Chapter 2
An Introduction to Differential Forms

In this chapter we introduce one of the fundamental ideas of this book, the differential one-form. Later chapters will discuss differential k-forms for $k > 1$. We slowly and systematically build up to the concept of a one-forms by first considering a variety of necessary ingredients. We begin in section one by introducing the Cartesian coordinate functions, which play a surprisingly important role in understanding exactly what a one-form is, as well as playing a role in the notation used for one-forms.

In section two we discuss manifolds, tangent spaces, and vector fields in enough detail to give you a good intuitive idea of what they are, leaving a more mathematically rigorous treatment for Chap. 10. In section three we return to the concept of directional derivatives that you are familiar with from vector calculus. The way directional derivatives are usually defined in vector calculus needs to be tweaked just a little for our purposes. Once this is done a rather surprising identification between the standard Euclidian unit vectors and differential operators can be made. This equivalence plays a central role going forward.

Finally in section four differential one-forms are defined. Thoroughly understanding differential one-forms requires pulling together a variety of concepts; vector spaces and dual spaces, the equivalence between Euclidian vectors and differential operators, directional derivatives and the Cartesian coordinate functions. The convergence of all of these mathematical ideas in the definition of the one-form is the goal of the chapter.

2.1 Coordinate Functions

Now that we have reviewed the necessary background we are ready to start getting into the real meat of the course, differential forms. For the first part of this course, until we understand the basics, we will deal with differential forms on $\mathbb{R}^n$, with $\mathbb{R}^2$ and $\mathbb{R}^3$ being our primary examples. Only after we have developed some intuitive feeling for differential forms will we address differential forms on more general manifolds. You can very generally think of a manifold as a space which is locally Euclidian - that means that if you look closely enough at one small part of a manifold then it basically looks like $\mathbb{R}^n$ for some n.

For the time being we will just work with our familiar old Cartesian coordinate system. We will discuss other coordinate systems later on. We are used to seeing $\mathbb{R}^2$ pictured as a plane with an x and a y axis as in Fig. 2.1. Here we have plotted the points $(2, 1)$, $(-4, 3)$, $(-2, -2)$, and $(1, -4)$ on the xy-plane. Similarly, we are used to seeing $\mathbb{R}^3$ pictured with an x, y, and z axis as show in Fig. 2.2. Here we have plotted the points $(2, 3, 4)$, $(-3, -4, 2)$, and $(4, 1, -3)$. Also, notice that we follow the "right-hand rule" convention in this book when drawing $\mathbb{R}^3$.

Notice the difference in the notations between points in $\mathbb{R}^n$ and vectors in $\mathbb{R}^n$. For example, points in $\mathbb{R}^2$ or $\mathbb{R}^3$ are denoted by $p = (x, y)$ or $p = (x, y, z)$, where $x, y, z \in \mathbb{R}$, whereas vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ are denoted by

$$v = \begin{bmatrix} x \\ y \end{bmatrix} \text{ or } v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

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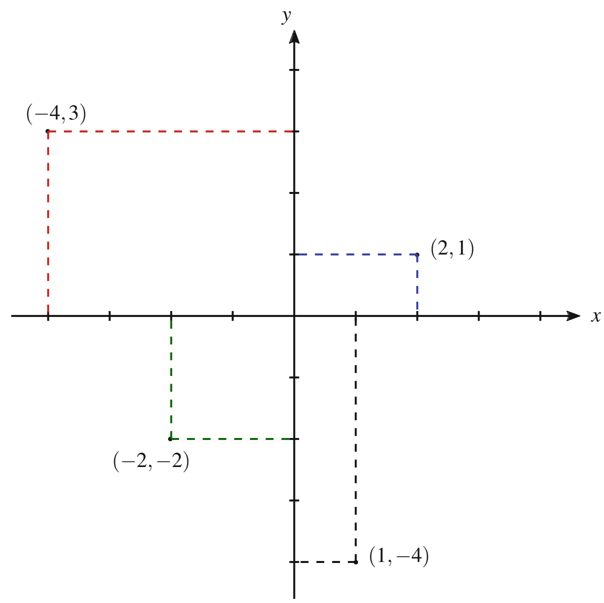

Fig. 2.1 The Cartesian coordinate system for $\mathbb{R}^2$ with the points (2, 1), (−4, 3), (−2, −2), and (1, −4) shown

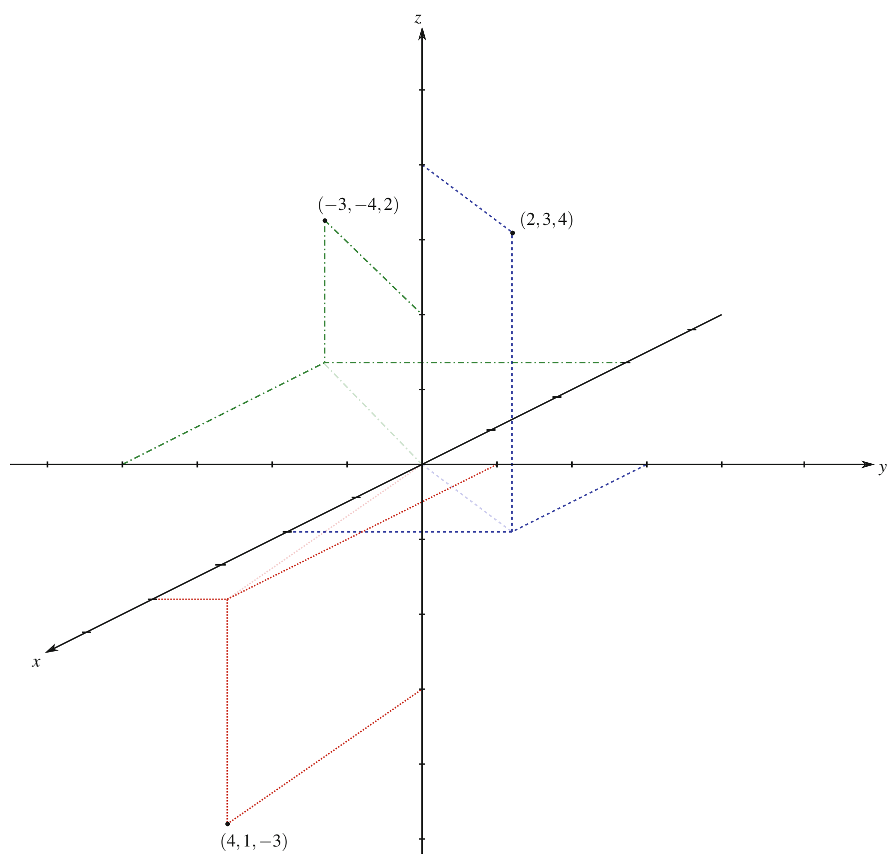

Fig. 2.2 The Cartesian coordinate system for $\mathbb{R}^3$ with the points (2, 3, 4), (−3, −4, 2), and (4, 1, −3) shown

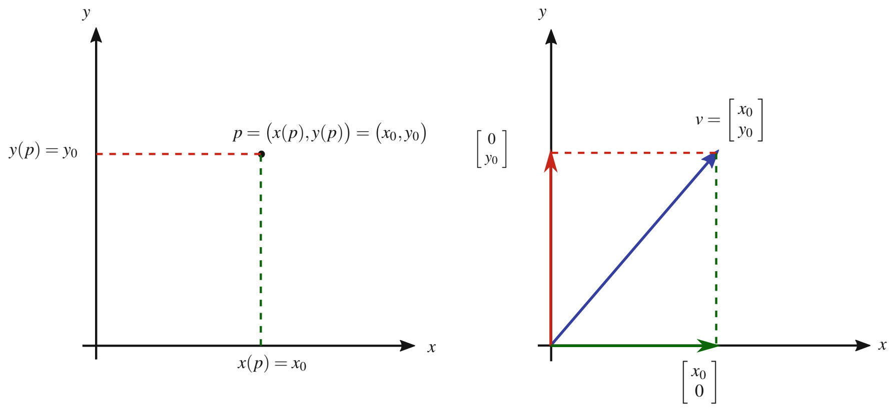

Fig. 2.3 The manifold $\mathbb{R}^2$ (left) and the vector space $\mathbb{R}^2$ (right)

where $x, y, z \in \mathbb{R}$. Points and vectors in $\mathbb{R}^n$ for arbitrary n are denoted similarly. Here comes a subtle distinction that is almost always glossed over in calculus classes, especially in multivariable or vector calculus, yet it is a distinction that is very important.

- We will call the collection of points $p = (x_1, x_2, \ldots, x_n)$, with $x_1, x_2, \ldots, x_n \in \mathbb{R}$, the *manifold* $\mathbb{R}^n$.
- We will call the collection of vectors $v = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, where $x_1, x_2, \ldots, x_n \in \mathbb{R}$, the *vector space* $\mathbb{R}^n$.

One of the reasons this distinction is glossed over in calculus classes is that most calculus students have not yet taken a course in linear algebra and do not yet know what a vector space is. Also, it is certainly possible to "do" vector calculus without knowing the technical definition of a vector space by being a little vague and imprecise mathematically speaking. The manifold $\mathbb{R}^2$ and the vector space $\mathbb{R}^2$ look exactly alike yet are totally different spaces. They have been pictured in Fig. 2.3. The manifold $\mathbb{R}^2$ contains points and the vector space $\mathbb{R}^2$ contains vectors. Of course, in the case of $\mathbb{R}^2$, $\mathbb{R}^3$, or $\mathbb{R}^n$ for that matter, these spaces can be naturally identified. That is, they are really the same. So why is making the distinction so important? It is important to make the distinction for a couple of reasons. First, keeping clear track of our spaces will help us understand the theory of differential forms better, and second, when we eventually get to more general manifolds, most manifolds will not have a vector space structure so we do not want to get into the bad habit of thinking of our manifolds as vector spaces too.

Now let us spend a moment reviewing good old-fashioned functions on $\mathbb{R}^2$ and $\mathbb{R}^3$. We will consider real-valued functions, which simply means that the range (or codomain) is the reals $\mathbb{R}$ or some subset of the reals. Real-valued functions from $\mathbb{R}^2$ are the easiest to visualize since they can be graphed in three dimensional space. Consider the real-valued functions $f : \mathbb{R}^2 \to \mathbb{R}$ defined by the following $f(x, y) = x^3 + y^2 - 3xy$. For example, $f(2, 3)$ is given by $(2)^3 + (3)^2 - 3(2)(3) = -1$, $f(5, -2)$ is given by $(5)^3 + (-2)^2 - 3(5)(-2) = 159$ and $f(-3, -1)$ is given by $(-3)^3 + (-1)^2 - 3(-3)(-1) = -35$. A function from $\mathbb{R}^2$ to $\mathbb{R}$ is graphed in $\mathbb{R}^3$ by the set of points $(x, y, f(x, y))$. The function $f(x, y) = x^3 + y^2 - 3xy$ is shown in Fig. 2.4. Sometimes we will write $f(p)$ where $p = (x, y)$. By doing this we are emphasizing that the input (x, y) is a point in the domain.

As a further illustration, the graphs of the functions $f(x, y) = xy$ and $f(x, y) = x^2 + y^2$ are shown in Fig. 2.5. As before, the set of points that are actually graphed are the points $(x, y, f(x, y))$ in $\mathbb{R}^3$, thus we can see that it requires three dimensions to draw the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$. That means that we can not accurately draw the graph of any function $f : \mathbb{R}^n \to \mathbb{R}$ for $n \geq 3$. However, we often continue to draw inaccurate cartoons similar to Figs. 2.4 and 2.5 when we want to try to visualize or depict functions $f : \mathbb{R}^n \to \mathbb{R}$ where $n \geq 3$.

We will use different name for "functions" on the manifold $\mathbb{R}^n$ and "functions" on the vector space $\mathbb{R}^n$. Functions on the manifold $\mathbb{R}^n$ will simply be called **functions**, or sometimes, if we want to emphasize that range is the set of real numbers, **real-valued functions**. However, our "functions" on the vector space $\mathbb{R}^n$ are called **functionals** when the range is the set of real values and **transformations** when the range is another vector space. We have already spent some time considering a

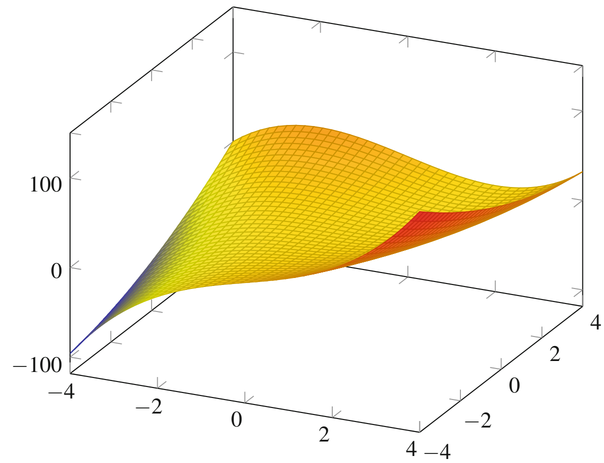

Fig. 2.4 The graph of function $f(x, y) = x^3 + y^2 - 3xy$ is displayed

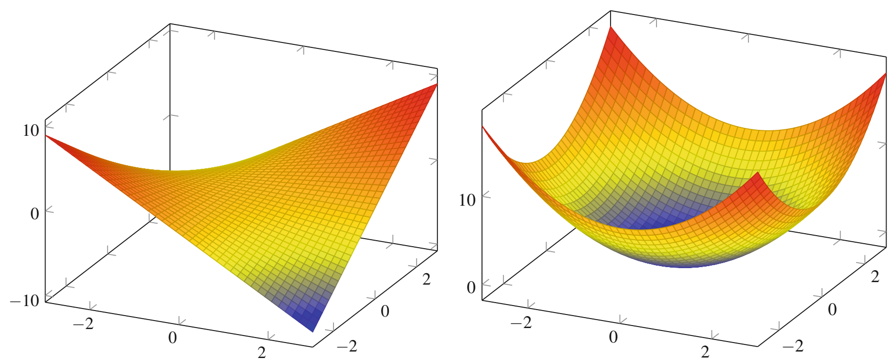

Fig. 2.5 The graphs of the function $f(x, y) = xy$ (left) and $f(x, y) = x^2 + y^2$ (right) as an illustration of the graphs of functions in two variables depicted in $\mathbb{R}^3$

certain class of functionals, the linear functionals that satisfy

$$f(v + w) = f(v) + f(w)$$
$$f(cv) = cf(v)$$

where v and w are vectors and c is a real number. So, the difference between manifolds $\mathbb{R}^n$ and vector spaces $\mathbb{R}^n$ is implicit in our language:

$$\text{function } f : (\text{manifold})\ \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$\text{functional } f : (\text{vector space})\ \mathbb{R}^n \longrightarrow \mathbb{R}.$$

Now, we will consider a very special function from $\mathbb{R}^n$ to $\mathbb{R}$. We will start by using $\mathbb{R}^2$ as an example. The special functions we want to look at are called the **coordinate functions**. For the moment we will look at the **Cartesian coordinate functions**.

Consider a point p in the manifold $\mathbb{R}^2$ given by the Cartesian coordinates $(3, 5)$. Then the coordinate function

$$x : \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$p \longmapsto x(p)$$

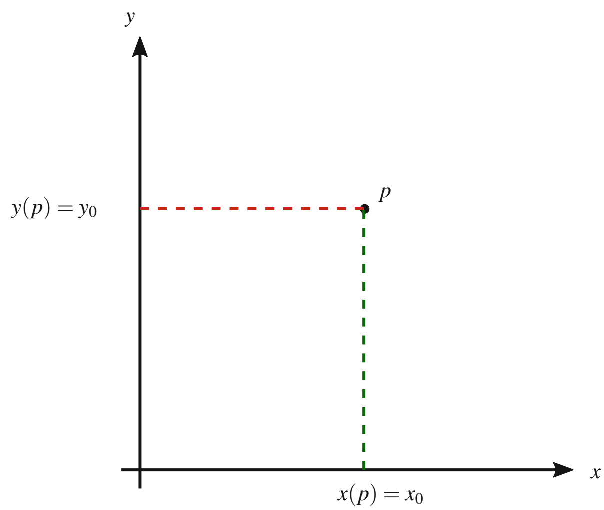

Fig. 2.6 The coordinate functions $x : \mathbb{R}^2 \to \mathbb{R}$ and $y : \mathbb{R}^2 \to \mathbb{R}$ picking off the coordinate values x_0 and y_0, respectively, of the point $p \in \mathbb{R}^2$

picks off the first coordinate, that is, $x(p) = x\big((3, 5)\big) = 3$, and the coordinate function

$$\begin{aligned} y : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ p &\longmapsto y(p) \end{aligned}$$

picks off the second coordinate, $y(p) = y\big((3, 5)\big) = 5$. This is illustrated in Fig. 2.6.

The fact of the matter is, even though you are used to thinking of (3, 5) as a point, a point is actually more abstract than that. A point exists independent of its coordinates. For example, you will sometimes see something like this,

$$p = \big(x(p), y(p)\big)$$

where, as an example, for the point we were given above we have $x(p) = 3$ and $y(p) = 5$.

There is some ambiguity that we need to get used to here. Often x and y are used to represent the outputs of the coordinate functions x and y. So, we could end up with an ambiguous statement like this

$$\begin{aligned} x(p) &= x \\ y(p) &= y \end{aligned}$$

where the x and the y on the left hand sides of the equations are coordinate functions from $\mathbb{R}^2$ to $\mathbb{R}$, and the x and y on the right hand side are real numbers, that is, elements of $\mathbb{R}$. Learning to recognize when x and y are coordinate functions and when they are real values requires a little practice, you have to look carefully at how they are being used. This distinction may sometimes be pointed out, but it often is glossed over.

Even though points are abstract and independent of coordinates, we can not actually talk about a particular point without giving it an "address." This is actually what coordinate functions do, they are used to give a point an "address."

Coordinate functions on $\mathbb{R}^3$ (and $\mathbb{R}^n$ for that matter) are completely analogous. For example, on $\mathbb{R}^3$ we have the three coordinate functions

$$\begin{aligned} x : \mathbb{R}^3 &\longrightarrow \mathbb{R} \\ p &\longmapsto x(p) \end{aligned} \qquad \begin{aligned} y : \mathbb{R}^3 &\longrightarrow \mathbb{R} \\ p &\longmapsto y(p) \end{aligned} \qquad \begin{aligned} z : \mathbb{R}^3 &\longrightarrow \mathbb{R} \\ p &\longmapsto z(p) \end{aligned}$$

which pick off the x, the y and the z coordinates of the point p as illustrated in Fig. 2.7.

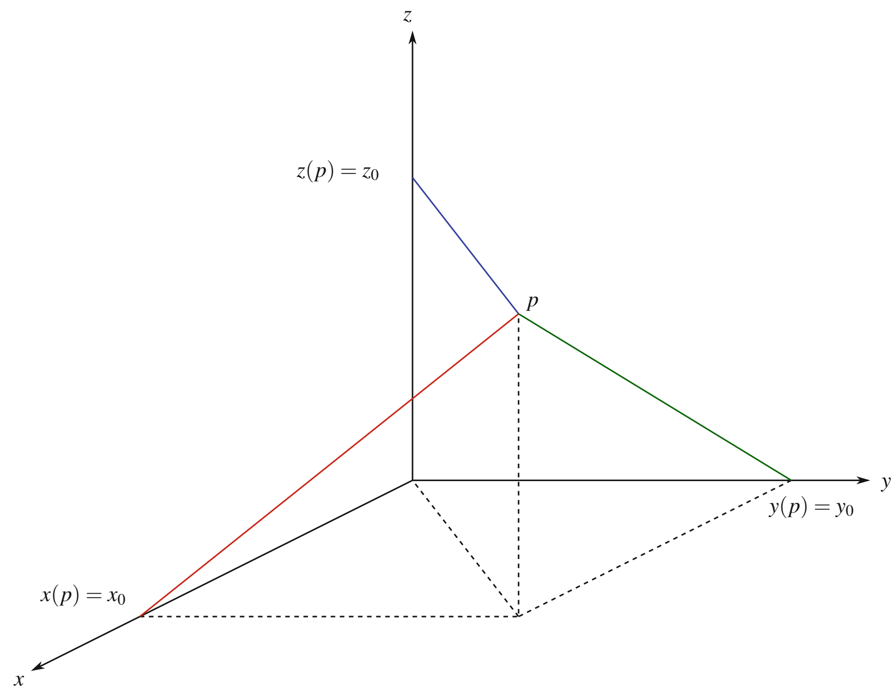

Fig. 2.7 The coordinate functions $x : \mathbb{R}^3 \to \mathbb{R}$, $y : \mathbb{R}^3 \to \mathbb{R}$, and $z : \mathbb{R}^3 \to \mathbb{R}$ picking off the coordinate values x_0, y_0, and z_0 respectively, of the point $p \in \mathbb{R}^3$

Of course Cartesian coordinate functions are not the only coordinate functions there are. Polar coordinate, spherical coordinate, and cylindrical coordinate functions are some other examples you have likely encountered before. We will talk about these coordinate systems in more detail later on but for the moment we consider one example. Consider the point $p = p\big(x(p), y(p)\big) = (3, 5)$. Polar coordinates r and θ are often given by $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Suppose we want to find the "address" of point p in terms of polar coordinates. That is, we want to find $p = \big(r(p), \theta(p)\big)$.

Since we know p in terms of x and y coordinates, we can write $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan\left(\frac{y}{x}\right)$. Then we have

$$\begin{aligned}
p = \Big(r(p), \theta(p)\Big) &= \Big(r\big(x(p), y(p)\big), \theta\big(x(p), y(p)\big)\Big) \\
&= \left(\sqrt{x(p)^2 + y(p)^2}, \arctan\left(\frac{y(p)}{x(p)}\right)\right) \\
&= \left(\sqrt{3^2 + 5^2}, \arctan\left(\frac{5}{3}\right)\right) \\
&= \Bigg(\underbrace{\sqrt{34}}_{r}, \underbrace{\arctan\left(\frac{5}{3}\right)}_{\theta}\Bigg) \\
&\approx \big(5.831, 59.036\big).
\end{aligned}$$

Again, the same ambiguity applies, r and θ can either be coordinate functions or the values given by the coordinate functions.

2.2 Tangent Spaces and Vector Fields

Let us recap what we have done so far. We have reviewed vector spaces and talked about $\mathbb{R}^n$ as a vector space, that is, as a collection of vectors of the from

$$v = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

where $x_1, \ldots, x_n \in \mathbb{R}$. Then we talked about $\mathbb{R}^n$ as a manifold, that is, as a collection of points $(x_1, \ldots, x_n)$ where $x_1, \ldots, x_n \in \mathbb{R}$. We have also reviewed linear functionals on the vector space $\mathbb{R}^n$ and functions on the manifold $\mathbb{R}^n$ and introduced coordinate functions. We will now see how the manifold $\mathbb{R}^n$ is related to vector spaces $\mathbb{R}^n$. Our motivation for this is our desire to do calculus. Well, for the moment to take directional derivatives anyway.

However, to help us picture what is going on a little better we are going to start by considering three manifolds, S^1, S^2, and T. The manifold S^1 is simply a circle, the manifold S^2 is the sphere, and the manifold T is the torus, which looks like a donut. For the moment we are going to be quite imprecise. We want you to walk away with a general feeling for what a manifold is and not overwhelm you with technical details. For the moment we will simply say a **manifold** is a space that is **locally Euclidian**. What we mean by that is if we look at a very small portion of the manifold that portion looks like $\mathbb{R}^n$ for some natural number n.

We will illustrate this idea for the three manifolds we have chosen to look at. The circle S^1 is locally like $\mathbb{R}^1$, which explains the superscript 1 on the S. In Fig. 2.8 we zoom in on a small portion of the one-sphere S^1 a couple of times to see that locally the manifold S^1 looks like $\mathbb{R}^1$. Similarly, in Fig. 2.9 we zoom in on a small portion of the two-sphere S^2 to see that it looks like $\mathbb{R}^2$ locally and in Fig. 2.10 we zoom in on a small portion of the torus T see that it also looks locally like $\mathbb{R}^2$.

The point with these examples is that even though these example manifolds *locally* look like $\mathbb{R}^1$ and $\mathbb{R}^2$, *globally* they do not. That is, their global behavior is more complex; these spaces somehow twist around and reconnect with themselves in a way that Euclidian space does not.

Now we will introduce the idea of a **tangent space**. Again, we want to give you an general feeling for what a tangent space is and not overwhelm you with the technical details, which will be presented in 10.2. From calculus you should recall what the tangent line to a curve at some point of the curve is. Each point of the curve has its own tangent line, as is shown in Fig. 2.11. We can see that the one dimensional curve has tangent spaces that are lines, that is, that are also one dimensional.

Similarly, a two dimensional surface has a tangent plane at each point of the surface, as is pictured in Fig. 2.12. The tangent space to a manifold at the point p is basically the set of all lines that are tangent to smooth curves that go through

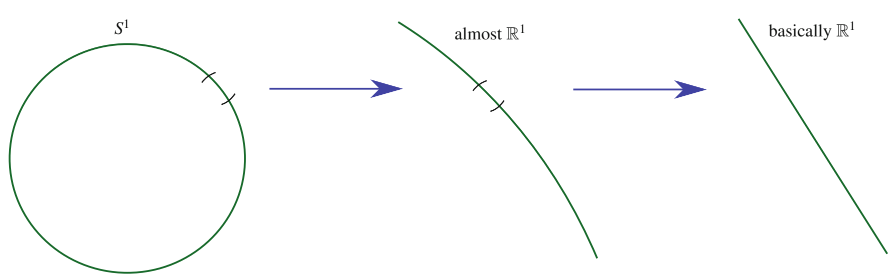

Fig. 2.8 Here a small portion of a one-sphere, or circle, S^1 is zoomed in on to show that locally the manifold S^1 looks like $\mathbb{R}^1$

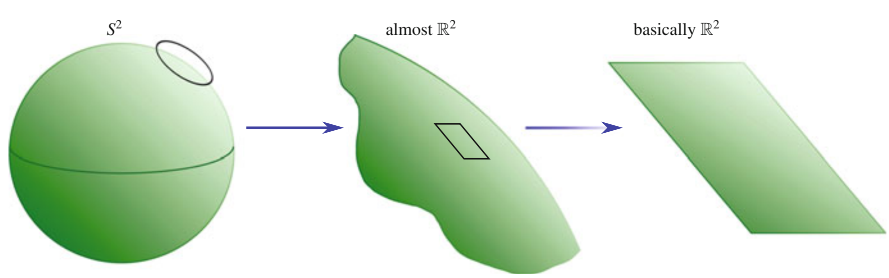

Fig. 2.9 Here a small portion of the two-sphere S^2 is zoomed in on to show that locally the manifold S^2 looks like $\mathbb{R}^2$

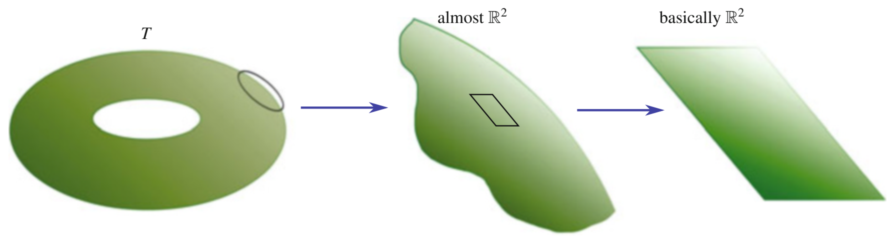

Fig. 2.10 Here a small portion of the torus T is zoomed in on to show that locally the manifold T looks like $\mathbb{R}^2$

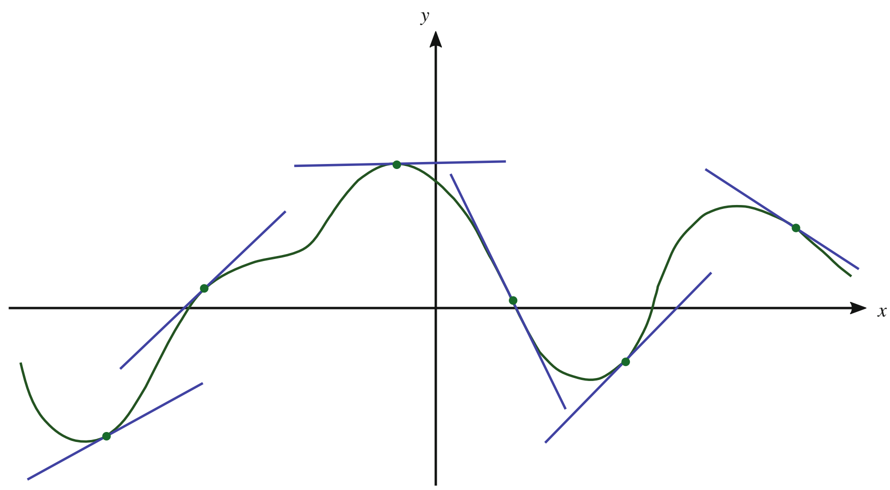

Fig. 2.11 Here a curve is pictured with tangent lines drown at several points

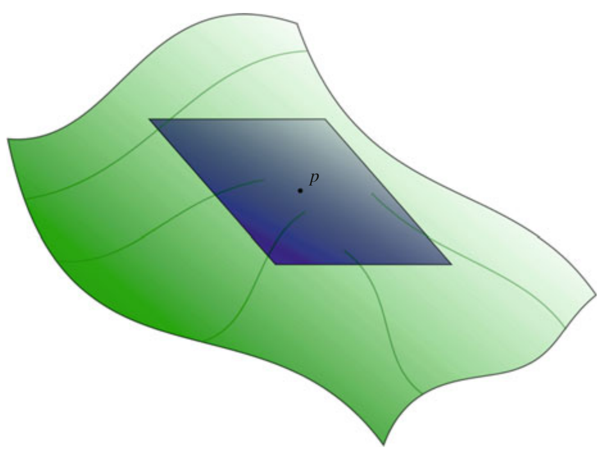

Fig. 2.12 Here a two-dimensional surface is pictured along with a tangent plane drown at one point

the manifold at the point p. This is illustrated in Fig. 2.13 where we have drawn a point p with five smooth curves going through it along with the five tangent lines to the curves at the point p. In both of these pictures the tangent space is a plane.

If you imagine a point in a three dimensional manifold and all the possible curves that can go through that point, and then imagine all the tangent lines to those curves at that point, you will see that the tangent space at that point is three dimensional. It turns out that the tangent space at a "nice" point is the same dimension as the manifold at that point. In this book we will not discuss manifolds with points that are not "nice" in this sense. In fact, the tangent spaces turn out to be nothing more than the vector spaces $\mathbb{R}^n$ for the correct value of n. Because the tangent spaces are vector spaces we think of the tangent space at a point p to be the set of all tangent vectors to the point p. We use special notation for the tangent space. Either T_pM or $T_p(M)$ denotes the tangent space to the manifold M at the point p.

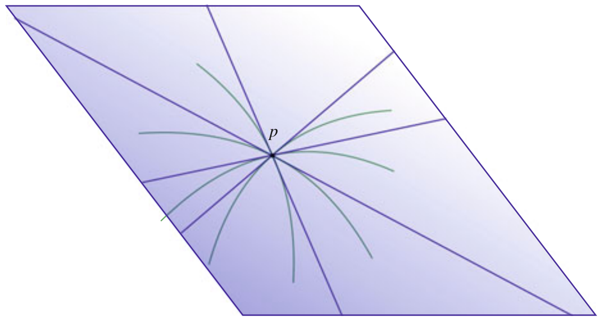

Fig. 2.13 The tangent plane to a two-dimensional surface at the point p is the set of all tangent lines at p to curves in that surface that go through p

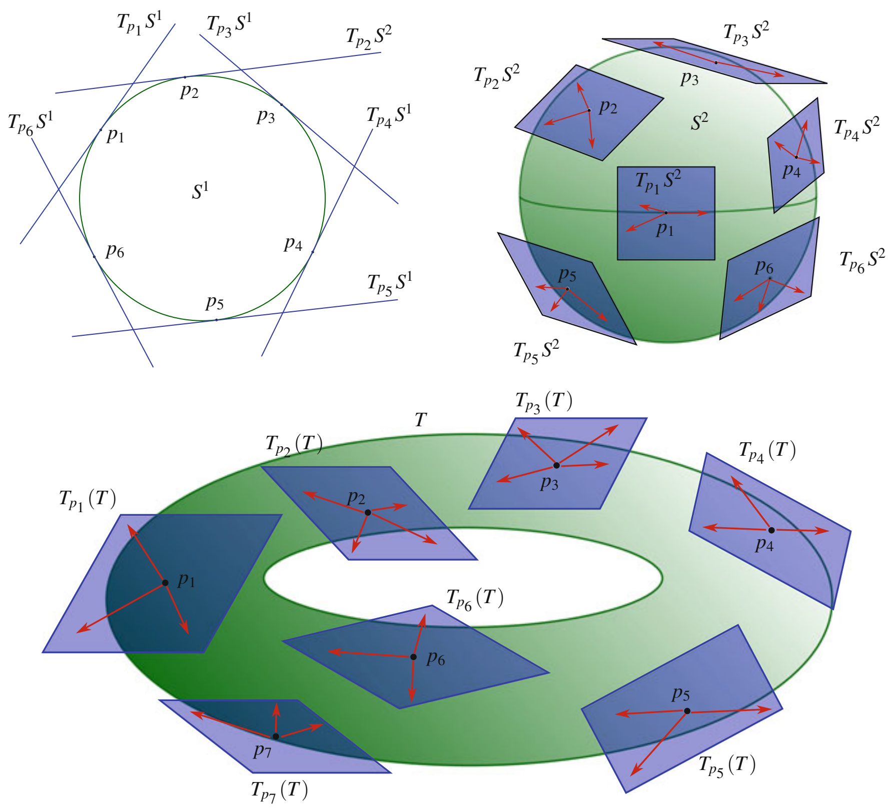

Fig. 2.14 The manifold S^1 along with a number of tangent lines depicted (top left). The manifold S^2 along with a number of tangent planes depicted (top right). The manifold T along with a number of tangent planes depicted (bottom). For both manifolds S^2 and T several vectors in each tangent plane are also depicted

To get a better idea of what the set of tangent spaces to a manifold look like, we have drawn a few tangent spaces to the three simple manifolds we had looked at earlier in Fig. 2.14, the circle S^1, the sphere S^2, and the torus T. For the two dimensional tangent spaces we have also drawn some elements of the tangent spaces, that is, vectors emanating from the point p. The tangent spaces of the circle are denoted by $T_p S^1$, the tangent spaces of the sphere are denoted by $T_p S^2$, and the tangent spaces of the torus are denoted by $T_p(T)$. We use the parenthesis here simply because $T_p T$ looks a little odd.

Even though we are quite familiar with the manifolds $\mathbb{R}^2$ and $\mathbb{R}^3$, imagining the tangent spaces to these manifolds is a little strange, though is it something you have been implicitly doing since vector calculus. The tangent space of the manifold $\mathbb{R}^2$ at a point p is the set of all vectors based at the point p. In vector calculus we dealt with vectors with different base points a lot, but we always simply thought of these vectors as being *in* the manifold $\mathbb{R}^2$ or $\mathbb{R}^3$. Now we think of these vectors as belonging to a separate copy of the vector space $\mathbb{R}^2$ or $\mathbb{R}^3$ attached at the point and called the tangent space.

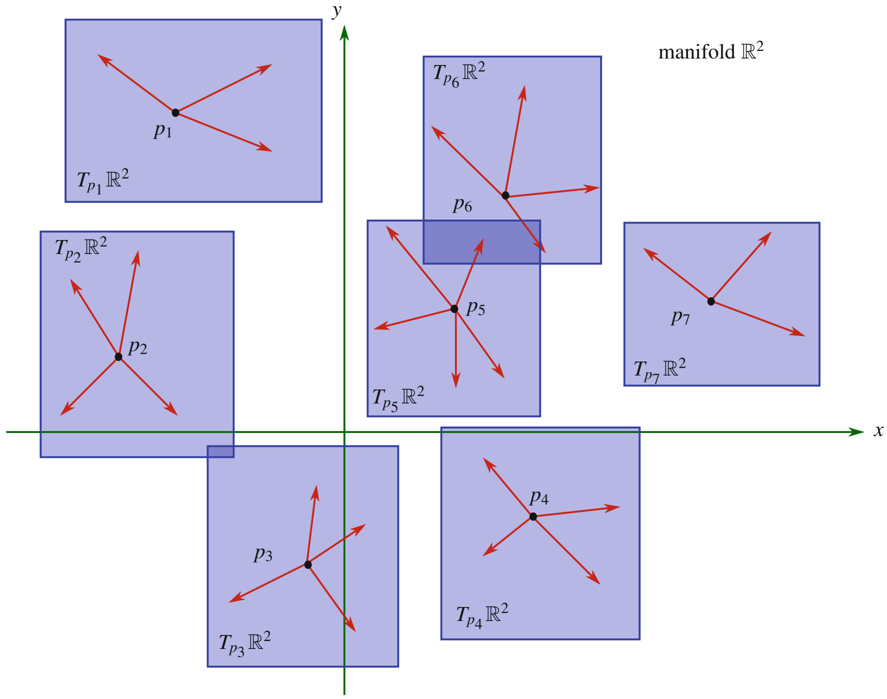

Fig. 2.15 The manifold $\mathbb{R}^2$ along with the tangent spaces $T_{p_i}\mathbb{R}^2$ at seven different points. A few vectors in each tangent space are shown

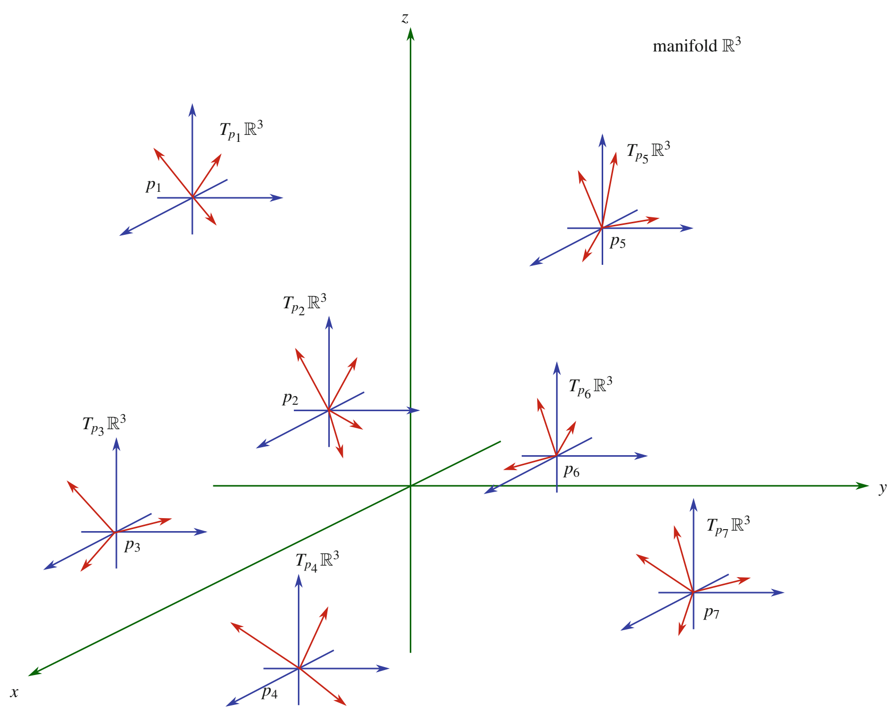

Fig. 2.16 The manifold $\mathbb{R}^3$ along with the tangent spaces $T_{p_i}\mathbb{R}^3$ at seven different points. A few vectors in each tangent space are shown

Figure 2.15 tries to help you imagine what is going on. The manifold $\mathbb{R}^2$ is pictured along with the tangent spaces $T_{p_i}\mathbb{R}^2$ at seven different points $p_1, \ldots, p_7$. Each of the tangent spaces $T_p\mathbb{R}^2$ is **isomorphic** to ("the same as") $\mathbb{R}^2$. Though we have drawn the tangent spaces at only seven points, there is in fact a separate tangent space $T_p\mathbb{R}^2$ for every single point $p \in \mathbb{R}^2$.

Figure 2.16 tries to help you imagine the tangent spaces to $\mathbb{R}^3$. Again, the manifold $\mathbb{R}^3$ is pictured along with the tangent spaces $T_{p_i}\mathbb{R}^3$ at seven different points $p_1, \ldots, p_7$. Each of the tangent spaces $T_p\mathbb{R}^3$ is naturally isomorphic to $\mathbb{R}^3$. Again, though we have drawn the tangent spaces at only seven points, there is in fact a separate tangent space $T_p\mathbb{R}^3$ for every single point $p \in \mathbb{R}^3$.

From now on we will often include the base point of a vector in our notation. For example, the vector v_p means vector v at the base point p of the manifold M. Clearly, the vector v_p belongs to, or is an element of, the tangent space T_pM. So, for example, we have

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{(-2,-3)} \in T_{(-2,-3)}\mathbb{R}^2, \qquad \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{(-1,4)} \in T_{(-1,4)}\mathbb{R}^2, \qquad \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{(2,4)} \in T_{(2,4)}\mathbb{R}^2,$$

and

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{(-2,-3,-5)} \in T_{(-2,-3,-5)}\mathbb{R}^3, \qquad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{(-1,4,3)} \in T_{(-1,4,3)}\mathbb{R}^3, \qquad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{(2,4,-1)} \in T_{(2,4,-1)}\mathbb{R}^3.$$

The manifold M, along with a copy of the vector space T_pM attached at every point of the manifold is called the **tangent bundle** of the manifold M which is denoted as TM or $T(M)$. Often a cartoon version of the tangent bundle of a manifold M is drawn as in Fig. 2.17 using a one dimensional representation for both the manifold and the tangent spaces.

If M is an n dimensional manifold then TM is $2n$ dimensional. That makes intuitive sense since to specify exactly any element (vector) in TM you need to specify the vector part, which requires n numbers, and the base point, which also requires n numbers. Thus, $T\mathbb{R}^2$ has four dimensions and $T\mathbb{R}^3$ has six dimensions. In fact, the tangent bundle TM of a manifold M is itself also a manifold. Thus, if M is an n dimensional manifold then TM is a $2n$ dimensional manifold.

Now we will very briefly introduce the concept of a **vector field**. Doubtless you have been exposed to the idea of a vector field in vector calculus where it is usually defined on $\mathbb{R}^2$ or $\mathbb{R}^3$ as a vector-valued function that assigns to each point $(x, y) \in \mathbb{R}^2$ or $(x, y, z) \in \mathbb{R}^3$ a vector

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_{(x,y)} \text{ or } \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}_{(x,y,z)}.$$

An example of a vector field on $\mathbb{R}^2$ is shown in Fig. 2.18.

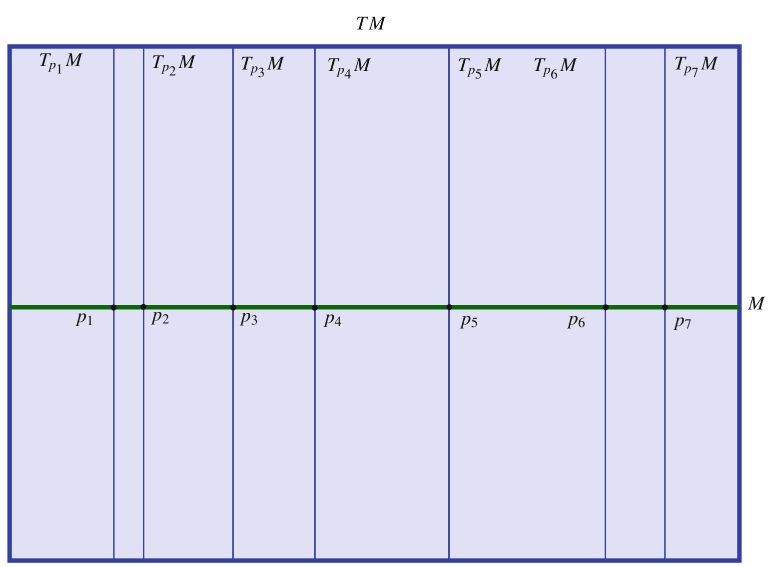

Fig. 2.17 A cartoon of a tangent bundle TM over a manifold M. Seven tangent spaces $T_{p_i}M$ are shown

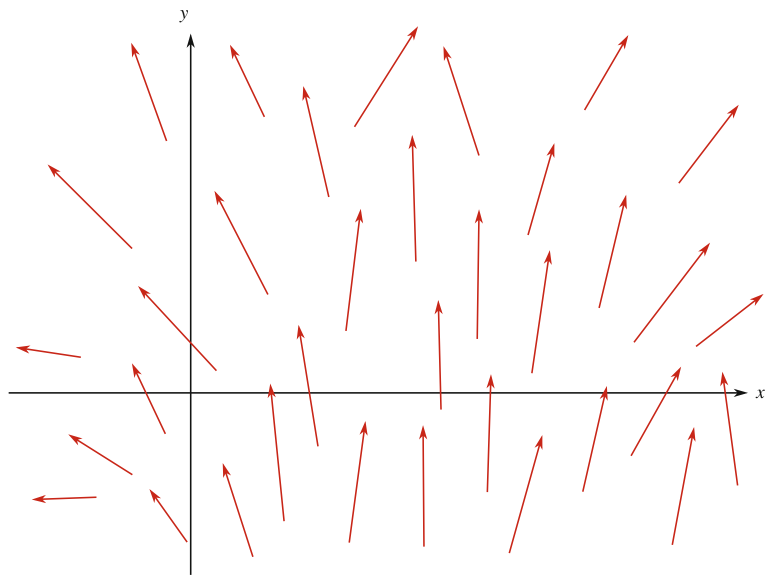

Fig. 2.18 A vector field drawn on the manifold $\mathbb{R}^2$. Recall, each of the vectors show are an element of a tangent space and not actually in the manifold $\mathbb{R}^2$

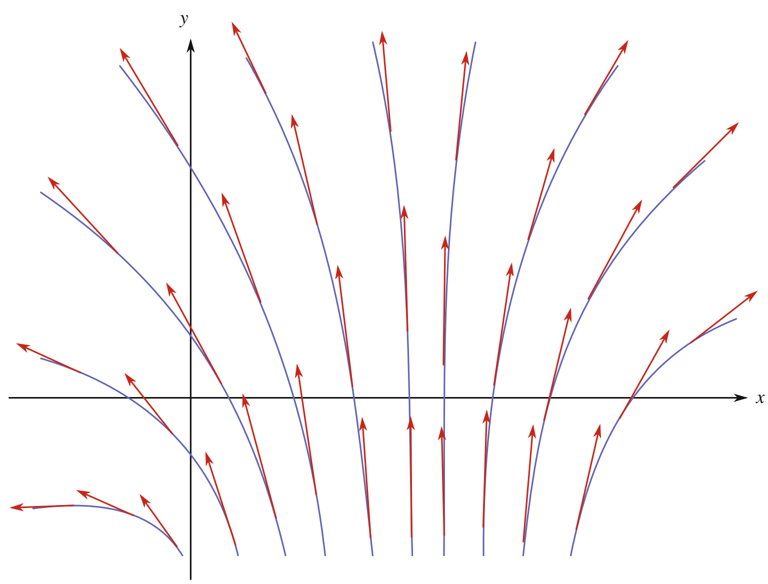

Fig. 2.19 A smooth vector field drawn on the manifold $\mathbb{R}^2$. Again, recall, each of the vectors shown are an element of a tangent space and not actually in the manifold $\mathbb{R}^2$

A vector field is called **smooth** if it is possible to find curves on the manifold such that all the vectors in the vector field are tangent to the curves. These curves are called the **integral curves** of the vector field, and finding them is essentially what differential equations are all about. Figure 2.19 shows several integral curves of a smooth vector field in $\mathbb{R}^3$.

While we certainly are not going to get into differential equations here, what we want to do is introduce a new way of thinking about vector fields. Notice, a vector field on a manifold such as $\mathbb{R}^2$ or $\mathbb{R}^3$ gives us a vector at each point of the manifold. But a vector v_p at a point p of a manifold M is an element of the tangent space at that point, T_pM. So being given a vector field is the same thing as being given an element of each tangent space of M. This is often called a **section** of the tangent bundle TM. Graphically, we often visualize sections of tangent bundles as in Fig. 2.20.

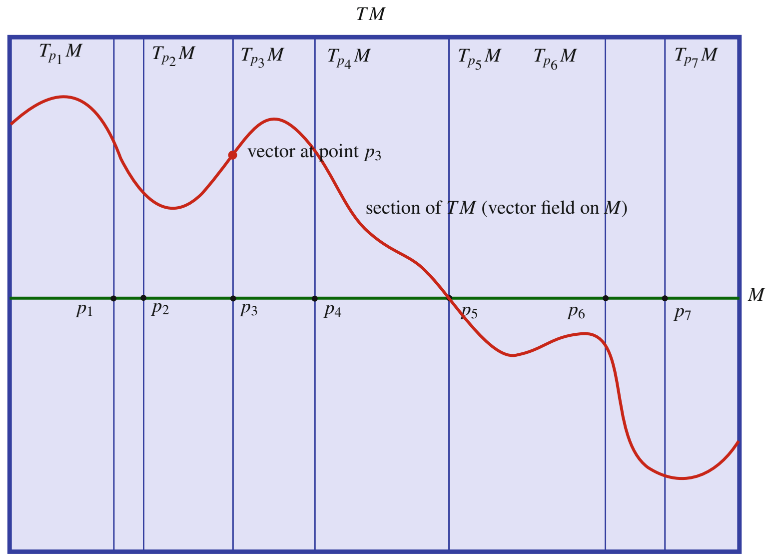

Fig. 2.20 A cartoon of a section of a tangent bundle over the manifold M. The section of TM is simply a vector field on M. The vector in the vector field on M at point p_3 is depicted as the point in $T_{p_3}M$ that is in the section. Notice, in this cartoon representation the vector at the point p_5 would be the zero vector

Question 2.1 Before we get into the next section let's take just a moment to review what derivatives were all about. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{x^2}{2} + 2$.

(a) Sketch the graph of the function $f(x) = \frac{x^2}{2} + 2$.
(b) Find the derivative of $f(x)$.
(c) What is $f(2)$? What is $f(3)$? Draw a line between the points $\big(2, f(2)\big)$ and $\big(3, f(3)\big)$.
(d) What is the slope of this secant line?
(e) Write down the limit of the difference quotient definition of derivative.
(f) Use this definition to find the derivative of f at $x = 2$. Does it agree with what you found above?
(g) The derivative of f at $x = 2$ is a number. Explain what this number represents.

From the above question you should now remember that derivatives are basically slopes of lines tangent to the graph of the function at a certain point.

2.3 Directional Derivatives

Now the time comes to explore why we might actually want to use all of this. Suppose we were given a real-valued function f on the manifold $\mathbb{R}^n$. That is, $f : \mathbb{R}^n \to \mathbb{R}$. How might we use a vector v_p at some given point p on the manifold? One natural thing we could do is take the **directional derivative** of f at the point p in the direction of v_p. Consider Fig. 2.21 where a real-valued function $f : \mathbb{R} \to \mathbb{R}$ is shown. Given a point p in the domain there are two directions that it is possible to move in, left or right. Similarly, if you had a vector based at that point it could either point left, as v_p does, or right, as w_p does. We could ask how the function varies in either of these directions. In Fig. 2.22 a real-valued function $f : \mathbb{R}^2 \to \mathbb{R}$ is depicted. Given a point $p = (x, y)$ in the domain one can go any direction in two dimensions, and a vector based at that point could point in any direction in two dimensions as well. And similarly, we could ask how the function varies in any of these directions. In Fig. 2.23 we choose one of those directions, the direction given by the vector

$$v_p = \begin{bmatrix} a \\ b \end{bmatrix}_{(x,y)}.$$

We are able to use this vector to find the directional derivative of f in the direction v_p at the point p. In other words, we are able to find how f is changing in the direction v_p at the point p.

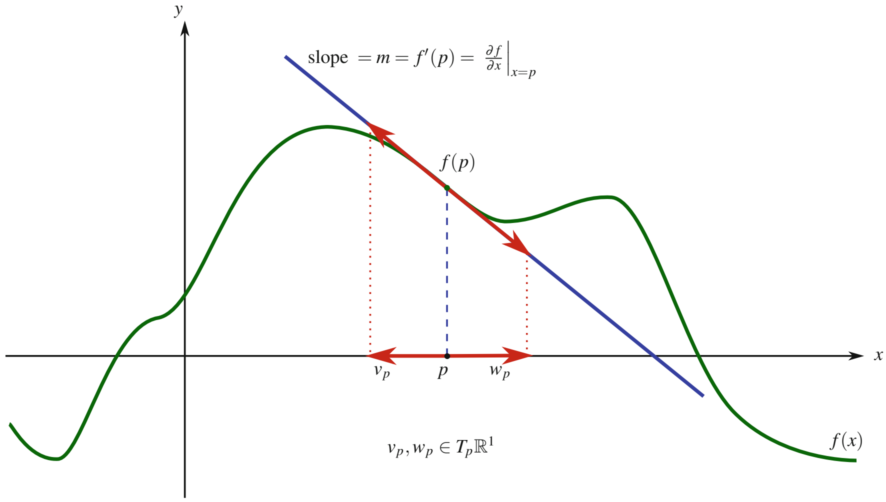

Fig. 2.21 Starting at $p \in \mathbb{R}$ one can go in any direction in one dimension. In other words, one can go either right or left

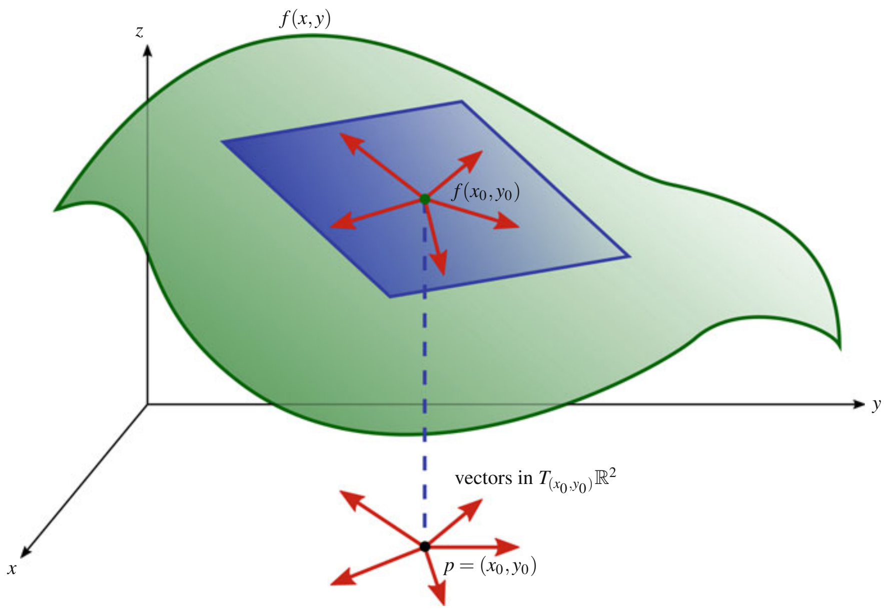

Fig. 2.22 Starting at $p = (x, y) \in \mathbb{R}^2$ one can go in any direction in two dimensions. A number of different directions are pictured

Notice, for the moment that we are going back to the vector calculus way of thinking, of thinking of the vector v_p as sitting inside the manifold $\mathbb{R}^2$ at the point p instead of in the tangent space $T_p\mathbb{R}^2$. Even though we just made a big deal about the vectors not being in the manifold but being in the tangent spaces instead, since we are reviewing the idea of directional derivatives from the viewpoint of vector calculus in this section and the next we will go back to thinking of vectors as being in the manifold based at a certain point. That way we can use the vector calculus notation and focus on the relevant ideas without being bogged down with notational nuance.

We start by considering how directional derivatives are generally introduced in most multivariable calculus courses. The reason we do this is that the definition of directional derivative that we will want to use will be slightly different than that used in most vector calculus classes and we do not want there to be any confusion. The definition that you probably saw was something like this.

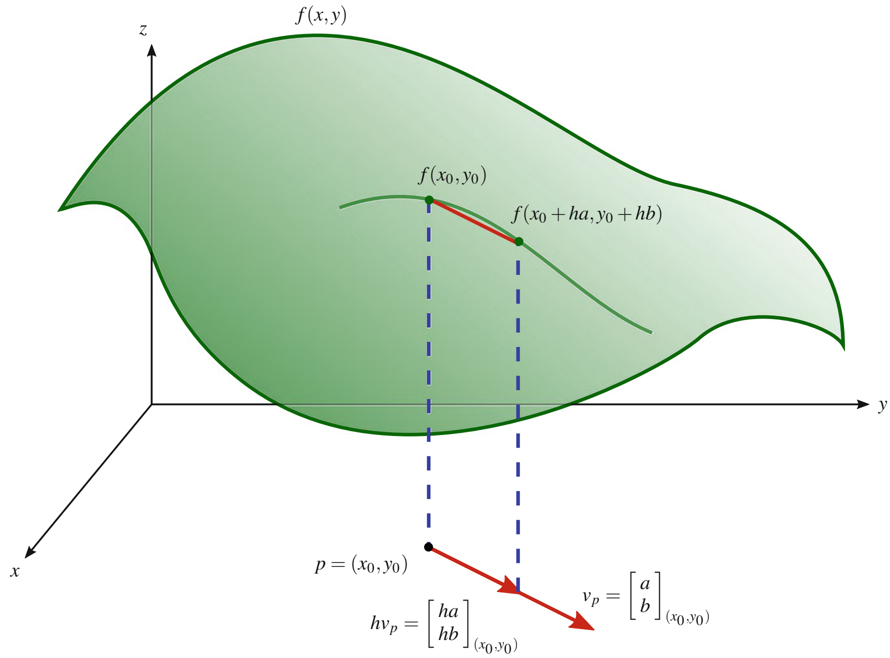

Fig. 2.23 Finding the directional derivative of a function $f : \mathbb{R}^2 \to \mathbb{R}$ requires a unit vector in some direction and the associated values for $f(t_0 + ha, y_0 + tb)$ and $f(x_0, y_0)$, which are necessary for the difference quotient

Definition 2.3.1 The **directional derivative** of $f : \mathbb{R}^2 \to \mathbb{R}$ at (x_0, y_0) in the direction of the unit vector $u = \begin{bmatrix} a \\ b \end{bmatrix}$ is

$$D_u f(x_0, y_0) = \lim_{t \to 0} \frac{f\big(x_0 + ta, y_0 + tb\big) - f\big(x_0, y_0\big)}{t}$$

if this limit exists.

Definition 2.3.2 The **directional derivative** of $f : \mathbb{R}^3 \to \mathbb{R}$ at (x_0, y_0, z_0) in the direction of the unit vector $u = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is

$$D_u f(x_0, y_0, z_0) = \lim_{t \to 0} \frac{f\big(t_0 + ta, y_0 + tb, z_0 + tc\big) - f\big(x_0, y_0, z_0\big)}{t}$$

if this limit exists.

Often vector calculus textbooks will write vectors as row vectors, sometimes like $[a, b]$ or sometimes like $\langle a, b\rangle$. We will never do that in this book. We will always write vectors as column vectors with square brackets, $v = \begin{bmatrix} a \\ b \end{bmatrix}$. Also, we will never use the angle brackets $\langle \cdot, \cdot \rangle$ for vectors or points, which one sometimes sees. Finally, notice that in both of the definitions that the vector u had to be a unit vector, that means the length of the vector is one unit. In other words, for the first definition we have to have $\sqrt{a^2 + b^2} = 1$ and for the second we have to have $\sqrt{a^2 + b^2 + c^2} = 1$.

To remind ourselves of some other equivalent notations, notice that if we let $p = (x_0, y_0)$ then we can also write

$$\lim_{t \to 0} \frac{f\big(x_0 + ta, y_0 + tb\big) - f\big(x_0, y_0\big)}{t} = \frac{d}{dt}\Big(f\big(p + tu\big)\Big)\Big|_{t=0}$$

and if we let $p = (x_0, y_0, z_0)$ then we can also write

$$\lim_{t\to 0} \frac{f\big(x_0 + ta, y_0 + tb, z_0 + tc\big) - f\big(x_0, y_0, z_0\big)}{t} = \frac{d}{dt}\Big(f\big(p + tu\big)\Big)\Big|_{t=0}.$$

That means that we could also have written the above definitions as

$$D_u f(x_0, y_0) = \frac{d}{dt}\Big(f\big(p + tu\big)\Big)\Big|_{t=0}$$

and

$$D_u f(x_0, y_0, z_0) = \frac{d}{dt}\Big(f\big(p + tu\big)\Big)\Big|_{t=0}.$$

Suppose we had a function $f : \mathbb{R}^2 \to \mathbb{R}$ and the unit vectors

$$u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

then it is easy to see that $D_u f = \frac{\partial f}{\partial x}$ and $D_v f = \frac{\partial f}{\partial y}$. Similarly, for a function $g : \mathbb{R}^3 \to \mathbb{R}$ and the unit vectors

$$u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

we get $D_u g = \frac{\partial g}{\partial x}$, $D_v g = \frac{\partial g}{\partial y}$, and $D_w g = \frac{\partial g}{\partial z}$.

Question 2.2 Find the derivative of $f(x, y) = x^3 - 3xy + y^2$ at $p = (1, 2)$ in the direction of $u = \begin{bmatrix} \cos(\pi/6) \\ \sin(\pi/6) \end{bmatrix}$.

Proceeding as you probably did in multivariable calculus, after the above definitions of directional derivatives were made the following theorems were derived.

Theorem 2.1 *If $f : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function of x and y, then f has directional derivatives in the direction of any unit vector $u = \begin{bmatrix} a \\ b \end{bmatrix}$ and*

$$D_u f(x, y) = \frac{\partial f}{\partial x}(x, y) \cdot a + \frac{\partial f}{\partial y}(x, y) \cdot b.$$

Sketch of proof: First define a one variable function $g(h)$ by $g(h) = f(x_0 + ha, y_0 + hb)$ and use this to rewrite $D_u f$. Then use the chain rule to write the derivative of $g(h)$. Finally, let $h = 0$ and combine.

Theorem 2.2 *If $f : \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function of x, y and z, then f has directional derivatives in the direction of any unit vector $u = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and*

$$D_u f(x, y, z) = \frac{\partial f}{\partial x}(x, y, z) \cdot a + \frac{\partial f}{\partial y}(x, y, z) \cdot b + \frac{\partial f}{\partial z}(x, y, z) \cdot c.$$

Question 2.3 Let $p = (2, 0, -1)$ and $v = \frac{1}{\sqrt{14}}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$. Find $D_{v_p} f$ if

(a) $f(x, y, z) = y^2 z$
(b) $f(x, y, z) = e^x \cos(y)$
(c) $f(x, y, z) = x^7$

Now we want to explore the reason why the stipulation that

$$u = \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{or} \quad u = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

be a unit vector is made. We begin by noting that the equation of a plane through the origin is given by

$$z = m_x x + m_y y$$

as shown in Fig. 2.24. From this it is straightforward to show that the equation of a plane through the point $(x_0, y_0, f(x_0, y_0))$ on the graph of f is given by

$$z - f(x_0, y_0) = m_x(x - x_0) + m_y(y - y_0).$$

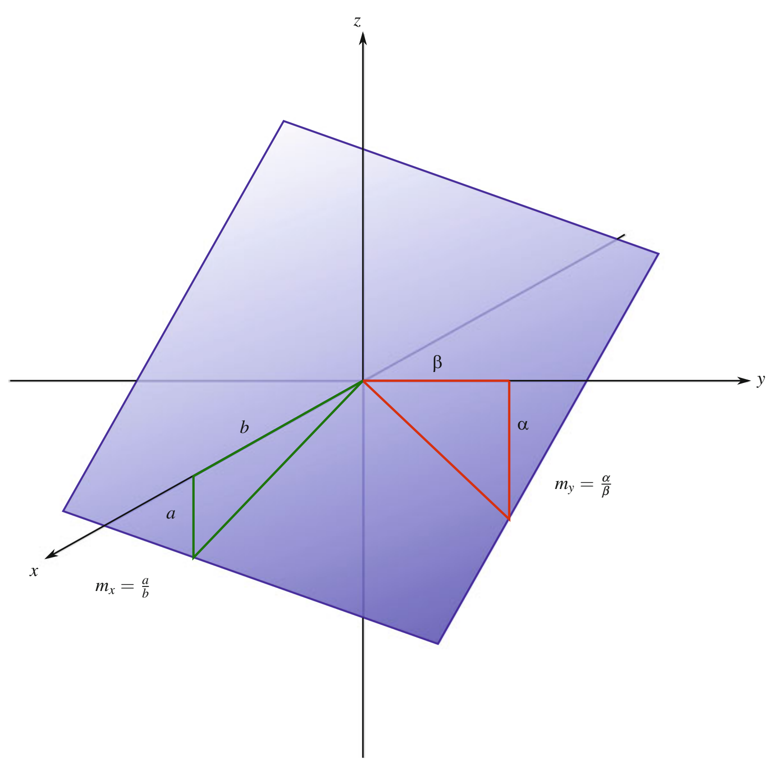

Fig. 2.24 The equation of a plane through the origin is given by $z = m_x x + m_y y$ where m_x is the slope of the plane along the x-axis and m_y is the slope of the plane along the y-axis

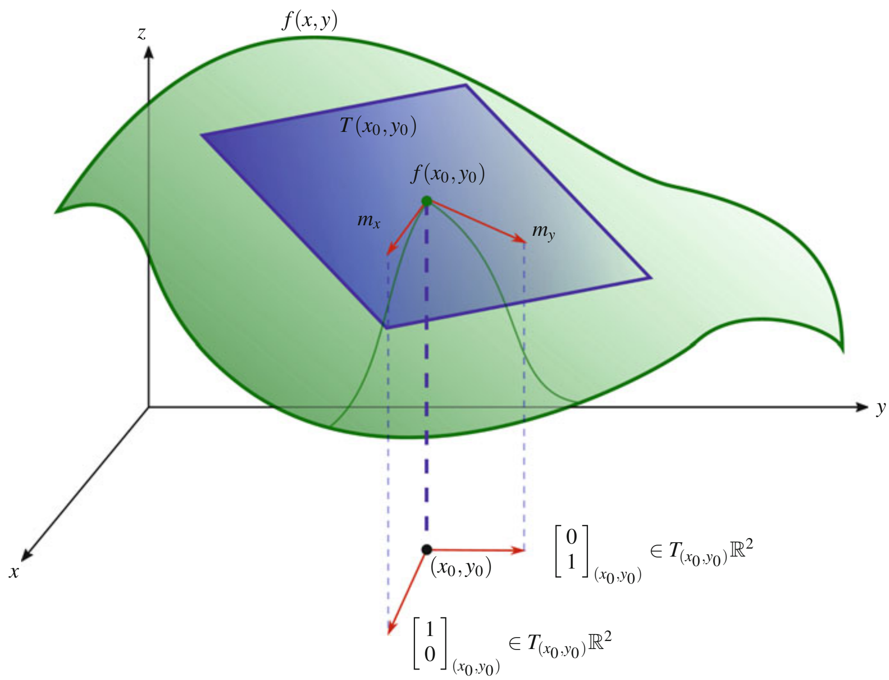

Fig. 2.25 The tangent plane $T(x_0, y_0)$ to $f(x, y)$ at point $(x_0, y_0, f(x_0, y_0))$

To get the equation of the tangent plane $T(x_0, y_0)$ to $f(x, y)$ at point $(x_0, y_0, f(x_0, y_0))$ we notice that

$$m_x = \frac{\partial f}{\partial x}, \quad m_y = \frac{\partial f}{\partial y}$$

$$\Rightarrow T(x, y) - f(x_0, y_0) = \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} \cdot (x - x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \cdot (y - y_0),$$

$$\Rightarrow T(x, y) = f(x_0, y_0) + \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} \cdot (x - x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \cdot (y - y_0),$$

which is shown in Fig. 2.25.

Consider the vector $u = \begin{bmatrix} a \\ b \end{bmatrix}$ at the point (x_0, y_0), that is, $u = \begin{bmatrix} a \\ b \end{bmatrix}_{(x_0,y_0)}$ as shown in Fig. 2.26. How would we find the slope of the line tangent to the graph of f through point $f(x_0, y_0)$? We would use $m = \frac{rise}{run}$, where clearly the "run" is the length of the vector $\begin{bmatrix} a \\ b \end{bmatrix}$. Our rise, or change in height z, can be calculated from the equation of the tangent plane to the point that we developed above,

$$\begin{aligned}
\frac{rise}{run} &= \frac{T(x_0 + a, y_0 + b) - T(x_0, y_0)}{\sqrt{a^2 + b^2}} \\
&= \frac{\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} \cdot a + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \cdot b}{\sqrt{a^2 + b^2}} \\
&= \frac{\frac{\partial f}{\partial x} \cdot a + \frac{\partial f}{\partial y} \cdot b}{\sqrt{a^2 + b^2}}.
\end{aligned}$$

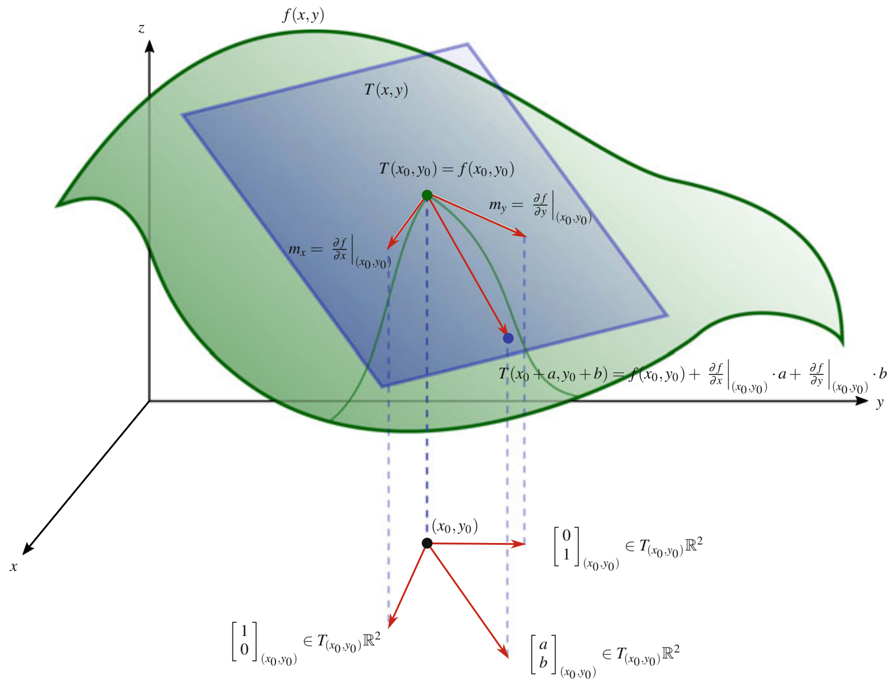

Fig. 2.26 The tangent plane to $f(x, y)$ at the point (x_0, y_0) shown with the tangent line in the direction $v = ae_1 + be_2$

Notice how close this is to the formula given for $D_u f$ in multivariable calculus,

$$D_u f = \frac{\partial f}{\partial x} \cdot a + \frac{\partial f}{\partial y} \cdot b.$$

As long as the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ is a unit vector then these two formulas are the same. Thus if u is a unit vector the formula for $D_u f$ also gives the slope of the tangent line that lies in the tangent plane and is in the direction of $\begin{bmatrix} a \\ b \end{bmatrix}$. So by stipulating that $u = \begin{bmatrix} a \\ b \end{bmatrix}$ be a unit vector ***we basically maintain that the definition of derivative also be the slope of the tangent line while simultaneously retaining a nice simple formula for the derivative***.

We will now loosen this requirement and allow any vector, not just unit vectors. However, we want to keep the nice formula. This means that we need *let go* of the idea that the derivative is the slope of the tangent line. So we will define the directional derivative exactly as before but we will drop the stipulation that u must be a unit vector. In order to obtain the slope of the tangent line we will have to divide the value given by our new definition of directional derivative by the length of the vector. So what exactly will our new definition of directional derivative actually represent? It will represent the "rise" portion of the above equation - how much "rise" the tangent line has over the length of the vector u.

Consider Fig. 2.26. By taking away the requirement that u be a unit vector the directional derivative

$$D_u f = \frac{\partial f}{\partial x} \cdot a + \frac{\partial f}{\partial y} \cdot b.$$

tells us how much the tangent plane to the graph of the function f at the point (x_0, y_0) "rises" as we move along u. Thus the directional derivative actually tells us something about the tangent plane to the function at the point. We also take a moment to point out that the tangent plane to the function f at the point (x_0, y_0) is actually the closest linear approximation of the graph of the function f at (x_0, y_0).

Now take a moment to consider our review of how directional derivatives are generally introduced in multivariable calculus classes. At no point was there any mention of the tangent space of the manifolds $\mathbb{R}^2$ or $\mathbb{R}^3$. One can, and does, mix-up, or merge, the tangent spaces with the underlying manifold in multivariable calculus because they are essentially the same space. However, eventually when we start to deal with other kinds of manifolds we will not be able to do this. That is why we make such an effort to always make the distinction.

Definition 2.3.3 Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real-valued function on the manifold $\mathbb{R}^n$ and let v_p be a vector tangent to manifold $\mathbb{R}^n$, that is, $v_p \in T_p(\mathbb{R}^n)$. The number

$$v_p[f] \equiv \frac{d}{dt}\Big(f\big(p + tv_p\big)\Big)\Big|_{t=0}$$

is called the **directional derivative** of f with respect to v_p, if it exists.

This is exactly the same definition as in the multivariable calculus case, only without the requirement that v_p be a unit vector. Thus, the number that we get from this definition of directional derivative will *not* be the slope of the tangent line if v_p is not a unit vector but will instead be the "rise" of the tangent line to the graph of f at the point p as we move the length of v_p.

Also notice the new notation, $v_p[f]$. This notation is very similar to the functional notation $f(x)$ where the function f "takes in" an input value x and gives out a number. In the notation $v_p[f]$ the vector v_p "takes in" a function f and gives out a number; the directional derivative of f in the direction v_p (at the point p.) So, the vector v_p becomes what is called an **operator** on the function f. Basically, an operator is a function that takes as inputs other functions.

Question 2.4 Let $v = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $p = (2, 0, -1)$.

(a) Find $v_p[f]$ where
   (i) $f(x) = x$,
   (ii) $f(x) = x^2 - x$,
   (iii) $f(x) = \cos(x)$.

(b) For each of the functions above, find $w_p[f]$ where
   (i) $w_p = 2v_p$,
   (ii) $w_p = \frac{1}{2}v_p$,
   (i) $w_p = -5v_p$,
   (ii) $w_p = \frac{-1}{5}v_p$.

Suppose that we have a function $f : \mathbb{R}^3 \to \mathbb{R}$ and a vector $v_p = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}_p \in T_p(\mathbb{R}^3)$ at the point $p = (p_1, p_2, p_3)$ and we want to find an expression for $v_p[f]$. First, write $p + tv_p$ as $p + tv_p = (p_1 + tv_1, p_2 + tv_2, p_3 + tv_3)$, which then gives us $f(p + tv_p) = f\big(p_1 + tv_1, p_2 + tv_2, p_3 + tv_3\big)$. Next we notice that if we define the functions $x_i(t) = p_i + tv_i$, for $i = 1, 2, 3$, then we have $\frac{dx_i}{dt} = \frac{d}{dt}\big(p_i + tv_i\big) = v_i$. Putting all of this together and using the chain rule we find

$$\begin{aligned}
v_p[f] &= \frac{d}{dt} f(p + tv)\Big|_{t=0} \\
&= \frac{d}{dt} f\big(\underbrace{p_1 + tv_1}_{x_1(t)}, \underbrace{p_2 + tv_2}_{x_2(t)}, \underbrace{p_3 + tv_3}_{x_3(t)}\big)\Big|_{t=0} \\
&= \frac{\partial f}{\partial x_1}\Big|_{x_1(0)} \cdot \frac{dx_1}{dt}\Big|_{t=0} + \frac{\partial f}{\partial x_2}\Big|_{x_2(0)} \cdot \frac{dx_2}{dt}\Big|_{t=0} + \frac{\partial f}{\partial x_3}\Big|_{x_3(0)} \cdot \frac{dx_3}{dt}\Big|_{t=0} \\
&= \frac{\partial f}{\partial x_1}\Big|_{p_1} \cdot v_1 + \frac{\partial f}{\partial x_2}\Big|_{p_2} \cdot v_2 + \frac{\partial f}{\partial x_3}\Big|_{p_3} \cdot v_3 \\
&= \sum_{i=1}^{3} v_i \cdot \frac{\partial f}{\partial x_i}\Big|_p .
\end{aligned}$$

In summary, in this example we have just found that

$$v_p[f] = \sum_{i=1}^{3} v_i \cdot \left.\frac{\partial f}{\partial x_i}\right|_p .$$

In particular, suppose we had $f : \mathbb{R}^3 \to \mathbb{R}$ a real-valued function on the manifold $\mathbb{R}^3$ along with the standard Euclidian vectors

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

For the moment we will suppress the base point from the notation since it plays no particular role. Since $v[f] = \sum_{i=1}^{3} v_i \cdot \frac{\partial f}{\partial x_i}$ we have $e_1[f] = \frac{\partial f}{\partial x_1}$, $e_2[f] = \frac{\partial f}{\partial x_2}$, and $e_3[f] = \frac{\partial f}{\partial x_3}$ We have actually just shown something very interesting. With a very slight change in notation it becomes even more obvious

$$e_1[f] = \frac{\partial}{\partial x_1}\big(f\big), \ e_2[f] = \frac{\partial}{\partial x_2}\big(f\big), \ e_3[f] = \frac{\partial}{\partial x_3}\big(f\big).$$

We have just equated the operator $\frac{\partial}{\partial x_i}$ with the euclidian vector e_i. In a single short paragraph we have shown one of the most important ideas in this book.

$$\boxed{\text{The Euclidian vectors } e_i \text{ can be identified with the partial differential operators } \frac{\partial}{\partial x_i}}$$

In other words, we can think of the Euclidian vector e_i as actually being the partial differential operator $\frac{\partial}{\partial x_i}$.

Question 2.5 Redo the above calculation putting the base point p into the notation in order to convince yourself that the base point p does not play any role in the calculation.

Question 2.6 Suppose that $a, b \in \mathbb{R}$, p a point on manifold $\mathbb{R}^3$, $v_p, w_p \in T_p(\mathbb{R}^3)$ and $f, g : \mathbb{R}^3 \to \mathbb{R}$.

(a) Show that $(av_p + bw_p)[f] = av_p[f] + bw_p[f]$.
(b) Show that $v_p[af + bg] = av_p[f] + bv_p[f]$.

Besides the identities in the above question that hold, something else, called the **Leibnitz rule**, holds. The Leibnitz rule says that $v_p[fg] = v_p[f] \cdot g(p) + f(p) \cdot v_p[g]$. In essence the Leibnitz rule is the product rule, which is actually used in proving this identity,

$$\begin{aligned}
v_p[fg] &= \sum_{i=1}^{3} v_i \cdot \left.\frac{\partial fg}{\partial x_i}\right|_{p_i} \\
&= \sum_{i=1}^{3} v_i \cdot \left(\left.\frac{\partial f}{\partial x_i}\right|_p \cdot g(p) + f(p) \cdot \left.\frac{\partial g}{\partial x_i}\right|_p \right) \\
&= \left(\sum_{i=1}^{3} v_i \left.\frac{\partial f}{\partial x_i}\right|_p \right) g(p) + f(p) \left(\sum_{i=1}^{3} v_i \left.\frac{\partial g}{\partial x_i}\right|_p \right) \\
&= v_p[f] \cdot g(p) + f(p) \cdot v_p[g].
\end{aligned}$$

First, we take a moment to point out a notational ambiguity. When we are giving a vector v we usually use the subscript to indicate the base point the vector is at, so v_p means that the vector v is based at point p. But here, for the Euclidian unit vectors e_1, e_2, e_3 the subscript clearly does not refer to a point, it refers to which of the Cartesian coordinates are non-zero (in fact, are one.) The Euclidian unit vectors are so common and the notation so useful we usually omit the base point p when using them. One could of course write e_{1_p}, e_{2_p}, and e_{3_p} but that starts getting a little silly after a while. Usually it should be clear from the context what the base point actually is.

Now that we have that little notational comment made, we can bask in the glow of what we have just accomplished, and try to wrap our heads around it at the same time. This last few pages, culminating in the last example, is actually a lot more profound than it seems, so we will take a moment to reiterate what has just been done.

First, you have seen vectors before and are probably fairly comfortable with them, at least vectors in n-dimensional Euclidian space. Of course, now we are viewing vectors as being elements of the tangent space, but they are still just good old-fashioned vectors. Second, you have seen differentiation before and are probably comfortable with differential operators, things like $\frac{d}{dx}, \frac{d}{dt}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}$, though it is also possible you have not seen the word operator applied to them before.

Given a function $f : \mathbb{R}^3 \to \mathbb{R}$ we have used the act of taking the directional derivative of f at the point $p \in \mathbb{R}^3$ in the direction of $v_p \in T_p\mathbb{R}^3$ to turn the vector $v_p \in T_p\mathbb{R}^3$ into an operator on f. This has given the following formula

$$v_p[f] = \sum_{i=1}^{3} v_i \cdot \left.\frac{\partial f}{\partial x_i}\right|_{p_i}.$$

This formula was then used to see exactly what the operator for the Euclidian unit vectors e_1, e_2, e_3 would be at the point p and we determined that we can make the following identifications

$$e_1 \equiv \frac{\partial}{\partial x_1},$$
$$e_2 \equiv \frac{\partial}{\partial x_2},$$
$$e_3 \equiv \frac{\partial}{\partial x_3}.$$

This is, if you have not seen it before, really a surprising identification. It doubtless feels a bit like a notational slight-of-hand. But this is one of the powers of just the right notation, it helps you understand relationships that may be difficult to grasp otherwise. By viewing a vector in the tangent space as acting on a function via the directional derivative we have equated vectors with partial differential operators. See Fig. 2.27. This is an identification that we will continue to make for the rest of the book.

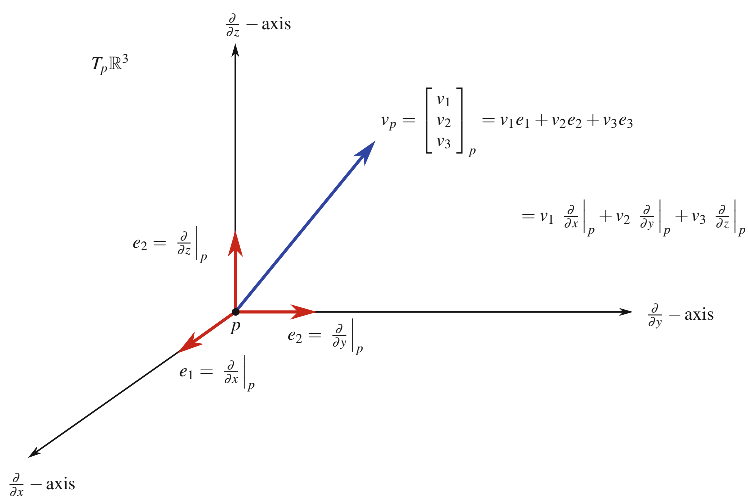

Fig. 2.27 Identifying the euclidian vectors e_i with the partial differential operators $\partial/\partial x_i$

Putting this identification into action consider the vector

$$v = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}.$$

When v is written using the Euclidian unit vectors e_1, e_2, e_2 we have

$$v = 2e_1 - 3e_2 + e_3$$

and when it is written using the differential operators we have

$$v = 2\frac{\partial}{\partial x_1} - 3\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}.$$

From now on you we shall make little or no distinction between e_1, e_2, e_3 and $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}$, using either notation as warranted. You should also note, often you may see $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}$ written as $\partial_1, \partial_2, \partial_3$, or $\partial_{x_1}, \partial_{x_2}, \partial_{x_3}$, or even $\partial_x, \partial_y, \partial_z$. If we want to take into account the base point, say

$$v_p = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}_p \in T_p(\mathbb{R}^3)$$

then we would write

$$v_p = v_1 \left.\frac{\partial}{\partial x_1}\right|_p + v_2 \left.\frac{\partial}{\partial x_2}\right|_p + v_3 \left.\frac{\partial}{\partial x_3}\right|_p .$$

2.4 Differential One-Forms

Let us recap what we have done. First we reviewed vector spaces and dual spaces. After that we made a distinction between manifolds (manifold $\mathbb{R}^3$) and vector spaces (vector space $\mathbb{R}^3$) and introduced the concept of the tangent space $T_p(\mathbb{R}^3)$, which is basically a vector space $\mathbb{R}^3$ attached to each point p of manifold $\mathbb{R}^3$. After that we reviewed directional derivatives and used them to discover an identity between vectors and partial differential operators. We discovered that each vector $v_p \in T_p(\mathbb{R}^3)$ was exactly a differential operator

$$v_p = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}_p = v_1 \left.\frac{\partial}{\partial x_1}\right|_p + v_2 \left.\frac{\partial}{\partial x_2}\right|_p + v_3 \left.\frac{\partial}{\partial x_3}\right|_p .$$

Recognizing that each tangent space $T_p(\mathbb{R}^n)$ is itself a vector space it should be obvious that each tangent space $T_p(\mathbb{R}^n)$ has its own dual space, which is denoted $T_p^*(\mathbb{R}^n)$. It is this space that we are now going to look at.

We are now ready for the definition we have all been waiting for, the definition of a differential one-form.

Definition 2.4.1 A **differential one-form** α on manifold $\mathbb{R}^n$ is a linear functional on the set of tangent vectors to the manifold $\mathbb{R}^n$. That is, at each point p of manifold $\mathbb{R}^n$, $\alpha : T_p(\mathbb{R}^n) \to \mathbb{R}$ and

$$\alpha(v_p + w_p) = \alpha(v_p) + \alpha(w_p),$$
$$\alpha(av_p) = a\alpha(v_p)$$

for all $v_p, w_p \in T_p(\mathbb{R}^n)$ and $a \in \mathbb{R}$.

There are a number of comments that need to be made to help you understand this definition better. First of all, we have already defined linear functionals when we discussed vector spaces. The set of all linear functionals of a vector space is called the **dual space** of that vector space. In the definition of one-forms the vector space was $T_p(\mathbb{R}^n)$. The dual space of $T_p(\mathbb{R}^n)$ is generally denoted as $T_p^*(\mathbb{R}^n)$. Thus, we have $\alpha \in T_p^*(\mathbb{R}^n)$.

The next thing to notice is that in the first required identity in the definition the addition on the left of the equal sign takes place in the vector space $T_p(\mathbb{R}^n)$. In other words, $v_p + w_p \in T_p(\mathbb{R}^n)$. The addition on the right of the equal sign takes place in the reals $\mathbb{R}$. That is, $\alpha(v_p) + \alpha(w_p) \in \mathbb{R}$. There is also a slight peculiarity with the terminology; α is called a **differential one-form on the manifold** $\mathbb{R}^n$ even though its inputs are vectors, elements of the manifold's tangent space $T_p(\mathbb{R}^n)$ at some point p in the manifold $\mathbb{R}^n$. So, even though α eats elements of the tangent spaces $T_p(\mathbb{R}^n)$ of manifold $\mathbb{R}^n$ it is still called a differential one-form **on** manifold $\mathbb{R}^n$. If we want to specify the point the differential form is at, we will use either α_p or $\alpha(p)$. Clearly we have $\alpha_p \in T_p^*(\mathbb{R}^n)$, the dual space of $T_p(\mathbb{R}^n)$.

Now, why is it called a differential one-form? The one- is added because it takes as its input only one tangent vector. Later on we will meet two-forms, three-forms, and general k-forms that take as inputs two, three, or k tangent vectors. The word differential is used because of the intimate way forms are related to the idea of exterior differentiation, which we will study later. Finally, the word form is a somewhat generic mathematical term that gets applied to a fairly wide range of objects (bilinear forms, quadratic forms, multilinear forms, the first and second fundamental forms, etc.) Thus, a **differential** k**-form** is a mathematical object whose input is k vectors at a point and which has something to do with an idea about differentiation. Very often the word differential is dropped and differential k-forms are simply referred to as k-**forms**. So far this definition is a little abstract so in order to gain a better understanding we will now consider some concrete but simple examples of one-forms on the manifold $\mathbb{R}^3$. But before we can write down concrete examples of one-forms on manifold $\mathbb{R}^3$ we need to decide on a basis for $T_p^*(\mathbb{R}^3)$.

Based on our review of vector spaces and the notation we used there, the **dual basis** is was written as the vector space basis with superscripts instead of subscripts. Suppose we wrote the basis of $T_p(\mathbb{R}^3)$ as $\left\{e_{1_p}, e_{2_p}, e_{3_p}\right\}$ then we could possibly try to write the basis of $T_p^*(\mathbb{R}^3)$ as $\left\{e_p^1, e_p^2, e_p^3\right\}$. But recalling the identification between the standard Euclidian unit vectors and the differential operators, we could also write the bases of $T_p(\mathbb{R}^3)$ as

$$\left\{ \left.\frac{\partial}{\partial x_1}\right|_p, \left.\frac{\partial}{\partial x_2}\right|_p, \left.\frac{\partial}{\partial x_3}\right|_p \right\}.$$

But how would we write the dual basis now? We will write the dual basis, that is, the basis of the dual space $T_p^*(\mathbb{R}^3)$, as

$$\left\{dx_{1_p}, dx_{2_p}, dx_{3_p}\right\}.$$

Yes, this basis looks like the differentials that you are familiar with from calculus. There is a reason for that. Once we have worked a few problems and you are a little more comfortable with the notation we will explore the reasons for the notation. Dropping the base point p from the notation we will write

$$\left\{dx_1, dx_2, dx_3\right\}$$

as the basis of $T^*(\mathbb{R}^3)$ dual to the basis of $T(\mathbb{R}^3)$,

$$\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right\}.$$

Figure 2.28 is an attempt to pictorially draw the manifold $\mathbb{R}^3$, a tangent space at a point p, $T_p\mathbb{R}^3$, and its dual space $T_p^*\mathbb{R}^3$. Often we think of the tangent space $T_p\mathbb{R}^3$ and the dual space $T_p^*(\mathbb{R}^3)$ as being "attached" to the manifold $\mathbb{R}^3$ at the point p, but we have drawn the dual space $T_p^*\mathbb{R}^3$, directly above the tangent space $T_p\mathbb{R}^3$. The dual space $T_p^*\mathbb{R}^3$ is very often called the **cotangent space** at p.

Since any element of $T_p^*\mathbb{R}^3$ can be written as a linear combination of $\{dx_1, dx_2, dx_3\}$ and elements of $T_p^*\mathbb{R}^3$ are one-forms this of course implies that the one-forms on $\mathbb{R}^3$ can be written as linear combinations of $\{dx_1, dx_2, dx_3\}$. In particular, the basis elements dx_1, dx_2, and dx_3 are themselves one-forms on $\mathbb{R}^3$.

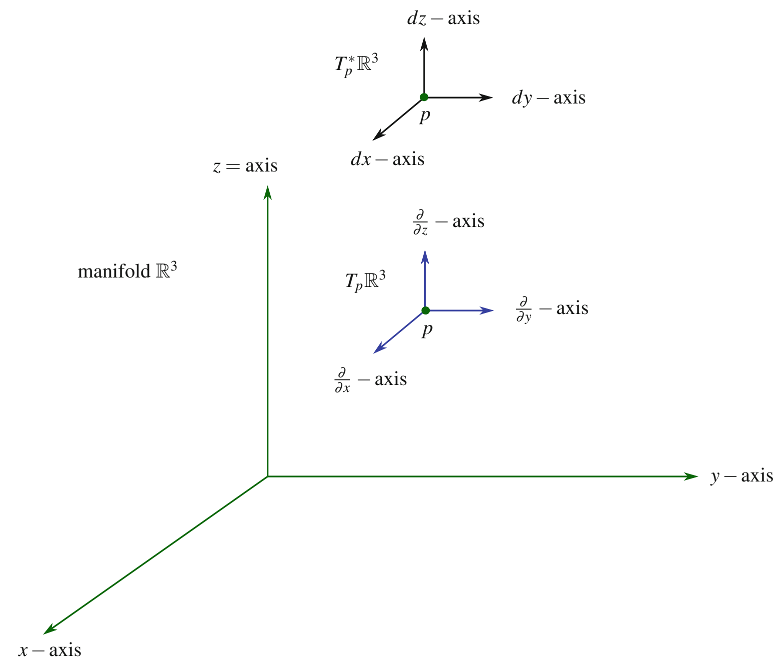

Fig. 2.28 An illustration of the manifold $\mathbb{R}^3$ along with the tangent space $T_p\mathbb{R}^3$ "attached" to the manifold at point p. The dual space $T_p^*\mathbb{R}^3$ is drawn above the tangent space which it is dual to. Notice the different ways the axis are labeled

Now we will see exactly how the dual basis, that is, the one-forms, $\{dx_1, dx_2, dx_3\}$ act on the tangent space basis elements $\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right\}$. From the definition of dual basis, we have

$$
\begin{aligned}
dx_1\left(\frac{\partial}{\partial x_1}\right) &= 1, \quad & dx_2\left(\frac{\partial}{\partial x_1}\right) &= 0, \quad & dx_3\left(\frac{\partial}{\partial x_1}\right) &= 0,\\
dx_1\left(\frac{\partial}{\partial x_2}\right) &= 0, \quad & dx_2\left(\frac{\partial}{\partial x_2}\right) &= 1, \quad & dx_3\left(\frac{\partial}{\partial x_2}\right) &= 0,\\
dx_1\left(\frac{\partial}{\partial x_3}\right) &= 0, \quad & dx_2\left(\frac{\partial}{\partial x_3}\right) &= 0, \quad & dx_3\left(\frac{\partial}{\partial x_3}\right) &= 1.
\end{aligned}
$$

Now let us consider a vector, say

$$
\begin{bmatrix} -1 \\ 3 \\ -4 \end{bmatrix} = -\frac{\partial}{\partial x_1} + 3\frac{\partial}{\partial x_2} - 4\frac{\partial}{\partial x_3},
$$

and see how the differential forms dx_1, dx_2 and dx_3 act on it. First we see how dx_1 acts on the vector:

$$
\begin{aligned}
& dx_1\left(-\frac{\partial}{\partial x_1} + 3\frac{\partial}{\partial x_2} - 4\frac{\partial}{\partial x_3}\right)\\
&= dx_1\left(-\frac{\partial}{\partial x_1}\right) + dx_1\left(3\frac{\partial}{\partial x_2}\right) + dx_1\left(-4\frac{\partial}{\partial x_3}\right)\\
&= -dx_1\left(\frac{\partial}{\partial x_1}\right) + 3dx_1\left(\frac{\partial}{\partial x_2}\right) - 4dx_1\left(\frac{\partial}{\partial x_3}\right)\\
&= -1(1) + 3(0) - 4(0)\\
&= -1.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
& dx_2\left(-\frac{\partial}{\partial x_1}+3\frac{\partial}{\partial x_2}-4\frac{\partial}{\partial x_3}\right) \\
&= dx_2\left(-\frac{\partial}{\partial x_1}\right)+dx_2\left(3\frac{\partial}{\partial x_2}\right)+dx_2\left(-4\frac{\partial}{\partial x_3}\right) \\
&= -dx_2\left(\frac{\partial}{\partial x_1}\right)+3dx_2\left(\frac{\partial}{\partial x_2}\right)-4dx_2\left(\frac{\partial}{\partial x_3}\right) \\
&= -1(0)+3(1)-4(0) \\
&= 3
\end{aligned}
$$

and

$$
\begin{aligned}
& dx_3\left(-\frac{\partial}{\partial x_1}+3\frac{\partial}{\partial x_2}-4\frac{\partial}{\partial x_3}\right) \\
&= dx_3\left(-\frac{\partial}{\partial x_1}\right)+dx_3\left(3\frac{\partial}{\partial x_2}\right)+dx_3\left(-4\frac{\partial}{\partial x_3}\right) \\
&= -dx_3\left(\frac{\partial}{\partial x_1}\right)+3dx_3\left(\frac{\partial}{\partial x_2}\right)-4dx_3\left(\frac{\partial}{\partial x_3}\right) \\
&= -1(0)+3(0)-4(1) \\
&= -4.
\end{aligned}
$$

In essence, the differential one-forms dx_1, dx_2, and dx_3 find the projections of the vector onto the appropriate axis. Reverting to more traditional x, y, z notation, the one-form dx finds the projection of v_p onto the $\frac{\partial}{\partial x}$ axis, the one-form dy finds the projection of v_p onto the $\frac{\partial}{\partial y}$ axis, and the one-form dz finds the projection of v_p onto the $\frac{\partial}{\partial z}$ axis. Pictorially this is shown by Fig. 2.29.

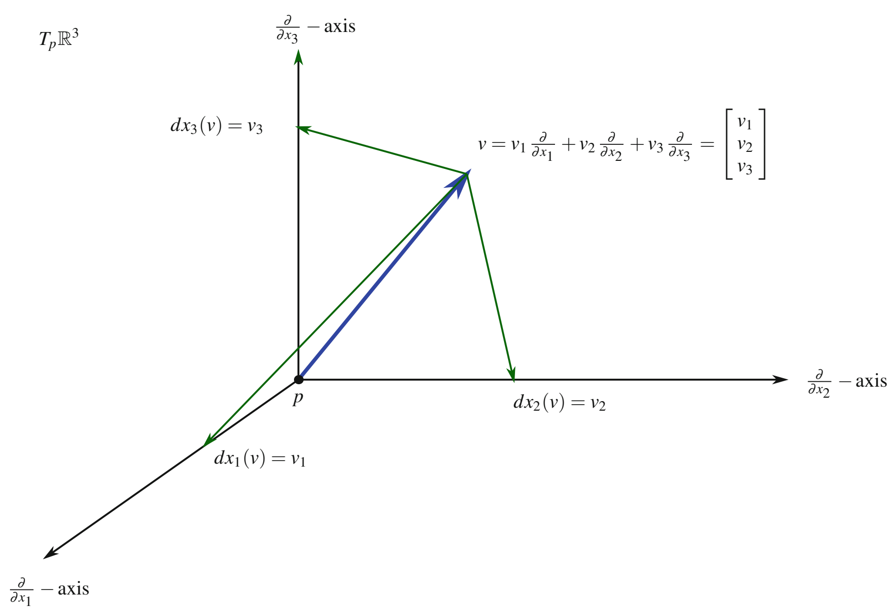

Fig. 2.29 An illustration of how the dual basis elements act on a vector v. The diagram does not include the point p in the notation

Question 2.7 Find the following:

(a) $dx_1\Big(2\frac{\partial}{\partial x_1} + 3\frac{\partial}{\partial x_1}\Big)$
(b) $dx_2\Big(-\frac{\partial}{\partial x_1} + 3\frac{\partial}{\partial x_2} - 4\frac{\partial}{\partial x_3}\Big)$
(c) $dx_3\Big(7\frac{\partial}{\partial x_2} - 5\frac{\partial}{\partial x_3}\Big)$

Since the set $\{dx, dy, dz\}$ is the basis for $T_p^*\mathbb{R}^3$ then any element $\alpha \in T_p^*\mathbb{R}^3$ can be written in the form $adx + bdy + cdz$ where $a, b, c \in \mathbb{R}$. These one-forms behave exactly as one would expect them too. Consider, for example, how the one form $2dx - 3dy + 5dz$ acts on the vector $-\frac{\partial}{\partial x_1} + 3\frac{\partial}{\partial x_2} - 4\frac{\partial}{\partial x_3}$:

$$
\begin{aligned}
&(2dx - 3dy + 5dz)\left(-\frac{\partial}{\partial x} + 3\frac{\partial}{\partial y} - 4\frac{\partial}{\partial z}\right)\\
&= 2dx\left(-\frac{\partial}{\partial x} + 3\frac{\partial}{\partial y} - 4\frac{\partial}{\partial z}\right)\\
&\quad - 3dy\left(-\frac{\partial}{\partial x} + 3\frac{\partial}{\partial y} - 4\frac{\partial}{\partial z}\right)\\
&\quad + 5dz\left(-\frac{\partial}{\partial x} + 3\frac{\partial}{\partial y} - 4\frac{\partial}{\partial z}\right)\\
&= 2dx\left(-\frac{\partial}{\partial x}\right) + 2dx\Big(3\frac{\partial}{\partial y}\Big) + 2dx\Big(-4\frac{\partial}{\partial z}\Big)\\
&\quad - 3dy\left(-\frac{\partial}{\partial x}\right) - 3dy\left(3\frac{\partial}{\partial y}\right) - 3dy\left(-4\frac{\partial}{\partial z}\right)\\
&\quad 5dz\left(-\frac{\partial}{\partial x}\right) + 5dz\left(3\frac{\partial}{\partial y}\right) + 5dz\left(-4\frac{\partial}{\partial z}\right)\\
&= (2)(-1)dx\left(\frac{\partial}{\partial x}\right) + (2)(3)dx\left(\frac{\partial}{\partial y}\right) + (2)(-4)dx\Big(\frac{\partial}{\partial z}\Big)\\
&\quad (-3)(-1)dy\left(\frac{\partial}{\partial x}\right) + (-3)(3)dy\left(\frac{\partial}{\partial y}\right) + (-3)(-4)dy\left(\frac{\partial}{\partial z}\right)\\
&\quad (5)(-1)dz\left(\frac{\partial}{\partial x}\right) + (5)(3)dz\left(\frac{\partial}{\partial y}\right) + (5)(-4)dz\left(\frac{\partial}{\partial z}\right)\\
&= (2)(-1)(1) + (2)(3)(0) + (2)(-4)(0)\\
&\quad (-3)(-1)(0) + (-3)(3)(1) + (-3)(-4)(0)\\
&\quad (5)(-1)(0) + (5)(3)(0) + (5)(-4)(1)\\
&= -31.
\end{aligned}
$$

We have gone to extra care to show what happens at every step of this computation, something we certainly will not always do.

Question 2.8 Find the following:

(a) $\Big(dx_1 + dx_2\Big)\Big(5\frac{\partial}{\partial x_1}\Big)$
(b) $\Big(2dx_2 - 3dx_3\Big)\Big(3\frac{\partial}{\partial x_1} - 7\frac{\partial}{\partial x_2} + 5\frac{\partial}{\partial x_3}\Big)$
(c) $\Big(-dx_1 - 4dx_2 + 6dx_3\Big)\Big(-2\frac{\partial}{\partial x_1} + 5\frac{\partial}{\partial x_2} - 3\frac{\partial}{\partial x_3}\Big)$

In order to make the computations simpler, one-forms are often written as row vectors, exactly as we wrote elements of the dual space as row vectors in our review of vector spaces. For example, the one-form $4dx_1 - 2dx_2 + 5dx_3$ can be written as the row vector $[4, -2, 5]$. Recall, we said the dual space at p is often called the cotangent space at p. Thus, when a one-form, which is an element of the cotangent space, is written as a row vector it is often called a **co-vector**. This allows us to do the above computation as a matrix multiplication. Let us redo the last computation to see how,

$$\begin{aligned}
(2dx - 3dy + 5dz)&\left(-\frac{\partial}{\partial x} + 3\frac{\partial}{\partial y} - 4\frac{\partial}{\partial z}\right) \\
&= [2, -3, 5]\begin{bmatrix} -1 \\ 3 \\ -4 \end{bmatrix} \\
&= (2)(-1) + (-3)(3) + (5)(-4) \\
&= -31.
\end{aligned}$$

This is certainly a more straight forward computation.

Question 2.9 Do the following computations as matrix multiplication by first writing the one-form as a row vector (co-vector) and the vector element as a column vector.

(a) $\Big(-7dx_1 + dx_2\Big)\Big(2\frac{\partial}{\partial x_2}\Big)$
(b) $dx_3\Big(7\frac{\partial}{\partial x_2} - 5\frac{\partial}{\partial x_3}\Big)$
(c) $\Big(2dx_2 - 3dx_3\Big)\Big(3\frac{\partial}{\partial x_1} - 7\frac{\partial}{\partial x_2} + 5\frac{\partial}{\partial x_3}\Big)$
(d) $\Big(-dx_1 - 4dx_2 + 6dx_3\Big)\Big(-2\frac{\partial}{\partial x_1} + 5\frac{\partial}{\partial x_2} - 3\frac{\partial}{\partial x_3}\Big)$

Notice that we now have two different mental images that we can use to try to picture the differential one-form. Consider Fig. 2.30. The top is an image of a differential one-form as a row vector, which is the easiest image to have. Differential one-forms of the form $adx + bdy + cdz$, $a, b, c \in \mathbb{R}$, are elements in the cotangent space at some point, $T_p^*\mathbb{R}^3$, and so imagining them as a row vector (called a co-vector) is natural. The second image is to view the one-forms dx, dy, dz as ways to find the projection of a vector v_p on different axes. In that case the one-form $adx + bdy + cdz$ can be viewed as scaling the respective projections by the factors a, b, and c and then summing these different scalings.

In the last few questions we implicitly assumed that the one-forms and vectors were all at the same point of the manifold. But just as vectors gave us vector fields, we can think of one-form "fields" on the manifold $\mathbb{R}^3$. Actually though, we generally won't call them fields, we simply refer to them as one-forms. Remember that slightly odd terminology, "one-forms *on* the manifold $\mathbb{R}^3$"? A one-form *on* a manifold is actually a **one-form field** that gives a particular one-form at each point p of the manifold, which acts on tangent vectors v_p that are based at that point.

Let us consider an example. Given the following real-valued functions on the manifold $\mathbb{R}^3$, $f(x, y, z) = x^2y$, $g(x, y, z) = \frac{x}{2} + yz$, and $h(x, y, z) = x + y + 3$ define, at each point at each point $p = (x, y, z) \in \mathbb{R}^3$, the one-form ϕ on manifold $\mathbb{R}^3$ by $\phi_{(x,y,z)} = f(x, y, z)dx + g(x, y, z)dy + h(x, y, z)dz$. The one-form ϕ on manifold $\mathbb{R}^3$ defines a different one-form at each point $p \in \mathbb{R}^3$. For example, at the point $(1, 2, 3)$ the one-form $\phi_{(1,2,3)}$ is given by

$$\begin{aligned}
\phi_{(1,2,3)} &= f(1, 2, 3)dx + g(1, 2, 3)dy + h(1, 2, 3)dz \\
&= 2dx + \frac{13}{2}dy + 6dz.
\end{aligned}$$

If we were given the vector

$$v_p = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}_p$$

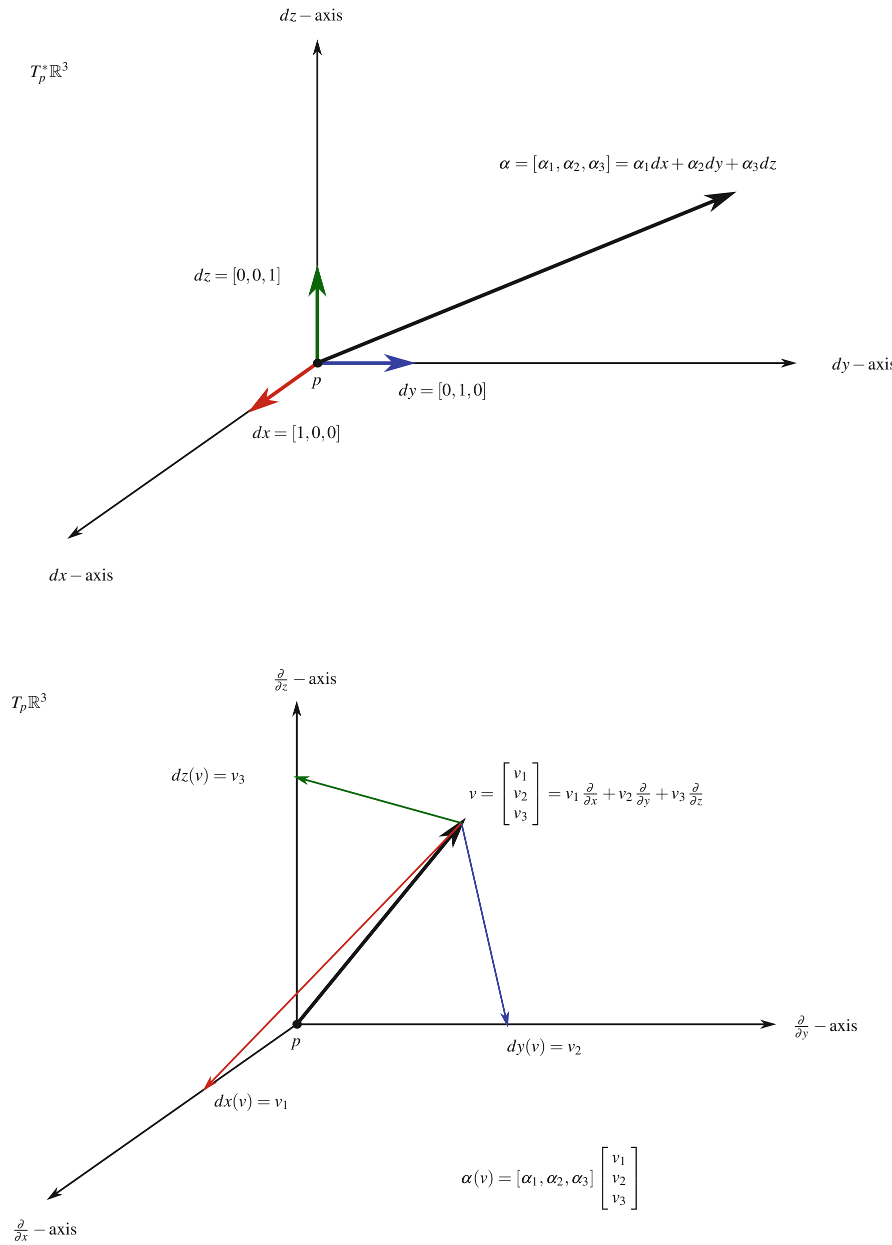

Fig. 2.30 Two different ways to imagine or visualize a differential one-form. As co-vectors in $T_p^*\mathbb{R}^3$ (top) or as a linear combination of the projections onto the axes in $T_p\mathbb{R}^3$ (bottom)

at $p = (1, 2, 3)$ we would have

$$
\begin{aligned}
\phi_p(v_p) &= \left(2dx + \frac{13}{2}dy + 6dz\right)\left(2\frac{\partial}{\partial x} - 1\frac{\partial}{\partial y} - 2\frac{\partial}{\partial z}\right) \\
&= 2dx\left(2\frac{\partial}{\partial x} - 1\frac{\partial}{\partial y} - 2\frac{\partial}{\partial z}\right) + \frac{13}{2}dy\left(2\frac{\partial}{\partial x} - 1\frac{\partial}{\partial y} - 2\frac{\partial}{\partial z}\right) + 6dz\left(2\frac{\partial}{\partial x} - 1\frac{\partial}{\partial y} - 2\frac{\partial}{\partial z}\right) \\
&= \left[2, \frac{13}{2}, 6\right]_{(1,2,3)} \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}_{(1,2,3)}
\end{aligned}
$$

$$
\begin{aligned}
&= 2(2) + \frac{13}{2}(-1) + 6(-2) \\
&= \frac{-29}{2}.
\end{aligned}
$$

Thus, given a one-form ϕ on the manifold $\mathbb{R}^3$, at each point $p \in \mathbb{R}^3$ we have a mapping $\phi_p : T_p(\mathbb{R}^3) \to \mathbb{R}$. If we were given a vector field v on manifold $\mathbb{R}^3$, that is, a section on the tangent bundle $T\mathbb{R}^3$, then for each $p \in \mathbb{R}^3$ we have $\phi_p(v_p) \in \mathbb{R}$. So what would $\phi(v)$ be? Notice we have not put in a point. We could consider $\phi(v)$ to be a function on manifold $\mathbb{R}^3$. That is, its inputs are points p on the manifold and its outputs are real numbers, like so

$$
\begin{aligned}
\phi(v) : \mathbb{R}^3 &\longrightarrow \mathbb{R} \\
p &\longmapsto \phi_p(v_p).
\end{aligned}
$$

Question 2.10 For the one-form $\phi_{(x,y,z)} = f(x,y,z)dx + g(x,y,z)dy + h(x,y,z)dz$ where $f(x,y,z) = xy^2$, $g(x,y,z) = \frac{xy}{3} + x^2$, and $h(x,y,z) = xy + yz + xz$, and the vector $v_p = \begin{bmatrix} x \\ x^2y \\ xz \end{bmatrix}_p$ find

(a) (i) $\phi_{(-1,2,1)}$
(ii) $\phi_{(0,-1,2)}$
(iii) $\phi_{(8,4,-3)}$
(b) Find $\phi(v)$
(c) (i) Find $\phi_p(v_p)$ for $p = (-1,2,1)$
(ii) Find $\phi_p(v_p)$ for $p = (0,-1,2)$
(iii) Find $\phi_p(v_p)$ for $p = (8,4,-3)$

Now that we are a little more comfortable with what one-forms actually do, we are ready to try to understand the notation better. In essence, we want to understand our reason for denoting dx, dy, dz as the basis of $T_p^*(\mathbb{R}^3)$ dual to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ of $T_p(\mathbb{R}^3)$. First we make another definition that, at first glance, may seem a little circular to you.

Definition 2.4.2 Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function on the manifold $\mathbb{R}^n$. The **differential** df of f is defined to be the one-form on $\mathbb{R}^n$ such that for all vectors v_p we have

$$
df(v_p) = v_p[f].
$$

Let's just take a moment to try to unpack and digest all this considering the manifold $\mathbb{R}^3$. First of all, given a function $f : \mathbb{R}^3 \to \mathbb{R}$ on manifold $\mathbb{R}^3$ and a vector $v_p \in T_p(\mathbb{R})$ at point p in manifold $\mathbb{R}^3$ we know how to take the directional derivative of this function at the point p the direction v_p

$$
\lim_{t \to 0} \frac{f\big(p + tv_p\big) - f\big(p\big)}{t},
$$

which can also be written as

$$
\frac{d}{dt}\Big(f\big(p + tv_p\big)\Big)\Big|_{t=0}.
$$

We then thought of this as using the vector v_p to perform an operation on the given function f, and so introduced a slightly new notation for this

$$
v_p[f] = \frac{d}{dt}\Big(f\big(p + tv_p\big)\Big)\Big|_{t=0},
$$

which in turn we have just used to define the differential of f, written as df, by $df(v_p) = v_p[f]$. Letting $p = (x_0, y_0, z_0)$ we put this all together to get

$$
\begin{aligned}
df(v_p) &= v_p[f] \\
&= \frac{d}{dt}\Big(f\big(p + tv_p\big)\Big)\Big|_{t=0} \\
&= \frac{d}{dt} f\big(\underbrace{x_0 + tv_1}_{x(t)}, \underbrace{y_0 + tv_2}_{y(t)}, \underbrace{z_0 + tv_3}_{z(t)}\big)\Big|_{t=0} \\
&= \frac{\partial f}{\partial x}\bigg|_p \cdot \frac{dx(t)}{dt}\bigg|_{t=0} + \frac{\partial f}{\partial y}\bigg|_p \cdot \frac{dy(t)}{dt}\bigg|_{t=0} + \frac{\partial f}{\partial z}\bigg|_p \cdot \frac{dz(t)}{dt}\bigg|_{t=0} \\
&= \frac{\partial f}{\partial x}\bigg|_p \cdot v_1 + \frac{\partial f}{\partial y}\bigg|_p \cdot v_2 + \frac{\partial f}{\partial z}\bigg|_p \cdot v_3.
\end{aligned}
$$

In summary we have

$$
df(v_p) = \frac{\partial f}{\partial x}\bigg|_p \cdot v_1 + \frac{\partial f}{\partial y}\bigg|_p \cdot v_2 + \frac{\partial f}{\partial z}\bigg|_p \cdot v_3.
$$

Since this formula works for any function $f : \mathbb{R}^3 \to \mathbb{R}$ it works for the special Cartesian coordinate functions $x, y, z : \mathbb{R}^3 \to \mathbb{R}$. Recall the Cartesian coordinate functions; if $p \in \mathbb{R}^3$, then $x(p)$ is the Cartesian x-coordinate value of p, $y(p)$ is the Cartesian y-coordinate value of p, and $z(p)$ is the Cartesian z-coordinate value of p. For example, if we could write the point p as $(3, 2, 1)$ in Cartesian coordinates, then $x(p) = 3$, $y(p) = 2$, and $z(p) = 1$. Now let us see exactly what happens to each of these Cartesian coordinate functions. Suppose that

$$
v_p = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}_p = v_1 \frac{\partial}{\partial x}\bigg|_p + v_2 \frac{\partial}{\partial y}\bigg|_p + v_3 \frac{\partial}{\partial z}\bigg|_p .
$$

Then for the Cartesian coordinate function $x : \mathbb{R}^3 \to \mathbb{R}$ we have that

$$
\begin{aligned}
dx(v_p) &= \frac{\partial x}{\partial x}\bigg|_p \cdot v_1 + \frac{\partial x}{\partial y}\bigg|_p \cdot v_2 + \frac{\partial x}{\partial z}\bigg|_p \cdot v_3 \\
&= (1) \cdot v_1 + (0) \cdot v_2 + (0) \cdot v_3 \\
&= v_1.
\end{aligned}
$$

But what do we mean when we write $\frac{\partial x}{\partial x}$? The x in the numerator is the Cartesian coordinate function while the x in the denominator represents the variable we are differentiating with respect to. Thus $\frac{\partial x}{\partial x}$ gives the rate of change of the Cartesian coordinate function x is in the x-direction.

Question 2.11 Explain that the rate of change of the Cartesian coordinate function x in the x-direction is indeed one, thereby showing that $\frac{\partial x}{\partial x} = 1$. It may be helpful to consider how $x(p)$ changes as the point p moves in the x-direction. What is the rate of change of the Cartesian coordinate function x in the y and z-directions? Explain how your answer shows that $\frac{\partial x}{\partial y} = 0$ and $\frac{\partial x}{\partial z} = 0$.

Similarly, we have

$$
\begin{aligned}
dy(v_p) &= \frac{\partial y}{\partial x}\bigg|_p \cdot v_1 + \frac{\partial y}{\partial y}\bigg|_p \cdot v_2 + \frac{\partial y}{\partial z}\bigg|_p \cdot v_3 \\
&= (0) \cdot v_1 + (1) \cdot v_2 + (0) \cdot v_3 \\
&= v_2
\end{aligned}
$$

and

$$\begin{aligned} dz(v_p) &= \left.\frac{\partial z}{\partial x}\right|_p \cdot v_1 + \left.\frac{\partial z}{\partial y}\right|_p \cdot v_2 + \left.\frac{\partial z}{\partial z}\right|_p \cdot v_3 \\ &= (0) \cdot v_1 + (0) \cdot v_2 + (1) \cdot v_3 \\ &= v_3. \end{aligned}$$

Question 2.12 Find the rate of change of the Cartesian coordinate function y in the x, y, and z-directions and use this to show that $\frac{\partial y}{\partial x} = 0$, $\frac{\partial y}{\partial y} = 1$, and $\frac{\partial y}{\partial z} = 0$. Then find the rate of change of the Cartesian coordinate function z in the x, y, and z-directions to show that $\frac{\partial z}{\partial x} = 0$, $\frac{\partial z}{\partial y} = 0$, and $\frac{\partial z}{\partial z} = 1$.

Question 2.13 Writing $e_1 = \frac{\partial}{\partial x}$, $e_2 = \frac{\partial}{\partial y}$ and $e_3 = \frac{\partial}{\partial z}$ show that the differentials of the Cartesian coordinate functions, dx, dy, and dz give

$$\begin{aligned} &dx\left(\frac{\partial}{\partial x}\right) = 1, \quad dy\left(\frac{\partial}{\partial x}\right) = 0, \quad dz\left(\frac{\partial}{\partial x}\right) = 0, \\ &dx\left(\frac{\partial}{\partial y}\right) = 0, \quad dy\left(\frac{\partial}{\partial y}\right) = 1, \quad dz\left(\frac{\partial}{\partial y}\right) = 0, \\ &dx\left(\frac{\partial}{\partial z}\right) = 0, \quad dy\left(\frac{\partial}{\partial z}\right) = 0, \quad dz\left(\frac{\partial}{\partial z}\right) = 1. \end{aligned}$$

Now we can compare the behavior of the differentials dx, dy, and dz of the Cartesian coordinate functions x, y, and z with the behavior of the dual basis elements, that is, the duals of the basis elements $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$ of $T_p(\mathbb{R}^3)$. We see that they behave exactly the same. This explains the choice of notation for the dual basis elements at the beginning of the section. ***The dual basis elements are exactly the differentials of the Cartesian coordinate functions.***

Given a specific function f we would like to know how to actually write the differential df. Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is a function on manifold $\mathbb{R}^3$ and v_p is an arbitrary vector at an arbitrary point. Since

$$df(v_p) = \left.\frac{\partial f}{\partial x}\right|_p \cdot v_1 + \left.\frac{\partial f}{\partial y}\right|_p \cdot v_2 + \left.\frac{\partial f}{\partial z}\right|_p \cdot v_3$$

and

$$v_1 = dx(v_p), \ v_2 = dy(v_p), \ v_3 = dz(v_p).$$

We put this together to get

$$\begin{aligned} df(v_p) &= \left.\frac{\partial f}{\partial x}\right|_p \cdot v_1 + \left.\frac{\partial f}{\partial y}\right|_p \cdot v_2 + \left.\frac{\partial f}{\partial z}\right|_p \cdot v_3 \\ &= \left.\frac{\partial f}{\partial x}\right|_p \cdot dx(v_p) + \left.\frac{\partial f}{\partial y}\right|_p \cdot dy(v_p) + \left.\frac{\partial f}{\partial z}\right|_p \cdot dz(v_p) \\ &= \left(\left.\frac{\partial f}{\partial x}\right|_p \cdot dx + \left.\frac{\partial f}{\partial y}\right|_p \cdot dy + \left.\frac{\partial f}{\partial z}\right|_p \cdot dz\right)(v_p) \end{aligned}$$

and so we have

$$df = \left.\frac{\partial f}{\partial x}\right|_p dx + \left.\frac{\partial f}{\partial y}\right|_p dy + \left.\frac{\partial f}{\partial z}\right|_p dz.$$

We have written the differential of the function f as a linear combination of the dual basis elements, thereby showing that $df \in T_p^*(\mathbb{R}^3)$. The differential of a function f is a one-form.

The final thing we want to do is to consider another way to think of the differential df of f. In order to draw the necessary pictures we will consider the manifold $\mathbb{R}^2$ since we can picture the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$ in three dimensions, while picturing the graph of a function $f : \mathbb{R}^3 \to \mathbb{R}$ properly would require four dimensions, but the underlying idea for all functions $f : \mathbb{R}^n \to \mathbb{R}$ is the same. Letting $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ we have

$$\begin{aligned} v[f] &= \left(v_1 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} \right) [f] \\ &= \frac{\partial f}{\partial x} v_1 + \frac{\partial f}{\partial y} v_2 \\ &= \frac{\partial f}{\partial x} dx(v) + \frac{\partial f}{\partial y} dy(v) \\ &= \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right) (v) \\ &= df(v). \end{aligned}$$

Question 2.14 Repeat the above calculation putting in the base point. The differential of f at p can be written either as df_p or as $df(p)$.

But now for the real question, what exactly is this differential of f, df? The key lies in Fig. 2.31. The differential df of f takes in a vector v_p, which is at some point p of the manifold $\mathbb{R}^2$, and gives out a number that is the "rise" of the tangent plane to the graph of the function f as one moves from p along v_p. This output can be viewed as also being a point on the tangent plane. Thus df in a sense "encodes" the "rises" that occur in the tangent plane to the function f as one moves along different vectors with base point p. This should not be too surprising. The formula for the tangent plane T to $f(x, y)$ at the point (x_0, y_0) is given by

$$T(x, y) = f(x_0, y_0) + \left. \frac{\partial f}{\partial x} \right|_{(x_0, y_0)} \cdot (x - x_0) + \left. \frac{\partial f}{\partial y} \right|_{(x_0, y_0)} \cdot (y - y_0).$$

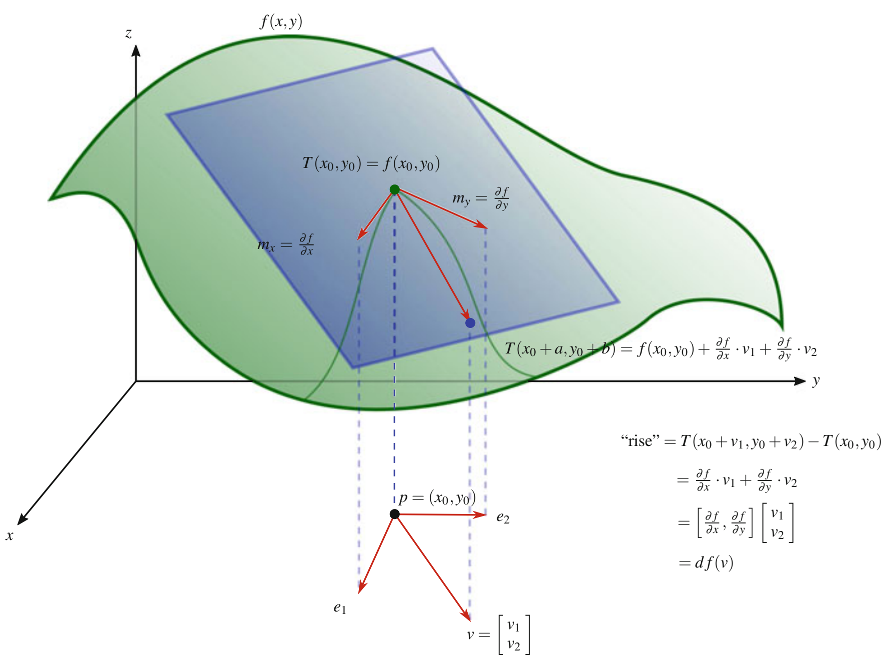

Fig. 2.31 The differential df_p is the linear approximation of the function f at the point p. In other words, the differential df_p "encodes" the tangent plane of f at p

As the picture indicates, given the vector $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_p$, this vector's endpoint (in $\mathbb{R}^2$) is $(x_0 + v_1, y_0 + v_2)$. The point in the tangent plane to f (at the point $p = (x_0, y_0)$) that lies above $(x_0 + v_1, y_0 + v_2)$ is given by

$$\begin{aligned} T(x_0 + v_1, y_0 + v_2) &= f(x_0, y_0) + \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} \cdot (x_0 + v_1 - x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \cdot (y_0 + v_2 - y_0) \\ &= f(x_0, y_0) + \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} \cdot v_1 + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \cdot v_2 \end{aligned}$$

so the "rise" in the tangent plane from $f(x_0, y_0)$ is given by

$$\begin{aligned} \text{"rise"} &= T(x_0 + v_1, y_0 + v_2) - T(x_0, y_0) \\ &= f(x_0, y_0) + \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} \cdot v_1 + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \cdot v_2 - f(x_0, y_0) \\ &= \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} \cdot v_1 + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \cdot v_2 \\ &= df_p(v_p). \end{aligned}$$

Thus the differential of f at p, df_p, which gives the "rise" of the tangent plane to f at p as you move along the vector v_p, essentially encodes how this tangent plane behaves. The tangent plane is the closest linear approximation of f at p, so in essence *df_p can be thought of as the linear approximation of the function f at the point p.*

As one last comment, does the differential one-form df written as a row vector

$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \\ &= \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right] \end{aligned}$$

in the $\mathbb{R}^2$ case or

$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \\ &= \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] \end{aligned}$$

in the $\mathbb{R}^3$ case remind you of anything from vector calculus? It should remind you of $grad(f)$ or $\nabla(f)$. In fact, df and $grad(f)$ are closely related. We will explore this relationship in Chap. 9.

We now consider a couple of examples. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x, y, z) = x^2y^3z$ and consider the vector $v_p = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_p$ at each point $p = (x, y, z) \in \mathbb{R}$. First let us find $df(v_p)$. Since v_p is the same at every point p we can drop the p from the notation. We have

$$\begin{aligned} df(v) &= v[f] \\ &= \left(\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y} + 3\frac{\partial}{\partial z}\right)(x^2y^3z) \\ &= \frac{\partial(x^2y^3z)}{\partial x} + 2\frac{\partial(x^2y^3z)}{\partial y} + 3\frac{\partial(x^2y^3z)}{\partial z} \\ &= 2xy^3z + 2(3x^2y^2z) + 3(x^2y^3). \end{aligned}$$

Now suppose we were given the specific point $p = (-1, 2, -2)$. Finding $df(v_p)$ at $p = (-1, 2, -2)$ simply requires us to substitute the numbers in

$$df(v_p) = 2(-1)^2(2)^3(-2) + 2(3(-1)^2(2)^2(-2)) + 3(-1)^2(2)^3 = 8.$$

Question 2.15 Suppose $f(x, y, z) = (x^2 - 1)y + (y^2 + 2)z$.

(i) Find df.

(ii) Let $v_p = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}_{(p_1,p_2,p_3)}$. Find $df(v_p)$.

(iii) Let $v_p = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}_{(-1,0,2)}$. Find $df(v_p)$.

Question 2.16 Let f and g be functions on $\mathbb{R}^3$. Show that $d(fg) = gdf + fdg$.

Question 2.17 Suppose $f : \mathbb{R}^3 \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$, and $h(f) : \mathbb{R}^3 \to \mathbb{R}$ are all defined. Show that $d(h(f)) = h'(f)df$.

2.5 Summary, References, and Problems

2.5.1 Summary

Because the distinction is important for more abstract spaces we first made a big deal between the collection of points in $\mathbb{R}^n$ as being the manifold $\mathbb{R}^n$ and the collection of vectors in $\mathbb{R}^n$ as being the vector space $\mathbb{R}^n$. Then we defined the Cartesian functions $x_i : \mathbb{R}^n \to \mathbb{R}$ that take a point p in the manifold $\mathbb{R}^n$ to its ith-coordinate value.

The tangent space of an n-dimensional manifold M at a point $p \in M$ was introduced as the space of all tangent vectors to M at the point p and was denoted as T_pM. The tangent space T_pM could essentially be viewed as a copy of the vector space $\mathbb{R}^n$ attached to M at the point p. The collection of all tangent spaces for a manifold was called the tangent bundle and was denoted by TM. A vector field was defined as a function v that assigns to each point p of M an element v_p of T_pM. A vector field is sometimes called a section of the tangent bundle.

The directional derivative of a function $f : M \to \mathbb{R}$ at the point p in the direction v_p was defined as

$$v_p[f] \equiv \frac{d}{dt}\Big(f\big(p + tv_p\big)\Big)\Big|_{t=0},$$

which is exactly the same definition from vector calculus only without the requirement that v_p be a unit vector. Dropping this requirement slightly changes the geometrical meaning of the directional derivative but allows us to retain the simple formula

$$v_p[f] = \sum_{i=1}^{n} v_i \cdot \frac{\partial f}{\partial x_i}\Big|_{p_i} \qquad \text{where} \qquad v_p = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

This formula is then used to make one of the must useful and unexpected identifications,

$$\boxed{\text{The Euclidian vectors } e_i \text{ can be identified with the partial differential operators } \frac{\partial}{\partial x_i}.}$$

In other words, we think of the Euclidian vector e_i as being the partial differential operator $\frac{\partial}{\partial x_i}$.

This directional derivative is then used to define a particular linear functional associated with the function f,

$$df(v_p) = v_p[f].$$

This linear functional is called the differential of f. Since $v_p \in T_pM$, which is a vector space, the linear functional df is in the dual space of T_pM, which is called the cotangent space. We write $df \in T_p^*M$. Furthermore, one can use the Cartesian coordinate functions x_i to define the differentials dx_i, which turn out to be the basis of T_p^*M dual to the Euclidian basis of T_pM. Thus, using the identification above, we have $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$ as the standard basis for T_pM and $\{dx_1, \dots, dx_n\}$ as the standard dual basis of T_p^*M. Elements of T_p^*M are also called differential one-forms. The differential one-forms dx_i act similar to the Cartesian coordinate functions and pick off the ith-component of the vector v_p, that is, $dx_i(v_p) = v_i$. Geometrically we can think of the dx_i as finding the length of the vector which is the projection of v_p onto the ith axis.

2.5.2 *References and Further Reading*

Vectors and vector fields show up all over multivariable calculus, see for example Stewart [43] and even Marsden and Hoffmen [31], but tangent spaces and tangent bundles are generally not encountered or made explicit at this level. That is usually left for more advanced courses, often in differential geometry. There are quite a number of interesting and rigorous introductions to this material. Naming only a few, Munkres [35], O'Neill [36], Tu [46], Spivak [41], and Thorpe [45] are all good introductions to analysis on manifolds and differential geometry and thus contain good expositions on the tangent space and tangent bundles. Again, Stewart [43] and Marsden and Hoffmen [31] are excellent references to directional derivatives and multivariable calculus. This chapter also begins a gentle introduction to differential forms, that will take several chapters to complete, by looking at differential one-forms. The material on differential one-forms in this chapter has generally followed the spirit of Bachman [4] and O'Neill [36], though Munkres [35], Edwards [18], and Martin [33] were also consulted.

2.5.3 *Problems*

Question 2.18 Let $p_1 = (2, 9, -2)$, $p_2 = (1, 0, -3)$, and $p_3 = (12, -7, 4)$ be points in $\mathbb{R}^3$ and let x,y, and z be the Cartesian coordinate functions on $\mathbb{R}^3$. Find

$a)\ x(p_1)$,	$d)\ x(p_2)$,	$g)\ x(p_3)$,
$b)\ y(p_1)$,	$e)\ y(p_2)$,	$h)\ y(p_3)$,
$c)\ z(p_1)$,	$f)\ z(p_2)$,	$i)\ z(p_3)$.

Question 2.19 Consider the manifold $\mathbb{R}^3$ and the vector field $v = xye_1 + \left(z - \frac{x}{3}\right) e_2 - (x + z - 4)e_3$, where e_1, e_2, e_3 are the Euclidian vectors. If $p_1 = (1, -3, 2)$, $p_2 = (0, 1, 4)$, and $p_3 = (-5, -2, 2)$ find the vectors v_{p_1}, v_{p_2}, and v_{p_3}. What space is each of these vectors in?

Question 2.20 Find the directional derivative of $f(x, y) = 5x^2y - 4xy^3$ in the directions $v_1 = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$ and $v_2 = \frac{1}{13}\begin{bmatrix} 5 \\ 12 \end{bmatrix}$ at the point $(1, 2)$. Explain what the difference between $D_{v_1}f$ and $D_{v_2}f$ is.

Question 2.21 Find the directional derivative of $f(x, y) = x\ln(y)$ in the directions $v_1 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ and $v_2 = \frac{1}{5}\begin{bmatrix} -4 \\ 3 \end{bmatrix}$ at the point $(1, -3)$. Explain what the difference between $D_{v_1}f$ and $D_{v_2}f$ is.

Question 2.22 Find the directional derivative of $f(x, y, z) = xe^{yz}$ in the directions $v_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ and $v_2 = \frac{1}{3}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ at the point $(3, 0, 2)$. Explain what the difference between $D_{v_1}f$ and $D_{v_2}f$ is.

Question 2.23 Given $f(x, y, z) = \sqrt{x + yz}$ and $v = 2\frac{\partial}{\partial x} + 3\frac{\partial}{\partial y} + 6\frac{\partial}{\partial z}$ find $v[f]$ at the point $(1, 3, 1)$.

Question 2.24 Given $f(x, y, z) = xe^y + ye^z + ze^x$ and $v = 5\frac{\partial}{\partial x} + 1\frac{\partial}{\partial y} - 2\frac{\partial}{\partial z}$ find $v[f]$ at the point $(0, 0, 0)$.

Question 2.25 Given $f(x, y, z) = \sqrt{xyz}$ and $v = -1\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y} + 2\frac{\partial}{\partial z}$ find $v[f]$ at the point $(3, 2, 6)$.

Question 2.26 Given $f(x, y, z) = x + y^2 + z^3$ and $v = -3\frac{\partial}{\partial x} + 4\frac{\partial}{\partial y} - 2\frac{\partial}{\partial z}$ find $v[f]$ at the point $(2, 5, -1)$.

Question 2.27 Given the one-forms $\alpha = (x^3 + 2yz^2)dx + (xy - 2)dy + x^4 dz$ on the manifold $\mathbb{R}^3$ find the one-forms α_{p_i} for $p_1 = (-1, 3, 2)$, $p_2 = (2, -2, 1)$, and $p_3 = (3, -5, -2)$.

Question 2.28 Given the one-forms $\alpha = (x^3 - 2yz^2)dx + dy + x^4 dz$ on the manifold $\mathbb{R}^3$ find the one-forms α_{p_i} for $p_1 = (-1, 3, 2)$, $p_2 = (2, -2, 1)$, and $p_3 = (3, -5, -2)$.

Question 2.29 Given the one-forms $\alpha = (x^3 + 2yz^2)dx + (xy - 2)dy + x^4 dz$ and $\beta = (x^3 - 2yz^2)dx + dy + x^4 dz$ on $\mathbb{R}^3$ find the one-forms $\alpha + \beta$ and $\alpha - \beta$. Then find $(\alpha + \beta)_{p_i}$ and $(\alpha - \beta)_{p_i}$ for the points p_i in the last question.

Question 2.30 Given the one-form $\alpha = x^3 dx - xyz dy + y^2 dz$ find the one-forms 2α and $\frac{1}{3}\alpha$.

Question 2.31 Given the one-form $\omega = \frac{xy}{2}dx - \sqrt{yz}dy + (x + z)^{\frac{2}{3}}dz$ and $p = (2, -1, 3)$ find $\alpha_p(v_p)$ for the following vectors:

$a)\ \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ $\quad d)\ 4e_1 + 7e_2 - 3e_3$ $\quad g)\ 2\dfrac{\partial}{\partial x} - 3\dfrac{\partial}{\partial y} + 4\dfrac{\partial}{\partial z}$

$b)\ \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix}$ $\quad e)\ \dfrac{1}{2}e_1 + \dfrac{7}{4}e_2 - \dfrac{3}{5}e_3$ $\quad h)\ -2\dfrac{\partial}{\partial x} + \dfrac{5}{3}\dfrac{\partial}{\partial y} + 3\dfrac{\partial}{\partial z}$

$c)\ \begin{bmatrix} x + y \\ z - 2 \\ xz \end{bmatrix}$ $\quad f)\ -x^y\, e_1 + \sqrt{xz}\, e_2 - 3y\, e_3$ $\quad i)\ \sqrt{x + y + z}\dfrac{\partial}{\partial x} - x\ln(y)\dfrac{\partial}{\partial y} + z^y\dfrac{\partial}{\partial z}$

Question 2.32 Find df, the differential of f, for the following functions:

$a)\ f(x, y) = x^2 y^3$ $\quad d)\ f(x, y, z) = x^2 + y^3 + z^4$ $\quad g)\ f(x, y, z) = \sqrt{x + yz}$

$b)\ f(x, y) = \sqrt{x^2 + y^2}$ $\quad e)\ f(x, y, z) = \sqrt{xyz}$ $\quad h)\ f(x, y, z) = x\ln(y) + y\ln(z) + z\ln(x)$

$c)\ f(x, y) = x + y^3$ $\quad f)\ f(x, y, z) = xe^y + ye^z + ze^x$ $\quad i)\ f(x, y, z) = \left(x^3 + y^3 + z^3\right)^{\frac{2}{3}}$

Question 2.33 Find $df[v_p]$ for the differentials you found in the previous problem and the vectors in the problem before that at the point $p = (2, -1, 3)$.

Chapter 3
The Wedgeproduct

In this chapter we introduce a way to "multiply" one-forms which is called the wedgeproduct. By wedgeproducting two one-forms together we get two-forms, by wedgeproducting three one-forms together we get three-forms, and so on. The geometrical meaning of the wedgeproduct and how it is computed is explained in section one. In section two general two-, three-, and k-forms are introduced and the geometry behind them is also explored.

The algebraic properties and several different formulas for the wedgeproduct are explored in depth in section three. Different books introduce the wedgeproduct in different ways. Often some of these formulas are given as definitions of the wedgeproduct. Doing this, however, obscures the geometric meaning behind the wedgeproduct, which is why we have taken a different approach in this book.

The fourth section is rather short and simply introduces something called the interior product and proves a couple of identities relating the interior product and the wedgeproduct. We will need these identities later on, but as they rely on the wedgeproduct this was the appropriate time to introduce and prove them.

3.1 Area and Volume with the Wedgeproduct

We begin this chapter by emphasizing an important point. Consider a manifold $\mathbb{R}^n$. At each point p of the manifold $\mathbb{R}^n$ there is a tangent space $T_p\mathbb{R}^n$ that is a vector space which is isomorphic to the vector space $\mathbb{R}^n$. Two vector spaces V and W are called **isomorphic**, denoted by $V \simeq W$, if there is a one-to-one and onto mapping $\phi : V \to W$ such that for $v_1, v_2, v \in V$ and $c \in \mathbb{R}$ we have

1. $\phi(v_1 + v_2) = \phi(v_1) + \phi(v_2)$,
2. $\phi(cv) = c\phi(v)$.

This is not a course in abstract algebra so we do not want to make too much of isomorphisms other than to emphasize that if two vector spaces are isomorphic we can think of them as essentially two copies of the same vector space. In the case of $\mathbb{R}^n$ the underlying manifold is also a vector space, which is isomorphic to the tangent space at each point.

Making the distinction between the underlying manifold and the vector spaces is extremely important when we eventually come to more general manifolds instead of manifolds given by the vector spaces $\mathbb{R}^n$. This will also eventually help us place all of vector calculus into a broader context. Vector calculus is a powerful subject, but it is implicitly built on $\mathbb{R}^3$ and for the manifold $\mathbb{R}^3$ we have

$$\text{manifold } \mathbb{R}^3 \simeq T_p\mathbb{R}^3 \simeq T_p^*\mathbb{R}^3 \simeq \text{ vector space } \mathbb{R}^3.$$

These isomorphisms, which were never made explicit in vector calculus, are what allowed you to think of vectors as being inside the manifold $\mathbb{R}^3$, thereby allowing you to take directional derivatives. Because of these isomorphisms all of these spaces are rather sloppily and imprecisely lumped together and combined. Moving to more manifolds requires us to tease out and understand all of the differences, which is a major goal of this book.

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