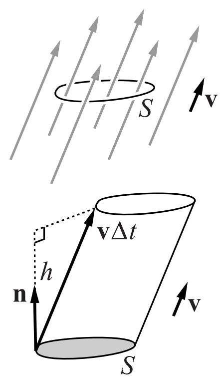
Chapter 10
Surface Integrals

Abstract We turn now to integrals over curved surfaces in space. They are analogous, in several ways, to integrals over curved paths. Both arise in scientific problems as ways to express the product of quantities that vary. The first surface integral we consider measures *flux*, the amount of fluid flowing through a surface. The integrand of a surface integral, like a path integral, can be either a scalar or a vector function: flux is the integral of a vector function, whereas area—another surface integral—is the integral of a scalar. Also, orientation matters, at least when the integrand is a vector function.

10.1 Measuring flux



Flux density
and total flux

How much fluid will pass through a plane region S in space? If fluid moves with constant velocity $\mathbf{v}$, then during a time interval Δt it will fill out an oblique cylinder with base S and generator $\mathbf{v}\Delta t$. The volume of that cylinder is the product of the area of its base with the height h perpendicular to that base. Now h equals the length of the projection of the generator on $\mathbf{n}$, the unit normal to S in the direction of flow: $h = \mathbf{v}\Delta t \cdot \mathbf{n}$. Therefore, if we denote the area of S by ΔA, then the volume of fluid is

$$\text{volume} = \mathbf{v} \cdot \mathbf{n}\, \Delta A\, \Delta t.$$

To determine the amount of fluid—that is, its mass—we just need to factor in its mass density ρ:

$$\text{mass} = \rho \mathbf{v} \cdot \mathbf{n}\, \Delta A\, \Delta t.$$

The vector quantity $\mathbb{V} = \rho \mathbf{v}$ is called the **flux density** (or *flow density*) of the fluid. Flux density is a *rate*; when ρ is measured in kilograms per cubic meter and velocity in meters per second, flux density is measured in kilograms per square meter per second. Its magnitude is the mass of fluid, in kilograms, that flows perpendicularly through a unit area in unit time. The mass of fluid that crosses the region S in unit time is called the **total flux** (or *total flow*) **through *S***; it is the product

J.J. Callahan, *Advanced Calculus: A Geometric View*, Undergraduate Texts in Mathematics,
DOI 10.1007/978-1-4419-7332-0_10, © Springer Science+Business Media, LLC 2010

$$\text{total flux} = \mathbb{V} \cdot \mathbf{n}\,\Delta A \ \text{ kilograms per second.}$$

In general, we allow flux density to vary continuously from point to point, but require it to be constant in time at any given point: $\mathbb{V} = \mathbb{V}(x,y,z)$. Physically, $\mathbb{V}$ is called a *steady flow*; mathematically, it is a continuous *vector field* on (a portion of) $\mathbb{R}^3$. We usually call $\mathbb{V}$ a **flow field**.

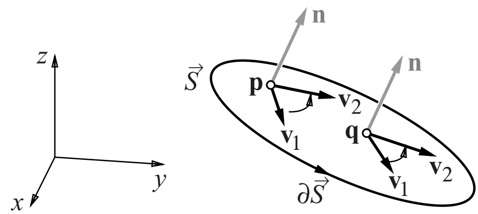
From which side does the fluid cross S?

Our expression for total flux does not yet tell us from which side the fluid crosses S. However, if we fix one of the two unit normals $\mathbf{n}$ in advance—that is, before we consider any given fluid flow—

then total flux $\mathbb{V} \cdot \mathbf{n}\,\Delta A$ becomes a signed quantity whose value is negative precisely when the fluid crosses S in the direction opposite $\mathbf{n}$.

Normals and orientation

Assigning a unit normal to a plane region S in space is equivalent to orienting it. To make the connection, we must first explain what it means to orient S in space. Essentially, it is the same as orienting it in the plane (p. 353): assign to each point $\mathbf{p}$ of S an ordered pair $\{\mathbf{v}_1(\mathbf{p}), \mathbf{v}_2(\mathbf{p})\}$ of linearly independent vectors that vary continuously with $\mathbf{p}$. The vectors are now in $\mathbb{R}^3$, of course, but we constrain them to be tangent to S at the point $\mathbf{p}$. Following earlier practice, we let $\vec{S}$ denote S with an orientation.



Orientation determines the normal

Next, we must make the connection between orienting S and choosing a unit normal for it. Suppose the ordered pair $\{\mathbf{v}_1(\mathbf{p}), \mathbf{v}_2(\mathbf{p})\}$ orients $\vec{S}$ at $\mathbf{p}$. Then, as in the figure above, we choose

$$\mathbf{n}(\mathbf{p}) = \frac{\mathbf{v}_1(\mathbf{p}) \times \mathbf{v}_2(\mathbf{p})}{\|\mathbf{v}_1(\mathbf{p}) \times \mathbf{v}_2(\mathbf{p})\|}$$

to be the unit normal to $\vec{S}$ at $\mathbf{p}$. On any pathwise-connected component of $\vec{S}$, both $\mathbf{n}(\mathbf{p})$ and the orientation of $\vec{S}$ are constant (Theorem 9.12, p. 354).

If we think of orientation as defining a "sense of rotation" on $\vec{S}$ (cf. p. 353), then, from the side of $\vec{S}$ on which $\mathbf{n}$ lies, that rotation is counterclockwise. This assumes that the coordinate frame in $\mathbb{R}^3$ is right-handed, for then the sense of rotation in the (x,y)-plane, as viewed from the positive z-axis, is counterclockwise.

The normal determines the orientation

It is equally straightforward to connect the choice of a unit normal to the choice of an orientation. Once the unit normal $\mathbf{n}$ for S is given, choose any two linearly independent vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ that are perpendicular to $\mathbf{n}$ and such that

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{n}\} \text{ or, equivalently } \{\mathbf{n}, \mathbf{v}_1, \mathbf{v}_2\},$$

is a positively oriented triple of vectors in $\mathbb{R}^3$. Then $\mathbf{v}_1$ and $\mathbf{v}_2$ are everywhere tangent to S, so we can orient $\vec{S}$ by assigning $\mathbf{v}_i(\mathbf{p}) = \mathbf{v}_i$, $i = 1, 2$, at every point $\mathbf{p}$ in $\vec{S}$ where $\mathbf{n}$ is the orienting normal.

Induced orientation on the boundary

The figure above also indicates that the orientation of $\vec{S}$ induces an orientation of $\partial\vec{S}$, just as in $\mathbb{R}^2$. When we view $\vec{S}$ from the side toward which the orienting normal $\mathbf{n}$ points, then $\vec{S}$ lies on the left as $\partial\vec{S}$ is traversed in the positive direction.

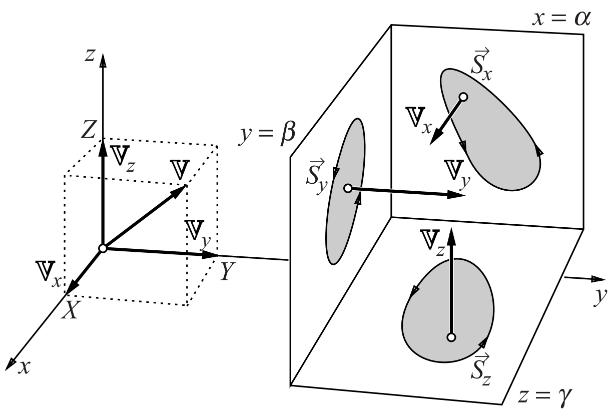
Definition 10.1 *Let a fluid have constant flux density $\mathbb{V}$, and let $\vec{S}$ be a plane region in space that has finite area ΔA and orientation given by the unit normal* $\mathbf{n}$. *The* ***total flux of the fluid through $\vec{\mathbf{S}}$*** *in unit time is*

$$\Phi = \mathbb{V} \cdot \mathbf{n}\,\Delta A.$$

Regions parallel to the coordinate planes

This formula has important special cases. Let $\mathbb{V} = (X, Y, Z)$, and suppose $\vec{S} = \vec{S}_x$ lies in the plane $x = \alpha$, has area $\Delta A = \Delta A_x$, and is oriented by the positive x-axis: $\mathbf{n} = (1, 0, 0)$. Then total flux through $\vec{S}_x$ is

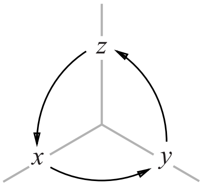
$$\Phi = \Phi_x = (X, Y, Z) \cdot (1, 0, 0)\,\Delta A_x = X\,\Delta A_x.$$



Here total flux depends only on the component X of the flow field $\mathbb{V}$ that is perpendicular to the plane $x = \alpha$; the other two components, Y and Z, in directions parallel to that plane, have no effect. In other words, $\mathbb{V}$ and $\mathbb{V}_x = (X, 0, 0)$ have the same total flux through $\vec{S}_x$. For a region $\vec{S}_y$ in the plane $y = \beta$ or $\vec{S}_z$ in $z = \gamma$, we find, respectively,

$$\Phi_y = Y\,\Delta A_y, \quad \Phi_z = Z\,\Delta A_z.$$

The figure above also shows how a region parallel to each coordinate plane is oriented when the remaining positive axis is used as the defining normal (and the three axes together have their usual right-hand orientation):



Plane	Normal	Order of Axes
(y,z)	x-axis	$y \to z$
(x,z)	y-axis	$z \to x$
(y,z)	z-axis	$y \to z$

In two cases, the plane is oriented by its axes in alphabetical order, but in the third, by the opposite order. As the figure in the margin shows, the correct order is the cyclic one $\cdots \to x \to y \to z \to x \to \cdots$ that the three coordinate axes have when they are viewed from the positive orthant (i.e., from the region where $x > 0$, $y > 0$, and $z > 0$).

Total flux through a parallelogram

If $\vec{S}$ is the oriented parallelogram $\mathbf{p} \wedge \mathbf{q}$, then its orienting unit normal is

$$\mathbf{n} = \frac{\mathbf{p} \times \mathbf{q}}{\|\mathbf{p} \times \mathbf{q}\|}$$

(if $\mathbf{p} \times \mathbf{q} \neq \mathbf{0}$) and its area is $\Delta A = \|\mathbf{p} \times \mathbf{q}\|$. Therefore, $\mathbf{n}\,\Delta A = \mathbf{p} \times \mathbf{q}$, and total flux through $\mathbf{p} \wedge \mathbf{q}$ takes the simple form

$$\Phi = \mathbb{V} \cdot \mathbf{p} \times \mathbf{q} = \mathbf{p} \times \mathbf{q} \cdot \mathbb{V},$$

the scalar triple product of $\mathbf{p}$, $\mathbf{q}$, and $\mathbb{V}$ (cf. p. 43). To compute $\mathbf{n}$, we need $\mathbf{p} \times \mathbf{q} \neq \mathbf{0}$; however, if $\mathbf{p} \times \mathbf{q} = \mathbf{0}$, then $\Delta A = 0$ and $\Phi = 0$, so the formula $\Phi = \mathbb{V} \cdot \mathbf{p} \times \mathbf{q}$ is still valid. If

$$\mathbb{V} = (X, Y, Z), \quad \mathbf{p} = (p_1, p_2, p_3), \quad \mathbf{q} = (q_1, q_2, q_3),$$

then (e.g., from the proof of Theorem 2.11, p. 43), we have

$$\mathbf{p} \times \mathbf{q} = \left(\begin{vmatrix} p_2 & p_3 \\ q_2 & q_3 \end{vmatrix}, \begin{vmatrix} p_3 & p_1 \\ q_3 & q_1 \end{vmatrix}, \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} \right),$$

$$\Phi = X \begin{vmatrix} p_2 & p_3 \\ q_2 & q_3 \end{vmatrix} + Y \begin{vmatrix} p_3 & p_1 \\ q_3 & q_1 \end{vmatrix} + Z \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix}.$$

Components of total flux

Suppose we project $\vec{S}$ onto each of the coordinate planes $x = 0$, $y = 0$, and $z = 0$; the images are parallelograms $\vec{S}_x$, $\vec{S}_y$, and $\vec{S}_z$, respectively, whose areas are the 2×2 determinants that appear as the components of the vector $\mathbf{p} \times \mathbf{q}$ (see p. 44):

$$\Delta A_x = \begin{vmatrix} p_2 & p_3 \\ q_2 & q_3 \end{vmatrix}, \quad \Delta A_y = \begin{vmatrix} p_3 & p_1 \\ q_3 & q_1 \end{vmatrix}, \quad \Delta A_z = \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix}.$$

Each of these is the *signed* area of an oriented parallelogram whose orientation is determined by the coordinate plane in which it lies. From the discussion above, we know that the value of the total flux through each of $\vec{S}_x$, $\vec{S}_y$, and $\vec{S}_z$ can be written as

$$\Phi_x = X\,\Delta A_x, \quad \Phi_y = Y\,\Delta A_y, \quad \Phi_z = Z\,\Delta A_z.$$

These are the **components of total flux through $\vec{S}$**: $\Phi = \Phi_x + \Phi_y + \Phi_z$.

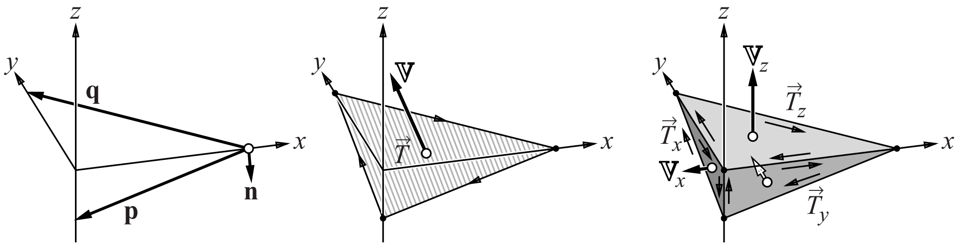
Example: flow through a tetrahedron

An example helps clarify these ideas. To simplify the picture as much as possible, we work with triangles instead of parallelograms. Let $\vec{T}$ be the triangle spanned by a pair of vectors $\mathbf{p}$ and $\mathbf{q}$ and oriented by $\mathbf{p} \times \mathbf{q}$. In the figure below,

$$\mathbf{p} = (-6,0,-2), \quad \mathbf{q} = (-6,4,0), \text{ and } \mathbb{V} = (-1,1,3);$$

$\mathbf{p}$ and $\mathbf{q}$ are placed so that each edge of $\vec{T}$ lies in one of the coordinate planes. Consequently, $\vec{T}$ and its projections $\vec{T}_x$, $\vec{T}_y$, and $\vec{T}_z$ form a tetrahedron. We have

$$\mathbf{p} \times \mathbf{q} = \left(\begin{vmatrix} 0 & -2 \\ 4 & 0 \end{vmatrix}, \begin{vmatrix} -2 & -6 \\ 0 & -6 \end{vmatrix}, \begin{vmatrix} -6 & 0 \\ -6 & 4 \end{vmatrix} \right) = (8,12,-24) = 4(2,3,-6),$$



The triangle $\vec{T}$ has half the area of the parallelogram $\mathbf{p} \wedge \mathbf{q}$; therefore total flux through $\vec{T}$ is

$$\Phi = \tfrac{1}{2}\, \mathbb{V} \cdot \mathbf{p} \times \mathbf{q} = \tfrac{1}{2}(-8 + 12 - 72) = -34.$$

Notice that, in the figure, the boundary of $\vec{T}$ has clockwise, or negative, orientation as we view it from the side on which $\mathbb{V}$ lies. This confirms that Φ must be negative. Furthermore, $\|\mathbf{p} \times \mathbf{q}\| = 28$, so

$$\mathbf{n} = \tfrac{1}{7}(2,3,-6) \text{ and } \Delta A = \text{area}\, T = \|\mathbf{p} \times \mathbf{q}\|/2 = 14.$$

We can read off the signed areas of $\vec{T}_x$, $\vec{T}_y$, and $\vec{T}_z$ as one-half of the corresponding component of $\mathbf{p} \times \mathbf{q}$:

$$\Delta A_x = 4, \quad \Delta A_y = 6, \quad \Delta A_z = -12.$$

The signs here confirm our direct observations: the boundaries of $\vec{T}_x$ and $\vec{T}_y$ have positive (counterclockwise) orientation with respect to the positive x- and y-axes, but $\vec{T}_z$ has negative (clockwise) orientation with respect to the positive z-axis. The total flux through each of these faces is

$$\Phi_x = -1 \times \Delta A_x = -4, \quad \Phi_y = +1 \times \Delta A_y = +6, \quad \Phi_z = +3 \times \Delta A_z = -36.$$

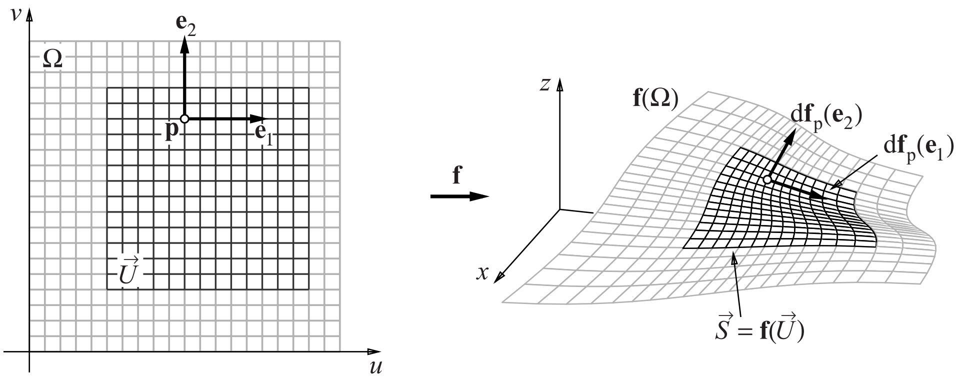
Of the three faces, total flux is positive only through $\vec{T}_y$, because only on that face does the component of $\mathbb{V}$ (shown in outline in the figure, lying inside the tetrahedron) point in the same direction as the orienting normal. Finally, because flux density $\mathbb{V}$ is constant, no fluid accumulates in the tetrahedron: the fluid that flows through the one face $\vec{T}$ must equal the total that flows through the other three that have the same boundary as $\vec{T}$:

$$\Phi = \Phi_x + \Phi_y + \Phi_z.$$

Flux through a *curved* surface

Now suppose that the oriented surface $\vec{S}$ is curved rather than flat. To be definite, let $\vec{S}$ be a parametrized surface patch. Thus we begin with a continuously differentiable 1–1 immersion $\mathbf{f} : \Omega \to \mathbb{R}^3$, where Ω is an open set in $\mathbb{R}^2$. The condition that $\mathbf{f}$ be an immersion means (Definition 6.8, p. 212) that the derivative $d\mathbf{f}_{(a,b)}$ is itself 1–1 (or, in this case, has maximal rank 2) for every (a,b) in Ω. This guarantees that the image of $\mathbf{f}$ is fully 2-dimensional everywhere; see the discussion on page 128.

Let $\vec{U}$ be a closed, bounded, and positively oriented subset of Ω that has area. Because $\mathbf{f}$ is a 1–1 immersion, the orientation on $\vec{U}$ will induce an orientation on its image $\mathbf{f}(\vec{U})$, exactly as on page 355.



Induced orientation

Theorem 10.1. *If the vectors* $\{\mathbf{v}_1(\mathbf{p}), \mathbf{v}_2(\mathbf{p})\}$ *determine the orientation of* $\vec{U}$*, then their images*

$$\{d\mathbf{f}_{\mathbf{p}}(\mathbf{v}_1(\mathbf{p})), d\mathbf{f}_{\mathbf{p}}(\mathbf{v}_2(\mathbf{p}))\}$$

determine an orientation of $\mathbf{f}(\vec{U})$ *that is called the* ***induced orientation****.*

Proof. First of all, the image vectors are tangent to $\mathbf{f}(\vec{U})$ at $\mathbf{f}(\mathbf{p})$. Second, they vary continuously with the point $\mathbf{f}(\mathbf{p})$ because $\mathbf{v}_i(\mathbf{p})$ vary continuously with $\mathbf{p}$, and $\mathbf{f}$ is continuously differentiable. Third, they are linearly independent because $\mathbf{f}$ is an immersion at $\mathbf{p}$. $\square$

Oriented surface patch

Definition 10.2 *We say* $\vec{S}$ ***is an oriented surface patch*** *if* $\vec{S} = \mathbf{f}(\vec{U})$*, where the map* $\mathbf{f} : \Omega \to \mathbb{R}^3$ *is a continuously differentiable* 1–1 *immersion on an open set* Ω *in* $\mathbb{R}^2$*,* $\vec{U} \subset \Omega$ *is a closed, bounded, positively oriented set with area, and* $\vec{S}$ *has the induced orientation.*

By the discussion on pages 388–389, we can always replace an ordered pair $\{d\mathbf{f}_{(a,b)}(\mathbf{v}_1), d\mathbf{f}_{(a,b)}(\mathbf{v}_2)\}$ of tangent vectors by the normal

$$d\mathbf{f}_{(a,b)}(\mathbf{v}_1) \times d\mathbf{f}_{(a,b)}(\mathbf{v}_2),$$

to orient $\vec{S}$ at the point (a,b). Let us orient $\vec{U}$ in $\mathbb{R}^2$ using just the standard basis vectors, by assigning the pair $\{\mathbf{e}_1, \mathbf{e}_2\}$ to every point $\mathbf{p}$ (as in the figure above). Then the vectors $d\mathbf{f}_{(a,b)}(\mathbf{e}_1)$ and $d\mathbf{f}_{(a,b)}(\mathbf{e}_2)$ that orient $\vec{S}$ are the columns of the matrix $d\mathbf{f}_{(a,b)}$, in that order. Hence, if

$$\mathbf{f}: \begin{cases} x = x(u,v), \\ y = y(u,v), \\ z = z(u,v), \end{cases} \qquad d\mathbf{f}_{(a,b)} = \begin{pmatrix} x_u(a,b) & x_v(a,b) \\ y_u(a,b) & y_v(a,b) \\ z_u(a,b) & z_v(a,b) \end{pmatrix},$$

parametrizes the oriented surface patch $\vec{S}$, then the cross-product of the column vectors of $d\mathbf{f}_{(a,b)}$ determines the orienting normal for $\vec{S}$ at $\mathbf{f}(a,b)$:

Orienting normal of the parametrization

$$N_{\mathbf{f}}(a,b) = \left(\begin{vmatrix} y_u(a,b) & y_v(a,b) \\ z_u(a,b) & z_v(a,b) \end{vmatrix}, \begin{vmatrix} z_u(a,b) & z_v(a,b) \\ x_u(a,b) & x_v(a,b) \end{vmatrix}, \begin{vmatrix} x_u(a,b) & x_v(a,b) \\ y_u(a,b) & y_v(a,b) \end{vmatrix} \right)$$

$$= \left(\frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)} \right)_{(a,b)}.$$

Because $\mathbf{f}$ is an immersion everywhere on $\boldsymbol{\Omega}$, the columns of $d\mathbf{f}_{(a,b)}$ are linearly independent and, therefore, $N_{\mathbf{f}}(a,b) \neq \mathbf{0}$.

Parametrizing $-\vec{S}$

From a parametrization $\mathbf{f}$ of $\vec{S}$ we can always construct a parametrization $\mathbf{f}^*$ of the oppositely oriented patch $-\vec{S}$ by reversing the order of the parameters. Specifically, let $L : \mathbb{R}^2 \to \mathbb{R}^2 : (s,t) \to (u,v)$ be the reflection

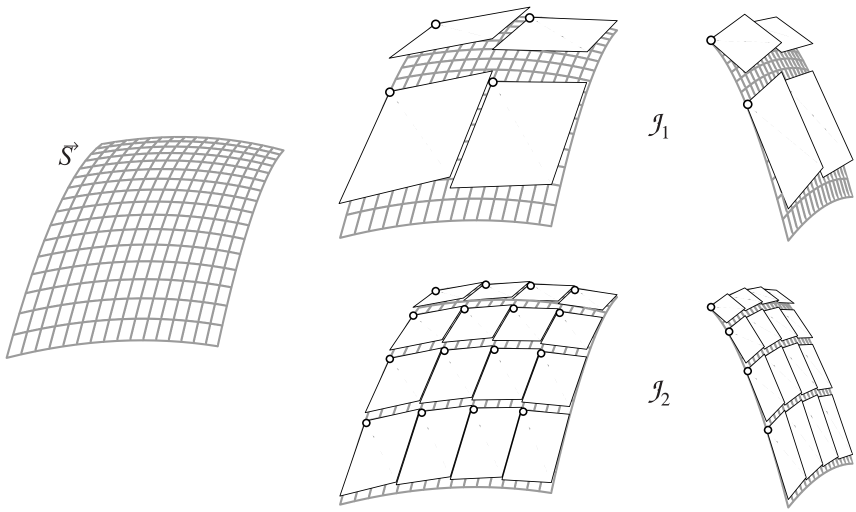
$$L: \begin{cases} u = t, \\ v = s, \end{cases} \qquad L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and let $\boldsymbol{\Omega}^* = L^{-1}(\boldsymbol{\Omega})$ and $U^* = L^{-1}(U)$ as sets. Let $\vec{U}$ and $\vec{U}^*$ both be positively oriented; then, because L reverses orientation, $L(\vec{U}^*) = -\vec{U}$. The final step is to let $\mathbf{f}^* = \mathbf{f} \circ L$; then $\mathbf{f}^*$ is defined on $\boldsymbol{\Omega}^*$ and

$$\mathbf{f}^*(\vec{U}^*) = \mathbf{f} \circ L(\vec{U}^*) = \mathbf{f}(-\vec{U}) = -\vec{S}.$$

Estimating total flux

Now let $\vec{S}$ be an oriented surface patch in (x,y,z)-space, and suppose a fluid with continuously varying flux density $\mathbb{V}(x,y,z)$ flows through $\vec{S}$. Our goal is to determine the total flux of $\mathbb{V}$ through $\vec{S}$. If we first approximate $\vec{S}$ by a collection of oriented parallelograms, then total flux through those parallelograms gives us an estimate of the total flux through $\vec{S}$. To get the parallelograms, partition the parameter domain $\vec{U}$ with one of the grids $\mathcal{J}_k$ that are used to define Jordan content in the plane (cf. p. 281), and let $\vec{Q}$ be the square cell of $\mathcal{J}_k$ whose lower-left corner is at the point (a,b), positively oriented as a part of $\vec{U}$. The image of $\vec{Q}$ under the linear map $d\mathbf{f}_{(a,b)}$ is an oriented parallelogram $\vec{P}$ in $\mathbb{R}^3$ that is tangent to $\vec{S}$ at $\mathbf{f}(a,b)$ and has one corner there. (If k is large enough, every $\vec{Q}$ that meets $\vec{U}$ will lie entirely within $\boldsymbol{\Omega}$, so $d\mathbf{f}_{(a,b)}$ will be defined.) See the next figure.



Approximating $\vec{S}$ by parallelograms

As $\vec{Q}$ ranges over all the cells of $\mathcal{J}_k$ that meet $\vec{U}$, the image parallelograms make up a collection of plates attached to $\vec{S}$ at their points of tangency, as in the figure above. The plates resemble the scales that cover the skin of a reptile or armadillo. The figure shows the surface patch $\vec{S}$ first by itself, then with the parallelograms from $\mathcal{J}_1$ attached, and finally with the parallelograms from $\mathcal{J}_2$ attached. We see each set of parallelograms from two different viewpoints. The figures suggest that the plates give us a rough approximation to the surface, an approximation that improves as the plates become smaller and more numerous, that is, as k increases.

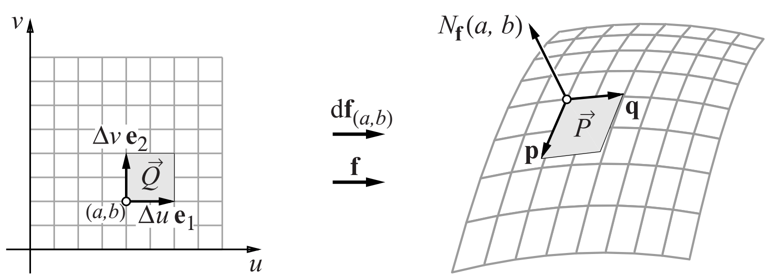
Total flux through a single parallelogram

To estimate the total flux through a single parallelogram $\vec{P} = \mathbf{df}_{(a,b)}(\vec{Q})$, note that the edges of $\vec{Q}$ are multiples of the basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ in $\mathbb{R}^2$. If we write those edges as

$$\Delta u\,\mathbf{e}_1 \quad \text{and} \quad \Delta v\,\mathbf{e}_2,$$

where $\Delta u = \Delta v = 1/2^k > 0$ when $\vec{Q}$ is a cell in $\mathcal{J}_k$, then we can then write the edges of $\vec{P}$ as

$$\mathbf{p} = \Delta u\,\mathbf{df}_{(a,b)}(\mathbf{e}_1) = \Delta u \begin{pmatrix} x_u(a,b) \\ y_u(a,b) \\ z_u(a,b) \end{pmatrix}, \quad \mathbf{q} = \Delta v\,\mathbf{df}_{(a,b)}(\mathbf{e}_2) = \Delta v \begin{pmatrix} x_v(a,b) \\ y_v(a,b) \\ z_v(a,b) \end{pmatrix}.$$



Therefore,

$$\mathbf{p} \times \mathbf{q} = N_{\mathbf{f}}(a,b)\,\Delta u\,\Delta v = \mathbf{n}\,\Delta A,$$

where

$$N_{\mathbf{f}}(a,b) = \left(\frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)}\right)_{(a,b)}$$

is the orienting normal for $\vec{S}$ at $\mathbf{f}(a,b)$ (p. 393). By the geometric definition of the cross product $\mathbf{p} \times \mathbf{q}$, $\mathbf{n}$ is the unit normal in the same direction as $N_{\mathbf{f}}(a,b)$ and ΔA is the ordinary area of the unoriented parallelogram P.

It remains for us to apply the formula $\Phi = \mathbb{V} \cdot \mathbf{p} \times \mathbf{q}$ on $\vec{P}$, but this requires the flow field $\mathbb{V}$ to be constant. We can get a constant by replacing

$$\mathbb{V}(x,y,z) = (X(x,y,z), Y(x,y,z), Z(x,y,z))$$

everywhere on $\vec{P}$ by the single value $\mathbb{V}(\mathbf{f}(a,b))$ that $\mathbb{V}$ takes on at the corner where $\vec{P}$ is attached to $\vec{S}$. By hypothesis, $\mathbb{V}(x,y,z)$ is continuous, so the error caused by this replacement can be made as small as we wish by taking $\vec{Q}$ sufficiently small. We now find

$$\Phi \approx \left(X\,\frac{\partial(y,z)}{\partial(u,v)} + Y\,\frac{\partial(z,x)}{\partial(u,v)} + Z\,\frac{\partial(x,y)}{\partial(u,v)}\right)_{(a,b)} \Delta u\,\Delta v.$$

The right-hand side is a constant determined by (a,b): the three Jacobians are evaluated at $(u,v) = (a,b)$, and X, Y, and Z are evaluated at the point $(x,y,z) = (x(a,b), y(a,b), z(a,b))$.

We can now estimate total flux through the oriented surface patch $\vec{S}$ by adding up the contributions from all the cells $\vec{Q}_1, \ldots, \vec{Q}_I$ that meet the domain $\vec{U}$. Let the lower-left corner of $\vec{Q}_i$ be at the point (u_i, v_i); then

Total flux for a given parametrization of $\vec{S}$

$$\Phi \approx \sum_{i=1}^{I} \left(X\,\frac{\partial(y,z)}{\partial(u,v)} + Y\,\frac{\partial(z,x)}{\partial(u,v)} + Z\,\frac{\partial(x,y)}{\partial(u,v)}\right)_{(u_i,v_i)} \Delta u\,\Delta v.$$

This is a Riemann sum for the oriented integral

$$\Phi_{\mathbf{f}} = \iint_{\vec{U}} \left(X\,\frac{\partial(y,z)}{\partial(u,v)} + Y\,\frac{\partial(z,x)}{\partial(u,v)} + Z\,\frac{\partial(x,y)}{\partial(u,v)}\right)_{(u,v)} du\,dv.$$

Because the integrand is continuous, the integral exists and the Riemann sums converge to it as $k \to \infty$ (Theorem 8.35, p. 305).

Definition 10.3 *Suppose $(x,y,z) = \mathbf{f}(u,v)$ parametrizes the oriented surface patch $\vec{S} = \mathbf{f}(\vec{U})$, and $\mathbb{V} = (X,Y,Z)$ is a continuous flow field defined on an open set containing $\vec{S}$, then the **total flux of $\mathbb{V}$ through $\vec{S}$ for the given parametrization** is the oriented integral*

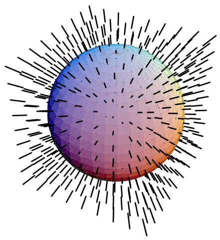
$$\Phi_{\mathbf{f}} = \iint_{\vec{U}} \left(X\,\frac{\partial(y,z)}{\partial(u,v)} + Y\,\frac{\partial(z,x)}{\partial(u,v)} + Z\,\frac{\partial(x,y)}{\partial(u,v)}\right)_{(u,v)} du\,dv.$$

The notation suggests that the value of $\Phi_{\mathbf{f}}$ depends on the parametrization $\mathbf{f}$. However, if total flux through a surface is to be physically meaningful, its value should be independent of the parametrization of that surface. Before we show that it is, we calculate Φ for two examples.

Example 1: radial flow out of a sphere

Let us determine the total flux of $\mathbb{V} = (X,Y,Z) = (Cx,Cy,Cz)$ (where C is a constant) through the unit sphere $\vec{S}$, parametrized as

$$\mathbf{f}: \begin{cases} x = \cos\theta\cos\varphi, \\ y = \sin\theta\cos\varphi, \\ z = \sin\varphi; \end{cases} \qquad \vec{U}: \begin{array}{c} -\pi \le \theta \le \pi, \\ -\pi/2 \le \varphi \le \pi/2. \end{array}$$



(Strictly, speaking, a surface patch can cover only a portion of the sphere; $\mathbf{f}$ is not 1–1. However, no essential error is introduced by using this parametrization; see pages 417–419. It is simpler to compute flux through the whole sphere.) The flow field $\mathbb{V}$ is radial; each vector points away from the origin with a magnitude proportional to its distance from the origin. Thus, although $\mathbb{V}$ varies, it is everywhere normal to the sphere and has constant magnitude $\|\mathbb{V}\| = C$ there. It follows directly—without calculating the integral—that

$$\Phi = \|\mathbb{V}\| \times \text{area}\, S = 4\pi C.$$

Let us compare this with the value provided by the integral. Because

$$\frac{\partial(y,z)}{\partial(\theta,\varphi)} = \begin{vmatrix} \cos\theta\cos\varphi & -\sin\theta\sin\varphi \\ 0 & \cos\varphi \end{vmatrix} = \cos\theta\cos^2\varphi,$$

$$\frac{\partial(z,x)}{\partial(\theta,\varphi)} = \begin{vmatrix} 0 & \cos\varphi \\ -\sin\theta\cos\varphi & -\cos\theta\sin\varphi \end{vmatrix} = \sin\theta\cos^2\varphi,$$

$$\frac{\partial(x,y)}{\partial(\theta,\varphi)} = \begin{vmatrix} -\sin\theta\cos\varphi & -\cos\theta\sin\varphi \\ \cos\theta\cos\varphi & -\sin\theta\sin\varphi \end{vmatrix} = \sin\varphi\cos\varphi,$$

the integrand is

$$\begin{aligned} &C\cos\theta\cos\varphi\cdot\cos\theta\cos^2\varphi + C\sin\theta\cos\varphi\cdot\sin\theta\cos^2\varphi + C\sin\varphi\cdot\sin\varphi\cos\varphi \\ &\quad = C(\cos^2\theta\cos^3\varphi + \sin^2\theta\cos^3\varphi + \sin^2\varphi\cos\varphi) \\ &\quad = C(\cos^2\varphi + \sin^2\varphi)\cos\varphi = C\cos\varphi, \end{aligned}$$

and therefore the integral equals

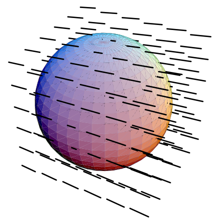
$$\iint_{\vec{U}} C\cos\varphi\,d\theta\,d\varphi = \int_{-\pi}^{\pi} C\left(\int_{-\pi/2}^{\pi/2}\cos\varphi\,d\varphi\right)d\theta = \int_{-\pi}^{\pi} 2C\,d\theta = 4\pi C.$$

Note that the orientation normal given by $\mathbf{f}$ at the point (x,y,z) is

$$N_{\mathbf{f}} = \left(\frac{\partial(y,z)}{\partial(\theta,\varphi)}, \frac{\partial(z,x)}{\partial(\theta,\varphi)}, \frac{\partial(x,y)}{\partial(\theta,\varphi)} \right) = \cos\varphi \cdot (x,y,z).$$

This is, of course, a multiple of the radius vector (x,y,z). Moreover, it is a positive multiple (at least when $-\pi/2 < \varphi < \pi/2$), so $N_{\mathbf{f}}$ is an *outward* normal on the sphere.

Example 2: constant flow through a sphere



For a second example, consider a constant flow $\mathbb{V} = (A,B,C)$ through the same sphere with the same parametrization. Because the flow is constant, all the fluid that enters on one side of the sphere exits on the other. That is, *inflow* equals *outflow*, so we expect the net flux through the whole sphere to be zero: $\Phi = 0$. The integral is

$$\int_{-\pi}^{\pi} \int_{-\pi/2}^{\pi/2} (A\cos\theta\cos^2\varphi + B\sin\theta\cos^2\varphi + C\sin\varphi\cos\varphi)\, d\varphi\, d\theta,$$

and can be dealt with one term at a time. The first is

$$A \int_{-\pi}^{\pi} \cos\theta\, d\theta \int_{-\pi/2}^{\pi/2} \cos^2\varphi\, d\varphi = A \times 0 \times \pi/2 = 0.$$

For similar reasons, the second and third terms also equal zero, so $\Phi = 0$.

Does Φ depend on the parametrization of S?

Our calculation of Φ for a given surface is tied to a parametrization of that surface. If we change the parametrization, will Φ change as well? Consider what happens when we revisit Example 1 with a different parametrization for the sphere. Let $\mathbf{g}: \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$\mathbf{g}: \begin{cases} x = \dfrac{2u}{1+u^2+v^2}, \\[2mm] y = \dfrac{2v}{1+u^2+v^2}, \\[2mm] z = \dfrac{1-u^2-v^2}{1+u^2+v^2}. \end{cases}$$

Because $\|\mathbf{g}(u,v)\|^2 = 1$ for every (u,v) in $\mathbb{R}^2$, the image of $\mathbf{g}$ is some part of the unit sphere. In fact, $\mathbf{g}(\mathbb{R}^2)$ covers the entire sphere except for the south pole $(x,y,z) = (0,0,-1)$ (see Exercise 10.6). After some calculations (and setting $D = 1+u^2+v^2$ to simplify the expressions), we find

$$\frac{\partial(y,z)}{\partial(u,v)} = \frac{8u}{D^3}, \quad \frac{\partial(z,x)}{\partial(u,v)} = \frac{8v}{D^3}, \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{4(1-u^2-v^2)}{D^3}.$$

This means that the orienting normal for $\mathbf{g}$ at the point (x,y,z) is

$$N_{\mathbf{g}} = \left(\frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)} \right) = \frac{4}{D^2} \cdot (x,y,z).$$

Because $4/D^2 > 0$, $N_{\mathbf{g}}$ is a positive multiple of the radius vector (x,y,z) and is thus, like $N_{\mathbf{f}}$, an outward normal. That is, $\mathbf{g}$ and $\mathbf{f}$ induce the same orientation of the sphere.

Let us now calculate Φ using $\mathbf{g}$ instead of $\mathbf{f}$. The integrand is

$$\frac{16Cu^2+16Cv^2+4C(1-u^2-v^2)^2}{D^4}=\frac{4C(1+u^2+v^2)^2}{D^4}=\frac{4C}{D^2},$$

so the integral itself is

$$\iint_{\mathbb{R}^2}\frac{4C\,du\,dv}{\left(1+(u^2+v^2)^2\right)^2}=\int_0^{2\pi}d\theta\int_0^{\infty}\frac{4C\rho\,d\rho}{(1+\rho^2)^2}=2\pi\left.\frac{-2C}{1+\rho^2}\right|_0^{\infty}=4\pi C,$$

under a change to polar coordinates. We find that the two parametrizations of $\vec{S}$ give the same value for Φ.

Preview: invariance of Φ

What we have just seen is true in general: total flux is independent of the parametrization, at least for parametrizations that induce the same orientation. To prove this, we first show that an orientation-preserving coordinate change in the source gives a new parametrization that induces the same orientation on $\vec{S}$ and yields the same value for total flux. Then we show that any two parametrizations that induce the same orientation on $\vec{S}$ are related by an orientation-preserving coordinate change in their sources (and thus give the same total flux).

Comparing parametrizations

In the following theorems, $\mathbf{f}:\Omega\to\mathbb{R}^3$ and $\mathbf{g}:\Omega^*\to\mathbb{R}^3$ both parametrize the oriented surface patch $\vec{S}$; they have coordinate functions

$$\mathbf{f}:\begin{cases}x=x(u,v),\\ y=y(u,v),\\ z=z(u,v),\end{cases}\qquad \mathbf{g}:\begin{cases}x=\xi(s,t),\\ y=\eta(s,t),\\ z=\zeta(s,t),\end{cases}$$

and orienting normals

$$N_{\mathbf{f}}(u,v)=\left(\frac{\partial(y,z)}{\partial(u,v)},\frac{\partial(z,x)}{\partial(u,v)},\frac{\partial(x,y)}{\partial(u,v)}\right),\quad N_{\mathbf{g}}(s,t)=\left(\frac{\partial(\eta,\zeta)}{\partial(s,t)},\frac{\partial(\zeta,\xi)}{\partial(s,t)},\frac{\partial(\xi,\eta)}{\partial(s,t)}\right).$$

Thus $\mathbf{f}$ and $\mathbf{g}$ are 1–1 immersions on their domains, $\mathbf{f}(\Omega)=\mathbf{g}(\Omega^*)$, and $\vec{S}=\mathbf{f}(\vec{U})=\mathbf{g}(\vec{U}^*)$, where $\vec{U}$ and $\vec{U}^*$ both have area and are closed, bounded, and positively oriented subsets of Ω and Ω^*, respectively. For the flow field $\mathbb{V}=(X(x,y,z),Y(x,y,z),Z(x,y,z))$, we define

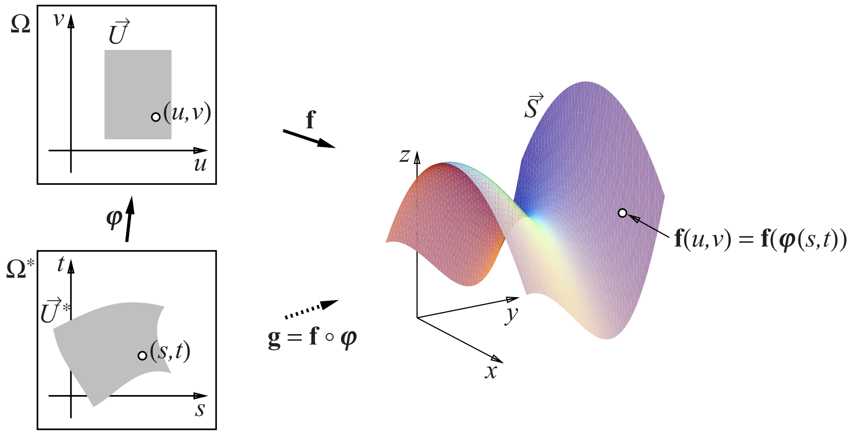
$$\Phi_{\mathbf{f}}=\iint_{\vec{U}}\left(X(\mathbf{f}(u,v))\frac{\partial(y,z)}{\partial(u,v)}+Y(\mathbf{f}(u,v))\frac{\partial(z,x)}{\partial(u,v)}+Z(\mathbf{f}(u,v))\frac{\partial(x,y)}{\partial(u,v)}\right)du\,dv,$$

$$\Phi_{\mathbf{g}}=\iint_{\vec{U}^*}\left(X(\mathbf{g}(s,t))\frac{\partial(\eta,\zeta)}{\partial(s,t)}+Y(\mathbf{g}(s,t))\frac{\partial(\zeta,\xi)}{\partial(s,t)}+Z(\mathbf{g}(s,t))\frac{\partial(\xi,\eta)}{\partial(s,t)}\right)ds\,dt.$$

In the first theorem, $\mathbf{g}$ is constructed from $\mathbf{f}$ by a coordinate change, that is, by a map $\boldsymbol{\varphi}:\Omega^*\to\Omega:(s,t)\to(u,v)$ that is continuously differentiable on an open set Ω^* in $\mathbb{R}^2$ and has a continuously differentiable inverse on Ω.

Theorem 10.2. *Suppose* $\mathbf{f} : \Omega \to \mathbb{R}^3 : (u,v) \to (x,y,z)$ *parametrizes the surface patch* $\vec{S} = \mathbf{f}(\vec{U})$*, and let* $\boldsymbol{\varphi} : \Omega^* \to \Omega : (s,t) \to (u,v)$ *be a coordinate change. Set* $\mathbf{g}(s,t) = \mathbf{f}(\boldsymbol{\varphi}(s,t))$ *on* Ω^**. If the Jacobian of* $\boldsymbol{\varphi}$*,* $\partial(u,v)/\partial(s,t)$*, is everywhere positive on* Ω^**, then* $\vec{U}^* = \boldsymbol{\varphi}^{-1}(\vec{U})$ *is positively oriented,* $\mathbf{g}$ *parametrizes* $\vec{S}$ *as an oriented surface patch, and* $\Phi_{\mathbf{g}} = \Phi_{\mathbf{f}}$.

Parametrizations from coordinate changes



Proof. Because $\boldsymbol{\varphi}$ is a coordinate change, $\mathbf{g}$ is a 1–1 immersion on Ω^* and $\vec{U}^* = \boldsymbol{\varphi}^{-1}(\vec{U})$ has area. Furthermore, $\boldsymbol{\varphi}$ preserves orientation because its Jacobian is positive; hence $\vec{U}^*$ is positively oriented (Theorem 9.13, p. 355). Because $\mathbf{g}(\vec{U}^*) = \mathbf{f}(\vec{U}) = \vec{S}$, $\mathbf{g}$ parametrizes $\vec{S}$ as an oriented surface patch.

It remains for us to prove that $\Phi_{\mathbf{g}} = \Phi_{\mathbf{f}}$. For this, it is helpful to write $(x,y,z) = \mathbf{g}(s,t) = \mathbf{f}(\boldsymbol{\varphi}(s,t))$ in terms of coordinates:

$$\begin{cases} x = \xi(s,t) = x(u(s,t),v(s,t)), \\ y = \eta(s,t) = y(u(s,t),v(s,t)), \\ z = \zeta(s,t) = z(u(s,t),v(s,t)). \end{cases}$$

The chain rule then implies the following about the various Jacobians:

$$\frac{\partial(\eta,\zeta)}{\partial(s,t)} = \frac{\partial(y,z)}{\partial(u,v)}\,\frac{\partial(u,v)}{\partial(s,t)}, \qquad \frac{\partial(\zeta,\xi)}{\partial(s,t)} = \frac{\partial(z,x)}{\partial(u,v)}\,\frac{\partial(u,v)}{\partial(s,t)},$$

$$\frac{\partial(\xi,\eta)}{\partial(s,t)} = \frac{\partial(x,y)}{\partial(u,v)}\,\frac{\partial(u,v)}{\partial(s,t)}.$$

We now show that the first terms of $\Phi_{\mathbf{g}}$ and $\Phi_{\mathbf{f}}$ are equal:

$$\iint_{\vec{U}^*} X(\mathbf{g}(s,t))\,\frac{\partial(\eta,\zeta)}{\partial(s,t)}\,ds\,dt = \iint_{\vec{U}} X(\mathbf{f}(u,v))\,\frac{\partial(y,z)}{\partial(u,v)}\,du\,dv.$$

Equality of the other two pairs of terms can be established the same way. We begin with the substitutions

$$\iint_{\vec{U}^*} X(\mathbf{g}(s,t))\,\frac{\partial(\eta,\zeta)}{\partial(s,t)}\,ds\,dt = \iint_{\vec{U}^*} X(\mathbf{f}(\boldsymbol{\varphi}(s,t)))\,\frac{\partial(y,z)}{\partial(u,v)}\,\frac{\partial(u,v)}{\partial(s,t)}\,ds\,dt.$$

The oriented change of variables formula (Theorem 9.14, p. 357) then implies

$$\iint_{\vec{U}^*} X(\mathbf{f}(\boldsymbol{\varphi}(s,t)))\,\frac{\partial(y,z)}{\partial(u,v)}\,\frac{\partial(u,v)}{\partial(s,t)}\,ds\,dt = \iint_{\boldsymbol{\varphi}(\vec{U}^*)} X(\mathbf{f}(u,v))\,\frac{\partial(y,z)}{\partial(u,v)}\,du\,dv.$$

Because $\boldsymbol{\varphi}(\vec{U}^*) = \vec{U}$, the proof is complete, by what we said above. □

Coordinate changes from parametrizations

Theorem 10.3. *Suppose* $\mathbf{f} : \Omega \to \mathbb{R}^3$ *and* $\mathbf{g} : \Omega^* \to \mathbb{R}^3$ *both parametrize the oriented surface patch* $\vec{S}$*. Then there is an orientation-preserving coordinate change* $\boldsymbol{\varphi} : \Omega^* \to \Omega$ *for which* $\mathbf{g}(s,t) = \mathbf{f}(\boldsymbol{\varphi}(s,t))$ *for all* (s,t) *in* Ω^*.

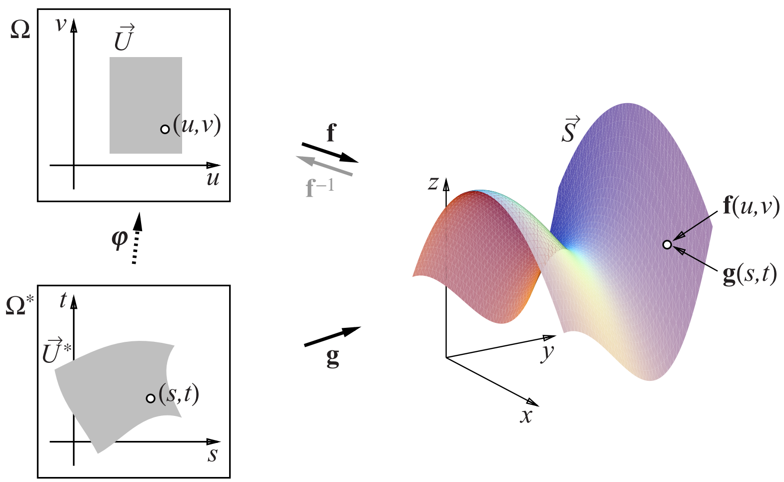
Proof. The map $\mathbf{f}$ is 1–1 everywhere on Ω, so its inverse $\mathbf{f}^{-1}$ is defined on $\mathbf{f}(\Omega) = \mathbf{g}(\Omega^*)$. Consequently, we can define

$$\boldsymbol{\varphi}(s,t) = \mathbf{f}^{-1}(\mathbf{g}(s,t))$$

for every (s,t) in Ω^*. Although $\boldsymbol{\varphi}$ is 1–1 because $\mathbf{f}^{-1}$ and $\mathbf{g}$ are, it is not obvious that it is also differentable. The chain rule,

$$d\boldsymbol{\varphi}_{(s,t)} = d\mathbf{f}^{-1}_{\mathbf{g}(s,t)} \circ d\mathbf{g}_{(s,t)}$$

fails here, because the needed derivative of $\mathbf{f}^{-1}$ is not available. To see why, recall that derivatives are linearizations. Because $\mathbf{g}$ maps an open subset of $\mathbb{R}^2$ to $\mathbb{R}^3$, its linearization $d\mathbf{g}_{(s,t)}$ at any point (s,t) is a map from $\mathbb{R}^2$ to $\mathbb{R}^3$. For the chain rule to work, the linearization $d\mathbf{f}^{-1}_{\mathbf{g}(s,t)}$ would have to map $\mathbb{R}^3$ back to $\mathbb{R}^2$, and that would require $\mathbf{f}^{-1}$ itself to be defined on an open subset of $\mathbb{R}^3$. Unfortunately, $\mathbf{f}^{-1}$ is undefined off $\mathbf{f}(\Omega)$: $d\mathbf{f}^{-1}_{\mathbf{g}(s,t)}$ does not exist.



But differentiability is a local condition; let us give $\boldsymbol{\varphi}$ a new local formulation that makes its differentiability evident. Fix a point (s_0,t_0) in Ω^*, and let $(u_0,v_0) = \boldsymbol{\varphi}(s_0,t_0)$ and $(x_0,y_0,z_0) = \mathbf{g}(s_0,t_0) = \mathbf{f}(u_0,v_0)$. Because $\mathbf{f}$ is an immersion at (u_0,v_0), Theorem 6.20 (p. 212) provides a coordinate change $\mathbf{h} : N^3 \to \mathbb{R}^3$ defined on a neighborhood of (x_0,y_0,z_0) that makes $\mathbf{h}\circ\mathbf{f}$ an *injection*. That is, for every (u,v) near (u_0,v_0),

$$\mathbf{h}\circ\mathbf{f}(u,v) = (u,v,0).$$

If $\Pi : \mathbb{R}^3 \to \mathbb{R}^2$ is the linear projection map that discards the third coordinate (i.e., $\Pi(x,y,z) = (x,y)$), then

$$\Pi\circ\mathbf{h}\circ\mathbf{f}(u,v) = (u,v).$$

In other words, $\Pi\circ\mathbf{h} : N^3 \to \mathbb{R}^2$ plays the role of the inverse of $\mathbf{f}$ on $\vec{S}$ near (x_0,y_0,z_0), but with the advantage that it is defined on a full 3-dimensional neighborhood of (x_0,y_0,z_0). Therefore, we set

$$\boldsymbol{\varphi}(s,t) = \Pi\circ\mathbf{h}\circ\mathbf{g}(s,t)$$

for all (s,t) near (s_0,t_0), and then have

$$d\boldsymbol{\varphi}_{(s_0,t_0)} = \Pi\circ d\mathbf{h}_{(x_0,y_0,z_0)}\circ d\mathbf{g}_{(s_0,t_0)}.$$

Because $\mathbf{h}$ and $\mathbf{g}$ are continuously differentiable and (s_0,t_0) was an arbitrary point in Ω^*, $\boldsymbol{\varphi}$ is continuously differentiable on Ω^*. A similar argument shows that $\boldsymbol{\varphi}^{-1}$ is continuously differentiable. Because $\vec{U}^*$ and $\vec{U} = \boldsymbol{\varphi}(\vec{U}^*)$ are both positively oriented (by definition of $\vec{S}$), $\boldsymbol{\varphi}$ preserves orientation (Theorem 9.13, p. 355). $\square$

Corollary 10.4 *Suppose* $\mathbf{f} : \Omega \to \mathbb{R}^3$ *and* $\mathbf{g} : \Omega^* \to \mathbb{R}^3$ *both parametrize the oriented surface patch* $\vec{S}$*. Then* $\Phi_\mathbf{f} = \Phi_\mathbf{g}$*; total flux of a fluid through* $\vec{S}$ *is independent of the parametrization.* $\square$ Invariance of Φ

Although the formula for Φ gives the same value no matter which parametrization is used to compute it, that formula is nevertheless bound to a parametrization. The invariance of Φ would be reflected better by an intrinsic formula, one not bound to a parametrization. The existing expression,

$$\iint_{\vec{U}} \left(X(\mathbf{f}(u,v))\,\frac{\partial(y,z)}{\partial(u,v)} + Y(\mathbf{f}(u,v))\,\frac{\partial(z,x)}{\partial(u,v)} + Z(\mathbf{f}(u,v))\,\frac{\partial(x,y)}{\partial(u,v)} \right) du\,dv,$$

is a double integral over a portion of the (u,v) parameter plane; an intrinsic formula would eliminate those parameters. The three Jacobians that appear here are the sort that would be "transformed away" when we change variables in an oriented double integral (e.g., using Theorem 9.14, p. 357). For example,

$$\frac{\partial(y,z)}{\partial(u,v)}\,du\,dv \ \text{ would be replaced by } \quad dy\,dz.$$

If we make that replacement here, and similarly replace

$$\frac{\partial(z,x)}{\partial(u,v)}\,du\,dv \text{ by } \quad dz\,dx, \quad \frac{\partial(x,y)}{\partial(u,v)}\,du\,dv \text{ by } \quad dx\,dy,$$

$\mathbf{f}(u,v)$ by (x,y,z), and $\vec{U}$ by $\vec{S}$, then no trace of the original parameters remains, and we are left with

$$\Phi = \iint_{\vec{S}} X(x,y,z)\,dy\,dz + Y(x,y,z)\,dz\,dx + Z(x,y,z)\,dx\,dy.$$

Surface integrals

This is a new kind of object called a *surface integral*. It provides the intrinsic formula we seek, expressing Φ solely in terms of the oriented surface patch $\vec{S}$ and the flow field $\mathbb{V}$ (by its component functions X, Y, and Z).

Definition 10.4 *Suppose the vector field $\mathbb{V}(x,y,z) = (X,Y,Z)$ is defined and continuous on the oriented surface patch $\vec{S}$; the* ***surface integral of $\mathbb{V}$ over $\vec{S}$*** *is the expression*

$$\iint_{\vec{S}} X(x,y,z)\,dy\,dz + Y(x,y,z)\,dz\,dx + Z(x,y,z)\,dx\,dy$$

whose value is given by the double integral

$$\iint_{\vec{U}} \left(X(\mathbf{f}(u,v))\,\frac{\partial(y,z)}{\partial(u,v)} + Y(\mathbf{f}(u,v))\,\frac{\partial(z,x)}{\partial(u,v)} + Z(\mathbf{f}(u,v))\,\frac{\partial(x,y)}{\partial(u,v)} \right) du\,dv,$$

where $\mathbf{f} : \Omega \to \mathbb{R}^3$ is any parametrization of $\vec{S} = \mathbf{f}(\vec{U})$.

In effect, the parametrization pulls back the surface integral from $\vec{S}$ in $\mathbb{R}^3$ to a double integral on $\vec{U}$ in $\mathbb{R}^2$. Corollary 10.4 implies that the value of the surface integral is independent of the paramatrization of $\vec{S}$.

Theorem 10.5. *When the orientation of $\vec{S}$ is reversed, the surface integral changes sign:*

$$\iint_{-\vec{S}} X\,dy\,dz + Y\,dz\,dx + Z\,dx\,dy = -\iint_{\vec{S}} X\,dy\,dz + Y\,dz\,dx + Z\,dx\,dy.$$

Proof. Suppose $\mathbf{f} : \Omega \to \mathbb{R}^3$ parametrizes the oriented surface patch $-\vec{S}$; then, by definition, $-\vec{S} = \mathbf{f}(\vec{U})$ for some positively oriented region $\vec{U} \subset \Omega$. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the orientation-reversing linear map (reflection)

$$(u,v) = L(s,t) = (t,s),$$

and let $\Omega^* = L^{-1}(\Omega)$. Because L^{-1} reverses orientation, the induced orientation (p. 355) on the image $\vec{U}^* = L^{-1}(-\vec{U})$ is positive. Define $\mathbf{g} = \mathbf{f} \circ L$; then $\mathbf{g} : \Omega^* \to \mathbb{R}^3$ parametrizes

$$\mathbf{g}(\vec{U}^*) = \mathbf{f}(L(\vec{U}^*)) = \mathbf{f}(-\vec{U}) = -\mathbf{f}(\vec{U}) = \vec{S}$$

itself, because $\vec{U}^*$ is positively oriented.

The expressions involved in proving the two surface integrals equal are long and complicated. To simplify our work, we deal only with the first terms of the integrals; the second and third terms can be dealt with the same way. First use the parametrization $\mathbf{f}$ to get

$$\iint_{-\vec{S}} X\,dy\,dz = \iint_{\vec{U}} X(\mathbf{f}(u,v))\,\frac{\partial(y,z)}{\partial(u,v)}\,du\,dv.$$

Now use the change of variables $(u,v) = L(s,t)$, $\mathbf{f}(u,v) = \mathbf{f}(L(s,t)) = \mathbf{g}(s,t)$ and the oriented change of variables formula (Theorem 9.14, p. 357) to write

$$\begin{aligned}\iint\limits_{\vec{U}} X(\mathbf{f}(u,v))\,\frac{\partial(y,z)}{\partial(u,v)}\,du\,dv &= \iint\limits_{L^{-1}(\vec{U})} X(\mathbf{f}(L(s,t)))\,\frac{\partial(y,z)}{\partial(u,v)}\,\frac{\partial(u,v)}{\partial(s,t)}\,ds\,dt \\ &= \iint_{-\vec{U}^*} X(\mathbf{g}(s,t))\,\frac{\partial(y,z)}{\partial(s,t)}\,ds\,dt = -\iint_{\vec{U}^*} X(\mathbf{g}(s,t))\,\frac{\partial(y,z)}{\partial(s,t)}\,ds\,dt.\end{aligned}$$

The last integral is a parametric representation of the surface integral

$$-\iint_{\vec{S}} X\,dy\,dz;$$

by what was said above, this proves the theorem. □

The form of path and surface integrals

Note the similarity in form between a surface integral and a path integral (for a path that lies in space):

$$\iint_{\vec{S}} X\,dy\,dz + Y\,dz\,dx + Z\,dx\,dy \quad \text{versus} \quad \int_{\vec{C}} P\,dx + Q\,dy + R\,dz.$$

In these integrals, the physical vectors, *flux density* $\mathbb{V} = (X,Y,Z)$ and *force* $\mathbf{F} = (P,Q,R)$, are represented by their components. However, for the path integral there is an alternate form in which $\mathbf{F}$ itself appears:

$$\text{work} = \int_{\vec{C}} P\,dx + Q\,dy + R\,dz = \int_{C} \mathbf{F}\cdot\mathbf{t}\,ds.$$

Here $\mathbf{t}$ is the unit tangent that orients the path $\vec{C}$ (see p. 19), and ds is the "element of arc length" for the unoriented path C. This alternate form is an *unoriented* path integral; information about the orientation of $\vec{C}$ has been transferred to the integrand, to the factor $\mathbf{t}$.

An alternate form for a surface integral

The surface integral also has an alternate form that is analogous to the second path integral. We can derive that new form by reconstructing our estimates for total flux through $\vec{S}$. In the original construction, we began with a collection of oriented parallelograms $\vec{P}_1,\ldots,\vec{P}_I$ that approximated $\vec{S}$; $\Phi(\vec{P}_i)$ was the total flux through $\vec{P}_i$, and the sum

$$\sum_{i=1}^{I} \Phi(\vec{P}_i)$$

estimated total flux through $\vec{S}$ itself. If $\vec{P}_i = \mathbf{p}_i \wedge \mathbf{q}_i$ and $\mathbb{V}_i$ was the value of $\mathbb{V}$ at the corner (u_i,v_i) of $\vec{P}_i$, then (p. 395)

$$\Phi(\vec{P}_i) \approx \mathbb{V}_i \cdot \mathbf{p}_i \times \mathbf{q}_i = \left(X \frac{\partial(y,z)}{\partial(u,v)} + Y \frac{\partial(z,x)}{\partial(u,v)} + Z \frac{\partial(x,y)}{\partial(u,v)} \right)_{(u_i,v_i)} \Delta u \, \Delta v.$$

The expressions on the right are the terms in a convergent Riemann sum; in the limit they give the surface integral

$$\Phi = \iint_{\vec{S}} X\,dy\,dz + Y\,dz\,dx + Z\,dx\,dy.$$

Rewriting the cross-product

To construct the second, alternate, form of the surface integral, note that our estimate for $\Phi(\vec{P}_i)$ used the component form of the cross-product: $\mathbf{p}_i \times \mathbf{q}_i$. We now switch to the geometric form,

$$\mathbf{p}_i \times \mathbf{q}_i = \mathbf{n}_i \, \Delta A_i,$$

in which $\Delta A_i \geq 0$ is the absolute area of $\vec{P}_i$ and $\mathbf{n_i}$ is its orienting unit normal. In terms of these geometric variables,

$$\sum_{i=1}^{I} \Phi(\vec{P}_i) = \sum_{i=1}^{I} \mathbb{V}_i \cdot \mathbf{n}_i \, \Delta A_i.$$

This is the Riemann sum; if we follow the usual pattern in expressing its limit as an integral, we get

$$\Phi = \lim_{\substack{I \to \infty \\ \Delta A_i \to 0}} \sum_{i=1}^{I} \mathbb{V}_i \cdot \mathbf{n}_i \, \Delta A_i = \iint_S \mathbb{V} \cdot \mathbf{n} \, dA.$$

This is the alternate form for a surface integral; that is,

$$\iint_{\vec{S}} X\,dy\,dz + Y\,dz\,dx + Z\,dx\,dy = \iint_S \mathbb{V} \cdot \mathbf{n} \, dA.$$

The integrand $\mathbb{V} \cdot \mathbf{n}$ is the normal component of flux density on $\vec{S}$; the domain of integration is the *unoriented* surface patch S. We call dA the **element of surface area** for S. Information about the orientation of $\vec{S}$ has been transferred from the domain of integration to the integrand. Compare the new integral to the original expession $\mathbb{V} \cdot \mathbf{n} \, \Delta A$ for total flux through a plane region (cf. pp. 387–389 and Definition 10.1).

Area and scalar integrals

In the next section, we discuss integrals of the general form

$$\iint_S f(\mathbf{x})\,dA,$$

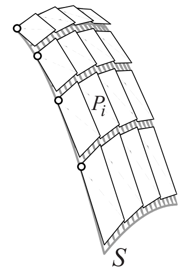
where f is a scalar function defined on a region in space that contains the surface patch S. In particular, $f(\mathbf{x}) \equiv 1$ leads to a notion of area for S and indicates why we think of dA as the element of surface area for S.

10.2 Surface area and scalar integrals

In this section, we define the area of a curved surface in space as an integral. Using that as a basis, we then define the integral of a scalar function over a curved surface. Surface area is analogous to arc length, and scalar integrals over surfaces are similarly analogous to scalar integrals over curved paths.

Approximating a surface patch

To define the area of an unoriented surface patch S, we begin with a parametrization $\mathbf{f} : \Omega \to \mathbb{R}^3$ of S. Thus $\mathbf{f}$ is a continuously differentiable 1–1 immersion on an open set Ω, $U \subset \Omega$ is a closed, bounded, unoriented set with area, and $S = \mathbf{f}(U)$. Using $\mathbf{f}$ and its derivative, we can approximate S by a collection of parallelograms whose total area will give us an estimate for the area of S. This is essentially the procedure described beginning on page 393. Pick one of the grids $\mathcal{J}_k$ used to define Jordan content in the plane, and select the square cells $Q_1, \ldots, Q_I$ of $\mathcal{J}_k$ that meet U. Let (u_i, v_i) be the lower-left corner of Q_i, and let



$$P_i = \mathbf{df}_{(u_i,v_i)}(Q_i).$$

This is a parallelogram tangent to S at $\mathbf{f}(u_i, v_i)$; the parallelograms $P_1, \ldots, P_I$ together give us an approximation of S that improves as $k \to \infty$. The edges of Q_i are multiples of the standard basis vectors that we can write as

$$\Delta u\, \mathbf{e}_1 \quad \text{and} \quad \Delta v\, \mathbf{e}_2,$$

where $\Delta u = \Delta v = 1/2^k$. The corresponding edges of P_i are

$$\mathbf{p}_i = \Delta u\, \mathbf{df}_{(u_i,v_i)}(\mathbf{e}_1), \quad \mathbf{q}_i = \Delta v\, \mathbf{df}_{(u_i,v_i)}(\mathbf{e}_2);$$

they are multiples of the columns of $\mathbf{df}_{(u_i,v_i)}$. If

$$\mathbf{f} : \begin{cases} x = x(u,v), \\ y = y(u,v), \\ z = z(u,v), \end{cases} \qquad \mathbf{df}_{(a,b)} = \begin{pmatrix} x_u(a,b) & x_v(a,b) \\ y_u(a,b) & y_v(a,b) \\ z_u(a,b) & z_v(a,b) \end{pmatrix}$$

then (p. 393)

$$\mathbf{p}_i \times \mathbf{q}_i = \left(\frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)} \right)_{(u_i,v_i)} \Delta u\, \Delta v,$$

and the area of P_i is

Area of P_i

$$\|\mathbf{p}_i \times \mathbf{q}_i\| = \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(z,x)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2}\, \Bigg|_{(u_i,v_i)} \Delta u\, \Delta v.$$

This is the ordinary nonnegative area of an unoriented region. The total area of the parallelograms that approximate S is therefore

$$\sum_{i=1}^{I} \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(z,x)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2}\,\Bigg|_{(u_i,v_i)} \Delta u\,\Delta v.$$

Area for a given parametrization

This is a Riemann sum for the double integral

$$\text{area}_{\mathbf{f}}(S) = \iint_U \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(z,x)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2}\, du\,dv.$$

Because $\mathbf{f}$ has continuous first derivatives, the integrand is continuous and the Riemann sums converge to the integral as $k \to \infty$. If we consider

$$M(U) = \text{area}_{\mathbf{f}}(S) = \iint_U \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(z,x)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2}\, du\,dv$$

to be a set function defined on closed bounded subsets $U \subset \Omega$ that have area (cf. pp. 310–312), then its derivative is

$$M'(u,v) = \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(z,x)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2}$$

Local area multiplier

(Theorem 8.39, p. 312). This implies that

$$\frac{\text{area}_{\mathbf{f}}(S)}{A(U)} \approx \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(z,x)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2}$$

as closely as we wish by making the diameter of U sufficiently small. It is for this reason that we defined (Definition 4.9, p. 139)

$$\sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(z,x)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2}$$

to be the local area multiplier for $\mathbf{f}$.

Does the value of the integral for surface area depend on the parametrization? Let $\mathbf{g} : \Omega^* \to \mathbb{R}^3$ be another parametrization of S, where

$$(x,y,z) = \mathbf{g}(s,t) = (\xi(s,t), \eta(s,t), \zeta(s,t)).$$

There is a closed bounded set $U^* \subset \Omega^*$ with area, for which $\mathbf{g}(U^*) = S$ and

$$\text{area}_{\mathbf{g}}(S) = \iint_{U^*} \sqrt{\left[\frac{\partial(\eta,\zeta)}{\partial(s,t)}\right]^2 + \left[\frac{\partial(\zeta,\xi)}{\partial(s,t)}\right]^2 + \left[\frac{\partial(\xi,\eta)}{\partial(s,t)}\right]^2}\, ds\,dt.$$

Invariance of surface area

Theorem 10.6. *Surface area is independent of the parametrization used to compute it:* $\text{area}_{\mathbf{g}}(S) = \text{area}_{\mathbf{f}}(S)$.

Proof. If we set orientation aside, the proof of Theorem 10.3 provides a coordinate change $\boldsymbol{\varphi} : \Omega^* \to \Omega$, $(u,v) = \boldsymbol{\varphi}(s,t)$, for which $\mathbf{g}(U^*) = U$ and $\mathbf{g}(s,t) = \mathbf{f}(\boldsymbol{\varphi}(s,t))$ for all (s,t) in Ω^*. The chain rule implies

$$\frac{\partial(\eta,\zeta)}{\partial(s,t)} = \frac{\partial(y,z)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(s,t)}, \quad \frac{\partial(\zeta,\xi)}{\partial(s,t)} = \frac{\partial(z,x)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(s,t)}, \quad \frac{\partial(\xi,\eta)}{\partial(s,t)} = \frac{\partial(x,y)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(s,t)}.$$

Therefore, on Ω^* we have

$$\begin{aligned}\sqrt{\left[\frac{\partial(\eta,\zeta)}{\partial(s,t)}\right]^2 + \left[\frac{\partial(\zeta,\xi)}{\partial(s,t)}\right]^2 + \left[\frac{\partial(\xi,\eta)}{\partial(s,t)}\right]^2}& \\ = \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(z,x)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2}&\;\left|\frac{\partial(u,v)}{\partial(s,t)}\right|\end{aligned}$$

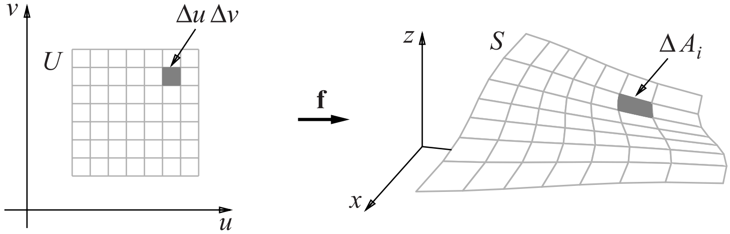
Now make this substitution in the formula for $\text{area}_{\mathbf{g}}(S)$, and then use the basic change of variables formula (Theorem 9.11, p. 350) to get

$$\begin{aligned}\text{area}_{\mathbf{g}}(S) &= \iint_{U^*} \sqrt{\left[\frac{\partial(\eta,\zeta)}{\partial(s,t)}\right]^2 + \left[\frac{\partial(\zeta,\xi)}{\partial(s,t)}\right]^2 + \left[\frac{\partial(\xi,\eta)}{\partial(s,t)}\right]^2}\, ds\,dt \\ &= \iint_{U^*} \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(z,x)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2}\;\left|\frac{\partial(u,v)}{\partial(s,t)}\right| ds\,dt \\ &= \iint_{U} \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(z,x)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2}\, du\,dv = \text{area}_{\mathbf{f}}(S). \qquad \square\end{aligned}$$

Definition 10.5 *Let* $\mathbf{f} : \Omega \to \mathbb{R}^3 : (u,v) \to (x,y,z)$ *be any parametrization of the surface patch* $S = \mathbf{f}(U)$*; then the* ***surface area of S*** *is* Surface area of S

$$A(S) = \iint_{U} \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(z,x)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2}\, du\,dv.$$

Although the value of $A(S)$ is independent of the parametrization used to compute it, our expression for $A(S)$ is still bound to a parametrization. As with total flux, the invariance of $A(S)$ would be reflected better by an intrinsic formula, one not bound to a parametrization. We can get that formula by looking at the areas of small cells on S.



Let $\mathbf{f} : \Omega \to \mathbb{R}^3$ be a parametrization of S with $\mathbf{f}(U) = S$, and let Q_i be a cell of the grid $\mathcal{J}_k$ that meets U. Suppose the image $\mathbf{f}(Q_i)$ has area ΔA_i as given by Definition 10.5. Then, using the local area muliplier for $\mathbf{f}$ (p. 406), we have

$$\Delta A_i \approx \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(z,x)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2}\,\Bigg|_{(u_i,v_i)} \Delta u\, \Delta v,$$

$$\text{so} \quad \sum_{i=1}^{I} \Delta A_i \approx \sum_{i=1}^{I} \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(z,x)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2}\,\Bigg|_{(u_i,v_i)} \Delta u\, \Delta v.$$

An intrinsic formula for surface area

These sums both converge to $A(S)$. We write the limit on the left as an integral, following the usual pattern:

$$\lim_{\substack{I\to\infty \\ \Delta A_i \to 0}} \sum_{i=1}^{I} \Delta A_i = \iint_S dA.$$

This gives us the simple intrinsic expression

$$A(S) = \iint_S dA$$

for the surface area of S. Comparing the intrinsic with the parametric expression for $A(S)$, we can see why

$$dA = \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(z,x)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2}\, du\, dv$$

is described as the *element of surface area* on S.

Comparison with arc length

There are striking similarities between surface area and arc length. If the path C in $\mathbb{R}^3$ is parametrized by

$$\mathbf{f}(u) = (x(u), y(u), z(u)), \quad a \le u \le b,$$

then the *element of arc length* on C is

$$ds = \sqrt{\left[\frac{dx}{du}\right]^2 + \left[\frac{dy}{du}\right]^2 + \left[\frac{dz}{du}\right]^2}\, du,$$

and

$$\text{arc length of } C = \int_C ds = \int_a^b \sqrt{\left[\frac{dx}{du}\right]^2 + \left[\frac{dy}{du}\right]^2 + \left[\frac{dz}{du}\right]^2}\, du,$$

For the integral of a scalar function $H(x,y,z)$ over the path C (Definition 1.6 and Theorem 1.5, p. 18), we have

$$\int_C H(x,y,z)\,ds = \int_a^b H(\mathbf{f}(u))\sqrt{\left[\frac{dx}{du}\right]^2 + \left[\frac{dy}{du}\right]^2 + \left[\frac{dz}{du}\right]^2}\,du.$$

As noted after the proof of Theorem 1.5, the value of the integral on the left does not depend on either the orientation of C or the parametrization of C used in the integral on the right.

Surface integral of a scalar function

Definition 10.6 *Let S be a surface patch in $\mathbb{R}^3$, and let $H(x,y,z)$ be a continuous function defined on S. We set*

$$\iint_S H(x,y,z)\,dA = \iint_U H(\mathbf{f}(u,v))\sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(z,x)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2}\,du\,dv,$$

where $\mathbf{f} : \Omega \to \mathbb{R}^3$ is a parametrization of $S = \mathbf{f}(U)$.

For the surface integral on the left to be well defined, its value must be independent of the parametrization used for S on the right.

Invariance of the scalar integral

Theorem 10.7. *Let $\mathbf{f} : \Omega \to \mathbb{R}^3$ and $\mathbf{g} : \Omega^* \to \mathbb{R}^3$ be two parametrizations of $S = \mathbf{f}(U) = \mathbf{g}(U^*)$, with $U \subset \Omega$ and $U^* \subset \Omega^*$. Then*

$$\iint_U H(\mathbf{f}(u,v))\sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(z,x)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2}\,du\,dv$$
$$= \iint_{U^*} H(\mathbf{g}(s,t))\sqrt{\left[\frac{\partial(y,z)}{\partial(s,t)}\right]^2 + \left[\frac{\partial(z,x)}{\partial(s,t)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(s,t)}\right]^2}\,ds\,dt,$$

for any continuous function $H(x,y,z)$ defined on S.

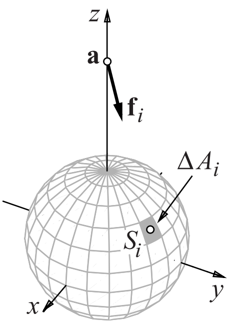
Proof. See Exercise 10.7. □

For example, if $H = \rho$ is mass density at a point of S, then the integral of H is the total mass of S. If H is density of electric charge (a signed quantity that may be negative), then the integral of H is the total electric charge on S. If $H = \mathbb{V} \cdot \mathbf{n}$, where $\mathbb{V}$ is a flux density and $\mathbf{n}$ is an orienting unit normal for the oriented surface patch $\vec{S}$, then the integral of H over S is total flux of $\mathbb{V}$ through $\vec{S}$.

The gravitational field of a hollow sphere

The gravitational field of a hollow sphere is yet another example of a surface integral, one that we now construct by adapting the work we did in Chapter 8.1 on the field of a flat plate. Let the sphere be a unit sphere S centered at the origin of (x,y,z)-space; suppose it has negligible thickness and has uniform density ρ (mass per unit area). By symmetry, it is enough to determine the vertical component of the gravitational field at a test point $\mathbf{a} = (0,0,a)$ on the positive z-axis. Symmetry guarantees that the x- and y-components of the field at $\mathbf{a}$ are zero; as we find, it matters whether the test point is inside or outside the sphere.

Let S be divided into small regions $S_1,\ldots,S_I$; suppose S_i has area ΔA_i and contains the point $\mathbf{p}_i = (x_i,y_i,z_i)$. Let



$$\mathbf{r}_i = \mathbf{p}_i - \mathbf{a} = (x_i, y_i, z_i - a)$$

be the vector from the test point to $\mathbf{p}_i$. Then the gravitational field on the test point that is due to the region S_i is approximately

$$\mathbf{f}_i \approx \frac{G\rho\,\Delta A_i}{\|\mathbf{r}_i\|^3}\,\mathbf{r}_i = \frac{G\rho\,\Delta A_i}{(x_i^2 + y_i^2 + (z_i - a)^2)^{3/2}}(x_i, y_i, z_i - a),$$

where G is the usual gravitational constant. We can therefore approximate the z-component of the gravitational field for the whole sphere S by the (scalar) sum

$$\text{field} \approx G\rho \sum_{i=1}^{I} \frac{z_i - a}{(x_i^2 + y_i^2 + (z_i - a)^2)^{3/2}}\,\Delta A_i.$$

Now let $I \longrightarrow \infty$ and let the maximum diameter of S_i tend to zero; in the limit, the sum becomes the surface integral

$$\text{field} = G\rho \iint_S \frac{z - a}{(x^2 + y^2 + (z - a)^2)^{3/2}}\,dA = G\rho \iint_S \frac{z - a}{(1 + a^2 - 2az)^{3/2}}\,dA$$

(because $x^2 + y^2 + z^2 = 1$ on S).

To evaluate the surface integral, we use the parametrization

$$\begin{cases} x = \cos\theta\cos\varphi, \\ y = \sin\theta\cos\varphi, \\ z = \sin\varphi; \end{cases} \qquad U: \quad \begin{array}{c} -\pi \le \theta \le \pi, \\ -\pi/2 \le \varphi \le \pi/2, \end{array}$$

of the unit sphere, keeping in mind the caveat we made when using the parametrization calculate total flux through the sphere (p. 396). From that earlier work, we find

$$dA = \sqrt{\left[\frac{\partial(y,z)}{\partial(\theta,\varphi)}\right]^2 + \left[\frac{\partial(z,x)}{\partial(\theta,\varphi)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(\theta,\varphi)}\right]^2}\,d\theta\,d\varphi = \cos\varphi\,d\theta\,d\varphi.$$

We can now compute the field (e.g., using a table of integrals or a computer algebra system):

$$\begin{aligned} \text{field} &= G\rho \int_{-\pi}^{\pi} d\theta \int_{-\pi/2}^{\pi/2} \frac{(\sin\varphi - a)\cos\varphi}{(1 + a^2 - 2a\sin\varphi)^{3/2}}\,d\varphi \\ &= 2\pi G\rho \cdot \left.\frac{1 - a\sin\varphi}{a^2\sqrt{1 + a^2 - 2a\sin\varphi}}\right|_{-\pi/2}^{\pi/2} \\ &= \frac{2\pi G\rho}{a^2}\left(\frac{1 - a}{\sqrt{(1 - a)^2}} - \frac{1 + a}{\sqrt{(1 + a)^2}}\right) = \frac{2\pi G\rho}{a^2}\left(\frac{1 - a}{|1 - a|} - \frac{1 + a}{|1 + a|}\right). \end{aligned}$$

By assumption, $a \geq 0$; therefore $|1+a| = 1+a$ and $(1+a)/|1+a| = 1$. The first term is more interesting:

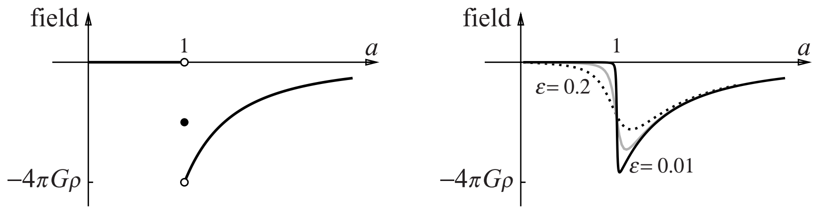
$$\frac{1-a}{|1-a|} = \begin{cases} 1 & \text{if } a < 1, \\ -1 & \text{if } a > 1. \end{cases}$$

Therefore,

$$\text{field} = \begin{cases} 0 & \text{if } a < 1, \\ -\dfrac{4\pi G\rho}{a^2} & \text{if } a > 1. \end{cases}$$

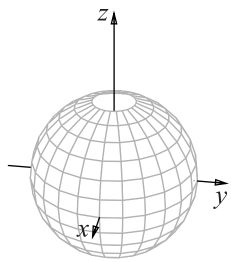
Inside, the sphere induces no gravitational field whatsoever. Outside, the sphere acts as if all its mass $4\pi\rho$ were concentrated at the origin. On the sphere itself, the field is discontinuous. When $a = 1$, the value of the field is the average of its inside and outside values; see Exercise 10.8.

The field vanishes inside the sphere



The discontinuity occurs where the z-axis passes through the sphere. If we put small holes at the north and south poles, then there is no matter on the z-axis, so the field should become continuous. The graphs on the right, above, show what happens. To determine the new field, we can use the same surface integral, parametrized the same way, but with φ restricted to

A sphere with a small hole at each pole



$$-\pi/2 + \varepsilon \leq \varphi \leq \pi/2 - \varepsilon,$$

where $\varepsilon > 0$ is some small number. Thus,

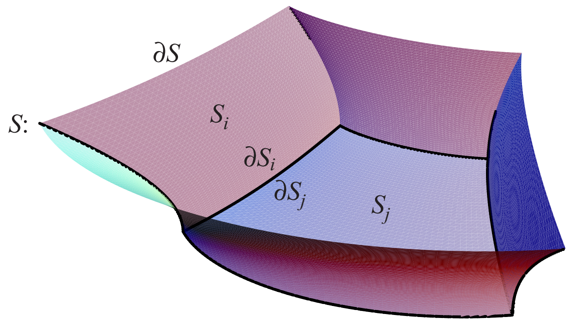
$$\begin{aligned} \text{field} &= 2\pi G\rho \left. \frac{1 - a\sin\varphi}{a^2\sqrt{1+a^2-2a\sin\varphi}} \right|_{-\pi/2+\varepsilon}^{\pi/2-\varepsilon} \\ &= \frac{2\pi G\rho}{a^2}\left(\frac{1-a\cos\varepsilon}{\sqrt{1+a^2-2a\cos\varepsilon}} - \frac{1+a\cos\varepsilon}{\sqrt{1+a^2+2a\cos\varepsilon}} \right). \end{aligned}$$

(Note that $\sin(\pm(\pi/2-\varepsilon)) = \pm\sin(\pi/2-\varepsilon) = \pm\cos\varepsilon$.) This is indeed a continuous function of a; the graphs show $\varepsilon = 0.2$, 0.075, and 0.01. It is evident that the field strength fades away inside the sphere as the holes close up (i.e., as $\varepsilon \to 0$), and the field develops a discontinuity where the z-axis meets the sphere.

There is also an ingenious geometric argument (see, for e.g., the Feynman Lectures [6]) that explains why the field vanishes inside a hollow sphere. However, that argument relies on the symmetry of the sphere and cannot be easily modified when we break the symmetry with holes at the poles. By contrast, the surface integral for the field still works.

Limitations of surface patches

Not every surface can be represented as a single surface patch. For one thing, the parametrization that defines a patch is 1–1, and the domain of the parametrization has a boundary; therefore the patch itself must have a boundary. A surface without boundary (e.g., a sphere or a torus) cannot be a surface patch. Furthermore, because a surface patch has a well-defined tangent plane at each point, no surface with edges or corners (e.g., a cube) can be a surface patch, either. However, because each of these examples can be assembled from a finite collection of surface patches, we are led to define a surface S as a union of surface patches $S_1, \ldots, S_k$ satisfying certain conditions.



A union of surface patches

To identify those conditions, we can be guided by the surface S illustrated above. For a start, we must have

$$S = S_1 \cup S_2 \cup \cdots \cup S_k,$$

where each S_i is a surface patch defined by a continuously differentiable parametrization

$$\mathbf{f}_i : \Omega_i \to \mathbb{R}^3, \quad U_i \subset \Omega_i, \quad S_i = \mathbf{f}_i(U_i), \quad \partial S_i = \mathbf{f}_i(\partial U_i).$$

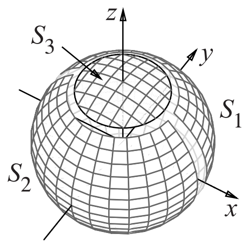
Recall (Definition 10.2, p. 392; see also p. 405) that the domain of a parametrization is always an open set Ω_i that extends beyond the closed bounded set U_i that is mapped to the patch S_i.

How the patches fit together

To ensure that the patches fit together properly, we require that each boundary ∂U_i, and hence each ∂S_i, is a piecewise-smooth closed curve (cf. p. 9), made up of a finite number of smooth arcs. Then we require that any two patches S_i and S_j meet only along their boundaries, and that any three patches meet in, at most, a finite number of isolated points. As the figure shows, some arcs are part of the boundary of two patches; the remaining arcs, each of which lies in only one patch, together form the boundary, ∂S, of S. In the following definition, no orientation is assumed, and "smooth" is used in the sense of *continuously differentiable*.

Definition 10.7 *A subset S of $\mathbb{R}^3$ is a* ***piecewise-smooth surface*** *if it can be decomposed into a finite number of surface patches that fit together as above.*

For example, we can decompose the unit sphere into four surface patches: S_1, eastern belt; S_2, western belt; S_3, northern cap; S_4, southern cap. The figure in the margin shows the first three patches, separated slightly for clarity. Latitude and longitude parametrize the belts S_1 and S_2; only the longitude ranges differ:



$$\mathbf{f}_{1,2}:\begin{cases} x=\cos\theta\cos\varphi, \\ y=\sin\theta\cos\varphi, \\ z=\sin\varphi; \end{cases} \qquad U_1: \begin{array}{c} 0\le\theta\le\pi, \\ -\alpha\le\varphi\le\alpha; \end{array} \qquad U_2: \begin{array}{c} -\pi\le\theta\le 0, \\ -\alpha\le\varphi\le\alpha. \end{array}$$

The angle $0<\alpha<\pi/2$ gives the latitude of the northern boundary of the belts. The polar caps are just the graphs of

$$z=\pm\sqrt{1-x^2-y^2}, \quad U_{3,4}: x^2+y^2\le\cos^2\alpha.$$

A surface can be decomposed into a finite number of surface patches in many ways. Another way to decompose the unit sphere (cf. p. 397) uses just its northern and southern hemispheres ($S_\pm$):

$$\mathbf{g}_\pm:\begin{cases} x=\dfrac{2u}{1+u^2+v^2}, \\ y=\dfrac{2v}{1+u^2+v^2}, \\ z=\pm\dfrac{1-u^2-v^2}{1+u^2+v^2}; \end{cases} \qquad U_\pm: u^2+v^2\le 1.$$

Integrating a scalar function on a piecewise smooth surface

Because the definition of the integral of a scalar function on a surface patch (Definition 10.6) does not depend on any orientation of the patch, we can use it to define the integral of a scalar function on an unoriented piecewise-smooth surface.

Definition 10.8 *Suppose $H(x,y,z)$ is a continuous function defined on a piecewise-smooth surface S; if $S=S_1\cup\cdots\cup S_k$ is a decomposition of S into surface patches, then*

$$\iint_S H(x,y,z)\,dA=\sum_{i=1}^{k}\iint_{S_i}H(x,y,z)\,dA.$$

In particular, if $H(x,y,z)\equiv 1$, then the integrals define surface area:

$$\iint_S dA=\operatorname{area}S=\sum_{i=1}^{k}\operatorname{area}S_i.$$

The integral is independent of the decomposition

A piecewise-smooth surface S has many different decompositions into surface patches; therefore an integral over S will be well defined only if its value is independent of that decomposition.

Theorem 10.8. *Suppose $S=S_1\cup\cdots\cup S_k=T_1\cup\cdots\cup T_m$ gives two decompositions of the piecewise-smooth surface S into surface patches. Then*

$$\sum_{i=1}^{k}\iint_{S_i}H(x,y,z)\,dA=\sum_{j=1}^{m}\iint_{T_j}H(x,y,z)\,dA.$$

Proof. Suppose S_i is parametrized by $(x,y,z) = \mathbf{f}_i(u,v)$, with $\mathbf{f}_i(U_i) = S_i$ and

$$J_i(u,v) = \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(z,x)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2},$$

the local area magnification factor for $\mathbf{f}_i$. Then, by definition,

$$\iint_{S_i} H\,dA = \iint_{U_i} H(\mathbf{f}_i(u,v))\,J_i(u,v)\,du\,dv.$$

Let $R_{ij} = S_i \cap T_j$; this is a "common refinement" (cf. p. 300) of the decompositions given by S_i and by T_j. (Of course, some sets R_{ij} may be empty.) Because $\mathbf{f}_i$ is continuous and 1–1 on U_i, the sets

$$(U_i)_j = \mathbf{f}_i^{-1}(R_{ij}) \subseteq U_i$$

are closed, bounded, and have area. Also, because $S_i = \cup_{j=1}^m R_{ij}$ and the R_{ij} are nonoverlapping,

$$U_i = \cup_{j=1}^m (U_i)_j$$

is a decomposition into nonoverlapping sets. Therefore, when i is fixed, each R_{ij} with $j = 1,\ldots,m$ is a (possibly empty) surface patch parametrized by $\mathbf{f}_i$, with $R_{ij} = \mathbf{f}_i((U_i)_j)$. Hence,

$$\begin{aligned}\iint_{U_i} H(\mathbf{f}_i(u,v))\,J_i(u,v)\,du\,dv &= \sum_{j=1}^m \iint_{(U_i)_j} H(\mathbf{f}_i(u,v))\,J_i(u,v)\,du\,dv \\ &= \sum_{j=1}^m \iint_{R_{ij}} H\,dA,\end{aligned}$$

and thus

$$\sum_{i=1}^k \iint_{S_i} H\,dA = \sum_{i=1}^k \sum_{j=1}^m \iint_{R_{ij}} H\,dA.$$

A similar argument, beginning with T_j, shows that

$$\sum_{j=1}^m \iint_{T_j} H\,dA = \sum_{j=1}^m \sum_{i=1}^k \iint_{R_{ij}} H\,dA. \qquad \square$$

From piecewise-smooth to smooth

By definition, a piecewise-smooth surface can have edges and corners; at such points, the surface fails to be smooth, that is, to have a well-defined tangent plane. But edges and corners can occur only where two surface patches of a decomposition meet. Thus, in our first decomposition above, the unit sphere could fail to be smooth only at a point on the boundary of one of the belts (S_1, S_2) or one of the caps (S_3, S_4). However, these are all interior points of the hemispheres $S_\pm$ of our second decomposition (except for the two points on the x-axis), so they must be smooth points of the sphere, after all. (The two points on the x-axis are interior points of a third decomposition that uses the polar caps and the equatorial belts rotated 90° around the z-axis.) Because every point on the sphere is interior to some surface patch in a decomposition, the sphere has no edges or corners: it is smooth.

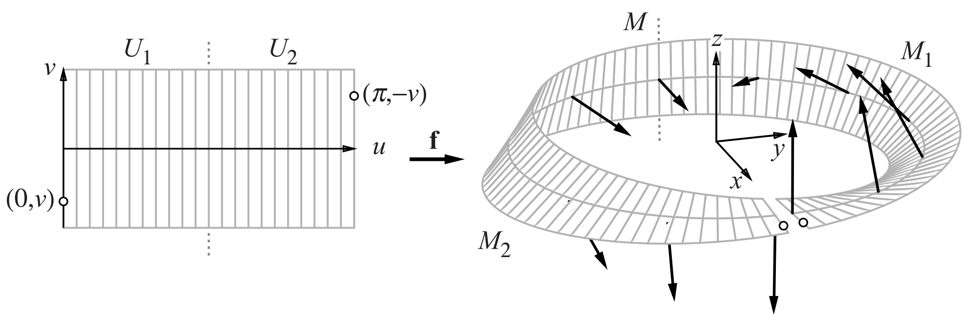
Definition 10.9 *A piecewise-smooth surface S is* **smooth** *if every point of S that is not in ∂S is an interior point of a surface patch that appears in some decomposition of S.* Smooth surfaces

We turn now to the question of *orientation*. On pages 388–389, we described two equivalent ways to orient a surface S in space. First, to each point $\mathbf{p}$ in S we assigned an ordered pair of linearly independent tangent vectors $\{\mathbf{v}_1(\mathbf{p}), \mathbf{v}_2(\mathbf{p})\}$ to S at $\mathbf{p}$ in such a way that each $\mathbf{v}_i(\mathbf{p})$ varied continuously with $\mathbf{p}$. Second, we assigned to $\mathbf{p}$ one of the unit normals $\mathbf{n}(\mathbf{p})$ to S at $\mathbf{p}$ in such a way that $\mathbf{n}(\mathbf{p})$ varied continuously with $\mathbf{p}$. Ways to orient a surface

A particular surface may admit no orientation whatsoever. One impediment is the presence of an edge or a corner, where there is no well-defined tangent plane or normal vector. It would appear, then, that a surface that is piecewise smooth but not smooth cannot be oriented. In fact, we see below that this impediment can sometimes be overcome. A different sort of impediment affects even some smooth surfaces; the overall shape of the surface may preclude orientation. Perhaps the simplest example is the *Möbius strip*. Impediments to orientation

The following **Möbius strip** is the smooth surface M formed by the union of two surface patches M_1 and M_2 parametrized by the same functions and with adjoining domains U_1 and U_2: The Möbius strip

$$\mathbf{f}: \begin{cases} x = (5 - v\cos u)\cos 2u, \\ y = (5 - v\cos u)\sin 2u, \\ z = -v\sin u; \end{cases} \qquad U_1: \begin{array}{c} 0 \le u \le \pi/2, \\ -1 \le v \le 1; \end{array} \qquad U_2: \begin{array}{c} \pi/2 \le u \le \pi, \\ -1 \le v \le 1. \end{array}$$



The boundary points $(0, v)$ and $(\pi, -v)$ have the same image (although the figure shows them separated slightly for clarity):

$$\mathbf{f}(0, v) = (5 - v, 0, 0) = \mathbf{f}(\pi, -v).$$

In effect, the rectangles U_1 and U_2 form a single ribbon that $\mathbf{f}$ bends into a loop, joining one end to the other after giving the ribbon a half-twist. Thus, if we follow a continuously varying normal vector along the center of the ribbon, we find that the normals at the two ends, on the seam where the ends of the ribbon join, point in opposite directions.

The Möbius strip cannot be oriented

This direction reversal is shown by the parametrization normal $N_{\mathbf{f}}(u,v)$, whose components are the continuous functions

$$\begin{aligned}\frac{\partial(y,z)}{\partial(u,v)} &= -v\sin 2u - 2(5-v\cos u)\sin u\cos 2u,\\ \frac{\partial(z,x)}{\partial(u,v)} &= v\cos 2u - 2(5-v\cos u)\sin u\sin 2u,\\ \frac{\partial(x,y)}{\partial(u,v)} &= 2(5-v\cos u)\cos u.\end{aligned}$$

At each point $\mathbf{f}(0,v)=\mathbf{f}(\pi,-v)$ on the seam, the parametrization normals are in conflict:

$$N_{\mathbf{f}}(\pi,-v) = (0,-v,-2(5-v)) = -N_{\mathbf{f}}(0,v).$$

It is therefore impossible to define a nonzero normal that varies continuously over all of M: the Möbius strip cannot be oriented.

Nevertheless, it is still possible to integrate a scalar function over M. In particular, M has a well-defined area (that we can compute using a numerical integrator, for example):

$$\begin{aligned}\operatorname{area} M = \iint_M dA &= \iint_{U_1+U_2} \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(z,x)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2}\, du\,dv\\ &= \int_{-1}^{1}\int_0^{\pi} \sqrt{v^2+4(5-v\cos u)^2}\, du\, dv\\ &\approx 62.9377.\end{aligned}$$

Unit normals on a smooth surface

Now suppose that S is a smooth surface and $\mathbf{p}$ is a point in the interior of S—that is, in $S\setminus\partial S$. By definition, $\mathbf{p}$ is in the interior of a surface patch S^* in some decomposition of S. Suppose $\mathbf{f}:\Omega\to\mathbb{R}^3$ parametrizes that patch, with $U\subset\Omega$ and $\mathbf{f}(U)=S^*$. If $f(a,b)=\mathbf{p}$, then the parametrization normal

$$N_{\mathbf{f}}(a,b) = \left(\frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)}\right)_{(a,b)}$$

is nonzero and is normal to S at $\mathbf{p}$. From it we can construct the two unit normals $\pm\mathbf{n}(\mathbf{p})$ to S at $\mathbf{p}$.

Definition 10.10 *Suppose S is a smooth surface that is pathwise connected; S is said to be* ***orientable*** *if a unit normal vector $\mathbf{n}(\mathbf{p})$ can be chosen that varies continuously over all points $\mathbf{p}$ in $S \setminus \partial S$. Such an assignment is called* ***an orientation of S****; we say that* ***S is oriented****, and write $\vec{\mathbf{S}}$.*

Orientability of a smooth surface

A surface is pathwise connected if any two points can be joined by a continuous path that lies on the surface. Because an orientable surface S is pathwise connected, the orienting normal at any point determines the orienting normal at every other point, by continuity. Thus, S has two just orientations, which we can denote as $\vec{S}$ and $-\vec{S}$.

Every surface patch is orientable, by definition. Any surface patch S_i in a decomposition of a smooth oriented surface $\vec{S}$ has an orientation induced as a subset of that surface. We write

$$\vec{S} = \vec{S}_1 + \cdots + \vec{S}_k$$

to represent a decomposition of a smooth oriented surface into surface patches with their induced orientations. We can now define the total flux of of a vector field through a smooth oriented surface.

Definition 10.11 *If $\vec{S}$ is a smooth oriented surface and $\mathbb{V} = (X,Y,Z)$ is a vector field defined on $\vec{S}$, then the* ***surface integral of*** $\mathbb{V}$ ***over*** $\vec{\mathbf{S}}$ *is*

Surface integrals

$$\iint_{\vec{S}} X\,dy\,dz + Y\,dz\,dx + Z\,dx\,dy = \sum_{i=1}^{k} \iint_{\vec{S}_i} X\,dy\,dz + Y\,dz\,dx + Z\,dx\,dy,$$

where $\vec{S} = \vec{S}_1 + \cdots + \vec{S}_k$ is a decomposition into oriented surface patches.

An individual surface integral on the right is computed using a parametrization of the given oriented surface patch. However, the surface integral over $\vec{S}$ will be well defined only if the sum on the right is independent of the decompositon of $\vec{S}$ into oriented surface patches. The following theorem ensures this; it is similar to Theorem 10.8 and can be proven that same way.

Theorem 10.9. *Suppose $\vec{S} = \vec{S}_1 + \cdots + \vec{S}_k = \vec{T}_1 + \cdots + \vec{T}_m$ gives two decompositions of the smooth oriented surface S into oriented surface patches. Then*

$$\sum_{i=1}^{k} \iint_{\vec{S}_i} X\,dy\,dz + Y\,dz\,dx + Z\,dx\,dy = \sum_{j=1}^{m} \iint_{\vec{T}_j} X\,dy\,dz + Y\,dz\,dx + Z\,dx\,dy. \qquad \square$$

To illustrate, let us go back and recalculate the total flux of the field $\mathbb{V} = (Cx, Cy, Cz)$ through the unit sphere $\vec{S}$ oriented by its outward normal (Example 1, p. 396), decomposing $\vec{S}$ (as on p. 412) into an eastern belt $\vec{S}_1$, a western belt $\vec{S}_2$, a northern cap $\vec{S}_3$, and a southern cap $\vec{S}_4$. For the belts $\vec{S}_1$ and $\vec{S}_2$, we can just use the parametrization from Example 1, replacing the domain $-\pi/2 \le \varphi \le \pi/2$ by $-\alpha \le \varphi \le \alpha$:

Radial flow out of a sphere, again

$$\iint\limits_{\vec{S}_1+\vec{S}_2} Cx\,dy\,dz + Cy\,dz\,dx + Cz\,dx\,dy = \int_{-\pi}^{\pi} C\,d\theta \int_{-\alpha}^{\alpha} \cos\varphi\,d\varphi = (2\pi C)(2\sin\alpha).$$

We parametrize the northern cap $\vec{S}_3$ as

$$\mathbf{f}_3 : \begin{cases} x = u, \\ y = v, \\ z = \sqrt{1-u^2-v^2}, \end{cases} \qquad \vec{U}_3 : u^2+v^2 \le \cos^2\alpha;$$

then

$$\frac{\partial(y,z)}{\partial(u,v)} = \begin{vmatrix} 0 & 1 \\ -u/z & -v/z \end{vmatrix} = \frac{u}{z}, \quad \frac{\partial(z,x)}{\partial(u,v)} = \begin{vmatrix} -u/z & -v/z \\ 1 & 0 \end{vmatrix} = \frac{v}{z}, \quad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$$

and

$$Cx\,dy\,dz + Cy\,dz\,dx + Cz\,dx\,dy = \left(\frac{Cu^2}{z} + \frac{Cv^2}{z} + Cz\right) du\,dv = \frac{C}{z}\,du\,dv.$$

Therefore (introducing polar coordinates $u = \rho\cos\theta$, $v = \rho\sin\theta$),

$$\iint_{\vec{S}_3} Cx\,dy\,dz + Cy\,dz\,dx + Cz\,dx\,dy = \iint_{u^2+v^2 \le \cos^2\alpha} \frac{C}{\sqrt{1-u^2-v^2}}\,du\,dv$$
$$= C\int_0^{2\pi} d\theta \int_0^{\cos\alpha} \frac{\rho\,d\rho}{\sqrt{1-\rho^2}} = 2\pi C\left(-\sqrt{1-\rho^2}\,\Big|_0^{\cos\alpha}\right) = 2\pi C(1-\sin\alpha).$$

If we were to parametrize the southern cap $\vec{S}_4$ by just changing the sign of z,

$$\mathbf{g}_4 : \begin{cases} x = u, \\ y = v, \\ z = -\sqrt{1-u^2-v^2}, \end{cases} \qquad \vec{U}_4 : u^2+v^2 \le \cos^2\alpha,$$

then we would have

$$\frac{\partial(y,z)}{\partial(u,v)} = \begin{vmatrix} 0 & 1 \\ -u/z & -v/z \end{vmatrix} = \frac{u}{z}, \quad \frac{\partial(z,x)}{\partial(u,v)} = \begin{vmatrix} -u/z & -v/z \\ 1 & 0 \end{vmatrix} = \frac{v}{z}, \quad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

These are the components of the orientation normal

$$N_{\mathbf{g}_4} = \left(\frac{u}{z}, \frac{v}{z}, 1\right) = \frac{1}{z}(u,v,z) = \frac{1}{z}(x,y,z).$$

$\mathbf{g}_4$ parametrizes $-\vec{S}_4$, not $+\vec{S}_4$

But because $1/z < 0$, this is a negative multiple of the radius vector (x,y,z) at a point on $\vec{S}_4$, and hence points inward. However, the orientation of $\vec{S}$ requires an outward normal here. The remedy is to reverse u and v:

$$\mathbf{f}_4 : \begin{cases} x = v, \\ y = u, \\ z = -\sqrt{1-u^2-v^2}, \end{cases} \qquad \vec{U}_4 : u^2+v^2 \le \cos^2\alpha.$$

Now

$$\frac{\partial(y,z)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ -u/z & -v/z \end{vmatrix} = -\frac{v}{z}, \quad \frac{\partial(z,x)}{\partial(u,v)} = \begin{vmatrix} -u/z & -v/z \\ 0 & 1 \end{vmatrix} = -\frac{u}{z},$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1,$$

and

$$N_{\mathbf{f}_4} = \left(-\frac{v}{z}, -\frac{u}{z}, -1\right) = -\frac{1}{z}(v,u,z) = -\frac{1}{z}(x,y,z).$$

This a positive multiple of the radius vector (x,y,z), and hence an outward normal. Furthermore,

$$\begin{aligned} Cx\,dy\,dz + Cy\,dz\,dx + Cz\,dx\,dy &= \left(-\frac{Cv^2}{z} - \frac{Cu^2}{z} - Cz\right) du\,dv \\ &= -\frac{C}{z}\,du\,dv = +\frac{C}{\sqrt{1-u^2-v^2}}\,du\,dv, \end{aligned}$$

as with $\vec{S}_3$, so

$$\iint_{\vec{S}_4} Cx\,dy\,dz + Cy\,dz\,dx + Cz\,dx\,dy = 2\pi C(1-\sin\alpha).$$

Total flux through $\vec{S}_4$ equals total flux through $\vec{S}_3$, as is already evident by symmetry. Total flux out of the whole sphere is therefore

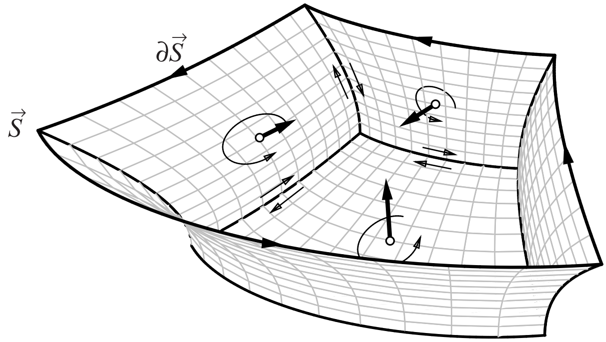
$$\sum_{i=1}^{4} \iint_{\vec{S}_i} Cx\,dy\,dz + Cy\,dz\,dx + Cz\,dx\,dy = 4\pi C,$$

Why the earlier computation worked

precisely the value we found earlier, when we assumed the whole sphere could be treated as if it were a single surface patch. Although this analysis does not justify that assumption, it does show why the earlier computation worked. As $\alpha \to \pi/2$, $\sin\alpha \to 1$; therefore, total flux through the polar caps approaches 0 and total flux through the two belts approaches $4\pi C$. These conclusions are also clear on physical grounds.

Orientation with edges and corners

At an edge or a corner, a piecewise-smooth surface does not have a well-defined normal (or a pair of linearly independent tangent vectors), so it cannot be oriented the same way as a smooth surface. However, a surface patch is always orientable, and because the boundary of any surface patch used in a decomposition of a piecewise-smooth surface is a piecewise-smooth curve, the orientation of the patch induces an orientation of its boundary. As we see in the figure below, it may be possible to orient all the surface patches in a decomposition so that their common boundary arcs have opposite orientations and thus cancel each other.



Orientability of a piecewise-smooth surface

Definition 10.12 *Suppose S is a piecewise-smooth surface that is pathwise connected, and $S = \vec{S}_1 \cup \cdots \cup \vec{S}_k$ is a decomposition into oriented surface elements. Suppose that whenever two surface elements $\vec{S}_i$ and $\vec{S}_j$ have a common boundary arc, $\partial\vec{S}_i$ and $\partial\vec{S}_j$ have opposite orientations there. Then we say **S is orientable** and is **oriented by those surface elements**. We write*

$$\vec{S} = \vec{S}_1 + \cdots + \vec{S}_k, \quad \partial\vec{S} = \partial\vec{S}_1 + \cdots + \partial\vec{S}_k.$$

The equation for $\partial\vec{S}$ reflects the cancellations that occur on the arcs that pairs of different $\partial\vec{S}_i$ have in common. The unpaired arcs that remain have a well-defined orientation and make up the oriented boundary of $\vec{S}$. The surface integral of a vector field over a piecewise-smooth oriented surface is defined exactly as for a smooth oriented surface; moreover, the definition is independent of the way the surface is decomposed into patches.

Surface integrals

Definition 10.13 *If $\vec{S}$ is a piecewise-smooth oriented surface and $\mathbb{V} = (X,Y,Z)$ is a vector field defined on $\vec{S}$, then the **surface integral of** $\mathbb{V}$ **over $\vec{\mathbf{S}}$** is*

$$\iint_{\vec{S}} X\,dy\,dz + Y\,dz\,dx + Z\,dx\,dy = \sum_{i=1}^{k} \iint_{\vec{S}_i} X\,dy\,dz + Y\,dz\,dx + Z\,dx\,dy,$$

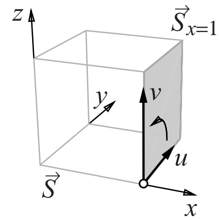
where $\vec{S} = \vec{S}_1 + \cdots + \vec{S}_k$ is a decomposition into oriented surface patches.

Theorem 10.10. *Suppose $\vec{S} = \vec{S}_1 + \cdots + \vec{S}_k = \vec{T}_1 + \cdots + \vec{T}_m$ gives two decompositions of the piecewise-smooth oriented surface S into oriented surface patches. Then*

$$\sum_{i=1}^{k} \iint_{\vec{S}_i} X\,dy\,dz + Y\,dz\,dx + Z\,dx\,dy = \sum_{j=1}^{m} \iint_{\vec{T}_j} X\,dy\,dz + Y\,dz\,dx + Z\,dx\,dy. \qquad \square$$

To illustrate, let us determine the total flux of $\mathbb{V} = (x+y, y-x, 0)$ out of the unit cube in (x,y,z)-space. Thus, we orient each face of the cube with the outward normal, show this gives us an orientation of the surface $\vec{S}$ of the entire cube, integrate $\mathbb{V}$ over each face, and then add the results.

We can parametrize the six faces with affine functions, but it takes some care to get the orientations right. To parametrize $\vec{S}_{x=1}$, the face that lies in the plane $x=1$, let $\vec{U}$ be the positively oriented unit square in the (u,v)-plane, and let $\mathbf{f}_{x=1} : \vec{U} \to \mathbb{R}^3$ be $(x,y,z) = (1,u,v)$. The orientation normal is



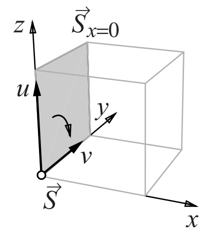
$$N_{x=1} = \left(\frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)} \right) = (1,0,0);$$

it does indeed point out of the cube on $\vec{S}_{x=1}$. Total flux through $\vec{S}_{x=1}$ is

$$\iint_{\vec{S}_{x=1}} (x+y)\,dy\,dz + (y-x)\,dz\,dx = \iint_{\vec{U}} (1+u) \times 1\,du\,dv = \int_0^1 dv \int_0^1 (1+u)\,du = \frac{3}{2}.$$

To parametrize the face $\vec{S}_{x=0}$ that lies in the plane $x=0$, suppose we were to use $\mathbf{g} : (x,y,z) = (0,u,v)$ with (u,v) again in the oriented unit square $\vec{U}$. The orientation normal is the same as on $\vec{S}_{x=1}$,

$$N_{\mathbf{g}} = (1,0,0),$$



but $\vec{S}_{x=0}$ is on the other side of the cube, so $N_{\mathbf{g}}$ points into the cube. Thus $\mathbf{g}$ gives the wrong orientation. We get the correct orientation by reversing u and v: let $\mathbf{f}_{x=0}$: $(x,y,z) = (0,v,u)$. Then

$$\frac{\partial(y,z)}{\partial(u,v)} = -1,$$

leading to an orientation normal

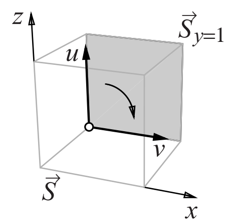
$$N_{x=0} = (-1,0,0)$$

that points out of the cube. With $\mathbf{f}_{x=0}$ we find that total flux through $\vec{S}_{x=0}$ is

$$\iint_{\vec{S}_{x=0}} (x+y)\,dy\,dz + (y-x)\,dz\,dx = \iint_{\vec{U}} (0+v) \times -1\,du\,dv = \int_0^1 du \int_0^1 -v\,dv = -\frac{1}{2}.$$

To parametrize the face $\vec{S}_{y=1}$, a bit of experimentation suggests that we use $\mathbf{f}_{y=1}$: $(x,y,z) = (v,1,u)$. Then

$$\frac{\partial(z,x)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$$



giving an orientation normal

$$N_{y=1} = (0,1,0)$$

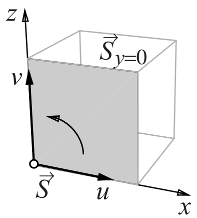
that points out of the cube. Total flux through $\vec{S}_{y=1}$ is

$$\iint_{\vec{S}_{y=1}} (x+y)\,dy\,dz + (y-x)\,dz\,dx = \iint_{\vec{U}} (1-v)\times 1\,du\,dv = \int_0^1 du \int_0^1 (1-v)\,dv = \frac{1}{2}.$$

The parametrization $\mathbf{f}_{y=0} : (x,y,z) = (u,0,v)$ of the face $\vec{S}_{y=0}$ has the orientation normal

$$N_{y=0} = (0,-1,0)$$

that points out of the cube. Total flux through $\vec{S}_{y=0}$ is



$$\iint_{\vec{S}_{y=0}} (x+y)\,dy\,dz + (y-x)\,dz\,dx = \iint_{\vec{U}} (0-u)\cdot -1\,du\,dv = \int_0^1 dv \int_0^1 u\,du = \frac{1}{2}.$$

We can parametrize $\vec{S}_{z=1}$ with $\mathbf{f}_{z=1} : (x,y,z) = (u,v,1)$ and $\vec{S}_{z=0}$ with $\mathbf{f}_{z=0} : (x,y,z) = (v,u,0)$. Then

$$N_{z=1} = (0,0,1), \quad N_{z=0} = (0,0,-1),$$

as required. However total flux is zero through both faces:

$$\iint_{\vec{S}_{z=1,0}} (x+y)\,dy\,dz + (y-x)\,dz\,dx = \iint_{\vec{U}} 0\,du\,dv = 0.$$

This is already clear because the flow is everywhere parallel to the (x,y)-plane, so $\mathbb{V}\cdot N_{z=1,0} = 0$. Addition now gives us the total flux out of the whole cubical surface $\vec{S}$:

$$\iint_{\vec{S}} (x+y)\,dy\,dz + (y-x)\,dz\,dx = 2.$$

Alternate form for a surface integral

The integral of the vector field $\mathbb{V} = (X,Y,Z)$ over an oriented piecewise-smooth surface $\vec{S}$ has the alternate form

$$\iint_S \mathbb{V}\cdot\mathbf{n}\,dA = \iint_{\vec{S}} X\,dy\,dz + Y\,dz\,dx + Z\,dx\,dy$$

that integrates the scalar function $\mathbb{V}\cdot\mathbf{n}$ over the unoriented surface S. If S is smooth, the unit normal $\mathbf{n}$ that appears here is the one that defines the orientation of $\vec{S}$. If S is only piecewise-smooth, then $\mathbf{n}$ is not defined everywhere. But if $\vec{S} = \vec{S}_1 + \cdots + \vec{S}_k$ is a decomposition into oriented surface patches, then we define

$$\iint_S \mathbb{V}\cdot\mathbf{n}\,dA = \sum_{i=1}^{k} \iint_{S_i} \mathbb{V}\cdot\mathbf{n}_i\,dA,$$

where $\mathbf{n}_i$ is the orienting unit normal on $\vec{S}_i$. (On the interior of $\vec{S}_i$, $\mathbf{n}_i$ is the orienting unit normal on $\vec{S}$, as well.)

10.3 Differential forms

The integrands of path and surface integrals, and of oriented single and double integrals, are *differential forms*. We generate new forms using algebraic operations and differentiation; in particular, these operations give us a simple connection between the forms that appear in the path and double integrals of Green's theorem. The books by H. Flanders [7] and H. Edwards [5] provide more extensive treatments of differential forms.

To fix ideas, we begin with differential forms in $\mathbb{R}^3$. In (x,y,z)-space, there are three "basic" differentials: dx, dy, and dz. A **differential *k*-form**, or an **exterior *k*-form**, or just a ***k*-form** $\alpha = \alpha(x,y,z)$, is a sum of "monomials" that contain exactly k of these differentials, as follows: Forms in $\mathbb{R}^3$

k	general k-form
0	$g(x,y,z)$
1	$P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$
2	$X(x,y,z)\,dy\,dz + Y(x,y,z)\,dz\,dx + Z(x,y,z)\,dx\,dy$
3	$H(x,y,z)\,dx\,dy\,dz$
>3	0

A general k-form is thus a linear combination of certain **basic *k*-forms**

$$1,\ dx,\ dy,\ dz,\ dy\,dz,\ dz\,dx,\ dx\,dy,\ dx\,dy\,dz;$$

we require the coefficient functions to have continuous second derivatives. A 1-form is the integrand of a path integral, so it is integrated over an oriented 1-dimensional domain. A 2-form is integrated over an oriented 2-dimensional domain, and a 3-form is integrated over an oriented 3-dimensional domain. Even a 0-form fits this pattern; see below, pages 428–429.

The sum of two k-forms is another k-form in the usual way, and the product of a k-form by a function is another k-form. We do not define the sum of a k-form and an l-form when $k \neq l$; for one thing, such a sum could not be an integrand. However, we can define the product of a k-form α and an l-form β. It is a $(k+l)$-form $\alpha \wedge \beta$, called the **exterior**, or **wedge**, **product** of α and β. On the basic differentials, the exterior product is *anticommutative*: Algebra; exterior product

$$\begin{aligned} dx \wedge dy &= -dy \wedge dx = dx\,dy, \\ dy \wedge dz &= -dz \wedge dy = dy\,dz, \\ dz \wedge dx &= -dx \wedge dz = dz\,dx, \\ dx \wedge dx = dy \wedge dy &= dz \wedge dz = 0. \end{aligned}$$

Anticommutativity implies the last line; for example, interchanging the first dx with the second gives $dx \wedge dx = -dx \wedge dx$, so $2(dx \wedge dx) = 0$.

The definition says that each basic 2-form is just an exterior product; for example, $dx\,dy$ stands for $dx \wedge dy$. For the basic 3-form, we have

$$\begin{aligned} dx\,dy\,dz &= dx \wedge dy \wedge dz = dy \wedge dz \wedge dx = dz \wedge dx \wedge dy \\ &= -dy \wedge dx \wedge dz = -dz \wedge dy \wedge dx = -dx \wedge dz \wedge dy. \end{aligned}$$

Because anticommutativity forces the exterior product of basic differentials to be zero unless they are distinct, there is no nonzero k-form in $\mathbb{R}^3$ when $k > 3$. Note that the exterior product is not anticommutative in all cases:

$$(dy\,dz) \wedge dx = dx\,dy\,dz = dx \wedge (dy\,dz).$$

For completeness, we define the exterior product with a 0-form—that is, an ordinary function $g = g(x,y,z)$—as

$$g \wedge \alpha = \alpha \wedge g = g(x,y,z)\,\alpha(x,y,z)$$

for any k-form α.

General products

We can now compute the exterior product of any two forms by using the distributive law. For the 1-forms

$$\alpha = \alpha(x,y,z) = P\,dx + Q\,dy + R\,dz, \quad \theta = \theta(x,y,z) = F\,dx + G\,dy + H\,dz,$$

we have

$$\begin{aligned} \alpha \wedge \theta &= (P\,dx + Q\,dy + R\,dz) \wedge (F\,dx + G\,dy + H\,dz) \\ &= PG\,dx \wedge dy + PH\,dx \wedge dz + QF\,dy \wedge dx \\ &\quad + QH\,dy \wedge dz + RF\,dz \wedge dx + RG\,dz \wedge dy \\ &= (QH - RG)\,dy\,dz + (RF - PH)\,dz\,dx + (PG - QF)\,dx\,dy. \end{aligned}$$

A similar calculation of $\theta \wedge \alpha$ would then show that $\theta \wedge \alpha = -\alpha \wedge \theta$. For the 2-form

$$\beta = X\,dy\,dz + Y\,dz\,dx + Z\,dx\,dy,$$

the wedge product $\alpha \wedge \beta$ has only three nonzero terms:

$$\alpha \wedge \beta = (PX + QY + RZ)\,dx\,dy\,dz = \beta \wedge \alpha.$$

Integrating a differential

Suppose $\vec{C}$ is an oriented curve that we parametrize as

$$\mathbf{x}(t) = (x(t), y(t), z(t)), \quad a \le t \le b.$$

Consider the simple 1-form $\alpha = dx$ on $\vec{C}$; we have

$$\int_{\vec{C}} \alpha = \int_{\vec{C}} dx = \int_a^b x'(t)\,dt = x(t)\Big|_a^b = x(b) - x(a),$$

which is the change in x along $\vec{C}$:

$$\int_{\vec{C}} dx = \Delta x \text{ along } \vec{C}.$$

Now let $\alpha = g_x\,dx + g_y\,dy + g_z\,dz$, where g_x, g_y, and g_z are the partial derivatives of a continuously differentiable function $g(x,y,z)$. (We call g a *potential function* for α; cf. p. 25.) Then, using the chain rule to convert $g_x x' + g_y y' + g_z z'$ into $dg(\mathbf{x}(t))/dt$, we have

$$\begin{aligned}\int_{\vec{C}} g_x\,dx + g_y\,dy + g_z\,dz &= \int_a^b (g_x x' + g_y y' + g_z z')\,dt \\ &= \int_a^b \frac{d}{dt} g(\mathbf{x}(t))\,dt = g(\mathbf{x}(t))\Big|_a^b = g(\mathbf{x}(b)) - g(\mathbf{x}(a))\end{aligned}$$

which is the change in g along $\vec{C}$. Analogy with the simple differential dx suggests we set

Differential of a function

$$g_x\,dx + g_y\,dy + g_z\,dz = dg,$$

and call this the **differential of g**, because then we have

$$\int_{\vec{C}} dg = \Delta g \text{ along } \vec{C}.$$

Exterior derivative

Using the fact that dg is defined for any 0-form g, we now define the **differential**, or **exterior derivative**, $d\alpha$ of any k-form α. There are two rules. First, if $\alpha = g \wedge \beta$, where β is a basic k-form, then

$$d\alpha = dg \wedge \beta,$$

a $(k+1)$-form. Second, for any k-forms α and ω,

$$d(\alpha \pm \omega) = d\alpha \pm d\omega.$$

It follows that the exterior derivative of any k-form is a $(k+1)$-form.

For a general 1-form $\alpha = P\,dx + Q\,dy + R\,dz$, we have

$$\begin{aligned} d\alpha &= dP \wedge dx + dQ \wedge dy + dR \wedge dz \\ &= (P_x\,dx + P_y\,dy + P_z\,dz) \wedge dx + (Q_x\,dx + Q_y\,dy + Q_z\,dz) \wedge dy \\ &\quad + (R_x\,dx + R_y\,dy + R_z\,dz) \wedge dz \\ &= (R_y - Q_z)\,dy\,dz + (P_z - R_x)\,dz\,dx + (Q_x - P_y)\,dx\,dy. \end{aligned}$$

For a general 2-form $\omega = X\,dy\,dz + Y\,dz\,dx + Z\,dx\,dy$, the calculation is briefer:

$$d\omega = (X_x + Y_y + Z_z)\,dx\,dy\,dz.$$

For a 3-form $\gamma = H\,dx\,dy\,dz$, the exterior derivative $d\gamma$ is a 4-form, and hence is automatically 0.

$d^2 = 0$

Theorem 10.11. *For every k-form α in $\mathbb{R}^3$, $d^2\alpha = d(d\alpha) = 0$.*

Proof. Suppose α is a 0-form, $\alpha = g$; then

$$\begin{aligned} d^2\alpha &= d(g_x\,dx + g_y\,dy + g_z\,dz) \\ &= (g_{xy}\,dy + g_{xz}\,dz)\,dx + (g_{yx}\,dx + g_{yz}\,dz)\,dy + (g_{zx}\,dx + g_{zy}\,dy)\,dz \\ &= (g_{zy} - g_{yz})\,dy\,dz + (g_{xz} - g_{zx})\,dz\,dx + (g_{yx} - g_{xy})\,dx\,dy \\ &= 0. \end{aligned}$$

All the coefficients vanish by the "equality of mixed partials" for functions with continuous second derivatives.

Suppose α is a 1-form: $\alpha = P\,dx + Q\,dy + R\,dz$; then (see above)

$$d\alpha = (R_y - Q_z)\,dy\,dz + (P_z - R_x)\,dz\,dx + (Q_x - P_y)\,dx\,dy.$$

This is a 2-form whose exterior derivative is

$$\begin{aligned} d^2\alpha &= \big((R_y - Q_z)_x + (P_z - R_x)_y + (Q_x - P_y)_z\big)\,dx\,dy\,dz \\ &= \big(R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz}\big)\,dx\,dy\,dz \\ &= 0. \end{aligned}$$

Again the "equality of mixed partials" implies that the coefficient vanishes. Finally, if α is a k-form with $k \geq 2$, then $d^2\alpha$ is a $(k+2)$-form and hence vanishes automatically. Thus $d^2 = 0$ on all differential forms. □

Anticommutativity in $d^2 = 0$

Note that $d^2 = 0$ is a consequence of the anticommutativity of the exterior product on basic differentials. For example, in

$$d^2 g = d(g_x\,dx + g_y\,dy + g_z\,dz),$$

the first term contributes

$$g_{xy}\,dy \wedge dx$$

and the second contributes

$$g_{yx}\,dx \wedge dy = g_{xy}\,dx \wedge dy = -g_{xy}\,dy \wedge dx.$$

The anticommutativity, in turn, is a reflection of the fact that $dy \wedge dx$ represents an oriented element of area for a double integral, and $dx \wedge dy$ represents the element of area for the opposite orientation. All these relations are nicely illustrated by differential forms in the plane.

Differential forms in $\mathbb{R}^2$

In the (x, y)-plane, there are just four "basic differentials,"

$$1, \quad dx, \quad dy, \quad dx\,dy,$$

and the general 1-form and 2-form are, respectively,

$$\alpha(x,y) = P(x,y)\,dx + Q(x,y)\,dy \quad \text{and} \quad \theta(x,y) = H(x,y)\,dx\,dy.$$

The exterior derivative is defined as for forms in $\mathbb{R}^3$; the exterior derivative of the 1-form $\alpha = P\,dx + Q\,dy$ is

$$\begin{aligned} d\alpha &= dP \wedge dx + dQ \wedge dy \\ &= (P_x\,dx + P_y\,dy) \wedge dx + (Q_x\,dx + Q_y\,dy) \wedge dy \\ &= (Q_x - P_y)\,dx\,dy. \end{aligned}$$

This means that Green's theorem,

Green's theorem: $\oint_{\partial\vec{R}} \alpha = \iint_{\vec{R}} d\alpha$

$$\oint_{\partial\vec{R}} P\,dx + Q\,dy = \iint_{\vec{R}} (Q_x - P_y)\,dx\,dy,$$

becomes, in the language of differential forms,

$$\oint_{\partial\vec{R}} \alpha = \iint_{\vec{R}} d\alpha.$$

We can write this equation in an even more striking way by regarding an oriented integral as a function, or map, that assigns a number (the value of the integral) to each pair of objects of a particular sort: the first object is an oriented k-dimensional region $\vec{D}$; the second is a k-form ω. To emphasize how an integral is the "pairing" of a region and a form, let us write it in symbolic fashion as

$$\langle \vec{D}, \omega \rangle.$$

For example, $\vec{D}$ could be the interval $[a,b]$ and $\omega(x) = g(x)\,dx$; then

$$\langle \vec{D}, \omega \rangle = \langle [a,b], g(x)\,dx \rangle = \int_a^b g(x)\,dx.$$

But $\vec{D}$ could just as well be a piecewise-smooth oriented surface in space and $\omega(x,y,z) = X\,dy\,dz + Y\,dz\,dx + Z\,dx\,dy$; then (Definition 10.13)

$$\langle \vec{D}, \omega \rangle = \iint_{\vec{D}} X\,dy\,dz + Y\,dz\,dx + Z\,dx\,dy.$$

In terms of this symbolic pairing, Green's theorem has the form

Green's theorem: $\langle \partial\vec{R}, \alpha \rangle = \langle \vec{R}, d\alpha \rangle$

$$\langle \partial\vec{R}, \alpha \rangle = \langle \vec{R}, d\alpha \rangle.$$

The operator d assigns the 2-form $d\alpha$ to the 1-form α; the operator ∂ (the symbol is a cursive "d" in the Cyrillic alphabet) assigns the 1-dimensional region $\partial\vec{R}$ to the 2-dimensional region $\vec{R}$. The symbolic content of Green's theorem is that each of these operators turns into the other when it "moves across" the pairing. In this context, we say that the operators d and ∂ are **adjoints**. The boundary operator is written as a d (albeit as a Russian ∂) because it is the adjoint of the exterior derivative, d.

The adjoint relation between ∂ and d supports the fact that $d^2 = 0$ (Theorem 10.11), because it is independently clear that $\partial^2 = 0$ (i.e., the boundary of a boundary is empty: $\partial(\partial D) = \emptyset$).

Green's theorem in dimension 1

Green's theorem, in its symbolic form $\langle \partial\vec{R}, \alpha\rangle = \langle \vec{R}, d\alpha\rangle$, has a remarkable 1-dimensional analogue. On the x-axis, there are just two basic differential forms, 1 and dx, generating the 0-forms $G(x)$ and 1-forms $g(x)\,dx$. In the fundamental theorem of calculus,

$$\int_a^b G'(x)\,dx = G(b) - G(a),$$

the left-hand side is the integral of the 1-form $d\alpha = G'\,dx$, where α itself is the 0-form $\alpha = G$. Can we make the right-hand side into a kind of "0-dimensional integral" of α?

0-dimensional regions

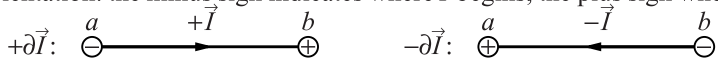
The basic 0-dimensional object is a single point. But the fundamental theorem involves a pair of points, the boundary points of $[a,b]$. To include this case, we take a 0-dimensional "region" to be any finite collection of points $D : \{a_1, a_2, \ldots, a_n\}$. Because a 0-form $G(x)$ takes a value on each of these points—and integrals represent sums—one possibility is to define the symbolic 0-dimensional integral to be

$$\langle D, G\rangle = G(a_1) + G(a_2) + \cdots + G(a_n).$$

However, this fails to produce the minus sign we see in $G(b) - G(a)$.

Orientation

To get the needed minus sign, we introduce orientation. Because $\vec{I} = [a,b]$ is itself oriented, we say it induces the orientation $\{-a, +b\}$ on $\partial\vec{I}$. The signs convey the orientation: the minus sign indicates where $\vec{I}$ begins; the plus sign where it ends.



The oppositely oriented $-\vec{I}$ induces the correct orientation of $-\partial\vec{I} : \{+a, -b\}$, regarded as $\partial\vec{I}$ with the opposite orientation. We convey the same information about $\partial\vec{I}$ if we write it not as a set but as a sum:

$$\partial\vec{I} = -a + b.$$

To see the advantage of this change, let

$$\vec{J} = [b,c], \quad \vec{K} = \vec{I} + \vec{J} = [a,b] + [b,c] = [a,c].$$

Then $\partial\vec{J} = -b + c$ and

$$\partial\vec{I} + \partial\vec{J} = -a + b - b + c = -a + c = \partial\vec{K}.$$

Also, $\partial(\vec{I} - \vec{I}) = -a + b + a - b = 0$, which is consistent with $\vec{I} - \vec{I} = 0$. We can therefore convert a general $D : \{a_1, \ldots, a_l\}$ into an oriented 0-dimensional region by attaching an integer m_i to each a_i, and writing the result as

$$\vec{D} = m_1 a_1 + \cdots + m_l a_l.$$

If we change either a point a_i or a "weight" m_i, then $\vec{D}$ is a different oriented 0-dimensional region. Finally, if G is a 0-form, we define the **0-dimensional oriented integral**

$$\langle \vec{D}, G \rangle = m_1 G(a_1) + \cdots + m_l G(a_l).$$

In these terms, the fundamental theorem of calculus takes the form

Fundamental theorem: $\langle \vec{I}, d\alpha \rangle = \langle \partial \vec{I}, \alpha \rangle$

$$\langle \vec{I}, d\alpha \rangle = \langle \partial \vec{I}, \alpha \rangle,$$

where $\alpha = G$ is a 0-form and $\vec{I}$ is an oriented interval. The fundamental theorem and Green's theorem thus make the same assertion about a k-dimensional region and a k-form, only for different values of k. They are instances of a more general result, called Stokes' theorem, that we consider in the next chapter.

Differential forms in polar coordinates

What happens to a differential form under a change of variables? For example, consider the change to polar coordinates with

$$\boldsymbol{\varphi} : \begin{cases} x = r\cos\theta, \\ y = r\sin\theta. \end{cases}$$

Then, because we can treat x and y as functions of r and θ,

$$\begin{cases} dx = \cos\theta\, dr - r\sin\theta\, d\theta, \\ dy = \sin\theta\, dr + r\cos\theta\, d\theta. \end{cases}$$

The element of area, $dx\,dy$, is transformed into

$$\begin{aligned} dx\,dy &= (\cos\theta\, dr - r\sin\theta\, d\theta) \wedge (\sin\theta\, dr + r\cos\theta\, d\theta) \\ &= r\cos^2\theta\; dr \wedge d\theta - r\sin^2\theta\; d\theta \wedge dr \\ &= r\,dr\,d\theta, \end{aligned}$$

the element of area in polar coordinates. For a second example, consider the 1-form

$$\alpha(x,y) = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy.$$

If we use $\boldsymbol{\varphi}^*\alpha(r,\theta)$ to denote the new form after the polar coordinate change $\boldsymbol{\varphi}$ is applied, then

$$\begin{aligned} \boldsymbol{\varphi}^*\alpha(r,\theta) &= \frac{-r\sin\theta}{r^2}(\cos\theta\, dr - r\sin\theta\, d\theta) + \frac{r\cos\theta}{r^2}(\sin\theta\, dr + r\cos\theta\, d\theta) \\ &= \frac{-\sin\theta\cos\theta + \sin\theta\cos\theta}{r}\,dr + (\sin^2\theta + \cos^2\theta)\,d\theta \\ &= d\theta. \end{aligned}$$

In other words, $\alpha(x,y)$ is the differential of the function

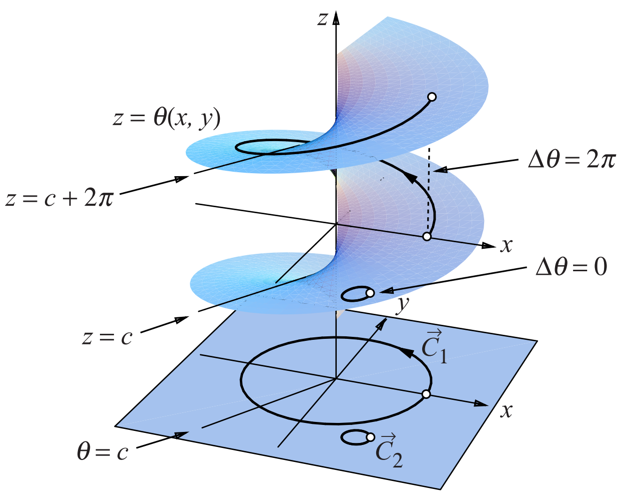
$$\theta(x,y) = \arctan\frac{y}{x}, \quad (x,y) \neq (0,0),$$

as we can verify directly:

$$\frac{\partial\theta}{\partial x} = \frac{-y/x^2}{1+(y/x)^2} = \frac{-y}{x^2+y^2}, \quad \frac{\partial\theta}{\partial y} = \frac{1/x}{1+(y/x)^2} = \frac{x}{x^2+y^2}.$$

The graph of $\theta(x,y)$ is an infinite spiral ramp

The graph of the function $z = \theta(x,y)$ is a spiral ramp, as shown below. The z-axis is not part of the graph, because $(x,y) \neq (0,0)$. The ray $\theta = c$ in the (x,y)-plane is carried to the level $z = c$ but also to $z = c + 2n\pi$ for every integer n. The polar angle θ is therefore *multiple-valued* in a particular way; the graph reflects this by spiraling around the origin infinitely many times, with successive levels separated by $\Delta z = 2\pi$.



Because $\alpha = d\theta$,

$$\int_{\vec{C}} \alpha = \int_{\vec{C}} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = \Delta\theta \text{ on } \vec{C}.$$

Ordinarily, the integral of the differential dg of a function g is zero on a closed path, because $\Delta g = 0$ there. The same is true of the integral of $d\theta$ on a path like $\vec{C}_2$ that does not enclose the origin. However, on a path like $\vec{C}_1$ that does enclose the origin, $z = \theta$ does not return to its starting value (it "climbs the ramp"), and $\Delta\theta \neq 0$.

Winding number

Definition 10.14 *Let $\vec{C}$ be an oriented closed path that does not meet the origin; the **winding number of $\vec{C}$** is*

$$W(\vec{C}) = \frac{1}{2\pi}\int_{\vec{C}} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy.$$

By what we've said, $W(\vec{C}) = \Delta\theta/2\pi$ is an integer; it counts the net number of times $\vec{C}$ "winds around" the origin in the positive sense minus the number of times in the negative sense. Furthermore, $W(-\vec{C}) = -W(\vec{C})$.

How a mapping transforms a k-form

We now want to determine how a differential form is transformed when a map introduces new variables (possibly more general than an invertible change of variables). First, consider a map from (an open set in) $\mathbb{R}^2$ to $\mathbb{R}^2$:

$$\mathbf{f}: \begin{cases} x = x(u,v), \\ y = y(u,v), \end{cases} \qquad \begin{cases} dx = x_u\,du + x_v\,dv, \\ dy = y_u\,du + y_v\,dv. \end{cases}$$

We assume that $\mathbf{f}$ is continuously differentiable, but do not assume, for the moment, that it is invertible. In other words, $\mathbf{f}$ need not be a coordinate change. A general 0-form $\alpha(x,y) = g(x,y)$ is transformed into

$$\mathbf{f}^*\alpha(u,v) = g^*(u,v) = g(x(u,v), y(u,v)).$$

A general 1-form $\alpha(x,y) = P(x,y)\,dx + Q(x,y)\,dy$ is transformed into

$$\begin{aligned} \mathbf{f}^*\alpha(u,v) &= P(x(u,v),y(u,v))(x_u\,du + x_v\,dv) \\ &\qquad + Q(x(u,v),y(u,v))(y_u\,du + y_v\,dv) \\ &= (P^*x_u + Q^*y_u)\,du + (P^*x_v + Q^*y_v)\,dv. \end{aligned}$$

For some purposes, the functions in differential forms should have continuous second derivatives. For $\mathbf{f}^*\alpha$ to meet that requirement, the components of $\mathbf{f}$ should have continuous third derivatives, because $\mathbf{f}^*\alpha$ contains the first derivatives of those components.

The basic 2-form $\alpha(x,y) = dx\,dy$ is transformed into

$$\begin{aligned} \mathbf{f}^*\alpha(u,v) &= (x_u\,du + x_v\,dv) \wedge (y_u\,du + y_v\,dv) \\ &= x_u y_v\,du \wedge dv + x_v y_u\,dv \wedge du = \frac{\partial(x,y)}{\partial(u,v)}\,du\,dv. \end{aligned}$$

Therefore, if $\alpha(x,y) = g(x,y)\,dx\,dy$, the general 2-form in $\mathbb{R}^2$, then

$$\mathbf{f}^*\alpha(u,v) = g^*(u,v)\,\frac{\partial(x,y)}{\partial(u,v)}\,du\,dv.$$

The pullback on differential forms

Notice that, although $\mathbf{f}$ maps the (u,v)-plane to the (x,y)-plane, the map $\mathbf{f}^*$ "goes the other way:" it maps differential forms on the (x,y)-plane to differential forms on the (u,v)-plane. Thus $\mathbf{f}^*$ **pulls back** forms from the (x,y)-plane to the (u,v)-plane; we call it the **pullback on forms** defined by $\mathbf{f}$.

$$\begin{array}{ccc} \text{forms in } (u,v) & \xleftarrow{\;\mathbf{f}^*\;} & \text{forms in } (x,y) \\ (u,v) & \xrightarrow{\;\mathbf{f}\;} & (x,y) \end{array}$$

Pulling back a map $\mathbf{f}:\mathbb{R}^2\to\mathbb{R}^3$

The map $\mathbf{f}$ need not preserve dimension; indeed, $\mathbf{f}:\mathbb{R}^2\to\mathbb{R}^3$ is an important case:

$$\mathbf{f}:\begin{cases} x = x(u,v), \\ y = y(u,v), \\ z = z(u,v). \end{cases} \qquad \begin{cases} dx = x_u\,du + x_v\,dv, \\ dy = y_u\,du + y_v\,dv, \\ dz = z_u\,du + z_v\,dv. \end{cases}$$

Pullbacks of 0- and 1-forms are similar to those for maps from $\mathbb{R}^2$ to $\mathbb{R}^2$. To be specific, if $\alpha(x,y,z) = g(x,y,z)$, then

$$\mathbf{f}^*\alpha(u,v) = g^*(u,v) = g(x(u,v),y(u,v),z(u,v));$$

and if $\alpha = P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$, then

$$\mathbf{f}^*\alpha(u,v) = (P^*x_u + Q^*y_u + R^*z_u)\,du + (P^*x_v + Q^*y_v + R^*z_v)\,dv.$$

There are similarities for 2-forms, as well; we just need to take into account that there are now three basic 2-forms in $\mathbb{R}^3$: $dx\,dy$, $dy\,dz$, and $dz\,dx$. However, in $\mathbb{R}^2$ there is only one basic 2-form, so the previous case of maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ need merely be applied three times:

$$\mathbf{f}^*dx\,dy = \frac{\partial(x,y)}{\partial(u,v)}\,du\,dv, \quad \mathbf{f}^*dy\,dz = \frac{\partial(y,z)}{\partial(u,v)}\,du\,dv, \quad \mathbf{f}^*dz\,dx = \frac{\partial(z,x)}{\partial(u,v)}\,du\,dv.$$

Thus, for the general 2-form $\alpha = X\,dy\,dz + Y\,dz\,dx + Z\,dx\,dy$, the pullback is

$$\begin{aligned} \mathbf{f}^*\alpha &= \left(X^*\frac{\partial(y,z)}{\partial(u,v)} + Y^*\frac{\partial(z,x)}{\partial(u,v)} + Z^*\frac{\partial(x,y)}{\partial(u,v)}\right)du\,dv \\ &= \left(X(\mathbf{f}(u,v))\frac{\partial(y,z)}{\partial(u,v)} + Y(\mathbf{f}(u,v))\frac{\partial(z,x)}{\partial(u,v)} + Z(\mathbf{f}(u,v))\frac{\partial(x,y)}{\partial(u,v)}\right)du\,dv. \end{aligned}$$

The final case to consider is the general 3-form $\alpha = H(x,y,z)\,dx\,dy\,dz$, but its pullback is zero because every 3-form in two variables must reduce to zero.

Surface integrals reformulated

The language of differential forms and pullbacks gives us a vivid and succinct way to reformulate the definition of a surface integral (Definition 10.4, p. 402). Thus we are given an oriented surface patch $\vec{S}$ and a 2-form

$$\omega = X\,dy\,dz + Y\,dz\,dx + Z\,dx\,dy$$

that is defined and continuous on $\vec{S}$. If $\mathbf{f}:\Omega^2\to\mathbb{R}^3:\vec{U}^2\to\vec{S}$ parametrizes $\vec{S}$ (Definition 10.2, p. 392), then

$$\iint_{\vec{S}}\omega = \iint_{\mathbf{f}(\vec{U})}\omega \quad \text{is by definition equal to } \iint_{\vec{U}}\mathbf{f}^*\omega.$$

The pullback on $\mathbb{R}^3$

Let us now determine the pullback map for a continuously differentiable map from $\mathbb{R}^3$ to $\mathbb{R}^3$:

$$\mathbf{f}:\begin{cases} x = x(u,v,w), \\ y = y(u,v,w), \\ z = z(u,v,w), \end{cases} \qquad \begin{cases} dx = x_u\,du + x_v\,dv + x_w\,dw, \\ dy = y_u\,du + y_v\,dv + y_w\,dw, \\ dz = z_u\,du + z_v\,dv + z_w\,dw. \end{cases}$$

Again, pullbacks of 0- and 1-forms are similar to what we have already seen. For the general 0-form $\alpha(x,y,z) = g(x,y,z)$,

$$\mathbf{f}^* g = g^*(u,v,w) = g(x(u,v,w), y(u,v,w), z(u,v,w));$$

for the general 1-form $\alpha = P\,dx + Q\,dy + R\,dz$,

$$\begin{aligned} \mathbf{f}^*\alpha(u,v,w) = {} & (P^* x_u + Q^* y_u + R^* z_u)\,du + (P^* x_v + Q^* y_v + R^* z_v)\,dv \\ & + (P^* x_w + Q^* y_w + R^* z_w)\,dw. \end{aligned}$$

With 2-forms, there are complications we have not seen before, because there are three basic 2-forms in the source. We begin with the basic 2-form $dx\,dy$ in the target, and write

$$\begin{aligned} \mathbf{f}^* dx\,dy &= (x_u\,du + x_v\,dv + x_w\,dw) \wedge (y_u\,du + y_v\,dv + y_w\,dw) \\ &= x_u y_v\,du \wedge dv + x_u y_w\,du \wedge dw + x_v y_u\,dv \wedge du \\ &\quad + x_v y_w\,dv \wedge dw + x_w y_u\,dw \wedge du + x_w y_v\,dw \wedge dv \\ &= \frac{\partial(x,y)}{\partial(u,v)}\,du\,dv + \frac{\partial(x,y)}{\partial(v,w)}\,dv\,dw + \frac{\partial(x,y)}{\partial(w,u)}\,dw\,du. \end{aligned}$$

Using similar results for the other basic 2-forms (i.e., $dy\,dz$ and $dz\,dx$; see the exercises), we find that the general 2-form

$$\alpha = X\,dy\,dz + Y\,dz\,dx + Z\,dx\,dy$$

is transformed into

$$\begin{aligned} \mathbf{f}^*\alpha(u,v,w) = {} & \left(X^* \frac{\partial(y,z)}{\partial(v,w)} + Y^* \frac{\partial(z,x)}{\partial(v,w)} + Z^* \frac{\partial(x,y)}{\partial(v,w)} \right) dv\,dw \\ & + \left(X^* \frac{\partial(y,z)}{\partial(w,u)} + Y^* \frac{\partial(z,x)}{\partial(w,u)} + Z^* \frac{\partial(x,y)}{\partial(w,u)} \right) dw\,du \\ & + \left(X^* \frac{\partial(y,z)}{\partial(u,v)} + Y^* \frac{\partial(z,x)}{\partial(u,v)} + Z^* \frac{\partial(x,y)}{\partial(u,v)} \right) du\,dv. \end{aligned}$$

The pullback of the general 3-form $\alpha = H\,dx\,dy\,dz$ is straightforward:

$$\mathbf{f}^*\alpha(u,v,w) = H^* \frac{\partial(x,y,z)}{\partial(u,v,w)}\,du\,dv\,dw.$$

Comparing integrals of α and $\boldsymbol{\varphi}^*\alpha$

Differential forms are involved in the calculation of physical quantities (e.g., work and total flux) whose values should be independent of the coordinate frames in which they are computed. This prompts us to determine how the integrals of α and its pullback $\boldsymbol{\varphi}^*\alpha$ are related when $\boldsymbol{\varphi}$ is a coordinate change, that is, an invertible map with a continuously differentiable inverse.

Transforming the integral of a 1-form

Theorem 10.12. *If* $\alpha(\mathbf{x}) = P\,dx + Q\,dy + R\,dz$ *is a 1-form,* $\vec{C}$ *is an oriented curve in* $\mathbf{u}$*-space, and* $\mathbf{x} = \boldsymbol{\varphi}(\mathbf{u})$ *is a coordinate change, then*

$$\int_{\boldsymbol{\varphi}(\vec{C})} \alpha = \int_{\vec{C}} \boldsymbol{\varphi}^*\alpha.$$

Proof. For simplicity, take $\alpha = P\,dx$; the other terms $Q\,dy$ and $R\,dz$ can be handled the same way. Then we know that

$$\boldsymbol{\varphi}^*\alpha(\mathbf{u}) = P^* x_u\,du + P^* x_v\,dv + P^* x_w\,dw,$$

where $P^*(\mathbf{u}) = P(\boldsymbol{\varphi}(\mathbf{u}))$. Let $\vec{C}$ be parametrized as $(u,v,w) = \mathbf{u}(t)$, with $a \le t \le b$. Then $\boldsymbol{\varphi}(\vec{C})$ is parametrized by $(x,y,z) = \boldsymbol{\varphi}(\mathbf{u}(t))$, $a \le t \le b$, and we have

$$\begin{aligned}\int_{\boldsymbol{\varphi}(\vec{C})} \alpha &= \int_a^b P(\boldsymbol{\varphi}(\mathbf{u}(t))) \cdot \frac{d}{dt} x(\mathbf{u}(t))\,dt \\ &= \int_a^b P^*(\mathbf{u}(t))(x_u \cdot u' + x_v \cdot v' + x_w \cdot w')\,dt = \int_{\vec{C}} \boldsymbol{\varphi}^*\alpha. \qquad \square\end{aligned}$$

An abbreviated version of the same proof shows that the theorem is true for 1-forms $\alpha(x,y)$ on the plane.

Transforming the integral of a 2-form

If $\alpha(x,y) = g\,dx\,dy$ is a 2-form, $\vec{D}$ an oriented region in the (u,v)-plane, and $\boldsymbol{\varphi}$ is an invertible coordinate change, then

$$\iint_{\boldsymbol{\varphi}(\vec{D})} \alpha = \iint_{\boldsymbol{\varphi}(\vec{D})} g\,dx\,dy = \iint_{\vec{D}} g^* \frac{\partial(x,y)}{\partial(u,v)}\,du\,dv = \iint_{\vec{D}} \boldsymbol{\varphi}^*\alpha.$$

The second equality is the change of variables formula for oriented double integrals (Theorem 9.14, p. 357). The next theorem deals with 2-forms in space.

Theorem 10.13. . *If* $\alpha(\mathbf{x}) = X\,dy\,dz + Y\,dz\,dx + Z\,dx\,dy$ *is a 2-form,* $\vec{S}$ *is an oriented surface patch in* $\mathbf{u}$*-space, and* $\mathbf{x} = \boldsymbol{\varphi}(\mathbf{u})$ *is a coordinate change, then* $\boldsymbol{\varphi}(\vec{S})$ *is an oriented surface patch in* $\mathbf{x}$*-space and*

$$\iint_{\boldsymbol{\varphi}(\vec{S})} \alpha = \iint_{\vec{S}} \boldsymbol{\varphi}^*\alpha.$$

Proof. Because $\vec{S}$ is an oriented surface patch in (u,v,w)-space, it has a parametrization $\mathbf{f} : \Omega \to \mathbb{R}^3 : (s,t) \to (u,v,w)$ with $\vec{S} = \mathbf{f}(\vec{U})$ for some closed, bounded, positively oriented set $\vec{U} \subset \Omega$ with area (Definition 10.2, p. 392). Because $\boldsymbol{\varphi}$ is a coordinate change in $\mathbb{R}^3$, the map

$$\boldsymbol{\varphi} \circ \mathbf{f} : \Omega \to \mathbb{R}^3 : (s,t) \to (x,y,z)$$

serves to parametrize $\boldsymbol{\varphi}(\vec{S}) = (\boldsymbol{\varphi}\circ\mathbf{f})(\vec{U})$, which is therefore an oriented surface patch in (x,y,z)-space. The two surface integrals that appear in the theorem can therefore be defined using the pullbacks of $\mathbf{f}$ and $\boldsymbol{\varphi}\circ\mathbf{f}$. The following lemma indicates how these pullbacks are related.

Lemma 10.1. *For any 2-form α, $(\boldsymbol{\varphi}\circ\mathbf{f})^*\alpha = (\mathbf{f}^*\circ\boldsymbol{\varphi}^*)\alpha = \mathbf{f}^*(\boldsymbol{\varphi}^*\alpha)$.*

Proof. For simplicity, take $\alpha = X\,dy\,dz$; the other terms $Y\,dz\,dx$ and $Z\,dx\,dy$ can be analyzed similarly. We have

$$(\boldsymbol{\varphi}\circ\mathbf{f})^*\alpha(\mathbf{s}) = X(\boldsymbol{\varphi}(\mathbf{f}(\mathbf{s})))\,\frac{\partial(y,z)}{\partial(s,t)}\,ds\,dt.$$

We also have

$$\boldsymbol{\varphi}^*\alpha(\mathbf{u}) = X(\boldsymbol{\varphi}(\mathbf{u}))\left(\frac{\partial(y,z)}{\partial(v,w)}\,dv\,dw + \frac{\partial(y,z)}{\partial(w,u)}\,dw\,du + \frac{\partial(y,z)}{\partial(u,v)}\,du\,dv\right),$$

from which it follows that

$$\begin{aligned}&\mathbf{f}^*(\boldsymbol{\varphi}^*\alpha)(\mathbf{s})\\ &= X(\boldsymbol{\varphi}(\mathbf{f}(\mathbf{s})))\left(\frac{\partial(y,z)}{\partial(v,w)}^{*}\frac{\partial(v,w)}{\partial(s,t)} + \frac{\partial(y,z)}{\partial(w,u)}^{*}\frac{\partial(w,u)}{\partial(s,t)} + \frac{\partial(y,z)}{\partial(u,v)}^{*}\frac{\partial(u,v)}{\partial(s,t)}\right)ds\,dt.\end{aligned}$$

The three Jacobians marked with asterisks (which are usually not written explicitly) are understood to be functions of s and t via pullbacks. By Exercise 10.27,

$$\frac{\partial(y,z)}{\partial(v,w)}\frac{\partial(v,w)}{\partial(s,t)} + \frac{\partial(y,z)}{\partial(w,u)}\frac{\partial(w,u)}{\partial(s,t)} + \frac{\partial(y,z)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(s,t)} = \frac{\partial(y,z)}{\partial(s,t)}. \qquad \square$$

To complete the proof of the theorem, we use the lemma and twice invoke the new formulation (p. 432) of the definition of a surface integral as the ordinary double integral of a pullback:

$$\iint\limits_{\boldsymbol{\varphi}(\vec{S})} \alpha = \iint\limits_{\boldsymbol{\varphi}(\mathbf{f}(\vec{U}))} \alpha = \iint\limits_{\vec{U}} (\boldsymbol{\varphi}\circ\mathbf{f})^*\alpha = \iint\limits_{\vec{U}} \mathbf{f}^*\circ\boldsymbol{\varphi}^*\alpha = \iint\limits_{\mathbf{f}(\vec{U})} \boldsymbol{\varphi}^*\alpha = \iint\limits_{\vec{S}} \boldsymbol{\varphi}^*\alpha. \qquad \square$$

Corollary 10.14 *Suppose that $\vec{S}$ is a piecewise-smooth oriented surface; then so is $\boldsymbol{\varphi}(\vec{S})$, and*

Allow $\vec{S}$ to be piecewise smooth

$$\iint_{\boldsymbol{\varphi}(\vec{S})} \alpha = \iint_{\vec{S}} \boldsymbol{\varphi}^*\alpha.$$

Proof. Let $\vec{S} = \vec{S}_1 + \cdots + \vec{S}_k$ be a decomposition into oriented surface patches, and suppose $\vec{S}_i$ is parametrized by $\mathbf{f}_i : \Omega_i \to \mathbb{R}^3$, with $\mathbf{f}_i(\vec{U}_i) = \vec{S}_i$. Then, by the proof of the theorem, $\boldsymbol{\varphi}(\vec{S}_i)$ is an oriented surface patch parametrized by $\boldsymbol{\varphi}\circ\mathbf{f}_i$. Therefore, because $\boldsymbol{\varphi}$ is 1–1,

$$\boldsymbol{\varphi}(\vec{S}) = \boldsymbol{\varphi}(\vec{S}_1) + \cdots + \boldsymbol{\varphi}(\vec{S}_k)$$

is a piecewise-smooth oriented surface. Finally, using the theorem on each pair of surface patches $\boldsymbol{\varphi}(\vec{S}_i)$ and $\vec{S}_i$, we obtain

$$\iint\limits_{\boldsymbol{\varphi}(\vec{S})} \alpha = \sum_{i=1}^{k} \iint\limits_{\boldsymbol{\varphi}(\vec{S}_i)} \alpha = \sum_{i=1}^{k} \iint\limits_{\vec{S}_i} \boldsymbol{\varphi}^* \alpha = \iint\limits_{\vec{S}} \boldsymbol{\varphi}^* \alpha. \qquad \square$$

Transforming the integral of a 3-form

The final possibility we must consider is how the integral of a 3-form $\alpha = H(\mathbf{x})\,dx\,dy\,dz$ transforms under a coordinate change $\mathbf{x} = \boldsymbol{\varphi}(\mathbf{u})$. We know

$$\boldsymbol{\varphi}^* \alpha = H^*(\mathbf{u}) \frac{\partial(x,y,z)}{\partial(u,v,w)}\,du\,dv\,dw,$$

where $H^*(\mathbf{u}) = H(\boldsymbol{\varphi}(\mathbf{u}))$. The integrals here are triple integrals over an oriented region $\vec{D}$ in $\mathbf{u}$-space and its image $\boldsymbol{\varphi}(\vec{D})$ in $\mathbf{x}$-space. The change of variables formula for oriented triple integrals (Theorem 9.16, p. 363) yields

$$\begin{aligned} \iiint_{\boldsymbol{\varphi}(\vec{D})} \alpha &= \iiint_{\boldsymbol{\varphi}(\vec{D})} H(\mathbf{x})\,dx\,dy\,dz \\ &= \iiint_{\vec{D}} H(\boldsymbol{\varphi}(\mathbf{x})) \frac{\partial(x,y,z)}{\partial(u,v,w)}\,du\,dv\,dw = \iiint_{\vec{D}} \boldsymbol{\varphi}^* \alpha. \end{aligned}$$

Theorem 10.15. *The integral of a differential k-form in n variables (where $k \le n$ and $n = 1,2,3$) is invariant under a coordinate change.* $\square$

$\boldsymbol{\varphi}$ and $\boldsymbol{\varphi}^*$ are adjoints

In terms of the symbolic integral pairing (p. 427), all the statements about the invariance of integrals under coordinate changes have the form

$$\langle \boldsymbol{\varphi}(\vec{D}), \alpha \rangle = \langle \vec{D}, \boldsymbol{\varphi}^* \alpha \rangle,$$

where α is a k-form in n variables, $n = 1,2,3$. In other words, the map $\boldsymbol{\varphi}$ and its pullback $\boldsymbol{\varphi}^*$ are adjoints (p. 428). *This is the essential content of the change of variables formulas* (wherein we take $k = n$). Incidentally, it is easy to check that the pairings are also equal when $n = 0$.

d and $\mathbf{f}^*$ commute

How does exterior differentiation interact with a pullback? Does it make a difference if we apply the exterior derivative before or after applying a map? In other words, do d and $\mathbf{f}^*$ commute? To explore this question, let us return to differential forms in just two variables. First consider the 0-form $\alpha = g(x,y)$; then

$$\mathbf{f}^* \alpha = g(x(u,v), y(u,v)),$$

so

$$\begin{aligned}
d(\mathbf{f}^*\alpha) &= \frac{\partial}{\partial u} g(x(u,v),y(u,v))\,du + \frac{\partial}{\partial v} g(x(u,v),y(u,v))\,dv \\
&= \big(g_1(x(u,v),y(u,v))x_u + g_2(x(u,v),y(u,v))y_u\big)\,du \\
&\quad + \big(g_1(x(u,v),y(u,v))x_v + g_2(x(u,v),y(u,v))y_v\big)\,dv \\
&= (g_1^* x_u + g_2^* y_u)\,du + (g_1^* x_v + g_2^* y_v)\,dv.
\end{aligned}$$

Here g_i is the partial derivative of $g(x,y)$ with respect to its ith variable, and $g_i^* = (g_i)^* = \mathbf{f}^*(g_i)$ (so $g_i^* \neq (g^*)_i$, in general). On the other hand, $d\alpha = g_1\,dx + g_2\,dy$, and

$$\begin{aligned}
\mathbf{f}^*(d\alpha) &= g_1^* \cdot (x_u\,du + x_v\,dv) + g_2^* \cdot (y_u\,du + y_v\,dv) \\
&= (g_1^* x_u + g_2^* y_u)\,du + (g_1^* x_v + g_2^* y_v)\,dv \\
&= d(\mathbf{f}^*\alpha).
\end{aligned}$$

Next, consider the 1-form $\alpha = P\,dx$; then

$$\mathbf{f}^*\alpha = P^* x_u\,du + P^* x_u\,dv,$$

so

$$\begin{aligned}
d(\mathbf{f}^*\alpha) &= \frac{\partial}{\partial v}\big(P^* x_u\big)\,dv \wedge du + \frac{\partial}{\partial u}\big(P^* x_v\big)\,du \wedge dv \\
&= \big[-(P_1^* x_v + P_2^* y_v)x_u - P^* x_{uv} + (P_1^* x_u + P_2^* y_u)x_v + P^* x_{vu}\big]\,du\,dv \\
&= -P_2^*(y_v x_u - y_u x_v)\,du\,dv \\
&= -P_2^* \frac{\partial(x,y)}{\partial(u,v)}\,du\,dv.
\end{aligned}$$

On the other hand, $d\alpha = -P_2\,dx\,dy$, and (from p. 431)

$$\mathbf{f}^*(d\alpha) = -P_2^* \frac{\partial(x,y)}{\partial(u,v)}\,du\,dv = d(\mathbf{f}^*\alpha).$$

Analysis of $\alpha = Q\,dy$ is similar. If $\alpha(x,y)$ is a k-form with $k \geq 2$, then $d\alpha = 0 = d(\mathbf{f}^*\alpha)$.

Theorem 10.16. *For any differentiable map $(x,y) = \mathbf{f}(u,v)$ and k-form $\alpha(x,y)$, $\mathbf{f}^*(d\alpha) = d(\mathbf{f}^*\alpha)$.* □

In fact, this theorem holds for differential forms $\alpha(x_1,\ldots,x_n)$ in any number of variables. In Exercise 10.28, you are asked to give a proof for $n = 3$.

The change of variables formula with integral pairings

The language of differential forms and symbolic pairings for integrals gives us a new way to look at the proofs of some earlier theorems. For example, consider the change of variables via Green's theorem (Theorem 9.21, p. 370):

Suppose $\mathbf{f}(s,t) = (x(s,t),y(s,t))$ has continuous second derivatives on a bounded open set Ω in $\mathbb{R}^2$. Let $\vec{S} \subset \Omega$ and $\vec{T} = \mathbf{f}(\vec{S})$ be closed oriented sets whose boundaries $\partial\vec{S}$ and $\partial\vec{T}$ are simple closed curves. Assume that Green's theorem holds for both

$\vec{S}$ and $\vec{T}$, that $\partial\vec{S}$ and $\mathbf{f}(\partial\vec{S})$ have common decompositions into smooth oriented curves, and that $\mathbf{f}(\partial\vec{S}) = k\cdot\partial\vec{T}$ as oriented paths. Then, for any continuous function $g(x,y)$ on $\vec{T}$,

$$k\iint_{\vec{T}} g(x,y)\,dx\,dy = \iint_{\vec{S}} g(x(s,t),y(s,t))\,\frac{\partial(x,y)}{\partial(s,t)}\,ds\,dt.$$

Proof. As in the original proof, choose $G(x,y)$ so that $G_x(x,y) = g(x,y)$, and let $\alpha = G\,dy$, $d\alpha = g\,dx\,dy$. A key step there was to transfer the path integral of α over $\mathbf{f}(\partial\vec{S})$ to the path integral of

$$\mathbf{f}^*\alpha(s,t) = G(x(s,t),y(s,t))(y_s\,ds + y_t\,dt) = G^*y_s\,ds + G^*y_t\,dt$$

over $\partial\vec{S}$. In the language of symbolic pairings, $\langle\mathbf{f}(\partial\vec{S}),\alpha\rangle = \langle\partial\vec{S},\mathbf{f}^*\alpha\rangle$, indicating that $\mathbf{f}$ and $\mathbf{f}^*$ are adjoints. The original proof invoked the results of an exercise. For future use, we restate these results in the language of differential forms and pullbacks.

$\mathbf{f}$ and $\mathbf{f}^*$ are adjoints on 1-forms

Lemma 10.2. *Let $\mathbf{f}: U^2 \to \mathbb{R}^2$ be continuously differentiable, and suppose $\vec{C}$ and $\mathbf{f}(\vec{C})$ are piecewise-smooth oriented curves with a common decomposition into smooth oriented curves:*

$$\vec{C} = \vec{C}_1 + \cdots + \vec{C}_m, \quad \mathbf{f}(\vec{C}) = \mathbf{f}(\vec{C}_1) + \cdots + \mathbf{f}(\vec{C}_m).$$

Then, for any 1-form α, $\langle\mathbf{f}(\vec{C}),\alpha\rangle = \langle\vec{C},\mathbf{f}^*\alpha\rangle$.

Proof. See Exercise 4.37, page 149. □

The proof of the change of variables theorem using Green's theorem now follows from this sequence of equalities.

$$\begin{aligned}
k\iint_{\vec{T}} g(x,y)\,dx\,dy &= k\,\langle\vec{T},d\alpha\rangle && \\
&= k\,\langle\partial\vec{T},\alpha\rangle && \text{Green's theorem on } \vec{T}\\
&= \langle k\,\partial\vec{T},\alpha\rangle && \\
&= \langle\mathbf{f}(\partial\vec{S}),\alpha\rangle && \text{hypothesis}\\
&= \langle\partial\vec{S},\mathbf{f}^*\alpha\rangle && \mathbf{f} \text{ and } \mathbf{f}^* \text{ are adjoints}\\
&= \langle\vec{S},d(\mathbf{f}^*\alpha)\rangle && \text{Green's theorem on } \vec{S}\\
&= \langle\vec{S},\mathbf{f}^*(d\alpha)\rangle && d \text{ and } \mathbf{f}^* \text{ commute}\\
&= \iint_{\vec{S}} g(x(s,t),y(s,t))\,\frac{\partial(x,y)}{\partial(s,t)}\,ds\,dt, &&
\end{aligned}$$

because $\mathbf{f}^*(d\alpha) = g^*(s,t)\,\dfrac{\partial(x,y)}{\partial(s,t)}\,ds\,dt.$ □

Differential forms in n variables

It is possible to construct differential k-forms in any number of variables. In $(x_1, x_2, \ldots, x_n)$-space, there are n basic differentials, dx_1, dx_2, $\ldots$, dx_n. For each multi-index $I = (i_1, \ldots, i_k)$ with $1 \le i_1 < \cdots < i_k \le n$, we define the basic k-form

$$d\mathbf{x}_I = dx_{i_1}\, dx_{i_2} \cdots dx_{i_k} = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}.$$

There are as many basic k-forms as there are ways of choosing k distinct elements from a set of n elements; this number is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

("n choose k"). A general k-form is a linear combination

$$\alpha = \sum_I P_I(x_1, \ldots, x_n)\, d\mathbf{x}_I$$

of the $\binom{n}{k}$ basic k-forms in which the coefficient functions P_I all have continuous second derivatives. Products can then be calculated using the anticommutativity relations on the basic 1-forms:

$$dx_i \wedge dx_j = -dx_j \wedge dx_i, \quad i, j = 1, \ldots, n.$$

As we pointed out above, anticommutativity here implies

$$dx_i \wedge dx_i = 0, \quad i = 1, \ldots, n.$$

If σ is a k-form and τ is an l-form, then anticommutativity on the basic 1-forms implies

$$\sigma \wedge \tau = (-1)^{kl}\, \tau \wedge \sigma.$$

The exterior derivative of the 0-form $g(x_1, \ldots, x_n)$ is the 1-form

$$dg = \sum_{i=1}^{n} g_i\, dx_i,$$

where $g_i = \partial g / \partial x_i$. For a general k-form

$$\alpha = \sum_I P_I\, d\mathbf{x}_I,$$

the exterior derivative is the $(k+1)$-form

$$d\alpha = \sum_I dP_I \wedge d\mathbf{x}_I,$$

obtained using the exterior derivatives dP_I of the coefficient functions of α.

Theorem 10.17. *For every k-form α in $\mathbb{R}^n$, $d^2\alpha = d(d\alpha) = 0$.* $d^2 = 0$

Proof. As in the essentially identical Theorem 10.11, the proof reduces to the "equality of mixed partials" for functions with continuous second derivatives. If $\alpha = \sum_I P_I \, d\mathbf{x}_I$, then

$$d\alpha = \sum_I dP_I \wedge d\mathbf{x}_I = \sum_I \left(\sum_{i=1}^n \frac{\partial P_I}{\partial x_i} dx_i \right) \wedge d\mathbf{x}_I,$$

and

$$d^2\alpha = \sum_I \left(\sum_{i=1}^n d\left(\frac{\partial P_I}{\partial x_i} \right) dx_i \right) \wedge d\mathbf{x}_I = \sum_I \left(\sum_{i,j=1}^n \frac{\partial^2 P_I}{\partial x_j \, \partial x_i} dx_j \wedge dx_i \right) \wedge d\mathbf{x}_I.$$

The inner sum consists of n^2 terms. For each of the n terms with $j = i$,

$$\frac{\partial^2 P_I}{\partial x_i \, \partial x_i} dx_i \wedge dx_i = 0.$$

Now pair each remaining term with the term in which i and $j \neq i$ are interchanged:

$$\frac{\partial^2 P_I}{\partial x_j \, \partial x_i} dx_j \wedge dx_i + \frac{\partial^2 P_I}{\partial x_i \, \partial x_j} dx_i \wedge dx_j.$$

Because

$$\frac{\partial^2 P_I}{\partial x_j \, \partial x_i} = \frac{\partial^2 P_I}{\partial x_i \, \partial x_j} \quad \text{and} \quad dx_j \wedge dx_i = -dx_i \wedge dx_j,$$

each pair sums to zero, so the entire inner sum equals zero. □

The product rule for differentials

Theorem 10.18 (Product Rule). *Suppose α and θ are differential forms in $\mathbb{R}^n$, and α is a k-form. Then $d(\alpha \wedge \theta) = d\alpha \wedge \theta + (-1)^k \alpha \wedge d\theta$.*

Proof. It is sufficient to show this for "monomials"

$$\alpha = P \, d\mathbf{x}_I \quad \text{and} \quad \theta = Q \, d\mathbf{x}_J$$

that have disjoint multi-indices I and J. Then $\alpha \wedge \theta = PQ \, d\mathbf{x}_I \wedge d\mathbf{x}_J$, and we have

$$\begin{aligned} d(\alpha \wedge \theta) &= \sum_m (P_m Q + P Q_m) \, dx_m \wedge d\mathbf{x}_I \wedge \mathbf{x}_J \\ &= \sum_m P_m Q \, dx_m \wedge d\mathbf{x}_I \wedge \mathbf{x}_J + \sum_m P Q_m \, dx_m \wedge d\mathbf{x}_I \wedge \mathbf{x}_J, \end{aligned}$$

where $P_m = \partial P / \partial x_m$ and the summation can be restricted to those indices m that do not occur in either I or J. The first sum is $d\alpha \wedge \theta$, because

$$d\alpha = \sum_m P_m \, dx_m \wedge d\mathbf{x}_I \quad \text{and} \quad d\alpha \wedge \theta = \sum_m P_m Q \, dx_m \wedge d\mathbf{x}_I \wedge d\mathbf{x}_J.$$

The second sum is $(-1)^k \alpha \wedge d\theta$, because $d\theta = \sum_m Q_m \, dx_m \wedge d\mathbf{x}_J$ and

$$\alpha \wedge d\theta = \sum_m PQ_m \, d\mathbf{x}_I \wedge dx_m \wedge d\mathbf{x}_J = (-1)^k \sum_m PQ_m \, dx_m \wedge d\mathbf{x}_I \wedge d\mathbf{x}_J.$$

The last equality is a consequence of the anticommutativity of basic 1-forms:

$$\begin{aligned} d\mathbf{x}_I \wedge \underline{dx_m} &= dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}} \wedge dx_{i_k} \wedge \underline{dx_m} \\ &= (-1)\, dx_{i_1} \wedge \ldots dx_{i_{k-1}} \wedge \underline{dx_m} \wedge dx_{i_k} \\ &= (-1)^2 \, dx_{i_1} \wedge \cdots \wedge \underline{dx_m} \wedge dx_{i_{k-1}} \wedge dx_{i_k} \\ &\vdots \\ &= (-1)^k \, \underline{dx_m} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}} \wedge dx_{i_k} \\ &= (-1)^k \, \underline{dx_m} \wedge d\mathbf{x}_I. \end{aligned}$$ □

Coefficients of $d\alpha$

Let us see how the coefficients of a $(k-1)$-form α in n variables determine the coefficients of its exterior derivative $\omega = d\alpha$, a k-form. We use the k-multi-index $I = (i_1, \ldots, i_k)$, $1 \le i_1 < \cdots < i_k \le n$ for the coefficients of ω,

$$\omega = \sum_I P_I \, d\mathbf{x}_I.$$

For the coefficients of α, we use the $(k-1)$-multi-index

$$\widehat{I_s} = (i_1, \ldots, \widehat{i_s}, \ldots, i_k), \quad s = 1, \ldots, k;$$

the circumflex over i_s means that i_s is deleted, so each $\widehat{I_s}$ contains only $k-1$ indices. Thus we can write

$$\alpha = \sum_{\widehat{I_s}} A_{\widehat{I_s}} \, d\mathbf{x}_{\widehat{I_s}}, \quad d\alpha = \sum_{\widehat{I_s}} d(A_{\widehat{I_s}}) \, d\mathbf{x}_{\widehat{I_s}};$$

There are $\binom{n}{k}$ k-mult-indices I and $\binom{n}{k-1}$ $(k-1)$-multi-indices $\widehat{I_s}$.

Given that $d\alpha = \omega$, we must determine what contribution the various terms of $d(A_{\widehat{I_s}}) \, d\mathbf{x}_{\widehat{I_s}}$ make to the term $P_I \, d\mathbf{x}_I$. We have

$$d(A_{\widehat{I_s}}) \, d\mathbf{x}_{\widehat{I_s}} = \sum_j \frac{\partial A_{\widehat{I_s}}}{\partial x_j} \, dx_j \, dx_{i_1} \cdots \widehat{dx_{i_s}} \cdots dx_{i_k},$$

and the sum has only one nonzero term, the one in which the summation index j equals i_s. It then takes $s-1$ successive transpositions to move $dx_j = dx_{i_s}$ from its initial position in that term to its proper position in the basic differential; that is,

$$dx_{i_s} \, dx_{i_1} \cdots \widehat{dx_{i_s}} \cdots dx_{i_k} = (-1)^{s-1} dx_{i_1} \cdots dx_{i_s} \cdots dx_{i_k} = (-1)^{s-1} d\mathbf{x}_I.$$

This proves the following theorem.

Theorem 10.19. *If* $\alpha = \sum_{\widehat{I}_s} A_{\widehat{I}_s}\, d\mathbf{x}_{\widehat{I}_s}$, *then* $d\alpha = \sum_I \left(\sum_{s=1}^{k} (-1)^{s-1} \frac{\partial A_{\widehat{I}_s}}{\partial x_{i_s}} \right) d\mathbf{x}_I$. □

Action of the pullback

Now let us consider how a differential form is transformed by the pullback of a differentiable map $\mathbf{f}: U^n \to \mathbb{R}^p$ with component functions

$$\mathbf{f}: \begin{cases} x_1 = x_1(u_1, \ldots, u_n), \\ x_2 = x_2(u_1, \ldots, u_n), \\ \quad \vdots \\ x_p = x_p(u_1, \ldots, u_n). \end{cases}$$

Here U^n is an open subset of $\mathbb{R}^n$, and we allow $n \neq p$. For a 0-form, $\alpha(\mathbf{x}) = g(\mathbf{x})$, the pullback is

$$\mathbf{f}^*\alpha(\mathbf{u}) = g(\mathbf{f}(\mathbf{u})) = g^*(\mathbf{u}).$$

For a basic 1-form dx_i, the pullback is

$$\mathbf{f}^* dx_i = \sum_{j=1}^{n} \frac{\partial x_i}{\partial u_j} du_j, \quad i = 1, \ldots, p.$$

For the basic 2-form $dx_1\, dx_2$, we have

$$\begin{aligned} \mathbf{f}^*(dx_1 \wedge dx_2) = \mathbf{f}^* dx_1 \wedge \mathbf{f}^* dx_2 &= \sum_{j \neq m} \frac{\partial x_1}{\partial u_j} \frac{\partial x_2}{\partial u_m} du_j \wedge du_m \\ &= \sum_{j<m} \frac{\partial x_1}{\partial u_j} \frac{\partial x_2}{\partial u_m} du_j \wedge du_m + \sum_{j>m} \frac{\partial x_1}{\partial u_j} \frac{\partial x_2}{\partial u_m} du_j \wedge du_m \end{aligned}$$

Anticommutivity simplifies this. If we transpose the dummy summation indices (i.e., $j \leftrightarrow m$) in the second sum, then that sum becomes

$$\sum_{m>j} \frac{\partial x_1}{\partial u_m} \frac{\partial x_2}{\partial u_j} du_m \wedge du_j = \sum_{j<m} -\frac{\partial x_1}{\partial u_m} \frac{\partial x_2}{\partial u_j} du_j \wedge du_m.$$

Recombining this with the first sum, we get

$$\begin{aligned} \mathbf{f}^*(dx_1\, dx_2) &= \sum_{j<m} \left(\frac{\partial x_1}{\partial u_j} \frac{\partial x_2}{\partial u_m} - \frac{\partial x_1}{\partial u_m} \frac{\partial x_2}{\partial u_j} \right) du_j \wedge du_m \\ &= \sum_{j<m} \frac{\partial(x_1, x_2)}{\partial(u_j, u_m)} du_j\, du_m. \end{aligned}$$

This is essentially the same as the earlier calculation of $\mathbf{f}^*(dx\,dy)$ on page 433. More generally, if $I = (i_1, i_2)$ is any pair with $1 \leq i_1 < i_2 \leq p$, then