$$I^2_{(2,0)} = \left\{ (x_1, 0) \mid (x_1, 0) \in I^2 \right\},$$
$$I^2_{(2,1)} = \left\{ (x_1, 1) \mid (x_1, 1) \in I^2 \right\}.$$

The picture of the 2-cube with boundary is given in Fig. 11.11. We now give the faces of the 2-cube the following orientations:

$$
\begin{aligned}
&I^2_{(1,0)} \text{ has orientation determined by } (-1)^{1+0} = -1 \ \Rightarrow \ \text{negative},\\
&I^2_{(1,1)} \text{ has orientation determined by } (-1)^{1+1} = 1 \ \Rightarrow \ \text{positive},\\
&I^2_{(2,0)} \text{ has orientation determined by } (-1)^{2+0} = 1 \ \Rightarrow \ \text{positive},\\
&I^2_{(2,1)} \text{ has orientation determined by } (-1)^{2+1} = -1 \ \Rightarrow \ \text{negative},
\end{aligned}
$$

which gives us the picture of the 2-cube with oriented boundary as in Fig. 11.12. Notice that by giving the faces $I^2_{(i,a)}$ the orientations determined by $(-1)^{i+a}$ our boundary orientations are somehow consistent, they "go around" the 2-cube in a consistent way. We denote the boundary of the two-cube by

$$
\begin{aligned}
\partial I^2 &= \sum_{i=1}^{2} \sum_{a=0}^{1} (-1)^{i+a} I^2_{(i,a)}\\
&= -I^2_{(1,0)} + I^2_{(1,1)} + I^2_{(2,0)} - I^2_{(2,1)}.
\end{aligned}
$$

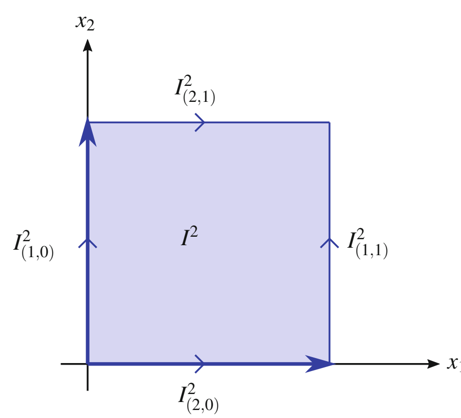


Fig. 11.11 The two-cube I^2 along with its boundary, which is given by the four faces $I^2_{(1,0)}$, $I^2_{(1,1)}$, $I^2_{(2,0)}$, and $I^2_{(2,1)}$. The four faces are shown with positive orientation

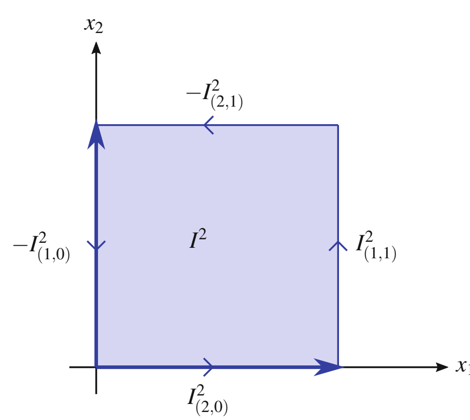


Fig. 11.12 The two-cube I^2 along with its four boundary faces, here shown with the orientations determined by the formula. For example, face $I^2_{(i,a)}$ has orientation determined by whether $(-1)^{i+a}$ is positive or negative

Question 11.3 The 2-cube in Fig. 11.11 is drawn in such a way that the four boundaries are show with positive orientation by the arrows. Explain why the arrows are the direction they are.

Finding $\partial\partial I^1$

Taking a step back we can see that that $\partial\partial I^1 = 0$ as well. We had $\partial I^1 = \{(1)\} - \{(0)\}$, where (0) and (1) are points on $\mathbb{R}^1$. Since a point does not have a boundary we say $\partial\{(0)\} = 0$ and $\partial\{(1)\} = 0$ so we get

$$\partial\partial I^1 = \partial\{(1)\} - \partial\{(0)\} = 0 - 0 = 0.$$

Finding $\partial\partial I^2$

Now, we will go one step further and find the boundaries of the boundary elements. That is, we will find $\partial\left(-I^2_{(1,0)}\right)$, $\partial I^2_{(1,1)}$, $\partial I^2_{(2,0)}$, and $\partial\left(-I^2_{(2,1)}\right)$. We begin with $\partial I^2_{(2,0)}$. We see that

$$\left(I^2_{(2,0)}\right)^1_{(1,0)} = \big\{(0,0)\big\} \text{ which has orientation } (-1)^{1+0} = -1 \;\Rightarrow\; \text{negative},$$
$$\left(I^2_{(2,0)}\right)^1_{(1,1)} = \big\{(1,0)\big\} \text{ which has orientation } (-1)^{1+1} = 1 \;\Rightarrow\; \text{positive}.$$

We show $\partial I^2_{(2,0)}$ in Fig. 11.13.

In order to find $\partial I^2_{(1,1)}$ we need to pay special attention to the subscripts. Recall that

$$I^2_{(1,1)} = \left\{ (\underbrace{1}_{x_1=1}, x_2) \;\middle|\; (1, x_2) \in I^2 \right\}.$$

Notice that $I^2_{(1,1)}$ is now in fact a one-cube, that is, $\left(I^2_{(1,1)}\right)^1$. So when we want to take the $(1,0)$ boundary of this one-cube we have that the variable this 1 refers to is the first variable in the one-cube, which is actually still being labeled as x_2,

$$\partial\left(I^2_{(1,1)}\right)^1_{(1,0)} = \left\{ (\underbrace{1}_{x_1=1}, \underbrace{x_2}_{x_1=0}) \right\} = \big\{(1,0)\big\}.$$

Similarly,

$$\partial\left(I^2_{(1,1)}\right)^1_{(1,1)} = \left\{ (\underbrace{1}_{x_1=1}, \underbrace{x_2}_{x_1=1}) \right\} = \big\{(1,1)\big\}.$$

For the orientations,

$$\left(I^2_{(1,1)}\right)^1_{(1,0)} = \big\{(1,0)\big\} \text{ which has orientation } (-1)^{1+0} = -1 \Rightarrow \text{negative},$$
$$\left(I^2_{(1,1)}\right)^1_{(1,1)} = \big\{(1,1)\big\} \text{ which has orientation } (-1)^{1+1} = 1 \Rightarrow \text{positive}.$$

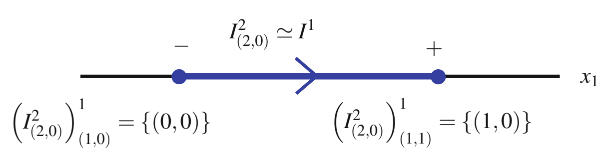


Fig. 11.13 The boundary of $I^2_{(2,0)}$ is given by $\left(I^2_{(2,0)}\right)^1_{(1,0)}$ with negative orientation and $\left(I^2_{(2,0)}\right)^1_{(1,1)}$ with positive orientation

We can then draw $\partial I^2_{(1,1)}$ as in Fig. 11.14. Now for $\partial\left(-I^2_{(2,1)}\right)$ we see that

$$\left(-I^2_{(2,1)}\right)^1_{(1,0)} = \{(0,1)\} \text{ which has orientation } -(-1)^{1+0} = 1 \Rightarrow \text{ positive},$$
$$\left(-I^2_{(2,1)}\right)^1_{(1,1)} = \{(1,1)\} \text{ which has orientation } -(-1)^{1+1} = -1 \Rightarrow \text{ negative},$$

thereby allowing us to draw $\partial\left(-I^2_{(2,1)}\right)$ as in Fig. 11.15. Finally, for $\partial\left(-I^2_{(1,0)}\right)$ we see that

$$\left(-I^2_{(1,0)}\right)^1_{(1,0)} = \{(0,0)\} \text{ which has orientation } -(-1)^{1+0} = 1 \Rightarrow \text{ positive},$$
$$\left(-I^2_{(1,0)}\right)^1_{(1,1)} = \{(0,1)\} \text{ which has orientation } -(-1)^{1+1} = -1 \Rightarrow \text{ negative},$$

thereby allowing us to draw $\partial\left(-I^2_{(1,0)}\right)$ as in Fig. 11.16.

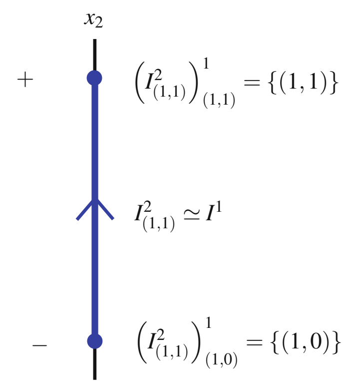


Fig. 11.14 The boundary of $I^2_{(1,1)}$ is given by $\left(I^2_{(1,1)}\right)^1_{(1,0)}$ with negative orientation and $\left(I^2_{(1,1)}\right)^1_{(1,1)}$ with positive orientation

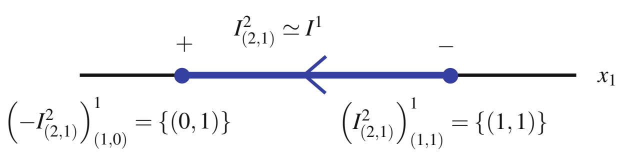


Fig. 11.15 The boundary of $-I^2_{(2,1)}$ is given by $\left(-I^2_{(2,1)}\right)^1_{(1,0)}$ with positive orientation and $\left(-I^2_{(2,1)}\right)^1_{(1,1)}$ with negative orientation

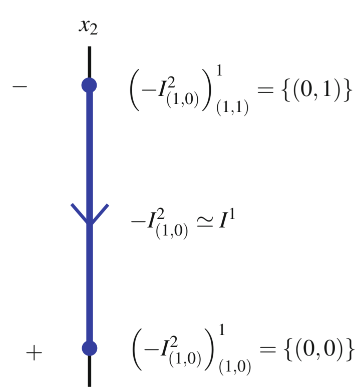


Fig. 11.16 The boundary of $-I^2_{(1,0)}$ is given by $\left(-I^2_{(1,0)}\right)^1_{(1,0)}$ with positive orientation and $\left(-I^2_{(1,0)}\right)^1_{(1,1)}$ with negative orientation

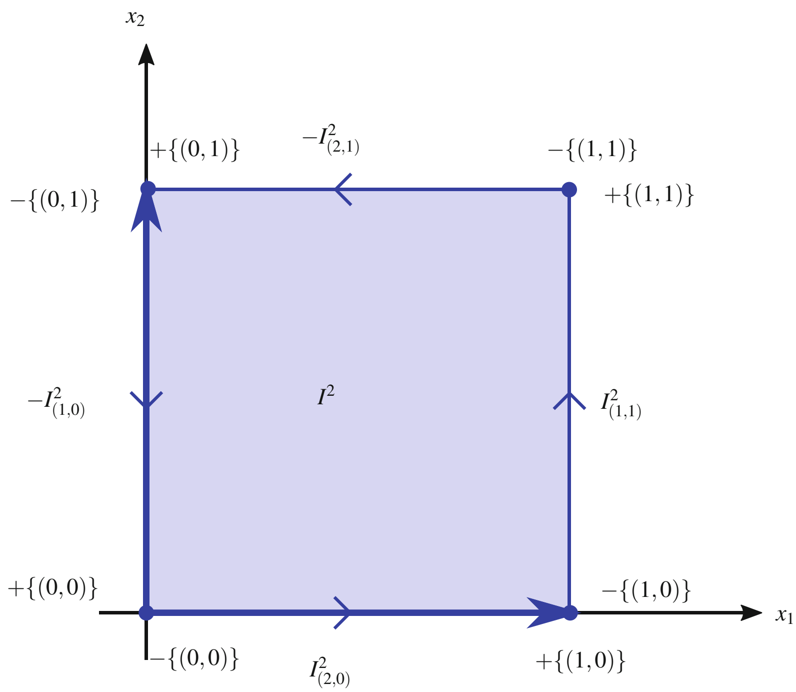


Fig. 11.17 To get a picture for $\partial\partial I^2$ we put together Figs. 11.13, 11.14, 11.15, and 11.16

Putting this all together, for the $\partial\partial I^2$ we have the picture in Fig. 11.17, which gives us

$$\begin{aligned}\partial\partial I^2 &= \partial\left(-I^2_{(1,0)}\right) + \partial I^2_{(1,1)} + \partial I^2_{(2,0)} + \partial\left(-I^2_{(2,1)}\right)\\ &= \{(0,0)\} - \{(0,1)\} - \{(1,0)\} + \{(1,1)\} - \{(0,0)\} + \{(1,0)\} + \{(0,1)\} - \{(1,1)\}\\ &= 0.\end{aligned}$$

Thus we see that the boundary elements and their orientations are defined in such a way that the boundary of the boundary is zero, which makes intuitive sense. After all, a closed curve does not have a boundary, which is exactly what the boundary of the two-cube is.

Finding ∂I^3

Now we will consider I^3. First we will find the faces of I^3 and then we will find the orientations of the faces, which we will use to find ∂I^3. I^3 clearly has six faces, given by

$$\begin{aligned}I^3_{(1,0)} &= \{(0,x_2,x_3) \mid 0 \le x_2, x_3 \le 1\} = \text{ back face,}\\ I^3_{(1,1)} &= \{(1,x_2,x_3) \mid 0 \le x_2, x_3 \le 1\} = \text{ front face,}\\ I^3_{(2,0)} &= \{(x_1,0,x_3) \mid 0 \le x_1, x_3 \le 1\} = \text{ left face,}\\ I^3_{(2,1)} &= \{(x_1,1,x_3) \mid 0 \le x_1, x_3 \le 1\} = \text{ right face,}\\ I^3_{(3,0)} &= \{(x_1,x_2,0) \mid 0 \le x_1, x_2 \le 1\} = \text{ bottom face,}\\ I^3_{(3,1)} &= \{(x_1,x_2,1) \mid 0 \le x_1, x_2 \le 1\} = \text{ top face,}\end{aligned}$$

and shown in Fig. 11.18.

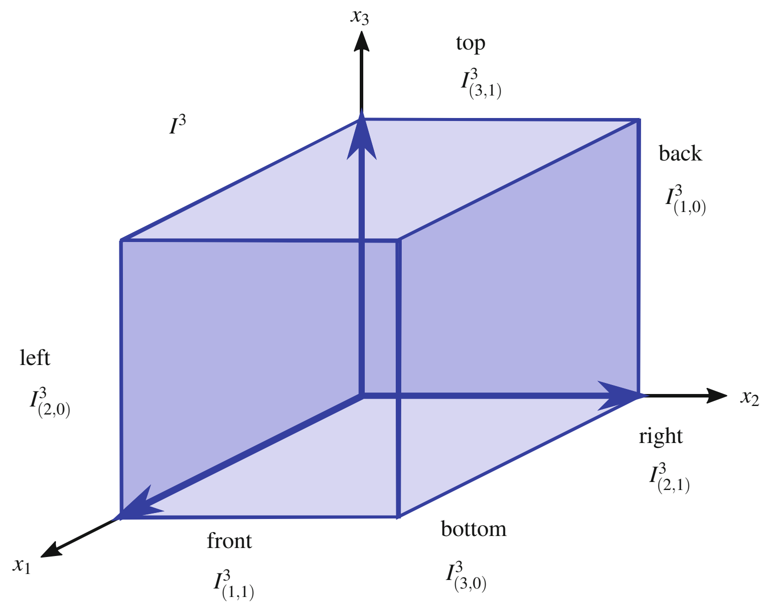


Fig. 11.18 The three-cube I^3 along with its six faces

We first show the positive orientations of all six faces of I^3. The volume form that determines the orientation of the front and back faces is $dx_2 \wedge dx_3$ since the front and back faces are in the (x_2, x_3)-plane. Consider the two vectors

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

at the point $p = (0, 0, 0)$ for the back face and the point $q = (1, 0, 0)$ for the front face. We have

$$dx_2 \wedge dx_3(p)\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_p, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_p\right) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$$

$$dx_2 \wedge dx_3(q)\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_q, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_q\right) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

for both points. The positive orientations of the front and back faces are shown in Fig. 11.19.

The volume form that determines positive orientation for the left and right is $dx_1 \wedge dx_3$ since the left and right faces are in the (x_1, x_3)-plane. Exactly as before consider the two vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

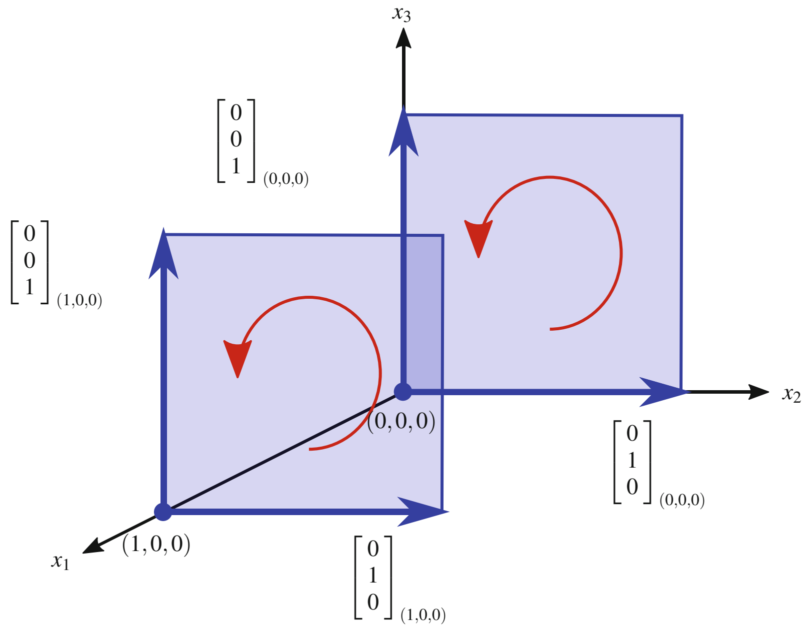


Fig. 11.19 The front and back face of I^3 shown with positive orientation

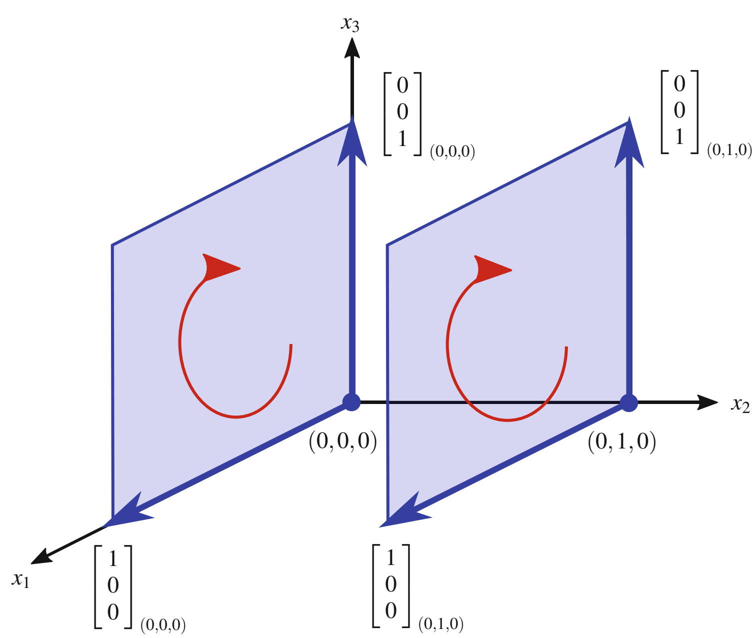


Fig. 11.20 The left and right face of I^3 shown with positive orientation

at the point $p = (0, 0, 0)$ for the left face and the point $q = (0, 1, 0)$ for the right face. We have

$$dx_1 \wedge dx_3(p)\left(\begin{bmatrix}1\\0\\0\end{bmatrix}_p, \begin{bmatrix}0\\0\\1\end{bmatrix}_p\right) = \begin{vmatrix}1 & 0\\0 & 1\end{vmatrix} = 1,$$

$$dx_1 \wedge dx_3(q)\left(\begin{bmatrix}1\\0\\0\end{bmatrix}_q, \begin{bmatrix}0\\0\\1\end{bmatrix}_q\right) = \begin{vmatrix}1 & 0\\0 & 1\end{vmatrix} = 1$$

for both points The positive orientations of the left and right faces are shown in Fig. 11.20.

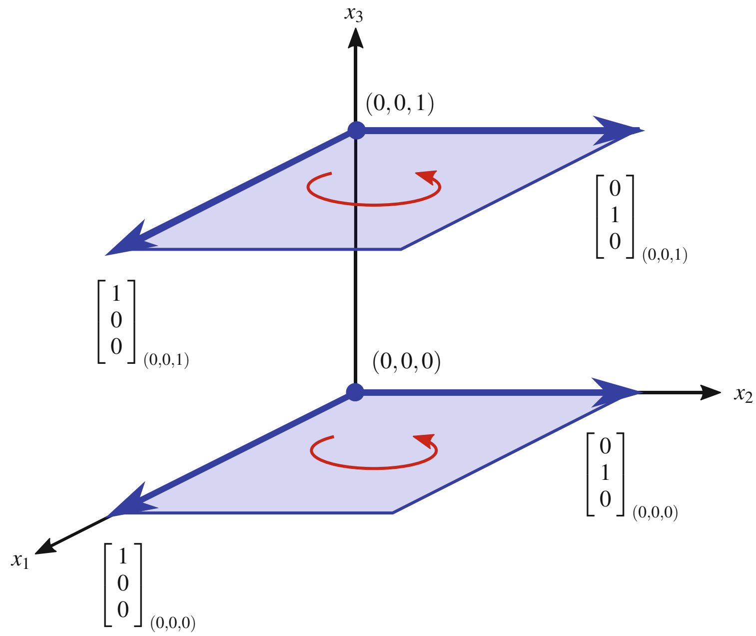


Fig. 11.21 The top and bottom face of I^3 shown with positive orientation

Question 11.4 Do this analysis for the top and bottom faces to get the positive orientations of those faces shown in Fig. 11.21.

Now we figure out the orientations of the six faces.

$$
\begin{aligned}
I^3_{(1,0)} &\text{ has orientation determined by } (-1)^{1+0} = -1 \;\Rightarrow\; \text{negative},\\
I^3_{(1,1)} &\text{ has orientation determined by } (-1)^{1+1} = +1 \;\Rightarrow\; \text{positive},\\
I^3_{(2,0)} &\text{ has orientation determined by } (-1)^{2+0} = +1 \;\Rightarrow\; \text{positive},\\
I^3_{(2,1)} &\text{ has orientation determined by } (-1)^{2+1} = -1 \;\Rightarrow\; \text{negative},\\
I^3_{(3,0)} &\text{ has orientation determined by } (-1)^{3+0} = -1 \;\Rightarrow\; \text{negative},\\
I^3_{(3,1)} &\text{ has orientation determined by } (-1)^{3+1} = +1 \;\Rightarrow\; \text{positive}.
\end{aligned}
$$

Putting everything together we can write down the boundary of I^3 as

$$
\begin{aligned}
\partial I^3 &= \sum_{i=1}^{3} \sum_{a=0}^{1} (-1)^{i+a} I^3_{(i,a)}\\
&= -I^3_{(1,0)} + I^3_{(1,1)} + I^3_{(2,0)} - I^3_{(2,1)} - I^3_{(3,0)} + I^3_{(3,1)}.
\end{aligned}
$$

Question 11.5 Sketch the unit cube I^3 including the appropriate orientations of the various faces. Do the orientations seem to fit together?

Finding $\partial\partial I^3$

Now we want to find the boundary of ∂I^3. Recall, ∂I^3 consists of six different two-cubes. We will begin by finding the edges of the bottom face $I^3_{(3,0)}$

$$
\begin{aligned}
\left(I^3_{(3,0)}\right)^2_{(1,0)} &= \{\, (0, x_2, 0) \mid 0 \le x_2 \le 1 \,\} \text{ with orientation } (-1)^{3+0}(-1)^{1+0} = 1,\\
\left(I^3_{(3,0)}\right)^2_{(1,1)} &= \{\, (1, x_2, 0) \mid 0 \le x_2 \le 1 \,\} \text{ with orientation } (-1)^{3+0}(-1)^{1+1} = -1,
\end{aligned}
$$

$$\left(I^3_{(3,0)}\right)^2_{(2,0)} = \{\ (x_1, 0, 0) \mid 0 \le x_1 \le 1\ \} \text{ with orientation } (-1)^{3+0}(-1)^{2+0} = -1,$$
$$\left(I^3_{(3,0)}\right)^2_{(2,1)} = \{\ (x_1, 1, 0) \mid 0 \le x_1 \le 1\ \} \text{ with orientation } (-1)^{3+0}(-1)^{2+1} = 1.$$

Similarly the edges of the top face $I^3_{(3,1)}$ are

$$\left(I^3_{(3,1)}\right)^2_{(1,0)} = \{\ (0, x_2, 1) \mid 0 \le x_2 \le 1\ \} \text{ with orientation } (-1)^{3+1}(-1)^{1+0} = -1,$$
$$\left(I^3_{(3,1)}\right)^2_{(1,1)} = \{\ (1, x_2, 1) \mid 0 \le x_2 \le 1\ \} \text{ with orientation } (-1)^{3+1}(-1)^{1+1} = 1,$$
$$\left(I^3_{(3,1)}\right)^2_{(2,0)} = \{\ (x_1, 0, 1) \mid 0 \le x_1 \le 1\ \} \text{ with orientation } (-1)^{3+1}(-1)^{2+0} = 1,$$
$$\left(I^3_{(3,1)}\right)^2_{(2,1)} = \{\ (x_1, 1, 1) \mid 0 \le x_1 \le 1\ \} \text{ with orientation } (-1)^{3+1}(-1)^{2+1} = -1.$$

We show the bottom and the top faces of I^3 along with the orientation of their boundary edges in Fig. 11.22.

Make sure that you understand how the notation works above. The bottom/top case is the easiest case. For the back/front and the left/right there is something a little bit tricky going on with the notation. Let us first of all consider the back edge $I^3_{(1,0)} = \{(0, x_2, x_1) | 0 \le x_2, x_3 \le 1\}$, which is a 2-cube. Let us first consider $\left(I^3_{(1,0)}\right)^2_{(1,0)}$. For the 3-cube the variables were (x_1, x_2, x_3). The $(1, 0)$ in $I^3_{(1,0)}$ means that the first variable, x_1, is replaced by 0. But now let us consider the back face of the 3-cube, which is a 2-cube. $\left(I^3_{(1,0)}\right)^2$. For this 2-cube the variables are (x_2, x_3), so the second $(1, 0)$ in $\left(I^3_{(1,0)}\right)^2_{(1,0)}$ means that the first variable is replaced by a 0, but in this case the first variable is x_2. So we have

$$\left(I^3_{(1,0)}\right)^2_{(1,0)} = \{\ (0, 0, x_3) \mid 0 \le x_3 \le 1\ \}.$$

The orientation is $(-1)^{1+0}(-1)^{1+0} = 1$, which is positive. Similarly, we have

$$\left(I^3_{(1,0)}\right)^2_{(1,1)} = \{\ (0, 1, x_3) \mid 0 \le x_3 \le 1\ \} \text{ with orientation } (-1)^{1+0}(-1)^{1+1} = -1 \ \Rightarrow\ \text{negative},$$
$$\left(I^3_{(1,0)}\right)^2_{(2,0)} = \{\ (0, x_2, 0) \mid 0 \le x_2 \le 1\ \} \text{ with orientation } (-1)^{1+0}(-1)^{2+0} = -1 \ \Rightarrow\ \text{negative},$$
$$\left(I^3_{(1,0)}\right)^2_{(2,1)} = \{\ (0, x_2, 1) \mid 0 \le x_2 \le 1\ \} \text{ with orientation } (-1)^{1+0}(-1)^{2+1} = 1 \ \Rightarrow\ \text{positive}.$$

The back face of the 3-cube along with the boundary orientations is shown in Fig. 11.23.

Question 11.6 Find the four boundary edges of the front $I^3_{(1,1)} = \{\ (1, x_2, x_3) \mid 0 \le x_2, x_3 \le 1\ \}$ and find the orientations of these edges. Your answers should agree with Fig. 11.23.

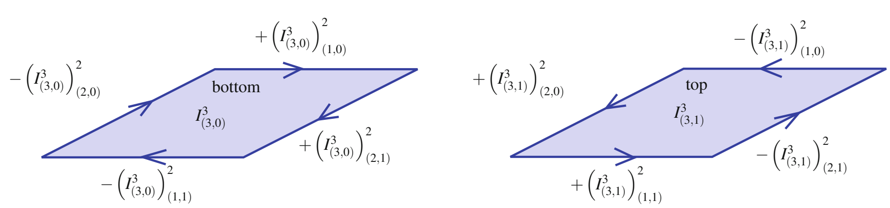


Fig. 11.22 The bottom and top faces of the three-cube shown along with the orientations of their boundaries

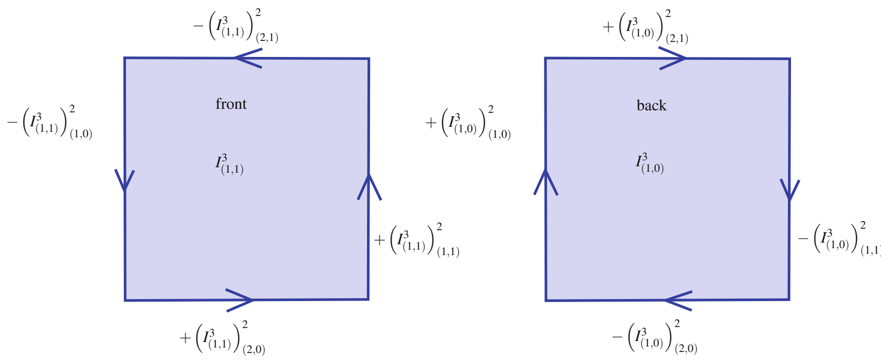


Fig. 11.23 The front and back faces of the three-cube I^3 shown along with the orientations of their boundaries

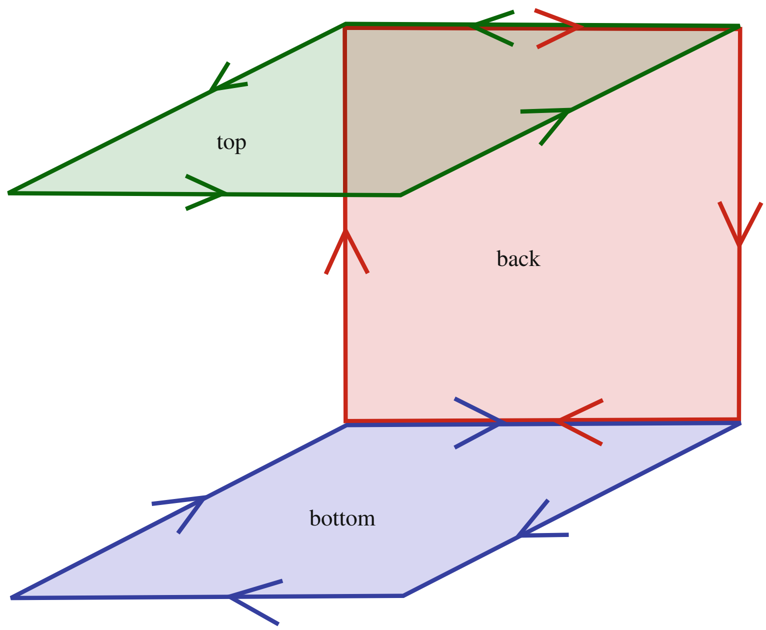


Fig. 11.24 The top, bottom, and back faces of the three-cube I^3 shown along with their boundaries. Notice where the boundaries of the three faces are the same, when the orientations are taken into account we have $\left(I^3_{(3,1)}\right)^2_{(1,0)} = -\left(I^3_{(1,0)}\right)^2_{(2,1)}$ and $\left(I^3_{(3,0)}\right)^2_{(1,0)} = -\left(I^3_{(1,0)}\right)^2_{(2,0)}$

Question 11.7 Show that when orientations are taken into account $\left(I^3_{(3,1)}\right)^2_{(1,0)} = -\left(I^3_{(1,0)}\right)^2_{(2,1)}$ and $\left(I^3_{(3,0)}\right)^2_{(1,0)} = -\left(I^3_{(1,0)}\right)^2_{(2,0)}$. See Fig. 11.24.

Question 11.8 Show that when orientations are taken into account $\left(I^3_{(3,1)}\right)^2_{(1,1)} = -\left(I^3_{(1,1)}\right)^2_{(2,1)}$ and $\left(I^3_{(3,0)}\right)^2_{(1,1)} = -\left(I^3_{(1,1)}\right)^2_{(2,0)}$. See Fig. 11.25.

Question 11.9 Find the four boundary edges of the left face $I^3_{(2,0)} = \{ (x_1, 0, x_3) \mid 0 \le x_1, x_3 \le 1 \}$ and the right face $I^3_{(2,1)} = \{ (x_1, 1, x_3) \mid 0 \le x_1, x_3 \le 1 \}$ and find the orientations of these eight boundary edges. Sketch the top, left, and bottom together and then sketch the top, right, and bottom together.

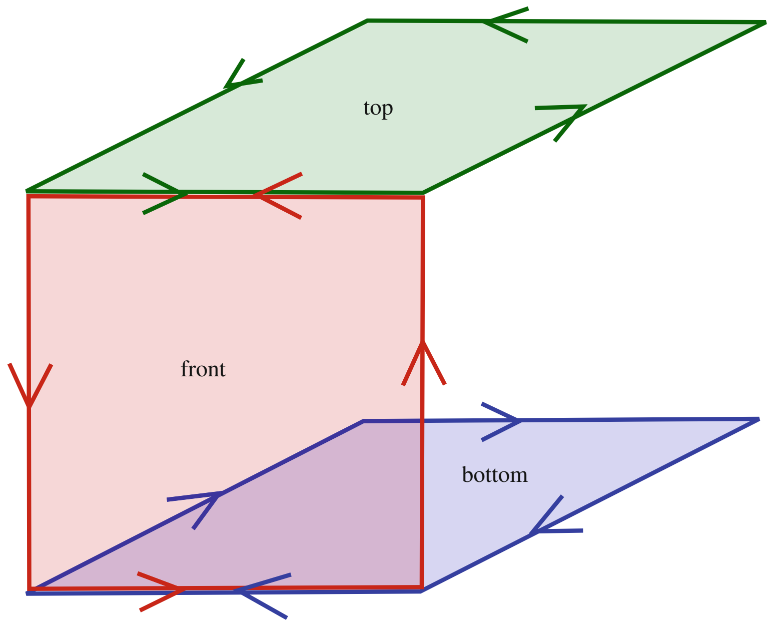


Fig. 11.25 The top, bottom, and back faces of the three-cube I^3 shown along with their boundaries. Notice where the boundaries of the three faces are the same, when the orientations are taken into account we have $\left(I^3_{(3,1)}\right)^2_{(1,1)} = -\left(I^3_{(1,1)}\right)^2_{(2,1)}$ and $\left(I^3_{(3,0)}\right)^2_{(1,1)} = -\left(I^3_{(1,1)}\right)^2_{(2,0)}$

You should be able to see, by inspecting the various sketches, that the following 1-cubes are identified

$$\left(I^3_{(2,0)}\right)^2_{(1,0)} = \left(I^3_{(1,0)}\right)^2_{(1,0)},$$
$$\left(I^3_{(2,1)}\right)^2_{(1,1)} = \left(I^3_{(1,1)}\right)^2_{(1,0)},$$
$$\left(I^3_{(2,0)}\right)^2_{(2,0)} = \left(I^3_{(3,0)}\right)^2_{(2,0)},$$
$$\left(I^3_{(2,0)}\right)^2_{(2,1)} = \left(I^3_{(3,1)}\right)^2_{(2,0)}.$$

Question 11.10 Put everything together to show that $\partial\partial I^3 = 0$.

Finding $\partial\partial I^k$

Consider the case where $i \le j \le k-1$ and $a, b = 0, 1$,

$$\begin{aligned}
\left(I^k_{(i,a)}\right)^{k-1}_{(j,b)} &= \left(\left\{(x_1,\dots,x_{i-1},a,x_{i+1},\dots,x_k)\right\}\right)^{k-1}_{(j,b)} \\
&= \left(\left\{(x_1,\dots,x_{i-1},a,\underbrace{x_i,\dots,x_{k-1}}_{\text{relabled}})\right\}\right)^{k-1}_{(j,b)} \quad \begin{array}{c}\text{relabeled to}\\(k-1)\text{-cube}\\\text{variable names}\end{array} \\
&= \left\{(x_1,\dots,x_{i-1},a,x_i,\dots,x_{j-1},b,x_{j+1},\dots,x_{k-1})\right\} \\
&= \left\{(x_1,\dots,x_{i-1},a,\underbrace{x_{i+1},\dots,x_j,b,x_{j+2},\dots,x_k}_{\text{relabled}})\right\} \quad \begin{array}{c}\text{relabeled to}\\k\text{-cube}\\\text{variable names}\end{array}
\end{aligned}$$

so b is in the $j+1$ slot when we use the k-cube variable names. We also have

$$\begin{aligned}\left(I^k_{(j+1,b)}\right)^{k-1}_{(i,a)} &= \left(\left\{(x_1,\ldots,x_j,b,x_{j+2},\ldots,x_k)\right\}\right)^{k-1}_{(i,a)} \\ &= \left\{(x_1,\ldots,x_{i-1},a,x_{i+1},\ldots,x_j,b,x_{j+2},\ldots,x_k)\right\}. \quad \begin{array}{c}\text{no relabeling}\\ \text{necessary}\\ \text{since } i\le j\end{array}\end{aligned}$$

Hence for $i \le j \le k-1$ we have

$$\left(I^k_{(i,a)}\right)^{k-1}_{(j,b)} = \left(I^k_{(j+1,b)}\right)^{k-1}_{(i,a)}.$$

Question 11.11 Explain why the condition that $i \le j \le k-1$ is not restrictive. That is, that every boundary element of ∂I^k is included.

Question 11.12 Show that for general I^k we have $\partial\partial I^k = 0$.

11.2 The Base Case: Stokes' Theorem on I^k

With all of this necessary background material covered, the actual proof of Stokes' theorem is not actually so difficult. As we stated earlier, we will be "bootstrapping" ourselves up to the fully general case. The first step is to prove Stoke's theorem on the k-cube I^k. Given a $(k-1)$-form α defined on a neighborhood of the unit k-cube I^k, we want to show that

$$\int_{I^k} d\alpha = \int_{\partial I^k} \alpha.$$

First, α is a $(k-1)$-form defined on some neighborhood of $I^k \subset \mathbb{R}^k$, so alpha has the general form

$$\begin{aligned}\alpha &= \alpha_1(x_1,\ldots,x_k)\, dx_2 \wedge dx_3 \wedge \cdots \wedge dx_k \\ &\quad + \alpha_2(x_1,\ldots,x_k)\, dx_1 \wedge dx_3 \wedge \cdots \wedge dx_k \\ &\quad + \cdots \\ &\quad + \alpha_k(x_1,\ldots,x_k)\, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{k-1} \\ &= \sum_{i=1}^{k} \alpha_i(x_1,\ldots,x_k)\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k,\end{aligned}$$

where $\widehat{dx_i}$ means that the dx_i is omitted and $\alpha_i : \mathbb{R}^k \to \mathbb{R}$, $0 \le i \le k$, are real-valued functions defined in a neighborhood of I^k. We know that exterior differentiation d is linear, which means that

$$d\left(\sum_{i=1}^{k} \alpha_i\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k\right) = \sum_{i=1}^{k} d\left(\alpha_i\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k\right).$$

We also know that integration is linear, so we have

$$\int_{I^k} \sum_{i=1}^{k} d\left(\alpha_i\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k\right) = \sum_{i=1}^{k} \int_{I^k} d\left(\alpha_i\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k\right).$$

Now let us consider the first term of the $(k-1)$-form α, which is $\alpha_1 \; dx_2 \wedge \ldots \wedge dx_k$. Then we have

$$\begin{aligned}
&\int_{I^k} d\left(\alpha_1 \; dx_2 \wedge \cdots \wedge dx_k\right) \\
&= \int_{I^k} \frac{\partial \alpha_1}{\partial x_1} \; dx_1 \wedge dx_2 \wedge \cdots \wedge dx_k \\
&= \int_{x_k=0}^{1} \cdots \int_{x_2=0}^{1} \int_{x_1=0}^{1} \frac{\partial \alpha_1}{\partial x_1} \; dx_1 \; dx_2 \; \cdots \; dx_k \quad \begin{array}{c}\text{traditional integration}\\ \text{(orientation does not matter)}\end{array} \\
&= \int_{x_k=0}^{1} \cdots \int_{x_2=0}^{1} \Big[\alpha_1(x_1, x_2, \ldots, x_k)\Big]_0^1 \; dx_2 \; \cdots \; dx_k \quad \begin{array}{c}\text{Fundamental}\\ \text{Theorem}\\ \text{of Calculus}\end{array} \\
&= \int_{x_k=0}^{1} \cdots \int_{x_2=0}^{1} \Big[\alpha_1(1, x_2, \ldots, x_k) - \alpha_1(0, x_2, \ldots, x_k)\Big] \; dx_2 \; \cdots \; dx_k \\
&= \underbrace{\int_{x_k=0}^{1} \cdots \int_{x_2=0}^{1} \alpha_1(1, x_2, \ldots, x_k) \; dx_2 \; \cdots \; dx_k}_{\int \text{ of } \alpha_1 \text{ restricted to face } I^k_{(1,1)}} \\
&\qquad\qquad - \underbrace{\int_{x_k=0}^{1} \cdots \int_{x_2=0}^{1} \alpha_1(0, x_2, \ldots, x_k) \; dx_2 \; \cdots \; dx_k}_{\int \text{ of } \alpha_1 \text{ restricted to face } I^k_{(1,0)}} \\
&= \int_{I^k_{(1,1)}} \alpha_1 \; dx_2 \wedge \cdots \wedge dx_k - \int_{I^k_{(1,0)}} \alpha_1 \; dx_2 \wedge \cdots \wedge dx_k.
\end{aligned}$$

As you notice, we switched notation from $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_k$ in the second line to $dx_1 \; dx_2 \; \cdots \; dx_k$ in the third line, and then switched back to $dx_2 \wedge \cdots \wedge dx_k$ in the final line. As you hopefully remember from an earlier chapter, our traditional integration that you learned in calculus class differs from the integration of differential forms in only one way, it does not keep track of the orientation of the surface. We make this switch in order to use the fundamental theorem of calculus, which we only know in terms of our traditional notation. Nothing that we have done while we use the traditional notation affects the orientation of the cube over which we are integrating. We recognize this because the ordering when we switch from the forms notation to the traditional notation and then when we switch from our traditional notation back to the forms notation is consistent. To see this compare the second line where we have the volume form $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_k$ with the last line where we have the volume form $dx_2 \wedge \cdots \wedge dx_k$. The volume form in the last line is one dimension lower than that in the first line, it is missing the dx_1 term because we integrated with respect to the dx_1 term, but other than that one omission the ordering of the other $k-1$ terms is unchanged. Because the ordering of the remaining terms is unchanged when we are ready to convert back to forms notation everything is consistent. For convenience sake we will rewrite this equality that we have just found denoting the first term of α as α_1 instead of $\alpha_1 \; dx_2 \wedge \ldots \wedge dx_k$, so we have

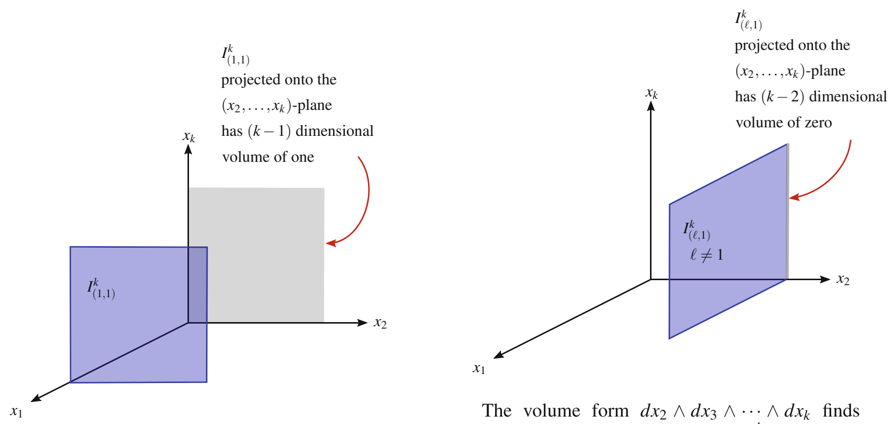
$$\int_{I^k} d\alpha_1 = \int_{I^k_{(1,1)}} \alpha_1 - \int_{I^k_{(1,0)}} \alpha_1.$$

This is admittedly a slight abuse of notation, but it does simplify things considerably.

Now we take a moment to show that the integral of the $(k-1)$-form $\alpha_1 dx^2 \wedge \cdots \wedge dx_k$ is zero on all of the other faces of ∂I^k and that the only contribution we would expect from integrating α_1 over the boundary of I^k is that which is given by the right hand side of the above equation. All the other faces of ∂I^k are of the form $I^k_{(i,a)}$ for $2 \le i \le k$ and $a = 0, 1$.

Now recall what $dx_2 \wedge dx_3 \wedge \cdots \wedge dx_k$ does, it projects parallelepipeds to the $(x_2 \cdots x_k)$-plane and finds the volume of that projection. We are interested in taking integrals of α_1 on the various faces of I^k. Consider $I^k_{(1,1)}$, which is a $(k-1)$-dimensional parallelepiped. Projecting $I^k_{(1,1)}$ onto the $(x_2 \cdots x_k)$-plane and finding the volume of this projection gives a volume of one. Similarly, projecting $I^k_{(1,0)}$ onto the $(x_2 \cdots x_k)$-plane also gives us a volume of one. See Fig. 11.26. Thus we are able to take integrals of α_1 over over this face.

But now consider $I^k_{(2,1)}$. When this face is projected onto the $(x_2 \cdots x_k)$-plane the $(k-1)$-dimensional volume is zero, since the projection of $I^k_{(2,1)}$ onto the $(x_2 \cdots x_k)$-plane is a $(k-2)$-dimensional set, which has zero $(k-1)$-dimensional



The volume form $dx_2 \wedge dx_3 \wedge \cdots \wedge dx_k$ finds the volume of the projection of $I^k_{(1,1)}$ onto the $(x_2 \cdots x_k)$-plane. The case of $I^k_{(1,0)}$ is similar. Both have a $(k-1)$-dimensional volume of one.

The volume form $dx_2 \wedge dx_3 \wedge \cdots \wedge dx_k$ finds the volume of the projection of $I^k_{(2,1)}$ onto the $(x_2 \cdots x_k)$-plane. The $(k-1)$-dimensional volume is zero. Similarly, the $(k-1)$ dimensional volumes of the projections of $I^i_{(j,a)}$ for $j = 2,\dots,k$ and $a = 0,1$ onto the $(x_2 \cdots x_k)$-plane are zero.

Fig. 11.26 Finding the volumes of the faces of I^k when they are projected to the $(x_2 \cdots x_k)$-plane

volume. This $\int_{I^k_{(2,1)}} \alpha_1 = 0$. Similarly we find that $\int_{I^k_{(j,a)}} \alpha_1 = 0$ for all $j = 2,\dots,k$ and $a = 0,1$, see Fig. 11.26. Thus, the only integral on the boundary of I^k that can contribute anything are those we already have, $\int_{I^k_{(1,1)}} \alpha_1$ and $\int_{I^k_{(1,0)}} \alpha_1$.

Also, notice what we found, that

$$\int_{I^k} d\alpha_1 = \int_{I^k_{(1,1)}} \alpha_1 - \int_{I^k_{(1,0)}} \alpha_1.$$

The integral over $I^k_{(1,1)}$ is positive, which is the orientation of the face $I^k_{(1,1)}$ in the boundary of I^k, since $(-1)^{1+1} = 1$. Similarly, the integral over $I^k_{(1,0)}$ is negative, which is the orientation of the face $I^k_{(1,0)}$ in the boundary of I^k, since $(-1)^{1+0} = -1$. Thus we could, and in fact do, make the following definition

$$\int_{I^k_{(1,1)}} \alpha_1 - \int_{I^k_{(1,0)}} \alpha_1 \equiv \int_{I^k_{(1,1)} - I^k_{(1,0)}} \alpha_1,$$

which allows us to write

$$\int_{I^k} d\alpha_1 = \int_{I^k_{(1,1)} - I^k_{(1,0)}} \alpha_1.$$

Now we will look at the second term of α. We have

$$\begin{aligned}
&\int_{I^k} d\left(\alpha_2\, dx_1 \wedge dx_3 \wedge \cdots \wedge dx_k\right) \\
&= \int_{I^k} \frac{\partial \alpha_2}{\partial x_2}\, dx_2 \wedge dx_1 \wedge dx_3 \wedge \cdots dx_k \\
&= (-1) \int_{I^k} \frac{\partial \alpha_2}{\partial x_2}\, dx_1 \wedge dx_2 \wedge dx_3 \wedge \cdots dx_k \\
&= (-1) \int_{x_k=0}^{1} \cdots \int_{x_2=0}^{1} \int_{x_1=0}^{1} \frac{\partial \alpha_2}{\partial x_2}\, dx_1\, dx_2\, \cdots\, dx_k \qquad \begin{array}{c}\text{traditional integration}\\ \text{(orientation not matter)}\end{array} \\
&= (-1) \int_{x_k=0}^{1} \cdots \int_{x_1=0}^{1} \int_{x_2=0}^{1} \frac{\partial \alpha_2}{\partial x_2}\, dx_2\, dx_1\, \cdots\, dx_k \qquad \begin{array}{c}\text{Fubini's}\\ \text{Theorem}\end{array}
\end{aligned}$$

$$
\begin{aligned}
&= (-1) \int_{x_k=0}^{1} \cdots \int_{x_1=0}^{1} \Big[\alpha_2(x_1, x_2, x_3 \dots, x_k)\Big]_0^1 \, dx_1 \, dx_3 \cdots dx_k \quad \begin{array}{c}\text{Fundamental}\\ \text{Theorem}\\ \text{of calculus}\end{array} \\
&= (-1) \int_{x_k=0}^{1} \cdots \int_{x_3=0}^{1} \int_{x_1=0}^{1} \Big[\alpha_2(x_1, 1, x_3 \dots, x_k) - \alpha_1(x_1, 0, x_3, \dots, x_k)\Big] \, dx_1 \, dx_3 \, \cdots \, dx_k \\
&= (-1) \underbrace{\int_{x_k=0}^{1} \cdots \int_{x_3=0}^{1} \int_{x_1=0}^{1} \alpha_2(x_1, 1, x_3, \dots, x_k) \, dx_1 \, dx_3 \, \cdots \, dx_k}_{\int \text{ of } \alpha_2 \text{ restricted to face } I^k_{(2,1)}} \\
&\qquad - (-1) \underbrace{\int_{x_k=0}^{1} \cdots \int_{x_3=0}^{1} \int_{x_1=0}^{1} \alpha_2(x_1, 0, x_3, \dots, x_k) \, dx_1 \, dx_3 \, \cdots \, dx_k}_{\int \text{ of } \alpha_2 \text{ restricted to face } I^k_{(2,0)}} \\
&= - \int_{I^k_{(2,1)}} \alpha_2 \, dx_1 \wedge dx_3 \wedge \cdots \wedge dx_k + \int_{I^k_{(2,0)}} \alpha_2 \, dx_1 \wedge dx_3 \wedge \cdots \wedge dx_k \\
&= \int_{-I^k_{(2,1)} + I^k_{(2,0)}} \alpha_2 \, dx_1 \wedge dx_3 \wedge \cdots \wedge dx_k .
\end{aligned}
$$

As you notice, we switched notation from $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_k$ in the third line to $dx_1 \, dx_2 \, \cdots \, dx_k$ in the fourth line, and then switched back to $dx_2 \wedge \cdots \wedge dx_k$ in the next to last line. We make this switch in order to use both the Fubini's Theorem and the fundamental theorem of calculus, which we only know in terms of our traditional notation. You may think that by switching the order of integration when applying Fubini's Theorem is changing our volume-orientation, but since traditional integration does not keep track of volume-orientation sign this change does not change the accompanying volume-orientation sign. This means that what matters is that the orientations of our volume forms when we switch to and from our traditional notation are consistent. For example, the volume form in the third line is $dx_1 \wedge dx_2 \wedge dx_3 \wedge \cdots \wedge dx_k$ and the orientation in the next to last line is $dx_1 \wedge dx_3 \wedge \cdots \wedge dx_k$. Though we are missing dx_2 in the next to last line because we integrated with respect to that term, the ordering of the remaining terms is unchanged and thus consistent.

Notice how the signs again match the orientations of the faces in the boundary of I^k. This allows us to rewrite the end result of this calculation. If we additionally use the same abuse of notation that we used earlier, writing α_2 for $\alpha_2 \, dx_1 \wedge dx_3 \wedge \cdots \wedge dx_k$, we have the equality

$$
\int_{I^k} d\alpha_2 = \int_{-I^k_{(2,1)} + I^k_{(2,0)}} \alpha_2 .
$$

Now that we have done this for the first two terms of the $(k-1)$-form you have some idea how we will proceed. Now we will do it for the full $\alpha = \sum \alpha_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k$, where $\widehat{dx_i}$ means that the dx_ith term is omitted. First we find $d\alpha$,

$$
\begin{aligned}
& d\left(\sum_{i=1}^{k} \alpha_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k \right) \\
&= \sum_{i=1}^{k} d\left(\alpha_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k \right) \\
&= \sum_{i=1}^{k} \frac{\partial \alpha_i}{\partial x_i} \, dx_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k \\
&= \sum_{i=1}^{k} (-1)^{i-1} \, \frac{\partial \alpha_i}{\partial x_i} \, \underbrace{dx_1 \wedge \cdots \wedge dx_i \wedge \cdots \wedge dx_k}_{i-1 \text{ transpositions gives } (-1)^{i-1}} \\
&= \left(\sum_{i=1}^{k} (-1)^{i-1} \, \frac{\partial \alpha_i}{\partial x_i} \right) \, dx_1 \wedge \cdots \wedge dx_i \wedge \cdots \wedge dx_k
\end{aligned}
$$

Now that we have $d\alpha$ we proceed with the integration,

$$
\begin{aligned}
\int_{I^k} d\alpha &= \int_{I^k} \left(\sum_{i=1}^{k} (-1)^{i-1} \frac{\partial \alpha_i}{\partial x_i} \right) dx_1 \wedge \cdots \wedge dx_k \\
&= \sum_{i=1}^{k} (-1)^{i-1} \int_{I^k} \frac{\partial \alpha_i}{\partial x_i} \, dx_1 \wedge \cdots \wedge dx_k \\
&= \sum_{i=1}^{k} (-1)^{i-1} \int_0^1 \cdots \int_0^1 \frac{\partial \alpha_i}{\partial x_i} \, dx_i dx_1 \cdots \widehat{dx_i} \cdots dx_k \qquad \text{traditional integration (orientation not matter)} \\
&= \sum_{i=1}^{k} (-1)^{i-1} \int_0^1 \cdots \int_0^1 \big[\alpha_i(x_1, \ldots, x_k)\big]_0^1 \, dx_1 \cdots \widehat{dx_i} \cdots dx_k \qquad \text{Fundamental Theorem of calculus} \\
&= \sum_{i=1}^{k} (-1)^{i-1} \int_0^1 \cdots \int_0^1 \Big[\alpha_i(x_1, \ldots, \underbrace{1}_{i^{th}}, \ldots, x_k) - \alpha_i(x_1, \ldots, \underbrace{0}_{i^{th}}, \ldots, x_k)\Big] \, dx_1 \cdots \widehat{dx_i} \cdots dx_k \\
&= \sum_{i=1}^{k} (-1)^{i-1} \Bigg[\int_{I^k_{(i,1)}} \alpha_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k \\
&\qquad\qquad - \int_{I^k_{(i,0)}} \alpha_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k \Bigg] \\
&= \sum_{i=1}^{k} \Bigg[\underbrace{(-1)^{i-1} \int_{I^k_{(i,1)}} \alpha_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k}_{\substack{I^k_{(i,1)} \text{ has orientation } (-1)^{i+1} \text{ but } (-1)^{i-1} = (-1)^{i+1} \\ \text{so sign of integral matches orientation of } I^k_{(i,1)} \text{ in } \partial I^k}} \\
&\qquad\qquad + \underbrace{(-1)^{i} \int_{I^k_{(i,0)}} \alpha_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k}_{\substack{I^k_{(i,0)} \text{ has orientation } (-1)^{i+0} \text{ but } (-1)^{i+0} = (-1)^{i} \\ \text{so sign of integral matches orientation of } I^k_{(i,0)} \text{ in } \partial I^k}} \Bigg] \\
&= \sum_{i=1}^{k} \Bigg[\int_{(-1)^{i-1} I^k_{(i,1)}} \alpha_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k \\
&\qquad\qquad + \int_{(-1)^{i+0} I^k_{(i,0)}} \alpha_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k \Bigg] \\
&= \sum_{i=1}^{k} \int_{(-1)^{i-1} I^k_{(i,1)} + (-1)^{i+0} I^k_{(i,0)}} \alpha_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k \\
&= \int_{\sum_{i=1}^{k} \left((-1)^{i-1} I^k_{(i,1)} + (-1)^{i+0} I^k_{(i,0)} \right)} \sum_{i=1}^{k} \alpha_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k \qquad \text{Only } \alpha_i \text{ term of } \alpha \text{ contributes to faces } I^i_{(i,1)} \text{ and } I^i_{(i,0)} \text{ so we can add this.} \\
&= \int_{\partial I^k} \alpha.
\end{aligned}
$$

Thus we have shown for a $(k-1)$-form α defined in a neighborhood of the unit k-cube that

$$\boxed{\int_{I^k} d\alpha = \int_{\partial I^k} \alpha.}$$

This is exactly Stokes' theorem for the case of the unit k-cube. Now that we have this all-important base case the other cases follow fairly quickly and easily.

11.3 Manifolds Parameterized by I^k

Now suppose we have a manifold with boundary M that is parameterized by the unit cube. That is, we have a mapping $\phi : \mathbb{R}^k \to \mathbb{R}^n$ where $\phi(I^k) = M$. See Fig. 11.27. Actually, ϕ need not even be one-to-one. The mapping ϕ is often called a **singular** k**-cube** on M. The word singular indicates that the mapping ϕ need not be one-to-one. Requiring ϕ to be non-singular is more restrictive and not necessary, though we will require that the ϕ be differentiable.

Whenever looking at the proofs of Stokes' theorem in other books recall that $T^*\phi$ is our somewhat nonstandard notation for ϕ^*. We will not be terribly rigorous at this point, but we we will also require that ϕ respect boundaries, and by that we mean if $M = \phi(I^k)$ then $\partial M = \phi(\partial I^k)$. In other words, we require that

$$\partial \phi(I^k) = \phi(\partial I^k).$$

This is sufficiently general and "obvious" that is does not cause a problem. We also need to recall the following identity regarding the integration of pull-backed forms,

$$\int_{\phi(R)} \alpha = \int_R T^*\phi \cdot \alpha$$

for some region R. In the previous chapter we would have written this with $\phi^{-1}(R)$, but if ϕ were singular, that is, not one-to-one, then ϕ would not be invertible so we avoid that notation here. Finally, recall that the exterior derivative commutes with pull-back, that is, that

$$d \circ T^*\phi = T^*\phi \circ d$$

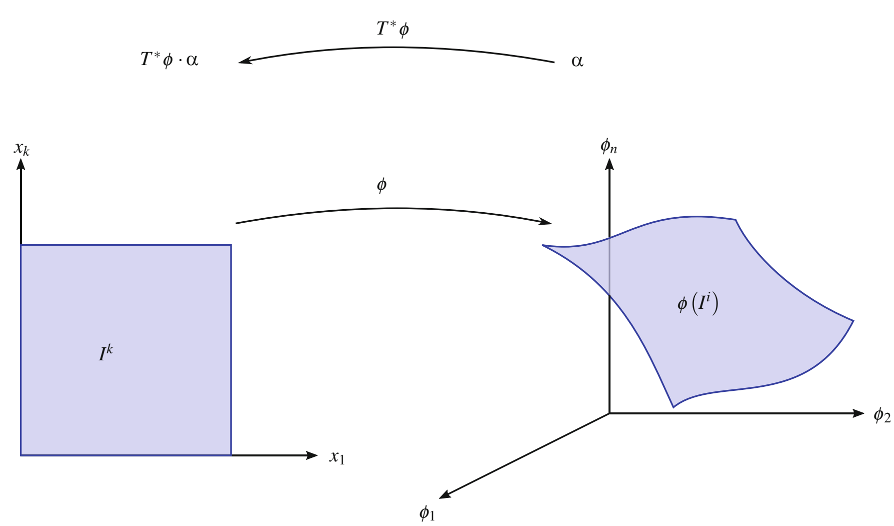


Fig. 11.27 The mapping $\phi : \mathbb{R}^k \to \mathbb{R}^n$ where $\phi(I^k) = M$. A differential form α defined on $\phi(I^k)$ can be pulled back by $T^*\phi$ to the differential form $T^*\phi \cdot \alpha$ defined on I^k

or $d \circ \phi^* = \phi^* \circ d$ in most differential geometry textbooks. Using all of this, the final step is almost anti-climactic,

$$\begin{aligned}
\int_M d\alpha &= \int_{\phi(I^i)} d\alpha && \text{notation} \\
&= \int_{I^k} T^*\phi \cdot d\alpha && \text{integration of pulled-back forms} \\
&= \int_{I^k} d(T^*\phi \cdot \alpha) && \text{exterior derivative commutes with pull-back} \\
&= \int_{\partial I^k} T^*\phi \cdot \alpha && \text{base case} \\
&= \int_{\phi(\partial I^k)} \alpha. && \text{integration of pulled-back forms} \\
&= \int_{\partial\phi(I^k)} \alpha && \phi \text{ respects boundaries} \\
&= \int_{\partial M} \alpha && \text{notation}
\end{aligned}$$

Thus, for a singular k-cube on M we have

$$\boxed{\int_M d\alpha = \int_{\partial M} \alpha.}$$

11.4 Stokes' Theorem on Chains

Now suppose that our manifold with boundary, M, is not nicely parameterized by a single map ϕ, but can be broken into cuboid regions, each of which can be parameterized by a map $\phi_i : I^k \to \phi_i(I^k)$, $i = 1, \dots, r$. We will assume a finite number of maps ϕ_i are sufficient. See Fig. 11.28. But a warning, here we are using the notation ϕ_i differently than we did in past chapters. Here each ϕ_i is a mapping $\phi_i : \mathbb{R}^k \to \mathbb{R}^n$ whereas before we had that a mapping $\phi : \mathbb{R}^k \to \mathbb{R}^n$ that was defined as $\phi = (\phi_1, \phi_2, \dots, \phi_k)$ where each ϕ_i was a real-valued function $\phi_i : \mathbb{R}^k \to \mathbb{R}$.

We choose the maps that are used to parameterize the manifold M such that the interiors of $\phi_i(I^k)$ are non-overlapping and the boundaries of $\phi_i(I^k)$ match up. We won't go into excessive technical details trusting that the above picture will give you the right intuitive idea. A **singular k-chain** C on M is defined to be

$$C = \phi_1 + \phi_2 + \cdots + \phi_r$$

where each ϕ_i, $i = 1, \dots, r$, is a singular k-cube on M. In other words, we have

$$\begin{aligned}
M &= \bigcup_{i=1}^{r} \phi_i(I^k) \\
&\equiv \phi_1(I^k) + \phi_2(I^k) + \cdots + \phi_r(I^k) \\
&\equiv C(I^k).
\end{aligned}$$

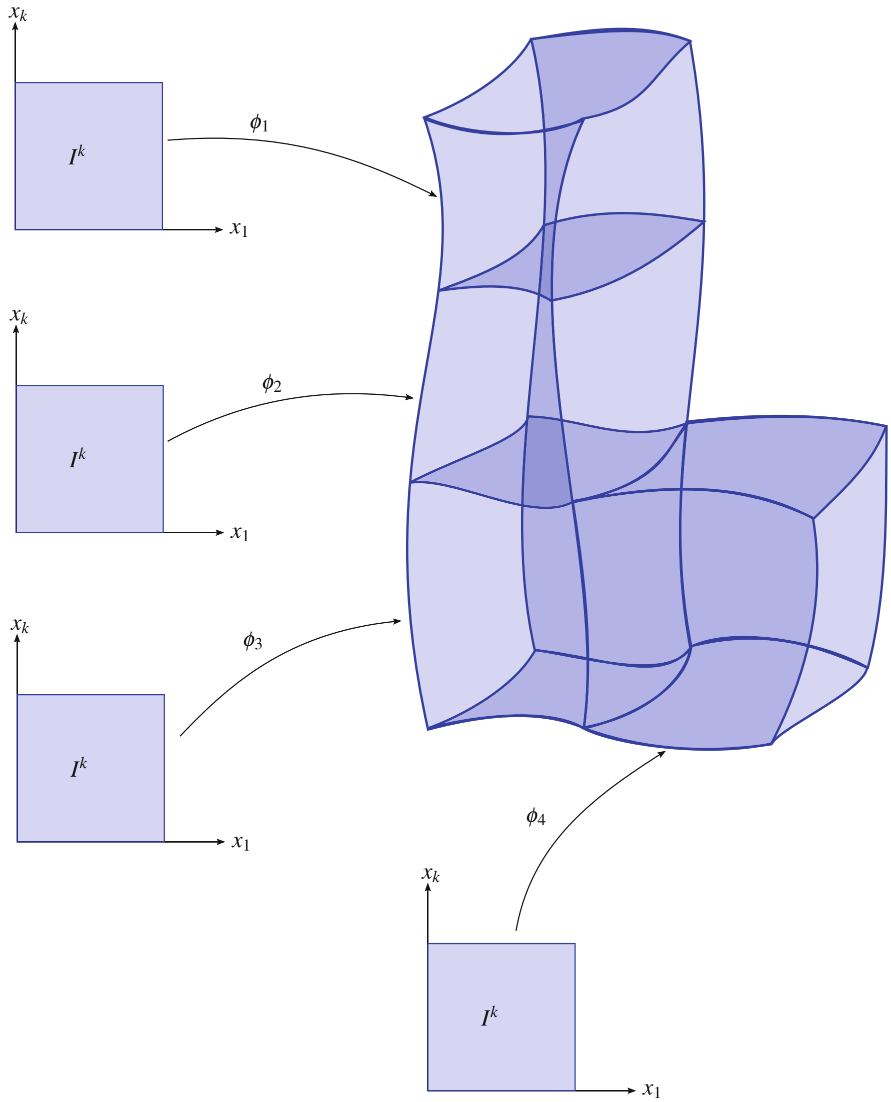


Fig. 11.28 The manifold M can be broken up into a finite number of cuboid regions each of which is parameterized by a map $\phi_i : I^k \to \phi(I^k) \subset M$. If the interiors of the $\phi_i(I^k)$ are non-overlapping, the boundaries of $\phi_i(I^k)$ match up, and $M = \bigcup_{i=1}^{r} \phi_i(I^k)$ then $\phi_1 + \phi_2 + \cdots + \phi_n$ is called a singular k-chain on M

We will also assume that all the interior boundaries of the singular k-chain C match up and have the opposite orientations as Fig. 11.29 shows. The orientation of $\phi_i(I^k_{(l,1)})$ is $(-1)^{l+1}$ and the orientation of $\phi_j(I^k_{(l,0)})$ is $(-1)^{l+0}$ so the contributions to the integral of α on $\phi_i(I^k_{(l,1)})$ is canceled by the contribution of the integral of α on $\phi_j(I^k_{(l,0)})$.

We have

$$
\begin{aligned}
\partial(M) &= \partial\big(C(I^k)\big) \\
&= \partial\big(\phi_1(I^k) + \phi_2(I^k) + \cdots + \phi_r(I^k)\big) \\
&= \partial\big(\phi_1(I^k)) + \partial(\phi_2(I^k)) + \cdots + \partial(\phi_r(I^k)\big) \\
&= \phi_1\big(\partial(I^k)\big) + \phi_2\big(\partial(I^k)\big) + \cdots + \phi_r\big(\partial(I^k)\big)
\end{aligned}
$$

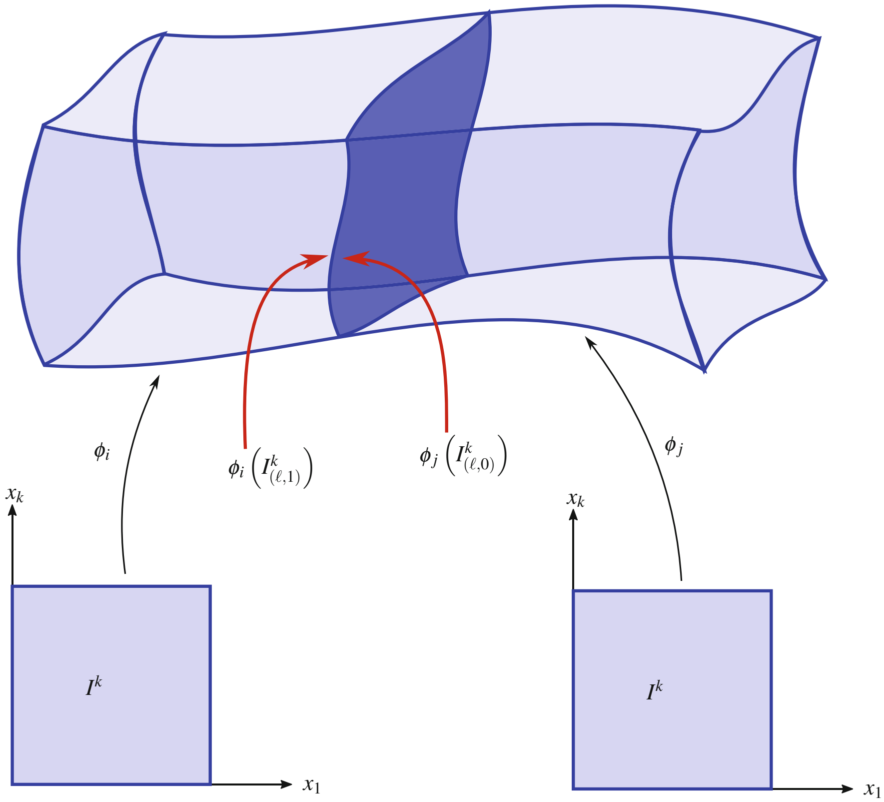


Fig. 11.29 Here the boundaries of two cuboid regions match up and have opposite orientations. That is, $\phi_i(I^k_{(l,1)}) = -\phi_j(I^k_{(l,0)})$

So proving Stokes' theorem on chains is easy,

$$
\begin{aligned}
\int_M d\alpha &= \int_{C(I^k)} d\alpha \\
&= \int_{\phi_1(I^k)+\cdots+\phi_r(I^k)} d\alpha \\
&= \int_{\phi_1(I^k)} d\alpha + \cdots + \int_{\phi_r(I^k)} d\alpha \\
&= \int_{I^k} T^*\phi_1 \cdot d\alpha + \cdots + \int_{I^k} T^*\phi_r \cdot d\alpha \\
&= \int_{I^k} d(T^*\phi_1 \cdot \alpha) + \cdots + \int_{I^k} d(T^*\phi_r \cdot \alpha) \\
&= \int_{\partial I^k} T^*\phi_1 \cdot \alpha + \cdots + \int_{\partial I^k} T^*\phi_r \cdot \alpha \\
&= \int_{\phi_1(\partial I^k)} \alpha + \cdots + \int_{\phi_r(\partial I^k)} \alpha \\
&= \int_{\phi_1(\partial I^k)+\cdots+\phi_r(\partial I^k)} \alpha \\
&= \int_{\partial\phi_1(I^k)+\cdots+\partial\phi_r(I^k)} \alpha \\
&= \int_{\partial(M)} \alpha .
\end{aligned}
$$

So we have

$$\boxed{\int_M d\alpha = \int_{\partial M} \alpha}$$

for singular k-chains on M. Sometimes Stokes' theorem is stated in terms of chains.

Theorem 11.2 (Stokes' Theorem (Chain Version)) *For any n-form α and $(n+1)$-chain c then*

$$\int_c d\alpha = \int_{\partial c} \alpha.$$

Question 11.13 Explain the rationale in each step of the above chain of equalities.

11.5 Extending the Parameterizations

Finally, while it may look like the cubical domain is special, it really isn't. We will only sketch this last step in the proof of Stokes' theorem, but it should be enough to convince you that it is true. We will leave it to you, in the exercises, to fill in more of the details. The most important thing is to realize that virtually any domain can be either reparameterized into unit cubes or broken up into pieces which can then be reparameterized into unit cubes. Reverting to two-dimensions consider the domain

$$D = \left\{ (u, v) \in \mathbb{R}^2 \;\middle|\; a \le u \le b,\ f(u) \le v \le g(u) \right\}$$

where we know $\phi : D \to \mathbb{R}^n$ is a nice parametrization of the manifold $M = \phi D$. See Fig. 11.30. Our intent is to show that having a parameterizations of our manifold M by domains like D is enough for Stokes' theorem. Consider the mapping

$$(s,t) \in I^2 \overset{\psi}{\longmapsto} \Big((1-s)a + sb,\ (1-t) f\big((1-s)a + sb\big) + t g\big((1-s)a + sb\big) \Big).$$

Question 11.14 Show that $\psi : I^2 \to D$ is one-to-one and onto.

Question 11.15 Show that $\phi \circ \psi : I^2 \to M$ is a surface parameterized by I^2 and hence the proof of Stokes' theorem applies to M.

A similar construction can be made for dimensions greater than two. Thus we have argued that Stokes' theorem

$$\boxed{\int_M d\alpha = \int_{\partial M} \alpha}$$

applies to any manifold whose parameterizations can reparameterized into unit cubes or broken up into pieces which can then be reparameterized into unit cubes. Without getting into the technical details this applies to essentially all manifolds.

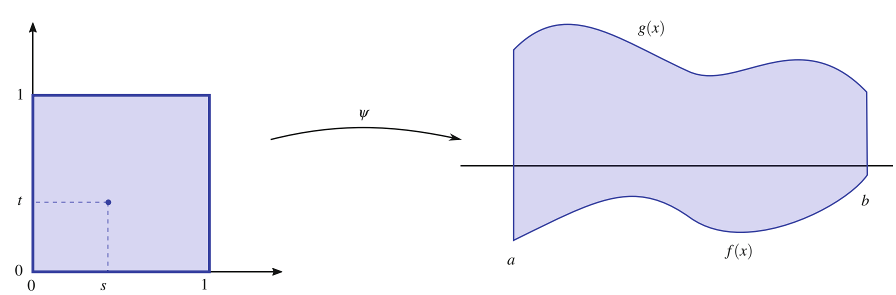


Fig. 11.30 Almost any domain can be reparameterized into a unit cube or broken into pieces that can be reparameterized into unit cubes

11.6 Visualizing Stokes' Theorem

In this section we will try to provide a cartoon image of what the generalized Stokes' theorem means, at least in three dimensions, based on the material in Chap. 5. This presentation is more in line with the way that physicists sometimes visualize differential forms than with the perspective that mathematicians generally take. This visualization technique is actually not general, there are manifolds and forms for which it breaks down and does not work, however, when appropriate it does provide a very nice geometrically intuitive way of looking at Stoke's theorem. In particular we will rely on the visualizations developed in Sect. 5.5. With the aid of Stokes' theorem we will be able to gain a slightly more nuanced view of these visualizations, at least in the cases where these visualizations are possible. We will tie together all of these ideas in this section.

First we recall that in $\mathbb{R}^3$ we can visualize one-forms as sheets filling $\mathbb{R}^3$, as shown in Fig. 5.36, two-forms as tubes filling $\mathbb{R}^3$, as shown in Fig. 5.40, and three-forms as cubes filling $\mathbb{R}^3$, as shown in Fig. 5.30. We will simply simply continue to view zero-forms f on $\mathbb{R}^3$ as we have always viewed functions, as a value $f(p)$ attached to each point $p \in \mathbb{R}^3$. We also will recall a differential form ω is called closed if $d\omega = 0$.

We will begin by considering a zero-form f. Clearly df is a one-form which we can imagine as sheets filling space. A one-dimensional manifold is a curve C, so in the case of the zero-form f Stokes' theorem gives us

$$\int_C df = \int_{\partial C} f.$$

From Sect. 9.5.2 we of course recognize this as simply the fundamental theorem of line integrals. As we move along C the integral of df counts the number of sheets of the one-form df the curve C goes through. See Fig. 11.31.

Question 11.16 Based on what you know about how one-forms can be written as sheets and how line integrals are calculated, argue that indeed $\int_C df$ counts the number of sheets of df that the curve C passes through.

Question 11.17 Suppose f is closed, that is, $df = 0$. What does this say about the function f? Use Fig. 11.31 as a guide.

Question 11.18 Based on the last two questions, how can we use f to help us draw the sheets of the one-form df?

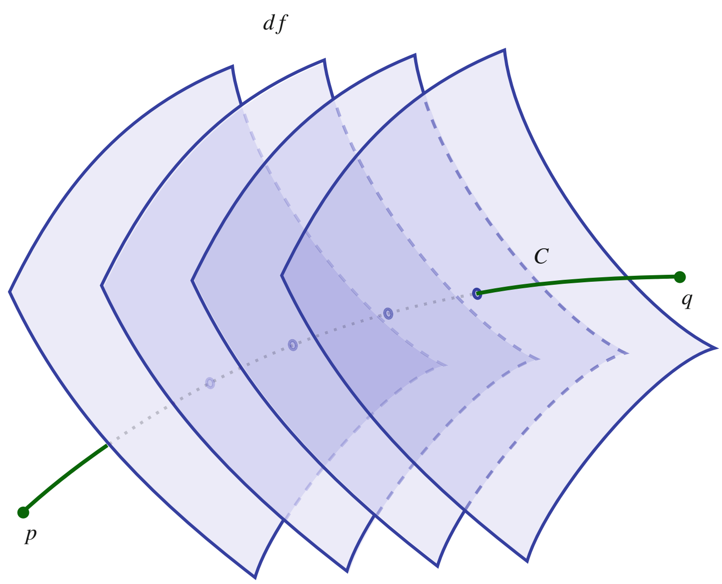


Fig. 11.31 A curve C passing through the sheets of df. In this figure $\int_C df = 4$. Thus, by Stokes' theorem we know $\int_{\partial C} f = f(q) - f(p) = 4$. We can think of the number of sheets as being given by the change in the value of the function between points p and q

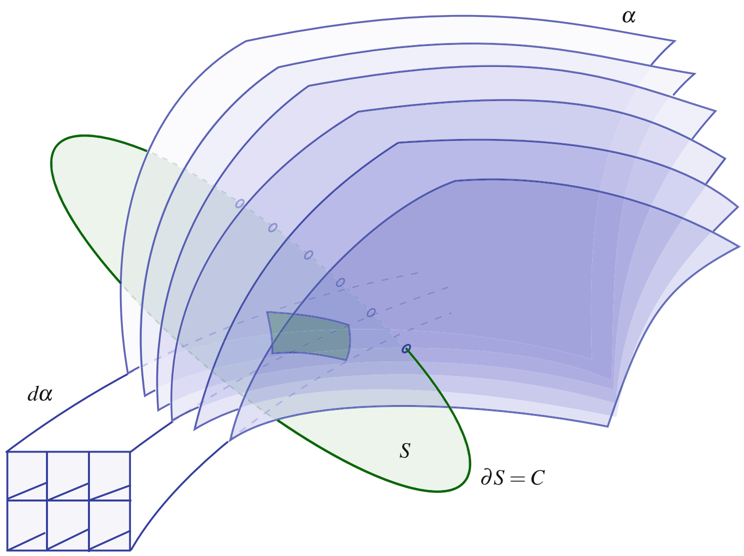


Fig. 11.32 A surface S with boundary $\partial S = C$. In the picture the one-form α is depicted as sheets and the two-form $d\alpha$ is depicted as tubes. Here $\int_{\partial S} \alpha$ counts the number of sheets that the curve C goes through and $\int_S d\alpha$ counts the number of tubes that go through the surface S. In this picture we have both $\int_S d\alpha = 6$ and $\int_{\partial S} \alpha = 6$. We can think of the sheets of α as emanating from the tubes of $d\alpha$, one sheet for each tube

Next we consider a one-form α. Since α is a one-form then we can visualize it as $\mathbb{R}^3$ filled with sheets whereas its exterior derivative $d\alpha$ is a two-form, which can be visualized as $\mathbb{R}^3$ filled with tubes. How exactly do these two pictures relate? A two-dimensional manifold is a surface S with boundary being a curve, $\partial S = C$. We will turn to Stokes' theorem

$$\int_S d\alpha = \int_{\partial S} \alpha.$$

From Sect. 9.5.3 we know this is simply the vector calculus version of Stokes' theorem. $\int_S d\alpha$ counts the number of tubes of $d\alpha$ that go through the surface S and, as before, $\int_{\partial S} \alpha$ counts the number of surfaces of α. From Stokes' theorem these are the same. See Fig. 11.32. Based on Stokes' theorem the number of tubes of $d\alpha$ going through S is the same as the number of sheets of α going through ∂S, so in a sense each tube of $d\alpha$ gives rise to additional sheets of α.

Question 11.19 Based on what you know about how two-forms are depicted as tubes and how surface integrals are calculated, argue that indeed $\int_S d\alpha$ counts the number of tubes of $d\alpha$ that pass through S.

Question 11.20 Suppose α is closed. What does that say about the number of tubes passing through S or the number of sheets passing through $\partial S = C$? Is it possible for α to still have sheets even though $d\alpha = 0$? Sketch a picture of this situation.

If α is a two-form then it can be visualized as $\mathbb{R}^3$ filled with tubes whereas the exterior derivative $d\alpha$ is a three-form, which can be visualized as $\mathbb{R}^3$ filled with small cubes. A three-dimensional manifold is a volume V with boundary being a surface, $\partial V = S$. Stokes' theorem becomes

$$\int_V d\alpha = \int_{\partial V} \alpha.$$

From Sect. 9.5.4 we know this is simply the divergence theorem from vector calculus. The integral $\int_V d\alpha$ counts the number of cubes of $d\alpha$ inside the volume V while $\int_{\partial V} \alpha$ counts the number of tubes of α that penetrate the surface S. This leads to a picture very similar to the last two, see Fig. 11.33. Since these are equal by Stokes' theorem then we can imagine that the tubes of α are emanating from the cubes of $d\alpha$, one tube for each cube.

Question 11.21 Based on what you know about how three-forms are depicted as cubes and how volume integrals are calculated, argue that $\int_V d\alpha$ indeed counts the number of cubes of $d\alpha$ inside V.

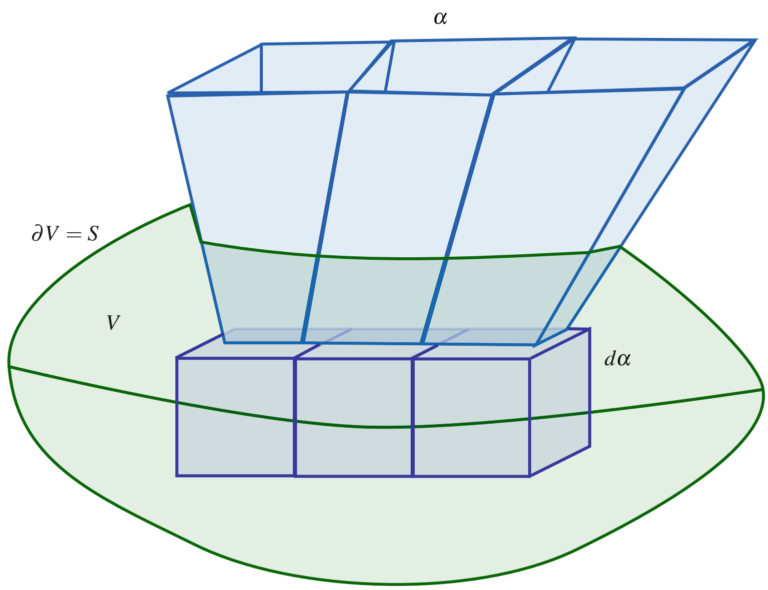


Fig. 11.33 A volume V with boundary $\partial V = S$. In the picture the two-form α is depicted as tubes and the three-form $d\alpha$ is depicted as cubes. Here $\int_{\partial V} \alpha$ counts the number of tubes that go through the surface S and $\int_V d\alpha$ counts the number of cubes that are contained in the volume V. In this picture we have both $\int_V d\alpha = 3$ and $\int_{\partial V} \alpha = 3$. We can think of the tubes of α as emanating from the cubes of $d\alpha$, one tube for each cube

Question 11.22 Suppose α is closed. What does this say about the number of cubes inside V or the number of tubes that pass though $\partial V = S$? Is it possible to still have tubes even though $d\alpha = 0$? Sketch a picture of this situation.

As you recall, in Chap. 4 we took great pains to introduce exterior differentiation from a variety of viewpoints. We did this because exterior differentiation plays a fundamental role in this book and in differential geometry and physics in general. In Sect. 4.5 we provided what we believe is one of the nicest geometrical meanings of exterior differentiation. However, that perspective relied on understanding integration of forms, which at that point we had not yet covered. We are now ready to return to the geometric picture of exterior differentiation presented in that section to understand the geometry of exterior differentiation, at least in three dimensions, a little more fully. In that section we had looked at the two-form ω and had obtained the following formula for $d\omega(v_1, v_2, v_3)$,

$$\begin{aligned}
d\omega(v_1, v_2, v_3) &= \lim_{h\to 0} \frac{1}{h^3} \int_{\partial(hP)} \omega \\
&= \lim_{h\to 0} \frac{1}{h^3} \left(\int_{hP_{(1,1)}} \omega - \int_{hP_{(1,0)}} \omega - \int_{hP_{(2,1)}} \omega + \int_{hP_{(2,0)}} \omega + \int_{hP_{(3,1)}} \omega - \int_{hP_{(3,0)}} \omega \right) \\
&= \underbrace{\lim_{h\to 0} \frac{1}{h^3} \left(\int_{hP_{(1,1)}} \omega - \int_{hP_{(1,0)}} \omega \right)}_{\text{see Fig. 11.34 left}} - \underbrace{\lim_{h\to 0} \frac{1}{h^3} \left(\int_{hP_{(2,1)}} \omega - \int_{hP_{(2,0)}} \omega \right)}_{\text{see Fig. 11.34 middle}} \\
&\quad + \underbrace{\lim_{h\to 0} \frac{1}{h^3} \left(\int_{hP_{(3,1)}} \omega - \int_{hP_{(3,0)}} \omega \right)}_{\text{see Fig. 11.34 right}}.
\end{aligned}$$

But now we have a much better idea of what this represents graphically. If ω is a two-form then we think of it as tubes filling space. The integral $\int_{hP_{(1,1)}} \omega$ is the number of tubes going through the $hP_{(1,1)}$ face of the parallelepiped hP. Similarly, the integral $\int_{hP_{(1,0)}} \omega$ is the number of tubes going through the $hP_{(1,0)}$ face of hP. The difference of these integrals is simply the difference in the number of tubes between these two faces in the v_1 direction. Of course we have to pay attention to the orientation of the tubes to get the sign of the integral correct, but this comes out when we actually do the calculation.

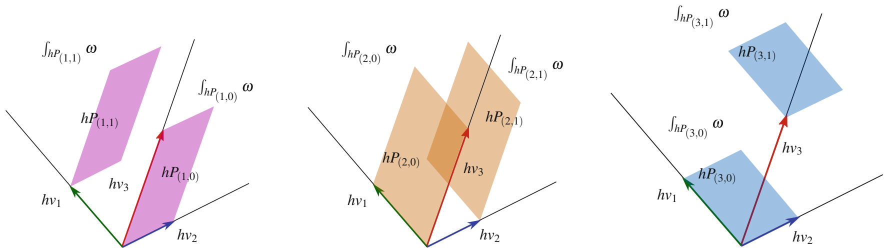


Fig. 11.34 The six faces of the cube $\partial(hP)$ which appear in the integral $\int_{\partial(hP)} \omega$. The integral $\int_{hP_{(1,1)}} \omega$ is the number of tubes going through the surface $hP_{(1,1)}$. The integral $\int_{hP_{(1,0)}} \omega$ is the number of tubes going through the surface $hP_{(1,0)}$, and so on. These six terms are added or subtracted consistent with the orientations of the face to get $\int_{\partial(hP)} \omega$

The other directions are handled similarly. When we sum the various terms we obtain the net change in the number of tubes occurring at a point in the directions v_1, v_2, and v_3 compatible with the orientations of the boundary of the parallelepiped $P = \text{span}\{v_1, v_2, v_3\}$. See Fig. 11.34.

Question 11.23 Repeat this analysis for a one-form α to explain the geometry of the exterior derivative $d\alpha$.

11.7 Summary, References, and Problems

11.7.1 Summary

Stokes' theorem is the major result regarding the integration of forms on manifolds. It states that if M is a smooth oriented n-dimensional manifold and α is an $(n-1)$-form on M then

$$\int_M d\alpha = \int_{\partial M} \alpha,$$

where ∂M is given the induced orientation.

11.7.2 References and Further Reading

There are probably no books on manifold theory or differential geometry that don't cover Stokes' theorem, but the presentation here largely followed Hubbard and Hubbard [27], Edwards [17], and Spivak [40]. While this presentation tried to be quite thorough and cover most cases, we have chosen to ignore several layers of technical details that make the theorem completely general. In particular, we have ignored the technical details related to manifolds with boundary. Also, as mentioned, it is possible to prove the whole thing on chains of k-simplices instead of k-cubes, the approach taken in Flanders [19]. In visualizing Stokes' theorem, at least in three dimensions, we have relied on the material of Chap. 5 as well as the paper by Warnick, Selfridge, and Arnold [49].

11.7.3 Problems

Question 11.24 Use Stokes' Theorem to find $\int_C \alpha$ where $\alpha = (2xy^3 + 4x)\, dx + (3x^2y^2 - 9y^2)\, dy$ is a one-form on $\mathbb{R}^2$ and C is the line segment from $(3, 2)$ to $(5, 4)$. Then find $\int_C \alpha$ for the line segment from $(5, 4)$ to $(3, 2)$.

Question 11.25 Use Stokes' Theorem to find $\int_C \alpha$ where $\alpha = (6x^2 - 3y)\, dx + (8y - 3x)\, dy$ is a one-form on $\mathbb{R}^2$ and C is the line segment from $(3, 3)$ to $(2, 2)$. Then find $\int_C \alpha$ for the line segment from $(2, 2)$ to $(3, 3)$.

Question 11.26 Let S be the region on $\mathbb{R}^2$ bounded by $y = 3x$ and $y = x^2$. Choose C_1 and C_2 such that $\partial S = C_1 + C_2$. Let $\alpha = x^2 y\, dx + y\, dy$ be a one-form on $\mathbb{R}^2$. Find $\int_S d\alpha$ directly. Then find $\int_{\partial S} \alpha$. Verify Stokes' theorem is satisfied.

Question 11.27 Let S be the quarter of the unit circle in the first quadrant of $\mathbb{R}^2$. Let C_1 be given by $\gamma_1(\theta) = \big(\cos(\theta), \sin(\theta)\big)$ for $0 \le \theta \le \frac{\pi}{2}$, let C_2 be the line segment from $(0, 1)$ to $(0, 0)$, and let C_3 be the line segment from $(0, 0)$ to $(1, 0)$. Hence $\partial S = C_1 + C_2 + C_3$. Let $\alpha = xy\, dx + x^2\, dy$. Use Stokes's theorem to find $\int_{C_1} \alpha$.

Question 11.28 Verify Stokes' theorem for M the unit disk $x^2 + y^2 \le 1$ and $\alpha = xy\, dy$.

Question 11.29 Verify Stokes' theorem for $M = \left\{(u + v, u - v^2, uv)\middle|0 \le u \le 1,\ 0 \le v \le 2\right\}$ and $\alpha = 3z^2\, dx$.

Question 11.30 Verify Stokes' theorem for $M = \left\{(u^2 \cos(v), u^3 \sin(v), u^2)\middle|0 \le u \le 2,\ 0 \le v \le 2\pi\right\}$ and $\alpha = -yz\, dx + xz\, dy$.

Question 11.31 Let V be a volume in $\mathbb{R}^3$. Show that the three-volume of V is given by $\int_{\partial V} \frac{1}{3}(z\, dx \wedge dy + y\, dz \wedge dx + x\, dy \wedge dz)$.

Question 11.32 Verify Stokes' theorem for M the unit cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$ and $\alpha = 2xz\, dx \wedge dy + x\, dy \wedge dz - y^2\, dz \wedge dz$.

Chapter 12
An Example: Electromagnetism

Electromagnetism is probably the first truly elegant and exciting application of differential forms in physics. Electromagnetism deals with both electrical fields and magnetic fields and Maxwell's equations are the four equations that describe how these fields act and how they are related to each other. Maxwell's complicated equations are rendered stunningly simple and beautiful when written in differential forms notation instead of the usual vector calculus notation.

We will begin the Electromagnetism chapter by first introducing the basic electromagnetic concepts and Maxwell's four equations in vector calculus notation. These four equations are generally known as

1. Gauss's law for electric fields,
2. Gauss's law for magnetic fields,
3. Faraday's law,
4. Ampère-Maxwell law (sometimes simply Ampère's law).

Since we do not assume you have already seen electromagnetism before we take our time. Section one introduces the first two of Maxwell's four equations and section two introduces the next two equations. After this introduction, in section three we will discuss Minkowski space, which is the four-dimensional space-time manifold of special relativity. The Minkowski metric, which gives the inner product on Minkowski space, is necessary for the Hodge star operator in the context of Minkowski space. We have already discussed both the inner product and the Hodge star operator in different contexts so other than the fact that we will be in four-dimensional Minkowski space there is fundamentally nothing new here. After that we derive the differential forms formulation of electromagnetism in section four.

This is certainly not meant to be an exhaustive introduction to electromagnetism. In fact, it will be a rather brief look at electromagnetism covering little more than the basics. It is meant to give you some idea of the power and utility of differential forms from the perspective of physics. Finally, please note, that in a typical mathematician's fashion we will not spend any time discussing units, which are, understandably, of vital importance to physicists and engineers. Hopefully you will not bare us too much ill-will for this omission.

12.1 Gauss's Laws for Electric and Magnetic Fields

Electric charge is an intrinsic conserved physical characteristic or property of subatomic particles. By conserved we are saying that this property remains fixed and does not change over time. On a very deep and fundamental level it may be impossible to truly understand what physical properties such as electric charge are, but on a more mundane operational and experiential level we can see, experience, test, categorize, measure, and write equations that describe these physical properties and how they interact with each other. After all, we all know an electrical shock when we get one.

There are two different kinds of electric fields. The first kind is the electrostatic field that is produced by an electric charge and the second kind of electric field is an induced electric field that is created or produced by a changing magnetic field. We will discuss electrostatic fields first.

There are two kinds of electrical charges, positive and negative, which are of course simply two completely arbitrary designations. Subatomic particles have a negative electric charge, a positive electric charge, or no electric charge. Electrons have a negative electric charge, protons have a positive electric charge, and neutrons are natural, meaning they have no electrical charge. When we discuss the charge of an object larger than a subatomic particle we are implying that there is a

J. P. Fortney, *A Visual Introduction to Differential Forms and Calculus on Manifolds*,
https://doi.org/10.1007/978-3-319-96992-3_12

net imbalance between the number of electrons and protons in the object. A negative charge means there are more negatively charged electrons than positively charged protons, and a positive charge means there are more positively charged protons than negatively charged electrons.

Particles that carry electrical charge interact with each other due to their charges and we can write equations that describe how they interact with each other. They can also interact with each other due to other reasons as well, such as gravitation or other forces, but we will not concern ourselves with these other interactions here. We will only concern ourselves with interactions that are due to electric charge. If two particles have the same electric charge, either positive or negative, they are repulsed by each other. If two particles have oppositive electric charges then they are attracted to each other.

Electrostatic fields are induced by, or created by, electric charges. We will denote electrical field by **E**. Right now we will assume space is equivalent to $\mathbb{R}^3$, though of course this is just a local approximation to general relativity's four-dimensional space-time manifold. In essence, electrical fields are basically just a vector field on space, that is

$$\mathbf{E}_p = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}_p$$

at each point $p \in \mathbb{R}^3$. Of course, we can't actually see this electric field, so how do we know it is there? Imagine you had a tiny positive "test particle." If you were to place this test particle at point p and then let go of it, the speed and direction that this tiny imaginary test particle moves when you let go of it determines the vector $\mathbf{E}_p$. Now imagine doing that at every point of space. At each point the movement of this tiny imaginary test particle gives the vector $\mathbf{E}_p$. Consider Fig. 12.1. If we were to place a tiny imaginary test particle somewhere close to a positive charge and let go, this test particle would move away from the positive charge. If we let go of the test particle very close to the positive charge it would move away very quickly. If we let go of the test particle further away from the positive charge it would move away more slowly. Similarly, if we were to place the test particle some distance away from the negative charge and let go it would move toward the negative charge slowly and if we were to put it very close to the negative charge and let go it would move toward the negative charge quickly.

Suppose we had a positive charge and a negative charge separated by a small distance as in Fig. 12.2. Wherever we released the test particle it would feel both a repulsion from the positive charge and an attraction to the negative charge. How strong the repulsion and attraction are relative to each other would depend on how far the test particle was from each charge. For example, a test particle released at points (a) or (c) would feel some attraction to the negative charge and some repulsion from the positive charge. A test particle released at (b) feels much more attraction to the negative charge than repulsion from the positive charge. And a test particle released at (d) feels much more repulsion from the positive charge than attraction to the negative charge.

Using the electric field we can draw electric field lines. The electric field lines are actually the integral curves of the electric field. Integral curves are the paths we get when we integrate vector fields, see for example Fig. 2.19 where the integral curves of a smooth vector field are shown. In fact, finding the integral curves of vector fields is one approach to integration on manifolds which we have not yet discussed. We will discuss it in a little more depth in Appendix B. At each point the electric field vector is given by the tangent vector to the electric field line. If a tiny positive test particle were released at a point it would travel along the electric field line with a velocity at each point given by the electric field vector at that point. The

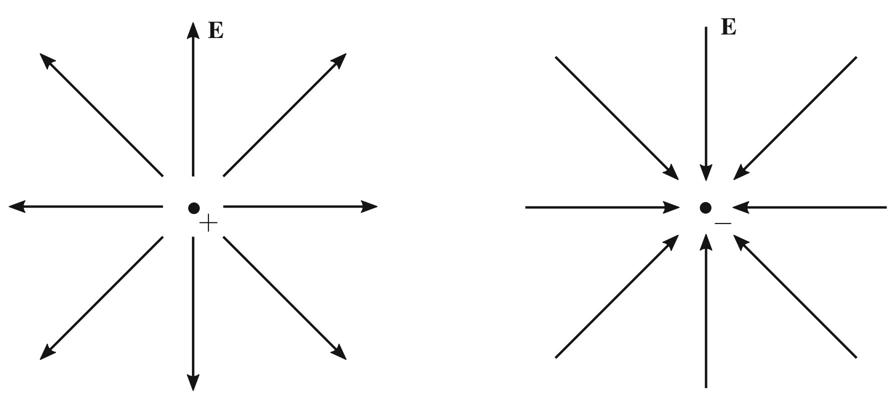


Fig. 12.1 A positively charged particle (left) and a negatively charged particle (right). Electric field lines emanating from the particles have been shown. Electric field lines are shown pointing away from positive charges and toward negative charges

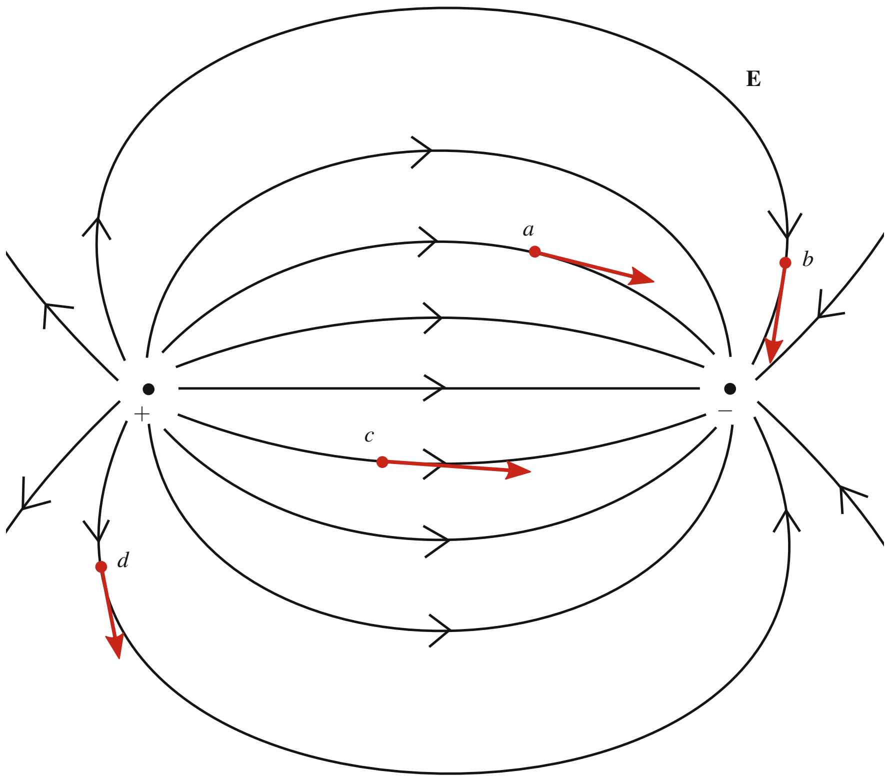


Fig. 12.2 A positively charged particle and a negatively charged particle that are close to each other. We can see some of the electric field lines that leave the positive charge go to the negative charge

length of the vector at a point tells how strong the field is at that point. Pictorially we try to show how strong the field is in a region of space by how close together we draw the electric field lines. So, in Fig. 12.2 we can see that the closer we are to the charges the denser the field lines are and so the stronger the field is. The further away we are from the charges the further apart the field lines are and the weaker the field is. Now we are ready to state **Gauss's law for electric fields**. This law concerns electrostatic fields.

An electric field is produced by an electric charge. The flux of this electric field through a closed surface is proportional to the amount of electric charge inside the closed surface.

This law can be expressed in vector calculus notation in two different ways, the integral version and the differential version. As the name implies the integral version involves an integral and the differential version involves a derivative. These two-forms are equivalent. First we give the integral form of Gauss's law for electric fields:

$$\int_S \mathbf{E} \cdot \hat{n}\, dS = \frac{q_{\text{enc.}}}{\varepsilon_0},$$

where

- $\mathbf{E}$ is the electrostatic field produced by a charge,
- S is the closed surface,
- $\hat{n}$ is the unit normal to surface S,
- dS is the area element of surface S,
- $q_{\text{enc.}}$ is the charge enclosed in surface S, and
- ε_0 is a physical constant of proportionality called the permittivity of free space.

We should recognize that $\int_S \mathbf{E} \cdot \hat{n} dS$ as the flux of the field $\mathbf{E}$ through the surface S. The flux of $\mathbf{E}$ through the closed surface S which is generated by $q_{\text{enc.}}$ is proportional to $q_{\text{enc.}}$ with a constant of proportionality given by $\frac{1}{\varepsilon_0}$. We also need to

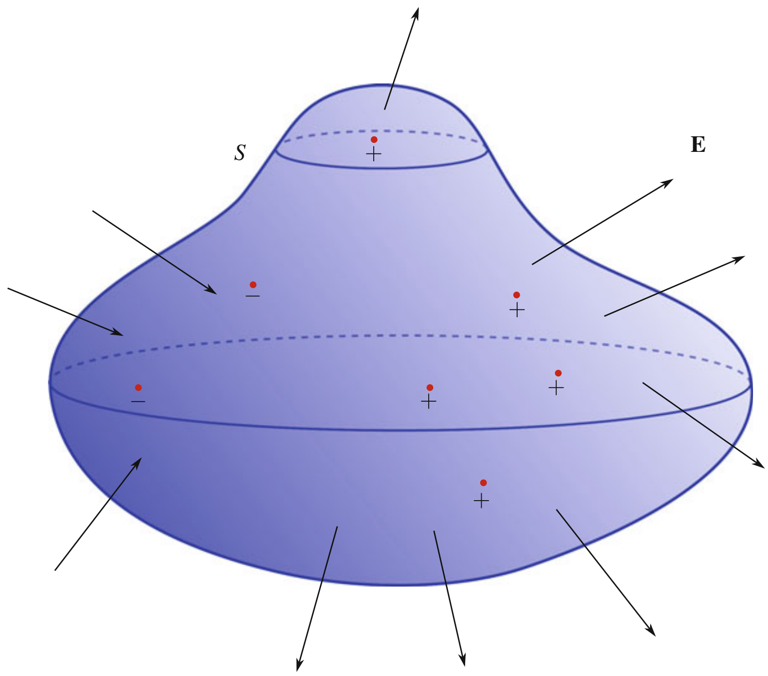


Fig. 12.3 A closed surface S enclosing some positive and negative charges. The net charge inside S is given by $q_{\text{enc.}}$. A few electric field lines penetrating S are drawn. Since there are more positive charges than negative charges there are more field lines going out of S than going into S

recognize that

$$q_{\text{enc.}} = \int_V \rho \, dV$$

where

- ρ is the charge density, a function on $\mathbb{R}^3$ that describes the amount of charge at each point.

We show what is going on in Fig. 12.3. A closed surface S encloses several point charges, some positive and some negative. There are more positive charges than negative enclosed and so when we integrate the electric vector field $\mathbf{E}$ over S we have a positive flux out of the surface.

We can move from the integral form of Gauss's law for electric fields to the differential form using the vector calculus divergence theorem,

$$\int_{S=\partial V} \mathbf{E} \cdot \hat{n} \, dS = \int_V \nabla \cdot \mathbf{E} \, dV.$$

Note that we are using the vector calculus notation of $\nabla \cdot \mathbf{E}$ for div $\mathbf{E}$. Putting this all together we get

$$\begin{aligned}
\int_V \nabla \cdot \mathbf{E} \, dV &= \int_S \mathbf{E} \cdot \hat{n} \, dS && \text{divergence theorem} \\
&= \frac{q_{\text{enc.}}}{\varepsilon_0} && \text{Gauss's law} \\
&= \frac{1}{\varepsilon_0} \int_V \rho \, dV && \text{definition of charge density} \\
&= \int_V \frac{\rho}{\varepsilon_0} \, dV
\end{aligned}$$

Equating the integrands of the first and the last terms we have the differential form of Gauss's law for electric fields,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}.$$

Question 12.1 Writing the electric field as $\mathbf{E} = E_1\hat{i} + E_2\hat{j} + E_3\hat{k}$ write Gauss's law for electric fields in terms of E_1, E_2, E_3. Do this twice, once for the integral form and once for the differential form of Gauss's law for electric fields.

Like electric charge, magnetism is an intrinsic property of subatomic particles, though in a number of ways it is a bit more complicated than electric charge and electric fields. It may be reasonable to expect that just as electric fields are produced by positive and negative electric charges, magnetic fields are produced by some sort of "magnetic charge." Unfortunately, this does not seem to be the case. Here we will consider the magnetic field produced by a spinning electron. The fields produced by spinning protons and neutrons are about a thousand times smaller than those produced by electrons and so can often be neglected.

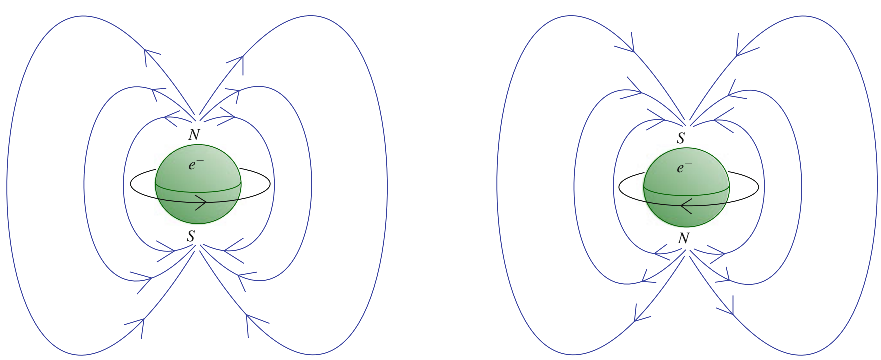
Just as with electric fields $\mathbf{E}$, magnetic fields, denoted by $\mathbf{B}$, exist at each point p in space $\mathbb{R}^3$,

$$\mathbf{B}_p = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}_p .$$

The magnetic field can be used to give us magnetic field lines, which are just the integral curves of the magnetic field $\mathbf{B}$. It is these magnetic field lines, or integral curves of $\mathbf{B}$, and not $\mathbf{B}$ itself, that are drawn in Figs. 12.4, 12.5, and 12.6. Like the electric fields, the only way to know that the magnetic field is there is by observing how it interacts with tiny test particles. Like before we consider a tiny positively charged particle. However, figuring out the magnetic field vector $\mathbf{B}_p$ at a point p is somewhat more complicated than the case of finding $\mathbf{E}_p$. We won't go into the details here but it involves watching how a moving positive test particle acts.

Two spinning electrons are shown in Fig. 12.4. The magnetic field lines generated by the electron depend on what direction the electron is spinning. Of course, our picture of an electron as a tiny ball that actually spins is in itself a cartoon that helps us both visualize what an electron is like and think about how the physical properties behave. The actual quantum mechanical nature of subatomic particles like electrons is a lot more complicated than this picture would have you believe. But for our purposes this picture is sufficient.

Larger objects that produce magnetic fields are often simply called permanent magnets or simply magnets. Often when one thinks of magnets one imagines bar magnets or horseshoe shaped magnets as in Fig. 12.5, or refrigerator magnets as shown in Fig. 12.6. The materials that make up these magnets are magnetized and thus create their own persistent magnetic fields. This happens because the chemical make-up and crystalline microstructure of the material cause the magnetic fields of the subatomic particles to line up thereby producing a stronger magnetic field.



Here an electron "spins" in one direction resulting in magnetic field lines.

Here an electrons "spins" in the other direction again resulting in magnetic field lines.

Fig. 12.4 A cartoon of an electron as a little ball that can spin. The direction of the magnetic field lines depends on the spin of the electron, which we often visualize as a spinning ball. But keep in mind, the real quantum mechanical nature of electrons is very much more complicated than this picture

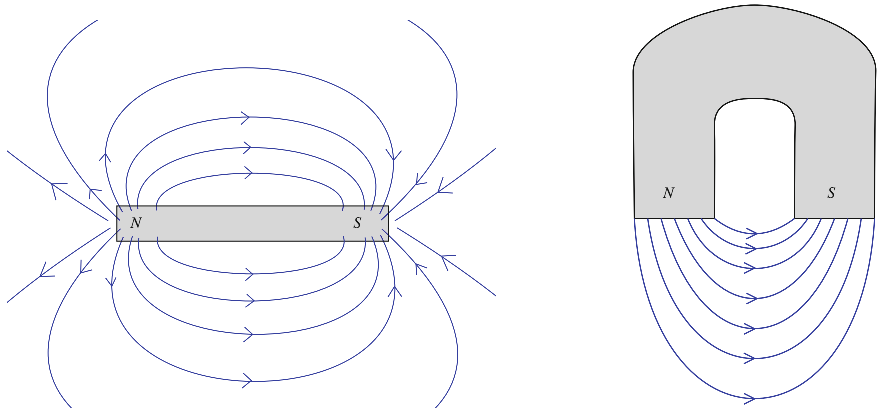


Fig. 12.5 A bar magnet (left) and a horseshoe magnet (right) with north and south poles depicted. The magnetic flux lines are shown in blue

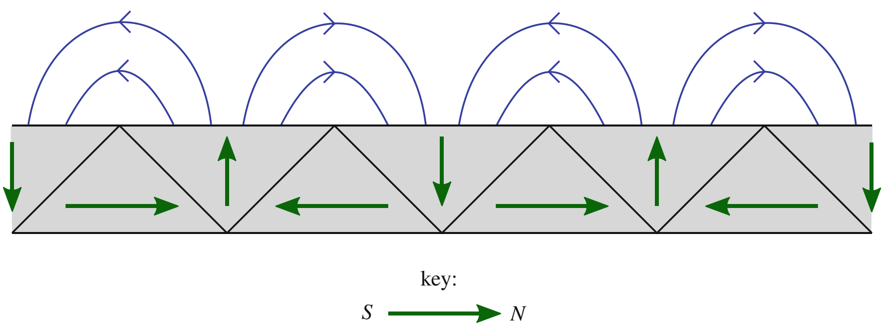


Fig. 12.6 A cross section of a refrigerator magnet. Notice the different domains where the magnetic north/south directions change. The green arrow points from the south pole to the north pole. The magnetic field lines are shown in blue. Due to the nature of the domains one side of the magnet has magnetic field lines and the other side does not

In some ways Gauss's law for magnetic fields is very similar to his law for electric fields, but its content is very different. This difference arises because it is possible to have a positive and a negative electric charge separated from each other. However, it is impossible to have a magnetic north pole separated from a magnetic south pole. That is, north poles and south poles always appear in pairs. Even a spinning electron has a north pole and a south pole, as in Fig. 12.4. No magnetic monopole, that is an isolated north pole or an isolated south pole, has ever been observed in nature or made in a laboratory. This means that the magnetic fields generated by a magnet, with both a north and a south pole, twist around back onto themselves, as we can see from Figs. 12.4, 12.5, and 12.6.

The end result of this twisting is that the total amount of magnetic field that exits a closed surface also enters that same closed surface so the net magnetic flux through the closed surface is always zero. This leads to **Gauss's law for magnetic fields**.

A magnetic field is produced by a magnet. The flux of this magnetic field through a closed surface zero.

Again, this law can all be expressed in vector calculus notation in two different ways, the integral version and the differential version. The integral version is given by

$$\int_S \mathbf{B} \cdot \hat{n} \, dS = 0,$$

where

- $\mathbf{B}$ is the magnetic field,
- S is the closed surface,

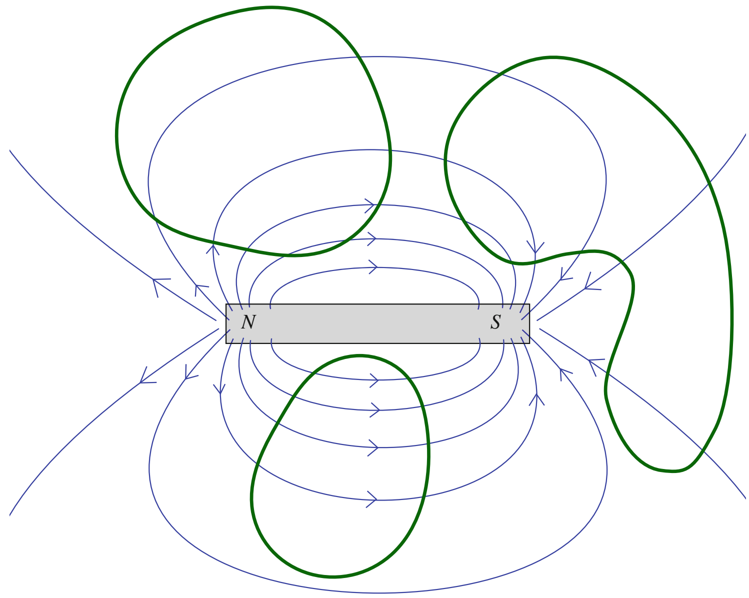


Fig. 12.7 A magnet with north and south poles is shown along with the magnetic field lines that "twist" back onto themselves. As can be seen with the green "closed surfaces," there are exactly as many magnetic field lines entering the surface as leaving the surface, and so the magnetic flux through any surface is always zero

- $\hat{n}$ is the unit normal to surface S, and
- dS is the area element of surface S.

We can see what Gauss's law for magnetic fields means by considering Fig. 12.7. The figure that is drawn is a two-dimensional cross-section of a magnet along with magnetic field lines that represent the vector field **B**. The closed blue curves represent closed surfaces. As we can see from the figure, each magnetic field line that enters a closed surface also leaves that closed surface.

Deriving the differential form of of Gauss's law for magnetic fields using the divergence theorem is then trivial

$$\begin{aligned}
\int_V \nabla \cdot \mathbf{B}\, dV &= \int_S \mathbf{B} \cdot \hat{n}\, dS && \text{divergence theorem} \\
&= 0 && \text{Gauss's law} \\
&= \int_V 0\, dV.
\end{aligned}$$

Equating the integrands of the first and last terms we have the differential form of Gauss's law for magnetic fields,

$$\nabla \cdot \mathbf{B} = 0.$$

Question 12.2 Writing the magnetic field as $\mathbf{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$ write Gauss's law for magnetic fields in terms of B_1, B_2, B_3. Do this twice, once for the integral form and once for the differential form of Gauss's law for magnetic fields.

12.2 Faraday's Law and the Ampère-Maxwell Law

Both Faraday's law and the Ampère-Maxwell law explain the relationship between electric and magnetic fields. We begin by looking at Faraday's law. The induced electric field **E** in Faraday's law is similar to the electrostatic field **E** in Gauss's law for electric fields, but it is different in its structure. Both fields act as forces to accelerate a charged particle but electrostatic fields have field lines that originate on positive charges and terminate on negative charges, while induced electric fields have field lines that loop back on themselves. The statement of **Faraday's law** is quite simple.