$$du \wedge dv \underbrace{\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{(0,0)}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{(0,0)} \right)}_{\text{vectors in } uv\text{-plane}} = 1$$

so $dx \wedge dy$ is the area form of the xy-plane and $du \wedge dv$ is the area form of the uv-plane.

First we will write $du \wedge dv$ in terms of $dx \wedge dy$. Part of this has already been done. Since $u = x + y$ and $v = x - y$ we have

$$\begin{aligned} du &= d(x + y) = dx + dy, \\ dv &= d(x - y) = dx - dy \end{aligned}$$

so we have

$$\begin{aligned} du \wedge dv &= (dx + dy) \wedge (dx - dy) \\ &= dx \wedge (dx - dy) + dy \wedge (dx - dy) \\ &= dx \wedge dx - dx \wedge dy + dy \wedge dx - dy \wedge dy \\ &= -dx \wedge dy - dx \wedge dy \\ &= -2dx \wedge dy. \end{aligned}$$

Next we will write $dx \wedge dy$ in terms of $du \wedge dv$. In the above question you showed that solving for x and y in terms of u and v gives us

$$\begin{aligned} x &= \frac{1}{2}(u + v), \\ y &= \frac{1}{2}(u - v) \end{aligned}$$

which in turn gives us

$$\begin{aligned} dx &= d\left(\frac{1}{2}(u + v)\right) = \frac{1}{2}du + \frac{1}{2}dv, \\ dy &= d\left(\frac{1}{2}(u - v)\right) = \frac{1}{2}du - \frac{1}{2}dv \end{aligned}$$

so we have

$$\begin{aligned} dx \wedge dy &= \left(\frac{1}{2}du + \frac{1}{2}dv\right) \wedge \left(\frac{1}{2}du - \frac{1}{2}dv\right) \\ &= \frac{1}{4}dv \wedge du - \frac{1}{4}du \wedge dv \\ &= -\frac{1}{2}du \wedge dv. \end{aligned}$$

In summary, what we have just done is compute the following relations between the xy and the uv area forms

$$\begin{aligned} du \wedge dv &= -2dx \wedge dy, \\ dx \wedge dy &= -\frac{1}{2}du \wedge dv. \end{aligned}$$

Compare what is happening in Figs. 6.2 and 6.3 with the relationships between the area forms. How do they relate to each other? First, we try to understand what the volume form relations are telling us. First consider Fig. 6.2 where the unit square in the xy-plane was mapped to the diamond in the uv-plane. The volume of the diamond in the uv-plane is -2 times the volume of the unit square in the xy-plane. We can see where the 2 comes from by noticing that the unit square in the xy-plane is mapped to the diamond in the uv-plane with twice the area. And where does the negative sign come from? From noticing that the counter-clockwise rotation in the xy-plane becomes a clockwise rotation in the uv-plane. This is our signed volume. So, what we are showing must have some relation to the identity

$$du \wedge dv = -2dx \wedge dy.$$

Next consider Fig. 6.3 where the unit square in the uv-plane was mapped to the diamond in the xy-plane. The area of the diamond in the xy-plane is $-\frac{1}{2}$ times the volume of the unit square in the uv-plane. This means that the unit square with (signed) volume 1 in the uv-plane is mapped to the diamond with (signed) volume $-\frac{1}{2}$ in the xy-plane, where the negative sign again indicates that a counter-clockwise rotation around the square in the uv-plane becomes a clockwise rotation around diamond in the xy-plane. Again, this clearly has some relation to the identity

$$dx \wedge dy = -\frac{1}{2}du \wedge dv.$$

Recall that when we were deriving the formula for the determinant in Sect. 1.2 we started out by specifying three fundamental and intuitive properties that we thought volumes should have:

(1) A unit cube should have volume one (in any dimension).
(2) Degenerate parallelepipeds (that is, parallelepipeds of less than n dimensions in $\mathbb{R}^n$) should have an n-dimensional volume of zero.
(3) Scaling an edge by a factor c should change the volume by the factor c.

By assuming our volume function D had only these three properties we discovered the fourth property, that by interchanging (switching) two edges of our parallelepiped we also changed the sign of the volume. In other words, fundamental to our intuitive notion of volume is the notion of orientation. In three dimensions orientation is often disguised as the "right-hand rule." See Fig. 6.4.

Over the last few pages we have been a little imprecise, but at least now you may be starting to see how these volume forms in different coordinate systems relate to each other - they somehow encode changes of volume as you move from one coordinate system to another. This is exactly the sort of thing that would be useful when we change variables during integration.

Now we want to look at volume forms a little more closely. The unit square in the xy-plane is given by (spanned by) the xy-plane vectors

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\substack{xy \\ (0,0)}} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\substack{xy \\ (0,0)}}.$$

In other words, the parallelepiped spanned by these two vectors is the unit square. Here we have used the xy in the subscript to indicate the coordinate plane that the vectors are in. The projection of these vectors under the change of coordinates $u = x + y$ and $v = x - y$ is given by

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\substack{xy \\ (0,0)}} \longmapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\substack{uv \\ (0,0)}},$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\substack{xy \\ (0,0)}} \longmapsto \begin{bmatrix} 1 \\ -1 \end{bmatrix}_{\substack{uv \\ (0,0)}}.$$

See Fig. 6.5. It is easy to compute that

$$dx \wedge dy \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\substack{xy \\ (0,0)}}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\substack{xy \\ (0,0)}} \right) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$$

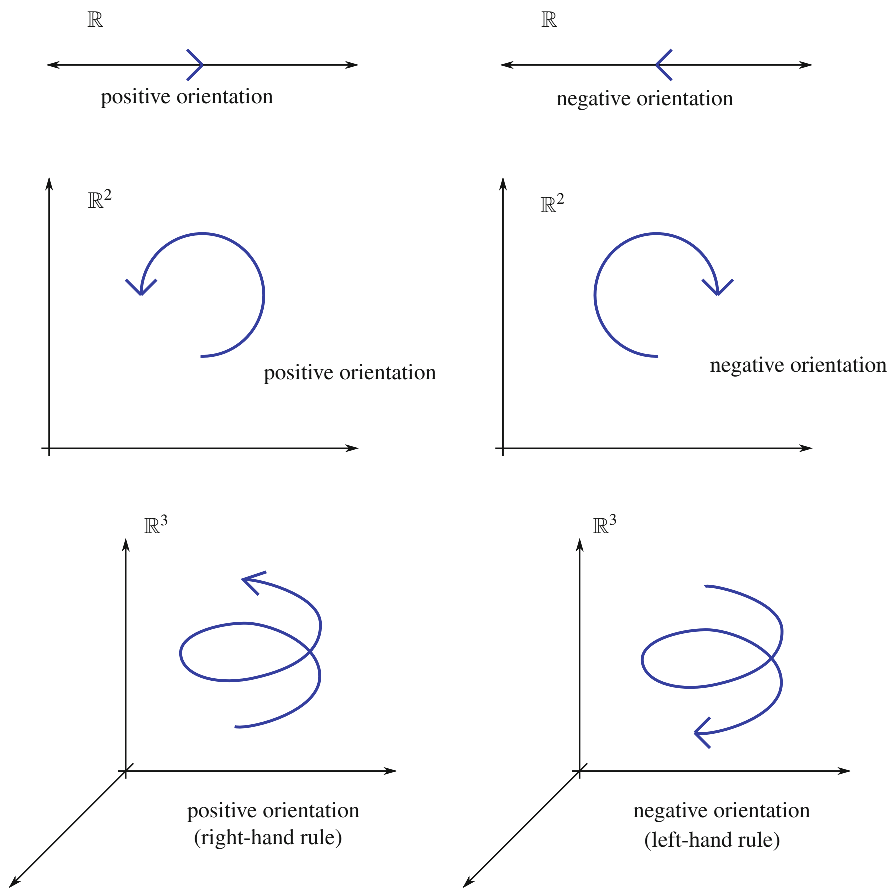

Fig. 6.4 The two orientations for $\mathbb{R}$ (top), for $\mathbb{R}^2$ (middle), and for $\mathbb{R}^3$ (bottom)

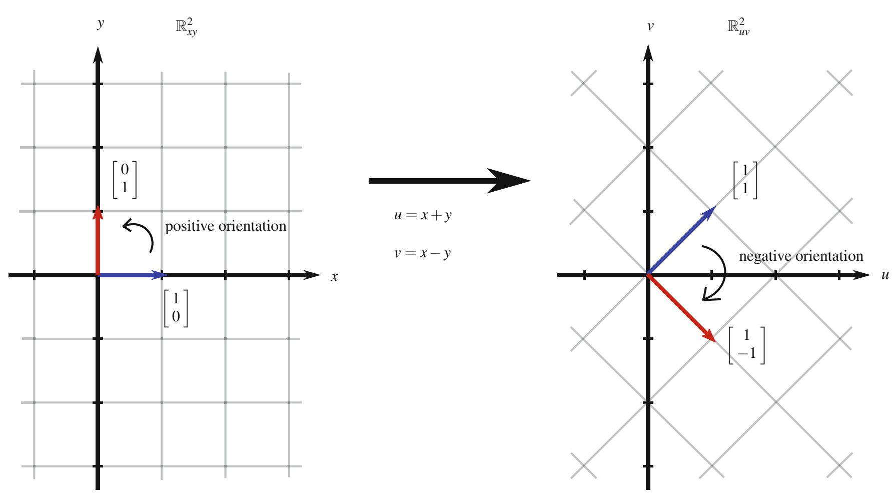

Fig. 6.5 The basis vectors in the xy-plane mapped to two vectors in the uv-plane. Notice the orientation changes

$$du \wedge dv \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\substack{uv \\ (0,0)}}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}_{\substack{uv \\ (0,0)}} \right) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2.$$

Recalling that the area form finds the area of the parallelepiped spanned by the input vectors, this coincides exactly with what we would expect from our picture. We also start to get a clearer picture of what is meant by the identity

$$dx \wedge dy = -\frac{1}{2} du \wedge dv.$$

Actually we have

$$dx \wedge dy \underbrace{\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\substack{xy \\ (0,0)}}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\substack{xy \\ (0,0)}} \right)}_{\text{vectors in } xy\text{-plane}} = -\tfrac{1}{2} du \wedge dv \underbrace{\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\substack{uv \\ (0,0)}}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}_{\substack{uv \\ (0,0)}} \right)}_{\text{projected vectors in } uv\text{-plane}}.$$

The area form $dx \wedge dy$ has to eat vectors from the xy-plane and the area form $du \wedge dv$ has to eat vectors from the uv-plane. Furthermore, the vectors that the area form $du \wedge dv$ is eating are the projections under the u and v mappings ($u = x + y$ and $v = x - y$) of the vectors that the area form $dx \wedge dy$ is eating.

Similarly, the unit square in the uv-plane is spanned by the vectors

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\substack{uv \\ (0,0)}} \text{ and } \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\substack{uv \\ (0,0)}}.$$

Under the change of coordinates given by the inverse of the above, $x = \frac{1}{2}(u+v)$ and $y = \frac{1}{2}(u-v)$, these vector's projections are given by

$$\begin{aligned} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\substack{uv \\ (0,0)}} &\longmapsto \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}_{\substack{xy \\ (0,0)}}, \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\substack{uv \\ (0,0)}} &\longmapsto \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}_{\substack{xy \\ (0,0)}}. \end{aligned}$$

See Fig. 6.6. Again, it is easy to compute

$$\begin{aligned} du \wedge dv \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\substack{uv \\ (0,0)}}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\substack{uv \\ (0,0)}} \right) &= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \\ dx \wedge dy \left(\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}_{\substack{xy \\ (0,0)}}, \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}_{\substack{xy \\ (0,0)}} \right) &= \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2}. \end{aligned}$$

Again this coincides with what we would expect from our picture. And again, we start to get a better idea of how the identity

$$du \wedge dv = -2 dx \wedge dy$$

actually works,

$$du \wedge dv \underbrace{\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\substack{uv \\ (0,0)}}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\substack{uv \\ (0,0)}} \right)}_{\text{vectors in } uv\text{-plane}} = -2 dx \wedge dy \underbrace{\left(\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}_{\substack{xy \\ (0,0)}}, \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}_{\substack{xy \\ (0,0)}} \right)}_{\text{projected vectors in } xy\text{-plane}}.$$

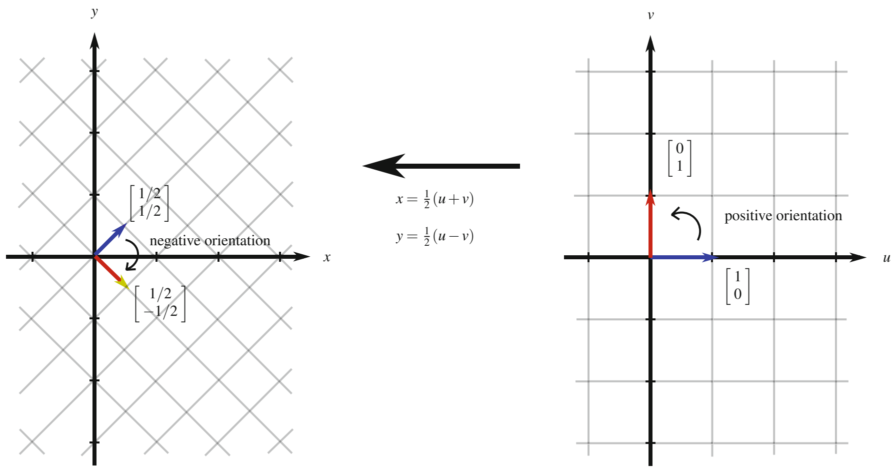

Fig. 6.6 The basis vectors in the uv-plane mapped to two vectors in the xy-plane. Notice the orientation changes

The area form $du \wedge dv$ eats vectors from the uv-plane and the area form $dx \wedge dy$ eats vectors from the xy-plane. Furthermore, the vectors that $dx \wedge dy$ is eating are projections under the x and y mappings ($x = \frac{1}{2}(u+v)$ and $y = \frac{1}{2}(u-v)$) of the vectors that $du \wedge dv$ is eating. So our identities

$$du \wedge dv = -2dx \wedge dy,$$
$$dx \wedge dy = -\frac{1}{2}du \wedge dv$$

hold, but only when there is a specific relationship between the vectors that they are eating, that is, when the vectors they are eating are related by the mappings between the two spaces.

Of course, in this section we have committed the cardinal vector calculus sin and have viewed the various vectors being eaten by the volume forms to be part of the manifold $\mathbb{R}^2$ and not elements of $T_{(0,0)}\mathbb{R}^2$ like they really are. We will correct this in the next section. Our main point in this section was to try to understand the subtleties involved with the identities between the volume forms a little better.

6.2 Push-Forwards of Vectors

In the last section we looked at the (linear change of coordinate) mapping $f(x, y) = \big(u(x, y), v(x, y)\big) : \mathbb{R}^2 \to \mathbb{R}^2$ given by $u(x, y) = x+y$ and $v(x, y) = x-y$. We then used these to find the identities $du\wedge dv = -2dx\wedge dy$ and $dx\wedge dy = -\frac{1}{2}du\wedge dv$ and we started to explore what these identities actually mean. They have to do with how the areas in the domain manifold $\mathbb{R}^2$ with xy-coordinates, which we will denote $\mathbb{R}^2_{xy}$, relate to the areas in the range manifold $\mathbb{R}^2$ with uv-coordinates, which we will denote $\mathbb{R}^2_{uv}$. We found that the identity $du \wedge dv = -2dx \wedge dy$ worked when the vectors that $du \wedge dv$ ate were the projections under the mapping $f(x, y) = \big(u(x, y), v(x, y)\big) = (x + y, x - y)$ of the vectors that $dx \wedge dy$ ate. Similarly, when we were considering the inverse mapping we found the identity $dx \wedge dy = -\frac{1}{2}du \wedge dv$ worked when the vectors that $dx \wedge dy$ ate were projections under the inverse mapping of the vectors that $du \wedge dv$ ate. However, when doing this we were rather imprecise with the vectors that we used to determine the parallelepipeds whose volumes we were finding. As we know vectors really live in tangent spaces, not in the manifold. As we will see, we were able to gloss over that fact here because the mapping was linear.

Not that we want to bore you to death with this example, but in this section we will continue to use this example while being more mathematically precise. This will serve as motivation and an introduction to the idea of **push-forwards** of vectors. Later in the chapter this example will also serve as motivation and an introduction to the idea of **pull-backs** of differential forms. Also, we want to work with a simple example before introducing more complicated examples like polar or spherical or cylindrical coordinates.

Next we recall that both the manifold $\mathbb{R}^2_{xy}$ (with xy-coordinates) and the manifold $\mathbb{R}^2_{uv}$ (with uv-coordinates), see Fig. 6.1, have a tangent space associated to each point similar to what was shown in Fig. 2.15. That is, if (x, y) is a point in $\mathbb{R}^2_{xy}$ then $T_{(x,y)}\mathbb{R}^2_{xy}$ is the tangent space of $\mathbb{R}^2_{xy}$ at the point (x, y). The point $f(x, y) = (x + y, x - y)$ is the projection of the point (x, y) to $\mathbb{R}^2_{uv}$ and $T_{(x+y,x-y)}\mathbb{R}^2_{uv}$ is the tangent space of $\mathbb{R}^2_{uv}$ at the projected point $(x + y, x - y)$.

Also, for the mapping $f(x, y) = \big(u(x, y), v(x, y)\big) = (x + y, x - y)$ we have the derivative of f at the point (x, y) given by the Jacobian matrix Df,

$$D_{(x,y)}f = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}_{(x,y)} .$$

Let us think about what the Jacobian matrix is and does. It is the closest linear approximation to the function $f(x, y)$ at the point (x, y). Suppose we have some vector $v = \begin{bmatrix} a \\ b \end{bmatrix}$ based at the point (x, y). This vector is really in the tangent space of $\mathbb{R}^2_{xy}$ at the point (x, y) but for now we still think of it as being in the manifold $\mathbb{R}^2_{xy}$. If we wanted to know how much f changed as we moved from (x, y) along the vector v we could find $f(x, y)$ and $f(x+a, y+b)$ to see the change. In fact, we could think of the change as being a vector from $f(x, y)$ to $f(x + a, y + b)$. Here both the points (x, y) and $(x + a, y + b)$ are in the manifold $\mathbb{R}^2_{xy}$ and the points $f(x, y)$ and $f(x + a, y + b)$ are in the manifold $\mathbb{R}^2_{uv}$. Or we could estimate the change using the closest linear approximation to f at the point (x, y), which is the derivative of f at the point (x, y), and which is given by the Jacobian matrix Df at the point (x, y). To do this we multiply the Jacobian matrix by the vector v, or find $Df \cdot v$. This gives us another vector. But where is this vector at? Since we are estimating a change in f from the point $f(x, y)$, this point is the base point for our new vector, so our new vector is in the tangent space of $\mathbb{R}^2_{uv}$ at the point $f(x, y)$. In summary, the Jacobian matrix is actually a mapping

$$D_{(x,y)}f : T_{(x,y)}\mathbb{R}^2_{xy} \longrightarrow T_{f(x,y)}\mathbb{R}^2_{uv}.$$

For reasons that we will explain a little bit later, we will generally use the notation $T_{(x,y)}f$ to denote the Jacobian matrix at the point (x, y) instead of the more common $D_{(x,y)}f$. For our particular mapping $f(x, y) = \big(u(x, y), v(x, y)\big) = (x+y, x-y)$ we have

$$T_{(x,y)}f = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}_{(x,y)} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}_{(x,y)} .$$

For example, at the point $(0, 0)$ the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{(0,0)}$ is sent to

$$T_{(0,0)}f \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{(0,0)} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}_{(0,0)} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{(0,0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{f(0,0)=(0,0)}$$

while the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}_{(0,0)}$ is sent to

$$T_{(0,0)}f \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{(0,0)} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}_{(0,0)} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{(0,0)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}_{f(0,0)=(0,0)} .$$

So, from the function

$$\begin{aligned} f : \mathbb{R}^2_{xy} &\longrightarrow \mathbb{R}^2_{uv} \\ p &\longmapsto f(p) \end{aligned}$$

we have *induced* a mapping

$$\begin{aligned} T_p f : T_p\mathbb{R}^2_{xy} &\longrightarrow T_{f(p)}\mathbb{R}^2_{uv} \\ v_p &\longmapsto T_p f \cdot v_p \end{aligned}$$

from the tangent bundle of the domain manifold $\mathbb{R}^2_{xy}$ to the tangent bundle of the range manifold $\mathbb{R}^2_{uv}$, which is nothing more than the Jacobian matrix. By *induced* we mean that we used f to derive another map Tf, or $T_p f$ when we take into account the base point p. We call $T_p f \cdot v_p$ the **push-forward** of v_p by $T_p f$ and $T_p f$ is sometimes called the **push-forward mapping**.

Let us do one more example, suppose we wanted to find $T_p f \cdot v_p$ when

$$v_p = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{(1,1)}.$$

First we find $f(1,1) = (1+1, 1-1) = (2,0)$. Next we have

$$T_{(1,1)} f \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{(1,1)} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}_{(1,1)} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{(1,1)} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}_{f(1,1)} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}_{(2,0)}.$$

Notice that on the left the base point is $(1,1)$ but on the right the base point is $f(1,1) = (2,0)$.

Question 6.4 Using the same function f, find $T_p f \cdot v_p$ of the following vectors.

(a) $v_p = \begin{bmatrix} -1 \\ -1 \end{bmatrix}_{(5,4)}$

(b) $v_p = \begin{bmatrix} -2 \\ 3 \end{bmatrix}_{(2,-3)}$

Since f is already a linear function then the Jacobian of f, that is the mapping $T_p f$, is the same at each point. We will soon meet examples where this is not true. But there is something else to notice as well. Consider a generic vector given by $v_p = \begin{bmatrix} x \\ y \end{bmatrix}_p$. Then

$$T_p f \cdot \begin{bmatrix} x \\ y \end{bmatrix}_p = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}_p \cdot \begin{bmatrix} x \\ y \end{bmatrix}_p = \begin{bmatrix} x+y \\ x-y \end{bmatrix}_{f(p)}.$$

We have that $T_p f$ acting on $\begin{bmatrix} x \\ y \end{bmatrix}_p \in T_p\mathbb{R}^2$, where $p \in$ manifold $\mathbb{R}^2$, looks exactly the same as the function f acting on $(x, y) \in$ manifold $\mathbb{R}^2$,

$$\begin{aligned} f(x,y) &= (x+y, x-y) \in \mathbb{R}^2 \\ T_p f \cdot \begin{bmatrix} x \\ y \end{bmatrix}_p &= \begin{bmatrix} x+y \\ x-y \end{bmatrix}_{f(p)} \in T_{f(p)}\mathbb{R}^2, \;\; p \in \mathbb{R}^2. \end{aligned}$$

Of course the domains and ranges of Tf and f are different spaces, but the way the functions Tf and f act on the input points *looks* the same. Very often *for linear functions* f one will see it written that

$$Tf = f.$$

Clearly, since the domains and ranges are different this can not be technically true, but this is one of those convenient so-called "abuses of notation." (Actually, abuses of notation are exceedingly common in mathematics. When you are first learning something these abuses of notation can often imped understanding, but once you understand what is going on they make doing computations and writing things down so much simpler and faster. To be absolutely precise all the time becomes overwhelming and difficult, and unnecessary to someone who really understands what is going on.)

Putting what we have just learned into use we will revisit what we did in the last section. Sticking with our same function $f(x, y) = \big(u(x, y), v(x, y)\big) = (x + y, x - y)$, consider the point $(5, 7) \in \mathbb{R}^2_{xy}$. We find that $f(5, 7) = (5 + 7, 5 - 7) = (12, -2)$. Next, we find the area of the parallelepiped spanned by the Euclidian unit vectors

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{(5,7)}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{(5,7)} \in T_{(5,7)}\mathbb{R}^2_{xy}$$

at that point. Technically, this parallelepiped lives in the tangent space $T_{(5,7)}\mathbb{R}^2_{xy}$, but given that we can naturally identify the tangent space with the underlying manifold $\mathbb{R}^2_{xy}$ we often also think of the parallelepiped as being in the manifold. Using our area form $dx \wedge dy$ we get the area as

$$(dx \wedge dy)_{(5,7)} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{(5,7)}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{(5,7)} \right) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

Next we use $T_{(5,7)}f$ to find the push-forwards of the vectors to the space $T_{(12,-2)}\mathbb{R}^2_{uv}$;

$$T_{(5,7)}f \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{(5,7)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{(5,7)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{(12,-2)} \in T_{(12,-2)}\mathbb{R}^2_{uv},$$

$$T_{(5,7)}f \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{(5,7)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{(5,7)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}_{(12,-2)} \in T_{(12,-2)}\mathbb{R}^2_{uv}.$$

Now, we use the area form $du \wedge dv$ to find the area of the parallelepiped spanned by these pushed-forward vectors,

$$(du \wedge dv)_{(12,-2)} \left(T_{(5,7)}f \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{(5,7)}, T_{(5,7)}f \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{(5,7)} \right) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1 - 1 = -2.$$

In summary, we have found for $p = (5, 7)$

$$(dx \wedge dy)_p \underbrace{\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}_p, \begin{bmatrix} 0 \\ 1 \end{bmatrix}_p \right)}_{\text{vectors in } T_p\mathbb{R}^2_{xy}} = 1$$

and

$$(du \wedge dv)_{f(p)} \underbrace{\left(T_pf \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}_p, T_pf \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}_p \right)}_{\substack{\text{push-forward of vectors in } T_p\mathbb{R}^2_{xy}, \\ \text{which are in } T_{f(p)}\mathbb{R}^2_{uv}}} = -2.$$

Combining everything we get the following equality

$$\underbrace{-2 \cdot \underbrace{(dx \wedge dy)_p \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}_p, \begin{bmatrix} 0 \\ 1 \end{bmatrix}_p \right)}_{=1}}_{=-2} = \underbrace{(du \wedge dv)_{f(p)} \left(T_pf \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}_p, T_pf \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}_p \right)}_{=-2}$$

which is exactly what we had from the previous section, only recognizing that the vectors that $du \wedge dv$ ate are the push-forwards of the vectors that $dx \wedge dy$ ate. So, in order to make rigorous sense of the identity $-2dx \wedge dy = du \wedge dv$ the concept that we need to work with is the push-forwards of vectors.

Question 6.5 Consider the same change in coordinates $f(x, y) = \big(u(x, y), v(x, y)\big) = (x + y, x - y)$.

(a) Find $f(3. - 2)$.

(b) Find $(dx \wedge dy)_{(3,-2)}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{(3,-2)}, \begin{bmatrix} 5 \\ -1 \end{bmatrix}_{(3,-2)}\right)$.

(c) Find $T_{(3,-2)}f \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{(3,-2)}$ and $T_{(3,-2)}f \cdot \begin{bmatrix} 5 \\ -1 \end{bmatrix}_{(3,-2)}$.

(d) Find $(du \wedge dv)_{f(3,-2)}\left(T_{(3,-2)}f \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{(3,-2)}, T_{(3,-2)}f \cdot \begin{bmatrix} 5 \\ -1 \end{bmatrix}_{(3,-2)}\right)$.

(e) Do the answers match what you would expect from the last example and the relationship $dx \wedge dy = \frac{-1}{2} du \wedge dv$?

Now we briefly look at push-forwards of vectors in a somewhat more general setting instead of in the context of our particular problem. Consider a map between two manifolds, $f : M \to N$. For the moment we will restrict our attention to the case where $M = N = \mathbb{R}^n$ for some n. If the map $f = (f_1, f_2, \ldots, f_n)$ is differentiable at a point $p \in M$ (that means each f_i is differentiable at p) and the coordinates of manifold M are given by $x_1, x_2, \ldots, x_n$, then the derivative of the map at the point $p \in M$ is given by the traditional Jacobian matrix

$$T_p f = \text{Jacobian of } f \text{ at } p = \begin{bmatrix} \left.\frac{\partial f_1}{\partial x_1}\right|_p & \left.\frac{\partial f_1}{\partial x_2}\right|_p & \cdots & \left.\frac{\partial f_1}{\partial x_n}\right|_p \\ \left.\frac{\partial f_2}{\partial x_1}\right|_p & \left.\frac{\partial f_2}{\partial x_2}\right|_p & \cdots & \left.\frac{\partial f_2}{\partial x_n}\right|_p \\ \vdots & \vdots & \ddots & \vdots \\ \left.\frac{\partial f_n}{\partial x_1}\right|_p & \left.\frac{\partial f_n}{\partial x_2}\right|_p & \cdots & \left.\frac{\partial f_n}{\partial x_n}\right|_p \end{bmatrix}.$$

The mapping $T_p f$ given by the Jacobian matrix is simply the closest linear approximation of the map $f : M \to N$. We have already spend a lot of time considering real-valued functions $g : \mathbb{R}^n \to \mathbb{R}$ and the differential dg. We know that the differential of g at the point p is in fact the closest linear approximation of g at the point p. In fact, we have said that dg_p in a sense "encodes" the tangent plane of g at the point p, and of course the tangent plane of g at p is the closest linear approximation to the graph of g. With Cartesian coordinates we can write the differential of g as

$$dg = \frac{\partial g}{\partial x_1}dx_1 + \frac{\partial g}{\partial x_2}dx_2 + \cdots + \frac{\partial g}{\partial x_n}dx_n = \left[\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \cdots, \frac{\partial g}{\partial x_n}\right].$$

In the case of our function $f = (f_1, f_2, \ldots, f_n)$ each component function f_i is itself a real-valued function $f_i : \mathbb{R}^n \to \mathbb{R}$ and hence we could consider the differentials of each f_i separately,

$$\begin{aligned} df_1 &= \frac{\partial f_1}{\partial x_1}dx_1 + \frac{\partial f_1}{\partial x_2}dx_2 + \cdots + \frac{\partial f_1}{\partial x_n}dx_n = \left[\frac{\partial f_1}{\partial x_1}, \frac{\partial f_1}{\partial x_2}, \cdots, \frac{\partial f_1}{\partial x_n}\right], \\ df_2 &= \frac{\partial f_2}{\partial x_1}dx_1 + \frac{\partial f_2}{\partial x_2}dx_2 + \cdots + \frac{\partial f_2}{\partial x_n}dx_n = \left[\frac{\partial f_2}{\partial x_1}, \frac{\partial f_2}{\partial x_2}, \cdots, \frac{\partial f_2}{\partial x_n}\right], \\ &\vdots \\ df_n &= \frac{\partial f_n}{\partial x_1}dx_1 + \frac{\partial f_n}{\partial x_2}dx_2 + \cdots + \frac{\partial f_n}{\partial x_n}dx_n = \left[\frac{\partial f_n}{\partial x_1}, \frac{\partial f_n}{\partial x_2}, \cdots, \frac{\partial f_n}{\partial x_n}\right]. \end{aligned}$$

Compare the co-vectors of the component functions of f with the rows of the Jacobian matrix. They are the same. Thus *the rows of the Jacobian matrix of f are nothing more than the differentials of the component functions of f, df_i, written as row-matrices.* The Jacobian Tf is the closest linear approximation of the mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ because its rows are the closest linear approximations of each $f_i : \mathbb{R}^n \to \mathbb{R}$.

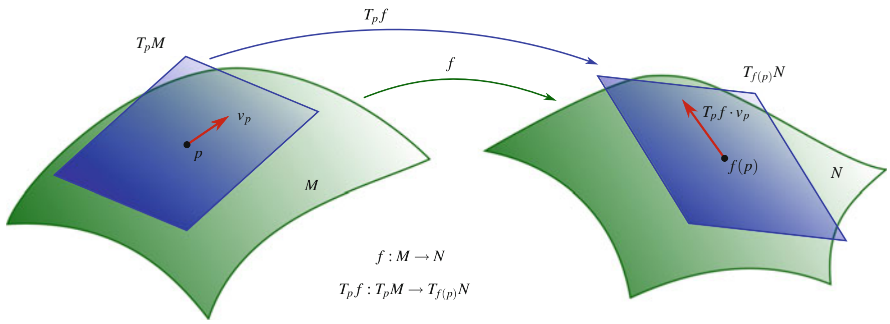

Fig. 6.7 The map $f : M \to N$ induces the tangent map $Tf : TM \to TN$. At a particular point $p \in M$ we get $T_p f : T_p M \to T_{f(p)} N$. Thus the vector $v_p \in T_p M$ gets pushed-forward to the vector $T_p f \cdot v_p \in T_{f(p)} N$

As before, suppose we have some vector $v_p \in T_p M$. $T_p f \cdot v_p$ gives us the linear approximation of the change in f as we move along v_p, which is another vector in $T_{f(p)} N$. Since we can find $T_p f : T_p M \to T_{f(p)} N$ at each point $p \in M$ where f is differentiable we end up with a mapping Tf from the tangent bundle of M to the tangent bundle of N. We say that the map $f : M \to N$ *induces* the map $Tf : TM \longrightarrow TN$. The mapping Tf **pushes-forward** vectors in $T_p M$ to vectors in $T_{f(p)} N$. This mapping is often called the tangent mapping induced by f. See Fig. 6.7. Often we would show this by writing something like this,

$$\begin{aligned}
TM &\xrightarrow{\substack{Tf \\ f_* \\ Df}} TN \\
v &\longmapsto Tf \cdot v \\
M &\xrightarrow{\;f\;} N \\
p &\longmapsto f(p).
\end{aligned}$$

There are three different notations you will encounter for this induced map, $D_p f$, $T_p f$, and $f_*(p)$. $D_p f$ is a pretty standard notation which you will frequently encounter, especially in calculus classes. Probably the most common notation, however, is f_*. If we want to include the base point it is often written as $f_*(p)$, which is a little cumbersome. This notation is most frequently encountered in differential geometry books. However, in general we will use the notation Tf or $T_p f$. Frankly, this is a fairly non-standard notation but we like it because as a notation it packs in a lot of information. The T in Tf tells us that we are dealing with the tangent map going from one tangent bundle to another, induced by the mapping f. The notation also can include the base point p nicely when we want it to in the same manner the base point is included in the tangent space notation. Thus $T_p f$ is used to specify a mapping from the tangent space $T_p M$ to the tangent space $T_{f(p)} N$.

6.3 Pull-Backs of Volume Forms

In this section we will define the **pull-backs** of volume forms. While most differential forms are not volume forms, the volume forms play a very important role in integration so we will concentrate on them in this section. We will explore the pull-backs of more general differential forms later. For the moment we will continue with our linear example. Our basic goal is to hone in on a more precise understanding of the relationship

$$\underbrace{-2 \cdot \underbrace{(dx \wedge dy)_p \left(\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}_p, \begin{bmatrix} 0 \\ 1 \end{bmatrix}_p}_{\text{vectors in } T_p\mathbb{R}^2_{xy}} \right)}_{=1}}_{=-2} = \underbrace{(du \wedge dv)_{f(p)} \left(\underbrace{T_p f \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}_p, T_p f \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}_p}_{\substack{\text{push-forward of vectors in } T_p\mathbb{R}^2_{xy}, \\ \text{which are in } T_{f(p)}\mathbb{R}^2_{uv}}} \right)}_{=-2}.$$

Recall, for the mapping $f(x, y) = (x + y, x - y)$ we had

$$\begin{aligned} du \wedge dv &= d(x+y) \wedge d(x-y) \\ &= (dx + dy) \wedge (dx - dy) \\ &= dx \wedge (dx - dy) + dy \wedge (dx - dy) \\ &= dx \wedge dx - dx \wedge dy + dy \wedge dx - dy \wedge dy \\ &= -dx \wedge dy - dx \wedge dy \\ &= -2dx \wedge dy. \end{aligned}$$

We reperform this computation in generality. That is, consider a change of variables $f : \mathbb{R}^2_{xy} \to \mathbb{R}^2_{uv}$ given by $f(x, y) = \big(u(x, y), v(x, y)\big) = (u, v)$. First we find

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \quad \text{and} \quad dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy,$$

and then using the properties of the wedgeproduct we get

$$\begin{aligned} du \wedge dv &= \left(\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy\right) \wedge \left(\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy\right) \\ &= \frac{\partial u}{\partial x} dx \wedge \left(\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy\right) + \frac{\partial u}{\partial y} dy \wedge \left(\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy\right) \\ &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} \underbrace{dx \wedge dx}_{=0} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} dx \wedge dy + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \underbrace{dy \wedge dx}_{-dx \wedge dy} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \underbrace{dy \wedge dy}_{=0} \\ &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} dx \wedge dy - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} dx \wedge dy \\ &= \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) dx \wedge dy \\ &= \underbrace{\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}}_{\substack{\text{Determinant of} \\ \text{Jacobian} \\ \text{matrix}}} dx \wedge dy. \end{aligned}$$

In essence it appears that the Jacobian matrix just "falls out" of the wedgeproduct when we are finding the relationship between two volume forms. In the last section we tried to get a better idea what this identity meant by introducing the induced tangent map Tf and the idea of push-forwards of vectors. The induced tangent map Tf operated according to

$$\begin{aligned} T_p\mathbb{R}^2 &\xrightarrow{T_p f} T_{f(p)}\mathbb{R}^2 \\ \mathbb{R}^2 &\xrightarrow{f} \mathbb{R}^2, \end{aligned}$$

where the tangent map at p, $T_p f$, is given by the Jacobian matrix evaluated at p,

$$T_p f = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}_p .$$

We defined the push-forward of the vector $v_p = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_p \in T_p\mathbb{R}^2$ by

$$T_p f \cdot v_p = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}_p \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_p = \begin{bmatrix} \frac{\partial u}{\partial x} v_1 + \frac{\partial u}{\partial y} v_2 \\ \frac{\partial v}{\partial x} v_1 + \frac{\partial v}{\partial y} v_2 \end{bmatrix}_{f(p)} \in T_{f(p)}\mathbb{R}^2.$$

Omitting the base point p from our calculations and using det to represent the determinant where convenient, we have

$$\begin{aligned}
du \wedge dv(Tf \cdot v, Tf \cdot w) &= \det \begin{bmatrix} du(Tf \cdot v) & du(Tf \cdot w) \\ dv(Tf \cdot v) & dv(Tf \cdot w) \end{bmatrix} \\
&= \det \begin{bmatrix} \frac{\partial u}{\partial x} v_1 + \frac{\partial u}{\partial y} v_2 & \frac{\partial u}{\partial x} w_1 + \frac{\partial u}{\partial y} w_2 \\ \frac{\partial v}{\partial x} v_1 + \frac{\partial v}{\partial y} v_2 & \frac{\partial v}{\partial x} w_1 + \frac{\partial v}{\partial y} w_2 \end{bmatrix} \\
&= \det \left(\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \right) \\
&= \det \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \cdot \det \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \\
&= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \cdot \begin{vmatrix} dx(v) & dx(w) \\ dy(v) & dy(w) \end{vmatrix} \\
&= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} dx \wedge dy(v, w)
\end{aligned}$$

giving us the identity

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} dx \wedge dy(v, w) = du \wedge dv(Tf \cdot v, Tf \cdot w).$$

Again, the Jacobian essentially "fell out" when we computed how the area form changes under a change of basis given by f. But also, the Jacobian is exactly our Tf that pushes vectors forward when we change basis, and vectors (or push-forwards of vectors) are obviously necessary to finding the areas of the parallelepiped spanned by the vectors. It is almost as if the Jacobian matrix "pulls through" in some sense.

Now we will define what the **pull-back** of a differential form is. Keep in mind, in this section the only differential form we have actually looked at is the volume form because of its importance to integration. But this basic definition actually applies to all differential forms. Suppose we have a mapping between manifolds $f : M \to N$. This mapping induces the tangent mapping $Tf : TM \to TN$. Now suppose that we have a differential k-form ω on N. Then we define the differential k-form $T^*f \cdot \omega$ as the pull-back of ω to M where

Definition of Pull-Back of Differential Form
$$\big(T^*f \cdot \omega\big)\big(v_1, v_2, \ldots, v_k\big) = \omega\big(Tf \cdot v_1, Tf \cdot v_2, \ldots, Tf \cdot v_k\big).$$

What exactly is this trying to say? If we are given a k-form ω on N, the range of f, then we are trying to find a k-form on M that is in some sense the "same" as ω. What do we mean by "the same"? The vectors that these two-forms will eat are from different tangent spaces. But we already know that tangent vectors on M can be pushed forward in a natural way to tangent vectors on N, so we decide to use this idea. When the k-form on M that we are trying to find eats k tangent vectors we want it to give the same value that the k-form ω on N gives when it eats the push-forwards of these same vectors. This k-form on M that we have found is denoted by $T^*f \cdot \omega$ or $f^*\omega$ and called the pull-back of ω. The most common notation in use is f^*,

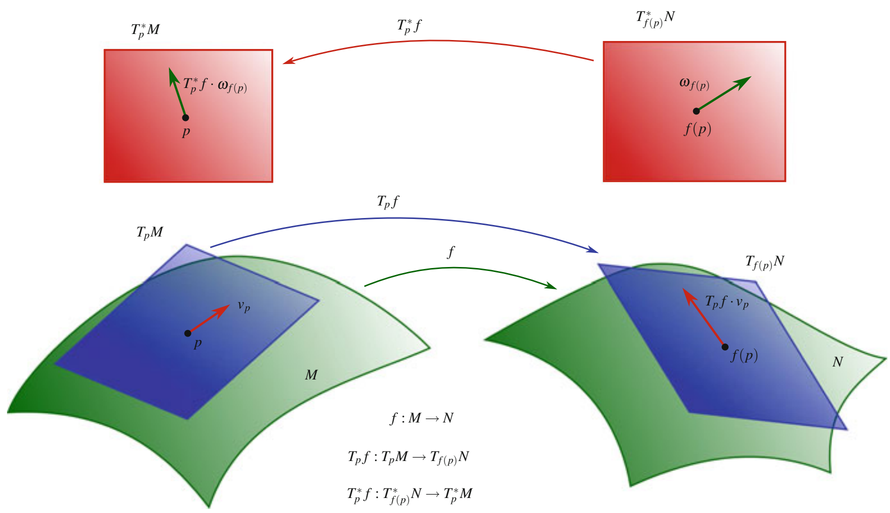

Fig. 6.8 The mappings f and Tf are shown as in Fig. 6.7, but now the pull-back mapping $T^*f : T^*N \to T^*M$ is included. At a particular point $f(p) \in N$ we have $T_p^*f : T_{f(p)}^*N \to T_p^*M$. Notice that the mapping T_p^*f is actually indexed by p, the point of the range. This is quite unusual, but doing so helps to keep the notation consistent

but for the same reasons as explained in the section on push-forwards of tangent vectors, we will use T^*f. The pull-back map is also sometimes called the **cotangent map**.

Here we show all the mappings together with base point notation included. See Fig. 6.8. There is one very important thing to notice, that when it comes to the pull-back map, *it is indexed by the base point in the image and not the domain!* This is quite unusual in mathematics, but in this situation it ends up simply being the best way to keep track of base points. We often write the mapping f and its induced mappings T_pf and T_p^*f as

$$
\begin{aligned}
\bigwedge\nolimits_p^k(M) &\xleftarrow{T_p^*f} \bigwedge\nolimits_{f(p)}^k(N) \\
T_p^*f \cdot \omega_{f(p)} &\longleftarrow\!\shortmid\ \omega_{f(p)} \\
T_pM &\xrightarrow{T_pf} T_{f(p)}N \\
v_p &\longmapsto T_{f(p)}f \cdot v_p \\
M &\xrightarrow{f} N \\
p &\longmapsto f(p).
\end{aligned}
$$

Now we are ready to understand what is going on in our example. Here everything is put together,

$$
\underbrace{\overbrace{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \cdot (dx \wedge dy)_p}^{\substack{T^*f \cdot (du \wedge dv), \\ \text{pull-back} \\ \text{of } du \wedge dv}} \underbrace{\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}_p, \begin{bmatrix} 0 \\ 1 \end{bmatrix}_p \right)}_{\text{vectors in } T_p\mathbb{R}^2_{xy}}}_{=(-2)(1)=-2} = \underbrace{(du \wedge dv)_{f(p)} \underbrace{\left(T_pf \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}_p, T_pf \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}_p \right)}_{\substack{\text{push-forward of vectors in } T_p\mathbb{R}^2_{xy}, \\ \text{which are in } T_{f(p)}\mathbb{R}^2_{uv}}}}_{=-2}.
$$

The *real* way to write the identity $-2dx \wedge dy = du \wedge dv$ is that the two-form $-2dx \wedge dy$ is the pull-back of $du \wedge dv$,

$$
T^*f \cdot \big(du \wedge dv\big) = -2dx \wedge dy.
$$

As we said earlier, in this section we are primarily concentrating on volume forms because of their role in integration. Now we will consider the pull-backs of volume in full generality. We want to find the pull-back of the volume form under a change of coordinate mapping $\phi : \mathbb{R}^n_{(x_1,\ldots,x_n)} \to \mathbb{R}^n_{(\phi_1,\ldots,\phi_n)}$ given by

$$\phi = \Big(\phi_1(x_1,\ldots,x_n), \phi_2(x_1,\ldots,x_n), \ldots, \phi_n(x_1,\ldots,x_n)\Big).$$

Now consider the volume one-form $\omega = d\phi_1 \wedge d\phi_2 \wedge \cdots \wedge d\phi_n$ on $\mathbb{R}^n_{(\phi_1,\ldots,\phi_n)}$. We want to find a general formula for $T^*\phi \cdot \omega$ on $\mathbb{R}^n_{(x_1,\ldots,x_n)}$. Clearly, we know that $T^*\phi \cdot \omega$ is going to be an n-form as well and hence will have to have the form $f\,dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ for some as of yet unknown function f. Letting

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

we have

$$dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n(e_1, e_2, \ldots, e_n) = 1.$$

Also, recalling that

$$T\phi = \begin{bmatrix} \frac{\partial\phi_1}{\partial x_1} & \frac{\partial\phi_1}{\partial x_2} & \cdots & \frac{\partial\phi_1}{\partial x_n} \\ \frac{\partial\phi_2}{\partial x_1} & \frac{\partial\phi_2}{\partial x_2} & \cdots & \frac{\partial\phi_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\phi_n}{\partial x_1} & \frac{\partial\phi_n}{\partial x_2} & \cdots & \frac{\partial\phi_n}{\partial x_n} \end{bmatrix}$$

we have

$$T\phi \cdot e_1 = \begin{bmatrix} \frac{\partial\phi_1}{\partial x_1} \\ \frac{\partial\phi_2}{\partial x_1} \\ \vdots \\ \frac{\partial\phi_n}{\partial x_1} \end{bmatrix}, \quad T\phi \cdot e_2 = \begin{bmatrix} \frac{\partial\phi_1}{\partial x_2} \\ \frac{\partial\phi_2}{\partial x_2} \\ \vdots \\ \frac{\partial\phi_n}{\partial x_2} \end{bmatrix}, \quad \ldots, \quad T\phi \cdot e_n = \begin{bmatrix} \frac{\partial\phi_1}{\partial x_n} \\ \frac{\partial\phi_2}{\partial x_n} \\ \vdots \\ \frac{\partial\phi_n}{\partial x_n} \end{bmatrix}$$

so we get

$$\begin{aligned} f &= \big(f\,dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n\big)\big(e_1, e_2, \ldots, e_n\big) \\ &= \Big(T^*\phi \cdot \omega\Big)\big(e_1, e_2, \ldots, e_n\big) \\ &= \omega\Big(T\phi \cdot e_1, T\phi \cdot e_2, \ldots, T\phi \cdot e_n\Big) \\ &= d\phi_1 \wedge d\phi_2 \wedge \cdots \wedge d\phi_n\Big(T\phi \cdot e_1, T\phi \cdot e_2, \ldots, T\phi \cdot e_n\Big) \end{aligned}$$

$$= \begin{vmatrix} d\phi_1(T\phi \cdot e_1) & d\phi_1(T\phi \cdot e_2) & \cdots & d\phi_1(T\phi \cdot e_n) \\ d\phi_2(T\phi \cdot e_1) & d\phi_2(T\phi \cdot e_2) & \cdots & d\phi_2(T\phi \cdot e_n) \\ \vdots & \vdots & \ddots & \vdots \\ d\phi_n(T\phi \cdot e_1) & d\phi_n(T\phi \cdot e_2) & \cdots & d\phi_n(T\phi \cdot e_n) \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} & \cdots & \frac{\partial \phi_1}{\partial x_n} \\ \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} & \cdots & \frac{\partial \phi_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \phi_n}{\partial x_1} & \frac{\partial \phi_n}{\partial x_2} & \cdots & \frac{\partial \phi_n}{\partial x_n} \end{vmatrix}.$$

Combining everything we get the general formula for the pull-back of a volume form by a change in basis $\phi : \mathbb{R}^n_{(x_1,\ldots,x_n)} \to \mathbb{R}^n_{(\phi_1,\ldots,\phi_n)}$ to be

Pull-back of Volume Form

$$\boxed{T^*\phi \cdot \Big(d\phi_1 \wedge d\phi_2 \wedge \cdots \wedge d\phi_n\Big) = \begin{vmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} & \cdots & \frac{\partial \phi_1}{\partial x_n} \\ \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} & \cdots & \frac{\partial \phi_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \phi_n}{\partial x_1} & \frac{\partial \phi_n}{\partial x_2} & \cdots & \frac{\partial \phi_n}{\partial x_n} \end{vmatrix} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n.}$$

At least when it comes to maps between manifolds of the same dimension, the pull-back of a volume form has a very nice representation in terms of the change of coordinate mapping ϕ. In essence, the Jacobian used in the push-forward of vector fields, $T\phi$, pulls through the volume form computation. However, we caution you that for more general k-forms there is no nice formula for finding pull-backs. This exceptionally wonderful formula involving the Jacobian matrix for the pull-back of volume forms *only applies to volume forms and to mappings between manifolds with the same dimension.*

6.4 Polar Coordinates

First we will explore the polar coordinate transformations $f : \mathbb{R}^2_{r\theta} \to \mathbb{R}^2_{xy}$ given by

$$f(r,\theta) = \big(x(r,\theta), y(r,\theta)\big) = \big(r\cos\theta, r\sin\theta\big)$$

which is often written in calculus textbooks as

$$x = r\cos\theta,$$
$$y = r\sin\theta.$$

The inverse transformation $f^{-1} : \mathbb{R}^2_{xy} \to \mathbb{R}^2_{r\theta}$ is given by the rather uglier

$$f^{-1}(x,y) = \big(r(x,y), \theta(x,y)\big) = \left(\pm\sqrt{x^2+y^2}, \arctan\left(\frac{y}{x}\right)\right),$$

where the $+$ is chosen if $x \geq 0$ and the $-$ is chosen if $x < 0$, and θ is restricted to $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, the interval where arctan is defined. This restriction also makes the original function one-to-one. This can also be written as

$$r = \pm\sqrt{x^2+y^2},$$
$$\theta = \arctan\left(\frac{y}{x}\right)$$

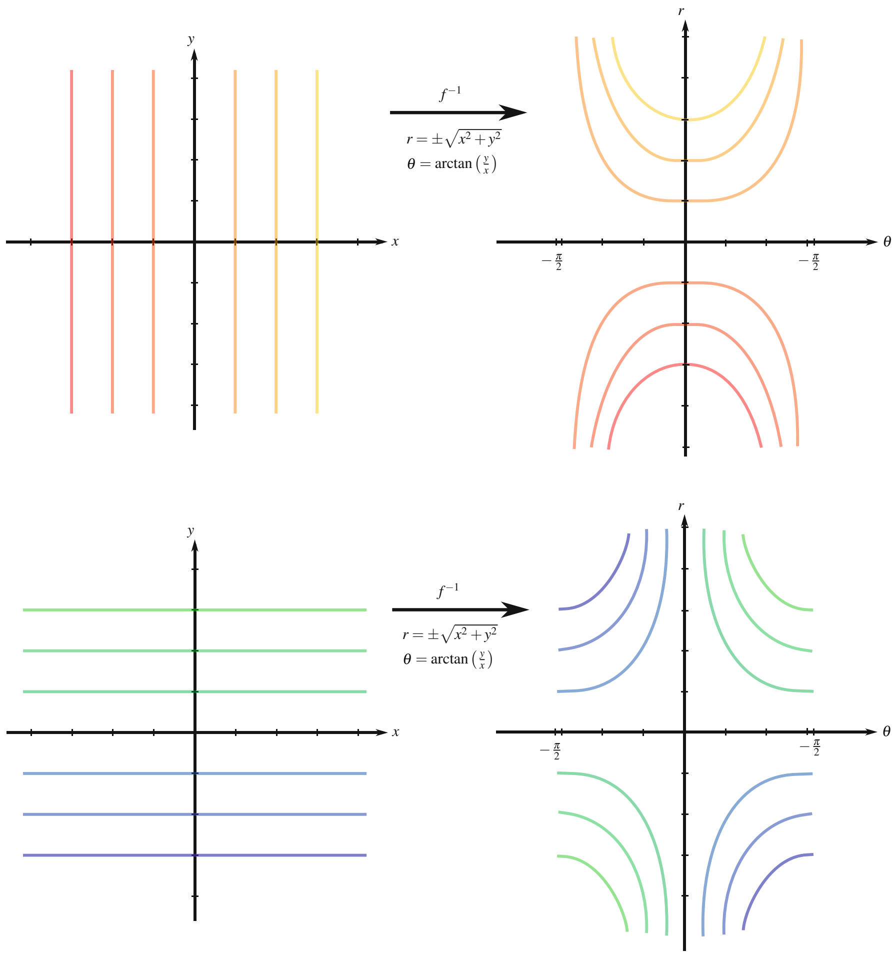

Fig. 6.9 The polar coordinate transformation $f^{-1} : \mathbb{R}^2_{xy} \to \mathbb{R}^2_{r\theta}$ restricted to $-\pi/2 < \theta \le \pi/2$. This restriction is necessary to make f^{-1} one-to-one. The image of several vertical lines is shown on the top and the image of several horizontal lines is shown on the bottom

and is shown in Fig. 6.9.

Question 6.6 Show that $f^{-1}(x, y)$ is indeed given by $f^{-1}(x, y) = \left(\pm\sqrt{x^2 + y^2}, \arctan\left(\frac{y}{x}\right)\right)$ where the $+$ is chosen if $x \ge 0$ and the $-$ is chosen if $x < 0$, and θ is restricted to $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Question 6.7 Similar to Fig. 6.9, sketch the mapping $f : \mathbb{R}^2_{r\theta} \to \mathbb{R}^2_{xy}$. Where do the lines $r = c$, where c is a constant, map to? Where do the lines $\theta = c$ map to? Compare this with what you learned in your calculus class about polar coordinates.

Actually, it is pretty easy to see how the horizontal and vertical lines on $\mathbb{R}^2_{xy}$ in Fig. 6.9 map to $\mathbb{R}^2_{r\theta}$, we just have to write the equations $x = c$ and $y = c$, where c is a constant, in polar coordinates,

$$x = c \;\Rightarrow\; c = r\cos\theta \;\Rightarrow\; r = \frac{c}{\cos\theta} = c\sec\theta$$

and the line $y = c$ can be written as

$$y = c \;\Rightarrow\; c = r\cos\theta \;\Rightarrow\; r = \frac{c}{\sin\theta} = c\csc\theta,$$

where $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$. For example, the line $x = 1$ can be written as $r = \sec\theta$, the line $x = 2$ is written as $r = 2\sec\theta$, etc.

The first thing that we will do in polar coordinates is "write the area form $dx \wedge dy$ in terms of $d\theta \wedge dr$." Using the mapping $f(r,\theta) = \big(x(r,\theta), y(r,\theta)\big) = \big(r\cos\theta, r\sin\theta\big)$ find dx and dy,

$$\begin{aligned} dx &= \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta \\ &= \frac{\partial (r\cos\theta)}{\partial r}dr + \frac{\partial (r\cos\theta)}{\partial \theta}d\theta \\ &= \cos\theta dr - r\sin\theta d\theta \end{aligned}$$

and

$$\begin{aligned} dy &= \frac{\partial y}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta \\ &= \frac{\partial (r\sin\theta)}{\partial r}dr + \frac{\partial (r\sin\theta)}{\partial \theta}d\theta \\ &= \sin\theta dr + r\cos\theta d\theta \end{aligned}$$

so

$$\begin{aligned} dx \wedge dy &= \Big(\cos\theta dr - r\sin\theta d\theta\Big) \wedge \Big(\sin\theta dr + r\cos\theta d\theta\Big) \\ &= \cos\theta\sin\theta \underbrace{dr \wedge dr}_{=0} - r\sin^2\theta d\theta \wedge dr \\ &\quad + r\cos^2\theta \underbrace{dr \wedge d\theta}_{-d\theta\wedge dr} - r^2\sin\theta\cos\theta \underbrace{d\theta \wedge d\theta}_{=0} \\ &= -r(\sin^2\theta + \cos^2\theta) d\theta \wedge dr \\ &= -r d\theta \wedge dr. \end{aligned}$$

Thus we get the "identity" $dx \wedge dy = -rd\theta \wedge dr$. Now that we have "written the area form $dx \wedge dy$ in terms of $d\theta \wedge dr$" let us analyze what we have actually done. We were given a mapping $f : \mathbb{R}^2_{r\theta} \to \mathbb{R}^2_{xy}$, which we learned induces the maps Tf and T^*f that behave as follows:

$$\begin{aligned} \bigwedge^2\big(\mathbb{R}^2_{r\theta}\big) &\xleftarrow{T^*f} \bigwedge^2\big(\mathbb{R}^2_{xy}\big) \\ T^*f \cdot \big(dx \wedge dy\big) &\longleftarrow\!\shortmid dx \wedge dy \\ T\mathbb{R}^2_{r\theta} &\xrightarrow{Tf} T\mathbb{R}^2_{xy} \\ \mathbb{R}^2_{r\theta} &\xrightarrow{f} \mathbb{R}^2_{xy}. \end{aligned}$$

The induced map T^*f finds the pull-back by f of the volume form on $\mathbb{R}^2_{xy}$. Thus, when we say that we have "written the area form $dx \wedge dy$ in terms of $d\theta \wedge dr$" what we have actually done is find the pull-back. Thus, the identity $dx \wedge dy = -rd\theta \wedge dr$ should actually be written as

$$\boxed{T^*f \cdot \big(dx \wedge dy\big) = -rd\theta \wedge dr.}$$

Question 6.8 Referring to Fig. 6.10 explain the formula $T^*f \cdot \big(dx \wedge dy\big) = -rd\theta \wedge dr$ in terms of the shaded regions.

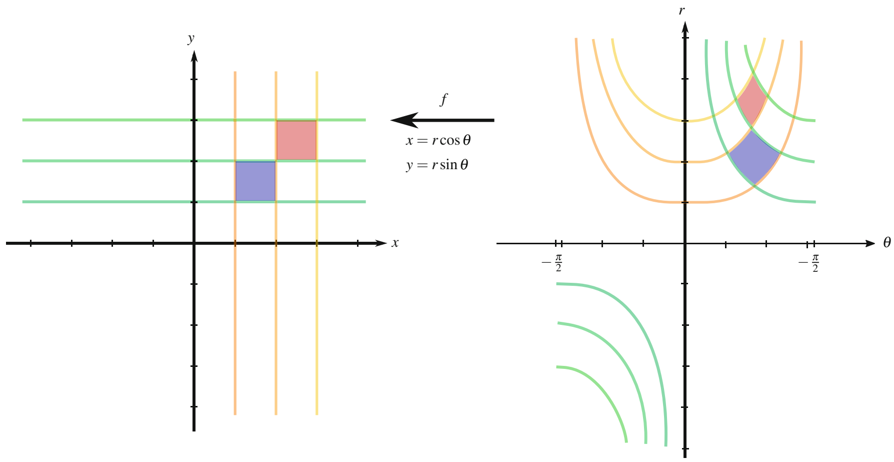

Fig. 6.10 The mapping $f : \mathbb{R}^2_{r\theta} \to \mathbb{R}^2_{xy}$. Notice what happens to the shaded regions

Similarly, we could have worked in the other direction to "write the area form $d\theta \wedge dr$ in terms of $dx \wedge dy$" using $f^{-1}(x, y) = \big(r(x, y), \theta(x, y)\big) = \big(\pm\sqrt{x^2 + y^2}, \arctan\big(\frac{y}{x}\big)\big)$. First we need to find $d\theta$ and dr,

$$\begin{aligned}
d\theta &= \frac{\partial \theta}{\partial x} dx + \frac{\partial \theta}{\partial y} dy \\
&= \frac{\partial \arctan(y/x)}{\partial x} dx + \frac{\partial \arctan(y/x)}{\partial y} dy \\
&\underset{\substack{\text{chain}\\\text{rule}}}{=} \frac{1}{1 + \left(\frac{y}{x}\right)^2} \frac{\partial}{\partial x}\left(\frac{y}{x}\right) dx + \frac{1}{1 + \left(\frac{y}{x}\right)^2} \frac{\partial}{\partial y}\left(\frac{y}{x}\right) dy \\
&= \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy
\end{aligned}$$

and

$$\begin{aligned}
dr &= \frac{\partial r}{\partial x} dx + \frac{\partial r}{\partial y} dy \\
&= \frac{\partial \sqrt{x^2 + y^2}}{\partial x} dx + \frac{\partial \sqrt{x^2 + y^2}}{\partial y} dy \\
&\underset{\substack{\text{chain}\\\text{rule}}}{=} \frac{x}{\sqrt{x^2 + y^2}} dx + \frac{y}{\sqrt{x^2 + y^2}} dy
\end{aligned}$$

so

$$\begin{aligned}
d\theta \wedge dr &= \left(\frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy\right) \wedge \left(\frac{x}{\sqrt{x^2 + y^2}} dx + \frac{y}{\sqrt{x^2 + y^2}} dy\right) \\
&= \frac{x^2}{(x^2 + y^2)\sqrt{x^2 + y^2}}\, dy \wedge dx - \frac{y^2}{(x^2 + y^2)\sqrt{x^2 + y^2}}\, dx \wedge dy \\
&= \frac{-(x^2 + y^2)}{(x^2 + y^2)\sqrt{x^2 + y^2}}\, dx \wedge dy \\
&= \frac{-1}{\sqrt{x^2 + y^2}}\, dx \wedge dy.
\end{aligned}$$

Thus we have obtained the identity $d\theta \wedge dr = \frac{-1}{\sqrt{x^2+y^2}} dx \wedge dy$. Like before, we had been given a mapping $f^{-1} : \mathbb{R}^2_{xy} \to \mathbb{R}^2_{r\theta}$, which induced mappings Tf^{-1} and $T^* f^{-1}$ such that

$$\bigwedge^2 \left(\mathbb{R}^2_{xy}\right) \xleftarrow{T^* f^{-1}} \bigwedge^2 \left(\mathbb{R}^2_{r\theta}\right)$$
$$T^* f \cdot \big(d\theta \wedge dr\big) \longleftarrow d\theta \wedge dr$$
$$T\mathbb{R}^2_{xy} \xrightarrow{Tf^{-1}} T\mathbb{R}^2_{r\theta}$$
$$\mathbb{R}^2_{xy} \xrightarrow{f^{-1}} \mathbb{R}^2_{r\theta}.$$

So when we said "write the area form $d\theta \wedge dr$ in terms of $dx \wedge dy$" we really mean find the pull-back. The identity $d\theta \wedge dr = \frac{-1}{\sqrt{x^2+y^2}} dx \wedge dy$ that we obtained should actually be written as

$$\boxed{T^* f^{-1} \cdot \big(d\theta \wedge dr\big) = \frac{-1}{\sqrt{x^2 + y^2}} dx \wedge dy.}$$

Since we are actually dealing with volume forms here, we could have done all of this by a different method, by using the general formula for pull-backs of volume forms derived in the last section. That is, given a mapping $\phi : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\phi = \Big(\phi_1(x_1, \ldots, x_n), \phi_2(x_1, \ldots, x_n), \ldots, \phi_n(x_1, \ldots, x_n)\Big)$$

the pull-back of the volume form is given by the following equation

$$T^*\phi \cdot \Big(d\phi_1 \wedge d\phi_2 \wedge \cdots \wedge d\phi_n\Big) = \begin{vmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} & \cdots & \frac{\partial \phi_1}{\partial x_n} \\ \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} & \cdots & \frac{\partial \phi_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \phi_n}{\partial x_1} & \frac{\partial \phi_n}{\partial x_2} & \cdots & \frac{\partial \phi_n}{\partial x_n} \end{vmatrix} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n.$$

Recall, volume forms are special. While the Jacobian matrix always gives $T\phi$ and so is always used to find the push-forwards of vectors, the Jacobian matrix can only be used to find the pull-back of volume forms, not other differential forms that are not volume forms.

First we find the Jacobian matrix of the mapping $f^{-1}(x, y) = \left(\arctan\left(\frac{y}{x}\right), \pm\sqrt{x^2 + y^2}\right)$ to be

$$\begin{bmatrix} \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \\ \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \\ \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \end{bmatrix}$$

which gives the determinant

$$\begin{vmatrix} \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \\ \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \end{vmatrix} = \frac{-y^2 - x^2}{(x^2 + y^2)^{3/2}} = \frac{-1}{\sqrt{x^2 + y^2}}$$

which gives

$$T^* f^{-1} \cdot \big(d\theta \wedge dr\big) = \frac{-1}{\sqrt{x^2 + y^2}} dx \wedge dy$$

DIRECTLY. The other direction is done similarly,

$$\begin{vmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial r} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial r} \end{vmatrix}_{(\theta,r)} = \begin{vmatrix} -r\sin\theta & \cos\theta \\ r\cos\theta & \sin\theta \end{vmatrix}_{(\theta,r)} = -r\sin^2\theta - r\cos^2\theta = -r$$

which gives

$$T^* f \cdot \big(dx \wedge dy\big) = -r d\theta \wedge dr.$$

Now we will work through an example for polar coordinates. Through this example and the following question we will explore one of the ways that changes between Cartesian and polar coordinates are quite different from the linear example we were looking at in earlier sections. Where the polar coordinate change differs from a linear coordinate change is that the relation between the two volume forms depends on what point we are at. That is, the pull-back equation depends on the base point.

For the polar coordinate transformations $f^{-1} : \mathbb{R}^2_{xy} \to \mathbb{R}^2_{r\theta}$ given by $f^{-1}(x, y) = \left(\arctan\left(\frac{y}{x}\right), \pm\sqrt{x^2 + y^2}\right)$ we will give θ in radians. Our goal is to relate

$$(dx \wedge dy)_{(1,1)} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{(1,1)}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{(1,1)} \right)$$

and

$$(d\theta \wedge dr)_{f^{-1}(1,1)} \left(T_{(1,1)} f^{-1} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{(1,1)}, T_{(1,1)} f^{-1} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{(1,1)} \right).$$

The first identity is simple, we get

$$(dx \wedge dy)_{(1,1)} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{(1,1)}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{(1,1)} \right) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

Here we used the xy-plane volume form $dx \wedge dy$ to compute the area of the parallelepiped spanned by the two Euclidian unit vectors. Not surprisingly we got a volume of 1, exactly what we would expect from a unit square. Now we move to the next identity. First we find that $f^{-1}(1, 1) \approx (0.785, 1.414)$. This is the base point of the push-forwards of the two Euclidian unit vectors. Next we find the general tangent mapping

$$T_{(x,y)} f^{-1} = \begin{bmatrix} \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \\ \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \\ \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \end{bmatrix}_{(x,y)}$$

which means that for the point $(x, y) = (1, 1)$ we have

$$T_{(1,1)} f^{-1} = \begin{bmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}_{(1,1)}.$$

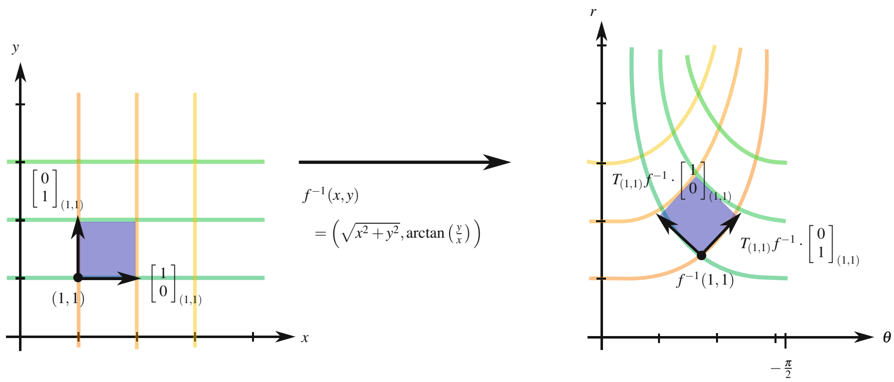

Fig. 6.11 The relationship between the volume forms $dx \wedge dy$ and $d\theta \wedge dr$ at the points $(x, y) = (1, 1)$ and $f^{-1}(1, 1)$

Next we find the push-forwards of the Euclidian unit vectors at $(1, 1)$ by f^{-1},

$$T_{(1,1)}f^{-1} \cdot \begin{bmatrix}1\\0\end{bmatrix}_{(1,1)} = \begin{bmatrix}\frac{-1}{2} & \frac{1}{2}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}_{(1,1)} \begin{bmatrix}1\\0\end{bmatrix}_{(1,1)} = \begin{bmatrix}\frac{-1}{2}\\ \frac{1}{\sqrt{2}}\end{bmatrix}_{(0.785,1.414)}$$

$$T_{(1,1)}f^{-1} \cdot \begin{bmatrix}0\\1\end{bmatrix}_{(1,1)} = \begin{bmatrix}\frac{-1}{2} & \frac{1}{2}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}_{(1,1)} \begin{bmatrix}0\\1\end{bmatrix}_{(1,1)} = \begin{bmatrix}\frac{1}{2}\\ \frac{1}{\sqrt{2}}\end{bmatrix}_{(0.785,1.414)}$$

which are needed in

$$(d\theta \wedge dr)_{(0.785,1.414)}\left(\begin{bmatrix}\frac{-1}{2}\\ \frac{1}{\sqrt{2}}\end{bmatrix}_{(0.785,1.414)}, \begin{bmatrix}\frac{1}{2}\\ \frac{1}{\sqrt{2}}\end{bmatrix}_{(0.785,1.414)}\right) = \begin{vmatrix}\frac{-1}{2} & \frac{1}{2}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{vmatrix} = \frac{-1}{\sqrt{2}}.$$

Here we used the $r\theta$-plane volume form to find the volume of the parallelepiped spanned by the pushed-forward vectors. The relationship between the areas computed in this way is in exact correspondence with the relationship between the volume forms $dx \wedge dy$ and $d\theta \wedge dr$ given by $T^* f^{-1} \cdot \big(d\theta \wedge dr\big) = \frac{-1}{\sqrt{x^2+y^2}} dx \wedge dy$ at the point $(1, 1) \in \mathbb{R}^2_{xy}$. That is,

$$\underbrace{(d\theta \wedge dr)_{f^{-1}(1,1)}\left(T_{(1,1)}f^{-1} \cdot \begin{bmatrix}1\\0\end{bmatrix}_{(1,1)}, T_{(1,1)}f^{-1} \cdot \begin{bmatrix}0\\1\end{bmatrix}_{(1,1)}\right)}_{=\frac{-1}{\sqrt{2}}} = \frac{-1}{\sqrt{2}} \underbrace{(dx \wedge dy)_{(1,1)}\left(\begin{bmatrix}1\\0\end{bmatrix}_{(1,1)}, \begin{bmatrix}0\\1\end{bmatrix}_{(1,1)}\right)}_{=1}.$$

In other words we have just seen, using two vectors that at the point $(1, 1)$, that

$$T^* f^{-1}(d\theta \wedge dr) = \frac{1}{\sqrt{2}}\, dx \wedge dy.$$

We attempt to show what is going on in Fig. 6.11. Now redo the last example, only at a different base point, to see what changes.

Question 6.9 For the polar coordinate transformations $f^{-1} : \mathbb{R}^2_{xy} \to \mathbb{R}^2_{r\theta}$ given by $f^{-1}(x, y) = \big(\arctan\left(\frac{y}{x}\right), \pm\sqrt{x^2 + y^2}\big)$,

(a) Find $f^{-1}(2, 2)$.

(b) Consider $\begin{bmatrix}1\\0\end{bmatrix}_{(2,2)}$ and $\begin{bmatrix}0\\1\end{bmatrix}_{(2,2)} \in T_{(1,1)}\mathbb{R}^2$. Find

$$(dx \wedge dy)_{(2,2)}\left(\begin{bmatrix}1\\0\end{bmatrix}_{(2,2)}, \begin{bmatrix}0\\1\end{bmatrix}_{(2,2)}\right).$$

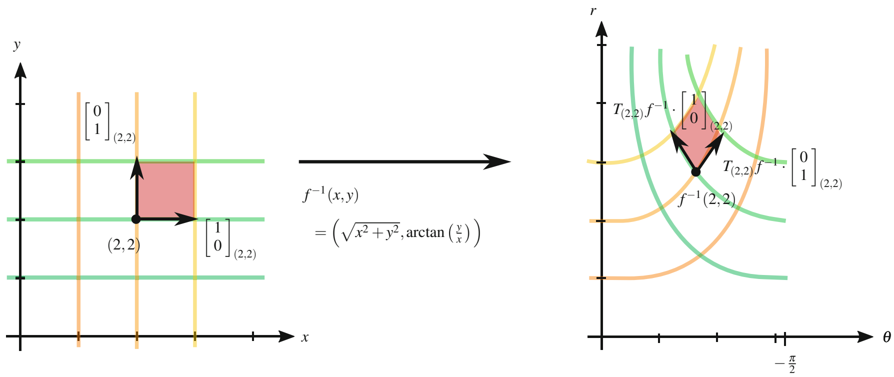

Fig. 6.12 The relationship between the volume forms $dx \wedge dy$ and $d\theta \wedge dr$ at the points $(x, y) = (2, 2)$ and $f^{-1}(2, 2)$

(c) Find $T_{(2,2)} f^{-1}$.

(d) Find $T_{(2,2)} f^{-1} \cdot \begin{bmatrix}1\\0\end{bmatrix}_{(2,2)}$ and $T_{(2,2)} f^{-1} \cdot \begin{bmatrix}0\\1\end{bmatrix}_{(2,2)}$ (These are the push-forwards of the Euclidian unit vectors at $(2, 2)$ by f^{-1} to $f^{-1}(2, 2)$.)

(e) Find $(d\theta \wedge dr)_{f^{-1}(2,2)} \left(T_{(2,2)} f^{-1} \cdot \begin{bmatrix}1\\0\end{bmatrix}_{(2,2)}, T_{(2,2)} f^{-1} \cdot \begin{bmatrix}0\\1\end{bmatrix}_{(2,2)} \right)$.

Figure 6.12 shows what is happening in this question. The relationship you should have ended up with in this question is

$$(d\theta \wedge dr)_{f^{-1}(2,2)} \left(T_{(1,1)} f^{-1} \cdot \begin{bmatrix}1\\0\end{bmatrix}_{(2,2)}, T_{(1,1)} f^{-1} \cdot \begin{bmatrix}0\\1\end{bmatrix}_{(2,2)} \right) = \frac{-1}{\sqrt{8}} (dx \wedge dy)_{(2,2)} \left(\begin{bmatrix}1\\0\end{bmatrix}_{(2,2)}, \begin{bmatrix}0\\1\end{bmatrix}_{(2,2)} \right).$$

The numerical coefficient has changed. This has happened because the point at which we are evaluating areas has changed. In other words, the relationship between the area forms in the xy-plane and the $r\theta$-plane depend on the point where the area form is being used. So, at the point $(1, 1)$ we have

$$T^*_{(1,1)} f^{-1} \cdot \big(d\theta \wedge dr\big) = \frac{-1}{\sqrt{2}} dx \wedge dy$$

and at the point $(2, 2)$ we have

$$T^*_{(2,2)} f^{-1} \cdot \big(d\theta \wedge dr\big) = \frac{-1}{\sqrt{8}} dx \wedge dy$$

which is exactly what we would expect from the actual identity we found,

$$T^*_{(x,y)} f^{-1} \cdot \big(d\theta \wedge dr\big) = \frac{-1}{\sqrt{x^2 + y^2}} dx \wedge dy.$$

6.5 Cylindrical and Spherical Coordinates

Cylindrical and Spherical coordinates are two coordinate systems that are very common for $\mathbb{R}^3$. Now that we have a pretty good idea of how push-forwards of vectors and pull-backs of volume forms work, we will provide less detail in this section. The cylindrical coordinate transformation $f : \mathbb{R}^3_{r\theta z} \to \mathbb{R}^3_{xyz}$ is given by

$$f(r, \theta, z) = \big(x(r, \theta, z), y(r, \theta, z), z(r, \theta, z)\big) = \big(r \cos\theta, r \sin\theta, z\big),$$

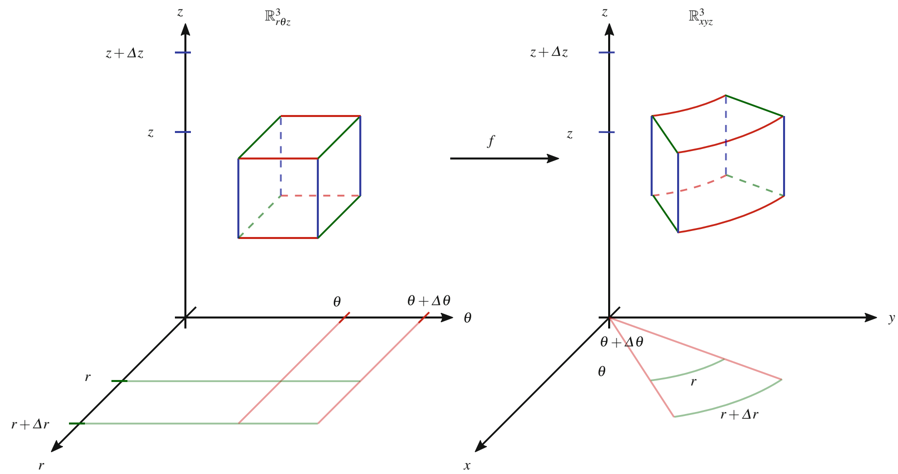

Fig. 6.13 The cylindrical coordinates mapping $f : \mathbb{R}^3_{r\theta z} \to \mathbb{R}^3_{xyz}$ given by $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$. A cube in $\mathbb{R}^3_{r\theta z}$ is mapped to a wedge in $\mathbb{R}^3_{xyz}$

which is generally written in calculus classes as

$$\begin{aligned}
x &= r\cos\theta,\\
y &= r\sin\theta,\\
z &= z.
\end{aligned}$$

See Fig. 6.13. The cylindrical coordinate transformation f induces the following maps:

$$\begin{array}{rcl}
\bigwedge^3\left(\mathbb{R}^2_{r\theta z}\right) & \xleftarrow{T^*f} & \bigwedge^3\left(\mathbb{R}^2_{xyz}\right)\\
T^*_{(r,\theta,z)}f\cdot\left(dx\wedge dy\wedge dz\right) & \longleftarrow\!\shortmid & dx\wedge dy\wedge dz\\
\mathbb{R}^3_{r\theta z} & \xrightarrow{Tf} & \mathbb{R}^3_{xyz}\\
v_{(r,\theta,z)} & \longmapsto & T_{(r,\theta,z)}f\cdot v_{(r,\theta,z)}\\
\mathbb{R}^3_{r\theta z} & \xrightarrow{f} & \mathbb{R}^3_{xyz}\\
(\theta, r, z) & \longmapsto & \big(r\cos\theta, r\sin\theta, z\big).
\end{array}$$

Proceeding as before and either using the properties of the wedgeproduct or the wedgeproduct formula in terms of the Jacobian from we can write $dx \wedge dy \wedge dz$ in terms of $d\theta \wedge dr \wedge dz$. Here we find

$$\begin{aligned}
dx &= \frac{\partial x}{\partial \theta}d\theta + \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial z}dz\\
&= -r\sin\theta d\theta + \cos\theta dr\\
dy &= \frac{\partial y}{\partial \theta}d\theta + \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial z}dz\\
&= r\cos\theta d\theta + \sin\theta dr\\
dz &= \frac{\partial z}{\partial \theta}d\theta + \frac{\partial z}{\partial r}dr + \frac{\partial z}{\partial z}dz\\
&= dz
\end{aligned}$$

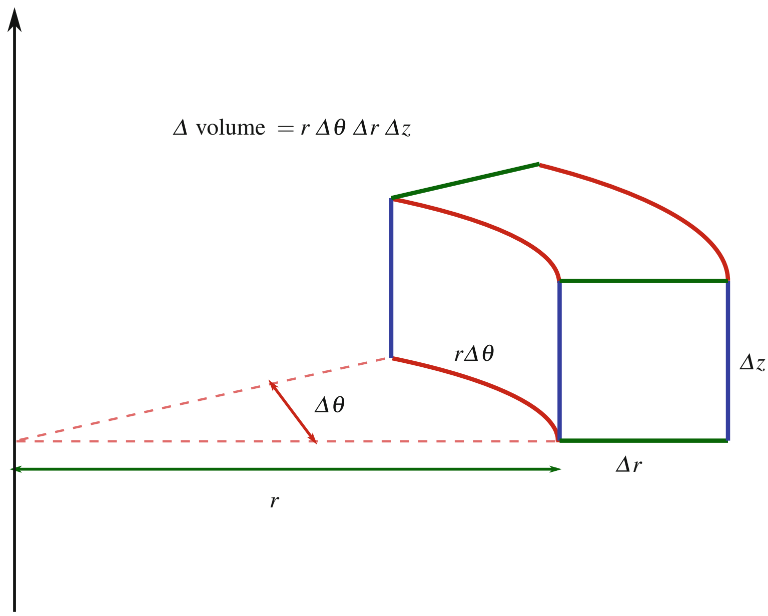

Fig. 6.14 The cylindrical "$\theta r z$ volume element" as shown in most calculus textbooks

which we then use to get

$$\begin{aligned} dx \wedge dy \wedge dz &= \Big(-r\sin\theta d\theta + \cos\theta dr \Big) \wedge \Big(r\cos\theta d\theta + \sin\theta dr \Big) \wedge dz \\ &= \Big(-r\sin^2\theta d\theta \wedge dr + r\cos^2\theta dr \wedge d\theta \Big) \wedge dz \\ &= -r d\theta \wedge dr \wedge dz. \end{aligned}$$

As we know by now, what this identity really means is

$$\boxed{T^*_{(r,\theta,z)} f \cdot \big(dx \wedge dy \wedge dz\big) = -r d\theta \wedge dr \wedge dz.}$$

What we have just done is find the "$\theta r z$ volume element" that is usually presented in calculus. A lot of calculus books give a picture of the "$\theta r z$ volume element" that looks like Fig. 6.14 and then use this picture to find the change in volume formula, though orientation is not taken into account. Thus, even though our presentation has been far more abstract and theoretical then anything you saw in calculus, you are already familiar with this from a different perspective.

Question 6.10 Find $T^*_{(r,\theta,z)} f \cdot \big(dx \wedge dy \wedge dz\big)$ using the formula for the pull-back of a volume form.

Question 6.11 Explain the relationship between the formula $T^*_{(r,\theta,z)} f \cdot \big(dx \wedge dy \wedge dz\big) = -r d\theta \wedge dr \wedge dz$ and the "$\theta r z$ volume element" shown in Fig. 6.14.

We now move to spherical coordinate changes. The spherical coordinate transformation $g : \mathbb{R}^3_{\rho\phi\theta} \to \mathbb{R}^3_{xyz}$ is given by

$$g(\rho, \phi, \theta) = \big(x(\rho, \phi, \theta), y(\rho, \phi, \theta), z(\rho, \phi, \theta)\big) = \big(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi\big),$$

which is generally written in calculus classes as

$$\begin{aligned} x &= \rho\sin\phi\cos\theta, \\ y &= \rho\sin\phi\sin\theta, \\ z &= \rho\cos\phi. \end{aligned}$$

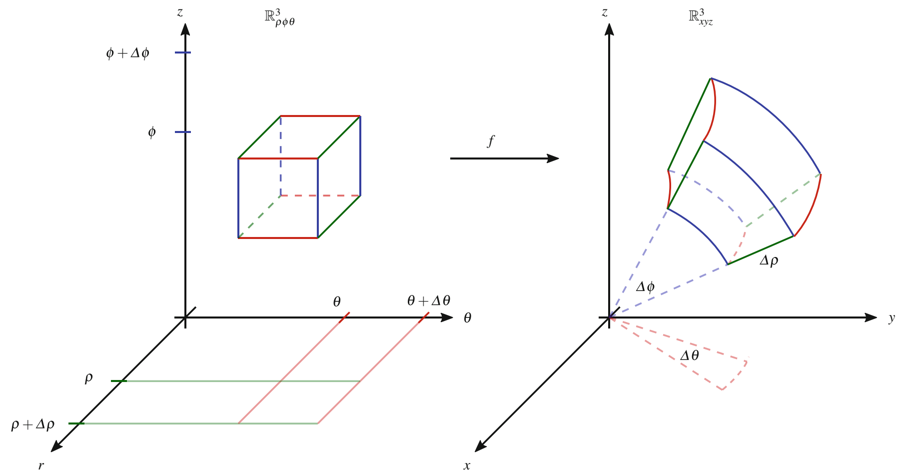

Fig. 6.15 The spherical coordinate transformation $g : \mathbb{R}^3_{\rho\phi\theta} \to \mathbb{R}^3_{xyz}$ is given by $x = \rho \sin\phi \cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$

The mapping $g : \mathbb{R}^3_{\rho\phi\theta} \to \mathbb{R}^3_{xyz}$ is illustrated in Fig. 6.15. The spherical coordinate transformation induces the following maps:

$$
\begin{array}{rcl}
\bigwedge^3\left(\mathbb{R}^2_{\rho\phi\theta}\right) & \xleftarrow{T^*g} & \bigwedge^3\left(\mathbb{R}^2_{xyz}\right) \\
T^*_{(\rho,\phi,\theta)}g \cdot \left(dx\wedge dy\wedge dz\right) & \longleftarrow & dx\wedge dy\wedge dz \\
\mathbb{R}^3_{\rho\phi\theta} & \xrightarrow{Tg} & \mathbb{R}^3_{xyz} \\
v_{(\rho,\phi,\theta)} & \longmapsto & T_{(\rho,\phi,\theta)}g\cdot v_{(\rho,\phi,\theta)} \\
\mathbb{R}^3_{\rho\phi\theta} & \xrightarrow{g} & \mathbb{R}^3_{xyz} \\
(\rho,\phi,\theta) & \longmapsto & \left(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi\right).
\end{array}
$$

Again, a straightforward computation shows that

$$
\begin{aligned}
dx &= \sin\phi\cos\theta d\rho - \rho\sin\phi\sin\theta d\theta + \rho\cos\phi\cos\theta d\phi, \\
dy &= \sin\phi\sin\theta d\rho + \rho\sin\phi\cos\theta d\theta + \rho\cos\phi\sin\theta d\phi, \\
dz &= \cos\phi d\rho - \rho\sin\phi d\phi.
\end{aligned}
$$

Another straightforward, though somewhat longer, computation that you should do yourself gives

$$dx\wedge dy\wedge dx = -\rho^2\sin\phi d\rho\wedge d\theta\wedge d\phi.$$

As we now know, this identity really means

$$\boxed{T^*_{(\rho,\phi,\theta)}g\cdot\left(dx\wedge dy\wedge dz\right) = -\rho^2\sin\phi d\rho\wedge d\theta\wedge d\phi.}$$

Since $dx\wedge dy\wedge dz$ is a volume form we could have found this using the pull-back of a volume form formula. A lot of calculus books give a picture of the "$\rho\phi\theta$ volume element" that looks like Fig. 6.16. Notice how we can pictorially find the change in volume from the picture, though again orientation is not taken into account. So again, though we have approached this from a much more abstract standpoint, it is something you have already been exposed to in calculus.

Question 6.12 Find $T^*_{(\rho,\phi,\theta)}g\cdot\left(dx\wedge dy\wedge dz\right)$ using the formula for the pull-back of a volume form.

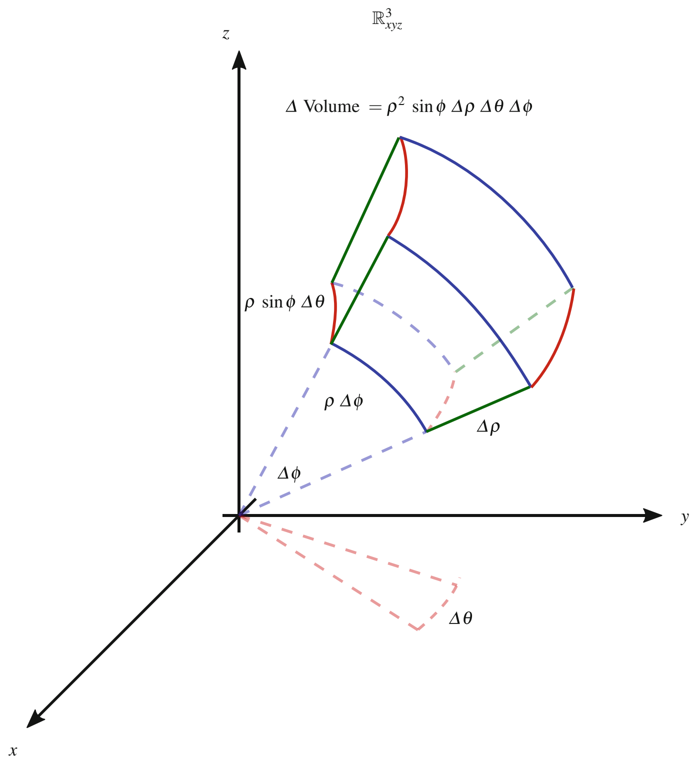

Fig. 6.16 The spherical "$\rho\phi\theta$ volume element" as shown in most calculus textbooks

Question 6.13 Explain the relationship between the formula $T^*_{(\rho,\phi,\theta)} g \cdot \big(dx \wedge dy \wedge dz\big) = -\rho^2 \sin\phi d\rho \wedge d\theta \wedge d\phi$ and the "$\rho\phi\theta$ volume element" shown in Fig. 6.16.

6.6 Pull-Backs of Differential Forms

Consider some of the relationships we have arrived at in the previous few sections

$$\begin{aligned}
T^* f \cdot (dx \wedge dy) &= \frac{-1}{2} du \wedge dv && \text{(linear example)}\\
T^* f \cdot (dx \wedge dy) &= -r d\theta \wedge dr && \text{(polar coordinates)}\\
T^* f \cdot (dx \wedge dy \wedge dz) &= -r d\theta \wedge dr \wedge dz && \text{(cylindrical coordinates)}\\
T^* f \cdot (dx \wedge dy \wedge dz) &= -\rho^2 \sin\phi d\rho \wedge d\theta \wedge d\phi && \text{(spherical coordinates)}
\end{aligned}$$

where f is the appropriate mapping $f : \mathbb{R}^{2/3}_{\text{Other coord}} \to \mathbb{R}^{2/3}_{\text{Euclidian}}$. Based on our definition of pull-backs of volume forms what we really mean by these are

$$\underbrace{\frac{-1}{2}(du \wedge dv)_p}_{= T_p^* f \cdot (dx \wedge dy)_{f(p)}} (v_p, w_p) = (dx \wedge dy)_{f(p)} (T_p f \cdot v_p, T_p f \cdot w_p),$$

$$\underbrace{-r(d\theta \wedge dr)_p}_{= T_p^* f \cdot (dx \wedge dy)_{f(p)}} (v_p, w_p) = (dx \wedge dy)_{f(p)} (T_p f \cdot v_p, T_p f \cdot w_p),$$

$$\underbrace{-r(d\theta \wedge dr \wedge dz)_p}_{=T_p^* f \cdot (dx \wedge dy \wedge dz)_{f(p)}}(u_p, v_p, w_p) = (dx \wedge dy \wedge dz)_{f(p)}(T_p f \cdot u_p, T_p f \cdot v_p, T_p f \cdot w_p),$$

$$\underbrace{-\rho^2 \sin\phi (d\rho \wedge d\theta \wedge d\phi)_p}_{=T_p^* f \cdot (dx \wedge dy \wedge dz)_{f(p)}}(u_p, v_p, w_p) = (dx \wedge dy \wedge dz)_{f(p)}(T_p f \cdot u_p, T_p f \cdot v_p, T_p f \cdot w_p).$$

Now we take a moment to give the definition of the pull-back of a differential form again in some detail. Suppose we have a map $\phi : \mathbb{R}^n \to \mathbb{R}^m$ given by

$$\phi(x_1, x_2, \ldots, x_n) = \Big(\phi_1(x_1, x_2, \ldots, x_n), \phi_2(x_1, x_2, \ldots, x_n), \ldots, \phi_m(x_1, x_2, \ldots, x_n)\Big).$$

ϕ defines a map, called the push-forward, or tangent map, of ϕ, which is denoted $T\phi : T\mathbb{R}^n \to T\mathbb{R}^m$ from the tangent bundle of $\mathbb{R}^n$ to the tangent bundle of $\mathbb{R}^m$. In coordinates the mapping $T\phi$ is given by the Jacobian matrix

$$T\phi = \begin{bmatrix} \frac{\partial \phi_1}{\partial x_1} & \cdots & \frac{\partial \phi_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi_m}{\partial x_1} & \cdots & \frac{\partial \phi_m}{\partial x_n} \end{bmatrix}.$$

As has been mentioned already, other notations for $T\phi$ that one often encounters are $D\phi$ and ϕ_*. We use this push-forward mapping $T\phi$ to help us define another mapping called the pull-back of a differential form by ϕ, or simply the pull-back by ϕ, which is denoted by $T^*\phi$ or ϕ^*. Actually, ϕ^* is the standard notation for the pull-back, but we will use $T^*\phi$.

Before actually defining $T^*\phi$ we will tell what spaces it operates on. Recall that the notation $\bigwedge^k \mathbb{R}^n$ is used to denote the vector space of k-forms on $\mathbb{R}^n$ while $\bigwedge \mathbb{R}^n = \bigoplus_{k=1}^{\infty} \bigwedge^k \mathbb{R}^n$ denotes the set of all differential forms on $\mathbb{R}^n$ of any size. Using this we generally show the maps ϕ and its induced maps $T\phi$ and $T^*\phi$ as

$$\bigwedge \mathbb{R}^n \xleftarrow{T^*\phi} \bigwedge \mathbb{R}^m$$

$$T\mathbb{R}^n \xrightarrow{T\phi} T\mathbb{R}^m$$

$$\mathbb{R}^n \xrightarrow{\phi} \mathbb{R}^m.$$

So, the pull-back map operates on the space $\bigwedge \mathbb{R}^m$ by taking a differential form on $\mathbb{R}^m$ and producing an element of $\bigwedge \mathbb{R}^n$, that is, a differential form on $\mathbb{R}^n$. Notice that this mapping is going in the opposite direction from ϕ and $T\phi$. This is the reason $T^*\phi$ is called the *pull-back* while $T\phi$ is called the *push-forward*. Next, even though $T^*\phi$ operates on a k-form of any order, we sometimes abuse notation and write

$$T^*\mathbb{R}^n \xleftarrow{T^*\phi} T^*\mathbb{R}^m,$$

writing $T^*\mathbb{R}^n$ and $T^*\mathbb{R}^m$ instead of $\bigwedge \mathbb{R}^n$ and $\bigwedge \mathbb{R}^m$. We will define the pull-back of the k-form α, which is denoted $T^*\phi \cdot \alpha \in \bigwedge^k \mathbb{R}^n$, to just be $\alpha \in \bigwedge^k \mathbb{R}^m$ acting on the pushed forward vectors, $T\phi \cdot v_i \in T\mathbb{R}^m$,

$$\boxed{\text{Pull-Back of } \alpha : \quad \Big(T^*\phi \cdot \alpha\Big)(v_1, v_2, \ldots, v_k) \equiv \alpha\Big(T\phi \cdot v_1, T\phi \cdot v_2, \ldots, T\phi \cdot v_k\Big).}$$

If we wanted to be careful and start keeping track of base points the notation starts to get a little more cumbersome, though there are times when you need to do that. The one odd part of this notation is that the map $T^*\phi$ is indexed NOT by the point it is coming from but instead by the point it is going to. This is done to keep the notation for the pull-back and push-forward mappings $T_p^*\phi$ and $T_p\phi$ consistent, that is, "dual" to each other. This is something that does not happen very often and certainly seems odd the first few times you see it.

$$\bigwedge\nolimits_p \mathbb{R}^n \xleftarrow{T_p^*\phi} \bigwedge\nolimits_{\phi(p)} \mathbb{R}^m$$

$$T_p\mathbb{R}^n \xrightarrow{T_p\phi} T_{\phi(p)}\mathbb{R}^m$$
$$\mathbb{R}^n \xrightarrow{\phi} \mathbb{R}^m.$$

With base points added we have the definition

$$\boxed{\text{Pull-Back of } \alpha: \quad \Big(T_p^*\phi \cdot \alpha_{\phi(p)}\Big)(v_{1_p}, v_{2_p}, \ldots, v_{k_p}) \equiv \alpha_{\phi(p)}\Big(T_p\phi \cdot v_{1_p}, T_p\phi \cdot v_{2_p}, \ldots, T_p\phi \cdot v_{k_p}\Big).}$$

Now, to actually see how this works let us consider a few examples. We have worked extensively with volume forms so we now turn our attention to k-forms for arbitrary k. Unfortunately there is no nice formula for finding the pull-backs of arbitrary forms, but there is a fairly standard procedure we can use. Consider the mapping $\phi : \mathbb{R}^2_{xy} \to \mathbb{R}^2_{uv}$ given by $\phi(x, y) = (x + y, x - y)$ and the one-form $\alpha = vdu + udv$ on $\mathbb{R}^2_{uv}$. We want to find $T^*\phi \cdot \alpha$ on $\mathbb{R}^2_{xy}$. Notice, we are now dealing with a one-form, not a volume form.

We will approach this problem in the following way. The pull-back of α, $T^*\phi \cdot \alpha$, will be a one-form on $\mathbb{R}^2_{xy}$ and thus will have the form $f(x, y)dx + g(x, y)dy$ for some, as of yet, unknown functions f and g. It is these functions that we want to find. First we notice that

$$(fdx + gdy)\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = fdx\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + gdy\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = f(1) + g(0) = f$$

and

$$(fdx + gdy)\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = fdx\left(\begin{bmatrix}0\\1\end{bmatrix}\right) + gdy\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = f(0) + g(1) = g.$$

We will use these identities to find out what f and g are. We will also need the identity

$$T\phi = \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}.$$

We now proceed to use the above identities to find f and g as follows,

$$\begin{aligned}
f &= \big(fdx + gdy\big)\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \big(T^*\phi \cdot \alpha\big)\left(\begin{bmatrix}1\\0\end{bmatrix}\right)\\
&= \alpha\left(T\phi \cdot \begin{bmatrix}1\\0\end{bmatrix}\right) = \big(vdu + udv\big)\left(\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\right)\\
&= \big(vdu + udv\big)\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = vdu\left(\begin{bmatrix}1\\1\end{bmatrix}\right) + udv\left(\begin{bmatrix}1\\1\end{bmatrix}\right)\\
&= u + v = (x + y) + (x - y) = 2x
\end{aligned}$$

and

$$\begin{aligned}
g &= \big(fdx + gdy\big)\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \big(T^*\phi \cdot \alpha\big)\left(\begin{bmatrix}0\\1\end{bmatrix}\right)\\
&= \alpha\left(T\phi \cdot \begin{bmatrix}0\\1\end{bmatrix}\right) = \big(vdu + udv\big)\left(\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\right)\\
&= \big(vdu + udv\big)\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = vdu\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) + udv\left(\begin{bmatrix}1\\-1\end{bmatrix}\right)\\
&= u - v = (x + y) - (x - y) = 2y.
\end{aligned}$$

Combining everything we have that

$$T^*\phi \cdot \big(vdu + udv\big) = 2xdx + 2ydy.$$

Notice in this example how we went about finding the exact form of $T^*\phi \cdot \alpha$. Knowing the pull-back was a one-form meant that we knew that it had to be a sum of functions multiplied by the basis elements dx and dy. We also knew that $dx\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = 1$ and $dy\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = 1$ so we could use the push-forwards of these vectors to find the desired functions.

Now Consider the same mapping, $\phi : \mathbb{R}^2_{xy} \to \mathbb{R}^2_{uv}$ given by $\phi(x, y) = (x + y, x - y)$, and the two-form $\alpha = du \wedge dv$. We want to find $T^*\phi \cdot \alpha$. I suspect you'll notice we are back to our area forms so you won't be surprised by the answer - we are just going to get it in a different way. Clearly $T^*\phi \cdot \alpha$ will be a two-form on $\mathbb{R}^2_{xy}$ so it will have the form $f(x, y)dx \wedge dy$ for some function f, since $dx \wedge dy$ is the only basis element of the two-forms on $\mathbb{R}^2$. Next we ask ourselves what two vectors, when "eaten" by $dx \wedge dy$ give 1? Clearly $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ work since

$$dx \wedge dy\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{vmatrix}1 & 0\\0 & 1\end{vmatrix} = 1.$$

Using this we have

$$\begin{aligned}
f &= \big(fdx \wedge dy\big)\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right)\\
&= \Big(T^*\phi \cdot \alpha\Big)\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right)\\
&= \alpha\Big(T\phi \cdot \begin{bmatrix}1\\0\end{bmatrix}, T\phi \cdot \begin{bmatrix}0\\1\end{bmatrix}\Big)\\
&= du \wedge dv\Big(\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\Big)\\
&= du \wedge dv\Big(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\Big)\\
&= \begin{vmatrix}1 & 1\\1 & -1\end{vmatrix}\\
&= -2.
\end{aligned}$$

Putting everything together we get

$$-2dx \wedge dy = T^*\phi \cdot \big(du \wedge dv\big).$$

Question 6.14 Now consider the inverse mapping $\phi^{-1} : \mathbb{R}^2_{uv} \to \mathbb{R}^2_{xy}$ given by $\phi^{-1}(u, v) = \Big(\frac{1}{2}(u + v), \frac{1}{2}(u - v)\Big)$ and suppose $\alpha = dx \wedge dy$. Find $T^*\phi^{-1} \cdot \alpha$ using this method.

Now we will consider a couple more examples of increasing complexity. Suppose $\phi : \mathbb{R}^2_{xy} \longrightarrow \mathbb{R}^3_{uvw}$ is given by $f(x, y) = \big(u(x, y), v(x, y), w(x, y)\big) = (x + y, x - y, xy)$ and consider the one-form $\alpha = vdu + wdv + udw$ on $\mathbb{R}^3$. We want to find $T^*\phi \cdot \alpha$. As before, $T^*\phi \cdot \alpha$ will have the form $fdx + gdy$ for some functions f and g, so as before we will use vectors $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$. Also,

$$T\phi = \begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y}\end{bmatrix} = \begin{bmatrix}1 & 1\\1 & -1\\y & x\end{bmatrix}$$

so

$$
\begin{aligned}
f &= \left(f\,dx + g\,dy\right)\left(\begin{bmatrix}1\\0\end{bmatrix}\right)\\
&= \left(T^*\phi\cdot\alpha\right)\left(\begin{bmatrix}1\\0\end{bmatrix}\right)\\
&= \alpha\left(T\phi\cdot\begin{bmatrix}1\\0\end{bmatrix}\right)\\
&= \left(v\,du + w\,dv + u\,dw\right)\left(\begin{bmatrix}1 & 1\\1 & -1\\y & x\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\right)\\
&= \left(v\,du + w\,dv + u\,dw\right)\left(\begin{bmatrix}1\\1\\y\end{bmatrix}\right)\\
&= v(1) + w(1) + u(y)\\
&= (x-y)(1) + (xy)(1) + (x+y)(y)\\
&= x - y + 2xy + y^2
\end{aligned}
$$

and

$$
\begin{aligned}
g &= \left(f\,dx + g\,dy\right)\left(\begin{bmatrix}0\\1\end{bmatrix}\right)\\
&= \left(T^*\phi\cdot\alpha\right)\left(\begin{bmatrix}0\\1\end{bmatrix}\right)\\
&= \alpha\left(T\phi\cdot\begin{bmatrix}0\\1\end{bmatrix}\right)\\
&= \left(v\,du + w\,dv + u\,dw\right)\left(\begin{bmatrix}1 & 1\\1 & -1\\y & x\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\right)\\
&= \left(v\,du + w\,dv + u\,dw\right)\left(\begin{bmatrix}1\\-1\\x\end{bmatrix}\right)\\
&= v(1) + w(-1) + u(x)\\
&= (x-y)(1) + (xy)(-1) + (x+y)(x)\\
&= x - y + x^2
\end{aligned}
$$

so we have

$$T^*\phi\cdot\left(v\,du + w\,dv + u\,dw\right) = \left(x - y + 2xy + y^2\right)dx + \left(x - y + x^2\right)dy.$$

Next suppose we have the mapping $\phi : \mathbb{R}^3_{abc} \longrightarrow \mathbb{R}^4_{xyzw}$ given by

$$
\begin{aligned}
\phi\left(a,b,c\right) &= \left(x(a,b,c),\, y(a,b,c),\, z(a,b,c),\, w(a,b,c)\right)\\
&= \left(a, b, c, abc\right).
\end{aligned}
$$

If $\omega = x^2 dy \wedge dz + y^2 dz \wedge dw$ is a two-form on $\mathbb{R}$, find $T^*\phi \cdot \omega$. First we find the push-forward map

$$T\phi = \begin{bmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c} \\ \frac{\partial w}{\partial a} & \frac{\partial w}{\partial b} & \frac{\partial w}{\partial c} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ bc & ac & ab \end{bmatrix}$$

and the push-forwards of the $\mathbb{R}^3$ basis vectors,

$$T\phi \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ bc \end{bmatrix}, \quad T\phi \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ ac \end{bmatrix}, \quad T\phi \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ ab \end{bmatrix}.$$

We know all two-forms on $\mathbb{R}^3$ are in $\text{span}\big\{da \wedge db, db \wedge dc, dc \wedge da\big\}$ and so

$$T^*\phi \cdot \omega = f da \wedge db + g db \wedge dc + h dc \wedge da$$

for some functions f, g, h. Also, we know

$$da \wedge db \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$$

$$db \wedge dc \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$$

$$dc \wedge da \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

All other combinations of basis two-forms and vectors equal zero. Following the same strategy as before we have

$$\begin{aligned} f &= \Big(f da \wedge db + g db \wedge dc + h dc \wedge da \Big) \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) \\ &= \Big(T^*\phi \cdot \omega \Big) \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) \\ &= \Big(x^2 dy \wedge dz + y^2 dz \wedge dw \Big) \left(T\Phi \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, T\Phi \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) \\ &= \Big(x^2 dy \wedge dz + y^2 dz \wedge dw \Big) \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ bc \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ ac \end{bmatrix} \right) \end{aligned}$$

$$
\begin{aligned}
&= x^2 dy \wedge dz \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ bc \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ ac \end{bmatrix} \right) + y^2 dz \wedge dw \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ bc \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ ac \end{bmatrix} \right) \\
&= x^2 \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} + y^2 \begin{vmatrix} 0 & 0 \\ bc & ac \end{vmatrix} \\
&= 0.
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
g = \ldots &= x^2 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + y^2 \begin{vmatrix} 0 & 1 \\ ac & ab \end{vmatrix} \\
&= x^2(1) + y^2(-ac) \\
&= a^2 - ab^2 c
\end{aligned}
$$

and

$$
\begin{aligned}
h = \ldots &= x^2 \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} + y^2 \begin{vmatrix} 1 & 0 \\ ab & bc \end{vmatrix} \\
&= x^2(0) + y^2(bc) \\
&= b^3 c.
\end{aligned}
$$

Thus

$$
T^*\phi \cdot \big(x^2 dy \wedge dz + y^2 dz \wedge dw \big) = (a^2 - ab^2 c) db \wedge dc + (b^3 c) dc \wedge da.
$$

Question 6.15 Fill in the ... for the two computations above.

Question 6.16 What is the pull-back of $dx \wedge dy \wedge dz \wedge dw$ under this same map?

Question 6.17 Suppose we have a map $\phi : \mathbb{R}^n_{x_1 \cdots x_n} \longrightarrow \mathbb{R}^m_{\phi_1 \cdots \phi_m}$ where $m > n$, $\phi = \big(\phi_1, \ldots, \phi_m\big)$ and ω is an n-form on $\mathbb{R}^m$. Show that

$$
T_p^*\phi \cdot \omega = \omega_{\phi(p)} \left(\frac{\partial \phi}{\partial x_1}, \ldots, \frac{\partial \phi}{\partial x_n} \right) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n
$$

where

$$
\frac{\partial \phi}{\partial x_i} = \begin{bmatrix} \frac{\partial \phi_1}{\partial x_i} \\ \frac{\partial \phi_2}{\partial x_i} \\ \vdots \\ \frac{\partial \phi_m}{\partial x_i} \end{bmatrix}.
$$

6.7 Some Useful Identities

We have discussed pull-backs in the context of volume forms pertaining to changes of variables, also called changes of coordinates. Suppose we have a change of coordinates $\phi : \mathbb{R}^n_{(x_1,\ldots,x_n)} \to \mathbb{R}^n_{(\phi_1,\ldots,\phi_n)}$ given by

$$
\phi(x_1, \ldots, x_n) = \Big(\phi_1(x_1, \ldots, x_n), \phi_2(x_1, \ldots, x_n), \ldots, \phi_n(x_1, \ldots, x_n) \Big)
$$

with a volume form on $\mathbb{R}^n_{(\phi_1,\dots,\phi_n)}$ given by $d\phi_1 \wedge d\phi_2 \wedge \cdots \wedge d\phi_n$. We found the pull-back of the volume form onto $\mathbb{R}^n_{(x_1,\dots,x_n)}$ by $T^*\phi \cdot \big(d\phi_1 \wedge d\phi_2 \wedge \cdots \wedge d\phi_n\big)$, which is given by the formula

$$T^*\phi \cdot \big(d\phi_1 \wedge d\phi_2 \wedge \cdots \wedge d\phi_n\big) = \begin{vmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} & \cdots & \frac{\partial \phi_1}{\partial x_n} \\ \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} & \cdots & \frac{\partial \phi_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \phi_n}{\partial x_1} & \frac{\partial \phi_n}{\partial x_m} & \cdots & \frac{\partial \phi_n}{\partial x_n} \end{vmatrix} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n,$$

where $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ is the volume form on $\mathbb{R}^n_{(x_1,\dots,x_n)}$. If you review the proof of this identity in Sect. 6.3 you will see that the Jacobian matrix $T\phi$ basically "pulls through" the computation because of the nature of the definition of the wedgeproduct as the determinant of a matrix. This gave us the very useful computational formula above. However, this formula only works for the pull-back of a volume form by a mapping ϕ between two manifolds of the same dimension.

As mentioned in Sect. 6.6 there are not any nice formulas like this for the pull-backs of arbitrary k-forms. However, given an arbitrary k-form there is still a lot we can do with it. There are three very important identities that give the relation between the pull-back and sums of forms, the wedgeproduct, and exterior differentiation. Letting α and β be differential forms and c be a constant, these identities are

$$\boxed{\begin{array}{ll} & \text{pull-back identities} \\ 1. & T^*\phi \cdot \Big(c\alpha + \beta\Big) = cT^*\phi \cdot \alpha + T^*\phi \cdot \beta, \\ 2. & T^*\phi \cdot \Big(\alpha \wedge \beta\Big) = T^*\phi \cdot \alpha \wedge T^*\phi \cdot \beta, \\ 3. & T^*\phi \cdot d\alpha = d\Big(T^*\phi \cdot \alpha\Big). \end{array}}$$

Using more traditional notation, these three identities are usually encountered as

$$\boxed{\begin{array}{ll} & \text{pull-back identities} \\ 1. & \phi^*\Big(c\alpha + \beta\Big) = c\phi^*\alpha + \phi^*\beta, \\ 2. & \phi^*\Big(\alpha \wedge \beta\Big) = \phi^*\alpha \wedge \phi^*\beta, \\ 3. & \phi^*\Big(d\alpha\Big) = d\Big(\phi^*\alpha\Big). \end{array}}$$

You will very often see the third identity written without the input argument α as just the composition of functions $\phi^* \circ d = d \circ \phi^*$ or $\phi^* d = d\phi^*$. Notice that the first identity really says that the pull-back is linear, meaning that we have both

$$\phi^*\big(\alpha + \beta\big) = \phi^*(\alpha) + \phi^*(\beta) \qquad \text{and} \qquad \phi^*(c\alpha) = c\phi^*(\alpha).$$

The proof of the first identity is not difficult,

$$\begin{aligned}
&\phi^*\big(c\alpha + \beta\big)\Big(v_1, \cdots, v_k\Big) \\
&= \big(c\alpha + \beta\big)\Big(\phi_* v_1, \cdots, \phi_* v_k\Big) \\
&= (c\alpha)\Big(\phi_* v_1, \cdots, \phi_* v_k\Big) + \beta\Big(\phi_* v_1, \cdots, \phi_* v_k\Big) \\
&= c\Big(\alpha\Big(\phi_* v_1, \cdots, \phi_* v_k\Big)\Big) + \beta\Big(\phi_* v_1, \cdots, \phi_* v_k\Big) \\
&= c\Big(\phi^*\alpha\Big)\Big(v_1, \cdots, v_k\Big) + \phi^*\beta\Big(v_1, \cdots, v_k\Big) \\
&= \Big(c\phi^*\alpha + \phi^*\beta\Big)\Big(v_1, \cdots, v_k\Big).
\end{aligned}$$

Thus we have proved our first identity.

Now we turn our attention to the second identity, which is quite easy when we use one of the definitions for the wedgeproduct from Sect. 3.3.3,

$$\begin{aligned}
&\big(\phi^*\alpha \wedge \phi^*\beta\big)(v_1, \ldots, v_{k+l}) \\
&= \sum_{\substack{\sigma \text{ is a} \\ (k,\ell)-\text{shuffle}}} \operatorname{sgn}(\sigma)\phi^*\alpha\big(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\big)\phi^*\beta\big(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}\big) \\
&= \sum_{\substack{\sigma \text{ is a} \\ (k,\ell)-\text{shuffle}}} \operatorname{sgn}(\sigma)\alpha\big(\phi_* v_{\sigma(1)}, \ldots, \phi_* v_{\sigma(k)}\big)\beta\big(\phi_* v_{\sigma(k+1)}, \ldots, \phi_* v_{\sigma(k+\ell)}\big) \\
&= \big(\alpha \wedge \beta\big)(\phi_* v_1, \ldots, \phi_* v_{k+\ell}) \\
&= \phi^*\big(\alpha \wedge \beta\big)(v_1, \ldots, v_{k+\ell}),
\end{aligned}$$

which gives us our second identity $\phi^*(\alpha \wedge \beta) = \phi^*\alpha \wedge \phi^*\beta$.

The third identity is a little more tricky. We first prove it for the special case of a zero-form f on $\mathbb{R}^n_{(\phi_1,\ldots,\phi_n)}$. Recall that a zero-form is simply a function, so df is a one-form on $\mathbb{R}^n_{(\phi_1,\ldots,\phi_n)}$. In this situation we have the following maps

$$\begin{gathered}
\bigwedge^0\left(\mathbb{R}^n_{(x_1,\ldots,x_n)}\right) \xleftarrow{\phi^*} \bigwedge^0\left(\mathbb{R}^n_{(\phi_1,\ldots,\phi_n)}\right) \\
T\mathbb{R}^n_{(x_1,\ldots,x_n)} \xrightarrow{\phi_*} T\mathbb{R}^n_{(\phi_1,\ldots,\phi_n)} \\
\mathbb{R}^n_{(x_1,\ldots,x_n)} \xrightarrow{\phi} \mathbb{R}^n_{(\phi_1,\ldots,\phi_n)} \xrightarrow{f} \mathbb{R}.
\end{gathered}$$

The way push-forwards of vectors and pull-back of the one-form df work in this situation is a little bit special due to the nature of the zero-form f. Let us first consider the case of $\phi_* v$, the push-forward of a vector $v \in T\mathbb{R}^n_{(x_1,\ldots,x_n)}$ by ϕ. We know $\phi_* v$ is a vector in $T\mathbb{R}^n_{(\phi_1,\ldots,\phi_n)}$ and as such it can be used to find the directional derivative of f, $\phi_* v[f]$. But this is exactly the same as finding the directional derivative of the function $f \circ \phi$ by the vector v, which means we have $v[f \circ \phi] = \phi_* v[f]$.

Question 6.18 Show $v[f \circ \phi] = \phi_* v[f]$ using an explicit calculation using the notation and formulas we have developed in this chapter.

Let us now consider the pull-back by ϕ of the zero-form $f \in \bigwedge^0\left(\mathbb{R}^n_{(\phi_1,\ldots,\phi_n)}\right)$. Our previous definition of the pull-back of a k-form can no longer work since zero-forms do not eat vectors, so we will define the pull-back of a zero-form, that is, a functions, by the only way that makes sense. Pulling the function f back to the manifold $\mathbb{R}^n_{(x_1,\ldots,x_n)}$ is defined to be $f \circ \phi$, which means we have $\phi^* f = f \circ \phi$,

$$\begin{aligned}
\phi^* df(v) &= df\big(\phi_* v\big) && \text{definition of pull-back} \\
&= \phi_* v[f] && \text{definition of differential} \\
&= v[f \circ \phi] && \text{question 6.18} \\
&= v[\phi^* f] && \text{definition of pull-back of zero-form} \\
&= d(\phi^* f)(v) && \text{definition of differential.}
\end{aligned}$$

Writing without the vector v we have, for a zero-form f, $\phi^* df = d\phi^* f$, thereby proving the third identity in the special case of a zero-form.

Now we prove the third identity in full generality. Let $\alpha = \sum_I a_I dx^I = \sum a_I d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k}$, where $i_1 < \cdots < i_k$. Noting that the ϕ_i are nothing more than zero-forms we have $\phi^*(d\phi_i) = d(\phi^*\phi_i)$,

$$\begin{aligned}
\phi^*\alpha &= \sum (\phi^* a_I)\ \phi^* d\phi_{i_1} \wedge \ldots \wedge \phi^* d\phi_{i_k} \\
&= \sum (\phi^* a_I)\ d(\phi^*\phi_{i_1}) \wedge \ldots \wedge d(\phi^*\phi_{i_k})
\end{aligned}$$

which leads to

$$\begin{aligned} d(\phi^*\alpha) &= d\left(\sum (\phi^* a_I)\, d(\phi^*\phi_{i_1}) \wedge \ldots \wedge d(\phi^*\phi_{i_k})\right) \\ &= \sum d(\phi^* a_I) \wedge d(\phi^*\phi_{i_1}) \wedge \ldots \wedge d(\phi^*\phi_{i_k}) \\ &= \sum \phi^* da_I \wedge \phi^* d\phi_{i_1} \wedge \ldots \wedge \phi^* d\phi_{i_k} \\ &= \phi^*\left(\sum da_I \wedge d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k}\right) \\ &= \phi^*\left(d \sum a_I\, d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k}\right) \\ &= \phi^*(d\alpha). \end{aligned}$$

Thus we have our third identity, $d\phi^*\alpha = \phi^* d\alpha$.

6.8 Summary, References, and Problems

6.8.1 Summary

Consider a map between two manifolds, $f : M \to N$ where $f = (f_1, f_2, \ldots, f_n)$ and each component function f_i is differentiable at a point $p \in M$. If the coordinates of manifold M are given by $x_1, x_2, \ldots, x_n$, then the derivative of the map f at the point $p \in M$ is given by the traditional Jacobian matrix

$$T_p f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}\Big|_p & \frac{\partial f_1}{\partial x_2}\Big|_p & \cdots & \frac{\partial f_1}{\partial x_n}\Big|_p \\ \frac{\partial f_2}{\partial x_1}\Big|_p & \frac{\partial f_2}{\partial x_2}\Big|_p & \cdots & \frac{\partial f_2}{\partial x_n}\Big|_p \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}\Big|_p & \frac{\partial f_n}{\partial x_2}\Big|_p & \cdots & \frac{\partial f_n}{\partial x_n}\Big|_p \end{bmatrix}.$$

Suppose we have some vector $v_p \in T_p M$. $T_p f \cdot v_p$ gives the linear approximation of the change in f as we move along v_p, which is another vector in $T_{f(p)}N$. Since we can find $T_p f : T_p M \to T_{f(p)}N$ at each point $p \in M$ where f is differentiable we end up with a mapping Tf from the tangent bundle of M to the tangent bundle of N. We say that the map $f : M \to N$ induces the map $Tf : TM \longrightarrow TN$. The mapping Tf pushes-forward vectors in $T_p M$ to vectors in $T_{f(p)}N$ and so is called the push-forward mapping. The push-forward map is also sometimes called the tangent mapping and is also denoted by f_* or Df.

Now suppose that we have a differential k-form ω on N. We define the differential k-form $T^* f \cdot \omega$ as the pull-back of ω to M where

Definition of Pull-Back of Differential Form	$\big(T^* f \cdot \omega\big)\big(v_1, v_2, \ldots, v_k\big) = \omega\big(Tf \cdot v_1, Tf \cdot v_2, \ldots, Tf \cdot v_k\big).$

The pull-back map is also sometimes called the cotangent map and is often denoted by f^*. Notice that when it comes to the pull-back map, it is indexed by the base point in the image and not the domain. This is quite unusual in mathematics, but is simply being the best way to keep track of base points,

$$\begin{gathered} {\textstyle\bigwedge}_p^k(M) \xleftarrow{T_p^* f} {\textstyle\bigwedge}_{f(p)}^k(N) \\ T_p M \xrightarrow{T_p f} T_{f(p)} N \\ M \xrightarrow{f} N. \end{gathered}$$

In general there is no single formula that one can use to find the pull-back of a differential form. However, in the case of the pull-back of a volume form using a change in basis given by $\phi : \mathbb{R}^n_{(x_1,\dots,x_n)} \to \mathbb{R}^n_{(\phi_1,\dots,\phi_n)}$ there is a single formula,

Pull-back of Volume Form	$$T^*\phi \cdot \Big(d\phi_1 \wedge d\phi_2 \wedge \cdots \wedge d\phi_n\Big) = \begin{vmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} & \cdots & \frac{\partial \phi_1}{\partial x_n} \\ \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} & \cdots & \frac{\partial \phi_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \phi_n}{\partial x_1} & \frac{\partial \phi_n}{\partial x_2} & \cdots & \frac{\partial \phi_n}{\partial x_n} \end{vmatrix} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n.$$

Finally, some useful pull-back identities are

pull-back identities
1. $T^*\phi \cdot \Big(c\alpha + \beta\Big) = cT^*\phi \cdot \alpha + T^*\phi \cdot \beta,$
2. $T^*\phi \cdot \Big(\alpha \wedge \beta\Big) = T^*\phi \cdot \alpha \wedge T^*\phi \cdot \beta,$
3. $T^*\phi \cdot d\alpha = d\Big(T^*\phi \cdot \alpha\Big).$

6.8.2 *References and Further Reading*

Push-forwards of vector fields and pull-backs of forms are covered in almost any introductory book on manifold theory or differential geometry, see for example Bachman [4], O'Neill [36], Renteln [37], Darling [12], Tu [46], Hubbard and Hubbard [27], or Edwards [18]. Indeed, all of these sources were used. However, we have attempted to present the material from a very down-to-earth point of view via a number of concrete situations. In the process we attempt to make clear the relationship between both push-forwards and pull-backs and material and formulas encountered in vector calculus books such as Stewart [43] or Marsden and Hoffman [31].

6.8.3 *Problems*

Question 6.19 Let the mapping $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $\varphi(u,v) = \Big(x(u,v), y(u,v), z(u,v)\Big) = \Big(u^2+v, 2u-v, u+v^3\Big)$. Find $T_p\varphi \cdot v_p$ for the following vectors,

a) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}_{(2,1)}$, *b)* $\begin{bmatrix} -4 \\ 5 \end{bmatrix}_{(3,-2)}$, *c)* $\begin{bmatrix} 7 \\ -3 \end{bmatrix}_{(-3,-1)}$, *d)* $\begin{bmatrix} -3 \\ -7 \end{bmatrix}_{(-4,8)}$, *e)* $\begin{bmatrix} 3 \\ 1 \end{bmatrix}_{(3,-2)}$.

Question 6.20 Let the mapping $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\varphi(u,v) = \Big(x(u,v), y(u,v)\Big) = \big(u^2+1, uv\big)$. Find $T_p\varphi \cdot v_p$ for the vectors from Question 6.19.

Question 6.21 Let the mapping $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\varphi(u,v) = \Big(x(u,v), y(u,v)\Big) = \big(u^2v, u-v\big)$. Find $T_p\varphi \cdot v_p$ for the vectors from Question 6.19.

Question 6.22 Let the mapping $\varphi : \mathbb{R}^2_{xy} \to \mathbb{R}^2_{uv}$ be defined by $\varphi(x,y) = \Big(u(x,y), v(x,y)\Big) = \big(x-y, xy\big)$. Find the pull-back of the volume form $du \wedge dv$. That is, find $T^*\varphi \cdot \big(du \wedge dv\big)$.

Question 6.23 Let the mapping $\varphi : \mathbb{R}^2_{xy} \to \mathbb{R}^2_{uv}$ be defined by $\varphi(x,y) = \Big(u(x,y), v(x,y)\Big) = \big(x+2y, x-2y\big)$. Find the pull-back of the volume form $du \wedge dv$. That is, find $T^*\varphi \cdot \big(du \wedge dv\big)$.

Question 6.24 Let the mapping $\varphi : \mathbb{R}^2_{uv} \to \mathbb{R}^2_{x,y}$ be defined by $\varphi(u,v) = \big(x(u,v), y(u,v)\big) = \big(u^2 - v^2, 2uv\big)$. Find the pull-back of the volume form $dx \wedge dy$. That is, find $T^*\varphi \cdot \big(dx \wedge dy\big)$.

Question 6.25 Let the mapping $\varphi : \mathbb{R}^2_{uv} \to \mathbb{R}^2_{x,y}$ be defined by $\varphi(u,v) = \big(x(u,v), y(u,v)\big) = \big(u^2 - v^2, u + v + 5\big)$. Find the pull-back of the volume form $dx \wedge dy$. That is, find $T^*\varphi \cdot \big(dx \wedge dy\big)$.

Question 6.26 Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $f(u,v) = \big(u^2 + v, 2u - v, u + v^3\big)$ and let $\alpha = (x + y + 1)\, dx + (3z - y)\, dy + x\, dz$ be a one-form on $\mathbb{R}^3$. Find the general formula for $T^* f \cdot \alpha$.

Question 6.27 Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $f(u,v) = \big(u^2 + v, 2u - v, u + v^3\big)$ and let $\alpha = -dx \wedge dy + z\, dy \wedge dz + (y + 2)\, dz \wedge dx$ be a one-form on $\mathbb{R}^3$. Find the general formula for $T^* f \cdot \alpha$.

Question 6.28 Let $f_1 = y^2 - x^2$ and $f_2 = 3xy$ be functions on $\mathbb{R}^2$ and let $\alpha_1 = x\, dx + xy\, dy$, $\alpha_2 = -3y\, dx + 2x\, dy$, and $\alpha_3 = (x^2 + y^2)\, dy$ be one-forms on $\mathbb{R}^2$. Given the mapping $\varphi : \mathbb{R} \to \mathbb{R}^2$ defined by $\varphi(t) = \big(x(t), y(t)\big) = \big(t, t^2\big)$ find

a) $T^*\varphi \cdot f_1$, *b)* $T^*\varphi \cdot f_2$, *c)* $T^*\varphi \cdot \alpha_1$, *d)* $T^*\varphi \cdot \alpha_2$, *e)* $T^*\varphi \cdot \alpha_3$.

Question 6.29 For the mapping and functions in Question 6.28 show that

a) $T^*\varphi \cdot \big(df_1\big) = d\big(T^*\varphi \cdot f_1\big)$, *b)* $T^*\varphi \cdot \big(df_2\big) = d\big(T^*\varphi \cdot f_2\big)$.

Question 6.30 Let $f_1 = y^2 - x^2$ and $f_2 = 3xy$ be functions on $\mathbb{R}^2$, $\alpha_1 = x\, dx + xy\, dy$, $\alpha_2 = -3y\, dx + 2x\, dy$, and $\alpha_3 = (x^2 + y^2)\, dy$ be one-forms on $\mathbb{R}^2$, and $\beta_1 = (xy - x)\, dx \wedge dy$ and $\beta_2 = (x + y^2)\, dx \wedge dy$ be two-forms on $\mathbb{R}^2$. Given the mapping $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\varphi(u,v) = \big(x(u,v), y(u,v)\big) = \big(u^2 + 1, uv\big)$ find

a) $T^*\varphi \cdot f_1$, *b)* $T^*\varphi \cdot f_2$, *c)* $T^*\varphi \cdot \alpha_1$, *d)* $T^*\varphi \cdot \alpha_2$, *e)* $T^*\varphi \cdot \alpha_3$, *f)* $T^*\varphi \cdot \beta_1$, *g)* $T^*\varphi \cdot \beta_2$.

Question 6.31 For the mapping, functions, and one-forms in Question 6.30 show that

a) $T^*\varphi \cdot \big(df_1\big) = d\big(T^*\varphi \cdot f_1\big)$, *b)* $T^*\varphi \cdot \big(df_2\big) = d\big(T^*\varphi \cdot f_2\big)$, *c)* $T^*\varphi \cdot \big(d\alpha_1\big) = d\big(T^*\varphi \cdot \alpha_1\big)$, *d)* $T^*\varphi \cdot \big(d\alpha_2\big) = d\big(T^*\varphi \cdot \alpha_2\big)$, *e)* $T^*\varphi \cdot \big(d\alpha_3\big) = d\big(T^*\varphi \cdot \alpha_3\big)$.

Question 6.32 For the mapping and one-forms in Question 6.30 show that

a) $T^*\varphi \cdot \big(\alpha_1 \wedge \alpha_2\big) = T^*\varphi \cdot \alpha_1 \wedge T^*\varphi \cdot \alpha_2$, *b)* $T^*\varphi \cdot \big(\alpha_1 \wedge \alpha_3\big) = T^*\varphi \cdot \alpha_1 \wedge T^*\varphi \cdot \alpha_3$, *c)* $T^*\varphi \cdot \big(\alpha_2 \wedge \alpha_3\big) = T^*\varphi \cdot \alpha_2 \wedge T^*\varphi \cdot \alpha_3$.

Question 6.33 Repeat Question 6.30 for the mapping $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\varphi(u,v) = \big(u^2 v, u - v\big)$.

Question 6.34 Repeat Question 6.31 for the mapping $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\varphi(u,v) = \big(u^2 v, u - v\big)$.

Question 6.35 Repeat Question 6.32 for the mapping $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\varphi(u,v) = \big(u^2 v, u - v\big)$.

Chapter 7
Changes of Variables and Integration of Forms

Integration is one of the most a fundamental concepts in mathematics. In calculus you began by learning how to integrate one-variable functions on $\mathbb{R}$. Then, you learned how to integrate two- and three-variable functions on $\mathbb{R}^2$ and $\mathbb{R}^3$. After this you learned how to integrate a function after a change-of-variables, and finally in vector calculus you learned how to integrate vector fields along curves and over surfaces. It turns out that differential forms are actually very nice things to integrate. Indeed, there is an intimate relationship between the integration of differential forms and the change-of-variables formulas you learned in calculus. In section one we define the integral of a two-form on $\mathbb{R}^2$ in terms of Riemann sums. Integrals of n-forms on $\mathbb{R}^n$ can be defined analogously. We then use the ideas from Chap. 6 along with the Riemann sum procedure to derive the change of coordinates formula from first principles. In section two we look carefully at a simple change of coordinates example. Section three continues by looking at changes from Cartesian coordinates to polar, cylindrical, and spherical coordinates. Finally in section four we consider a more general setting where we see how we can integrate arbitrary one- and two-forms on parameterized one- and two-dimensional surfaces.

7.1 Integration of Differential Forms

We begin by looking at the fundamental definition of integration in terms of Riemann sums. We will start out by finding the volume under the graph of a function, basically what you have seen in calculus. For now we assume traditional Cartesian coordinates on the Euclidian manifold $\mathbb{R}^2$. Let R be a region on the plane and let $f(x, y)$ be the function whose graph we are finding the volume under. From calculus class you should recall the following steps:

(1) Choose a lattice of evenly spaced points $\{x_i\}_{i=1}^n$ and $\{y_j\}_{j=1}^m$ that covers the region R. This lattice gives the points $(x_i, y_j) \in \mathbb{R}^2_{xy}$.
(2) Let $\Delta x_i = x_{i+1} - x_i$ and $\Delta y_j = y_{j+1} - y_j$.
(3) For each $i = 1, \ldots, n$ and $j = 1, \ldots, m$ find $f(x_i, y_j)\Delta x_i \Delta y_j$.
(4) Sum over i and j; $\sum_i \sum_j f(x_i, y_j)\Delta x_i \Delta y_j$.
(5) Take the limit of this double sum as $n, m \to \infty$, that is, as $\Delta x_i, \Delta y_j \to 0$, and define this to be $\int\int_R f(x, y)\, dx\, dy$.

In essence, we are approximating the volume under the graph of $f(x, y)$ and above the rectangle $\Delta x_i \Delta y_j$, Fig. 7.1, by the product $f(x_i, y_j)\Delta x_i \Delta y_j$, Fig. 7.2. Of course, this is only the theoretical definition, this is not how we actually compute integrals, we have a whole slew of techniques at our disposal for integration which you learned in calculus.

Now, we are essentially going to copy both the idea and the procedure in a way that is more appropriate for the differential forms setting, thereby giving a theoretical definition for the integral of a differential form $\alpha = f(x, y)dx \wedge dy$ on a region $R \subset \mathbb{R}^2$. However, we caution that the exact procedure we are giving here implicitly relies on the fact that we are still in the Euclidian manifold setting $\mathbb{R}^2$. A more general definition for $\mathbb{R}^n$ is certainly possible, but it does require greater attention to notation, which would obscure the fundamental ideas.

(1) Choose a lattice of evenly spaced points $\{x_i\}_{i=1}^n$ and $\{y_j\}_{j=1}^m$ that covers the region R. This lattice gives the points $(x_i, y_j) \in \mathbb{R}^2_{xy}$.
(2) For each i, j define the vectors

J. P. Fortney, *A Visual Introduction to Differential Forms and Calculus on Manifolds*,
https://doi.org/10.1007/978-3-319-96992-3_7

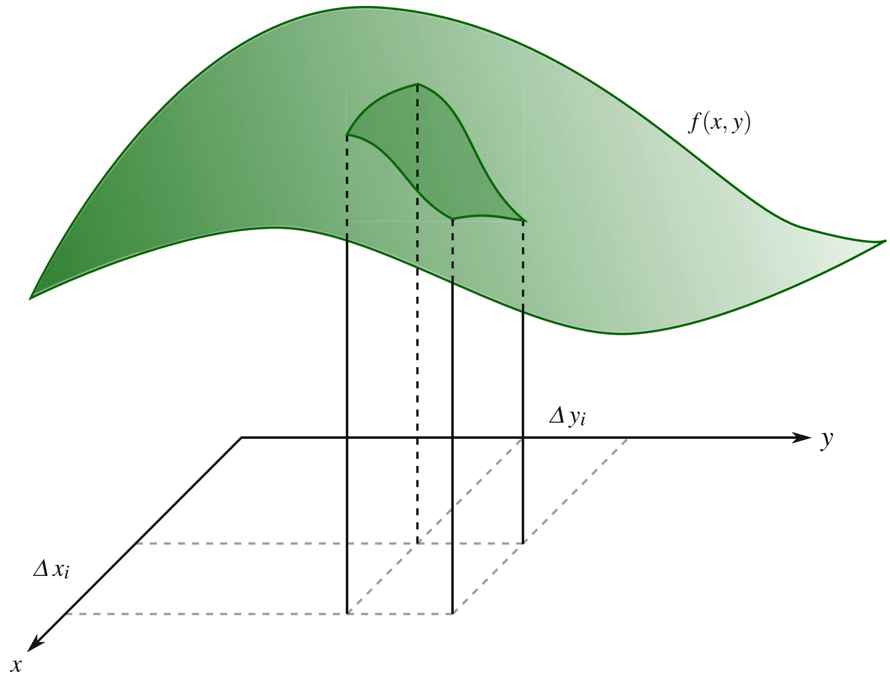

Fig. 7.1 The volume under the graph of $f(x, y)$ and above the rectangle $\Delta x_i \Delta y_j$

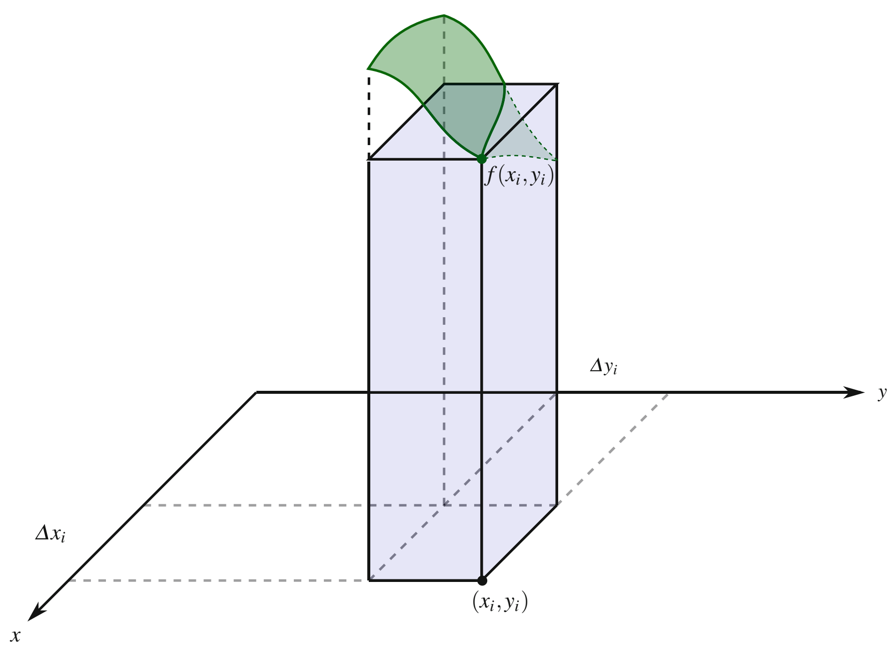

Fig. 7.2 The approximate volume under the graph of $f(x, y)$ and above the rectangle $\Delta x_i \Delta y_j$ given by the product $f(x_i, y_j)\Delta x_i \Delta y_j$, shown in blue

$$V_{i,j}^1 = \begin{bmatrix} x_{i+1} - x_i \\ 0 \end{bmatrix}_{(x_i, y_j)},$$

$$V_{i,j}^2 = \begin{bmatrix} 0 \\ y_{j+1} - y_j \end{bmatrix}_{(x_i, y_j)}.$$

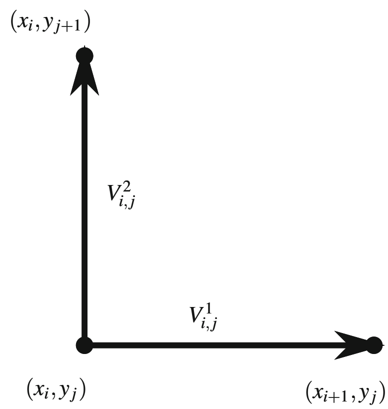

Fig. 7.3 The vectors $V_{i,j}^1$ and $V_{i,j}^2$. Despite the way they were defined we can consider $V_{i,j}^1, V_{i,j}^2 \in T_{(x_i,y_j)}\mathbb{R}^2$

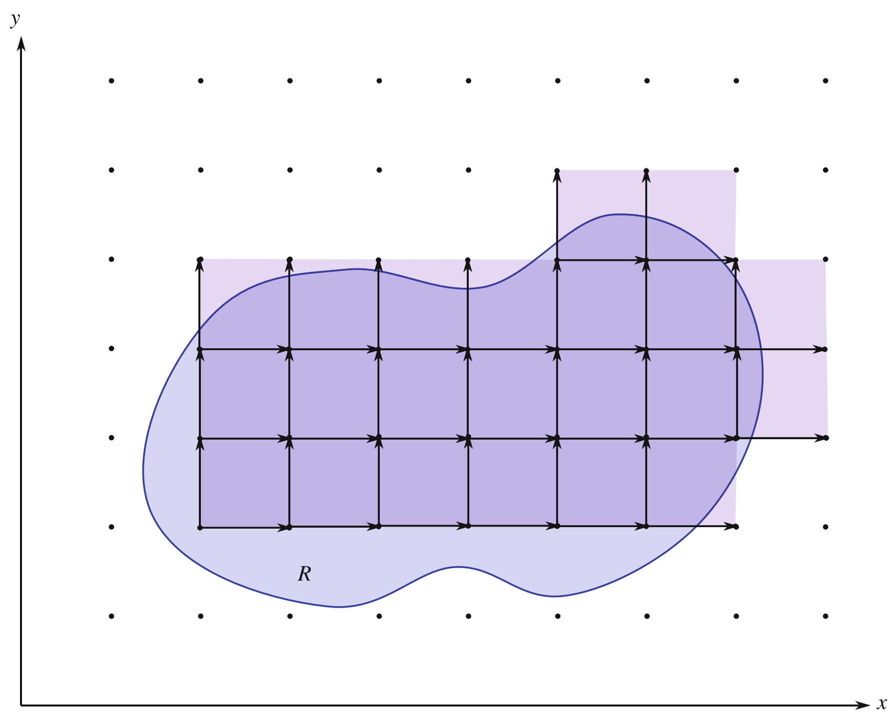

Fig. 7.4 A region $R \subset \mathbb{R}^2$ shown with the lattice of evenly spaced points $\{x_i\}_{i=1}^n$ and $\{y_j\}_{j=1}^m$ along with the vectors $V_{i,j}^1$ and $V_{i,j}^2$ for every point in the region R. As the lattice points get closer together, that is, as $|x_{i+1} - x_i|, |y_{j+1} - y_j| \to 0$, the region R is more accurately approximated

Note that we can view the vectors $V_{i,j}^1$ and $V_{i,j}^2$ as elements of the tangent space $T_{(x_i,y_j)}\mathbb{R}^2$. See Fig. 7.3. This is where we are implicitly assuming that we are in the Euclidian setting and can make this identification. Thus, we have something like Fig. 7.4.

(3) For each i, j compute $f(x_i, y_j)dx \wedge dy(V_{i,j}^1, V_{i,j}^2)$.

(4) Sum over i and j; $\sum_i \sum_j f(x_i, y_j)dx \wedge dy(V_{i,j}^1, V_{i,j}^2)$.

(5) Take the limit of this double sum as $m, n \to \infty$, that is, as $|x_{i+1} - x_i|, |y_{j+1} - y_j| \to 0$, and define this to be $\int\int_R f dx \wedge dy$. That is,

$$\int\int_R f dx \wedge dy \equiv \lim_{m,n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j)dx \wedge dy(V_{i,j}^1, V_{i,j}^2).$$