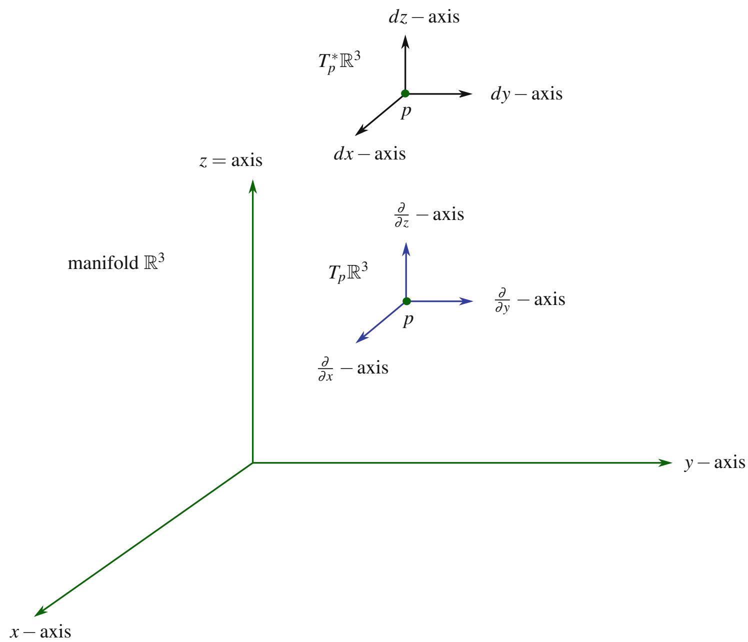

Fig. 3.1 The manifold $\mathbb{R}^3$, the tangent space $T_p\mathbb{R}^3$, and the cotangent space $T_p^*\mathbb{R}^3$, all shown together. Even though the cotangent space $T_p^*\mathbb{R}^3$ is actually attached to the manifold at the same point p that the tangent space $T_p\mathbb{R}^3$ is attached, it is shown above the tangent space. We will generally follow this convention in this book

In the last chapter we also discovered that the tangent space $T_p\mathbb{R}^n$ at each point p has a basis given by

$$\left\{ \left.\frac{\partial}{\partial x_1}\right|_p, \left.\frac{\partial}{\partial x_2}\right|_p, \ldots, \left.\frac{\partial}{\partial x_n}\right|_p \right\}$$

and at each point p there is a cotangent space $T_p^*\mathbb{R}^n$, which is the dual space to the vector space $T_p\mathbb{R}^n$, which has a basis given by

$$\left\{dx_1(p), dx_2(p), \ldots, dx_n(p)\right\}$$

or simply $\{dx_1, dx_2, \ldots, dx_n\}$ if we suppress the base point p in the notation. The picture we have built up so far is shown in Fig. 3.1 where we have drawn manifold $\mathbb{R}^3$ with the tangent space $T_p\mathbb{R}^3$ superimposed. The cotangent space $T_p^*\mathbb{R}^3$ is pictured above the tangent space.

The one-forms $dx_1(p), dx_2(p), \ldots, dx_n(p)$ are exactly the linear functionals on the vector space $T_p\mathbb{R}^n$. And in line with the above, we have $T_p\mathbb{R}^n \simeq \mathbb{R}^n$, with elements $v_p \in T_p\mathbb{R}^n$ usually written as column vectors, and $T_p^*\mathbb{R}^n \simeq \mathbb{R}^n$, with elements $\alpha_p \in T_p^*\mathbb{R}^n$ often written as row vectors and called co-vectors. Thus we can use matrix multiplication (a row vector multiplied by a column vector) for the one-form α_p acting on the vector v_p, $\alpha_p(v_p)$. Right now we will introduce one additional bit of notation. This is also sometimes written as

$$\begin{aligned}\alpha_p(v_p) &= \langle \alpha_p, v_p \rangle \\ &= \langle \alpha, v \rangle_p.\end{aligned}$$

Thus, the angle brackets $\langle \cdot, \cdot \rangle$ denotes the **canonical pairing** between differential forms and vectors. Sometimes authors do not even worry about the order in the canonical pairing and you will even see $\langle v_p, \alpha_p \rangle$ on occasion. Some calculus textbooks use the angle brackets to denote row vectors; we will not use that notation here.

A differential one-form at the point p is simply an element of $T_p^*\mathbb{R}^n$. Our goal is to figure out how to multiply one-forms to give two-, three-, and k-forms. We want to be able to multiply our differential one-forms in such a way that certain volume related properties will be preserved. (This will become clearer in a moment.)

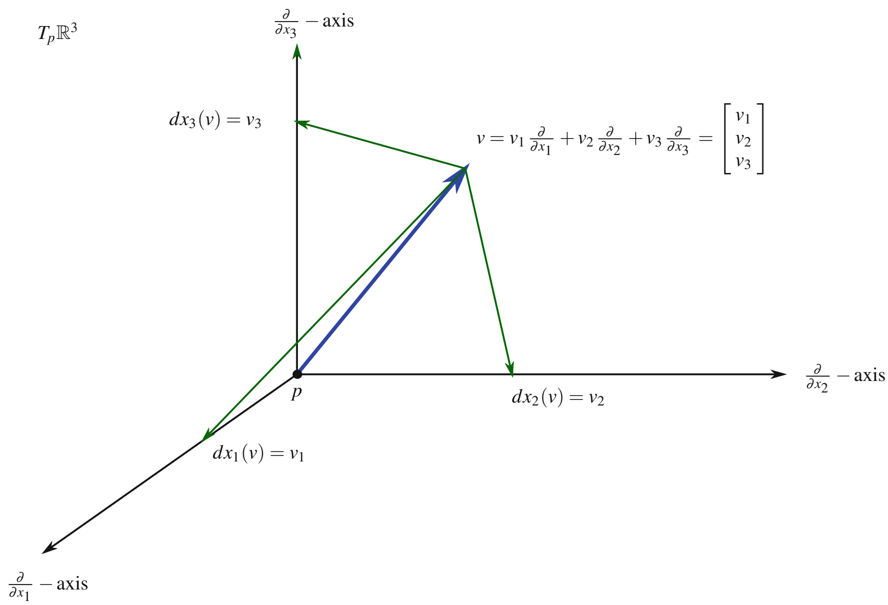

Fig. 3.2 An illustration of how the dual basis elements act on a vector v. The diagram does not include the point p in the notation

For the moment we will work with manifold $\mathbb{R}^3$ since it is easier to draw the relevant pictures. Then we will move to $\mathbb{R}^2$ and even $\mathbb{R}$ before moving to the more general $\mathbb{R}^n$ case. First we recall what we learned in the last chapter. Figure 3.2 shows what the linear functionals, or dual basis elements, dx_1, dx_2, dx_3, (dx, dy, dz), do to the vector $v_p \in T_p\mathbb{R}^3$; they find the projection of v_p onto the coordinate axes of $T_p\mathbb{R}^3$. In general, the base point p is suppressed from the notation most of the time.

Figure 3.3 top depicts the one-form $\alpha = \alpha_1 dx + \alpha_2 dy + \alpha_3 dz$ as an element of the cotangent space $T_p^*\mathbb{R}^3$. The image below shows the result of the one-form acting on the vector $v_p \in T_p\mathbb{R}^3$. Each basis element dx, dy, and dz finds the projection of v onto the appropriate axis. The projection of v onto the x-axis is found as $dx(v) = v_1$ which is then scaled, or multiplied, by α_1. The projection of v onto the y-axis is found as $dy(v) = v_2$, which is then multiplied by α_2. The projection of v onto the z-axis is found as $dz(v) = v_3$, which is then multiplied by α_3. These three scaled terms, $\alpha_1 v_1$, $\alpha_2 v_2$, and $\alpha_3 v_3$ are then added together to obtain $\alpha(v) = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$.

For example, since

$$dx_1(v_p) = dx_1\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = v_1$$

we can view dx_1 as finding the projection of the vector v_p onto the $\frac{\partial}{\partial x_1}\big|_p$ coordinate axis of $T_p\mathbb{R}^3$. Similarly for dx_2 and dx_3. In other words, dx_1 finds a length along the $\frac{\partial}{\partial x_1}\big|_p$ coordinate axis. And what is length but a one-dimensional "volume"? We will want the product of two one-forms to do something similar, to find a volume of a two-dimensional projection. In the end this will mean that the product of two one-forms will no longer be a one-form, or even a linear functional. That means it will no longer be an element of $T_p^*\mathbb{R}^3$ but will be something different, something we will call a two-form. We will denote the space of two-forms on $\mathbb{R}^3$ as $\bigwedge^2(\mathbb{R}^3)$.

Here we are trying to be very precise about notation and about keeping track of what spaces objects are in. It is easy to see how confusion can arise. Using Fig. 3.4 as a guide consider the vector

$$\begin{bmatrix} dx_1(v_p) \\ dx_2(v_p) \\ 0 \end{bmatrix}_p .$$

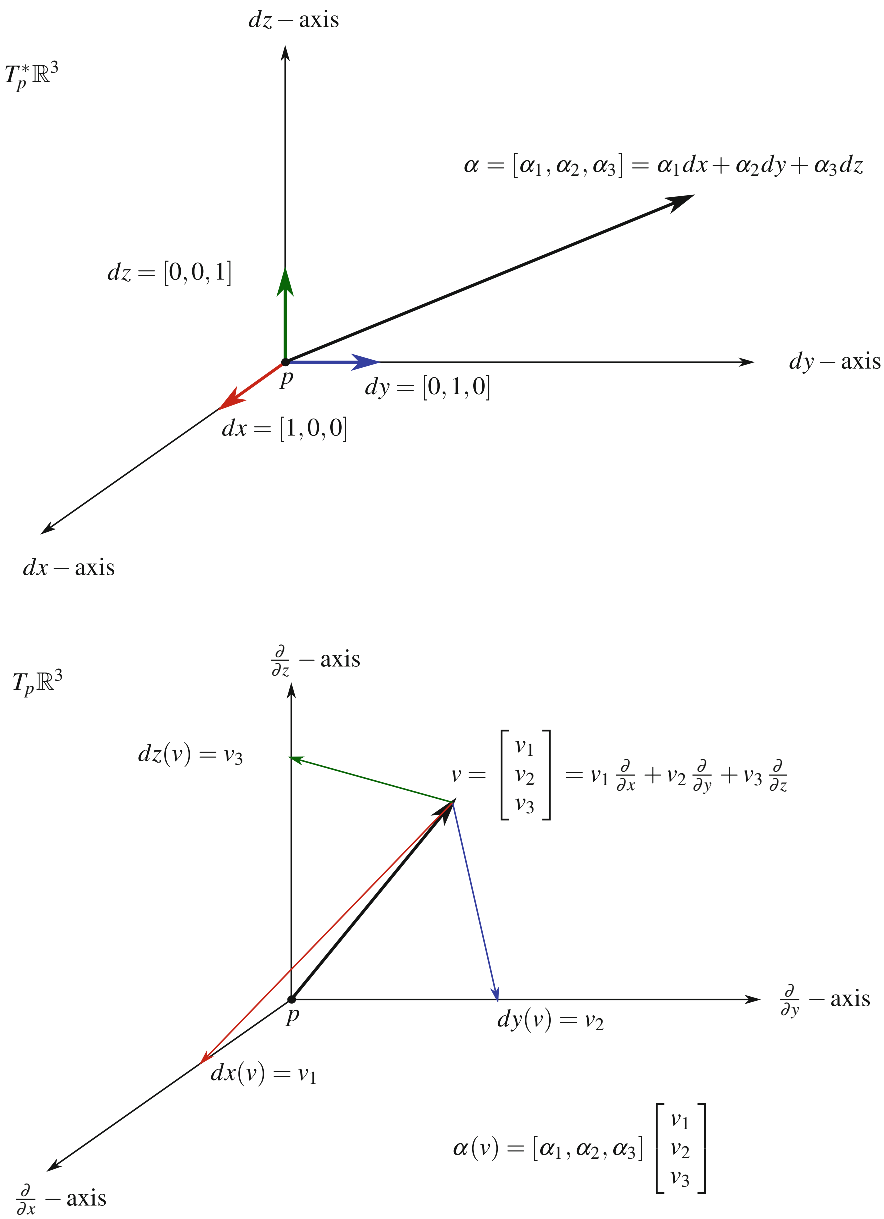

Fig. 3.3 Above the one-form $\alpha = \alpha_1 dx + \alpha_2 dy + \alpha_3 dz$ is shown as an element of the cotangent space $T_p^*\mathbb{R}^3$. Below the result of the one-form acting on the vector $v_p \in T_p\mathbb{R}^3$ is shown

This vector is in the tangent space of $\mathbb{R}^3$ at the point p. In other words, it is in $T_p\mathbb{R}^3$. But this vector is also the projection of v_p onto the $\frac{\partial}{\partial x_1}\big|_p \frac{\partial}{\partial x_2}\big|_p$-plane of $T_p\mathbb{R}^3$. Thus we can identify it with the vector

$$\begin{bmatrix} dx_1(v_p) \\ dx_2(v_p) \end{bmatrix}_p \in span\left\{ \left.\frac{\partial}{\partial x_1}\right|_p, \left.\frac{\partial}{\partial x_2}\right|_p \right\} \subset T_p\mathbb{R}^3.$$

This vector is generally called the projection of v_p onto the $\frac{\partial}{\partial x_1}\big|_p \frac{\partial}{\partial x_2}\big|_p$-plane of $T_p\mathbb{R}^3$.

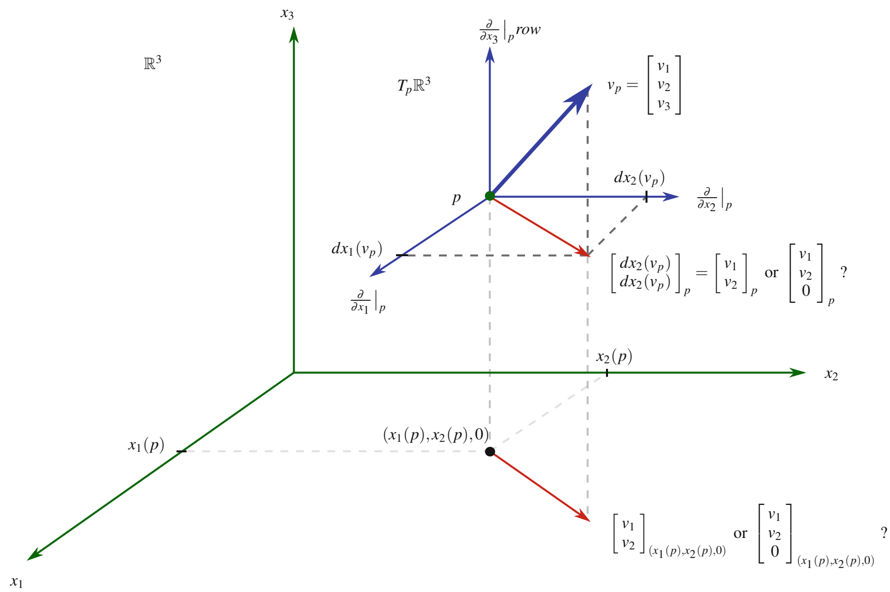

Fig. 3.4 Mixing things up

Looking again at Fig. 3.4 to guide you. Notice that if we are being imprecise how easy it is to confuse the vector

$$\begin{bmatrix} dx_1(v_p) \\ dx_2(v_p) \\ 0 \end{bmatrix}_p \in T_p\mathbb{R}^3$$

in the tangent space with the vector

$$\begin{bmatrix} dx_1(v_p) \\ dx_2(v_p) \end{bmatrix} \in \frac{\partial}{\partial x_1}\bigg|_p \frac{\partial}{\partial x_2}\bigg|_p \text{-plane of } T_p\mathbb{R}^3$$

with the vector

$$\begin{bmatrix} dx_1(v_p) \\ dx_2(v_p) \\ 0 \end{bmatrix}_p \in \mathbb{R}^3$$

at the point p in manifold $\mathbb{R}^3$ or even with the vector

$$\begin{bmatrix} dx_1(v_p) \\ dx_2(v_p) \\ 0 \end{bmatrix}_{(x_1(p),x_2(p),0)} \in \mathbb{R}^3$$

at the point $(x_1(p), x_2(p), 0)$ in manifold $\mathbb{R}^3$, or with the vector

$$\begin{bmatrix} dx_1(v_p) \\ dx_2(v_p) \end{bmatrix}_{(x_1(p),x_2(p),0)} \in xy\text{-plane of } \mathbb{R}^3$$

at the point $(x_1(p), x_2(p), 0)$ in the xy-plane of $\mathbb{R}^3$? In the case of $\mathbb{R}^3$ or $\mathbb{R}^n$ this may muddle things up in our minds but generally it does not result in any computational problems, which is the reason precise distinctions are genially not made in vector calculus. But in general we simply can not do this.

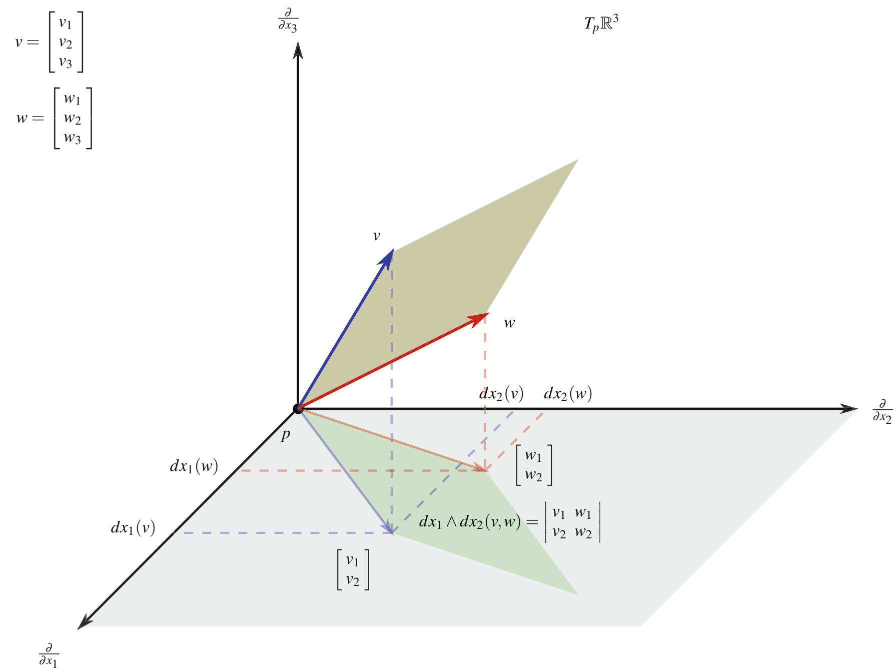

Fig. 3.5 The parallelepiped spanned by v and w (brown) is projected onto the $\partial_{x_1}\partial_{x_2}$-plane in $T_p\mathbb{R}^3$ (green). We want $dx_1 \wedge dx_2$ to find the volume of this projected area

Now that we have taken some effort to understand the spaces that various vectors are in, we turn our attention back our original problem of finding a product of two one-forms that does something similar to what a one-form does, that is, find some sort of a volume. Just like the one-form dx_1 took one vector v_p as input and give a one-dimensional volume as output we want the product of two one-forms, say dx_1 and dx_2, to take as input two vectors v_p and w_p and give as output a two-dimensional volume. What volume should it give? Consider Fig. 3.5.

We have shown the projection of vectors

$$v_p = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \text{ and } w_p = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

onto the $\partial_{x_1}\partial_{x_2}$-plane in $T_p\mathbb{R}^3$. Notice we changed notation and wrote $\left.\frac{\partial}{\partial x_i}\right|_p$ as ∂_{x_i}. The projection of v_p onto the $\partial_{x_1}\partial_{x_2}$-plane is given by the vector

$$\begin{bmatrix} dx_1(v_p) \\ dx_2(v_p) \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Similarly, the projection of w_p onto the same plane is given by

$$\begin{bmatrix} dx_1(w_p) \\ dx_2(w_p) \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

The most natural volume that we may want is the volume of the parallelepiped spanned by the projected vectors

$$\begin{bmatrix} dx_1(v_p) \\ dx_2(v_p) \end{bmatrix} \text{ and } \begin{bmatrix} dx_1(w_p) \\ dx_2(w_p) \end{bmatrix}$$

in the $\partial_{x_1}\partial_{x_2}$-plane in $T_p\mathbb{R}^3$. So, we want this "multiplication" or "product" of the two one-forms dx_1 and dx_2 to somehow give us this projected volume when we input the two vectors v_p and w_p.

To remind ourselves that this multiplication is actually different from anything we are used to so far we will use a special notation for it, called a wedge, $\wedge$, and call it something special as well, the **wedgeproduct**. So, this is what we want

$$dx_1 \wedge dx_2(v_p, w_p) = \begin{array}{c}\text{Volume of parallelepiped spanned} \\ \text{by the projection of} \\ v_p \text{ and } w_p \text{ onto } \partial_{x_1}\partial_{x_2}\text{-plane.}\end{array}$$

or,

$$dx_1 \wedge dx_2(v_p, w_p) = \begin{array}{c}\text{Volume of parallelepiped spanned} \\ \text{by } \begin{bmatrix} dx_1(v_p) \\ dx_2(v_p) \end{bmatrix} \text{ and } \begin{bmatrix} dx_1(w_p) \\ dx_2(w_p) \end{bmatrix}.\end{array}$$

But this is exactly how we derived the definition of determinant in the determinant section of chapter one. The volume of the parallelepiped spanned by the vectors

$$\begin{bmatrix} dx_1(v_p) \\ dx_2(v_p) \end{bmatrix} \text{ and } \begin{bmatrix} dx_1(w_p) \\ dx_2(w_p) \end{bmatrix}.$$

is given by the formula for the determinant of a matrix which has these vectors as columns. Thus we can use the determinant to help us define the wedgeproduct,

$$dx_1 \wedge dx_2(v_p, w_p) \equiv \begin{vmatrix} dx_1(v_p) & dx_1(w_p) \\ dx_2(v_p) & dx_2(w_p) \end{vmatrix}.$$

In summary, **the wedgeproduct of two one-forms is defined in terms of the determinant of the appropriate vector projections**. So we see that volumes, determinants, and projections (via the one-forms) are all mixed together and intimately related in the definition of the wedgeproduct. More generally, the wedgeproduct of two one-forms dx_i and dx_j is defined by

$$\boxed{\text{Wedgeproduct of two one-forms} \qquad dx_i \wedge dx_j(v_p, w_p) \equiv \begin{vmatrix} dx_i(v_p) & dx_i(w_p) \\ dx_j(v_p) & dx_j(w_p) \end{vmatrix}.}$$

Sticking with the manifold $\mathbb{R}^3$ let us try to get a better picture of what is going on. Refer back to Fig. 3.5 where two vectors v and w are drawn in a tangent space $T_p\mathbb{R}^3$ at some arbitrary point p. (The point p is henceforth omitted from the notation.) The projections of v and w onto the $\partial_{x_1}\partial_{x_2}$-plane are given by

$$\begin{bmatrix} dx_1(v_p) \\ dx_2(v_p) \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \text{ and } \begin{bmatrix} dx_1(w_p) \\ dx_2(w_p) \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

The area of the parallelepiped spanned by these projections is given by $dx_1 \wedge dx_2(v, w)$. As an example, suppose that

$$v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } w = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$

First we find $dx_1 \wedge dx_2(v, w)$ as

$$dx_1 \wedge dx_2\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\right) = \begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} = (1)(5) - (4)(2) = -3.$$

Notice we have an area of -3. This shouldn't be surprising since we know that areas are really signed. If instead we found $dx_2 \wedge dx_1(v, w)$ as

$$dx_2 \wedge dx_1 \left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right) = \begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} = (2)(4) - (5)(1) = 3$$

we have an area of 3. A word of caution, when calculating $dx_1 \wedge dx_2$ our projected vectors are

$$\begin{bmatrix} dx_1(v_p) \\ dx_2(v_p) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} dx_1(w_p) \\ dx_2(w_p) \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

while when calculating $dx_2 \wedge dx_1$ our projected vectors are

$$\begin{bmatrix} dx_2(v_p) \\ dx_1(v_p) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} dx_2(w_p) \\ dx_1(w_p) \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix},$$

which may look like different vectors. However, they are not different vectors. We have just written the vector components in a different order, that is all. Instead of writing the x_1 component first as you typically would we have written the x_2 component first.

In Fig. 3.6 the same two vectors v and w are drawn in the same tangent space $T_p\mathbb{R}^3$ at some arbitrary point p. (Again,the point p is henceforth omitted from the notation.) The projections of v and w onto the $\partial_{x_2}\partial_{x_3}$-plane are given by

$$\begin{bmatrix} dx_2(v_p) \\ dx_3(v_p) \end{bmatrix} = \begin{bmatrix} v_2 \\ v_3 \end{bmatrix} \text{ and } \begin{bmatrix} dx_2(w_p) \\ dx_3(w_p) \end{bmatrix} = \begin{bmatrix} w_2 \\ w_3 \end{bmatrix}.$$

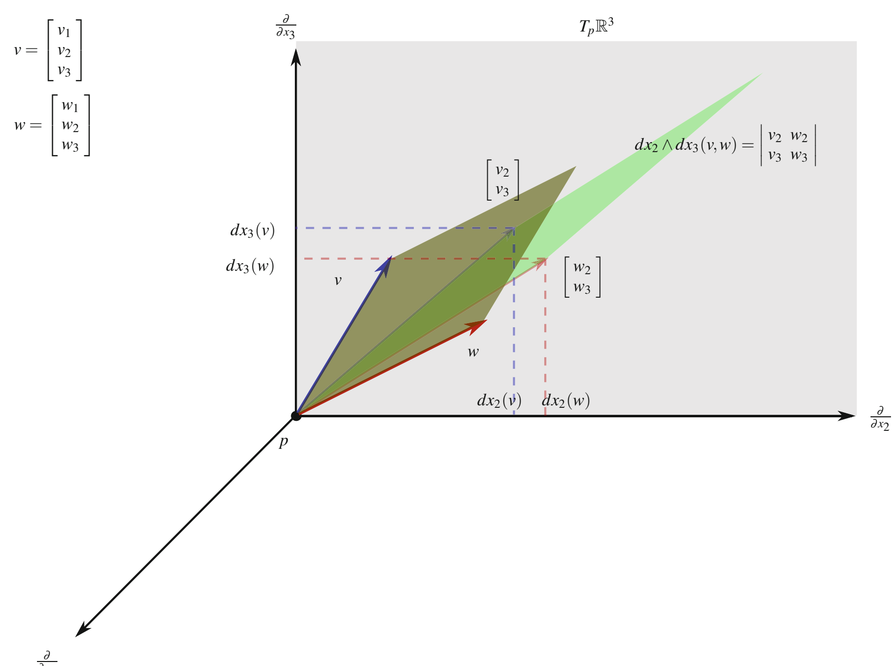

Fig. 3.6 The parallelepiped spanned by v and w (brown) is projected onto the $\partial_{x_2}\partial_{x_3}$-plane in $T_p\mathbb{R}^3$ (green). The wedgeproduct $dx_2 \wedge dx_3$ will find the volume of this projected area when the vectors v and w are its input

The area of the parallelepiped spanned by these projections is given by $dx_2 \wedge dx_3(v, w)$. Continuing with the same example, we find $dx_2 \wedge dx_3(v, w)$ as

$$dx_2 \wedge dx_3\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}\right) = \begin{vmatrix}2 & 5\\3 & 6\end{vmatrix} = (2)(6) - (3)(5) = -3.$$

whereas $dx_3 \wedge dx_2(v, w)$ gives

$$dx_3 \wedge dx_2\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}\right) = \begin{vmatrix}3 & 6\\2 & 5\end{vmatrix} = (3)(5) - (6)(2) = 3.$$

Again, the same two vectors v and w are drawn in the same tangent space $T_p\mathbb{R}^3$ and the projections of v and w onto the $\partial_{x_1}\partial_{x_3}$-plane are given by

$$\begin{bmatrix}dx_1(v_p)\\dx_3(v_p)\end{bmatrix} = \begin{bmatrix}v_1\\v_3\end{bmatrix} \text{ and } \begin{bmatrix}dx_1(w_p)\\dx_3(w_p)\end{bmatrix} = \begin{bmatrix}w_1\\w_3\end{bmatrix}.$$

This is shown in Fig. 3.7. The area of the parallelepiped spanned by these projections is given by $dx_1 \wedge dx_3(v, w)$.

Using the same example we find $dx_1 \wedge dx_3(v, w)$ as

$$dx_1 \wedge dx_3\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}\right) = \begin{vmatrix}1 & 4\\3 & 6\end{vmatrix} = (1)(6) - (3)(4) = -6.$$

whereas $dx_3 \wedge dx_1(v, w)$ gives

$$dx_3 \wedge dx_1\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}\right) = \begin{vmatrix}3 & 6\\1 & 4\end{vmatrix} = (3)(4) - (1)(6) = 6.$$

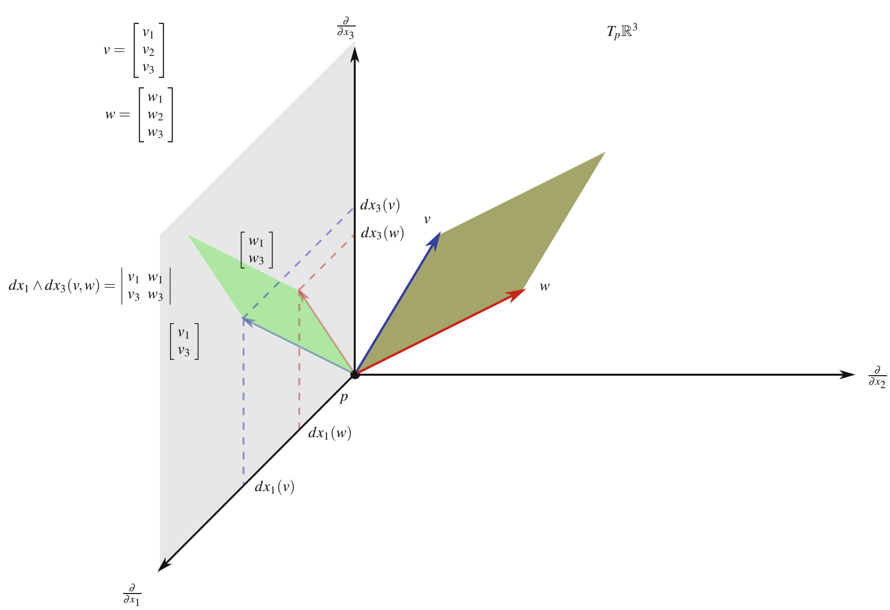

Fig. 3.7 The parallelepiped spanned by v and w (brown) is projected onto the $\partial_{x_1}\partial_{x_3}$-plane in $T_p\mathbb{R}^3$ (green). The wedgeproduct $dx_1 \wedge dx_3$ will find the volume of this projected area when the vectors v and w are its input

Based on these examples it appears that $dx_i \wedge dx_j = -dx_j \wedge dx_i$.Using the definition of wedgeproduct given above, it is simple to see that indeed $dx_i \wedge dx_j = -dx_j \wedge dx_i$. This property is called **skew symmetry** and it follows from the properties of the determinant. Recall, if you switch two rows in the determinant the sign of the determinant changes. That is essentially what is happening here. This implies that the order in which we "multiply" (via wedgeproduct) two one-forms matters. If we switch the order our answer changes by a sign.

Sticking with the same vectors $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $w = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ let us find $dx_1 \wedge dx_1(v_p, w_p)$,

$$dx_1 \wedge dx_1 \left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right) = \begin{vmatrix} 1 & 4 \\ 1 & 4 \end{vmatrix} = (1)(4) - (4)(1) = 0.$$

Similarly we have $dx_2 \wedge dx_2(v, w) = 0$ and $dx_3 \wedge dx_3(v, w) = 0$. This should not be too surprising since in general we would have

$$dx_i \wedge dx_i(v, w) = \begin{vmatrix} dx_i(v) & dx_i(w) \\ dx_i(v) & dx_i(w) \end{vmatrix} = dx_i(v) \cdot dx_i(w) - dx_i(v) \cdot dx_i(w) = 0.$$

Another way to see this is to note that since $dx_i \wedge dx_j = -dx_j \wedge dx_i$ that implies $dx_i \wedge dx_i = -dx_i \wedge dx_i$, which can only happen if $dx_i \wedge dx_i = 0$. Unfortunately there is no accurate picture that shows what is going on here. It is best to simply recognize that the parallelepiped spanned by the projected vectors is degenerate and thus has area zero.

So, given two vectors v_p and w_p in $T_p\mathbb{R}^3$ we now considered every possible wedgeproduct of two one-forms. We have considered

$$\begin{gathered} dx_1 \wedge dx_2, \\ dx_2 \wedge dx_3, \\ dx_1 \wedge dx_3. \end{gathered}$$

We have also discovered that

$$\begin{gathered} dx_1 \wedge dx_2 = -dx_2 \wedge dx_1, \\ dx_2 \wedge dx_3 = -dx_3 \wedge dx_2, \\ dx_1 \wedge dx_3 = -dx_3 \wedge dx_1, \end{gathered}$$

so once we know what the left hand side does to two vectors we automatically know what the right hand side does to these same vectors. And finally, we found that

$$\begin{gathered} dx_1 \wedge dx_1 = 0, \\ dx_2 \wedge dx_2 = 0, \\ dx_3 \wedge dx_3 = 0. \end{gathered}$$

That is all there is. The wedgeproducts $dx_1 \wedge dx_2$, $dx_2 \wedge dx_3$, and $dx_1 \wedge dx_3$ are all called two-forms. In fact, every single two-form can be written as linear combination of these three two-forms. For example, for any $a, b, c \in \mathbb{R}$ we have that

$$a\ dx_1 \wedge dx_2 + b\ dx_2 \wedge dx_3 + c\ dx_1 \wedge dx_3$$

is a two-form. For this reason, the two-forms $dx_1 \wedge dx_2$, $dx_2 \wedge dx_3$, and $dx_1 \wedge dx_3$ are called a basis of the space of two-form on $\mathbb{R}^3$, which is denoted as $\bigwedge^2(\mathbb{R}^3)$. Actually, there is a habit, or convention, that we use the two-forms $dx_1 \wedge dx_2$, $dx_2 \wedge dx_3$, and $dx_3 \wedge dx_1$ as the basis of $\bigwedge^2(\mathbb{R}^3)$. We have just substituted $dx_3 \wedge dx_1$ for $dx_1 \wedge dx_3$. The reason we do that is to maintain

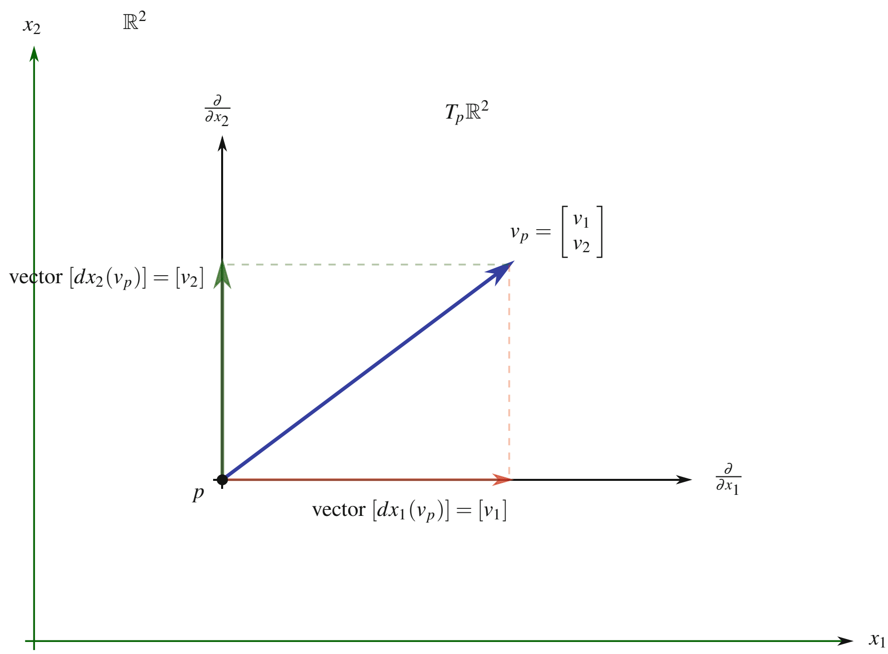

Fig. 3.8 The manifold $\mathbb{R}^2$ pictured with the tangent space $T_p\mathbb{R}^2$ superimposed on it. The vector v_p is an element of the tangent space $T_p\mathbb{R}^2$

a "cyclic ordering" when we write the basis down, something we can only do in three dimensions. But more about this in the next section.

Up to now we have been working with $\mathbb{R}^3$ because it is easy to draw picture and to see what we mean by projections. Now we step back and think about $\mathbb{R}^2$ as pictured in Fig. 3.8 for a few moments. Projections of $v_p = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_p$ onto the one-dimensional subspace ∂_{x_1} of $T_p\mathbb{R}^2$ is given by the vector $[dx_1(v_p)] = [v_1]$ and span and the projection of v_p onto the subspace ∂_{x_2} is given by the vector $[dx_2(v_p)] = [v_2]$ as pictured.

But now consider Fig. 3.9. It is similar to Fig. 3.8 except now there are two vectors, v_p and w_p in $T_p\mathbb{R}^2$, that form a parallelepiped. What does $dx_1 \wedge dx_2(v_p, w_p)$ represent here? What space is v_p and w_p being projected on in this case? We have

$$dx_1 \wedge dx_2(v_p, w_p) = \begin{vmatrix} dx_1(v_p) & dx_1(w_p) \\ dx_2(v_p) & dx_2(w_p) \end{vmatrix} = \begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix} = v_1 w_2 - w_1 v_2$$

which is exactly the area of the parallelepiped spanned by

$$\begin{bmatrix} dx_1(v_p) \\ dx_2(v_p) \end{bmatrix}_p = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_p = v_p$$

and

$$\begin{bmatrix} dx_1(w_p) \\ dx_2(w_p) \end{bmatrix}_p = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}_p = w_p$$

so the space that v_p and w_p are getting "projected" onto is the whole space $T_p\mathbb{R}^2$.

Question 3.1 Find $dx_2 \wedge dx_1(v_p, w_p)$. How does it relate to $dx_1 \wedge dx_2(v_p, w_p)$. What would the basis of $\bigwedge_p^2 \left(\mathbb{R}^2\right)$ be? What would a general element of $\bigwedge_p^2 \left(\mathbb{R}^2\right)$ look like?

Question 3.2 For the moment we will remain with our $\mathbb{R}^2$ example.

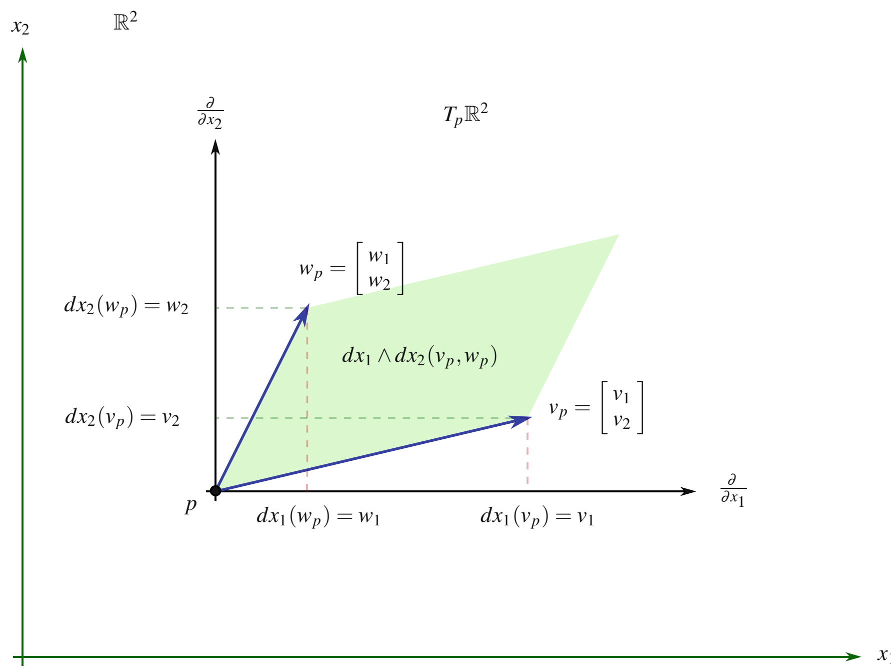

Fig. 3.9 The manifold $\mathbb{R}^2$ pictured with the tangent space $T_p\mathbb{R}^2$ superimposed on it. The parallelepiped spanned by v_p and w_p is shown. But what space does this get projected to when $dx_1 \wedge dx_2$ acts on the vectors v_p and w_p? It gets projected to the exact same space that it is already in, namely, $T_p\mathbb{R}^2$

(a) Consider $dx_1 \wedge dx_2 \wedge dx_1$. How many vectors do you think this would take as input?
(b) What would the three-dimensional volume of the parallelepiped spanned by three vectors be in a two-dimensional space?
(c) Suppose $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, and $s = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ and suppose that

$$dx_1 \wedge dx_2 \wedge dx_1(u, v, w) \equiv \begin{vmatrix} dx_1(u) & dx_1(v) & dx_1(w) \\ dx_2(u) & dx_2(v) & dx_2(w) \\ dx_1(u) & dx_1(v) & dx_1(w) \end{vmatrix}.$$

Explain why this is always zero.
(d) Based on this, what do you think the space $\bigwedge_p^3 (\mathbb{R}^2)$ will be?
(e) Based on this, what do you think the space $\bigwedge_p^n (\mathbb{R}^2)$ for $n > 3$ will be?

We will define the wedgeproduct of three one-forms as follows,

Wedgeproduct of three one-forms

$$dx_i \wedge dx_j \wedge dx_k(u, v, w) \equiv \begin{vmatrix} dx_i(u) & dx_i(v) & dx_i(w) \\ dx_j(u) & dx_j(v) & dx_j(w) \\ dx_k(u) & dx_k(v) & dx_k(w) \end{vmatrix}.$$

We will call the wedgeproduct of three one-forms $dx_i \wedge dx_j \wedge dx_k$ a three-form. What does this three-form $dx_i \wedge dx_j \wedge dx_k$ find? Suppose we are on manifold $\mathbb{R}^n$. Given vectors u, v, w at a point p we first find the projection of these vectors onto the $\partial_{x_i}\partial_{x_i}\partial_{x_k}$-subspace of $T_p\mathbb{R}^n$. These projections are given by the vectors

$$\begin{bmatrix} dx_i(u) \\ dx_j(u) \\ dx_k(u) \end{bmatrix}, \begin{bmatrix} dx_i(v) \\ dx_j(v) \\ dx_k(v) \end{bmatrix}, \begin{bmatrix} dx_i(w) \\ dx_j(w) \\ dx_k(w) \end{bmatrix}.$$

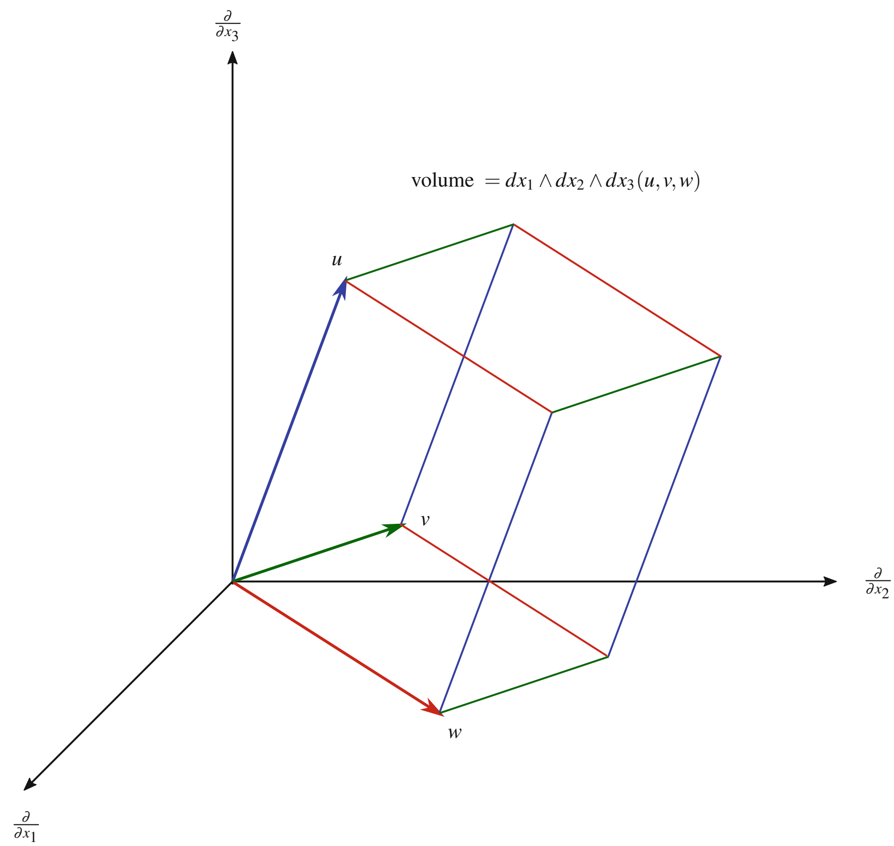

Fig. 3.10 The tangent space $T_p\mathbb{R}^3$ with the parallelepiped spanned by u, v and w shown. The three-form $dx_1 \wedge dx_2 \wedge dx_3$, shown here as $dx \wedge dy \wedge dz$, simply finds the volume of this parallelepiped

Next, we find the volume of the parallelepiped spanned by these projected vectors by using the determinant. Returning to $\mathbb{R}^3$ again we will consider the three-form $dx_1 \wedge dx_2 \wedge dx_3$. What subspace of $T_p\mathbb{R}^3$ does the three-form $dx_1 \wedge dx_2 \wedge dx_3$ project onto? The three-form $dx_1 \wedge dx_2 \wedge dx_3$ projects onto the whole space $T_p\mathbb{R}^3$. Thus the three-form $dx_1 \wedge dx_2 \wedge dx_3$ simply finds the volume of the parallelepiped spanned by the three input vectors, see Fig. 3.10.

Question 3.3 Consider the manifold $\mathbb{R}^3$ and the three-form $dx_1 \wedge dx_2 \wedge dx_3$.

(a) How does $dx_1 \wedge dx_2 \wedge dx_3$ relate to $dx_2 \wedge dx_1 \wedge dx_3$? How about to $dx_2 \wedge dx_3 \wedge dx_1$? Or $dx_3 \wedge dx_2 \wedge dx_1$? Do you notice a pattern?
(b) What is the basis for $\bigwedge_p^3 \left(\mathbb{R}^3\right)$?
(c) How do you think $dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_1(u, v, w, x)$ would be defined? What do you think it will be? Why? (Here the input x represents a fourth vector.)
(d) What is $\bigwedge_p^n \left(\mathbb{R}^3\right)$ for $n \geq 4$?

In general, letting $v_1, v_2, \ldots, v_n$ represent vectors, we will define the wedgeproduct of n one-forms as follows:

Wedgeproduct of n one-forms

$$dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_n}\big(v_1, v_2, \ldots v_n\big) \equiv \begin{vmatrix} dx_{i_1}(v_1) & dx_{i_1}(v_2) & \cdots & dx_{i_1}(v_n) \\ dx_{i_2}(v_1) & dx_{i_2}(v_2) & \cdots & dx_{i_2}(v_n) \\ \vdots & \vdots & \ddots & \vdots \\ dx_{i_n}(v_1) & dx_{i_n}(v_2) & \cdots & dx_{i_n}(v_n) \end{vmatrix}$$

Notice, if we have any two of the one-forms the same, that is, $i_j = i_k$ for some $j \neq k$ then we have two rows that are the same, which gives a value of zero.

To close this section we will consider the one-form dx on the manifold $\mathbb{R}$, the two-form $dx_1 \wedge dx_2$ on the manifold $\mathbb{R}^2$, the three-form $dx_1 \wedge dx_2 \wedge dx_3$ on the manifold $\mathbb{R}^3$, and the n-form $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ on the manifold $\mathbb{R}^n$. These are very special differential forms that are very often called ***volume forms***, though in two dimensions the two-form $dx_1 \wedge dx_2$ is often also called an ***area form***. After this section the reason behind this terminology is fairly obvious. The one-form dx finds the one-dimensional volume, or length, of vectors on $\mathbb{R}$. Similarly, the two-form $dx_1 \wedge dx_2$, also written as $dx \wedge dy$, finds the two-dimensional volume, or area, of parallelepipeds on $\mathbb{R}^2$ (Fig. 3.9), the three-form $dx_1 \wedge dx_2 \wedge dx_3$, also written as $dx \wedge dy \wedge dz$, finds the three-dimensional volume of parallelepipeds on $\mathbb{R}^3$ (Fig. 3.10), and obviously the n-form $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ finds the n-dimensional volume of parallelepipeds on $\mathbb{R}^n$. Technically speaking, the parallelepipeds whose volumes are found are actually in some tangent space and not in the manifold, but for Euclidian space we can actually also imagine the parallelepipeds as being in the manifolds themselves so we will not belabor the point too much. These volume forms will play a vital role when we get to integration of forms.

Question 3.4 Consider the n-form $dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_n}$. What happens if any one-forms $dx_{i_j} = dx_{i_k}$ for $i_j \neq i_k$? What happens if any vector $v_j = v_k$ for $j \neq k$?

Question 3.5 Show that if $i_j \neq i_k$ and we switch dx_{i_j} and dx_{i_k} that the wedgeproduct changes sign.

3.2 General Two-Forms and Three-Forms

Two-forms are built by wedgeproducting two one-forms together. Just as we denoted the cotangent space at the point p by $T_p^*\mathbb{R}^3$, which was the space of one-forms at p, we will denote the space of two-forms at p as $\bigwedge_p^2(\mathbb{R}^3)$. Keeping in line with this convention, you will occasionally see $T_p^*\mathbb{R}^3$ written as $\bigwedge_p^1(\mathbb{R}^3)$. As we have already mentioned, the basis of the space $\bigwedge_p^2(\mathbb{R}^3)$ is given by

$$\left\{dx_1 \wedge dx_2,\ dx_2 \wedge dx_3,\ dx_3 \wedge dx_1\right\}.$$

Why is the last element in the list written as $dx_3 \wedge dx_1$ instead of $dx_1 \wedge dx_3$? This is purely a matter of convention based on a desire to write the indexes cyclicly. Each of these basis elements $dx_1 \wedge dx_2,\ dx_2 \wedge dx_3,\ dx_3 \wedge dx_1$ finds the signed areas of the parallelepipeds formed by the projections of two vectors onto the appropriate planes of $T_p\mathbb{R}^3$, as depicted in Fig. 3.11.

Similarly, it is not difficult to see that the basis of the space $\bigwedge_p^2(\mathbb{R}^2)$ is given by

$$\left\{dx_1 \wedge dx_2\right\}$$

and the basis of the space $\bigwedge_p^2(\mathbb{R}^4)$ is given by

$$\left\{dx_1 \wedge dx_2,\ dx_1 \wedge dx_3,\ dx_1 \wedge dx_4,\ dx_2 \wedge dx_3,\ dx_2 \wedge dx_4,\ dx_3 \wedge dx_4\right\},$$

etc. The basis of $\bigwedge_p^2(\mathbb{R}^n)$ contains $\frac{(n)(n-1)}{2}$ elements.

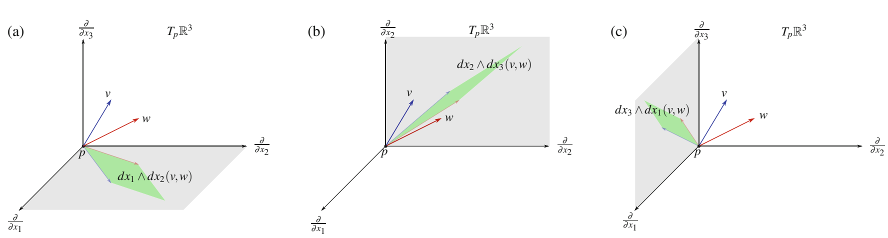

Fig. 3.11 The actions of the basis elements of $\bigwedge^2(\mathbb{R}^3)$ projecting v and w onto the appropriate planes in $T^3_{\mathbb{R}}$ and then finding the signed area of parallelepiped spanned by the projections, which is shown in green. (**a**) Action of $dx_1 \wedge dx_2$. (**b**) Action of $dx_2 \wedge dx_3$. (**c**) Action of $dx_3 \wedge dx_1$

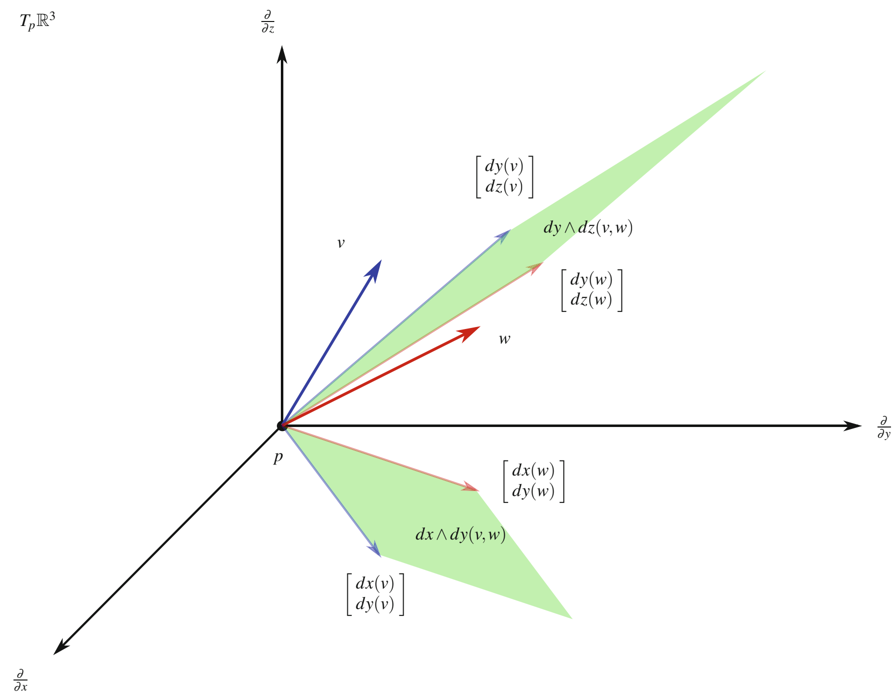

Fig. 3.12 The action of the two-form $dx \wedge dy + dy \wedge dz$ on the two vectors v and w. The two-form $dx \wedge dy + dy \wedge dz$ finds the sum of the areas of the two projected parallelepipeds

Question 3.6 Explain why $\{dx_1 \wedge dx_2\}$ is the basis of $\bigwedge_p^2(\mathbb{R}^2)$ and why $\{dx_1 \wedge dx_2,\ dx_1 \wedge dx_3,\ dx_1 \wedge dx_4,\ dx_2 \wedge dx_3,\ dx_2 \wedge dx_4,\ dx_3 \wedge dx_4\}$ is the basis of $\bigwedge_p^2(\mathbb{R}^4)$.

Question 3.7 Explain why the basis of $\bigwedge_p^2(\mathbb{R}^n)$ contains $\frac{(n)(n-1)}{2}$ elements.

Thus far we have simply looked at the basis elements of the two-forms. But what about the other elements in the spaces $\bigwedge_p^2(\mathbb{R}^2)$, $\bigwedge_p^2(\mathbb{R}^3)$, $\bigwedge_p^2(\mathbb{R}^4)$, or $\bigwedge_p^2(\mathbb{R}^n)$? What do they look like and what do they do? For the moment we will concentrate on the space $\bigwedge_p^2(\mathbb{R}^3)$ and so will resort to the notation x, y, z instead of x_1, x_2, x_3.

Consider the two-form $5dx \wedge dy$ acting on two vectors v_p and w_p for some p. What does the factor 5 do? $dx \wedge dy(v_p, w_p)$ gives the area of the parallelepiped spanned by the projection of v_p and w_p onto the $\partial_x\partial_y$-plane. The number five scales this area by multiplying the area by five. Thus the numerical factors in front of the basis elements $dx \wedge dy$, $dy \wedge dz$, $dz \wedge dx$ are scaling factors.

Figure 3.12 is an attempt to show what a two-form of the form $dx \wedge dy + dy \wedge dz$ does to two vectors v and w in $T_p\mathbb{R}^3$. The projections of v and w onto the $\partial_x\partial_y$-plane are found

$$\begin{bmatrix} dx(v) \\ dy(v) \end{bmatrix} \text{ and } \begin{bmatrix} dx(w) \\ dy(w) \end{bmatrix}$$

and the parallelepiped in the $\partial_x\partial_y$-plane formed from these two projected vectors is drawn. $dx \wedge dy(v, w)$ finds the signed area of this parallelepiped. Then the projections of v and w onto the $\partial_y\partial_z$-plane are found

$$\begin{bmatrix} dy(v) \\ dz(v) \end{bmatrix} \text{ and } \begin{bmatrix} dy(w) \\ dz(w) \end{bmatrix}$$

and the parallelepiped in the $\partial_y\partial_z$-plane formed from these two projected vectors is also drawn. $dx \wedge dy(v, w)$ finds the signed area of this parallelepiped. The one-form $dx \wedge dy + dy \wedge dz$ sums these two signed areas:

$$\Big(dx \wedge dy + dy \wedge dz\Big)(v, w) = dx \wedge dy(v, w) + dy \wedge dz(v, w).$$

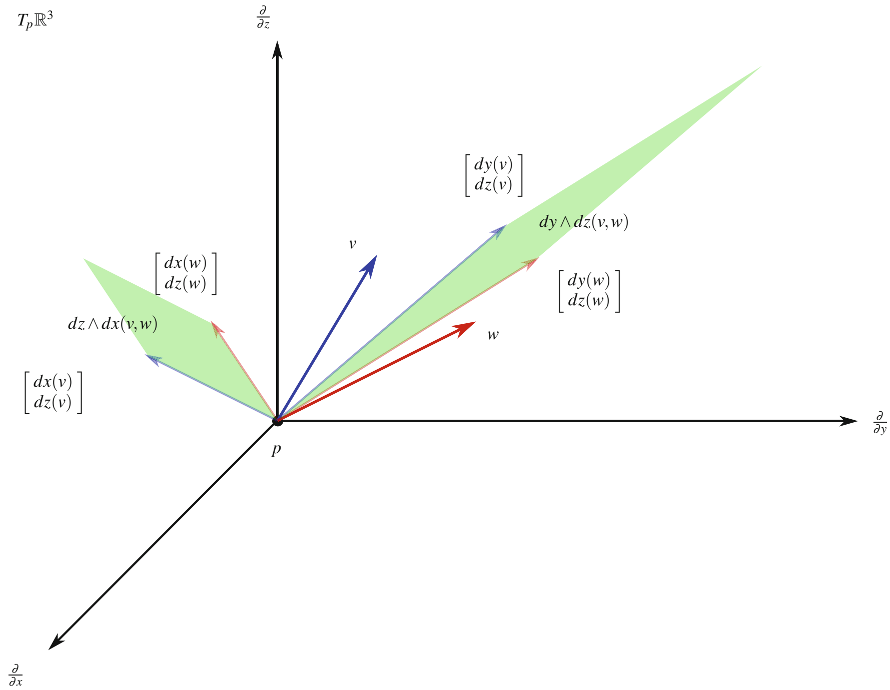

Fig. 3.13 The action of the two-form $dy \wedge dz + dz \wedge dx$ on the two vectors v and w. The two-form $dy \wedge dz + dz \wedge dx$ finds the sum of the areas of the two projected parallelepipeds

A two-form of the form $adx \wedge dy + bdy \wedge dz$, where $a, b \in \mathbb{R}$, multiplies the parallelepiped areas by the factors a and b before adding them:

$$\Big(adx \wedge dy + bdy \wedge dz\Big)(v, w) = adx \wedge dy(v, w) + bdy \wedge dz(v, w).$$

Figure 3.13 shows that the action of the two-form $dy \wedge dz + dz \wedge dx$ on two vectors v and w in $T_p\mathbb{R}^3$ is essentially similar. The projections of v and w onto the $\partial_y \partial_z$-plane are found

$$\begin{bmatrix} dy(v) \\ dz(v) \end{bmatrix} \text{ and } \begin{bmatrix} dy(w) \\ dz(w) \end{bmatrix}$$

and the parallelepiped in the $\partial_y \partial_z$-plane formed from these two projected vectors is drawn. $dy \wedge dz(v, w)$ finds the signed area of this parallelepiped. Then the projections of v and w onto the $\partial_x \partial_z$-plane are found

$$\begin{bmatrix} dx(v) \\ dz(v) \end{bmatrix} \text{ and } \begin{bmatrix} dx(w) \\ dz(w) \end{bmatrix}$$

and the parallelepiped in the $\partial_x \partial_z$-plane formed from these two projected vectors is also shown. $dz \wedge dx(v, w)$ finds the signed area of this parallelepiped. The one-form $dy \wedge dz + dz \wedge dx$ sums these two signed areas:

$$\Big(dy \wedge dz + dz \wedge dx\Big)(v, w) = dy \wedge dz(v, w) + dz \wedge dx(v, w).$$

A two-form of the form $ady \wedge dz + bdz \wedge dx$, where $a, b \in \mathbb{R}$, multiplies the parallelepiped areas by the factors a and b before adding them:

$$\Big(ady \wedge dz + bdz \wedge dx\Big)(v, w) = ady \wedge dz(v, w) + bdz \wedge dx(v, w).$$

Without going into all the details again, the two-forms $dx \wedge dy + dz \wedge dx$ and $adx \wedge dy + bdz \wedge dx$ behaves analogously when they act on two vectors $v, w \in T_p\mathbb{R}^3$, see Fig. 3.14.

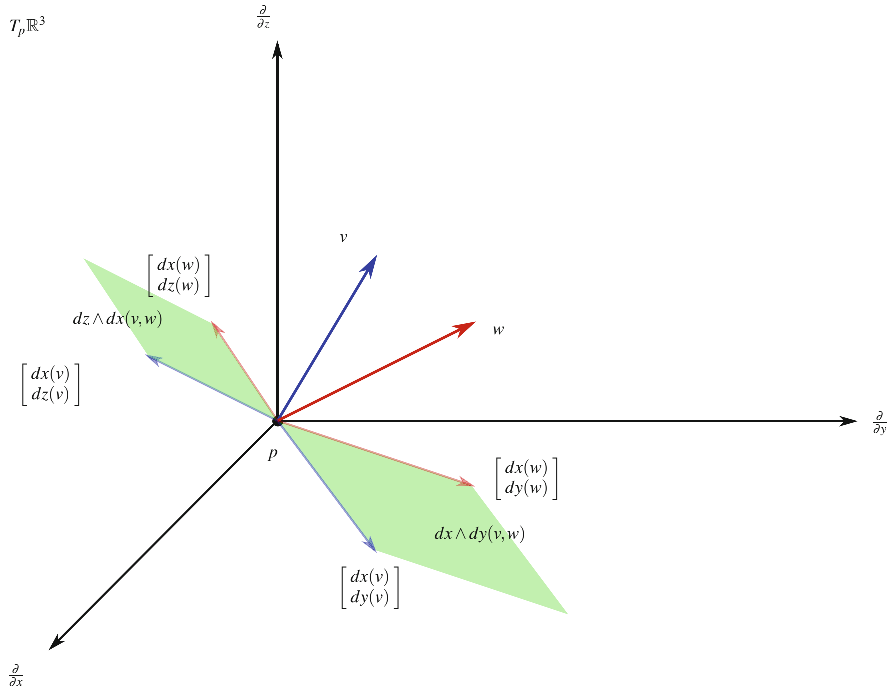

Fig. 3.14 The action of the two-form $dx \wedge dy + dz \wedge dx$ on the two vectors v and w. The two-form $dx \wedge dy + dz \wedge dx$ finds the sum of the areas of the two projected parallelepipeds

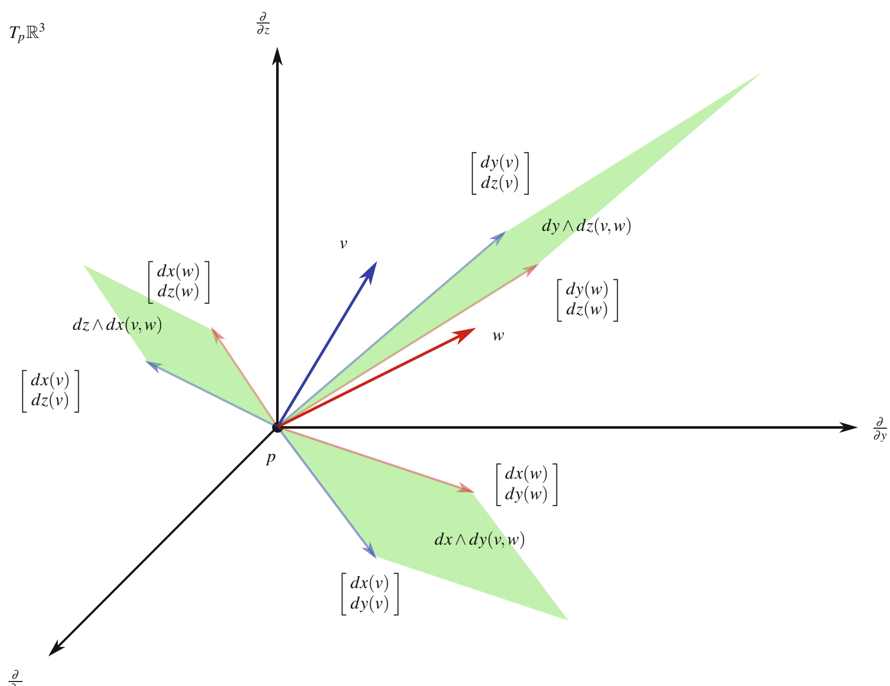

Fig. 3.15 The action of the two-form $dx \wedge dy + dy \wedge dz + dz \wedge dx$ on two vectors v and w, which are not shown

Finally, we consider two-forms of the form $dx \wedge dy + dy \wedge dz + dz \wedge dx$. This two-forms acts on two vectors $v, w \in T_p\mathbb{R}^3$ exactly as you would expect, as shown in Fig. 3.15. First, the projections onto the different planes in $T_p\mathbb{R}^3$ are found. Then the parallelepipeds formed from these projected vectors are drawn. $dx \wedge dy(v, w)$ finds the signed volume of the parallelepiped in the $\partial_x \partial_y$-plane, $dy \wedge dz(v, w)$ finds the signed volume of the parallelepiped in the $\partial_y \partial_z$-plane, and $dz \wedge dx(v, w)$ finds the signed volume of the parallelepiped in the $\partial_x\partial_z$-plane. These signed volumes are then summed up:

$$\Big(dx \wedge dy + dy \wedge dz + dz \wedge dx\Big)(v, w) = dx \wedge dy(v, w) + dy \wedge dz(v, w) + dz \wedge dx(v, w).$$

Two-forms of the form $adx \wedge dy + bdy \wedge dz + cdz \wedge dx$, where $a, b, c \in \mathbb{R}$ work analogously, only with the various signed volumes being scaled by the factors a, b, c before being summed together.

Two forms on n-dimensional manifolds, such as $\mathbb{R}^n$ where $n > 3$, behave in exactly analogous ways. Projections of two vectors v, w are found on the appropriate two dimensional subspaces of $T_p\mathbb{R}^n$, two-dimensional parallelepipeds are formed from these projected vectors, their volumes are first found and then scaled by the appropriate factor, and then the scaled volumes are summed. The only distinction is that $T_p\mathbb{R}^n$ has $\frac{n(n-1)}{2}$ distinct two dimensional subspaces.

We have already discussed the space $\bigwedge_p^3(\mathbb{R}^3)$ of three-forms on $\mathbb{R}^3$, but we will cover it again here for completeness. The basis of $\bigwedge_p^3(\mathbb{R}^3)$ is given by $\{dx \wedge dy \wedge dz\}$ since the three-form $dx \wedge dy \wedge dz$ projects three vectors u, v, w onto the $\partial_x\partial_y\partial_z$-subspace of $T_p\mathbb{R}^3$, which happens to be the whole space $T_p\mathbb{R}^3$. See Fig. 3.10. Thus, the three-form $dx \wedge dy \wedge dz$ simply finds the volume of the parallelepiped spanned by the vectors u, v, w. Elements of $\bigwedge_p^3(\mathbb{R}^3)$ of the form $adx \wedge dy \wedge dz$, where $a \in \mathbb{R}$, simply scales the volume by the factor a.

Three-forms only start to get interesting for manifolds of dimension four or higher. As an illustrative example let us consider $\bigwedge_p^3(\mathbb{R}^4)$, the three-forms on the manifold $\mathbb{R}^4$. A little bit of thought should convince you that the basis of $\bigwedge_p^3(\mathbb{R}^4)$ is given by

$$\Big\{dx_1 \wedge dx_2 \wedge dx_3,\ dx_1 \wedge dx_2 \wedge dx_4,\ dx_1 \wedge dx_3 \wedge dx_4,\ dx_2 \wedge dx_3 \wedge dx_4\Big\}$$

and all elements of $\bigwedge_p^3(\mathbb{R}^4)$ are of the form

$$adx_1 \wedge dx_2 \wedge dx_3 + bdx_1 \wedge dx_2 \wedge dx_4 + cdx_1 \wedge dx_3 \wedge dx_4 + ddx_2 \wedge dx_3 \wedge dx_4$$

for $a, b, c, d \in \mathbb{R}$.

Question 3.8 Explain why $\{dx_1 \wedge dx_2 \wedge dx_3,\ dx_1 \wedge dx_2 \wedge dx_4,\ dx_1 \wedge dx_3 \wedge dx_4,\ dx_2 \wedge dx_3 \wedge dx_4\}$ is the basis of $\bigwedge_p^3(\mathbb{R}^4)$. What are all the possible three-dimensional subspaces of $\mathbb{R}^4$?

The picture we will employ for "visualizing" the actions of these three-forms on three vectors $u, v, w \in T_p\mathbb{R}^4$ will, out of necessity, be a bit of a "cartoon," but it works well enough; see Fig. 3.16. In fact, similar cartoons can be used for all n-forms on m-dimensional manifolds. The projections of the four dimensional vectors

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix},\quad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix},\quad w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$$

onto the $\partial_{x_1}\partial_{x_2}\partial_{x_3}$-subspace are found to be

$$\begin{bmatrix} dx_1(u) \\ dx_2(u) \\ dx_3(u) \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix},\quad \begin{bmatrix} dx_1(v) \\ dx_2(v) \\ dx_3(v) \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix},\quad \begin{bmatrix} dx_1(w) \\ dx_2(w) \\ dx_3(w) \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}.$$

The three-form $dx_1 \wedge dx_2 \wedge dx_3(u, v, w)$ finds the signed volume of the parallelepiped spanned by the projected vectors. The three-form $adx_1 \wedge dx_2 \wedge dx_3(u, v, w)$ scales the signed volume by the factor a. The other three-forms work similarly and can be imagined using the same sort of cartoon. Finally, an arbitrary element of $\bigwedge_p^3(\mathbb{R}^4)$ finds the appropriate signed volumes, scales the signed volumes by the appropriate factors, and then sums the results together according to

$$\begin{aligned}&\Big(adx_1 \wedge dx_2 \wedge dx_3 + bdx_1 \wedge dx_2 \wedge dx_4 \\ &+ cdx_1 \wedge dx_3 \wedge dx_4 + ddx_2 \wedge dx_3 \wedge dx_4\Big)(u, v, w) \\ =& adx_1 \wedge dx_2 \wedge dx_3(u, v, w) + bdx_1 \wedge dx_2 \wedge dx_4(u, v, w) \\ &+ cdx_1 \wedge dx_3 \wedge dx_4(u, v, w) + ddx_2 \wedge dx_3 \wedge dx_4(u, v, w)\end{aligned}$$

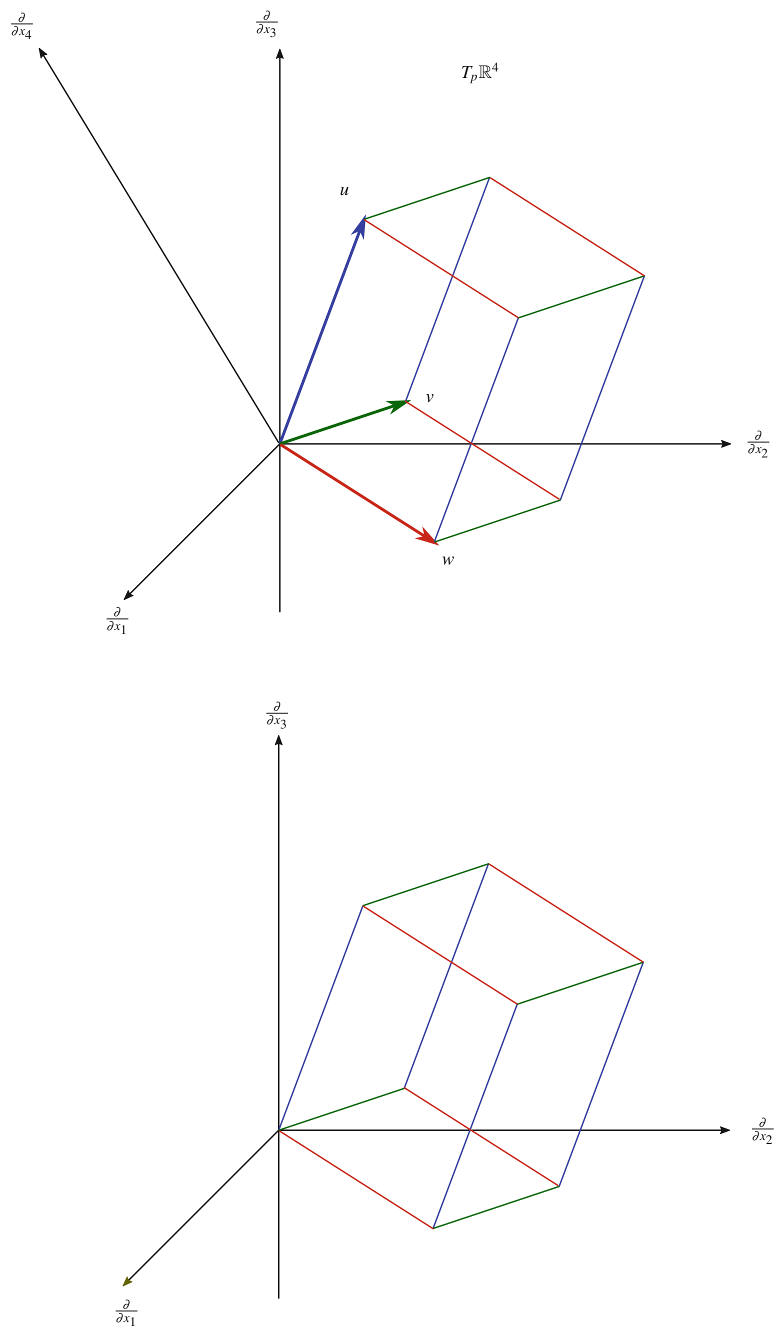

Fig. 3.16 An attempt is made to illustrate the tangent space $T_p\mathbb{R}^4$ and show the parallelepiped spanned by v_1, v_2, v_3 (top). The action of the three-form $dx_1 \wedge dx_2 \wedge dx_3$ on the three vectors u, v, w is shown as the projection of this parallelepiped to the three-dimensional $\partial_{x_1}\partial_{x_2}\partial_{x_3}$ subspace (below)

Question 3.9 Consider the manifold $\mathbb{R}^3$.

(a) Which of the following are one-forms on $\mathbb{R}^3$?

(i) $3dx_1$
(ii) $-4dx_2 + 7dx_3$
(iii) $5dx_1 + 3dx_2 - 6dx_1 + 4dx_3$
(iv) $2dx_1 \wedge dx_2 - 4dx_3$
(v) $dx_1 \wedge dx_2 - 3dx_2 \wedge dx_3$

(b) Which of the following are two-forms on $\mathbb{R}^3$?

(i) $-4dx_3 \wedge dx_2 \wedge dx_1$
(ii) $6dx_3 \wedge dx_2 + 8dx_3$
(iii) $-10dx_1 \wedge dx_3 + 5dx_2 \wedge dx_3 - dx_1 \wedge dx_2$
(iv) $-2dx_1 \wedge dx_2 + 3dx_3 \wedge dx_2 \wedge dx_1$
(v) $3dx_1 \wedge dx_3 - dx_2 \wedge dx_3$

(c) Which of the following are three-forms on $\mathbb{R}^3$?

(i) $5dx_3 \wedge dx_2 \wedge dx_1$
(ii) $-3dx_3 \wedge dx_2 + 2dx_3 \wedge dx_1 \wedge dx_2$
(iii) $5dx_2 \wedge dx_3 \wedge dx_1 + 5dx_1 \wedge dx_3 \wedge dx_2 - dx_1 \wedge dx_2 \wedge dx_3$
(iv) $-5dx_3 \wedge dx_2 + 3dx_1 \wedge dx_2 - dx_3 \wedge dx_1$
(v) $9dx_1 - 7dx_2 \wedge dx_3$

3.3 The Wedgeproduct of *n*-Forms

Now that we have an idea of what the wedgeproduct does geometrically we want to be able to manipulate wedgeproducts of forms quickly with algebraic formulas. In this section we learn some of the algebra related to differential forms. First of all we will see that differential forms essentially follows the rules we expect from algebra with only minor tweaks to take into account of the fact that $dx_i \wedge dx_j = -dx_j \wedge dx_i$. After that we will introduce a little bit of common notation, and then we will write out a general formula for the wedgeproduct.

3.3.1 Algebraic Properties

Suppose that $\omega, \omega_1, \omega_2$ are k-forms and η, η_1, η_2 are ℓ-forms. Then the following properties hold:

(1) $a\omega \wedge \eta = \omega \wedge a\eta = a\big(\omega \wedge \eta\big)$ where $a \in \mathbb{R}$,
(2) $\big(\omega_1 + \omega_2\big) \wedge \eta = \omega_1 \wedge \eta + \omega_2 \wedge \eta$,
(3) $\omega \wedge \big(\eta_1 + \eta_2\big) = \omega \wedge \eta_1 + \omega \wedge \eta_2$,
(4) $\omega \wedge \eta = (-1)^{k\ell} \eta \wedge \omega$.

We begin by showing these algebraic properties are reasonable by simply looking at a few easy examples on $\mathbb{R}^3$. We will begin with the first property and show that $(adx_1) \wedge dx_2 = dx_1 \wedge (adx_2) = a(dx_1 \wedge dx_2)$. First we have

$$\begin{aligned}
(adx_1) \wedge dx_2(u, v) &= \begin{vmatrix} adx_1(u) & adx_1(v) \\ dx_2(u) & dx_2(v) \end{vmatrix} \\
&= adx_1(u)dx_2(v) - adx_1(v)dx_2(u) \\
&= a\big(dx_1(u)dx_2(v) - dx_1(v)dx_2(u)\big) \\
&= a \begin{vmatrix} dx_1(u) & dx_1(v) \\ dx_2(u) & dx_2(v) \end{vmatrix} \\
&= a\big(dx_1 \wedge dx_2\big)(u, v).
\end{aligned}$$

Similarly,

$$
\begin{aligned}
dx_1 \wedge (adx_2)(u,v) &= \begin{vmatrix} dx_1(u) & dx_1(v) \\ adx_2(u) & adx_2(v) \end{vmatrix} \\
&= adx_1(u)dx_2(v) - adx_1(v)dx_2(u) \\
&= a\big(dx_1(u)dx_2(v) - dx_1(v)dx_2(u)\big) \\
&= a \begin{vmatrix} dx_1(u) & dx_1(v) \\ dx_2(u) & dx_2(v) \end{vmatrix} \\
&= a\big(dx_1 \wedge dx_2\big)(u,v).
\end{aligned}
$$

Putting all of this together and not writing the vectors gives us the identity we wanted. Next we look at an example that illustrates the second property; we show that $\big(dx_1 + dx_2\big) \wedge dx_3 = dx_1 \wedge dx_3 + dx_2 \wedge dx_3$.

$$
\begin{aligned}
&\Big(\big(dx_1 + dx_2\big) \wedge dx_3\Big)(u,v) \\
&= \begin{vmatrix} \big(dx_1 + dx_2\big)(u) & \big(dx_1 + dx_2\big)(v) \\ dx_3(u) & dx_3(v) \end{vmatrix} \\
&= \big(dx_1(u) + dx_2(u)\big)dx_3(v) - \big(dx_1(v) + dx_2(v)\big)dx_3(u) \\
&= dx_1(u)dx_3(v) + dx_2(u)dx_3(v) - dx_1(v)dx_3(u) - dx_2(v)dx_3(u) \\
&= dx_1(u)dx_3(v) - dx_1(v)dx_3(u) + dx_2(u)dx_3(v) - dx_2(v)dx_3(u) \\
&= \begin{vmatrix} dx_1(u) & dx_1(v) \\ dx_3(u) & dx_3(v) \end{vmatrix} + \begin{vmatrix} dx_2(u) & dx_2(v) \\ dx_3(u) & dx_3(v) \end{vmatrix} \\
&= \big(dx_1 \wedge dx_2\big)(u,v) + \big(dx_2 \wedge dx_3\big)(u,v).
\end{aligned}
$$

Notice that the third property $dx_1 \wedge \big(dx_2 + dx_3\big) = dx_1 \wedge dx_2 + dx_1 \wedge dx_3$ is essentially the same.

The last property is very straightforward. We already know that $dx_1 \wedge dx_2 = -dx_2 \wedge dx_1$. In fact, if $i \neq j$ we have $dx_i \wedge dx_j = -dx_j \wedge dx_i$. Suppose we had a three-form wedged with a two-form, $(dx_1 \wedge dx_2 \wedge dx_3) \wedge (dx_4 \wedge dx_5)$. We want to show that

$$
(dx_1 \wedge dx_2 \wedge dx_3) \wedge (dx_4 \wedge dx_5) = (-1)^{3 \cdot 2}(dx_4 \wedge dx_5) \wedge (dx_1 \wedge dx_2 \wedge dx_3).
$$

This involves nothing more than counting up how many switches are made to rearrange the terms.

Question 3.10 Rearrange $dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5$ to $dx_4 \wedge dx_5 \wedge dx_1 \wedge dx_2 \wedge dx_3$ using six switches. For example, moving from $dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5$ to $dx_1 \wedge dx_2 \wedge dx_4 \wedge dx_3 \wedge dx_5$ requires one switch, and hence

$$
dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5 = -dx_1 \wedge dx_2 \wedge dx_4 \wedge dx_3 \wedge dx_5
$$

since $dx_3 \wedge dx_4 = -dx_4 \wedge dx_3$.

The general definition of the wedgeproduct of n one-forms was given in Sect. 3.1 as

$$
dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_n}\big(v_1, v_2, \ldots v_n\big) \equiv \begin{vmatrix} dx_{i_1}(v_1) & dx_{i_1}(v_2) & \cdots & dx_{i_1}(v_n) \\ dx_{i_2}(v_1) & dx_{i_2}(v_2) & \cdots & dx_{i_2}(v_n) \\ \vdots & \cdots & \ddots & \vdots \\ dx_{i_n}(v_1) & dx_{i_n}(v_2) & \cdots & dx_{i_n}(v_n) \end{vmatrix},
$$

where $v_1, v_2, \ldots, v_n$ represented vectors. Geometrically speaking, the n-form $dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_n}$ found the volume of the parallelepiped spanned by the projections of $v_1, v_2, \ldots, v_n$ onto the $\frac{\partial}{\partial x_{i_1}} \frac{\partial}{\partial x_{i_2}} \cdots \frac{\partial}{\partial x_{i_n}}$-subspace. Now recalling our formula for the determinant that we found in Sect. 1.2,

$$\begin{vmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{vmatrix} = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{\sigma(i)i},$$

we can combine this with our definition for the wedgeproduct of n one-forms to get the following important formula for the wedgeproduct, where S_n is the set of permutations on n elements,

Wedgeproduct of n one-forms
$$dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_n}\big(v_1, v_2, \ldots v_n\big) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{j=1}^{n} dx_{\sigma(i_j)}(v_j).$$

In fact, occasionally you see the wedgeproduct defined in terms of this formula. The problem with defining the wedgeproduct this way to start with is that it is not at all clear what the formula is actually doing or why this definition is important. At least we now thoroughly understand exactly what the wedgeproduct finds. It is not difficult to use this formula to actually prove the algebraic properties in this section, as the following questions show.

Question 3.11 Let $\omega = a\ dx_{i_1} \wedge \cdots dx_{i_k}$ and let $\eta = b\ dx_{i_{k+1}}$, where $a, b \in \mathbb{R}$. Using the formula for the wedgeproduct of n one-forms and a procedure similar to the example above, show that

$$\omega \wedge \eta = ab\ dx_{i_1} \wedge \cdots dx_{i_k} \wedge dx_{i_{k+1}}.$$

Question 3.12 Let $\omega_1 = a_1\ dx_{i_1} \wedge \cdots dx_{i_k}$, $\omega_2 = a_2\ dx_{j_1} \wedge \cdots dx_{j_k}$, and $\eta = b\ dx_\ell$. Use the formula for the wedgeproduct of n one-forms to show that

$$\big(\omega_1 + \omega_2\big) \wedge \eta = \omega_1 \wedge \eta + \omega_2 \wedge \eta.$$

Explain how this also implies $\omega \wedge \big(\eta_1 + \eta_2\big) = \omega \wedge \eta_1 + \omega \wedge \eta_2$.

Question 3.13 Let $\omega = dx_{i_1} \wedge \cdots dx_{i_k}$ and $\eta = dx_{j_1} \wedge \cdots dx_{j_\ell}$. Show that

$$\omega \wedge \eta = (-1)^{k\ell} \eta \wedge \omega.$$

Question 3.14 Suppose that $\omega, \omega_1, \omega_2$ are k-forms and η, η_1, η_2 are ℓ-forms. Using the above three questions, complete the proofs of the algebraic properties in this section,

(1) $a\omega \wedge \eta = \omega \wedge a\eta = a\big(\omega \wedge \eta\big)$ where $a \in \mathbb{R}$,
(2) $\big(\omega_1 + \omega_2\big) \wedge \eta = \omega_1 \wedge \eta + \omega_2 \wedge \eta$,
(3) $\omega \wedge \big(\eta_1 + \eta_2\big) = \omega \wedge \eta_1 + \omega \wedge \eta_2$,
(4) $\omega \wedge \eta = (-1)^{k\ell} \eta \wedge \omega$.

3.3.2 Simplifying Notation

In the last section we looked at two-forms on $\mathbb{R}^2$, denoted by $\bigwedge^2(\mathbb{R}^3)$, and found that $dx_1 \wedge dx_2$, $dx_2 \wedge dx_3$, and $dx_3 \wedge dx_1$ was a basis of $\bigwedge^2(\mathbb{R}^3)$. Hence any two-form $\alpha \in \bigwedge^2(\mathbb{R}^3)$ was of the form

$$\alpha = a_{12} dx_1 \wedge dx_2 + a_{23} dx_2 \wedge dx_3 + a_{31} dx_3 \wedge dx_1$$

where a_{12}, a_{23}, and a_{31} are just constants from $\mathbb{R}$. We could just as easily have written

$$\alpha = a dx_1 \wedge dx^2 + b dx_2 \wedge dx_3 + c dx_3 \wedge dx_1$$

as we did in the last section, or even written

$$\alpha = a_1 dx_1 \wedge dx_2 + a_2 dx_2 \wedge dx_3 + a_3 dx_3 \wedge dx_1.$$

After all, what does it matter what we name a constant? But it turns out that indexing our constants with the same numbers that appear in the basis element that follows it is notationally convenient. Recall, we wrote the indices of the two-form basis elements on $\mathbb{R}^3$ in cyclic order, which meant that we wrote $dx_3 \wedge dx_1$ instead of $dx_1 \wedge dx_3$. This is the convention in vector calculus. But for manifolds $\mathbb{R}^n$, where $n > 3$ this convention no longer works and we simply write our indices in increasing order. If $\alpha \in \bigwedge^2(\mathbb{R}^4)$ we would have

$$\begin{aligned}\alpha = {} & a_{12} dx_1 \wedge dx_2 + a_{13} dx_1 \wedge dx_3 + a_{14} dx_1 \wedge dx_4 \\ & a_{23} dx_2 \wedge dx_3 + a_{24} dx_2 \wedge dx_4 + a_{34} dx_3 \wedge dx_4.\end{aligned}$$

The basis elements of $\bigwedge^3(\mathbb{R}^4)$ are

$$dx_1 \wedge dx_2 \wedge dx_3, \quad dx_1 \wedge dx_2 \wedge dx_4, \quad dx_1 \wedge dx_3 \wedge dx_4, \quad dx_2 \wedge dx_3 \wedge dx_4.$$

Again, note that the indices are all in increasing order. An arbitrary three-form $\alpha \in \bigwedge^3(\mathbb{R}^4)$ would be written as

$$\begin{aligned}\alpha = {} & a_{123} dx_1 \wedge dx_2 \wedge dx_3 + a_{124} dx_1 \wedge dx_2 \wedge dx_4 \\ & + a_{134} dx_1 \wedge dx_3 \wedge dx_4 + a_{234} dx_2 \wedge dx_3 \wedge dx_4.\end{aligned}$$

For arbitrary k-forms in $\bigwedge^k(\mathbb{R}^n)$ we would not want to actually write out all of the elements of the basis of $\bigwedge^k(\mathbb{R}^n)$, so instead we will write

$$\alpha = \sum_I a_I dx^I.$$

Here the I stands for our elements in the set of k increasing indices $i_1 i_2 \dots i_k$, where $1 \le i_1 < i_2 < \cdots < i_k \le n$. That is,

$$I \in \mathcal{I}_{k,n} = \Big\{ (i_1 i_2 \dots i_k) \Big| \; 1 \le i_1 < i_2 < \cdots < i_k \le n \Big\}.$$

So, for $k = 2$ and $n = 4$ we would have

$$I \in \mathcal{I}_{2,4} = \big\{12, 13, 14, 23, 24, 34\big\}$$

and for $k = 3$ and $n = 4$ we have

$$I \in \mathcal{I}_{3,4} = \big\{123, 124, 134, 234\big\}.$$

In this notation we would have $dx^{12} \equiv dx_1 \wedge dx_2$, $dx^{24} \equiv dx_2 \wedge dx_4$, and $dx^{134} \equiv dx_1 \wedge dx_3 \wedge dx_4$, and so on. In case you are wondering why the indices on the left are superscripts while the indices on the right are subscripts, this is so the notation is compatible with Einstein summation notation. We will explain Einstein summation notation in Sect. 9.4. For now don't worry about it.

Suppose that we have $\alpha \in \bigwedge^k(\mathbb{R}^n)$ and $\beta \in \bigwedge^l(\mathbb{R}^n)$ where $\alpha = \sum_I a_I dx^I$ and $\beta = \sum_J b_J dx^J$. Then $\alpha \wedge \beta$ is given by

Wedgeproduct of two arbitrary forms, Formula One	$\left(\sum_I a_I dx^I\right) \wedge \left(\sum_J b_J dx^J\right) = \sum a_I b_J dx^I \wedge dx^J.$

In the sum on the right hand side we have $dx^I \wedge dx^J = 0$ if I and J have any indices in common.

Question 3.15 Let $\alpha = \sum_I a_I dx^I \in \bigwedge^k(\mathbb{R}^n)$ and $\beta = \sum_J b_J dx^J \in \bigwedge^l(\mathbb{R}^n)$. By repeated applications of the algebraic properties of the wedgeproduct prove this formula for the wedgeproduct of two arbitrary forms.

If I and J are disjoint then we have $dx^I \wedge dx^J = \pm dx^K$ where $K = I \cup J$, but is reordered to be in increasing order and elements with repeated indices are dropped. Thus formula one automatically gives us formula two,

Wedgeproduct of two arbitrary forms, Formula Two	$$\left(\sum_I a_I dx^I\right) \wedge \left(\sum_J b_J dx^J\right) = \sum_K \left(\sum_{\substack{I \cup J \\ I, J \text{ disjoint}}} \pm a_I b_J\right) dx^K.$$

An example should hopefully make this clearer. Let α and β be the following forms on $\mathbb{R}^8$

$$\alpha = 5dx_1 \wedge dx_2 - 6dx_2 \wedge dx_4 + 7dx_1 \wedge dx_7 + 2dx_2 \wedge dx_8,$$
$$\beta = 3dx_3 \wedge dx_5 \wedge dx_8 - 4dx_5 \wedge dx_6 \wedge dx_8.$$

Then we have $\alpha = \sum_I a_I dx^I$ where the Is are the elements of the set $\{12, 24, 17, 28\} \subset \mathcal{I}_{2,8}$ and our coefficients are $a_{12} = 5$, $a_{24} = -6$, $a_{17} = 7$, and $a_{28} = 2$. Similarly, we have $\beta = \sum_J b_J dx^J$ where the Js are the elements of the set $\{358, 568\} \subset \mathcal{I}_{3,8}$ and the coefficients are $b_{358} = 3$ and $b_{568} = -4$. So, what are our elements K? They are given by the set $I \cup J$ but reordered into increasing order and with elements with repeated indices dropped. First we have

$$\begin{aligned} I \cup J = \{&12358, 24358, 17358, 28358 \\ &12568, 24568, 17568, 28568\}. \end{aligned}$$

The first thing that we notice is that two elements of this set have indices that are repeated. Both 28358 and 28568 repeat the 8. And since both $dx_2 \wedge dx_8 \wedge dx_3 \wedge dx_5 \wedge dx_8 = 0$ and $dx_2 \wedge dx_8 \wedge dx_5 \wedge dx_6 \wedge dx_8 = 0$ we can drop these elements.

To get K we need to put the other sets of indices in increasing order. The first element, 12358, is already in increasing order so nothing needs to be done. The coefficient in front of dx^{12358} is simply $a_{12}b_{358}$. To get 24358 into increasing order all we need to do is one switch, we need to switch the 4 and the 3 to get 23458. The one switch gives a negative sign which shows up in the coefficient in front of dx^{23458}, which is $-a_{24}b_{358} = -(-6)(3) = 18$. Similarly for the other terms. Thus we end up with

$$\begin{aligned} K = \{&12358, 23458, 13578, \cancel{23588} \\ &12568, 24568, 15678, \cancel{25688}\}. \end{aligned}$$

Notice that the way we have defined K means we are not keeping track of the sign changes that occur when we have to make the switches to get the indices in order. That is why we have the $\pm$ when we write $dx^I \wedge dx^J = \pm dx^K$.

Question 3.16 Finish writing out $\alpha \wedge \beta$.

Question 3.17 Let

$$\alpha = 4dx_1 \wedge dx_3 + 5dx_3 \wedge dx_5 - 7dx_3 \wedge dx_9$$

be a two-form on $\mathbb{R}^9$ and let

$$\beta = 7dx_1 \wedge dx_4 \wedge dx_6 \wedge dx_8 - 3dx_2 \wedge dx_3 \wedge dx_7 \wedge dx_9 + dx_5 \wedge dx_6 \wedge dx_8 \wedge dx_9$$

be a four-form on $\mathbb{R}^9$. Find $\alpha \wedge \beta$ and then find $\beta \wedge \alpha$.

From the preceding questions you can see that actually using this formula means we have to go through a lot of work rearranging the coefficients of I and J to get K and finding the correct sign. Where this formula really comes in useful is when we are doing general calculations, instead of working with specific examples, and are not so worried about the exact form the wedgeproduct takes.

3.3.3 The General Formula

This section introduces several new formulas for the wedgeproduct of two arbitrary forms. In fact, the following three general formulas are very often encountered as definitions of the wedgeproduct. We will discuss two of the three formulas in depth in this section. The third is provided here simply for completeness' sake. Suppose that α is a k-form and β is a ℓ-form. Then the wedgeproduct of α with β is given by

$$\boxed{\text{Wedgeproduct of two arbitrary forms, Formula Three} \quad (\alpha \wedge \beta)\big(v_1, \ldots, v_{k+\ell}\big) = \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}(\sigma)\alpha\big(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\big)\beta\big(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}\big),}$$

which can also be written as

$$\boxed{\text{Wedgeproduct of two arbitrary forms, Formula Four} \quad (\alpha \wedge \beta)\big(v_1, \ldots, v_{k+\ell}\big) = \sum_{\substack{\sigma \text{ is a} \\ (k+\ell)- \\ \text{shuffle}}} \operatorname{sgn}(\sigma)\alpha\big(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\big)\beta\big(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}\big).}$$

In books where tensors are introduced first and then differential forms are defined as a certain type of tensor, the wedgeproduct is often defined by the formula

$$\boxed{\text{Wedgeproduct of two arbitrary forms, Formula Five} \quad \alpha \wedge \beta = \frac{(k+\ell)!}{k!\ell!}\mathscr{A}(\alpha \otimes \beta)}$$

where $\otimes$ is the tensor product and $\mathscr{A}$ is the skew-symmetrization (or anti-symmetrization) operator. The explanation for this particular definition is given in the appendix on tensors, Sect. A.5.

We now turn our attention to understanding formulas three and four, and to seeing that they are indeed identical to our volume/determinant based definition of the wedgeproduct. The first step is to show that the right hand sides of formulas three and four are equal to each other. Once we have done that we will show that these formulas are in fact derivable from our definition of the wedgeproduct of n one-forms and thus can also be used as equations that define the wedgeproduct; that is, we will show that these formulas are indeed formulas for $(\alpha \wedge \beta)\big(v_1, \ldots, v_{k+\ell}\big)$.

To start with consider the formula given by the right hand side of formula three,

$$\frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}(\sigma)\alpha\big(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\big)\beta\big(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}\big).$$

Where does this term $\frac{1}{k!\ell!}$ in the front come from? Consider a permutation $\sigma \in S_{k+\ell}$. There are $k!$ different permutations τ in $S_{k+\ell}$ that permute the first k terms but leave the terms $k+1, \ldots, k+\ell$ fixed. This means that for any permutation σ we have $\sigma\tau(k+1) = \sigma(k+1), \ldots, \sigma\tau(k+\ell) = \sigma(k+\ell)$. That means we have

$$\begin{aligned}
&\operatorname{sgn}(\sigma\tau)\alpha\big(v_{\sigma\tau(1)}, \ldots, v_{\sigma\tau(k)}\big)\beta\big(v_{\sigma\tau(k+1)}, \ldots, v_{\sigma\tau(k+\ell)}\big) \\
&= \operatorname{sgn}(\sigma\tau)\operatorname{sgn}(\tau)\alpha\big(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\big)\beta\big(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}\big) \\
&= \operatorname{sgn}(\sigma)\underbrace{\operatorname{sgn}(\tau)\operatorname{sgn}(\tau)}_{=1}\alpha\big(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\big)\beta\big(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}\big) \\
&= \operatorname{sgn}(\sigma)\alpha\big(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\big)\beta\big(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}\big)
\end{aligned}$$

where the first equality follows from the fact that $\tau(1), \ldots \tau(k)$ is just a permutation of $1, \ldots, k$, which results in $\alpha\big(v_{\sigma\tau(1)}, \ldots, v_{\sigma\tau(k)}\big) = \operatorname{sgn}(\tau)\alpha\big(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\big)$, and that $\sigma\tau(k+1) = \sigma(k+1), \ldots, \sigma\tau(k+\ell) = \sigma(k+l)$, which gives $\beta\big(v_{\sigma\tau(k+1)},\dots,v_{\sigma\tau(k+\ell)}\big) = \beta\big(v_{\sigma(k+1)},\dots,v_{\sigma(k+\ell)}\big)$. This means that given a particular σ then there are $k!$ other terms in the sum

$$\sum_{\sigma\in S_{k+\ell}} \operatorname{sgn}(\sigma)\alpha\big(v_{\sigma(1)},\dots,v_{\sigma(k)}\big)\beta\big(v_{\sigma(k+1)},\dots,v_{\sigma(k+\ell)}\big)$$

that are exactly the same. We divide the sum by $k!$ to eliminate the $k!$ repeated terms.

Similarly, given a permutation $\sigma \in S_{k+\ell}$ there are $\ell!$ different permutations τ in $S_{k+\ell}$ that permute the last ℓ terms but hold the terms $1,\dots,k$ fixed. An identical arguments shows we also need to divide by $\ell!$ to eliminate the $\ell!$ repeated terms. Thus the factor $\frac{1}{k!\ell!}$ eliminates all the repeated terms in the sum over $\sigma \in S_{k+\ell}$. This means that the right hand side of our first definition of the wedgeproduct eliminates all of the repeated terms.

Next we turn our attention to seeing that

$$\begin{aligned}\frac{1}{k!\ell!}&\sum_{\sigma\in S_{k+\ell}} \operatorname{sgn}(\sigma)\alpha\big(v_{\sigma(1)},\dots,v_{\sigma(k)}\big)\beta\big(v_{\sigma(k+1)},\dots,v_{\sigma(k+\ell)}\big)\\ &= \sum_{\substack{\sigma \text{ is a}\\ (k,\ell)-\text{shuffle}}} \operatorname{sgn}(\sigma)\alpha\big(v_{\sigma(1)},\dots,v_{\sigma(k)}\big)\beta\big(v_{\sigma(k+1)},\dots,v_{\sigma(k+\ell)}\big).\end{aligned}$$

A permutation $\sigma \in S_{k+\ell}$ is called a (k,ℓ)-shuffle if

$$\sigma(1) < \cdots < \sigma(k) \quad \text{and} \quad \sigma(k+1) < \cdots < \sigma(k+\ell).$$

In other words, if both the first k terms of the permutation are in increasing order and the last ℓ terms of the permutation are in increasing order.

Question 3.18 Find the $(2,3)$-shuffles of 1 2 3 4 5.

When we are summing over $\sigma \in S_{k+\ell}$ on the left hand side, for each of these σ there are $k!\ell! - 1$ other permutations which give the same term. This is essentially the same argument that we just went through. Exactly one of these $k!\ell!$ different permutations that give the same term will actually be a (k,l)-shuffle. Thus, by choosing the particular permutation which is the (k,ℓ)-shuffle what we are doing is choosing one representative permutation from the $k!\ell!$ identical terms. So when we sum over all the (k,ℓ)-shuffles we are only summing over one term from each of the sets of identical terms that appear in the sum over all $\sigma \in S_{k+l}$, thereby eliminating the need to divide by $k!\ell!$. Hence we have shown that the two formulas on the right hand sides of formula three and formula four for the wedgeproduct of two differential forms are indeed equal.

Question 3.19 Fill in the details of this argument.

Now we will show that the right hand side of formula four is indeed equal to the left hand side of formula four. Once this is done formula three will automatically follow by the above. Before actually doing the general case we will show it is true for a simple example. Consider $\alpha = dx_2 \wedge dx_3 \in \bigwedge^2(\mathbb{R}^5)$ and $\beta = dx_5 \in \bigwedge^1(\mathbb{R}^5)$. We will first compute $\alpha\wedge\beta(v_1,v_2,v_3)$ from the volume/determinant-based definition of the wedgeproduct. We will then apply the right hand side of formula four to α and β and thereby see that they are equal, which is exactly what we want to show.

First we write out $\alpha\wedge\beta(v_1,v_2,v_3)$ using our volume/determinat based definition,

$$\begin{aligned}&\alpha\wedge\beta(v_1,v_2,v_3)\\ &= \big(dx_2\wedge dx_3\wedge dx_5\big)(v_1,v_2,v_3)\\ &= \begin{vmatrix} dx_2(v_1) & dx_2(v_2) & dx_2(v_3)\\ dx_3(v_1) & dx_3(v_2) & dx_3(v_3)\\ dx_5(v_1) & dx_5(v_2) & dx_5(v_3)\end{vmatrix}\\ &= dx_2(v_1)dx_3(v_2)dx_5(v_3) + dx_2(v_2)dx_3(v_3)dx_5(v_1)\\ &\quad + dx_2(v_3)dx_3(v_1)dx_5(v_2) - dx_2(v_3)dx_3(v_2)dx_5(v_1)\\ &\quad - dx_2(v_2)dx_3(v_1)dx_5(v_3) - dx_2(v_1)dx_3(v_3)dx_5(v_2)\\ &= \big(dx_2dx_3dx_5 + dx_5dx_2dx_3 + dx_3dx_5dx_2\\ &\quad - dx_5dx_3dx_2 - dx_3dx_2dx_5 - dx_2dx_5dx_3\big)(v_1,v_2,v_3).\end{aligned}$$

Now we will use the right hand side of formula four and find $\sum_\sigma \text{sgn}(\sigma)\alpha(v_{\sigma(1)}, v_{\sigma(2)})\beta(v_{\sigma(3)})$ where σ is a (2, 1)-shuffle. The (2, 1)-shuffles of 123 are 231, 132, and 123. Thus we have the three (2, 1)-shuffle permutations

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 3 & 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 1 & 3 & 2 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 \end{pmatrix}.$$

It is easy to show that $\text{sgn}(\sigma_1)$ and $\text{sgn}(\sigma_3)$ are positive and $\text{sgn}(\sigma_2)$ is negative, hence

$$\begin{aligned}
&\alpha \wedge \beta(v_1, v_2, v_3) \\
&= \sum \text{sgn}(\sigma)(dx_2 \wedge dx_3)(v_{\sigma(1)}, v_{\sigma(2)})dx_5(v_{\sigma(3)}) \\
&= (dx_2 \wedge dx_3)(v_2, v_3)dx_5(v_1) - (dx_2 \wedge dx_3)(v_1, v_3)dx_5(v_2) \\
&\quad + (dx_2 \wedge dx_3)(v_1, v_2)dx_5(v_3) \\
&= \begin{vmatrix} dx_2(v_2) & dx_2(v_3) \\ dx_3(v_2) & dx_3(v_3) \end{vmatrix} dx_5(v_1) - \begin{vmatrix} dx_2(v_1) & dx_2(v_3) \\ dx_3(v_1) & dx_3(v_3) \end{vmatrix} dx_5(v_2) \\
&\quad + \begin{vmatrix} dx_2(v_1) & dx_2(v_2) \\ dx_3(v_1) & dx_3(v_2) \end{vmatrix} dx_5(v_3) \\
&= dx_2(v_2)dx_3(v_3)dx_5(v_1) - dx_2(v_3)dx_3(v_2)dx_5(v_1) \\
&\quad - dx_2(v_1)dx_3(v_3)dx_5(v_2) + dx_2(v_3)dx_3(v_1)dx_5(v_2) \\
&\quad + dx_2(v_1)dx_3(v_2)dx_5(v_3) + dx_2(v_2)dx_3(v_1)dx_5(v_3) \\
&= \big(dx_5dx_2dx_3 - dx_5dx_3dx_2 - dx_2dx_5dx_3 \\
&\quad + dx_3dx_5dx_2 + dx_2dx_3dx_5 - dx_3dx_2dx_5\big)(v_1, v_2, v_3).
\end{aligned}$$

Upon rearrangement we see that indeed, $\alpha \wedge \beta(v_1, v_2, v_3)$ found using the volume/determinant-based definition of wedgeproduct is exactly the same as what we found when we used the right hand side of formula four, $\sum_\sigma \text{sgn}(\sigma)\alpha(v_{\sigma(1)}, v_{\sigma(2)})\beta(v_{\sigma(3)})$ where σ is a (2, 1)-shuffle. Thus, indeed, $\alpha \wedge \beta(v_1, v_2, v_3) = \sum_\sigma \text{sgn}(\sigma)\alpha(v_{\sigma(1)}, v_{\sigma(2)}) \beta(v_{\sigma(3)})$ where σ is a (2, 1)-shuffle

Now we turn our attention to the general case. The procedure we use will mirror that of the example. We want to show this equality for a k-form $\alpha \in \bigwedge^k(\mathbb{R}^n)$ and ℓ-form $\beta \in \bigwedge^\ell(\mathbb{R}^n)$. In general we have $\alpha = \sum_I a_I dx^I$ and $\beta = \sum_J b_J dx^J$, where

$$\begin{aligned}
I \in \mathfrak{I}_{k,n} &= \big\{ i_1 i_2 \cdots i_k \;\big|\; 1 \le i_1 < i_2 < \cdots < i_k \le n \big\}, \\
J \in \mathfrak{J}_{\ell,n} &= \big\{ j_1 j_2 \cdots j_\ell \;\big|\; 1 \le j_1 < j_2 < \cdots < j_\ell \le n \big\}.
\end{aligned}$$

The first thing to notice is that our definition of the wedgeproduct based on the volume/determinant only applies to elements of the form $dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_m}$, which are the basis elements of $\bigwedge^m(\mathbb{R}^n)$. If we had a general k-form such as $\alpha = \sum_I a_I dx^I$ and wanted to find $\alpha(v_1, \ldots, v_k)$ using this definition we first have to find the value of $dx^I(v_1, \ldots, v_k) = dx_{i_1} \wedge \cdots \wedge dx_{i_k}(v_1, \ldots, v_k)$ for each I by taking the determinant of the projected vectors, multiply what we found by the coefficient a_I, and then added all the terms up. Similarly for β.

Taking the wedgeproduct of α and β and using formula one proved in the last section we have

$$\begin{aligned}
&\alpha \wedge \beta \\
&= \left(\sum_I a_I dx^I \right) \wedge \left(\sum_J b_J dx^J \right) \\
&= \sum_{I,J} a_I b_J dx^I \wedge dx^J \\
&= \sum_{I,J} a_I b_J dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge dx_{j_\ell}.
\end{aligned}$$

So using our volume/determinant definition of wedgeproduct we have

$$
\begin{aligned}
&\alpha \wedge \beta(v_1, \cdots v_{k+\ell}) \\
&= \sum_{I,J} a_I b_J dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge dx_{j_\ell}(v_1, \cdots, v_{k+\ell}) \\
&= \sum_{\mathbf{I}} a_{\mathbf{I}} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{i_{k+1}} \wedge \cdots \wedge dx_{i_{k+\ell}}(v_1, \cdots, v_{k+\ell}) \\
&= \sum_{\mathbf{I}} a_{\mathbf{I}} \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}(\sigma) \prod_{m=1}^{k+\ell} dx_{i_{\sigma(m)}}(v_m),
\end{aligned}
$$

where we make a notation change to $\mathbf{I} = I \cup J$ in the second equality to make our indice labels line up nicely. Hence $j_1 = i_{k+1}, \ldots j_\ell = i_{k+\ell}$ and the $a_{\mathbf{I}}$ term is the appropriate $a_I b_J$ term; that is, $a_{i_1 \cdots i_{k+\ell}} = a_{i_1 \cdots i_k} b_{j_1 \cdots j_\ell}$.

Now, using our (k, ℓ)-shuffle formula we have

$$
\begin{aligned}
&\sum_{\substack{\sigma \text{ a} \\ (k,\ell)- \\ \text{shuffle}}} \operatorname{sgn}(\sigma)\alpha(v_{\sigma(1)}, \cdots v_{\sigma(k)})\beta(v_{\sigma(k+1)}, \cdots v_{\sigma(k+\ell)}) \\
&= \sum_{\substack{\sigma \text{ a} \\ (k,\ell)- \\ \text{shuffle}}} \operatorname{sgn}(\sigma) \left(\sum_I a_I dx^I \right) (v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \left(\sum_J b_J dx^J \right) (v_{\sigma(k+1)}, \cdots, v_{\sigma(k+\ell)}) \\
&= \sum_{\substack{\sigma \text{ a} \\ (k,\ell)- \\ \text{shuffle}}} \operatorname{sgn}(\sigma) \sum_{I,J} a_I b_J dx^I (v_{\sigma(1)}, \cdots, v_{\sigma(k)}) dx^J (v_{\sigma(k+1)}, \cdots, v_{\sigma(k+\ell)}) \\
&= \sum_{\mathbf{I}} a_{\mathbf{I}} \sum_{\substack{\sigma \text{ a} \\ (k,\ell)- \\ \text{shuffle}}} \operatorname{sgn}(\sigma) dx_{i_1} \wedge \cdots \wedge dx_{i_k}(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \underbrace{\cdot}_{\substack{\text{multiplication} \\ \text{not } \wedge}} dx_{i_{k+1}} \wedge \cdots \wedge dx_{i_{k+1}}(v_{\sigma(k+1)}, \cdots, v_{\sigma(k+\ell)})
\end{aligned}
$$

where in the last equality we made the same notation changes as before and the order of summation is changed. Our goal is to see that this is identical to what the volume/determinant definition gave us in the preceding formula.

First, using the determinant definition of wedgeproduct we have

$$
dx_{i_1} \wedge \cdots \wedge dx_{i_k}(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) = \sum_{\tilde{\sigma} \in S_k} \operatorname{sgn}(\tilde{\sigma}) \prod_{m=1}^{k} dx_{i_{\tilde{\sigma}(m)}}(v_{\sigma(m)})
$$

and

$$
dx_{i_{k+1}} \wedge \cdots \wedge dx_{i_{k+1}}(v_{\sigma(k+1)}, \cdots, v_{\sigma(k+\ell)}) = \sum_{\tilde{\tilde{\sigma}} \in S_\ell} \operatorname{sgn}(\tilde{\tilde{\sigma}}) \prod_{m=k+1}^{k+\ell} dx_{i_{\tilde{\tilde{\sigma}}(m)}}(v_{\sigma(m)}).
$$

Combining this we have

$$
\begin{aligned}
&\sum_{\mathbf{I}} a_{\mathbf{I}} \sum_{\substack{\sigma \text{ a} \\ (k,\ell)- \\ \text{shuffle}}} \operatorname{sgn}(\sigma) dx_{i_1} \wedge \cdots \wedge dx_{i_k}(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \underbrace{\cdot}_{\substack{\text{mult.} \\ \text{not } \wedge}} dx_{i_{k+1}} \wedge \cdots \wedge dx_{i_{k+1}}(v_{\sigma(k+1)}, \cdots, v_{\sigma(k+\ell)}) \\
&= \sum_{\mathbf{I}} a_{\mathbf{I}} \sum_{\substack{\sigma \text{ a} \\ (k,\ell)- \\ \text{shuffle}}} \operatorname{sgn}(\sigma) \left(\sum_{\tilde{\sigma} \in S_k} \operatorname{sgn}(\tilde{\sigma}) \prod_{m=1}^{k} dx_{i_{\tilde{\sigma}(m)}}(v_{\sigma(m)}) \right) \cdot \left(\sum_{\tilde{\tilde{\sigma}} \in S_\ell} \operatorname{sgn}(\tilde{\tilde{\sigma}}) \prod_{m=k+1}^{k+\ell} dx_{i_{\tilde{\tilde{\sigma}}(m)}}(v_{\sigma(m)}) \right)
\end{aligned}
$$

$$= \sum_{\mathbf{I}} a_{\mathbf{I}} \sum_{\substack{\sigma \text{ a} \\ (k,\ell)-\\ \text{shuffle}}} \sum_{\tilde{\sigma} \in S_k} \sum_{\tilde{\tilde{\sigma}} \in S_\ell} \text{sgn}(\sigma)\text{sgn}(\tilde{\sigma})\text{sgn}(\tilde{\tilde{\sigma}}) \left(\prod_{m=1}^{k} dx_{i_{\tilde{\sigma}(m)}}(v_{\sigma(m)}) \right) \left(\prod_{m=k+1}^{k+\ell} dx_{i_{\tilde{\tilde{\sigma}}(m)}}(v_{\sigma(m)}) \right)$$

$$= \sum_{\mathbf{I}} a_{\mathbf{I}} \sum_{\sigma \in S_{k+\ell}} \text{sgn}(\sigma) \prod_{m=1}^{k+\ell} dx_{i_{\sigma(m)}}(v_m),$$

which is exactly the same as $\alpha \wedge \beta(v_1, \cdots v_{k+\ell})$ given by volume/determinant definition. Hence, formula four is indeed a formula for $\alpha \wedge \beta(v_1, \cdots v_{k+\ell})$. Formula three then follows. A little thought may be required to convince yourself of the final equality.

Question 3.20 Explain the final equality above.

3.4 The Interior Product

We now introduce something called the **interior product**, or **inner product**, of a vector and a k-form. Given the vector v and a k-form α the interior product of v with α is denoted by $\iota_v \alpha$. The interior product of a k-form α with a vector v is a $(k-1)$-form defined by

$$\boxed{\text{Interior Product:} \qquad \iota_v \alpha(v_1, \ldots, v_{k-1}) = \alpha(v, v_1, \ldots, v_{k-1}).}$$

where $v_1, \ldots, v_{k-1}$ are any $k-1$ vectors. In other words, $\iota_v \alpha$ just puts the vector v into α's first slot. So, when α is a k-form then $\iota_v \alpha$ is a $(k-1)$-form.

We now point out one simple identity. If both α and β are k-forms then

$$\boxed{\iota_v \big(\alpha + \beta\big) = \iota_v \alpha + \iota_v \beta.}$$

This comes directly from how the sums of forms are evaluated,

$$\begin{aligned}
&\iota_v \big(\alpha + \beta\big)(v_1, \ldots, v_{k-1}) \\
&= \big(\alpha + \beta\big)(v, v_1, \ldots, v_{k-1}) \\
&= \alpha(v, v_1, \ldots, v_{k-1}) + \beta(v, v_1, \ldots, v_{k-1}) \\
&= \iota_v \alpha(v_1, \ldots, v_{k-1}) + \iota_v \beta(v_1, \ldots, v_{k-1}),
\end{aligned}$$

which gives us the identity. Now suppose α is a k-form and v, w are vectors. We also have

$$\boxed{\iota_{(v+w)} \alpha = \iota_v \alpha + \iota_w \alpha.}$$

Thus the interior product $\iota_v \alpha$ is linear in both v and α.

Question 3.21 If α is a k-form and v, w are vectors, show that $\iota_{(v+w)} \alpha = \iota_v \alpha + \iota_w \alpha$.

Let us see how this works for some simple examples. Let us begin by looking at two-forms on the manifold $\mathbb{R}^3$. First we will consider the most basic two-forms, $dx \wedge dy$, $dy \wedge dz$, and $dx \wedge dz$. For the moment we will use increasing notation instead of cyclic notation in order to stay consistent with the notation of the last section. Here this simply means we will write $dx \wedge dz$ instead of $dz \wedge dx$. Letting

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

we get

$$\begin{aligned}
\iota_v\big(dx \wedge dy\big) &= dx \wedge dy(v, \cdot) \\
&= \begin{vmatrix} dx(v) & dx(\cdot) \\ dy(v) & dy(\cdot) \end{vmatrix} \\
&= dx(v)dy(\cdot) - dy(v)dx(\cdot) \\
&= v_1 dy - v_2 dx,
\end{aligned}$$

which is clearly a one-form. Similarly, for the second term we have

$$\begin{aligned}
\iota_v\big(dy \wedge dz\big) &= dy \wedge dz(v, \cdot) \\
&= \begin{vmatrix} dy(v) & dy(\cdot) \\ dz(v) & dz(\cdot) \end{vmatrix} \\
&= dy(v)dz(\cdot) - dz(v)dy(\cdot) \\
&= v_2 dz - v_3 dy.
\end{aligned}$$

And for the third term we have

$$\begin{aligned}
\iota_v\big(dx \wedge dz\big) &= dx \wedge dz(v, \cdot) \\
&= \begin{vmatrix} dx(v) & dx(\cdot) \\ dz(v) & dz(\cdot) \end{vmatrix} \\
&= dx(v)dz(\cdot) - dz(v)dx(\cdot) \\
&= v_1 dz - v_3 dx.
\end{aligned}$$

Question 3.22 Show that $\iota_v\big(f dx \wedge dy\big) = f \iota_v\big(dx \wedge dy\big)$.

Now suppose we had the more general two-form,

$$\begin{aligned}
\alpha &= f(x, y, z)dx \wedge dy + g(x, y, z)dy \wedge dz + h(x, y, z)dx \wedge dz \\
&= f dx \wedge dy + g dy \wedge dz + h dx \wedge dz.
\end{aligned}$$

Using the above we can write

$$\begin{aligned}
\iota_v \alpha &= \iota_v\big(f dx \wedge dy + g dy \wedge dz + h dx \wedge dz\big) \\
&= \iota_v\big(f dx \wedge dy\big) + \iota_v\big(g dy \wedge dz\big) + \iota_v\big(h dx \wedge dz\big) \\
&= f \iota_v\big(dx \wedge dy\big) + g \iota_v\big(dy \wedge dz\big) + h \iota_v\big(dx \wedge dz\big) \\
&= f\big(v_1 dy - v_2 dx\big) + g\big(v_2 dz - v_3 dy\big) + h\big(v_1 dz - v_3 dx\big) \\
&= -\big(f v_2 + h v_3\big)dx + \big(f v_1 - g v_3\big)dy + \big(g v_2 + h v_1\big)dz.
\end{aligned}$$

To get the second line we used the fact that $\iota_v\big(\alpha + \beta\big) = \iota_v \alpha + \iota_v \beta$ and the third line came from the above question. Clearly, simply writing down what $\iota_v \alpha$ is can be a fairly complicated endeavor. So we give the general formula

$$\boxed{\iota_v\big(dx_1 \wedge \cdots \wedge dx_k\big) = \sum_{i=1}^{k} (-1)^{i-1} dx_i(v)\big(dx_1 \wedge \cdots \widehat{dx_i} \cdots \wedge dx_k\big),}$$

where $\widehat{dx_i}$ means that the element dx_i is omitted.

Question 3.23 Use the general formula for $\iota_v \alpha$ to find $\iota_v\big(dx \wedge dy\big)$, $\iota_v\big(dy \wedge dz\big)$, and $\iota_v\big(dx \wedge dz\big)$.

Question 3.24 (Requires some linear algebra) Using the determinant definition of the wedgeproduct,

$$dx_1 \wedge dx_2 \wedge \cdots \wedge dx_k\big(v_1, v_2, \dots v_k\big) = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \prod_{i=1}^{k} dx_{\sigma(i)}(v_i)$$

deduce the formula

$$\iota_v\big(dx_1 \wedge \cdots \wedge dx_k\big) = \sum_{i=1}^{k} (-1)^{i-1} dx_i(v) \big(dx_1 \wedge \cdots \widehat{dx_i} \cdots \wedge dx_k\big)$$

by expanding the determinant in terms of the elements of the first column and their cofactors.

We now turn toward showing two important identities. If α is a k-form and β is any form then

$$\boxed{\iota_v\big(\alpha \wedge \beta\big) = \big(\iota_v \alpha\big) \wedge \beta + (-1)^k \alpha \wedge \big(\iota_v \beta\big).}$$

We first show this for the basis elements and then use the linearity of the interior product to show it is true for arbitrary forms. Suppose $\alpha = dx_1 \wedge \cdots \wedge dx_k$ and $\beta = dy_1 \wedge \cdots \wedge dy_q$. Then we have

$$\begin{aligned}
\iota_v(\alpha \wedge \beta) &= \iota_v\big(dx_1 \wedge \cdots \wedge dx_k \wedge dy_1 \wedge \cdots \wedge dy_q\big) \\
&= \sum_{i=1}^{k} (-1)^{i-1} dx_i(v) \big(dx_1 \wedge \cdots \widehat{dx_i} \cdots \wedge dx_k \wedge dy_1 \wedge \cdots \wedge dy_q\big) \\
&\quad + \sum_{j=1}^{q} (-1)^{k+j-1} dy_j(v) \big(dx_1 \wedge \cdots \wedge dx_p \wedge dy_1 \wedge \cdots \widehat{dy_j} \cdots \wedge dy_q\big) \\
&= \big(\iota_v \alpha\big) \wedge \beta + (-1)^k \alpha \wedge \big(\iota_v \beta\big).
\end{aligned}$$

Question 3.25 Suppose we have the general k-form $\alpha = \sum \alpha_{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ and ℓ-from $\beta = \sum \beta_{j_1 \cdots j_\ell} dy_{j_1} \wedge \cdots \wedge dy_{j_\ell}$. Use the linearity of the interior product to show the identity $\iota_v\big(\alpha \wedge \beta\big) = \big(\iota_v \alpha\big) \wedge \beta + (-1)^k \alpha \wedge \big(\iota_v \beta\big)$ applies to general forms.

Next we prove

$$\boxed{\big(\iota_u \iota_v + \iota_v \iota_u\big)\alpha = 0,}$$

which is often simply written as $\iota_v \iota_w + \iota_w \iota_v = 0$. Again we begin by supposing $\alpha = dx_1 \wedge \cdots \wedge dx_k$. We start by finding $\iota_u \iota_v \alpha$,

$$\begin{aligned}
\iota_u \iota_v \alpha &= \iota_u \iota_v \big(dx_1 \wedge \cdots \wedge dx_k\big) \\
&= \sum_{i=1}^{k} (-1)^{i-1} dx_i(v) \iota_u \big(dx_1 \wedge \cdots \widehat{dx_i} \cdots \wedge dx_k\big) \\
&= \sum_{i=1}^{k-1} (-1)^{i-1} dx_i(v) \sum_{j=i+1}^{k} dx_j(u) \big(dx_1 \wedge \cdots \widehat{dx_i} \cdots \widehat{dx_j} \cdots \wedge dx_k\big) \\
&\quad + \sum_{i=2}^{k} (-1)^{i-1} dx_i(v) \sum_{j=1}^{i-1} dx_j(u) \big(dx_1 \wedge \cdots \widehat{dx_j} \cdots \widehat{dx_i} \cdots \wedge dx_k\big).
\end{aligned}$$

Since i and j are just dummy indices we can switch them in the second double summation and we get

$$\iota_u \iota_v \alpha = \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} (-1)^{i+j-3} dx_i(v) dx_j(u) \big(dx_1 \wedge \cdots \widehat{dx_i} \cdots \widehat{dx_j} \cdots \wedge dx_k \big)$$
$$+ \sum_{j=2}^{k} \sum_{i=1}^{j-1} (-1)^{i+j-2} dx_i(u) dx_j(v) \big(dx_1 \wedge \cdots \widehat{dx_i} \cdots \widehat{dx_j} \cdots \wedge dx_k \big).$$

To get the second term $\iota_v \iota_u \alpha$ we simply interchange the u and v.

Question 3.26 Using $\iota_u \iota_v \alpha$ and $\iota_v \iota_u \alpha$ found above, show that $(\iota_u \iota_v + \iota_v \iota_u)\alpha = 0$.

Question 3.27 Using the above identity, show that $\iota_v^2 = 0$.

3.5 Summary, References, and Problems

3.5.1 Summary

A way to "multiply" together one-forms in a way that gives a very precise geometric meaning is introduced. This "multiplication" product is called a wedgeproduct. In words, the wedgeproduct of two one-forms $dx_i \wedge dx_j$ eats two vectors v_p and w_p and finds the two-dimensional volume of the parallelepiped spanned by the projections of those two vectors onto the $\partial_{x_i} \partial_{x_j}$-plane in $T_p M$. First the projections of v_p and w_p onto the $\partial_{x_i} \partial_{x_j}$-plane are found to be

$$\begin{bmatrix} dx_i(v_p) \\ dx_j(v_p) \end{bmatrix} = \begin{bmatrix} v_i \\ v_j \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} dx_i(w_p) \\ dx_j(w_p) \end{bmatrix} = \begin{bmatrix} w_i \\ w_j \end{bmatrix}.$$

Recalling that the determinate of a matrix finds the volume of the parallelepiped spanned by the columns of a matrix we take the determinant of the 2×2 matrix with these vectors as its columns,

$$dx_i \wedge dx_j(v_p, w_p) = \begin{vmatrix} dx_i(v_p) & dx_i(w_p) \\ dx_j(v_p) & dx_j(w_p) \end{vmatrix}.$$

A two-form is defined to be a linear combination of elements of the form $dx_i \wedge dx_j$ and the space of two-forms on M at the point p is denoted by $\bigwedge_p^2(\mathbb{R}^2)$. The geometric meaning and formula for the wedgeproduct of n one-forms is defined analogously,

$$\boxed{\text{Wedgeproduct of } n \text{ one-forms} \quad dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_n}\big(v_1, v_2, \ldots v_n\big) \equiv \begin{vmatrix} dx_{i_1}(v_1) & dx_{i_1}(v_2) & \cdots & dx_{i_1}(v_n) \\ dx_{i_2}(v_1) & dx_{i_2}(v_2) & \cdots & dx_{i_2}(v_n) \\ \vdots & \vdots & \ddots & \vdots \\ dx_{i_n}(v_1) & dx_{i_n}(v_2) & \cdots & dx_{i_n}(v_n) \end{vmatrix}.}$$

An n-form is defined to be a linear combination of elements of the form $dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_n}$ and the space of n-forms on M at point p is denoted by $\bigwedge_p^n(M)$. Using the definition of the determinant the wedgeproduct of n one-forms can also be written as

$$\boxed{\text{Wedgeproduct of } n \text{ one-forms} \quad dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_n}\big(v_1, v_2, \ldots v_n\big) = \textstyle\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} dx_{\sigma(i_j)}(v_j).}$$

A number of different formulas for the wedgeproduct of an arbitrary k-form and an ℓ-form were derived:

Wedgeproduct of two arbitrary forms, Formula One	$\left(\sum_I a_I dx^I\right) \wedge \left(\sum_J b_J dx^J\right) = \sum a_I b_J dx^I \wedge dx^J,$

Wedgeproduct of two arbitrary forms, Formula Two	$\left(\sum_I a_I dx^I\right) \wedge \left(\sum_J b_J dx^J\right) = \sum_K \left(\sum_{\substack{I \cup J \\ I,J \text{ disjoint}}} \pm a_I b_J \right) dx^K,$

Wedgeproduct of two arbitrary forms, Formula Three	$(\alpha \wedge \beta)\big(v_1, \ldots, v_{k+\ell}\big) = \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}(\sigma) \alpha\big(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\big) \beta\big(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}\big),$

Wedgeproduct of two arbitrary forms, Formula Four	$(\alpha \wedge \beta)\big(v_1, \ldots, v_{k+\ell}\big) = \sum_{\substack{\sigma \text{ is a} \\ (k+\ell)- \\ \text{shuffle}}} \operatorname{sgn}(\sigma) \alpha\big(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\big) \beta\big(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}\big),$

Wedgeproduct of two arbitrary forms, Formula Five	$\alpha \wedge \beta = \frac{(k+\ell)!}{k!\ell!} \mathscr{A}(\alpha \otimes \beta).$

Properties of the wedgeproduct were also derived. If $\omega, \omega_1, \omega_2$ are k-forms and η, η_1, η_2 are ℓ-forms then:

(1) $a\omega \wedge \eta = \omega \wedge a\eta = a\big(\omega \wedge \eta\big)$ where $a \in \mathbb{R}$,
(2) $\big(\omega_1 + \omega_2\big) \wedge \eta = \omega_1 \wedge \eta + \omega_2 \wedge \eta$,
(3) $\omega \wedge \big(\eta_1 + \eta_2\big) = \omega \wedge \eta_1 + \omega \wedge \eta_2$,
(4) $\omega \wedge \eta = (-1)^{k\ell} \eta \wedge \omega$.

The interior product of a k-form α and a vector v was defined to be

$$\boxed{\text{Interior Product:} \quad \iota_v \alpha(v_1, \ldots, v_{k-1}) = \alpha(v, v_1, \ldots, v_{k-1}).}$$

And a number of interior product identities were found,

$$\boxed{\iota_v\big(\alpha + \beta\big) = \iota_v \alpha + \iota_v \beta,}$$

$$\boxed{\iota_{(v+w)} \alpha = \iota_v \alpha + \iota_w \alpha,}$$

$$\boxed{\iota_v\big(dx_1 \wedge \cdots \wedge dx_k\big) = \sum_{i=1}^{k} (-1)^{i-1} dx_i(v) \big(dx_1 \wedge \cdots \widehat{dx_i} \cdots \wedge dx_k\big),}$$

$$\boxed{\iota_v\big(\alpha \wedge \beta\big) = \big(\iota_v \alpha\big) \wedge \beta + (-1)^k \alpha \wedge \big(\iota_v \beta\big),}$$

$$\boxed{\big(\iota_u \iota_v + \iota_v \iota_u\big) \alpha = 0.}$$

3.5.2 References and Further Reading

The first two sections of this chapter are a slow and careful introduction to the wedgeproduct from a geometric point of view. As much as possible we have attempted to focus on the meanings behind the fairly complicated formula that defines the wedgeproduct of two differential forms. The general approach taken in the first few sections of this chapter generally follows Bachman [4], though Edwards [18] also uses this approach as well. But very often the opposite approaches taken, the geometric meaning of the wedgeproduct is deduced, or at least alluded to, based on the formula; see for example Darling [12], Arnold [3], and even Spivak [41]. In Spivak what we have referred to as the "scaling factors" for general k-forms are called "signed scalar densities," which is another good way of thinking about the scaling factors.

In section three we have attempted to clearly connect the geometrically motivated formula for the wedgeproduct with the various other formulas that are often used to define the wedgeproduct. In reality, most books take a much more formal approach to defining the wedgeproduct than we do. See for example Tu [46], Munkres [35], Martin [33], or just about any other book on manifold theory or differential geometry. In a sense this is understandable, in these books differential forms are simply one of many topics that need to be covered, and so a much briefer approach is needed. It is our hope that this section will provide the link to help readers of those books gain a deeper understanding of what the wedgeproduct actually is.

3.5.3 Problems

Question 3.28 Which of the following make sense? If the expression makes sense then evaluate it.

$$a)\ dx_1 \wedge dx_2\left(\begin{bmatrix}3\\-2\\1\end{bmatrix},\begin{bmatrix}5\\3\\-4\end{bmatrix}\right) \qquad d)\ dx_1 \wedge dx_2 \wedge dx_4\left(\begin{bmatrix}6\\1\\-3\end{bmatrix},\begin{bmatrix}5\\-2\\-1\end{bmatrix},\begin{bmatrix}5\\-2\\-1\end{bmatrix}\right)$$

$$b)\ dx_1 \wedge dx_3\left(\begin{bmatrix}-4\\-2\end{bmatrix},\begin{bmatrix}-3\\4\end{bmatrix}\right) \qquad e)\ dx_2 \wedge dx_3 \wedge dx_4\left(\begin{bmatrix}-1\\4\\5\\-2\end{bmatrix},\begin{bmatrix}3\\-1\\1\\-3\end{bmatrix},\begin{bmatrix}-2\\3\\4\\1\end{bmatrix}\right)$$

$$c)\ dx_1 \wedge dx_4\left(\begin{bmatrix}5\\-2\\3\\-1\end{bmatrix},\begin{bmatrix}1\\2\\-3\\-4\end{bmatrix}\right) \qquad f)\ (3\,dx_2 \wedge dx_4 - 2\,dx_1 \wedge dx_3)\left(\begin{bmatrix}-1\\4\\5\\-2\end{bmatrix},\begin{bmatrix}3\\-1\\1\\-3\end{bmatrix}\right)$$

Question 3.29 Compute the following numbers. Then explain in words what the number represents.

$$a)\ dx_1 \wedge dx_2\left(\begin{bmatrix}2\\3\\-5\\2\end{bmatrix},\begin{bmatrix}2\\3\\4\\-5\end{bmatrix}\right) \qquad d)\ dx_2 \wedge dx_3\left(\begin{bmatrix}2\\3\\-5\\2\end{bmatrix},\begin{bmatrix}2\\3\\4\\-5\end{bmatrix}\right)$$

$$b)\ dx_1 \wedge dx_3\left(\begin{bmatrix}2\\3\\-5\\2\end{bmatrix},\begin{bmatrix}2\\3\\4\\-5\end{bmatrix}\right) \qquad e)\ dx_2 \wedge dx_4\left(\begin{bmatrix}2\\3\\-5\\2\end{bmatrix},\begin{bmatrix}2\\3\\4\\-5\end{bmatrix}\right)$$

$$c)\ dx_1 \wedge dx_4\left(\begin{bmatrix}2\\3\\-5\\2\end{bmatrix},\begin{bmatrix}2\\3\\4\\-5\end{bmatrix}\right) \qquad f)\ dx_3 \wedge dx_4\left(\begin{bmatrix}2\\3\\-5\\2\end{bmatrix},\begin{bmatrix}2\\3\\4\\-5\end{bmatrix}\right)$$

Question 3.30 Using the results you obtained in problem 3.29 compute the following numbers. Then explain in words what the number represents.

$$a)\ 3\,dx_2 \wedge dx_3 \left(\begin{bmatrix} 2 \\ 3 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ -5 \end{bmatrix} \right) \qquad d)\ \left(-2\,dx_2 \wedge dx_3 + \frac{1}{4}\,dx_3 \wedge dx_4 \right) \left(\begin{bmatrix} 2 \\ 3 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ -5 \end{bmatrix} \right)$$

$$b)\ -2\,dx_1 \wedge dx_4 \left(\begin{bmatrix} 2 \\ 3 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ -5 \end{bmatrix} \right) \qquad e)\ (4\,dx_1 \wedge dx_2 - 3\,dx_3 \wedge dx_4) \left(\begin{bmatrix} 2 \\ 3 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ -5 \end{bmatrix} \right)$$

$$c)\ \frac{1}{2}\,dx_2 \wedge dx_4 \left(\begin{bmatrix} 2 \\ 3 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ -5 \end{bmatrix} \right) \qquad f)\ \left(\frac{4}{3}\,dx_1 \wedge dx_4 + 3\,dx_2 \wedge dx_4 \right) \left(\begin{bmatrix} 2 \\ 3 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ -5 \end{bmatrix} \right)$$

Question 3.31 Compute the following functions for the given forms on manifold $\mathbb{R}^n$ and constant vector fields. Then explain in words what the function represents.

$$a)\ xyz\,dz \wedge dx \left(\begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix} \right) \qquad d)\ \left(x\,dx \wedge dy + \sin(z)\,dy \wedge dz \right) \left(\begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \right)$$

$$b)\ y^z \sin(x)\,dy \wedge dz \left(\begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix} \right) \qquad e)\ \left(e^{xy}\,dx_1 \wedge dx_3 + \sqrt{|z|}\,dx_1 \wedge dx_4 \right) \left(\begin{bmatrix} 3 \\ 2 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} -4 \\ -3 \\ 2 \\ -1 \end{bmatrix} \right)$$

$$c)\ \left(x_1 + x_2^{x_4} \right) dx_2 \wedge dx_3 \left(\begin{bmatrix} -4 \\ 3 \\ -2 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 4 \\ -5 \end{bmatrix} \right) \qquad f)\ e_1^x \sqrt{|x_3|}\,dx_2 \wedge dx_3 \wedge dx_4 \left(\begin{bmatrix} 3 \\ 4 \\ -4 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \\ 4 \end{bmatrix} \right)$$

Question 3.32 Compute the following functions for the given forms on manifold $\mathbb{R}^n$ and the non-constant vector fields. Then explain in words what the function represents.

$$a)\ xyz\,dz \wedge dx \left(\begin{bmatrix} 3x \\ -y \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ z \\ x \end{bmatrix} \right) \qquad d)\ \left(x\,dx \wedge dy + \sin(z)\,dy \wedge dz \right) \left(\begin{bmatrix} x^y \\ -z \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ x^2 \\ z^3 \end{bmatrix} \right)$$

$$b)\ y^z \sin(x)\,dy \wedge dz \left(\begin{bmatrix} 2x \\ -y^z \\ z \end{bmatrix}, \begin{bmatrix} 3 \\ -x \\ -2y \end{bmatrix} \right) \qquad e)\ \left(e^{xy}\,dx_1 \wedge dx_3 + \sqrt{|z|}\,dx_1 \wedge dx_4 \right) \left(\begin{bmatrix} x_4 \\ x_2 + 3 \\ -2 \\ x_3 \end{bmatrix}, \begin{bmatrix} -x_2 \\ -3 \\ x_3^{x_1} \\ -1 \end{bmatrix} \right)$$

$$c)\ \left(x_1 + x_2^{x_4} \right) dx_2 \wedge dx_3 \left(\begin{bmatrix} -2x_2 \\ x_3 \\ x_4 - 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ x_2^4 \\ -x_4 \\ -5 \end{bmatrix} \right) \qquad f)\ e_1^x \sqrt{|x_3|}\,dx_2 \wedge dx_3 \wedge dx_4 \left(\begin{bmatrix} 1 \\ 2^{x_1} \\ 3 \\ -2x_3 \end{bmatrix}, \begin{bmatrix} x_3 \\ 2 \\ e^{x_4} \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ x_3 x_3 \\ x_3 x_4 \\ x_4 x_1 \end{bmatrix} \right)$$

Question 3.33 Given $\alpha = \cos(xz)\, dx \wedge dy$ a two-form on $\mathbb{R}^3$ and the constant vectors fields

$$v = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \text{and} \qquad w = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

on $\mathbb{R}^3$ find $\alpha(v, w)$. Then find $\alpha_p(w_p, v_p)$ for both points $(1, 2, \pi)$ and $\left(\frac{1}{2}, 2, \pi\right)$.

Question 3.34 Given $\beta = \sin(xz)\, dy \wedge dz$ a two-form on $\mathbb{R}^3$ and the constant vector fields v and w on $\mathbb{R}^3$ from problem 3.33, find $\beta(v, w)$. Then find $\beta_p(w_p, v_p)$ for both points $(1, 2, \pi)$ and $\left(\frac{1}{2}, 2, \pi\right)$.

Question 3.35 Given $\gamma = xyz\, dz \wedge dx$ a two-form on $\mathbb{R}^3$ and the constant vector fields v and w from problem 3.33, find $\gamma(v, w)$. Then find $\gamma_p(w_p, v_p)$ for both points $(1, 2, \pi)$ and $\left(\frac{1}{2}, 2, \pi\right)$.

Question 3.36 Given $\phi = \cos(xz)\, dx \wedge dy + \sin(xz)\, dy \wedge dz + xyz\, dz \wedge dx$ a two-form on $\mathbb{R}^3$ and the constant vector fields v and w from problem 3.33, find $\phi(v, w)$. Then find $\phi_p(w_p, v_p)$ for both points $(1, 2, \pi)$ and $\left(\frac{1}{2}, 2, \pi\right)$.

Question 3.37 Find the basis of $\bigwedge_p^2(\mathbb{R}^5)$. That is, list the elementary two-forms on $\mathbb{R}^5$. Then find the basis of $\bigwedge_p^3(\mathbb{R}^5)$, $\bigwedge_p^4(\mathbb{R}^5)$, and $\bigwedge_p^5(\mathbb{R}^5)$.

Question 3.38 Find the basis of $\bigwedge_p^1(\mathbb{R}^6)$, $\bigwedge_p^2(\mathbb{R}^6)$, $\bigwedge_p^3(\mathbb{R}^6)$, $\bigwedge_p^4(\mathbb{R}^6)$, $\bigwedge_p^5(\mathbb{R}^6)$, and $\bigwedge_p^6(\mathbb{R}^6)$.

Question 3.39 Using the algebraic properties of differential forms simplify the following expressions. Put indices in increasing order.

a) $\left(3\, dx + 2\, dy\right) \wedge dz$ *d)* $dz \wedge \left(4\, dx + 3\, dy\right)$ *g)* $(dx_1 \wedge dx_3) \wedge \left(3\, dx_2 - 4\, dx_4\right)$

b) $\left(z\, dx - x\, dy\right) \wedge dz$ *e)* $dx \wedge \left(6\, dy - z\, dz\right)$ *h)* $\left(x_3\, dx_2 \wedge dx_5 + x_1\, dx_4 \wedge dx_6\right) \wedge (-5\, dx_1 \wedge dx_3)$

c) $\left(x^y\, dz + 4z\, dx\right) \wedge (4\, dy)$ *f)* $-dy \wedge \left(e^z\, dz - x\, dx\right)$ *i)* $(2\, dx_3 \wedge dx_4) \wedge \left(x_4\, dx_1 \wedge dx_2 + e^{x_6}\, dx_1 \wedge dx_6\right)$

Question 3.40 Using the algebraic properties of differential forms simplify the following expressions. Put indices in increasing order.

a) $\left(3\, dx_1 \wedge dx_3 + 2\, dx_2 \wedge dx_4\right) \wedge \left(-dx_1 \wedge dx_2 + 3\, dx_2 \wedge dx_4 - 6\, dx_2 \wedge dx_3\right)$

b) $\left(-4\, dx_2 \wedge dx_4 \wedge dx_5\right) \wedge \left(3\, dx_1 \wedge dx_3 \wedge dx_7 + 5\, dx_1 \wedge dx_6 \wedge dx_7\right)$

c) $\left(x_2 x_5\, dx_3 \wedge dx_4 \wedge dx_8 + \sin(x_3)\, dx_3 \wedge dx_6 \wedge dx_8\right) \wedge \left(e^{x_7}\, dx_2 \wedge dx_4 \wedge dx_7\right)$

d) $\left(\sin(x_2)\, dx_2 \wedge dx_3 - e^{x_3} dx_5 \wedge dx_7\right) \wedge \left(x_2^{x_4}\, dx_1 \wedge dx_4 \wedge dx_6 + (x_3 + x_4)\, dx_4 \wedge dx_8 \wedge dx_9\right)$

Question 3.41 Let $v_1 = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -5 \\ 6 \\ -4 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 7 \\ -5 \\ -2 \end{bmatrix}$. Find

a) $\iota_{v_1}\left(3\, dx \wedge dy\right)$ *d)* $\iota_{v_1}\left(-6\, dx \wedge dy + 2\, dx \wedge dz\right)$ *g)* $\iota_{v_1}\left(x^y\, dx \wedge dz\right)$

b) $\iota_{v_2}\left(5\, dx \wedge dz\right)$ *e)* $\iota_{v_2}\left(3\, dx \wedge dz - 4\, dy \wedge dz\right)$ *h)* $\iota_{v_2}\left(y \sin(x) e^z\, dy \wedge dz\right)$

c) $\iota_{v_3}\left(-4\, dy \wedge dz\right)$ *f)* $\iota_{v_3}\left(2\, dx \wedge dy + 7\, dy \wedge dz\right)$ *i)* $\iota_{v_3}\left((x + y + z)\, dx \wedge dy\right)$

Question 3.42 Let $v_1 = e_1 - 2e_2 - 5e_3 + 5e_4 - 3e_5 + 6e_6$ and $v_2 = 4e_1 - 7e_2 - 3e_3 + 2e_4 + e_5 + 7e_6$. Find

$$a)\ \iota_{v_1}\Big(dx_1 \wedge dx_3 \wedge dx_5 \wedge dx_6\Big) \qquad c)\ \iota_{v_1}\Big(dx_1 \wedge dx_2 \wedge dx_4 \wedge dx_5 \wedge dx_6\Big)$$

$$b)\ \iota_{v_2}\Big(dx_2 \wedge dx_4 \wedge dx_5 \wedge dx_6\Big) \qquad d)\ \iota_{v_2}\Big(dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_6\Big)$$

Chapter 4
Exterior Differentiation

One of the central concepts of calculus, and in fact of all mathematics, is that of differentiation. One way to view differentiation is to view it as a mathematical object that measures how another mathematical object varies. For example, in one dimensional calculus the derivative of a function $f : \mathbb{R} \to \mathbb{R}$ is another function. In vector calculus you learned that the derivative of a transformation is the Jacobian matrix. You learned that both the divergence and curl are in some sense "derivatives" that measure how vector fields are varying. Now we want to consider the derivatives of forms.

From these examples you can see that the idea of differentiation can actually get rather complicated. Often the more complicated or abstract a space or object is, the more ways there are for the object to vary. This actually leads to different types of differentiations, each of which is useful in different circumstances. In this section we will introduce the most common type of differentiation for differential forms, the exterior derivative. There is also another way to define the derivative of a differential form, the Lie derivative of a form, which is introduced in Appendix A. Exterior differentiation is an extremely powerful concept, as we will come to see. Section one provides an overview of the four different approaches to exterior differentiation that books usually take. Since our goal in this chapter is to completely understand exterior differentiation each of these four approaches are then explored in detail in the following four sections. The chapter then concludes with a section devoted to some examples.

4.1 An Overview of the Exterior Derivative

In essence the basic objects that were studied in calculus were functions. You learned to take derivatives of functions and to integrate functions. In calculus on manifolds the basic objects are differential forms. We want to be able to take derivatives of differential forms and to integrate differential forms.

Derivatives of real-valued functions $f : \mathbb{R} \to \mathbb{R}$ and directional derivatives of multivariable real-valued functions $f : \mathbb{R}^n \to \mathbb{R}$ were both defined in terms of the limit of a difference quotient. This gave a very clean and precise geometrical meaning to the concept of the derivative of a function. Unfortunately, with the exterior derivative of differential form we do not have such a clean, precise, and easily understood geometrical meaning. Therefore, it is difficult to see right away why the definition of the exterior derivative of a differential form is the "right" definition to use.

In a sense, with exterior differentiation, we rely on the idea in the idiom "the proof is in the pudding." Basically, the end results of the definition of exterior derivative are just so amazingly "nice" that the definition has to be "right." What we mean is that by defining the exterior derivative of a differential form the way it is defined, we are able to perfectly generalize major ideas from vector calculus. For example, it turns out that the concepts of gradient, curl, and divergence from vector calculus all become special cases of exterior differentiation and the fundamental theorem of line integrals, Stokes theorem, and the divergence theorem all become special cases of what is called the generalized Stokes theorem. It allows us to summarize Maxwell's equations in electromagnetism into a single line, it gives us the Poncaré lemma, which is a powerful tool for the study of manifolds, and gives us the canonical symplectic form, which allows us to provide an abstract theoretical formulation of classical mechanics. We will study all of these consequences of the exterior derivative through the rest of this book. The amount of mathematics that works out perfectly because of this definition tells us that the definition has to be "right" even if it is difficult to see, geometrically, what the definition means.

In case all of this makes you uncomfortable, and feels a little circular to you, that is okay. In a sense it is. But you would be surprised by how often this sort of thing happens in mathematics. The definitions that prove the most useful are the ones

© Springer Nature Switzerland AG 2018

J. P. Fortney, *A Visual Introduction to Differential Forms and Calculus on Manifolds*,
https://doi.org/10.1007/978-3-319-96992-3_4

that survive to be taught in courses. Other, less useful, definitions eventually fall by the wayside. With all of this in mind we are now finally ready to take a look at exterior differentiation.

Most books introduce exterior differentiation in one of four different ways, and which approach a book's author takes depends largely on the flavor of the book.

1. The local ("in coordinates") formula is given. For example, if we have the one-form $\alpha = \sum f_i dx_i$ then the formula for the exterior derivative of α is simply given as
$$d\alpha = \sum df_i \wedge dx_i.$$
Notice that the coordinates that α is written in, here x_i, show up in the above formula. This is probably the most common approach taken in most introductory math and physics textbooks. From the formula the various properties are then derived. Frequently in these books the global formula is not given. The differential of a function df was defined in Definition 2.4.2 in Chap. 2 to be $df(v_p) = v_p[f] = \lim_{t\to 0} \frac{f(p+tv_p)-f(p)}{f}$ and thus df is related to our good old-fashioned directional derivative of f. Thus we can think of the exterior derivative as being, in some sense, a generalization of directional derivatives.
2. A list of the algebraic properties (or axioms) that the exterior derivative should have is given. These properties are then used to show that the exterior derivative is unique and to derive the (local and/or global) formula. Books that use this approach tend to be quite formal and axiomatic in nature, quite the antithesis of this book.
3. The global (also called invariant) formula is given. Most books that take this approach are quite advanced and most readers already have some knowledge of exterior differentiation. At this point in your education you have probably only ever seen the local ("in coordinates") form of any formula, so this concept is probably unfamiliar to you. The general idea is that you want to provide a formula but you do not want your formula to depend on which coordinates (Cartesian, polar, spherical, cylindrical, etc.) you are using at the time. In other words, you want your formula to be independent of the coordinates, or invariant. Without yet getting into what the formula means, if a one-form is written in-coordinates as $\alpha = \sum f_i dx_i$ (the x_i are the coordinates that are being used) then the in-coordinates formula for the exterior derivative of the one-form α is given by
$$d\alpha = \sum df_i \wedge dx_i.$$
The global formula for $d\alpha$, which is a two-form that acts on vectors v and w, is given by
$$d\alpha(v, w) = v\big[\alpha(w)\big] - w\big[\alpha(v)\big] - \alpha([v, w]).$$
Recall that $v[f]$ is one notation for the directional derivative of f in the direction of v. We also have that $[v, w]$ is the lie-bracket of two vector fields, which is defined by $[v, w] = vw - wv$. This will be explained later. Thus, the square brackets in the first two terms mean something different from the square brackets in the last term.
Notice that this formula is written entirely in terms of the one-form α and the vectors v and w. Nowhere in this formula do the particular coordinates that you are using (the x_i from the previous formula) show up. This type of formula is sometimes called a coordinate-free formula, a global formula, or an invariant formula. If α is an arbitrary k-form and $v_0, \ldots, v_k$ are $k+1$ vector fields then the global formula is given by
$$d\alpha(v_0, \ldots, v_k) = \sum_i (-1)^i v_i\big[\alpha(v_0, \ldots, \widehat{v_i}, \ldots, v_k)\big] + \sum_{i<j} (-1)^{i+j}\alpha\big([v_i, v_j], v_0, \ldots, \widehat{v_i}, \ldots, \widehat{v_j}, \ldots, v_k\big).$$
A hat over a vector means that the vector is omitted. Again, notice this formula is written entirely in terms of the k-form α and the vectors fields. It is also quite a complicated looking formula, not nearly as neat as the in-coordinates formula turns out to be.
4. A geometric definition in terms of the limit of the integral of the form over the boundary of a parallelepiped is given. This is a fairly infrequent approach that tends to show up in engineering or applied physics texts that want to emphasis the physical meaning of the exterior derivative. While this approach is really the most geometric of the approaches it either requires the book be structured so that integration is covered before differentiation or is necessarily somewhat imprecise.