf{x}), \mathbf{f}_2(\mathbf{x})) \\
&= (\mathbf{f}_1^{-1}(\mathbf{f}_1(\mathbf{x})), -\mathbf{f}_2(\mathbf{f}_1^{-1}(\mathbf{f}_1(\mathbf{x}))) + \mathbf{f}_2(\mathbf{x})) \\
&= (\mathbf{x}, -\mathbf{f}_2(\mathbf{x}) + \mathbf{f}_2(\mathbf{x})) \\
&= (\mathbf{x}, \mathbf{0}).
\end{aligned}
$$

This is an injection.

To prove the converse (the "if" part), assume $J = \mathbf{h} \circ \mathbf{f}$ is an injection. Rearrange the variables, if necessary, so that $J(\mathbf{x}) = (\mathbf{x}, \mathbf{0})$ in $\mathbb{R}^{n+q}$. Then

$$\mathbf{f}(\mathbf{x}) = \mathbf{h}^{-1}(\mathbf{x}, \mathbf{0}),$$

implying that $\mathbf{f}$ is continuously differentiable wherever it is defined. We must show it is defined on some open neighborhood of $\mathbf{c}$.

Because $\mathbf{h}(N^{n+q})$ is an open set containing $\mathbf{h}(\mathbf{f}(\mathbf{c}))$ (by the inverse function theorem), it contains an open "rectangle" $X^n \times Y^q$ centered at $\mathbf{h}(\mathbf{f}(\mathbf{c})) = (\mathbf{c}, \mathbf{0})$. Therefore, for any $\mathbf{x}$ in X^n, $\mathbf{h}^{-1}(\mathbf{x}, \mathbf{0}) = \mathbf{f}(\mathbf{x})$ is defined. Finally,

$$\mathrm{d}\mathbf{f}_{\mathbf{x}} = \mathrm{d}\mathbf{h}^{-1}_{J(\mathbf{x})} \circ dJ_{\mathbf{x}} = \mathrm{d}\mathbf{h}^{-1}_{(\mathbf{x},\mathbf{0})} \circ J$$

so $\mathrm{d}\mathbf{f}_{\mathbf{x}}$ is 1–1 because injections and invertible maps are 1–1. □

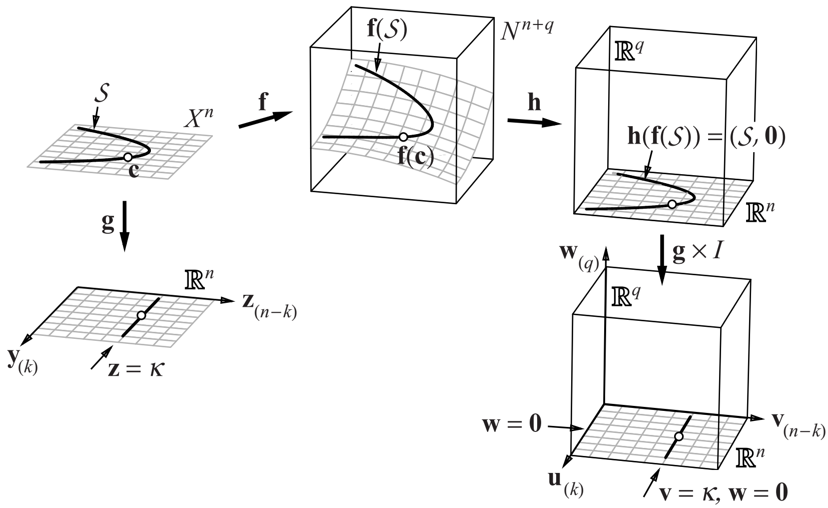
Thus, immersions are precisely the maps that are locally equivalent to injections. Moreover, because $\mathbf{h} \circ \mathbf{f} = J = \mathrm{d}\mathbf{h}_{\mathbf{f}(\mathbf{x})} \circ \mathrm{d}\mathbf{f}_{\mathbf{x}}$, we see that coordinate changes locally transform $\mathbf{f}$ into its linear approximation, so $\mathbf{f}$ "looks like" $\mathrm{d}\mathbf{f}$. The following corollary is an immediate consequence of the fact that an immersion is a local injection. **f "looks like" d$\mathbf{f}$**

Corollary 6.21 *If* $\mathbf{f} : X^n \to \mathbb{R}^{n+q}$ *is an immersion at* $\mathbf{c}$*, then* $\mathbf{f}$ *is* 1–1 *on a neighborhood of* $\mathbf{c}$. □

The next theorem, which says that the image of a surface patch under an immersion is still a surface patch of the same dimension, has a more complicated proof than its analogue for submersions because surface patches are naturally determined by submersions (under pullbacks).

Theorem 6.22. *Suppose* $\mathbf{f} : X^n \to \mathbb{R}^{n+q}$ *is an immersion at* $\mathbf{c}$*, and* $\mathcal{S}$ *is an embedded surface patch of dimension* k *at* $\mathbf{c}$ *in* X^n*. Then the image* $\mathbf{f}(\mathcal{S})$ *is an embedded surface patch of the same dimension* k *at* $\mathbf{f}(\mathbf{c})$ *in* $\mathbb{R}^{n+q}$. Immersions preserve embeddings under push-forward

Proof. By the definition of an embedded surface patch (Definition 6.6), there is a coordinate change $\mathbf{g} : X^n \to \mathbb{R}^n : \mathbf{x}_{(n)} \to (\mathbf{y}_{(k)}, \mathbf{z}_{(n-k)})$ that "straightens" $\mathcal{S}$ near $\mathbf{c}$. Let us suppose that $\mathbf{g}(\mathcal{S})$ is given by equations $z_1 = \kappa_1, \ldots, z_{n-k} = \kappa_{n-k}$ (i.e., $\mathbf{z} = \boldsymbol{\kappa}$) in the new coordinates. To prove that $\mathbf{f}(\mathcal{S})$ is an embedded surface patch of dimension k at $\mathbf{f}(\mathbf{c})$, it is sufficient to find new coordinates $(\mathbf{u}_{(k)}, \mathbf{v}_{(n-k)}, \mathbf{w}_{(q)})$ in a neighborhood N^{n+q} of $\mathbf{f}(\mathbf{c})$ in which $\mathbf{f}(\mathcal{S})$ is specified by the $n - k + q$ equations $\mathbf{v} = \boldsymbol{\kappa}$, $\mathbf{w} = \mathbf{0}$.



The figure shows how to build the new coordinates. First, use the hypothesis that $\mathbf{f}$ is an immersion at $\mathbf{c}$ to get (from Theorem 6.20) a coordinate change $\mathbf{h} : N^{n+q} \to \mathbb{R}^n \times \mathbb{R}^q : \mathbf{r}_{(n+q)} \to (\mathbf{s}_{(n)}, \mathbf{t}_{(q)})$ on a neighborhood N^{n+q} of $\mathbf{f}(\mathbf{c})$ that transforms $\mathbf{f}$ into an injection to the first n coordinates:

$$(\mathbf{s}, \mathbf{t}) = (\mathbf{h} \circ \mathbf{f})(\mathbf{x}) = (\mathbf{x}, \mathbf{0}); \quad (\mathbf{h} \circ \mathbf{f})(\mathcal{S}) = (\mathcal{S}, \mathbf{0}).$$

Next, use the coordinate change $\mathbf{g}$ already introduced to define

$$\mathbf{g} \times I : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^k \times \mathbb{R}^{n-k} \times \mathbb{R}^q : (\mathbf{s}, \mathbf{t}) \to (\mathbf{u}_{(k)}, \mathbf{v}_{(n-k)}, \mathbf{w}_{(q)})$$

on a neighborhood of $(\mathbf{c}, \mathbf{0})$ in $\mathbb{R}^p$. Then the composite coordinate change $(\mathbf{g} \times I) \circ \mathbf{h}$ "straightens" $\mathbf{f}(\mathcal{S})$ near $\mathbf{f}(\mathbf{c})$: $\mathbf{f}(\mathcal{S})$ is given by the $n - k + q$ equations $\mathbf{v} = \boldsymbol{\kappa}$, $\mathbf{w} = \mathbf{0}$. □

Corollary 6.23 *If* $\mathbf{f} : X^n \to \mathbb{R}^{n+q}$ *is an immersion at a point* $\mathbf{c}$*, then* $\mathbf{f}(X^n)$ *is an embedded surface patch of dimension n at the point* $\mathbf{f}(\mathbf{c})$ *in* $\mathbb{R}^{n+q}$.

Proof. Because X^n is an open set in $\mathbb{R}^n$, we can view it as a surface patch of dimension n (i.e., of codimension 0) at $\mathbf{c}$ in $\mathbb{R}^n$. It follows from the theorem that its image $\mathbf{f}(X^n)$ is a surface patch of dimension n at $\mathbf{f}(\mathbf{c})$ in $\mathbb{R}^{n+q}$. □

Parametrized curves and surfaces

vWe can think of the previous corollary as the basis for our study of curves (in Chapter 1.2) and surfaces in space (Chapter 4.3). To see the connection, let $\mathbf{f} : X^n \to \mathbb{R}^p$ be an arbitrary continuously differentiable map (i.e., not necessarily an immersion); it is given by p component functions of n real variables:

$$\mathbf{f} : \begin{cases} v_1 = f_1(x_1, \ldots, x_n), \\ \quad \vdots \\ v_p = f_p(x_1, \ldots, x_n). \end{cases}$$

If $n = 1$ then $\mathbf{f}$ defines the parametrization of a curve in $\mathbb{R}^p$. Our earlier definition (Definition 1.2, p. 7) is more restrictive: $\mathbf{f}$ must be smooth (i.e., have continuous derivatives of all orders) and $\mathbf{df}_t = \mathbf{f}'(t)$ must have a nonzero derivative at each interior point t. Smoothness is mainly just a technical convenience, but the requirement that $\mathbf{df}_t \neq \mathbf{0}$ means $\mathbf{f}$ must be an immersion at each interior point. Consequently, a curve as given by Definition 1.2 is actually embedded at each point: there are curvilinear coordinates $(x_1,\ldots,x_p)$ in $\mathbb{R}^p$ in which the curve is specified by the conditions $x_2 = \cdots x_n = 0$, and x_1 serves as a parameter along the curve.

If $n = 2$ and $p = 3$, then $\mathbf{f}$ parametrizes an ordinary surface in space. Such a map is an immersion at a point $\mathbf{c}$ only if the derivative $\mathbf{df_c}$ has rank 2, its maximal rank. In the examples in Chapter 4.3, $\mathbf{f}$ was indeed an immersion at most points, so the image surface was embedded there. That is, (by Corollary 6.23) we could introduce coordinates (p,q,r) in a neighborhood of such a point so that the surface was given locally by the equation $r = 0$ and (p,q) could serve as coordinates on the surface near the point.

The notable example of a nonimmersion is the crosscap. The crosscap map

$$\mathbf{f}: \begin{cases} x = u, \\ y = uv, \\ z = -v^2, \end{cases} \qquad \mathbf{df_u} = \begin{pmatrix} 1 & 0 \\ v & u \\ 0 & -2v \end{pmatrix}$$

fails to be an immersion at the origin (cf. pp. 127–128).

Exercises

6.1. a. Determine the location of the three saddle points and one relative maximum of $f(x,y) = (3y^2 - x^2)(x-1)$.

b. Plot together the graphs of $z = f(x,y)$ and $z = 0$ on the square for which $-0.3 \le x \le 1.3$, $-0.8 \le y \le 0.8$. Determine the locus $f(x,y) = 0$ from the intersection of the two graphs, and note the location of the four critical points of f in relation to this intersection.

6.2. Solve the equation $e^{xy} = 1$ for y near the point $(2,0)$. What is dy/dx at that point? Sketch the locus $e^{xy} = 1$.

6.3. Solve the equation $e^{xy} = e$ for y near the point $(2,1/2)$. What is dy/dx at that point? Sketch the locus $e^{xy} = e$.

6.4. Solve the equation $y^2 - 2y\cos x - \sin^2 x = 0$ for y, and determine dy/dx. Sketch the locus $y^2 - 2y\cos x - \sin^2 x = 0$.

6.5. Solve the equation $y^2 - 2y\cos x + \sin^2 x = 0$ for y; for which values of x is y undefined (as a function of x)? Determine dy/dx; where is $dy/dx = \infty$? Sketch the locus $y^2 - 2y\cos x + \sin^2 x = 0$.

6.6. a. Solve the equation $x^2+3xy+4y^2=14$ for x in terms of y. Determine dx/dy when $y=1$.

b. Sketch the locus $f(x,y)=x^2+3xy+4y^2=14$.

c. Determine the implicit function $y=\varphi(x)$ for which $f(x,\varphi(x))=14$ and $\varphi(2)=1$. Determine $\varphi'(2)$ and relate it to the value of dx/dy that you found in part (a).

6.7. Determine the linearization of the locus $f(x,y)=0$ at the given point (a,b). Indicate whether the linearization is the tangent line to the locus at that point.

a. $f(x,y)=y^2+x^2(x+1)$; $(a,b)=(-1,0)$.
b. $f(x,y)=y^2+x^2(x+1)$; $(a,b)=(0,0)$.
c. $f(x,y)=(3y^2-x^2)(x-1)$; $(a,b)=(1,0)$.
d. $f(x,y)=(3y^2-x^2)(x-1)$; $(a,b)=(1,1/\sqrt{3})$.
e. $f(x,y)=(3y^2-x^2)(x-1)$; $(a,b)=(0,0)$.
f. $f(x,y)=x^3+y^3$; $(a,b)=(0,0)$.
g. $f(x,y)=x^3+y^3$; $(a,b)=(1,-1)$.

6.8. a. Sketch representative level curves of $f(x,y)=xy^2$ in the window W for which $1\le x\le 2$, $1\le y\le 2$. Verify that every point of W is a regular point of f.

b. Obtain the map $\mathbf{h}: W\to\mathbb{R}^2$ that straightens the level curves of f, using the construction in the proof of Theorem 6.2. Then, using a suitable parametrization of a level curve, verify that it does indeed have a horizontal image under $\mathbf{h}$.

c. Show that the image of W is the set

$$1\le u\le 2,\quad u\le v\le 4u.$$

Sketch level curves of f that meet either the top or the bottom of W, and then sketch their images under $\mathbf{h}$. Where do those images meet the boundary of $\mathbf{h}(W)$?

d. Obtain the formula for the inverse of $\mathbf{h}$ on $\mathbf{h}(W)$.

6.9. a. Sketch representative level curves of $f(x,y)=x^2+y^2$ in window W for which $1\le x\le 2$, $1\le y\le 2$. Verify that every point of W is a regular point of f.

b. Obtain the map $\mathbf{h}: W\to\mathbb{R}^2$ that straightens the level curves of f, using the construction in the proof of Theorem 6.2. Then, using a suitable parametrization of a level curve, verify that it does indeed have a horizontal image under $\mathbf{h}$.

c. Show that the image of W is the set

$$0\le u\le 2,\quad u^2+1\le v\le u^2+4,$$

and sketch the image, including imges of the level curves of f.

6.10. a. Let $f(x,y) = x^2 + y^2$ and let Z be the window $0 \le x \le 2$, $-1 \le y \le 1$. Verify that every point of Z except the origin is a regular point of f. Sketch the level curves of f in Z. Note that $f_y = 0$ at the center of Z.

b. Show that the map $\mathbf{h} : Z \to \mathbb{R}^2$,

$$\mathbf{h} : \begin{cases} u = y, \\ v = f(x,y), \end{cases}$$

is a valid coordinate change near $(1,0)$; that is, show $\mathbf{h}$ is continuously differentiable with a continuously differentiable inverse in a neighborhood of $(1,0)$.

c. Show that $\mathbf{h}$ "straightens out" the level curves of f. Describe the salient geometric features of the action of $\mathbf{h}$; in particular, indicate what happens to a horizontal line in Z.

6.11. Consider the function

$$f(u,v,w) = \frac{1+w}{1-w}\,\frac{1-u}{1+u}\,\frac{1-v}{1+v},$$

for $-1 < u,v,w < 1$. Think of u, v, and w as speeds expressed as fractions of the speed of light. Note that $f(0,0,0) = 1$. This exercise studies the implicit function $w = \varphi(u,v)$ defined near $(u,v) = (0,0)$ by the equation $f(u,v,w) = 1$.

a. Use f to compute the partial derivatives $\partial\varphi/\partial u$ and $\partial\varphi/\partial v$, and deduce that $\varphi(u,v) = u + v + O(2)$.

b. Show that

$$w = \varphi(u,v) = \frac{u+v}{1+uv} = u \oplus v.$$

This defines a binary operation called the *law of addition of velocities* in special relativity. That is, if observer A is moving away from observer B with velocity u (as a fraction of the speed of light), and B is moving away from C along the same straight line with velocity v, then A will be moving away from C with velocity $w = u \oplus v$. According to part (a), if u and v are small, then $u \oplus v \approx u + v$, but not otherwise.

c. Show that $u \oplus v$ is defined for all $|u| < 1$ and $|v| < 1$, and that $|u \oplus v| < 1$.

d. Show that $\lim_{u \to 1} u \oplus v = 1$, allowing us to extend φ so that $1 \oplus v = 1$ (and, by symmetry, $u \oplus 1 = 1$).

Thus, if A now represents a photon (a light particle), it moves away from B and from C at the same speed, even though B is moving in relation to C. Special relativity is built on the premise that the speed of light is an invariant for all observers moving uniformly in relation to each other.

6.12. Prove Theorem 6.3 and Corollary 6.4. (Suggestion: Adapt the proofs of their 2-dimensional analogues.)

6.13. Show that any set of points in the (x,y)-plane that can occur as the zero-locus of a function of x and y can occur as the zero-locus of a suitably chosen function of x, y, and z in the $(x,y,0)$-plane. (Suggestion: consider $f(x,y,z) = [g(x,y)]^2 + z^2$.)

6.14. Sketch the intersection of the surfaces $\mathcal{S}_f : x^2 + y^3 - z = 0$, $\mathcal{S}_g : z = 0$. Is the intersection the graph of a continuously differentiable function $y = \varphi(x)$ within the plane $z = 0$? Explain. Address the same question using a function of the form $x = \psi(y)$.

6.15. Show that the surfaces defined by $(x+y)^3 - z = 0$ and $z = 0$ intersect in a straight line. Verify that the surfaces are not transverse at any intersection point.

6.16. Let $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $(s,t) = \mathbf{f}(x,y,z) = (x,(y-z)^3)$.

a. Determine the image of $d\mathbf{f}_\mathbf{0}$ and show thereby that it is 1-dimensional.
b. Show that $\mathbf{f}$ maps any window centered at $\mathbf{x} = (0,0,0)$ onto a small window centered at $\mathbf{s} = (0,0)$. In particular, show that, for any a, b near 0, the equation $\mathbf{f}(x,y,z) = (a,b)$ has a one-parameter family (i.e., a curve) of solutions. Determine that curve.
c. Conclude that $\mathbf{f}$ does not “look like” $d\mathbf{f}_\mathbf{0}$ near the origin.

6.17. In (x,y,z)-space, $x^2 + y^2 = r^2$ is a cylinder of radius $r > 0$ whose axis is the z-axis, and $x^2 + z^2 = 1$ is a cylinder of radius 1 whose axis is the y-axis.

a. Sketch the intersection of the two cylinders when $r^2 = 3/4$. Now let r be arbitrary, assuming only that $r < 1$. Find implicit functions $y = \varphi(x)$ and $z = \psi(x)$ determined by the equations $x^2 + y^2 = r^2$ and $x^2 + z^2 = 1$. Do this for each of the four seed points $(0, \pm r, \pm 1)$. What are the domains of definition of φ and ψ?
b. Sketch the intersection of the two cylinders when $r^2 = 4$. Now let r be arbitrary, assuming only that $r > 1$. Find implicit functions $y = \varphi(x)$ and $z = \psi(x)$ determined by the equations $x^2 + y^2 = r^2$ and $x^2 + z^2 = 1$ and the four seed points $(0, \pm r, \pm 1)$. Now what are the domains of definition of φ and ψ?
c. The implicit functions take simple forms when $r = 1$. What are those forms, and what is the shape of the intersection?