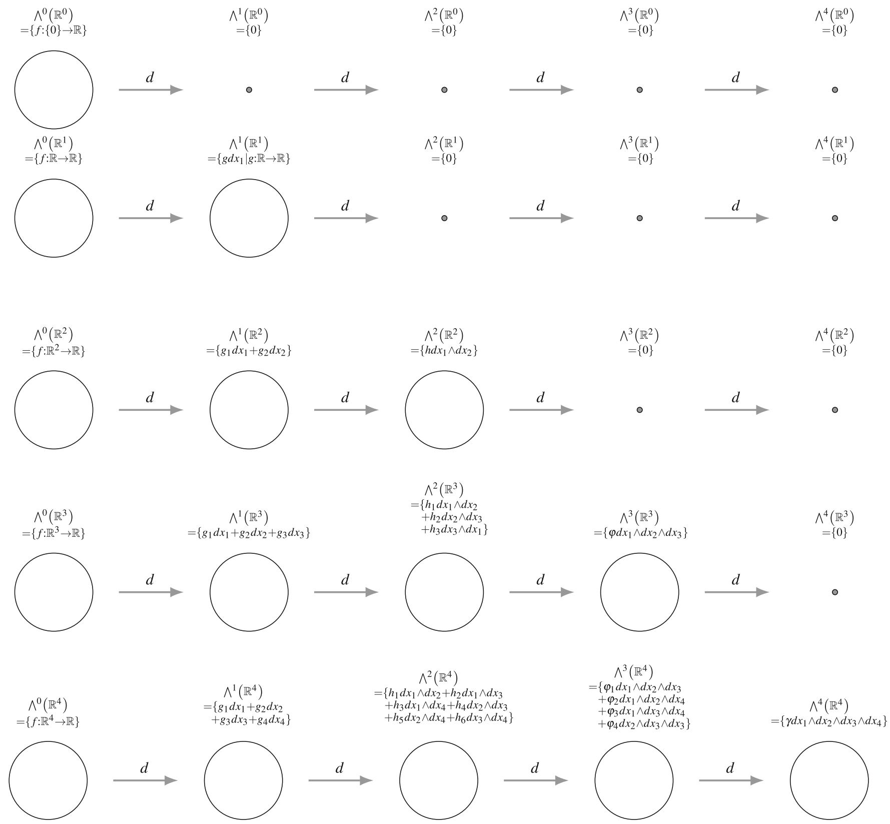


Fig. 8.6 Here the *columns* of 8.5 are shown as *rows* using Venn-like diagrams. The base case for the induction hypothesis is the first column of Fig. 8.5, which is shown as the top row here. The column of the commutative diagram in Fig. 8.5 that is associated with our mid-level sample case is shown as the fourth row here

In the first case, if $\alpha \in \bigwedge^1(\mathbb{R}^0)$ then we must have $\alpha = 0$ since the only form in $\bigwedge^1(\mathbb{R}^0)$ is the zero one-form. And clearly $d\alpha = d0 = 0$ and so it is closed. In order to show it is exact we must show that $\alpha = df$ for some zero-form f. But consider any function $f \in \bigwedge^0(\mathbb{R}^0)$. Since $f : \{0\} \to \mathbb{R}$ then $f(0)$ is simply an unchanging real number. The exterior derivative of f finds how f changes as we move in any direction. But there is no direction to move and no change that can happen so we must have $df = 0 = \alpha$ for every $f \in \bigwedge^0(\mathbb{R}^0)$. Thus the closed α is also exact, thereby proving the first case.

The second case is even easier. It is obvious that $\alpha \in \bigwedge^k(\mathbb{R}^0)$ for $k > 1$ is closed since $d\alpha = d0 = 0$. But is is also obvious that α must be exact since if $\beta \in \bigwedge^{k-1}(\mathbb{R}^0)$ then $d\beta = d0 = 0 = \alpha$. Thus we have also shown that any closed form in $\bigwedge^k(\mathbb{R}^0)$, for any $k > 1$, is also exact. Putting this together we have that any closed form on $\mathbb{R}^0$ is also exact, thereby proving our base case.

Now, instead of proceeding to the general case we will spend the rest of this section doing a somewhat mid-level sample case. We do this in order to help you become familiar with the strategy we will use in the general case. The general case is notationally cumbersome and so it is easier to understand the basic strategy when the notation is not so overwhelming. We shall show that closed one-forms on $\mathbb{R}^3$ are exact. The fourth row of Fig. 8.6 shows the part of the commutative diagram in Fig. 8.5 that is associated with $\mathbb{R}^3$. In order to do our sample case we will need to introduce several mappings. We will have more to say about these mappings after the sample case and before we do the general case. The first mapping we will denote by $\mathscr{Z}$ and the second mapping by $\mathscr{C}$;

$$\mathscr{Z} : \bigwedge^k(\mathbb{R}^n) \longrightarrow \bigwedge^k(\mathbb{R}^{n-1}),$$

$$\mathscr{C} : \bigwedge^k(\mathbb{R}^n) \longrightarrow \bigwedge^k(\mathbb{R}^{n+1}).$$

In essence the mapping $\mathscr{Z}$ "squishes" a k-form on $\mathbb{R}^n$ to a k-form on $\mathbb{R}^{n-1}$ while the mapping $\mathscr{C}$ "expands" a k-form on $\mathbb{R}^n$ to a k-form on $\mathbb{R}^{n+1}$.

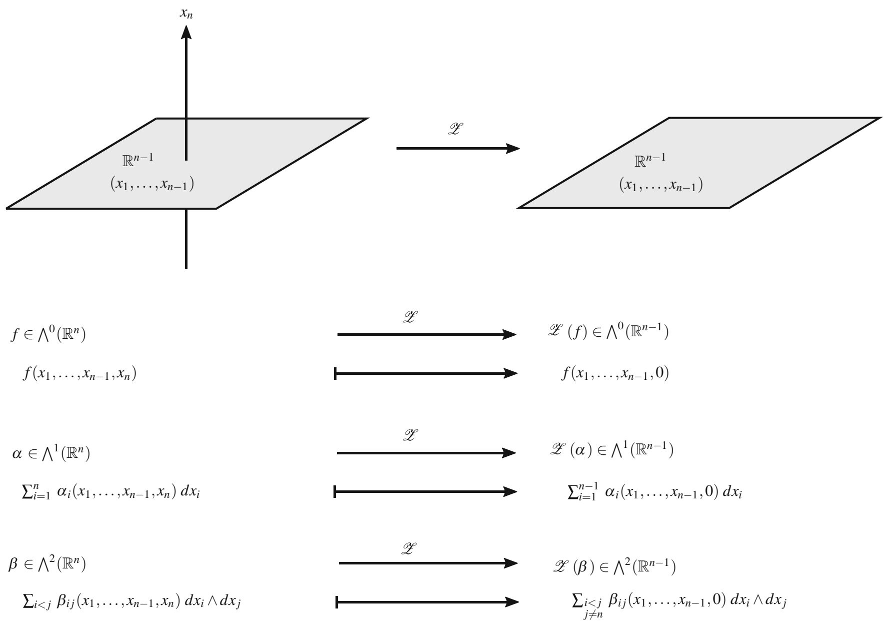


Fig. 8.7 The mapping $\mathscr{Z} : \bigwedge^k(\mathbb{R}^n) \to \bigwedge^k(\mathbb{R}^{n-1})$ is shown. The mapping "squishes" k-forms on $\mathbb{R}^n$ to k-forms on $\mathbb{R}^{n-1}$. This is shown concretely for zero-forms, one-forms, and two-forms. $\mathscr{Z}$ sets $x_n = 0$ and kills all terms that contain dx_n

Suppose that the coordinate functions of $\mathbb{R}^n$ are $x_1, \ldots, x_n$. As usual, we are of course being a bit imprecise and will use the same notation, $x_1, \ldots, x_n$, to both denote real-valued coordinate functions on $\mathbb{R}^n$ as well as the numerical values those functions give, which denotes the point. That is, if $p \in \mathbb{R}^n$ and x_i is a coordinate function $x_i : \mathbb{R}^n \to \mathbb{R}$, then we have $x_i(p) = x_i$, where the x_i on the left hand side is the coordinate function and the x_i on the right hand side is a numerical value. This allows us to write the point $p = (x_1, \ldots, x_n)$. We shall continue with this ambiguity throughout and hope that it will not cause you too much confusion.

A zero-form on $\mathbb{R}^n$ at the point $p = (x_1, \ldots, x_n)$ is simply a function $f(p) = f(x_1, \ldots, x_n)$, one-form α on $\mathbb{R}^n$ at a point p is written as $\alpha = \sum_{i=1}^n \alpha_i(p)\ dx_i = \sum_{i=1}^n \alpha_i(x_1, \ldots, x_n)\ dx_i$, a two-form at the point p can be written is $\beta = \sum_{i<j} \beta_{ij}(p)\ dx_i \wedge dx_j = \sum_{i<j} \beta_{ij}(x_1, \ldots, x_n)\ dx_i \wedge dx_j$, and so on. The mapping $\mathscr{Z} : \bigwedge^k(\mathbb{R}^n) \to \bigwedge^k(\mathbb{R}^{n-1})$ essentially restricts k-forms to the subspace $\mathbb{R}^{n-1}$ of $\mathbb{R}^n$. See Fig. 8.7. It does this by simply replacing the x_n value with 0 and killing every term that has dx_n in it. Thus we have

$$f(x_1, \ldots, x_{n-1}, x_n) \overset{\mathscr{Z}}{\longmapsto} f(x_1, \ldots, x_{n-1}, 0),$$

$$\sum_{i=1}^{n} \alpha_i(x_1, \ldots, x_{n-1}, x_n)\ dx_i \overset{\mathscr{Z}}{\longmapsto} \sum_{i=1}^{n-1} \alpha_i(x_1, \ldots, x_{n-1}, 0)\ dx_i,$$

$$\sum_{i<j} \beta_{ij}(x_1, \ldots, x_{n-1}, x_n)\ dx_i \wedge dx_j \overset{\mathscr{Z}}{\longmapsto} \sum_{\substack{i<j\\ j\neq n}} \beta_{ij}(x_1, \ldots, x_{n-1}, 0)\ dx_i \wedge dx_j,$$

and so on for k-forms where $k > 2$.

The mapping $\mathscr{C}$ "expands" a k-form on $\mathbb{R}^n$ to a k-form on $\mathbb{R}^{n+1}$,

$$f(x_1, \ldots, x_{n-1}, x_n) \overset{\mathscr{C}}{\longmapsto} f(x_1, \ldots, x_{n-1}, x_n),$$

$$\sum_{i=1}^{n} \alpha_i(x_1, \ldots, x_{n-1}, x_n)\ dx_i \overset{\mathscr{C}}{\longmapsto} \sum_{i=1}^{n} \alpha_i(x_1, \ldots, x_{n-1}, x_n)\ dx_i,$$

$$\sum_{i<j} \beta_{ij}(x_1, \ldots, x_{n-1}, x_n)\ dx_i \wedge dx_j \overset{\mathscr{C}}{\longmapsto} \sum_{i<j} \beta_{ij}(x_1, \ldots, x_{n-1}, x_n)\ dx_i \wedge dx_j.$$

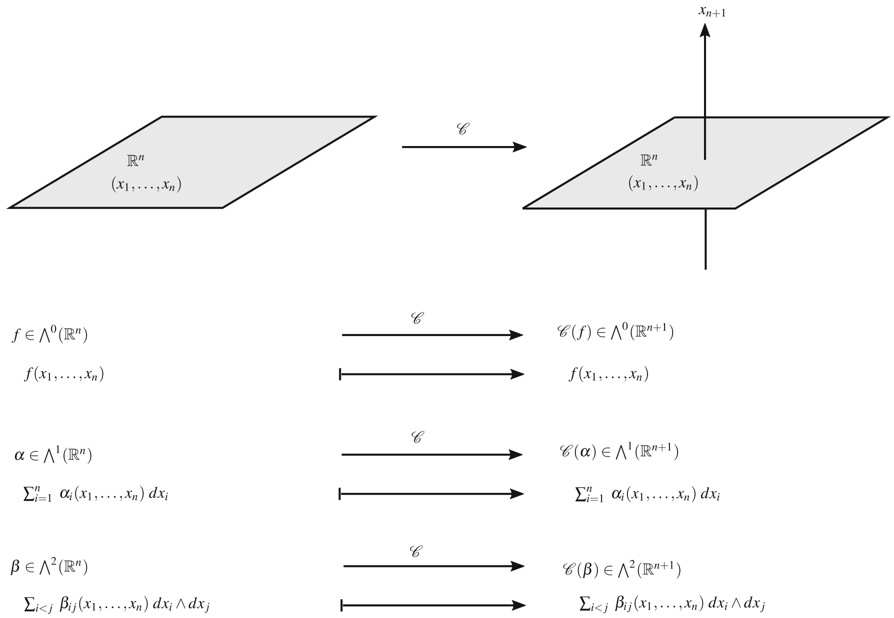


Fig. 8.8 The mapping $\mathscr{C} : \bigwedge^k(\mathbb{R}^n) \to \bigwedge^k(\mathbb{R}^{n+1})$ is shown. The mapping "expands" k-forms on $\mathbb{R}^n$ to k-forms on $\mathbb{R}^{n+1}$. The expanded k-form on $\mathbb{R}^{n+1}$ looks exactly like the k-form on $\mathbb{R}^n$. This is shown concretely for zero-forms, one-forms, and two-forms

See Fig. 8.8. Notice, the way that $f \in \bigwedge^0(\mathbb{R}^n)$ looks exactly the same as $\mathscr{C}(f) \in \bigwedge^1(\mathbb{R}^{n+1})$, which is a function that simply does not depend on the x_{n+1} variable. Similarly, $\alpha \in \bigwedge^1(\mathbb{R}^n)$ looks exactly like $C(\alpha) \in \bigwedge^1(\mathbb{R}^{n+1})$, which is a one-form on $\mathbb{R}^{n+1}$ that does not have a dx_{n+1} term and whose component functions α_i do not depend at all on the x_{n+1} variable. The same is true of the two-form $\beta \in \bigwedge^2(\mathbb{R}^n)$. $\mathscr{C}(\beta)$ looks exactly the same as β; it is two-form on $\mathbb{R}^{n+1}$ which does not have any terms involving dx_{n+1} and whose component functions β_{ij} do not depend on the x_{n+1} variable.

To see this better consider the one-form $\alpha = P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz \in \bigwedge^1(\mathbb{R}^3)$. Then

$$\mathscr{Z}(\alpha) = P(x, y, 0)dx + Q(x, y, 0)dy \in \bigwedge^1(\mathbb{R}^2).$$

Notice that we killed the dz term, which was $R(x, y, z)dz$. Now consider the one-form $\alpha = P(x, y)dx + Q(x, y)dy \in \bigwedge^1(\mathbb{R}^2)$ then

$$\mathscr{C}(\alpha) = P(x, y)dx + Q(x, y)dy \in \bigwedge^1(\mathbb{R}^3).$$

This one-form on $\mathbb{R}^3$ looks exactly like the one-form from $\mathbb{R}^2$, even though it is actually on $\mathbb{R}^3$. It is a one-form on $\mathbb{R}^3$ without a dz term and whose component functions do not depend on the variable z.

Question 8.1 Using $\alpha = P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz \in \bigwedge^1(\mathbb{R}^3)$, show that $d(\mathscr{Z}(\alpha)) = \mathscr{Z}(d(\alpha))$. We often write this as $d\mathscr{Z} = \mathscr{Z}d$.

Question 8.2 Using $\alpha = P(x, y)dx + Q(x, y)dy \in \bigwedge^1(\mathbb{R}^2)$, show that $d(\mathscr{C}(\alpha)) = \mathscr{C}(d(\alpha))$. We often write this as $d\mathscr{C} = \mathscr{C}d$.

Question 8.3 Using $\alpha = P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz \in \bigwedge^1(\mathbb{R}^3)$, find $\mathscr{Z}(\alpha)$ and then find $\mathscr{C}\big(\mathscr{Z}(\alpha)\big)$.

Besides $\mathscr{Z}$ and $\mathscr{C}$ we need to define one more mapping $\mathscr{K} : \bigwedge^1(\mathbb{R}^3) \to \bigwedge^0(\mathbb{R}^3)$. If $\alpha = P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz$ is a one-form on $\mathbb{R}^3$ then $\mathscr{K}(\alpha)$ is a zero-form, that is, a function on $\mathbb{R}^3$, defined by

$$\mathscr{K}(\alpha) = \mathscr{K}\Big(P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz\Big) = \int_0^z R(x, y, t)\, dt.$$

In other words, $\mathscr{K}(\alpha) : \mathbb{R}^3 \to \mathbb{R}$. How does this actually work? If we are given a point $p = (x_0, y_0, z_0)$ on $\mathbb{R}^3$ then

$$\mathscr{K}(\alpha)(p) = \mathscr{K}(\alpha)(x_0, y_0, z_0) = \int_0^{z_0} R(x_0, y_0, t)\, dt \in \mathbb{R}.$$

Since the Poincaré lemma is about showing closed forms are exact, in our proof we will need to make use of some facts that are related to closed forms. In this mid-level sample case we are dealing with a closed one-form α on $\mathbb{R}^3$. We will use the fact that $d\alpha = 0$ to find some identities that will be needed during the course of the proof.

$$\begin{aligned} d\alpha &= d\big(P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz\big) \\ &= dP \wedge dx + dQ \wedge dy + dR \wedge dz \\ &= \frac{\partial P}{\partial y} dy \wedge dx + \frac{\partial P}{\partial z} dz \wedge dx + \frac{\partial Q}{\partial x} dx \wedge dy + \frac{\partial Q}{\partial z} dz \wedge dy + \frac{\partial R}{\partial x} dx \wedge dz + \frac{\partial R}{\partial y} dy \wedge dz \\ &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy + \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) dx \wedge dz + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz \\ &= 0 \end{aligned}$$

thus the coefficients of each term of the two-form are equal to zero, thereby giving us the following three identities

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}, \quad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}.$$

Now that we have all the necessary preliminaries finally pinned down we are ready to proceed with the proof of the Poincaré lemma in our sample case. We will begin by giving the overall strategy. Once we have done that we will proceed by giving the details of each step. Suppose that we know $\alpha \in \bigwedge^1(\mathbb{R}^3)$ is closed. Our goal is to show that α is also exact. That is, we want to show there is some $\beta \in \bigwedge^0(\mathbb{R}^3)$ such that $d\beta = \alpha$.

First we give the general idea of what we will do. A schematic of what we want to do is presented in Fig. 8.9. We will "squish" α down with $\mathscr{Z}$ to get $\mathscr{Z}(\alpha) \in \bigwedge^1(\mathbb{R}^2)$ which we can then show is exact by the induction hypothesis. We will and then "expand" the exact form $\mathscr{Z}(\alpha)$ with $\mathscr{C}$ to get $\mathscr{C}\big(\mathscr{Z}(\alpha)\big) \in \bigwedge^1(\mathbb{R}^3)$, which we can then also show is exact. Since $\mathscr{C}\big(\mathscr{Z}(\alpha)\big)$ is in some sense part of α we want to show that the rest of α, namely $\alpha - \mathscr{C}\big(\mathscr{Z}(\alpha)\big)$, is also exact. We can do this by using $\mathscr{K}$ on α to get $\mathscr{K}(\alpha) \in \bigwedge^0(\mathbb{R}^3)$ and then taking the exterior derivative of that to get $d\big(\mathscr{K}(\alpha)\big) \in \bigwedge^1(\mathbb{R}^3)$, which

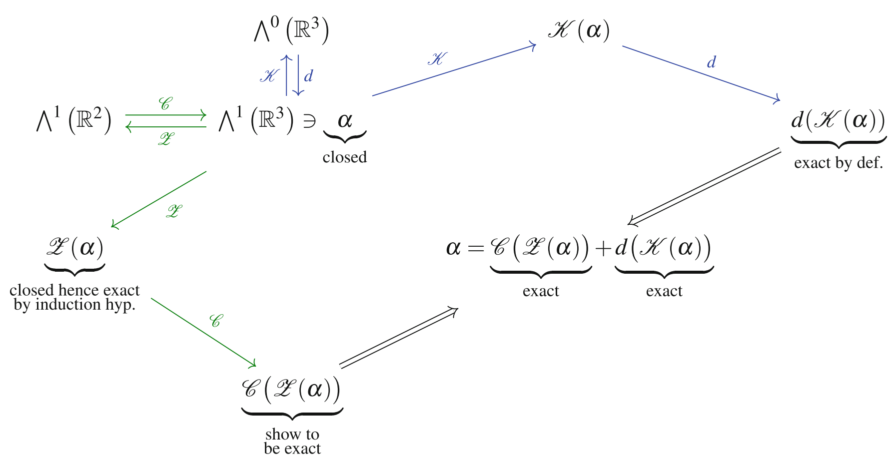


Fig. 8.9 The general idea of our proof for the sample case. For $\alpha \in \bigwedge^1(\mathbb{R}^3)$ we compute $\mathscr{K}(\alpha) \in \bigwedge^0(\mathbb{R}^3)$ and then $d\big(\mathscr{K}(\alpha)\big) \in \bigwedge^1(\mathbb{R}^3)$, which is clearly exact. We also compute $\mathscr{Z}(\alpha) \in \bigwedge^1(\mathbb{R}^2)$, which is closed and hence exact by our induction hypothesis. We then compute $\mathscr{C}\big(\mathscr{Z}(\alpha)\big) \in \bigwedge^1(\mathbb{R}^3)$, which can be shown to be exact. It is then possible to show that $\alpha = \mathscr{C}\big(\mathscr{Z}(\alpha)\big) + d\big(\mathscr{K}(\alpha)\big)$, the sum of two exact terms, which means that α itself is exact

is clearly exact. It turns out that $d\big(\mathscr{K}(\alpha)\big) = \alpha - \mathscr{C}\big(\mathscr{Z}(\alpha)\big)$. In other words we have that

$$\underbrace{d\big(\mathscr{K}(\alpha)\big)}_{\substack{\text{clearly exact}\\ \text{by definition}}} = \alpha - \underbrace{\mathscr{C}\big(\mathscr{Z}(\alpha)\big)}_{\substack{\text{exacty by}\\ \text{induction hyp.}}} .$$

But this means that

$$\begin{aligned}
\alpha &= \underbrace{\mathscr{C}\big(\mathscr{Z}(\alpha)\big)}_{\substack{\text{exact so there}\\ \text{exists some }\omega\\ \text{s.t. }\mathscr{C}(\mathscr{Z}(\alpha))=d\omega}} + \underbrace{d\big(\mathscr{K}(\alpha)\big)}_{\substack{\text{clearly exact}\\ \text{by definition}}} \\
&= d\big(\omega\big) + d\big(\mathscr{K}(\alpha)\big) \\
&= d\big(\omega + \mathscr{K}(\alpha)\big) \qquad \text{by linearity of } d
\end{aligned}$$

and hence α is exact itself, which is what we wanted to show. Notice that we never explicitly showed what ω was. In other words, while we logically reasoned that ω must exist we never actually found an explicit formula for ω.

Now we have given you the overall strategy that we will employ we are ready for the details. We now have to show two things, first that $\mathscr{C}\big(\mathscr{Z}(\alpha)\big)$ is exact by the induction hypotheses and second that $d\big(\mathscr{K}(\alpha)\big) = \alpha - \mathscr{C}\big(\mathscr{Z}(\alpha)\big)$. Showing these two things is actually the guts of the proof. Since α is closed by definition of closed we have that $d\alpha = 0$. Using the fact that $d\mathscr{Z} = \mathscr{Z}d$ we have

$$d\big(\mathscr{Z}(\alpha)\big) = \mathscr{Z}(d\alpha) = \mathscr{Z}(0) = 0 \quad \Rightarrow \quad \mathscr{Z}(\alpha) \in \bigwedge^1(\mathbb{R}^2) \text{ is closed.}$$

The ***induction hypothesis*** *is that all closed one-forms on* $\mathbb{R}^2$ *are already known to be exact*. Since $\mathscr{Z}(\alpha)$ is a one-form on $\mathbb{R}^2$ which is now known to be closed, we know, by the inductive hypothesis, that $\mathscr{Z}(\alpha)$ is exact. that means there is some zero-form ω on $\mathbb{R}^2$ such that $\mathscr{Z}(\alpha) = d\omega$. Now to show that $\mathscr{C}(\mathscr{Z}(\alpha)) \in \bigwedge^1(\mathbb{R}^3)$ is exact we use the fact that $d\mathscr{C} = \mathscr{C}d$,

$$\mathscr{C}\big(\mathscr{Z}(\alpha)\big) = \mathscr{C}(d\omega) = d(\mathscr{C}\omega) \quad \Rightarrow \quad \mathscr{C}\big(\mathscr{Z}(\alpha)\big) \in \bigwedge^1(\mathbb{R}^3) \text{ is exact.}$$

Now we will show that $d\big(\mathscr{K}(\alpha)\big)$ is the part of α that is left after subtracting the exact part $\mathscr{C}\big(\mathscr{Z}(\alpha)\big)$; that is, that $d\big(\mathscr{K}(\alpha)\big) = \alpha - \mathscr{C}\big(\mathscr{Z}(\alpha)\big)$. This is done using an explicit calculation

$$\begin{aligned}
d\big(\mathscr{K}(\alpha)\big) &= d\left(\mathscr{K}\Big(P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz\Big)\right) \\
&= d\left(\int_0^z R(x,y,t)\,dt\right) \\
&= \frac{\partial}{\partial x}\left(\int_0^z R(x,y,t)\,dt\right)dx + \frac{\partial}{\partial y}\left(\int_0^z R(x,y,t)\,dt\right)dy + \frac{\partial}{\partial z}\left(\int_0^z R(x,y,t)\,dt\right)dz \\
&= \underbrace{\left(\int_0^z \frac{\partial}{\partial x}R(x,y,t)\,dt\right)}_{\int \text{ has no depenance on } x}dx + \underbrace{\left(\int_0^z \frac{\partial}{\partial y}R(x,y,t)\,dt\right)}_{\int \text{ has no depenance on } y}dy + \underbrace{R(x,y,z)}_{\substack{\text{fundamental}\\ \text{theorem}\\ \text{of calculus}}}dz \\
&= \underbrace{\left(\int_0^z \frac{\partial P(x,y,t)}{\partial z}\,dt\right)}_{\alpha \text{ closed} \Rightarrow \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}}dx + \underbrace{\left(\int_0^z \frac{\partial Q(x,y,t)}{\partial z}\,dt\right)}_{\alpha \text{ closed} \Rightarrow \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}}dy + R(x,y,z)dz \\
&= \Big[P(x,y,z) - P(x,y,0)\Big]dx + \Big[Q(x,y,z) - Q(x,y,0)\Big]dy + R(x,y,z)dz \\
&= \Big(P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz\Big) - \Big(P(x,y,0)dx + Q(x,y,0)dy\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \alpha - \mathscr{C}\big(\mathscr{Z}(\alpha)\big) \\
\Rightarrow \quad & d(\mathscr{K}(\alpha)) = \alpha - \mathscr{C}(\mathscr{Z}(\alpha)) \\
\Rightarrow \quad & \alpha = \underbrace{d(\mathscr{K}(\alpha))}_{\text{exact by def.}} + \underbrace{\mathscr{C}(\mathscr{Z}(\alpha))}_{\text{exact from above}} \\
\Rightarrow \quad & \alpha \text{ is exact.}
\end{aligned}
$$

Thus we have shown the Poincaré lemma in our sample case. We showed that if $\alpha \in \bigwedge^1(\mathbb{R}^3)$ is closed then α is also exact.

A few comments are now in order. In the general proof of Poincaré's Lemma we will show that if α is a closed k-form on $\mathbb{R}^n$ then it is exact. Our proof requires us to assume what is called the **induction hypothesis**. In this case to show a closed k-form on $\mathbb{R}^n$ is exact we use the fact that a closed k-form on $\mathbb{R}^{n-1}$ is exact. By our base case we have already shown that any closed k-form on $\mathbb{R}^0$ is exact.

The general proof is for any n, so if we let $n = 1$ we can then show that any closed k-form on $\mathbb{R}^1$ is exact. We show this in the general proof of the Poincaré lemma, which requires us to use the induction hypothesis. In the case of $n = 1$ our induction hypothesis is that any closed k-form on $\mathbb{R}^0$ is exact, which we already know. Then, once we know this we can use it to prove that closed k-forms on $\mathbb{R}^1$ are exact. Once this is known the same argument can be used to show closed k-forms on $\mathbb{R}^2$ are exact, which can then be used to show that closed k-forms on $\mathbb{R}^3$ are exact, which can then be used to…and so on. In this way we can bootstrap ourselves up to show that closed k-forms on any $\mathbb{R}^n$ we want are exact.

Of course, we don't want to make the same argument over and over again an infinite number of times, so we make it only once, but do it in a way that is general enough that it holds for any value of k and n we want. This means that we have to be very general with our notation, which can make our argument a little notationally overwhelming. This is one of the reasons we went through a simple case first so you could get a feel for how the argument works without being too distracted by trying to follow all the notation. We will now also say that the general proof will be easier if we make a minor twist to what we did in the sample case. We will explain that when the time comes.

8.3 The General Case

In the general case in the proof of the Poincaré lemma we will show that closed k-forms on $\mathbb{R}^{n+1}$ are exact, for arbitrary k and n. But before launching into the proof of the general case let us revisit the $\mathscr{Z}$ and $\mathscr{C}$ mappings and make sure we understand what they are and how they work for any k and n. It turns out that $\mathscr{Z}$ and $\mathscr{C}$ are the pull-backs of two simple maps. We begin by examining $\mathscr{Z}$.

Consider the stretch mapping denoted by S shown in Fig. 8.10,

$$
\begin{aligned}
\mathbb{R}^n &\xrightarrow{S} \mathbb{R}^{n+1} \\
(x_1, \ldots, x_n) &\longmapsto (x_1, \ldots, x_n, 0)
\end{aligned}
$$

Notice that there is some ambiguity here. Besides having x_i being used as our notation for both a coordinate function and a numerical value, we are also using x_i as the coordinate functions for both the space $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$. We want to be just a little more careful and keep our spaces distinct in our minds, so we will say that $x_1, \ldots, x_n$ are the coordinate functions on $\mathbb{R}^n$

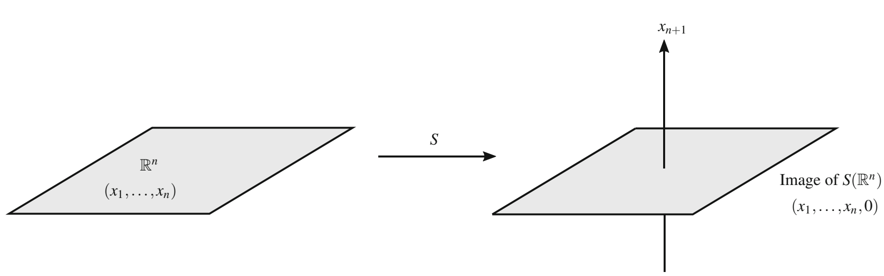


Fig. 8.10 The mapping $S : \mathbb{R}^n \to \mathbb{R}^{n+1}$

and $y_1, \ldots, y_{n+1}$ are the coordinate functions on $\mathbb{R}^{n+1}$. Being precise we would say that

$$\begin{aligned} \mathbb{R}^n &\xrightarrow{S} \mathbb{R}^{n+1} \\ (x_1, \ldots, x_n) &\longmapsto \big(y_1(x_1, \ldots, x_n), \ldots, y_{n+1}(x_1, \ldots, x_n)\big), \end{aligned}$$

where

$$\begin{aligned} y_1(x_1, \ldots, x_n) &= x_1, \\ &\vdots \\ y_n(x_1, \ldots, x_n) &= x_n, \\ y_{n+1}(x_1, \ldots, x_n) &= 0. \end{aligned}$$

Of course the mapping S gives rise to a pull-back mapping $T^*S \equiv S^*$ of k-forms,

$$\begin{gathered} \bigwedge^k(\mathbb{R}^n) \xleftarrow{T^*S} \bigwedge^k(\mathbb{R}^{n+1}) \\ \mathbb{R}^n \xrightarrow{S} \mathbb{R}^{n+1}. \end{gathered}$$

This is shown in Fig. 8.11.

The building blocks of k-forms on $\mathbb{R}^{n+1}$ are of the form $f_{i_1 \cdots i_k} dy_{i_1} \wedge dy_{i_2} \wedge \cdots \wedge dy_{i_k}$, where by convention we have $i_1 < i_2 < \cdots < i_k \leq n+1$. Relying on the second and third identities in Sect. 6.7 and knowing what the pull-back of a zero-form is, also covered in Sect. 6.7, we compute the pull-back for one of these k-form building blocks,

$$\begin{aligned} &T^*S \cdot \big(f_{i_1 \cdots 1_k} dy_{i_1} \wedge dy_{i_2} \wedge \cdots \wedge dy_{i_k}\big) \\ &= T^*Sf_{i_1 \cdots 1_k} \cdot T^*S dy_{i_1} \wedge T^*S dy_{i_2} \wedge \cdots \wedge T^*S dy_{i_k} \\ &= (f_{i_1 \cdots 1_k} \circ S) \cdot d(S^* y_{i_1}) \wedge d(S^* y_{i_2}) \wedge \cdots \wedge d(S^* y_{i_k}) \\ &= (f_{i_1 \cdots 1_k} \circ S) \cdot d(y_{i_1} \circ S) \wedge d(y_{i_2} \circ S) \wedge \cdots \wedge d(y_{i_k} \circ S) \end{aligned}$$

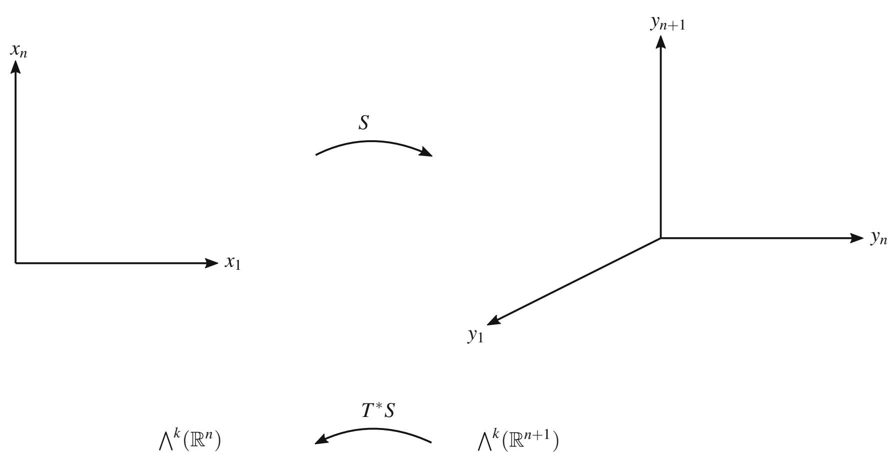


Fig. 8.11 The mapping $S : \mathbb{R}^n \to \mathbb{R}^{n+1}$ shown using the more appropriate coordinate function notation, along with the pullback mapping $T^*S \equiv S^* : \bigwedge^k(\mathbb{R}^{n+1}) \to \bigwedge^k(\mathbb{R}^n)$

$$
\begin{aligned}
&= (f_{i_1 \cdots i_k} \circ S) \cdot \left(\sum_{j=1}^{n} \frac{\partial y_{i_1}(x_1, \ldots, x_n)}{\partial x_j} dx_j \right) \wedge \cdots \wedge \left(\sum_{j=1}^{n} \frac{\partial y_{i_k}(x_1, \ldots, x_n)}{\partial x_j} dx_j \right) \\
&= \begin{cases} f_{i_1 \cdots i_k}(x_1, \ldots, x_n, 0) \cdot dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ \quad 0 \qquad \text{if some } i_\ell = n+1 \end{cases}
\end{aligned}
$$

The last equality follows from the definition of $y_{i_\ell}(x_1, \ldots, x_n)$. For each i_ℓ, $1 \leq i_l \leq n+1$, we have

$$
\sum_{j=1}^{n} \frac{\partial y_{i_\ell}(x_1, \ldots, x_n)}{\partial x_j} dx_j = \begin{cases} \sum_{j=1}^{n} \frac{\partial\, x_{i_\ell}}{\partial x_j} dx_j & \text{if } i_\ell \neq n+1 \\ \sum_{j=1}^{n} \frac{\partial\, 0}{\partial x_j} dx_j & \text{if } i_\ell = n+1 \end{cases} = \begin{cases} dx_{i_\ell} & \text{if } i_\ell \neq n+1 \\ 0 & \text{if } i_\ell = n+1 \end{cases}.
$$

This is exactly the map $\mathscr{Z}$ that we had before, hence $\mathscr{Z} = T^*S$. Pulling back a full k-form can be done using this formula and the linearity of the pull-back which is the first identity in Sect. 6.7. In other words, just apply this formula to each term of the k-form.

Question 8.4 For $S : \mathbb{R}^2 \to \mathbb{R}^3$ and $\alpha = P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz \in \bigwedge^1(\mathbb{R}^3)$ show that $T^*S \cdot \alpha$ as defined above is the same as $\mathscr{Z}(\alpha)$.

Now we turn our attention to $\mathscr{C}$. Now we consider a different map, the projection map, denoted by P and shown in Fig. 8.12,

$$
\begin{aligned}
\mathbb{R}^{n+1} &\xrightarrow{P} \mathbb{R}^n \\
(y_1, \ldots, y_{n+1}) &\longmapsto \big(x_1(y_1, \ldots, y_{n+1}), \ldots, x_n(y_1, \ldots, y_{n+1}) \big)
\end{aligned}
$$

where

$$
\begin{aligned}
x_1(y_1, \ldots, y_{n+1}) &= y_1, \\
&\vdots \\
x_n(y_1, \ldots, y_n, y_{n+1}) &= y_n.
\end{aligned}
$$

This gives us the pullback map $T^*P \equiv P^*$,

$$
\begin{gathered}
\bigwedge^k(\mathbb{R}^{n+1}) \xleftarrow{T^*P} \bigwedge^k(\mathbb{R}^n) \\
\mathbb{R}^{n+1} \xrightarrow{P} \mathbb{R}^n.
\end{gathered}
$$

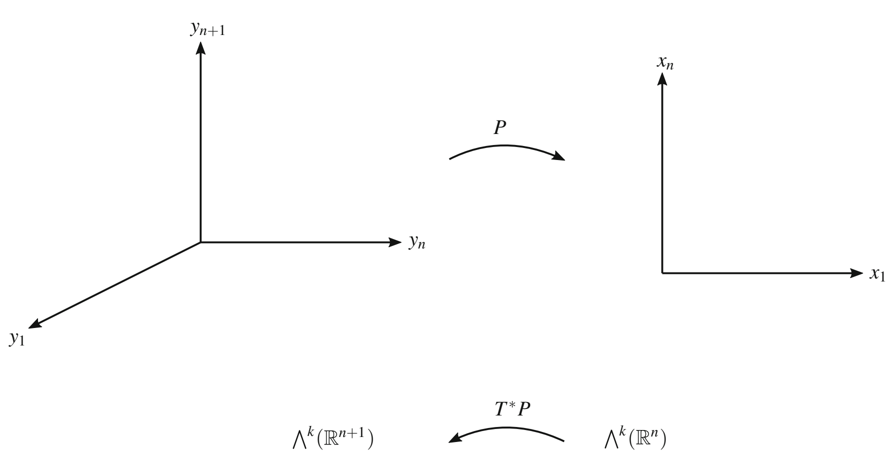


Fig. 8.12 The mapping $P : \mathbb{R}^{n+1} \to \mathbb{R}^n$ along with the pullback mapping $T^*P \equiv P^* : \bigwedge^k(\mathbb{R}^n) \to \bigwedge^k(\mathbb{R}^{n+1})$

The building blocks of k-forms on $\mathbb{R}^n$ are $f_{i_1\cdots 1_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ where $i_1 < i_2 < \cdots < i_k$. We compute

$$
\begin{aligned}
&T^*P \cdot \left(f_{i_1\cdots 1_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right) \\
&= (T^*Pf_{i_1\cdots 1_k}) \cdot T^*P dx_{i_1} \wedge \cdots \wedge T^*P dx_{i_k} \\
&= (f_{i_1\cdots 1_k} \circ P) \cdot d(T^*Px_{i_1}) \wedge \cdots \wedge d(T^*Px_{i_k}) \\
&= (f_{i_1\cdots 1_k} \circ P) \cdot d(x_{i_1} \circ P) \wedge \cdots \wedge d(x_{i_k} \circ P) \\
&= f_{i_1\cdots 1_k}(y_1, \ldots, y_{n+1})\, dy_{i_1} \wedge \cdots \wedge dy_{i_k}.
\end{aligned}
$$

Thus we see that our mapping is exactly $\mathscr{C} = P^*$.

Question 8.5 Show the last equality above, that is, show that $dx_{i_l} = dy_{i_l}$ for $1 \le l \le k \le n$.

Question 8.6 Show that $\mathscr{Z}\big(\mathscr{C}(\beta)\big) = \beta$ for any k-form β.

Since $\mathscr{C} = P^*$ and $\mathscr{Z} = S^*$ and since we already know from the properties of pull-backs and exterior derivatives that $dP^* = P^*d$ and $dS^* = S^*d$ we automatically have $d\mathscr{C} = \mathscr{C}d$ and $d\mathscr{Z} = \mathscr{Z}d$.

In the general case the mapping $\mathscr{K} : \bigwedge^k(\mathbb{R}^{n+1}) \to \bigwedge^{k-1}(\mathbb{R}^{n+1})$ is given by

$$
\mathscr{K}\Big(f(x_1, \ldots, x_{n+1}) dx_{i_1} \wedge \cdots \wedge dx_{i_k}\Big) = \begin{cases} 0 & \text{if } i_k \ne n+1, \\ \Big(\int_0^{x_{n+1}} f(x_1, \ldots, x_n, t) dt\Big)\, dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}} & \text{if } i_k = n+1. \end{cases}
$$

The mapping $\mathscr{K} : \bigwedge^{k+1}(\mathbb{R}^{n+1}) \to \bigwedge^k(\mathbb{R}^{n+1})$ is similar, but adjusted accordingly.

We are now ready for the proof of the general case of the Poincaré lemma. Suppose we have $\alpha \in \bigwedge^k(\mathbb{R}^{n+1})$, which is closed. That is, $d\alpha = 0$. The mappings we will use are shown at the top of Fig. 8.13 and the general schematic of what we

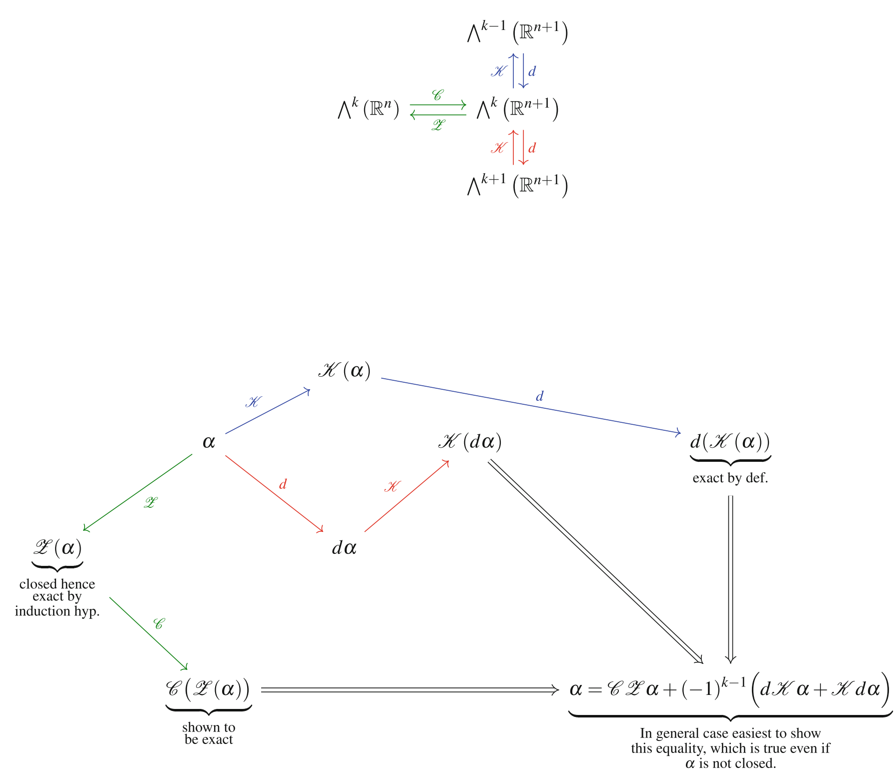


Fig. 8.13 The various mappings which are important in the proof of the general case of the Poincaré lemma are shown above along with a schematic of the general case of the proof of the Poincaré lemma, shown below. Recall that a k-form α has the form $\alpha = \sum \alpha_{i_1\cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, where each $\alpha_{i_1\cdots i_k}$ is a function on the manifold. While the full k-form α may be closed, each of the terms $\alpha_{i_1\cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ that makes up α may not be. The derived formula $\alpha = \mathscr{C}\mathscr{Z}\alpha + (-1)^{k-1}\big(d\mathscr{K}\alpha + \mathscr{K}d\alpha\big)$ applies to each of these terms. Using linearity the formula applies to all of α, and if α is closed then the formula becomes $\alpha = \mathscr{C}\mathscr{Z}\alpha + (-1)^{k-1}d\mathscr{K}\alpha$, which is shown to be exact

do is shown on the bottom. In the sample case we showed $\alpha = \mathscr{C}\mathscr{Z}\alpha + d(\mathscr{K}\alpha)$. We also showed that $\mathscr{C}\mathscr{Z}\alpha$ was exact and clearly $d(\mathscr{K}\alpha)$ is exact and so it followed that α was exact. It turns out that in the general case it is actually easier to prove something more general, that for any closed k-form α we have

$$\begin{aligned}\alpha &= \mathscr{C}\mathscr{Z}\alpha + (-1)^{k-1}\Big(d(\mathscr{K}\alpha) + \mathscr{K}d\alpha\Big)\\ &= \mathscr{C}\mathscr{Z}\alpha + (-1)^{k-1}\big(d\mathscr{K} + \mathscr{K}d\big)\alpha.\end{aligned}$$

Why would we do this? The problem is that in general the k-form α on $\mathbb{R}^{n+1}$ has the form

$$\alpha = \sum \alpha_{j_1\cdots j_k}(x_1, x_2, \ldots, x_{n+1})dx_{j_1} \wedge \cdots \wedge dx_{j_k},$$

where each $\alpha_{j_1\cdots j_k}$ is a real-valued function and of course $j_1 < \cdots < j_k$. While α overall may be closed each individual term in the sum may not be closed, and in fact probably is not closed. This is what creates the problem. For simplicity's sake we want to do the computation in our proof on just one term in the sum at a time. That is, we want to do our computation on each term $\alpha_{j_1\cdots j_k}(x_1, x_2, \ldots, x_{n+1})dx_{j_1} \wedge \cdots \wedge dx_{j_k}$ in the sum separately. Being able to do that makes the computations much simpler. We can then just use linearity to apply our result to all of α, which is just a sum of the terms.

The first part of the proof, showing $\mathscr{C}\mathscr{Z}\alpha$ is exact is the same as before. Since α is closed then by definition of closed we have $d\alpha = 0$. which gives us

$$d(\mathscr{C}\alpha) = \mathscr{C}(d\alpha) = \mathscr{C}(0) = 0,$$

so $\mathscr{C}\alpha$ is also closed. But $\mathscr{C}\alpha \in \bigwedge^k(\mathbb{R}^n)$ and our ***induction hypothesis*** *is that all closed forms on* $\mathbb{R}^n$ *are exact*. By the induction hypotheses since $\mathscr{C}\alpha$ is closed then it is also exact so $\mathscr{C}\alpha = d\beta$ for some β. Hence we have

$$\mathscr{Z}\mathscr{C}\alpha = \mathscr{Z}d\beta = d\mathscr{Z}\beta.$$

But this means that $\mathscr{Z}\mathscr{C}\alpha$ is the exterior derivative of the form $\mathscr{Z}\beta$, which by the definition of exactness means the $\mathscr{Z}\mathscr{C}\alpha$ is exact. Thus we have shown that if α is closed then $\mathscr{Z}\mathscr{C}\alpha$ is exact.

Now we turn our attention to the identity we want to show. We will need to break our general proof on a building block element of α into two cases, one where $j_k = n+1$ and one where $j_k \neq n+1$.

Case One: $j_k = n + 1$

We will consider a single term of the form $f(x_1, \ldots, x_{n+1})dx_{j_1} \wedge \cdots \wedge dx_{j_k}$, where $j_1 < j_2 < \cdots < j_k$ and $j_k = n+1$,

$$\begin{aligned}
& d\mathscr{K}\Big(f(x_1, \ldots, x_{n+1})dx_{j_1} \wedge \cdots \wedge dx_{j_k}\Big)\\
&= d\left(\left(\int_0^{x_{n+1}} f(x_1, \ldots, x_n, t)dt\right) dx_{j_1} \wedge \cdots \wedge dx_{j_{k-1}}\right)\\
&= \sum_{i=1}^{n+1} \frac{\partial}{\partial x_i}\left(\int_0^{x_{n+1}} f(x_1, \ldots, x_n, t)dt\right) dx_i \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{k-1}}\\
&= \sum_{i=1}^{n} \left(\int_0^{x_{n+1}} \frac{\partial}{\partial x_i} f(x_1, \ldots, x_n, t)dt\right) dx_i \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{k-1}}\\
&\qquad + \frac{\partial}{\partial x_{n+1}}\left(\int_0^{x_{n+1}} f(x_1, \ldots, x_n, t)dt\right) dx_{n+1} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{k-1}}\\
&= \sum_{i=1}^{n} \left(\int_0^{x_{n+1}} \frac{\partial}{\partial x_i} f(x_1, \ldots, x_n, t)dt\right) dx_i \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{k-1}}\\
&\qquad + f(x_1, \ldots, x_{n_1})dx_{n+1} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{k-1}}.
\end{aligned}$$

Next we compute

$$\begin{aligned}
&\mathscr{K} d\Big(f(x_1,\dots,x_{n+1})dx_{j_1}\wedge\cdots\wedge dx_{j_k}\Big)\\
&= \mathscr{K}\left(\sum_{i=1}^{n+1}\frac{\partial f(x_1,\dots,x_{n+1})}{\partial x_i}\,dx_i\wedge dx_{j_1}\wedge\cdots\wedge dx_{j_k}\right)\\
&= \mathscr{K}\left(\sum_{i=1}^{n}\frac{\partial f(x_1,\dots,x_{n+1})}{\partial x_i}\,dx_i\wedge dx_{j_1}\wedge\cdots\wedge dx_{j_k}\right)\\
&= \sum_{i=1}^{n}\mathscr{K}\left(\frac{\partial f(x_1,\dots,x_{n+1})}{\partial x_i}\,dx_i\wedge dx_{j_1}\wedge\cdots\wedge dx_{j_k}\right)\\
&= \sum_{i=1}^{n}\left(\int_0^{x_{n+1}}\frac{\partial f(x_1,\dots,x_n,t)}{\partial x_i}\,dt\right)\,dx_i\wedge dx_{j_1}\wedge\cdots\wedge dx_{j_k}.
\end{aligned}$$

Subtracting the second from the first, recalling that $j_k = n+1$, and then noting that

$$\mathscr{C}\mathscr{Z}\big(f(x_1,\dots,x_{n_1})dx_{j_1}\wedge\cdots\wedge dx_{j_{k-1}}\wedge dx_{n+1}\big)=0,$$

we get

$$\begin{aligned}
&(d\mathscr{K}-\mathscr{K}d)\big(f(x_1,\dots,x_{n+1})dx_{j_1}\wedge\cdots\wedge dx_{j_k}\big)\\
&= f(x_1,\dots,x_{n_1})dx_{n+1}\wedge dx_{j_1}\wedge\cdots\wedge dx_{j_{k-1}}\\
&= (-1)^{k-1}f(x_1,\dots,x_{n_1})dx_{j_1}\wedge\cdots\wedge dx_{j_{k-1}}\wedge dx_{n+1}-0\\
&= (-1)^{k-1}\big(1-\mathscr{C}\mathscr{Z}\big)\big(f(x_1,\dots,x_{n_1})dx_{j_1}\wedge\cdots\wedge dx_{j_{k-1}}\wedge dx_{n+1}\big).
\end{aligned}$$

A bit of rearrangement, and relabeling $\omega = f(x_1,\dots,x_{n+1})dx_{j_1}\wedge\cdots\wedge dx_{j_k}$, which may not actually itself be closed, gives us what we wanted, namely

$$\omega = \mathscr{C}\mathscr{Z}\omega + (-1)^{k-1}\big(d\mathscr{K}\omega - \mathscr{K}d\omega\big).$$

Question 8.7 Show that $\mathscr{C}\mathscr{Z}\big(f(x_1,\dots,x_{n_1})dx_{j_1}\wedge\cdots\wedge dx_{j_{k-1}}\wedge dx_{n+1}\big)=0$.

Case Two: $j_k \neq n+1$
We will consider a single term of the form $f(x_1,\dots,x_{n+1})dx_{j_1}\wedge\cdots\wedge dx_{j_k}$, where $j_1 < j_2 < \cdots < j_k$ but where $j_k \neq n+1$. Here, by definition of the mapping $\mathscr{K}$, we have

$$d\mathscr{K}\big(f(x_1,\dots,x_{n+1})dx_{j_1}\wedge\cdots\wedge dx_{j_k}\big)=0.$$

Now we compute

$$\begin{aligned}
&\mathscr{K} d\Big(f(x_1,\dots,x_{n+1})dx_{j_1}\wedge\cdots\wedge dx_{j_k}\Big)\\
&= \mathscr{K}\left(\sum_{i=1}^{n+1}\frac{\partial f(x_1,\dots,x_{n+1})}{\partial x_i}\,dx_i\wedge dx_{j_1}\wedge\cdots\wedge dx_{j_k}\right)\\
&= \mathscr{K}\left(\underbrace{\sum_{i=1}^{n}\frac{\partial f(x_1,\dots,x_{n+1})}{\partial x_i}\,dx_i\wedge dx_{j_1}\wedge\cdots\wedge dx_{j_k}}_{\mathscr{K}\text{ kills all these terms since they not have } dx_{n+1}}+\frac{\partial f(x_1,\dots,x_{n+1})}{\partial x_{n+1}}\,dx_{n+1}\wedge dx_{j_1}\wedge\cdots\wedge dx_{j_k}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \mathscr{K}\left((-1)^k \frac{\partial f(x_1, \ldots, x_{n+1})}{\partial x_{n+1}}\, dx_{j_1} \wedge \cdots \wedge dx_{j_k} \wedge dx_{n+1}\right) \\
&= (-1)^k \left(\int_0^{x_{n+1}} \frac{\partial f(x_1, \ldots, x_n, t)}{\partial x_{n+1}}\, dt\right) dx_{j_1} \wedge \cdots \wedge dx_{j_k} \\
&= (-1)^k \Big(f(x_1, \ldots, x_n, x_{n+1}) - f(x_1, \ldots, x_n, 0)\Big) dx_{j_1} \wedge \cdots \wedge dx_{j_k}. \qquad \text{fundamental theorem of calculus.}
\end{aligned}
$$

Subtracting the second equation from the first give us

$$
\begin{aligned}
&(d\mathscr{K} - \mathscr{K} d)\big(f(x_1, \ldots, x_{n+1}) dx_{j_1} \wedge \cdots \wedge dx_{j_k}\big) \\
&= 0 - (-1)^k \Big(f(x_1, \ldots, x_n, x_{n+1}) - f(x_1, \ldots, x_n, 0)\Big) dx_{j_1} \wedge \cdots \wedge dx_{j_k} \\
&= (-1)^{k-1} \Big(f(x_1, \ldots, x_n, x_{n+1}) dx_{j_1} \wedge \cdots \wedge dx_{j_k} - f(x_1, \ldots, x_n, 0) dx_{j_1} \wedge \cdots \wedge dx_{j_k}\Big).
\end{aligned}
$$

But if $j_k \neq n+1$ then

$$
\mathscr{C}\mathscr{Z}\big(f(x_1, \ldots, x_{n+1}) dx_{j_1} \wedge \cdots \wedge dx_{j_k}\big) = f(x_1, \ldots, x_n, 0) dx_{j_1} \wedge \cdots \wedge dx_{j_k}
$$

so letting $\omega = f(x_1, \ldots, x_{n+1}) dx_{j_1} \wedge \cdots \wedge dx_{j_k}$ we have

$$
\begin{aligned}
(d\mathscr{K} - \mathscr{K} d)\omega &= (-1)^{k-1}\big(\omega - \mathscr{C}\mathscr{Z}\omega\big) \\
\Rightarrow \omega &= \mathscr{C}\mathscr{Z}\omega + (-1)^{k-1}\big(d\mathscr{K}\omega - \mathscr{K} d\omega\big),
\end{aligned}
$$

which is exactly the formula we had in case one.

Thus, regardless of whether $j_k = n+1$ or if $j_k \neq n+1$ we end up with exactly the same identity for each term ω of the k-form α, namely that

$$
\mathscr{C}\mathscr{Z}\omega + (-1)^{k-1}\big(d\mathscr{K}\omega - \mathscr{K} d\omega\big).
$$

Thus this formula holds for each individual term in the k-form α. Now we are ready to complete the proof of the Poincaré lemma. We will leave a couple of the steps to you in the following questions.

Question 8.8 Show that the mappings $\mathscr{Z}$, $\mathscr{C}$, and $\mathscr{K}$ are linear. That is, show that if α and β are k-forms that $\mathscr{Z}(\alpha+\beta) = \mathscr{Z}(\alpha) + \mathscr{Z}(\beta)$, $\mathscr{C}(\alpha+\beta) = \mathscr{C}(\alpha) + \mathscr{C}(\beta)$, and $\mathscr{K}(\alpha+\beta) = \mathscr{K}(\alpha) + \mathscr{K}(\beta)$.

Question 8.9 Using linearity of $\mathscr{C}$, $\mathscr{Z}$, $\mathscr{K}$, and d show that the identity

$$
\omega = \mathscr{C}\mathscr{Z}\omega + (-1)^{k-1}\big(d\mathscr{K}\omega - \mathscr{K} d\omega\big)
$$

applies to a general k-form

$$
\alpha = \sum_i^{n+1} \alpha_{j_1 \cdots j_k} dx_{j_1} \wedge \cdots \wedge dx_{j_k}.
$$

From the above questions, for a general k-form α we have that $\alpha = \mathscr{C}\mathscr{Z}\omega + (-1)^{k-1}\big(d\mathscr{K}\alpha - \mathscr{K} d\alpha\big)$. To show that the closed k-forms on $\mathbb{R}^{n+1}$ are exact we are assuming by the ***induction hypothesis*** *that the closed k-forms on $\mathbb{R}^n$ are exact*. The induction hypothesis was used to show that if α is closed then $\mathscr{C}\mathscr{Z}\alpha$ is exact, so there exists a β such that $\mathscr{C}\mathscr{Z}\alpha = d\beta$. Also, since α is closed we have $\mathscr{K} d\alpha = 0$, so

$$
\begin{aligned}
\alpha &= \mathscr{C}\mathscr{Z}\alpha + (-1)^{k-1}\big(d\mathscr{K}\alpha - \mathscr{K} \underbrace{d\alpha}_{=0}\big) \\
&= d\beta + (-1)^{k-1}\big(d\mathscr{K}\alpha - 0\big)
\end{aligned}
$$

$$= d\beta + d\big((-1)^{k-1}\mathscr{K}\alpha\big)$$
$$= d\Big(\beta + (-1)^{k-1}\mathscr{K}\alpha\Big)$$

so α is exact.

Question 8.10 Explain in more detail, using the diagram in Figs. 8.5 and 8.13, how the induction hypothesis, along with the formula that was derived, allow us to conclude that every closed k-form on any $\mathbb{R}^n$, is exact.

8.4 Summary, References, and Problems

8.4.1 Summary

The Poincaré lemma states that *every closed form on* $\mathbb{R}^n$ *is exact*. A differential form α is called closed if $d\alpha = 0$. A differential form α is called exact if there is another differential form β such that $\alpha = d\beta$. Obviously, if α is an exact k-form then β must be a $(k-1)$-form. So, another way of phrasing the Poincaré lemma is to say that *if* α *is a* k*-form on* $\mathbb{R}^n$ *such that* $d\alpha = 0$, *then there exists some* $(k-1)$*-form* β *such that* $\alpha = d\beta$.

8.4.2 References and Further Reading

The Poincaré lemma is an absolutely essential material for any book that looks at calculus on manifolds. In particular, Munkres [35], Renteln [37], and Abraham, Marsden, and Ratiu [1] are nice presentations. As is often the case in mathematics, with the right mathematical concepts, machinery, and sophistication the proofs of many theorems can be reduced to a mere handful of lines. This is very much the case with the Poincaré lemma. The presentation here, while long, is meant to be readily understood by someone with little more mathematical background than calculus. Generally that is not the case with most other presentations of the Poincaré lemma.

8.4.3 Problems

Question 8.11 Show that each of the below one-forms α_i is closed; that is, show that $d\alpha_i = 0$. By the Poincaré lemma α_i must then be exact. Then find a function f_i such that $\alpha_i = df_i$.

a) $\alpha_1 = y^2\,dx + 2xy\,dy$

b) $\alpha_2 = (3x^2 + y)\,dx + (x + 2y)\,dy$

c) $\alpha_3 = 2x\,dx + 3y^2\,dy + 4z^3\,dz$

d) $\alpha_4 = 2xy\,dx + (x^2 + 2y + z)\,dy + (y - 3z^2)\,dz$

e) $\alpha_5 = (6x^2z - yz)\,dx + (3z^4 - xz)\,dy + (2x^3 + 12yz^3 - xy)\,dz$

f) $\alpha_6 = (3y^2 - 4z^4)\,dx + 6xy\,dy - 16xz^3\,dz$

Question 8.12 Show that each of the below two-forms β_i is closed; that is, show that $d\beta_i = 0$. By the Poincaré lemma β_i must then be exact. Find a one-form α_i such that $\beta_i = d\alpha_i$.

a) $\beta_1 = 24x^3y^2\,dx \wedge dy$

b) $\beta_2 = (6x^2y - 3xy^2)\,dx \wedge dy$

c) $\beta_3 = 8x^3y^3\,dy \wedge dz - 6x^2y^4\,dz \wedge dx$

d) $\beta_4 = (2x - 1)\,dx \wedge dy + (3y^2 - 2z)\,dy \wedge dz + (1 - 3x^2)\,dz \wedge dx$

e) $\beta_5 = -4xy^2\,dx \wedge dy + 2x^2y^2\,dy \wedge dz + 3xz^2\,dz \wedge dx$

f) $\beta_6 = -4xy\,dx \wedge dy + 2yz\,dy \wedge dz - 2xz\,dz \wedge dx$

Question 8.13 Show that each of the below three-forms γ_i is closed; that is, show that $d\gamma_i = 0$. By the Poincaré lemma γ_i must then be exact. Find a two-form β_i such that $\gamma_i = d\beta_i$.

a) $\gamma_1 = 12x^2y^3z^4\,dx \wedge dy \wedge dz$

b) $\gamma_2 = (x^3 - 4y + z^2)\,dx \wedge dy \wedge dz$

c) $\gamma_3 = (2xz + 3yz^2)\,dx \wedge dy \wedge dz$

d) $\gamma_4 = 2(x + y + z)\,dx \wedge dy \wedge dz$

Chapter 9
Vector Calculus and Differential Forms

In sections one through three we take a careful look at divergence, curl, and gradient from vector calculus, introducing and defining all three from a geometrical point of view. In section four we introduce some notation and consider two important operators, the sharp and flat operators. These operators are necessary to understand the relationship between divergence, curl, gradient and differential forms, which is looked at in section five. Also in section five we see how the fundamental theorem of line integrals, the divergence theorem, and the vector calculus version of Stokes' theorem can all be written in the same way using differential forms notation. These three theorems are all special cases of what is called the generalized Stokes' theorem. In more advanced mathematics classes Stokes' theorem always refers to the generalized Stokes' theorem and not to the version of Stokes' theorem you learned in vector calculus, while in physics Stokes' theorem may refer to either version depending on context. But keep in mind, they are actually the same theorem from a more abstract perspective.

9.1 Divergence

In vector calculus a vector field $\mathbf{F}$ on $\mathbb{R}^3$ was generally written as $\mathbf{F} = P(x, y, z)\hat{i} + Q(x, y, z)\hat{j} + R(x, y, z)\hat{k}$, where $P, Q, R : \mathbb{R}^3 \to \mathbb{R}$. Here $\hat{i}$, $\hat{j}$, and $\hat{k}$ are the unit vectors in the x, y, and z directions respectively. If we were to denote our variables as x_1, x_2, x_3 then the unit vectors are generally written as e_1, e_2, e_2. Recall that we always write a vector as a column vector,

$$\mathbf{F} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}.$$

In vector calculus classes the divergence of a vector field $\mathbf{F}$ is usually defined to be

$$\text{div}\, \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Often an operator ∇ is defined as

$$\nabla = \frac{\partial}{\partial x}e_1 + \frac{\partial}{\partial y}e_2 + \frac{\partial}{\partial z}e_3.$$

If we do this then we can consider the "dot product" of the operator ∇ with a vector $\mathbf{F}$ as

$$\text{div}\, \mathbf{F} = \nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}e_1 + \frac{\partial}{\partial y}e_2 + \frac{\partial}{\partial z}e_3\right) \cdot (Pe_1 + Qe_2 + Re_3) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

This does seem a little silly since dot products require two vectors and clearly ∇ is not really a vector, it is something else. It is generally called an operator and is said to "operate" on the vector $\mathbf{F}$. Similarly, one can think of the ∇ operator as being a

J. P. Fortney, *A Visual Introduction to Differential Forms and Calculus on Manifolds*,
https://doi.org/10.1007/978-3-319-96992-3_9

row vector which is matrix multiplied by the column vector **F**,

$$\nabla \cdot \mathbf{F} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \begin{bmatrix} P \\ Q \\ R \end{bmatrix}.$$

This so-called definition of div **F** as $\nabla \cdot \mathbf{F}$ is really more of a mnemonic device to help us remember the form that div **F** takes when using Cartesian coordinates.

A quick word about terminology. The symbol ∇ is actually called *nabla* after the Hebrew word for harp, a stringed musical instrument that has roughly a ∇ shape. When ∇ is used to represent the divergence operator then it is generally called *del*. One would read $\nabla \cdot \mathbf{F}$ as "del dot F," or as "div F," or as "divergence of F."

We will take a different approach here and define div **F** in a way that will allow the divergence theorem to just fall out of the definition. Using this we will derive the standard formula for div **F** in Cartesian coordinates. We take this approach because we believe it provides a somewhat more geometrical meaning behind the divergence. But before we do that we first need to understand what the surface integral of the normal component of a vector field is. Since we actually assume you have already taken a vector calculus course we will cover this material quickly and without too much detail; we are more interested in conceptual understanding than in mathematical rigor.

Suppose we are given a vector field $\mathbf{F} = P\hat{i} + Q\hat{j} + R\hat{k}$ defined on $\mathbb{R}^3$ along with a two dimensional surface S in $\mathbb{R}^3$ as in Fig. 9.1. We want to define the integral of **F** over the surface S. In a sense we are asking how much of the vector field **F** goes through the surface S. This makes more intuitive sense if **F** represents the velocity of fluid particles. Then we are basically asking the rate of fluid flow through the surface S. See Fig. 9.2.

Suppose for some fluid with density $\rho(x, y, z)$ we have that $\mathbf{v}(x, y, z)$ is the velocity at which the particle of fluid at point (x, y, z) is flowing. We want to find how much fluid flows through some small surface ΔS. If we suppose that **v** is perpendicular to ΔS. Then we have that over a small time period Δt the mass of the fluid that flows through ΔS is given by $\rho(x, y, z)\ \mathbf{v}(x, y, z)\ \Delta t\ \Delta S$, as is seen in Fig. 9.2. The rate of flow is given by the mass of the fluid that passes through the surface divided by the time over which this mass passed through the surface,

$$\begin{array}{c}\text{rate of}\\ \text{fluid flow}\\ \text{through}\\ \text{surface } \Delta S\end{array} = \frac{\rho\ \mathbf{v}\ \Delta t\ \Delta S}{\Delta t} = \rho\ \mathbf{v}\ \Delta S.$$

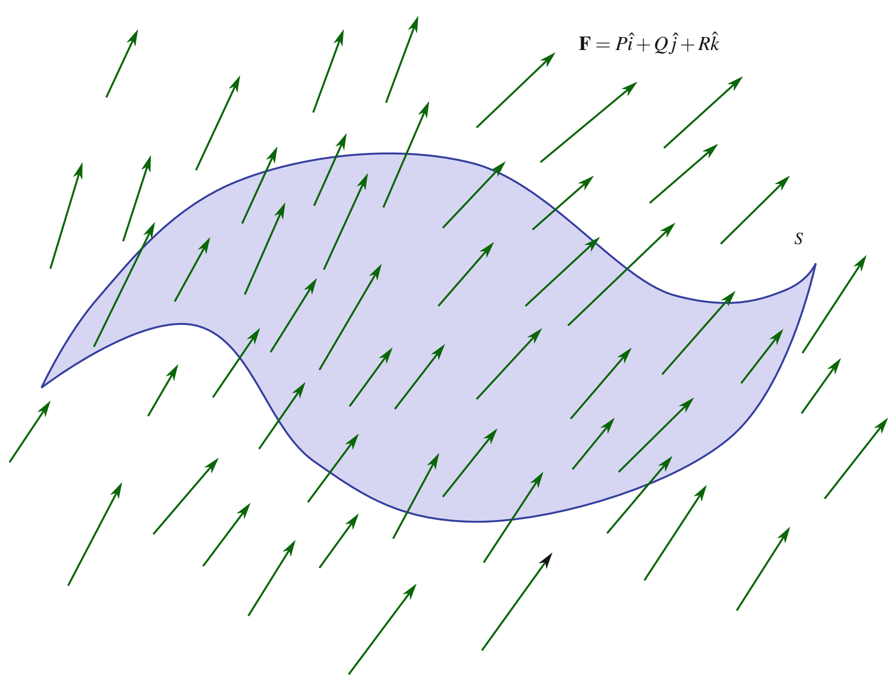
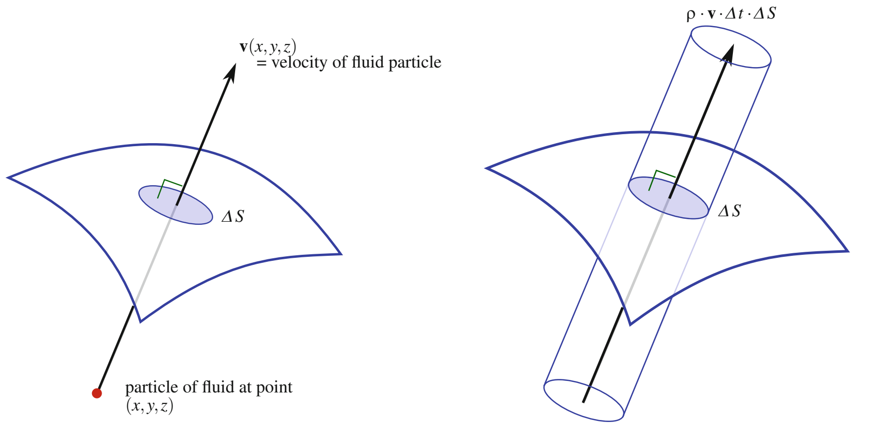


Fig. 9.1 A vector field **F** in $\mathbb{R}^3$ along with a two dimensional surface S



A vector $\mathbf{v}$ with base point (x,y,z) that "penetrates" the surface S. It is helpful to imagine a particle at point (x,y,z) traveling with velocity $\mathbf{v}$.

Estimating the mass of fluid flow through a small bit of surface, ΔS, over a small time period Δt assuming all particles are traveling with velocity $\mathbf{v}$ and the particles have density ρ.

Fig. 9.2 A natural way to think about integrating a vector field $\mathbf{v}$ over a surface S is to think of it as finding the rate of flow of a fluid over that surface traveling with velocity $\mathbf{v}$

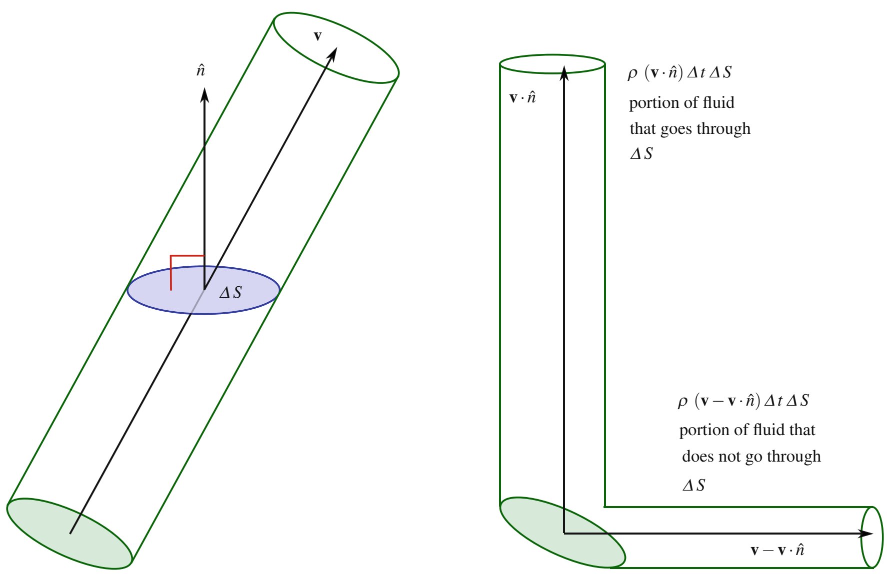


Fig. 9.3 Suppose $\mathbf{v}$ is not perpendicular to the surface S (left). Then we "decompose" the fluid flow into into two parts, a portion that goes through the surface S and a portion that does not go through S (right). This necessitates taking the dot product of $\mathbf{v}$ with the normal unit vector to S, denoted by $\hat{n}$

If $\mathbf{v}$ is not perpendicular to the surface ΔS then the flow through ΔS is given by $\rho(x, y, z)\ \mathbf{v}(x, y, z) \cdot \hat{n}\ \Delta t\ \Delta S$. That is, we replace $\mathbf{v}$ in the product with $\mathbf{v} \cdot \hat{n}$, the dot product with the unit vector normal to the surface element ΔS. This dot product gives us the component of $\mathbf{v}$ in the $\hat{n}$ direction. See Fig. 9.3. In other words, it gives us the velocity of the fluid flow in the $\hat{n}$ direction,

$$\begin{array}{c}\text{rate of}\\ \text{fluid flow}\\ \text{through}\\ \text{surface } \Delta S\end{array} = \frac{\rho\ (\mathbf{v} \cdot \hat{n})\ \Delta t\ \Delta S}{\Delta t} = \rho\ (\mathbf{v} \cdot \hat{n})\ \Delta S.$$

We can approximate the fluid flow through the whole surface S by breaking S into small pieces ΔS, finding the flow through each piece and then summing. Then to find the exact flow we can take the limit as the size of the ΔS shrinks to zero,

$|\Delta S| \to 0$,

$$\begin{array}{c}\text{rate of}\\ \text{fluid flow}\\ \text{through}\\ \text{surface } S\end{array} = \lim_{|\Delta S_i| \to 0} \sum_i \rho\,(\mathbf{v} \cdot \hat{n})\,\Delta S \;=\; \int_S \rho\,(\mathbf{v} \cdot \hat{n})\,dS.$$

We have been dealing with a concrete situation, a fluid flow, to help us understand what is going on better. But the same reasoning applies to any vector field $\mathbf{F}$; we can find the "flow" of the vector field $\mathbf{F}$ through the surface S. But when we take away our mental crutch of the fluid and deal just with a vector field there is actually no fluid density ρ, so we must drop the ρ from the above equations. But in this more pure situation where we just have a vector field and not a fluid then we are not actually finding the rate of a fluid flow through S. This admittedly abstract thing we are finding is called the *flux of the vector field* $\mathbf{F}$ *through the surface* S. Flux is the Latin word for flow. What flux actually represents in a physical situation depends upon the particular problem and on how the vector field is being interpreted. Is the vector a fluid flow? An electric field? A magnetic field? A force? In each case the flux through a surface would be interpreted differently. So the flux is actually an abstract mathematical concept that can be interpreted differently in different physical situations. Later on we will see examples where the field $\mathbf{F}$ represents an electric field or a magnetic field and we will find the electric and the magnetic flux through a surface. However, in general probably the best way to think about what a flux actually is is to imagine a fluid without a density. Thus we would have

$$\text{Flux of } \mathbf{F} = \lim_{|\Delta S_i| \to 0} \sum_i \mathbf{F} \cdot \hat{n}\,\Delta S \;=\; \int_S \mathbf{F} \cdot \hat{n}\,dS.$$

Note that this integral makes sense, at each point p on the surface S we have that $\mathbf{F}_p \cdot \hat{n}_p$ is a real number. That is, we can think of $\mathbf{F} \cdot \hat{n}$ as a real-valued function on S, $\mathbf{F} \cdot \hat{n} : S \to \mathbb{R}$, which means we can easily integrate it over the surface S. In vector calculus you learned various techniques for doing just this. In summary, the flux of the vector field $\mathbf{F}$ through the surface S is given by the integral

$$\text{Flux of } \mathbf{F} \;=\; \int_S \mathbf{F} \cdot \hat{n}\,dS.$$

A couple notes on notation, sometimes this integral is written as

$$\int_S \mathbf{F} \cdot dS \qquad \text{or} \qquad \int_S \mathbf{F} \cdot d\mathbf{S}.$$

So when you see either $\int_S \mathbf{F} \cdot dS$ or $\int_S \mathbf{F} \cdot d\mathbf{S}$ this means exactly $\int_S \mathbf{F} \cdot \hat{n}\; dS$. This notation is explained in more depth in Sect. 9.5.3. Also, since S is a surface and not function then clearly dS does not represent the exterior derivative of a zero-from. But dS comes from the ΔS, which represents the area of a small bit of surface S, so roughly you can think of dS as being an "area form" on the surface S, which allows you to find the surface area of the surface S. We will not delve into the details here.

Now we will define the divergence of $\mathbf{F}$ at a point (x_0, y_0, z_0). Suppose we have a small three dimensional region V about the point (x_0, y_0, z_0). The boundary of this region V is denoted by ∂V. By closed we mean that the surface S is like a sphere with no edges. We will also denote the volume of the region V by ΔV. Then we define the **divergence** of $\mathbf{F}$ to be given by

$$\boxed{\begin{array}{c}\text{Definition}\\ \text{of}\\ \text{divergence}\end{array} \qquad \operatorname{div} \mathbf{F} \;=\; \lim_{\Delta V \to 0} \frac{1}{\Delta V} \int_{\partial V} \mathbf{F} \cdot \hat{n}\,dS.}$$

Understanding what this actually represents may be a little easier if we write the right hand side as

$$\lim_{\Delta V \to 0} \frac{\int_{\partial V} \mathbf{F} \cdot \hat{n}\,dS}{\Delta V}$$

and consider what this means. In the numerator is the flux through ∂V, that is, the flow of the vector field through the surface ∂V. Since ∂V is a closed surface, if we choose the normal vector $\hat{n}$ to point outwards, the flux of $\mathbf{F}$ through ∂V is the net flux of $\mathbf{F}$ out of V minus the net flux of $\mathbf{F}$ into V. In other words, the net flux of $\mathbf{F}$ out of V through ∂V, that is, $\int_{\partial V} \mathbf{F} \cdot \hat{n}\,dS$, can either be positive, zero, or negative. By dividing net flux of $\mathbf{F}$ out of V by ΔV we are finding the net flux of $\mathbf{F}$ out of V per unit volume. When we let the limit as the volume ΔV around the point (x_0, y_0, z_0) go to zero then we are finding the net

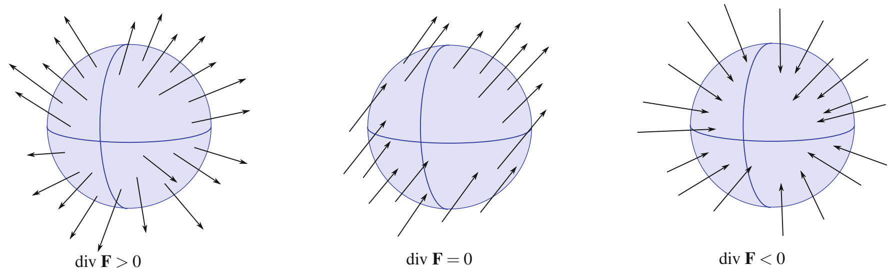


Fig. 9.4 Three examples of the flux of a vector field out of a small volume ΔV about a point (x_0, y_0, z_0). The cases where the flux is positive (left), zero (middle), and negative (right) are shown

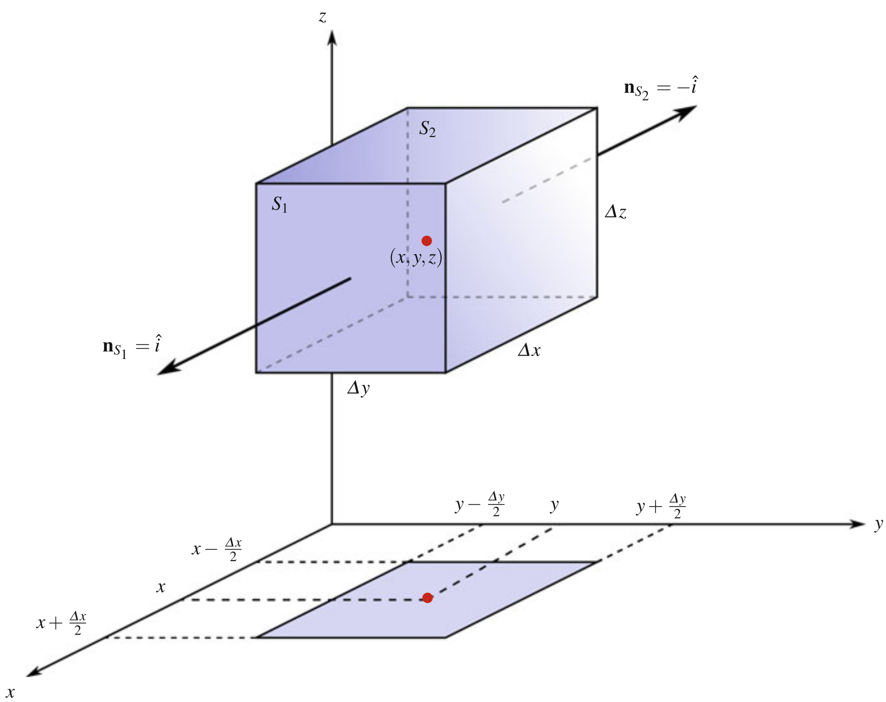


Fig. 9.5 The region V as a cube with sides Δx, Δy, and Δz. Thus we have $\Delta V = \Delta x \ \Delta y \ \Delta z$. The sides S_1 and S_2 (the front and the back of the cube) are perpendicular to the x-axis. The outward pointing unit normal to S_1 is $\hat{i}$ and the outward pointing unit normal to S_2 is $-\hat{i}$

flux out per unit volume at the point (x_0, y_0, z_0). In other words, this is how much the vector field $\mathbf{F}$ "diverges" at the point (x_0, y_0, z_0).

If div $\mathbf{F}$ is positive at the point (x_0, y_0, z_0) then that means that the vector field $\mathbf{F}$ is dissipating from the point (x_0, y_0, z_0). If div $\mathbf{F}$ is negative at the point (x_0, y_0, z_0) then that means that the vector field $\mathbf{F}$ is accumulating at the point (x_0, y_0, z_0). If div $\mathbf{F}$ is zero at the point (x_0, y_0, z_0) then $\mathbf{F}$ is neither accumulating nor dissipating at the point. See Fig. 9.4.

Now we want to use our definition of the divergence of $\mathbf{F}$ as $\lim_{\Delta V \to 0} \frac{1}{\Delta V} \int_S \mathbf{F} \cdot \hat{n} \ dS$ to get the standard formula $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$ for divergence that you learned in vector calculus. Using the cubical region V that is shown in Fig. 9.5 we have surfaces S_1 and S_2 which are parallel to the y-axis and z-axis and perpendicular to the x-axis. We think of S_1 as the front of the cube and S_2 as the back of the cube. The outward normal for S_1 is $\hat{n}_{S_1} = \hat{i}$ and the outward normal for S_2 is $\hat{n}_{S_2} = -\hat{i}$. We have

$$
\begin{aligned}
\int_{S_1} \mathbf{F} \cdot \hat{n}_{S_1} \, dS &\approx \mathbf{F}\left(x_0 + \frac{\Delta x}{2}, y_0, z_0\right) \cdot \hat{i} \ \Delta y \ \Delta z \\
&= P\left(x_0 + \frac{\Delta x}{2}, y_0, z_0\right) \ \Delta y \ \Delta z
\end{aligned}
$$