$$\mathbf{f}^*(dx_{i_1}\,dx_{i_2}) = \sum_{j<m} \frac{\partial(x_{i_1},x_{i_2})}{\partial(u_j,u_m)}\,du_j\,du_m.$$

We can write this in a way that is both more compact and more striking:

$\mathbf{f}^* d\mathbf{x}_I = \sum_J \frac{\partial \mathbf{x}_I}{\partial \mathbf{u}_J} d\mathbf{u}_J$

$$\mathbf{f}^* d\mathbf{x}_I = \sum_J \frac{\partial \mathbf{x}_I}{\partial \mathbf{u}_J}\,d\mathbf{u}_J.$$

The summation multi-index J consists of all pairs $J = (j_1, j_2)$ with $1 \le j_1 < j_2 \le n$, and

$$\frac{\partial \mathbf{x}_I}{\partial \mathbf{u}_J} = \frac{\partial(x_{i_1},x_{i_2})}{\partial(u_{j_1},u_{j_2})}.$$

In fact, the same formula holds when $d\mathbf{x}_I$ is a basic k-form (where $I = (i_1, \dots, i_k)$ and $1 \le i_1 < \cdots < i_k \le p$):

$$\mathbf{f}^* d\mathbf{x}_I = \sum_J \frac{\partial \mathbf{x}_I}{\partial \mathbf{u}_J}\,d\mathbf{u}_J,$$

where now $J = (j_1, \dots, j_k)$ with $1 \le j_1 < \cdots < j_k \le n$, and

$$\frac{\partial \mathbf{x}_I}{\partial \mathbf{u}_J} = \frac{\partial(x_{i_1},\dots,x_{i_k})}{\partial(u_{j_1},\dots,u_{j_k})}.$$

Pullback of a general k-form

Theorem 10.20. *If* $\alpha = \sum_I P_I(\mathbf{x})\,d\mathbf{x}_I$ *is a general k-form, then*

$$\mathbf{f}^*\alpha = \sum_I \sum_J P_I^*(\mathbf{u})\,\frac{\partial \mathbf{x}_I}{\partial \mathbf{u}_J}\,d\mathbf{u}_J. \qquad \square$$

Exercises

10.1. Suppose there is a steady flow of matter given by the vector $\mathbb{V} = (2, -7, 1)$ kilograms per second per square meter. (All space coordinates are given in meters.)

a. In 1 second, how much matter passes through a unit square in the (x,y)-plane in the positive z-direction? Through a unit square in the (y,z)-plane in the positive x-direction? Through a unit square in the (z,x)-plane in the positive y-direction?

b. How much matter passes through a triangle with area 12 meters^2 in the (x,y)-plane in the positive z-direction in 7 seconds?

c. In 10 seconds, how much matter passes through the rectangle with vertices

$$(5,0,0), \quad (5,3,0), \quad (5,3,6), \quad (5,0,6)$$

in the direction in which x increases.

d. In unit time, how much matter passes through a unit square in the plane $x+y+z=1$ in the "upward" direction (i.e., the direction in which z increases)?

e. Calculate how much matter passes through each of the six faces of the unit cube Q in the first octant, *in the outward direction on each face* in unit time. The sum of these six numbers is zero; why?

10.2. Determine the total flux of the flow field $\mathbb{V}=(0,z,x)$ through:

a. The unit square in the (y,z)-plane, oriented in the positive x-direction.
b. The unit square in the (x,y)-plane, oriented in the negative z-direction.
c. The triangle with vertices $(2,2,0)$, $(0,2,2)$, $(2,0,2)$, using this ordering of the vertices to orient the boundary and thus the triangle itself.

10.3. Determine the flux of the flow field $\mathbb{V}=(x,y,z)$ through the surface S given by

$$\begin{aligned} x &= u+v \\ y &= u^2-v^2 \\ z &= 2uv \end{aligned} \qquad \begin{aligned} 0 &\le u \le 3 \\ 0 &\le v \le 1 \end{aligned}$$

Assume that S inherits the positive orientation of the (u,v)-plane.

10.4. Calculate the flux of $\mathbb{V}=(x,y,z)$ out of the sphere S of radius R centered at the origin $(x,y,z)=(0,0,0)$ to show

$$\iint_S \mathbb{V}\cdot\mathbf{n}\,dA = 4\pi R^3.$$

10.5. Calculate the flux of $\mathbb{V}=(-y,x,0)$ out of the rectangular parallelepiped P in (x,y,z)-space given by $0\le x\le 5$, $0\le y\le 3$, $0\le z\le 2$.

10.6. Let $\mathbf{g}:\mathbb{R}^2\to\mathbb{R}^3:(u,v)\to(x,y,z)$ be the map defined on page 397:

$$x=\frac{2u}{1+u^2+v^2}, \quad y=\frac{2u}{1+u^2+v^2}, \quad z=\frac{1-u^2-v^2}{1+u^2+v^2}.$$

a. Let $\widehat{S}$ be the unit sphere $x^2+y^2+z^2=1$ minus the "south pole" $(0,0,-1)$. Show that $\mathbf{g}(\mathbb{R}^2)\subseteq\widehat{S}$.
b. Show that $\mathbf{g}$ maps the (u,v)-plane onto $\widehat{S}$ by expressing (u,v) in terms of (x,y,z) when $\mathbf{g}(u,v)=(x,y,z)$. (That is, "invert" $\mathbf{g}$ on $\widehat{S}$.)

10.7. Prove Theorem 10.7 (e.g., by modifying the proof of Theorem 10.6).

10.8. When $a=1$, the integral expression for the gravitational field of the hollow sphere (p. 410) involves the improper integral

$$\int_{-\pi/2}^{\pi/2} \frac{(\sin\varphi - 1)\cos\varphi}{(2 - 2\sin\varphi)^{3/2}}\, d\varphi.$$

Show that the improper integral converges and has the value -1, implying that the z-component of the field is $-2\pi G\rho$ when $a = 1$.

10.9. Determine the surface area of the torus

$$x = (R + a\cos v)\cos u, \quad y = (R + a\cos v)\sin u, \quad z = a\sin v,$$

where $R > a > 0$ and $0 \le u, v \le 2\pi$.

10.10. Calculate the differential dg when

a. $g(x,y) = x^3 - 3xy^2$;
b. $g(x,y) = \sin(xy)$;
c. $g(x,y) = x\cos y - y\sin x$;
d. $g(x,y,z) = \ln\sqrt{x^2+y^2}$;
e. $g(x,y,z) = xy + yz + zx$;
f. $g(\rho,\varphi,\theta) = \rho\sin\varphi\cos\theta$;
g. $g(x,y) = \arctan(y/x)$;
h. $g(x,y,u,v) = xu - yv$;
i. $g(x,y,u,v) = xu/yv$;
j. $g(x_1,x_2,\ldots,x_n) = x_1x_2\cdots x_n$.

10.11. Calculate the differential of each of the following k-forms.

a. $\omega(x,y) = y\,dx - x\,dy$;
b. $\omega(x,y) = (x^2 - y^2)\,dx - 2xy\,dy$;
c. $\omega(x,y) = dx/y - dy/x$;
d. $\omega(x,y,z) = (y-z)\,dx + (z-x)\,dy + (x-y)\,dz$;
e. $\omega(x_1,\ldots,x_n) = \sum_{j=2}^{n-1} (x_{j-1} - x_{j+1})\, dx_j$;
f. $\omega(u,v,w) = u^2\,dv\,dw + v^2\,dw\,du + w^2\,du\,dv$;
g. $\omega(x,y,u,v) = (ue^x - ve^y)\, dx\,dy + (x^2+y^2)\, du\,dv$
h. $\omega(x,y,u,v) = \sinh u\cosh v\, dx\,dy + \sin x\cos y\, du\,dv$;
i. $\omega(q_1,q_2,\ldots,q_n,p_1,p_2,\ldots,p_n)) = \sum_{j=1}^{n} p_j\,dq_j$.
j. $\omega(x_1,x_2,\ldots,x_n) = \sum_{j=1}^{n} (-1)^{j-1} x_j\, dx_1\cdots\widehat{dx_j}\cdots dx_n$;
k. $\omega(x_1,x_2,\ldots,x_n) = \sum_{j=1}^{n} x_j\, dx_1\cdots\widehat{dx_j}\cdots dx_n$;

10.12. Consider the 1-form dg that you obtained in each part of Exercise 10.10. Determine its differential, the 2-form $d^2g = d(dg)$, and confirm $d^2g = 0$ in each case.

10.13. Condsider the $(k+1)$-form $d\omega$ that you obtained in part of Exercise 10.11; confirm that $d^2\omega = 0$ in each case.

10.14. Calculate $du \wedge dv$ when

a. $du = dx, \quad dv = 2y\,dy$;
b. $du = \cos\theta\,dr - r\sin\theta\,d\theta, \quad dv = \sin\theta\,dr + r\cos\theta\,d\theta$;
c. $du = 2x\,dx - 2y\,dy, \quad dv = 2y\,dx + 2x\,dy$;
d. $du = 3(x^2-y^2)\,dx - 6xy\,dy, \quad dv = 6xy\,dx + 3(x^2-y^2)\,dy$.

10.15. For each of the following 1-forms ω, first show that $d\omega = 0$ and then find a function f for which $\omega = df$. That is, show ω is the differential of a 0-form (or function).

a. $\omega(x,y) = x\,dx + \cos y\,dy$.
b. $\omega(x,y) = f(x)\,dx + g(y)\,dy$.
c. $\omega(u,v) = 2v\,du + 2u\,dv$.
d. $\omega(x,y,z) = yz\,dx + zx\,dy + xy\,dz$.
e. $\omega(x,y,z) = (y+z)\,dx + (z+x)\,dy + (x+y)\,dz$
f. $\omega(x,y) = \dfrac{1}{y}\,dx - \dfrac{x}{y^2}\,dy$.
g. $\omega(x,y,u,v) = \dfrac{u}{yv}\,dx - \dfrac{xu}{y^2v}\,dy + \dfrac{x}{yv}\,du - \dfrac{xu}{yv^2}\,dv$.

10.16. For each of the following 2-forms α, first show that $d\alpha = 0$ and then find a 1-form ω for which $d\omega = \alpha$.

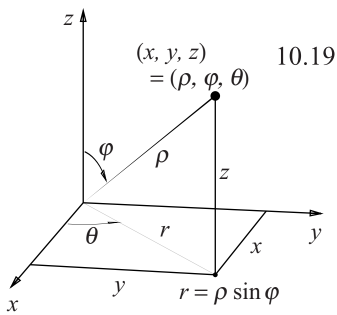
a. $\alpha(x,y) = (x-y)\,dx \wedge dy$.
b. $\alpha(x,y) = \varphi(x,y)\,dx \wedge dy$.
c. $\alpha(x,y,z) = dx \wedge dy + dy \wedge dz + dz \wedge dx$.

10.17. Let $\omega = (x^2+y^2)\,dx\,dy\,dz$. Determine $\alpha = P(x,y,z)\,dx\,dy$ and $\beta = Q(x,y,z)\,dy\,dz$ so that $\omega = d\alpha = d\beta$.

10.18. Let $\omega = \frac{1}{2}(-y\,dx + x\,dy)$, $\alpha = d\omega = dx\,dy$, and let (cf. p. 359)

$$\boldsymbol{\varphi} : \begin{cases} x = \sin s \cosh t, \\ y = \cos s \sinh t. \end{cases}$$

Determine the pullbacks $\boldsymbol{\varphi}^*\omega$ and $\boldsymbol{\varphi}^*\alpha$ and confirm that $d\boldsymbol{\varphi}^*\omega = \boldsymbol{\varphi}^*\alpha$.



10.19. (Spherical coordinates). Let

$$\boldsymbol{\sigma} : \begin{cases} x = \rho\sin\varphi\cos\theta \ (= r\cos\theta) \\ y = \rho\sin\varphi\sin\theta \ (= r\sin\theta), \\ z = \rho\cos\varphi. \end{cases}$$

These equations are similar to the *spherical coordinates* of Exercise 5.10, page 178. The difference is that here φ is **co-latitude**, measuring the angle down from the positive z-axis. In the earlier exercise, φ was **latitude**, measuring the angle up from the (x,y)-plane. Here $(\rho,\varphi,\theta) \to (x,y,z)$ is seen to be orientation-preserving. (With φ representing *latitude*, however, the order needs to be (ρ,θ,φ): in Exercise 5.10, $\partial(x,y,z)/\partial(\rho,\theta,\varphi) \geq 0$.)

a. Determine the Jacobian $\partial(x,y,z)/\partial(\rho,\varphi,\theta)$ as a function of ρ, φ, and θ.
b. Determine the differentials dx, dy, and dz in terms of ρ, φ, θ, and their differentials.
c. Determine the volume element $dx \wedge dy \wedge dz$ in terms of ρ, φ, θ, and the volume element $d\rho \wedge d\varphi \wedge d\theta$. Compare this to the Jacobian you obtained in part (a).

10.20. (Cylindrical coordinates: (r,θ,z)). These replace x and y by polar coordinates while leaving z unchanged: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

a. Determine the Jacobian $\partial(x,y,z)/\partial(r,\theta,z)$. Given the relation to polar coordinates in the plane, is this what you would expect?
b. Determine the volume element $dx \wedge dy \wedge dz$ in terms of r, θ, z, and the volume element $dr \wedge d\theta \wedge dz$. Again, is this what you would expect?

10.21. Determine the pullback $\boldsymbol{\sigma}^*\alpha$ where $\boldsymbol{\sigma}$ is the spherical coordinates map of Exercise 10.19 and $\alpha = x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$.

10.22. Let $\beta = x\,dy\,dz + y\,dz\,dx - 2z\,dx\,dy$, and let

$$\mathbf{f}: \begin{cases} x = \alpha\cos u\cosh v, \\ y = \alpha\sin u\cosh v, \\ x = v. \end{cases}$$

Determine the pullbacks $\mathbf{f}^*(dy\,dz)$, $\mathbf{f}^*(dz\,dx)$, $\mathbf{f}^*(dx\,dy)$, and $\mathbf{f}^*(\beta)$.

10.23. Let $\vec{S}$ be the surface defined parametrically by $x = u+v$, $y = u-v$, $z = v$, where $-1 \leq u \leq 3$, $0 \leq v \leq 2$ is positively oriented. Determine

$$\iint_{\vec{S}} xy\,dx\,dy + yz\,dy\,dz + zx\,dx\,dy.$$

10.24. Let $\vec{S}$ be parametrized by $x = a\cos u$, $y = a\sin u$, $z = v$, where $0 \leq u \leq 2\pi$, $0 \leq v \leq h$ is positively oriented.

a. Show that $\vec{S}$ is a *cylinder* of radius a whose axis is the z-axis. Sketch $\vec{S}$, showing where the images of the u- and v-axes lie on the cylinder, and show how this indicates the orientation of $\vec{S}$.
b. Determine the pullback of the 2-form $\alpha = (x^2+y^2)\,dy \wedge dz$ to the (u,v)-plane and then determine

$$\iint_{\vec{S}} \alpha.$$

c. Show that $\iint_{\vec{S}} f(x,y,z)\,dx\wedge dy = 0$ for any function $f(x,y,z)$.

10.25. Let $\vec{S} = \mathbf{m}(\vec{U})$ is the oriented surface in $\mathbb{R}^4$ parametrized by

$$\mathbf{m} : \begin{cases} p = x^2+y^2, \\ q = x-y, \\ r = xy, \\ s = x+y, \end{cases} \qquad \vec{U} : \begin{array}{l} 0 \le x \le 1, \\ 0 \le y \le 1. \end{array}$$

Let $\beta = pq\,dq\wedge dr + qr\,dp\wedge ds$.

a. Determine the pullback $\mathbf{m}^*(\beta)$.
b. Determine $\iint_{\vec{S}} \beta$.

10.26. Sketch the oriented curve $\vec{C}$,

$$(x(t),y(t)) = \mathrm{e}^{\sin(t/2)}(\cos t,\sin t), \quad 0 \le t \le 4\pi,$$

and determine its winding number (Definition 10.14, p. 430).

10.27. Prove the claim

$$\frac{\partial(y,z)}{\partial(v,w)}\frac{\partial(v,w)}{\partial(s,t)} + \frac{\partial(y,z)}{\partial(w,u)}\frac{\partial(w,u)}{\partial(s,t)} + \frac{\partial(y,z)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(s,t)} = \frac{\partial(y,z)}{\partial(s,t)}$$

made in the proof of Lemma 10.1.

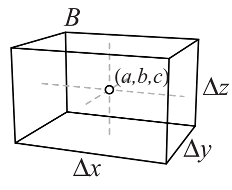
10.28. Prove Theorem 10.16 for differential forms in three variables.

Chapter 11
Stokes' Theorem

Abstract Stokes' theorem equates the integral of one expression over a surface to the integral of a related expression over the curve that bounds the surface. A similar result, called Gauss's theorem, or the divergence theorem, equates the integral of a function over a 3-dimensional region to the integral of a related expression over the surface that bounds the region. The similarities are not accidental. Using the language of differential forms, we show these two theorems are instances (along with Green's theorem and the fundamental theorem of calculus) of a single theorem that connects one integral over a domain to a related one over its boundary. To explore the connections, we combine the "modern" approach, using differential forms to clarify statements and proofs, with the "classical" appoach, using vector fields to understand the individual theorems in the physical terms in which they arose.

11.1 Divergence

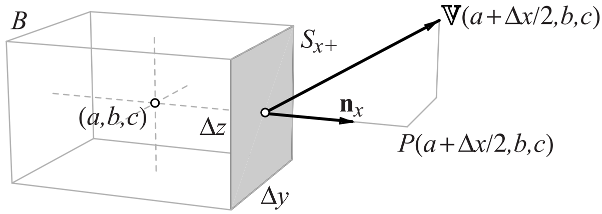
In this section, we analyze the flux of a continuously differentiable vector field $\mathbb{V} = (P,Q,R)$ through the boundary ∂D of a solid region D, in the direction of the normal on ∂D that points out of D. We saw, in an example worked out on pages 420–422, that the net flux could be nonzero. In other words, inward flow and outward flow need not always balance.



First take D to be a parallelpiped B whose edges are parallel to the coordinate axes (a box). Suppose B is centered at the point $(x,y,z) = (a,b,c)$ and has length Δx, width Δy, and height Δz. Its boundary ∂B consists of three pairs of plane parallel faces. The face S_{x+} of ∂B that lies in the plane $x = a + \Delta x/2$ has area $\Delta y\,\Delta z$ and outward normal $\mathbf{n}_x = (1,0,0)$. If the box is sufficiently small, we can approximate $\mathbb{V}$ everywhere on S_{x+} by its value at the center $(x,y,z) = (a + \Delta x/2, b, c)$ of S_{x+}. Under this assumption, total flux (Definition 10.1, p. 389) through S_{x+} is approximately

$$\Phi_{x+} \approx \mathbb{V}(a + \Delta x/2, b, c) \cdot \mathbf{n}_x\,\Delta y\,\Delta z = P(a + \Delta x/2, b, c)\,\Delta y\,\Delta z.$$

J.J. Callahan, *Advanced Calculus: A Geometric View*, Undergraduate Texts in Mathematics,
DOI 10.1007/978-1-4419-7332-0_11, © Springer Science+Business Media, LLC 2010



The parallel face S_{x-} that lies in the plane $x = a - \Delta x/2$ has the same area but the opposite outward normal $-\mathbf{n}_x$. Approximating $\mathbb{V}$ everywhere by its value at $(a - \Delta x/2, b, c)$, we can then write

$$\Phi_{x-} \approx \mathbb{V}(a - \Delta x/2, b, c) \cdot (-\mathbf{n}_x)\, \Delta y\, \Delta z = -P(a - \Delta x/2, b, c)\, \Delta y\, \Delta z.$$

Therefore,

$$\Phi_{x+} + \Phi_{x-} \approx \big(P(a + \Delta x/2, b, c) - P(a - \Delta x/2, b, c)\big)\, \Delta y\, \Delta z.$$

Approximate total flux through a pair of faces

By the microscope equation,

$$P(a + \Delta x/2, b, c) - P(a - \Delta x/2, b, c) \approx \frac{\partial P}{\partial x}(a, b, c)\, \Delta x$$

when $\Delta x \approx 0$, so

$$\Phi_{x+} + \Phi_{x-} \approx \frac{\partial P}{\partial x}(a, b, c)\, \Delta x\, \Delta y\, \Delta z = \frac{\partial P}{\partial x}(a, b, c)\, \Delta V,$$

where ΔV is the volume of the box B.

There are similar formulas for the other faces. For the pair $S_{y\pm}$ that lie in the planes $y = b \pm \Delta y/2$, the normals are $\pm\mathbf{n}_y = (0, \pm 1, 0)$, and we find

$$\begin{aligned} \Phi_{y+} + \Phi_{y-} &\approx \big(Q(a, b + \Delta y/2, c) - Q(a, b - \Delta y/2, c)\big)\, \Delta z\, \Delta x \\ &\approx \frac{\partial Q}{\partial y}(a, b, c)\, \Delta y\, \Delta z\, \Delta x = \frac{\partial Q}{\partial y}(a, b, c)\, \Delta V. \end{aligned}$$

Similarly, for $S_{z\pm}$ we have

$$\Phi_{z+} + \Phi_{z-} \approx \big(R(a, b, c + \Delta z/2) - R(a, b, c - \Delta y/2)\big)\, \Delta z\, \Delta x \approx \frac{\partial R}{\partial z}(a, b, c)\, \Delta V.$$

Approximate total flux out of the box

Therefore, we estimate the total flux through ∂B in the outward direction to be

$$\Phi \approx \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)_{(a,b,c)} \Delta V.$$

For boxes that are small enough for this formula to provide a good approximation, we find that total flux is proportional to the volume of the box. It is remarkable that any formula for Φ should involve 3-dimensional volume, because Φ measures flow through 2-dimensional surfaces. The proportionality factor that connects flux to volume is the scalar quantity

$$\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z},$$

evaluated at the center of the box. Thus, although total flux of $\mathbb{V}$ must certainly depend on $\mathbb{V}$, we find that when the surface is the complete boundary of a small box, total flux depends only on a certain scalar, called the *divergence*, derived from the components of $\mathbb{V}$.

The *divergence* of a vector field

Definition 11.1 *The* ***divergence*** *of the vector field* $\mathbb{V}=(P,Q,R)$ *is the scalar field (i.e., function)*

$$\operatorname{div}\mathbb{V}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}.$$

To illustrate, consider our earlier example (pp. 420–422) of the total flux Φ of the vector field $\mathbb{V}=(x+y,y-x,0)$ out of the unit cube. Here

$$\operatorname{div}\mathbb{V}=\frac{\partial}{\partial x}(x+y)+\frac{\partial}{\partial y}(y-x)+\frac{\partial}{\partial z}0=1+1=2,$$

a constant. The volume of the unit cube is $\Delta V=1$; therefeore we obtain the estimate $\Phi\approx 2\times 1=2$. In fact, we already found $\Phi=2$ by direct calculation. Even though the unit cube is not "small," the estimate still works well because $\operatorname{div}\mathbb{V}$ is the same at all points.

From $\operatorname{div}\mathbb{V}\,\Delta V$ to a triple integral

The product $\operatorname{div}\mathbb{V}\,\Delta V$ in a small box leads us to a triple integral in a larger region. On page 309, we introduced the triple integral of a function $f(x,y,z)$ over a region D in (x,y,z)-space that has volume (3-dimensional Jordan content). In particular, if $f(x,y,z)$ is bounded and continuous on D, then

$$\iiint_D f(x,y,z)\,dV$$

exists (Theorem 8.35, p. 305, adapted from double to triple integrals). Furthermover, if $\delta(D)$, the diameter of D (Definition 8.14, p. 291) is sufficiently small, then it follows from Corollary 8.30, p. 299 that

$$f(a,b,c)\,\Delta V\approx\iiint_D f(x,y,z)\,dV,$$

where (a,b,c) is a point in D. Hence, when $\mathbb{V}$ is continuously differentiable, so that $\operatorname{div}\mathbb{V}$ is continuous, and B is a box with small diameter,

$$\operatorname{div}\mathbb{V}(a,b,c)\,\Delta V\approx\iiint_B \operatorname{div}\mathbb{V}\,dV.$$

Connecting $\operatorname{div}\mathbb{V}\,dV$ and $\mathbb{V}\cdot\mathbf{n}\,dA$

But because we have just found that $\operatorname{div}\mathbb{V}(a,b,c)\,\Delta V$ approximates the total flux Φ through ∂B in the outward unit direction $\mathbf{n}$, we can also write

$$\operatorname{div}\mathbb{V}(a,b,c)\,\Delta V \approx \iint_{\partial B} \mathbb{V}\cdot\mathbf{n}\,dA, \quad \text{implying} \quad \iiint_B \operatorname{div}\mathbb{V}\,dV \approx \iint_{\partial B} \mathbb{V}\cdot\mathbf{n}\,dA.$$

In fact, we show that these two integrals are actually equal: they both represent the total flux. More generally, we show that, for a large class of regions D, the triple integral of $\operatorname{div}\mathbb{V}$ over D equals the surface integral of $\mathbb{V}\cdot\mathbf{n}$ over ∂D. This equality is called the *divergence theorem*, or *Gauss's theorem*.

Divergence theorem for a unit cube

Theorem 11.1. *Let B be the unit cube, $0 \le x,y,z \le 1$, and $\mathbf{n}$ the outward unit normal on ∂B. Let $\mathbb{V}$ be a continuously differentiable vector field defined on an open set containing B; then*

$$\iiint_B \operatorname{div}\mathbb{V}\,dV = \iint_{\partial B} \mathbb{V}\cdot\mathbf{n}\,dA.$$

Proof. To make the proof clearer, we convert the integrands to differential forms. If $\mathbb{V} = (P,Q,R)$, then (cf. pp. 403–404)

$$\begin{aligned} \mathbb{V}\cdot\mathbf{n}\,dA &= P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy, \\ \operatorname{div}\mathbb{V}\,dV &= (P_x + Q_y + R_z)\,dx\,dy\,dz. \end{aligned}$$

Now that the integrands are differential forms, the domains of integration must be oriented. Let $\vec{B}$ have the positive orientation given by the standard basis vectors of $\mathbb{R}^3$ in their usual order. Let $\partial\vec{B}$ have the orientation induced by $\vec{B}$; this is the orientation given by $\mathbf{n}$, the outward unit normal on $\partial\vec{B}$.

Now we show that

$$\iiint_{\vec{B}} P_x\,dx\,dy\,dz = \iint_{\partial\vec{B}} P\,dy\,dz.$$

A similar approach can be used to show the other two pairs of components are equal, thus completing the proof.

Let us label the faces of $\partial\vec{B}$ using the notation from the example on pages 420–422. Thus, for example, $\vec{S}_{x=0}$ is the face (properly oriented) that lies in the plane $x = 0$, and

$$\partial\vec{B} = \vec{S}_{x=0} + \vec{S}_{x=1} + \vec{S}_{y=0} + \vec{S}_{y=1} + \vec{S}_{z=0} + \vec{S}_{z=1}.$$

Because $dy = 0$ on the faces $\vec{S}_{y=0}$ and $\vec{S}_{y=1}$, and because $dz = 0$ on the faces $\vec{S}_{z=0}$ and $\vec{S}_{z=1}$, we find

$$\begin{aligned} \iint_{\partial\vec{B}} P\,dy\,dz &= \iint_{\vec{S}_{x=0}} P\,dy\,dz + \iint_{\vec{S}_{x=1}} P\,dy\,dz \\ &= \int_0^1\!\int_0^1 -P(0,y,z)\,dy\,dz + \int_0^1\!\int_0^1 P(1,y,z)\,dy\,dz \\ &= \int_0^1\!\int_0^1 \big(P(1,y,z) - P(0,y,z)\big)\,dy\,dz. \end{aligned}$$

As we saw with similar computations on page 421, to take the orientation of $S_{x=0}$ properly into account when we use y and z as parameters, we must include the minus sign in the integral of $P(0,y,z)$.

We can compute the triple integral as a simple (threefold) iterated integral:

$$\iiint_{\vec{B}} P_x\,dx\,dy\,dz = \int_0^1\int_0^1\left(\int_0^1 P_x\,dx\right)dy\,dz = \int_0^1\int_0^1\left(P(x,y,x)\Big|_{x=0}^{x=1}\right)dy\,dz$$
$$= \int_0^1\int_0^1 \big(P(1,y,z) - P(0,y,z)\big)\,dy\,dz.$$

Thus the surface integral and the triple integral are equal; by the remark made above, this completes the proof. □

Theorem 11.2. *Let B be the unit ball, $x^2+y^2+z^2 \le 1$, and $\mathbf{n}$ the outward unit normal on ∂B. Let $\mathbb{V}$ be a continuously differentiable vector field defined on an open set containing B; then*

Divergence theorem for a unit ball

$$\iiint_B \operatorname{div}\mathbb{V}\,dV = \iint_{\partial B} \mathbb{V}\cdot\mathbf{n}\,dA.$$

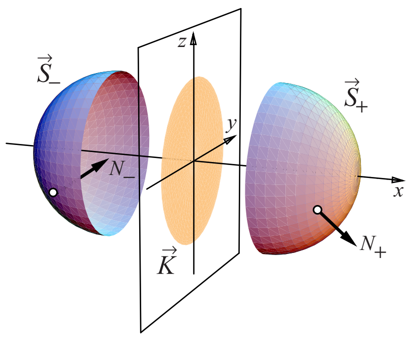
Proof. Again we convert the integrands to differential forms, orient the domains appropriately, and then show that

$$\iiint_{\vec{B}} P_x\,dx\,dy\,dz = \iint_{\partial\vec{B}} P\,dy\,dz;$$

similar arguments prove that the other two pairs of components are equal.

To determine the surface integral, let $\vec{S}_+$ and $\vec{S}_-$ be the graphs of the functions

$$\vec{S}_+ : x = +\sqrt{1-y^2-z^2} \quad\text{and}\quad \vec{S}_- : x = -\sqrt{1-y^2-z^2},$$



defined on the positively oriented disk $\vec{K} : y^2+z^2 \le 1$, and inheriting their orientations from $\vec{K}$. (In the figure, the surfaces are shown separated for clarity.) The orientation normals N_+ and N_- of both surfaces therefore point in the positive x-direction. Thus, N_+ points outward, but N_- points inward, so $\partial\vec{B} = \vec{S}_+ - \vec{S}_-$, and

$$\begin{aligned}
\iint_{\partial\vec{B}} P\,dy\,dz &= \iint_{\vec{S}_+} P\,dy\,dz - \iint_{\vec{S}_-} P\,dy\,dz \\
&= \iint_{\vec{K}} P(\sqrt{1-y^2-z^2},y,z)\,dy\,dz - \iint_{\vec{K}} P(-\sqrt{1-y^2-z^2},y,z)\,dy\,dz \\
&= \iint_{\vec{K}} \left(P(\sqrt{1-y^2-z^2},y,z) - P(-\sqrt{1-y^2-z^2},y,z)\right) dy\,dz
\end{aligned}$$

To determine the triple integral, let $\vec{B}$ be the positively oriented solid region given by the inequalties

$$\vec{B}: \quad \begin{array}{c} y^2+z^2 \le 1, \\ -\sqrt{1-y^2-z^2} \le x \le \sqrt{1-y^2-z^2}. \end{array}$$

Then

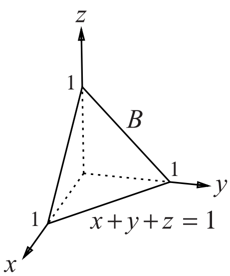
$$\begin{aligned}
\iiint_{\vec{B}} P_x\,dx\,dy\,dz &= \iint_{\vec{K}} \left(\int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} P_x(x,y,z)\,dx \right) dy\,dz \\
&= \iint_{\vec{K}} P(x,y,z) \Big|_{x=-\sqrt{1-y^2-z^2}}^{x=\sqrt{1-y^2-z^2}} dy\,dz \\
&= \iint_{\vec{K}} \left(P(\sqrt{1-y^2-z^2},y,z) - P(-\sqrt{1-y^2-z^2},y,z)\right) dy\,dz,
\end{aligned}$$

so the triple integral is equal to the surface integral. By what has been said above, this proves the theorem. □

Divergence theorem for a unit tetrahedron

Theorem 11.3. *Let B be the unit tetrahedron, $0 \le x,y,z,\ x+y+z \le 1$, and* $\mathbf{n}$ *the outward unit normal on ∂B. Let $\mathbb{V}$ be a continuously differentiable vector field defined on an open set containing B; then*

$$\iiint_B \operatorname{div}\mathbb{V}\,dV = \iint_{\partial B} \mathbb{V}\cdot\mathbf{n}\,dA.$$



Proof. In Exercise 11.8, you are asked to prove this theorem using differential forms, following the pattern of the last two proofs. For the sake of illustration, we take an alternate approach, integrating the scalar functions $\operatorname{div}\mathbb{V}$ and $\mathbb{V}\cdot\mathbf{n}$ directly over the unoriented domains B and ∂B, respectively. If $\mathbb{V} = (P,Q,R)$, then

$$\iiint_B \operatorname{div}\mathbb{V}\,dV = \iiint_B (P_x+Q_y+R_z)\,dV$$

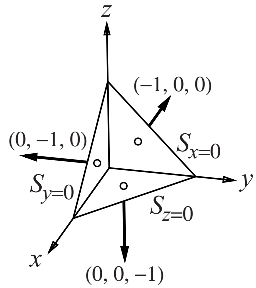
We convert each term into an iterated integral, describing B by inequalities in three different ways, each suited to the term being integrated. To integrate P_x, let

$$B: \quad \begin{array}{c} 0 \le y \le 1, \\ 0 \le z \le 1-y, \\ 0 \le x \le 1-y-z; \end{array}$$

then

$$\iiint_B P_x\, dV = \int_0^1 \int_0^{1-y} \left(\int_0^{1-y-z} P_x(x,y,z)\, dx \right) dz\, dy$$
$$= \int_0^1 \int_0^{1-y} P(x,y,z) \Big|_0^{1-y-z} dz\, dy$$
$$= \int_0^1 \int_0^{1-y} \big(P(1-y-z,y,z) - P(0,y,z)\big)\, dz\, dy.$$

By changing the description of B appropriately, we obtain similar expressions for the integrals of Q_y and R_z:



$$\iiint_B Q_y\, dV = \int_0^1 \int_0^{1-z} \big(Q(x,1-x-z,z) - Q(x,0,z)\big)\, dx\, dz,$$
$$\iiint_B R_z\, dV = \int_0^1 \int_0^{1-x} \big(R(x,y,1-x-y) - R(x,y,0)\big)\, dy\, dx.$$

(Recall that the order of the differentials here indicates merely the order in which the integrations are to be carried out, not the orientation of the domain of integration.)

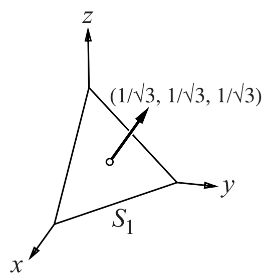
In the tetrahedral surface ∂B, three faces $S_{x=0}$, $S_{y=0}$, and $S_{z=0}$ lie in coordinate planes; the fourth, S_1, lies in the plane $x+y+z=1$. Because the outward unit normal on $S_{x=0}$ is $\mathbf{n} = (-1,0,0)$, we have

$$\iint\limits_{S_{x=0}} \mathbb{V}\cdot\mathbf{n}\, dA = \iint\limits_{S_{x=0}} (P,Q,R)\cdot(-1,0,0)\, dA = \int_0^1 \int_0^{1-y} -P(0,y,z)\, dz\, dy.$$

In a similar way, $\mathbf{n} = (0,-1,0)$ on $S_{y=0}$ and $\mathbf{n} = (0,0,-1)$ on $S_{z=0}$, so

$$\iint\limits_{S_{y=0}} \mathbb{V}\cdot\mathbf{n}\, dA = \int_0^1 \int_0^{1-z} -Q(x,0,z)\, dx\, dz$$
$$\iint\limits_{S_{z=0}} \mathbb{V}\cdot\mathbf{n}\, dA = \int_0^1 \int_0^{1-x} -R(x,y,0)\, dy\, dx$$

On the fourth face, S_1, the outward unit normal is $\mathbf{n} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, so



$$\iint_{S_1} \mathbb{V}\cdot\mathbf{n}\, dA = \iint_{S_1} \frac{P+Q+R}{\sqrt{3}}\, dA.$$

To integrate the first term, $P/\sqrt{3}$, treat S_1 as the graph of

$$x = 1-y-z \ \text{ on } S_{x=0}: \ \begin{array}{c} 0 \le y \le 1, \\ 0 \le z \le 1-y. \end{array}$$

To get an expression of dA, we must calculate (cf. Definition 10.6, p. 409)

$$\sqrt{\left[\frac{\partial(y,z)}{\partial(y,z)}\right]^2+\left[\frac{\partial(z,x)}{\partial(y,z)}\right]^2+\left[\frac{\partial(x,y)}{\partial(y,z)}\right]^2}=\sqrt{\begin{vmatrix}1&0\\0&1\end{vmatrix}^2+\begin{vmatrix}0&-1\\1&-1\end{vmatrix}^2+\begin{vmatrix}-1&1\\-1&0\end{vmatrix}^2}=\sqrt{3}.$$

Thus $dA=\sqrt{3}\,dy\,dz$, and

$$\iint_{S_1}\frac{P}{\sqrt{3}}\,dA=\int_0^1\int_0^{1-y}P(1-y-z,y,z)\,dz\,dy.$$

For $Q/\sqrt{3}$, treat S_1 as the graph of $y=1-x-z$ on $S_{y=0}$, and for $R/\sqrt{3}$, treat it as $z=1-x-y$ on $S_{z=0}$; then

$$\iint_{S_1}\frac{Q}{\sqrt{3}}\,dA=\int_0^1\int_0^{1-z}Q(x,1-x-z,z)\,dx\,dz,$$

$$\iint_{S_1}\frac{R}{\sqrt{3}}\,dA=\int_0^1\int_0^{1-x}R(x,y,1-x-y)\,dy\,dx.$$

The triple integral and the surface integral reduce to six iterated double integrals each, and these are equal in pairs. □

Symbolic form of the divergence theorem

We must still show that the divergence theorem applies to other regions. To do this, it is helpful if we think of the integrals in the theorem as *symbolic pairings* (cf. p. 427). Thus if $\mathbb{V}=(P,Q,R)$ and

$$\begin{aligned}\mathbb{V}\cdot\mathbf{n}\,dA&=P\,dy\,dz+Q\,dz\,dx+R\,dx\,dy=\alpha,\\ \operatorname{div}\mathbb{V}\,dV&=(P_x+Q_y+R_z)\,dx\,dy\,dz=d\alpha,\end{aligned}$$

we write

$$\iiint_B \operatorname{div}V\,dV=\iiint_{\vec{B}}d\alpha=\langle\vec{B},d\alpha\rangle,\qquad\iint_{\partial B}\mathbb{V}\cdot\mathbf{n}\,dA=\iint_{\partial\vec{B}}\alpha=\langle\partial\vec{B},\alpha\rangle.$$

Here $\vec{B}$ is the positively oriented unit cube, unit ball, or unit tetrahedron, and $\partial\vec{B}$ is its boundary with the induced orientation. In terms of symbolic pairings, the divergence theorem thus has the form

$$\langle\vec{B},d\alpha\rangle=\langle\partial\vec{B},\alpha\rangle.$$

Note that when Green's theorem and the fundamental theorem of calculus are expressed in terms of symbolic pairings (pp. 427–429), they have exactly the same form. The essential point of each theorem is that the exterior derivative d and the boundary operator ∂ are adjoints in the symbolic pairing.

Images under coordinate changes

We can now extend the divergence theorem to any region D that is the image, under a coordinate change, of a region (such as a unit cube) to which the divergence theorem already applies.

Theorem 11.4. *Let $\boldsymbol{\varphi} : \Omega \to \mathbb{R}^3$ be a coordinate change, and let $B \subset \Omega$ be a region for which the divergence theorem is known to hold. Suppose $D = \boldsymbol{\varphi}(B)$, $\mathbf{n}$ is the outward unit normal on ∂D, and $\mathbb{V}$ is a continuously differentiable vector field defined on $\boldsymbol{\varphi}(\Omega)$; then*

$$\iiint_D \operatorname{div}\mathbb{V}\, dV = \iint_{\partial D} \mathbb{V}\cdot\mathbf{n}\, dA.$$

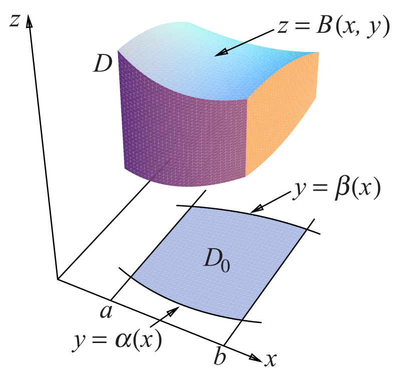
Proof. We convert to differential forms, as in the earlier proofs. Thus, let $\alpha = \mathbb{V}\cdot\mathbf{n}\, dA$, $d\alpha = \operatorname{div}\mathbb{V}\, dV$. Let $\vec{B}$ be B with its positive orientation, let $\partial\vec{B}$, $\vec{D} = \boldsymbol{\varphi}(\vec{B})$, and $\partial\vec{D} = \boldsymbol{\varphi}(\partial\vec{B})$ receive the appropriate induced orientations (cf. p. 355), and let $\boldsymbol{\varphi}^*$ be the pullback of $\boldsymbol{\varphi}$ on forms (cf. pp. 433ff.). Then

$$\begin{aligned}
\iiint_D \operatorname{div}\mathbb{V}\, dV = \iiint_{\vec{D}} d\alpha &= \langle \vec{D}, d\alpha\rangle \\
&= \langle \boldsymbol{\varphi}(\vec{B}), d\alpha\rangle && \text{definition of } \vec{D} \\
&= \langle \vec{B}, \boldsymbol{\varphi}^*(d\alpha)\rangle && \boldsymbol{\varphi} \text{ and } \boldsymbol{\varphi}^* \text{ are adjoints} \\
&= \langle \vec{B}, d(\boldsymbol{\varphi}^*\alpha)\rangle && d \text{ and } \boldsymbol{\varphi}^* \text{ commute} \\
&= \langle \partial\vec{B}, \boldsymbol{\varphi}^*\alpha\rangle && \text{divergence theorem for } \vec{B} \\
&= \langle \boldsymbol{\varphi}(\partial\vec{B}), \alpha\rangle && \boldsymbol{\varphi} \text{ and } \boldsymbol{\varphi}^* \text{ are adjoints} \\
&= \langle \partial\vec{D}, \alpha\rangle && \text{definition of } \partial\vec{D} \\
&= \iint_{\partial\vec{D}} \alpha = \iint_{\partial D} \mathbb{V}\cdot\mathbf{n}\, dA. && \square
\end{aligned}$$

At two key points in the proof, we use the fact that a coordinate change $\boldsymbol{\varphi}$ and its pullback $\boldsymbol{\varphi}^*$ on differential forms are adjoints (Theorem 10.15, p. 436). This is just the change of variables formula for integrals expressed in terms of symbolic pairings.

A good example of the image of a cube under a coordinate change in $\mathbb{R}^3$ is the region D in (x,y,z)-space between two graphs $z = A(x,y)$ and $z = B(x,y)$, when A and B have a common domain of the form

$$D_0 : \quad \begin{array}{c} a \le x \le b, \\ \alpha(x) \le y \le \beta(x). \end{array}$$



We assume that α and β are continuously differentiable on some open interval I containing $[a,b]$, and A and B are likewise continuously differentiable on some open set Ω_0 containing D_0 in the (x,y)-plane. The inequalites defining D allow us to transform a triple integral over D into a threefold iterated integral:

$$\iiint_D f(x,y,z)\,dV = \int_a^b \left(\int_{\alpha(x)}^{\beta(x)} \left(\int_{A(x,y)}^{B(x,y)} f(x,y,z)\,dz \right) dy \right) dx.$$

The image of a cube

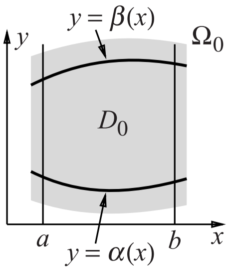
Theorem 11.5. *Suppose there are continuously differentiable functions $\alpha(x)$, $\beta(x)$, $A(x,y)$, and $B(x,y)$ for which $\alpha_0 < \alpha(x) < \beta(x) < \beta_0$ for every x in an open interval I, and $A_0 < A(x,y) < B(x,y) < B_0$ for every (x,y) in an open set Ω_0. Suppose $[a,b] \subset I$, and suppose $D_0 \subset \Omega_0$ is given by*

$$D_0: \qquad a \le x \le b, \qquad \alpha(x) \le y \le \beta(x).$$

If D is the region in (x,y,z)-space defined by

$$D: \qquad (x,y) \in D_0, \qquad A(x,y) \le z \le B(x,y),$$

then D is the image of the unit cube B under a coordinate change.



Proof. Let Ω be the open set in $\mathbb{R}^3$ constructed as the product of Ω_0 in the (x,y)-plane and the open interval (A_0,B_0) on the z-axis: $\Omega = \Omega_0 \times (A_0,B_0)$. Now define $\boldsymbol{\varphi} : \Omega \to \mathbb{R}^3$ as

$$\boldsymbol{\varphi} : \begin{cases} u = \dfrac{x-a}{b-a}, \\ v = \dfrac{y-\alpha(x)}{\beta(x)-\alpha(x)}, \\ w = \dfrac{z-A(x,y)}{B(x,y)-A(x,y)}. \end{cases}$$

Because the components are continuously differentiable and the denominators are never zero on Ω, $\boldsymbol{\varphi}$ is well-defined and continuously differentiable.

If $a \le x \le b$, then $0 \le u \le 1$. If, in addition, $\alpha(x) \le y \le \beta(x)$, then $0 \le v \le 1$. Finally, if $A(x,y) \le z \le B(x,y)$ as well, then $0 \le w \le 1$. Thus, $\boldsymbol{\varphi}(D) = B$, the unit cube. We can solve for x, y, and z to get the inverse:

$$\boldsymbol{\varphi}^{-1} : \begin{cases} x = a + (b-a)u, \\ y = \alpha(x) + \big(\beta(x) - \alpha(x)\big)v, \\ z = A(x,y) + \big(B(x,y) - A(x,y)\big)w, \end{cases}$$

understanding that x will be replaced by $a+(b-a)u$ in the formula for y, and these expressions will then replace x and y in the formula for z. Hence $\boldsymbol{\varphi}^{-1}$ is continuously differentiable, so $\boldsymbol{\varphi}^{-1}$ is a coordinate change for which $D = \boldsymbol{\varphi}^{-1}(B)$. □

Corollary 11.6 *The divergence theorem holds for the region D.* □

The divergence theorem on more general regions

The solid region D is the 3-dimensional analogue of the basic 2-dimensional region on which we established Green's theorem (cf. Theorem 9.18, p. 364, and pp. 367–368). In the final version of Green's theorem, we assumed the domain can be decomposed into a finite number of nonoverlapping regions on which Green's theorem is known to apply. Our final version of the divergence theorem is similar.

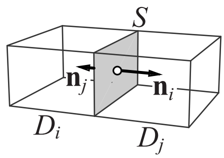
Theorem 11.7 (Divergence theorem). *Suppose $\mathbb{V}$ is a continuously differentiable vector field defined on an open set Ω in $\mathbb{R}^3$, and D is a closed bounded subset of Ω. Suppose $D_1,\ldots,D_k$ are nonoverlapping regions in $\mathbb{R}^3$ on which the divergence theorem applies, and $\vec{D} = \vec{D}_1 + \cdots + \vec{D}_k$ when all regions are positively oriented; then*

$$\iiint_D \operatorname{div}\mathbb{V}\, dV = \iint_{\partial D} \mathbb{V}\cdot\mathbf{n}\, dA.$$

Proof. By the additivity of triple integrals, we know immediately that

$$\iiint_D \operatorname{div}\mathbb{V}\, dV = \iiint_{D_1} \operatorname{div}\mathbb{V}\, dV + \cdots + \iiint_{D_k} \operatorname{div}\mathbb{V}\, dV.$$

The surface integrals combine in a more interesting way. If two cells $\vec{D}_i$ and $\vec{D}_j$ meet along a face S, then, at any point $\mathbf{p}$ on S, the outward normal $\mathbf{n}_i(\mathbf{p})$ from $\vec{D}_i$ is opposite the outward normal $\mathbf{n}_j(\mathbf{p})$ from $\vec{D}_j$: $\mathbf{n}_j(\mathbf{p}) = -\mathbf{n}_i(\mathbf{p})$. Therefore,



$$\underbrace{\iint_S \mathbb{V}\cdot\mathbf{n}_j\, dA}_{S \text{ as part of } \partial D_j} = \iint_S \mathbb{V}\cdot -\mathbf{n}_i\, dA = -\underbrace{\iint_S \mathbb{V}\cdot\mathbf{n}_i\, dA}_{S \text{ as part of } \partial D_i},$$

so the contributions that S makes to

$$\iint_{\partial D_i} \mathbb{V}\cdot\mathbf{n}\, dA \quad\text{and}\quad \iint_{\partial D_j} \mathbb{V}\cdot\mathbf{n}\, dA$$

exactly cancel. The only contributions that do not cancel are from the faces S that $\partial\vec{D}_i$ shares with $\partial\vec{D}$ itself. In those circumstances (and because $\vec{D}_i$ lies in $\vec{D}$), the outward normal on S is the same for $\vec{D}_i$ and for $\vec{D}$, so

$$\underbrace{\iint_S \mathbb{V}\cdot\mathbf{n}\, dA}_{\text{as part of } \partial D_i} = \underbrace{\iint_S \mathbb{V}\cdot\mathbf{n}_i\, dA}_{\text{as part of } \partial D}.$$

Therefore, after all the cancellations are taken into account,

$$\iint_{\partial D} \mathbb{V}\cdot\mathbf{n}\, dA = \iint_{\partial D_1} \mathbb{V}\cdot\mathbf{n}\, dA + \cdots + \iint_{\partial D_k} \mathbb{V}\cdot\mathbf{n}\, dA.$$

By hypothesis,

$$\iiint_{D_i} \operatorname{div}\mathbb{V}\, dV = \iint_{\partial D_i} \mathbb{V}\cdot\mathbf{n}\, dA$$

for each $i = 1,\ldots,k$, so the proof is complete. $\square$

11.2 Circulation and vorticity

The differential forms corresponding to a field

In the previous section we used the connection between a vector field and its divergence, on the one hand, and a corresponding 2-form and its exterior derivative. Quite generally, there is a natural way to make a vector field correspond to either a 1-form or a 2-form, and a scalar field (i.e., a function) to either a 0-form or a 3-form:

$$f \leftrightarrow \omega_f^0 = f,$$

$$\mathbb{F} = (A,B,C) \leftrightarrow \omega_{\mathbb{F}}^1 = A\,dx + B\,dy + C\,dz,$$

$$\mathbb{V} = (P,Q,R) \leftrightarrow \omega_{\mathbb{V}}^2 = P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy,$$

$$H \leftrightarrow \omega_H^3 = H\,dx\,dy\,dz.$$

$d\left(\omega_{\mathbb{V}}^2\right) = \omega_{\operatorname{div}\mathbb{V}}^3.$

The connection we made between the divergence and the exterior derivative can now be viewed in the following light. For the 2-form corresponding to $\mathbb{V}$,

$$\omega_{\mathbb{V}}^2 = P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy,$$

we have

$$d\left(\omega_{\mathbb{V}}^2\right) = (P_x + Q_y + R_z)\,dx\,dy\,dz = \omega_{\operatorname{div}\mathbb{V}}^3.$$

Hence, we can reformulate the divergence theorem as

$$\iiint_{\vec{B}} \omega_{\operatorname{div}\mathbb{V}}^3 = \iint_{\partial\vec{B}} \omega_{\mathbb{V}}^2, \quad \text{or} \quad \langle \vec{B}, \omega_{\operatorname{div}\mathbb{V}}^3 \rangle = \langle \partial\vec{B}, \omega_{\mathbb{V}}^2 \rangle,$$

for every suitable oriented region $\vec{B}$ in $\mathbb{R}^3$.

$d\left(\omega_f^0\right) = \omega_{\operatorname{grad} f}^1.$

Suppose instead that we begin with the 0-form ω_f^0 corresponding to a function f; then

$$d\left(\omega_f^0\right) = df = f_x\,dx + f_y\,dy + f_z\,dz = \omega_{\operatorname{grad} f}^1,$$

where $\operatorname{grad} f = (f_x, f_y, f_z)$ is the gradient vector field of f. For any piecewise-smooth oriented path $\vec{C}$, we have (cf. p. 425)

$$\int_{\vec{C}} \omega_{\operatorname{grad} f}^1 = \int_{\vec{C}} \operatorname{grad} f \cdot d\mathbf{x} = \int_{\vec{C}} df = f(\text{end of } \vec{C}) - f(\text{start of } \vec{C}).$$

The right-hand side of this equation is the "0-dimensional integral" of $f = \omega_f^0$ over $\partial\vec{C} = \text{end of } \vec{C} - \text{start of } \vec{C}$ (cf. pp. 428–429). Therefore, we can rewrite the equation itself as the symbolic pairing

$$\langle \vec{C}, \omega_{\operatorname{grad} f}^1 \rangle = \langle \partial\vec{C}, \omega_f^0 \rangle.$$

This is, in essence, the fundamental theorem of calculus; compare it to our reformulation of the divergence theorem, above.

The gradient and the divergence are differential operators. The gradient takes as input a scalar field and produces as output a vector field. The divergence does the reverse: the input is a vector field, the output a scalar. We have just noted that these two differential operators correspond to the exterior derivative operator on k-forms when $k=0$ and 2, respectively. What differential operator corresponds to the exterior derivative when $k=1$, and how is it defined? To answer these questions, we begin with the correspondence

Corresponding differential operators

$$\mathbb{F}=(A,B,C)\leftrightarrow\omega^1_{\mathbb{F}}=A\,dx+B\,dy+C\,dz.$$

A straightforward calculation (cf. p. 425) then gives

$$d\left(\omega^1_{\mathbb{F}}\right)=(C_y-B_z)\,dy\,dz+(A_z-C_x)\,dz\,dx+(B_x-A_y)\,dx\,dy.$$

This is a 2-form; it corresponds to the new vector field

$$\mathbb{V}=(C_y-B_z,A_z-C_x,B_x-A_y),$$

whose components are particular combinations of the derivatives of the components of $\mathbb{F}$. For reasons that emerge later, we call $\mathbb{V}$ the *curl* of $\mathbb{F}$.

Definition 11.2 *The **curl** of the vector field $\mathbb{F}=(A,B,C)$ is the vector field*

$\operatorname{curl}\mathbb{F}$

$$\operatorname{curl}\mathbb{F}=(C_y-B_z,A_z-C_x,B_x-A_y).$$

The curl is thus a differential operator whose input and output are both vector fields. It completes the trio of operators that correspond to the exterior derivative; we have

$d\left(\omega^1_{\mathbb{F}}\right)=\omega^2_{\operatorname{curl}\mathbb{F}}.$

$$d\left(\omega^1_{\mathbb{F}}\right)=\omega^2_{\operatorname{curl}\mathbb{F}}.$$

The gradient, divergence, and curl can all be expressed in terms of "nabla," the vector differential operator

nabla

$$\nabla=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)$$

introduced on page 93 for two variables and extended here to three. By treating nabla as if it were an ordinary vector, we can combine it with scalar and vector fields using scalar multiplication and the dot and cross-products. Scalar multiplication (by a function placed to the right of nabla) gives the gradient, a vector function:

$\nabla f=\operatorname{grad} f$
$\nabla\cdot\mathbb{V}=\operatorname{div}\mathbb{V}$
$\nabla\times\mathbb{F}=\operatorname{curl}\mathbb{F}$

$$\nabla f(x,y,z)=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)f=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)=\operatorname{grad} f.$$

The dot (or scalar) product with a vector field gives the divergence, a scalar function:

$$\nabla\cdot\mathbb{V}=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\cdot(P,Q,R)=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=\operatorname{div}\mathbb{V}.$$

The cross (or vector) product gives the curl, a vector function:

$$\nabla \times \mathbb{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (A,B,C) = \left(\begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ B & C \end{vmatrix}, \begin{vmatrix} \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial x} \\ C & A \end{vmatrix}, \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ A & B \end{vmatrix} \right)$$

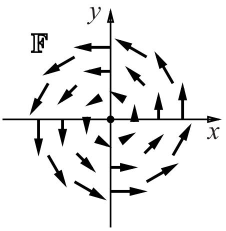
$$= \left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}, \frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}, \frac{\partial B}{\partial x} - \frac{\partial A}{\partial y},\right) = \operatorname{curl} \mathbb{F}.$$

Physical meaning of $\operatorname{curl} \mathbb{F}$

What is the physical meaning of $\operatorname{curl} \mathbb{F}$ when $\mathbb{F}$ describes a steady fluid flow? The answer to this question is complex and subtle, in part because $\operatorname{curl} \mathbb{F}$ is itself a vector rather than a scalar. To explore the question, we begin by looking at some examples.

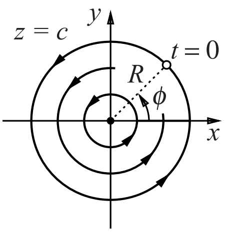
Example 1: $\mathbb{F} = (-\omega y, \omega x, 0)$

For the first example, take $\mathbb{F} = (-\omega y, \omega x, 0)$, where ω is a constant (not a differential form!). Because the z-component is zero, the field $\mathbb{F}$ is everywhere parallel to the (x,y)-plane, so fluid in the plane $z =$ constant stays in that plane. The figure shows $\mathbb{F}$ in one such plane; it suggests that the z-axis is a *vortex*. We now show, more exactly, that the fluid rotates around the z-axis with angular speed ω.



To begin, recall that $\mathbb{F}(x,y,z)$ represents the velocity of the fluid at the point (x,y,z). Thus, if $\mathbf{x}(t) = (x(t),y(t),z(t))$ represents the position of a particle of fluid at time t, then its velocity at that point is $\mathbb{F}(\mathbf{x}(t))$, so

$$\mathbf{x}'(t) = \mathbb{F}(\mathbf{x}(t)), \text{ or } x' = -\omega y, \quad y' = \omega x, \quad z' = 0.$$



It follows that $x'' = (-\omega y)' = -\omega^2 x$, implying that the solutions are sines and cosines. The general solution is the three-parameter family

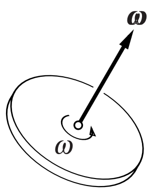
$$x(t) = R\cos(\omega t + \phi), \quad y(t) = R\sin(\omega t + \phi), \quad z(t) = c;$$

the parameters $R \geq 0$, ϕ, and c are the arbitrary constants of integration. These equations describe the motion of a fluid particle that is initially (i.e., when $t = 0$) at the point whose cylindrical coordinates are (R, ϕ, c) (cf. Exercise 5.11, p. 178). The particle remains in the plane $z = c$, moves on the circle of radius R centered on the z-axis, and makes an angle of $\theta = \omega t + \phi$ with the positive x-axis at time t. The angular speed is $\theta' = \omega$, as we wished to show.

Angular velocity vector

Any uniform rotation in space (such as we see in this example) is characterized by three elements:

1. Its axis of rotation
2. Its angular speed
3. The direction it rotates around its axis



We can use a vector, the **angular velocity vector**, $\boldsymbol{\omega}$, to represent these three elements. We take $\boldsymbol{\omega}$ to be parallel to the axis of rotation, to have magnitude $\omega = \|\boldsymbol{\omega}\|$ equal to the angular speed, and to have the direction that the thumb points when the fingers of the right hand curl in the direction of the rotation. Thus, a spinning disk with angular velocity $\boldsymbol{\omega}$ turns in the counterclockwise direction when viewed from the side toward which $\boldsymbol{\omega}$ points. (This is the definition to use when the coordinate frame itself is right-handed. If it is left-handed, then we would curl the fingers of the left hand to determine the direction of $\boldsymbol{\omega}$.) The angular velocity vector captures all aspects of a uniform rotation except the location—as distinct from the direction—of the axis of rotation in space.

Angular velocity and the curl

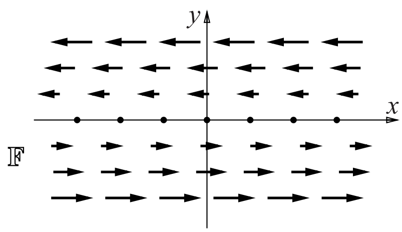
According to our analysis, the rotation of the flow $\mathbb{F}$ at any point on the z-axis is given by the angular velocity vector $\boldsymbol{\omega} = (0,0,\omega)$. A quick computation shows that

$$\operatorname{curl}\mathbb{F} = (0,0,2\omega) = 2\boldsymbol{\omega},$$

suggesting that $\operatorname{curl}\mathbb{F}$ essentially represents this uniform rotational motion (with $\|\operatorname{curl}\mathbb{F}\|/2$ equal to the angular speed). But there is a problem. Because $\operatorname{curl}\mathbb{F}$ is a field, it assigns a vector to each point (x,y,z) in space, namely, the constant vector $(0,0,2\omega)$. At any point $(0,0,z)$ on the z-axis, this vector appears to explain the rotation we see in the flow. But at no other point is the flow a rotation around that point. What is $\operatorname{curl}\mathbb{F}$ telling us there?

Example 2: $\mathbb{F} = (-ky,0,0)$

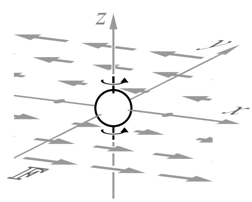
In fact, the curl does gives us information about rotation at every point, but the rotation is not the rotation of the fluid itself. To see what is actually involved, it is helpful to study a second flow that lacks the obvious vortex of Example 1. For Example 2 we take $\mathbb{F} = (-ky,0,0)$, $k > 0$.



The figure shows how $\mathbb{F}$ looks in the (x,y)-plane; it looks the same in every parallel plane. The fluid flows in straight lines parallel to the x-axis, moving left when $y > 0$ and right when $y < 0$. Everywhere on the (x,z)-plane ($y = 0$), the fluid is stationary. The fluid does not rotate. Nevertheless,

$$\operatorname{curl}\mathbb{F} = (0,0,k)$$

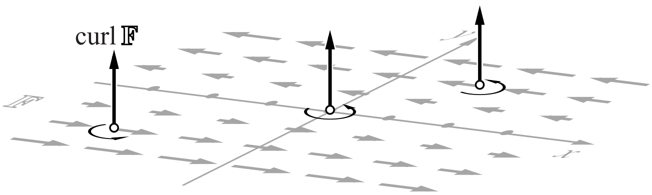
at every point (x,y,z). If this were an angular velocity, it would represent a counterclockwise rotation with angular speed $k/2$ around the vertical axis. What, if anything, is rotating?



Place at the origin a little ball with a rough surface like a tennis ball; use one whose density is the same as the fluid's, so it will have no tendency to float or sink. Because the fluid at the origin is motionless, the ball will stay put, but it will not remain motionless. The shearing action of the nearby fluid will make it spin in place around a vertical axis. The fluid at higher and lower levels (i.e., where $z > 0$ and $z < 0$) flows the same way as in the (x,y)-plane; therefore that fluid will not alter the way the ball moves. The net effect is a counterclockwise spin around the vertical; only the magnitude of the spin (its angular speed ω) remains undetermined. The angular velocity vector of the little ball is thus a positive multiple of curl$\mathbb{F}$.

The ball spins the same way everywhere

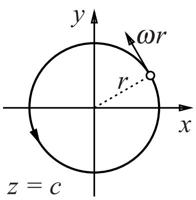
A similar test ball placed anywhere on the x-axis or, for that matter, anywhere in the vertical (x,z)-plane should behave the same way. Off the (x,z)-plane, the flow is nonzero, and the ball will be carried along by the fluid. But fluid particles at points farther from the (x,z)-plane move even faster, so they drag that side of the ball forward; particles closer to the (x,z)-plane move more slowly, dragging that side of the ball back. The fluid thus has the same shearing effect on the moving ball that it does on the stationary one: it spins the ball counterclockwise as it carries it along. So curl$\mathbb{F}$ describes the rotation of the test ball everywhere, at least qualitatively. Only the quantitative link between angular speed and $\|\,\text{curl}\,\mathbb{F}\|$ remains undetermined.



Vorticity

Let us call this tendency of a moving fluid to spin an object that is carried along with it the **vorticity** of the flow. Example 2 suggests that the vorticity of $\mathbb{F}$ is caused by its shearing action and is decribed by curl$\mathbb{F}$. In fact, by associating the curl with a flow's vorticity instead of its rotation, *per se*, we can clear up the puzzle of Example 1. In that example, fluid farther from the z-axis moves faster than fluid that is closer, but here the flow at one level $z =$ constant is the same as at any other. Therefore, as a test ball moves with the fluid around the z-axis, it also spins because of the shearing action of nearby fluid. At every point, the spin is counterclockwise around a vertical axis, a motion described qualitatively by curl$\mathbb{F} = (0,0,2\omega)$ at that point.

Quantifying vorticity



To describe vorticity quantitatively and not just qualitatively, we need some way to specify the magnitude of the spin induced by the shearing action of a flow. Return to Example 1 and its simple rotational motion around the z-axis. As the fluid moves around the circle of radius r centered at the origin in the plane $z = c$, its (linear) speed at any point is ωr. If we think of speed as a measure of the "motion" of a fluid, then the quantity

$$\text{speed} \times \text{length of path} = 2\omega\pi r^2$$

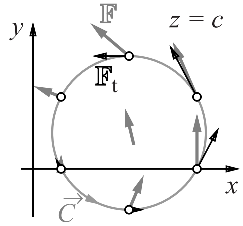
describes, in some sense, the total motion of the fluid as it travels around that circle. Note that this quantity, which we call the *circulation*, is proportional to the area of the circle. In fact,

$$\frac{\text{circulation}}{\text{area}} = \frac{2\omega\pi r^2}{\pi r^2} = 2\omega$$

is exactly the magnitude of the vector curl$\mathbb{F} = (0,0,2\omega)$.

Circulation as a path integral

Staying with Example 1, we ask: Can we determine the circulation of the fluid around a circle C in some plane $z = c$ but centered at a point away from the origin? Along C, the fluid flow is, in general, not longer tangent, so we need to decide how to measure the fluid's "motion." The figure in the margin suggests we replace the flow $\mathbb{F}$ by its tangential component $\mathbb{F}_{\mathbf{t}}$. Here $\mathbf{t}$ is the unit tangent vector to C in the counterclockwise direction. With this choice of $\mathbf{t}$, C becomes the oriented path that we denote as $\vec{C}$. Because the speed of the flow given by the tangential component is



$$\|\mathbb{F}_{\mathbf{t}}\| = \mathbb{F}\cdot\mathbf{t},$$

the "total motion" of this flow around $\vec{C}$ will be the integral of this scalar quantity with respect to arc length:

$$\text{circulation} = \oint_C \mathbb{F}\cdot\mathbf{t}\,ds.$$

(The domain of a scalar integral is an unoriented path; cf. Definition 1.6.) On page 19 we noted that this scalar integral has the same value as the vector integral

$$\oint_{\vec{C}} \mathbb{F}\cdot d\mathbf{x},$$

where $\vec{C}$ is C provided with the orientation given by the unit tangent $\mathbf{t}$. If we parametrize $\vec{C}$ as $(a+r\cos t, b+r\sin t, c)$ with $0 \le t \le 2\pi$, and recall that $\mathbb{F} = (-\omega y, \omega x, 0)$, then

$$\begin{aligned}
\text{circulation} &= \oint_{\vec{C}} \mathbb{F}\cdot d\mathbf{x} = \oint_{\vec{C}} -\omega y\,dx + \omega x\,dy \\
&= \omega r \int_0^{2\pi} \big((b+r\sin t)\sin t + (a+r\cos t)\cos t\big)\,dt \\
&= \omega r \int_0^{2\pi} \big(b\sin t + a\cos t + r\big)\,dt = 2\omega\pi r^2.
\end{aligned}$$

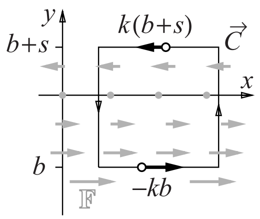
Thus, for every circle parallel to the (x,y)-plane, we have

$$\frac{\text{circulation}}{\text{area}} = \frac{2\omega\pi r^2}{\pi r^2} = 2\omega;$$

circulation per unit area equals the magnitude of the vorticity vector $\operatorname{curl}\mathbb{F}$ at every point.

Circulation for Example 2

Although we have not established that circulation per unit area measures vorticity in all cases, let us try it on the flow $\mathbb{F}$ of Example 2. This time we calculate the circulation around a square instead of a circle, because the calculation reduces to a simple product. Let $\vec{C}$ be the boundary of the square that lies in the plane $z = c$, has its lower-left corner at the point (a,b,c), and has sides of length s parallel to the x- and y-axes. Give $\vec{C}$ the counterclockwise orientation when seen from above (i.e., from where $z > c$).



The contributions to the circulation from the left and right sides are zero because the tangential velocity is zero there. On the bottom side (where $y = b$), the tangential velocity is $-kb$, so the contribution is $-kbs$. (In the figure, b is chosen to be negative, so $-kb$ is positive, as shown.) On the top side (where $y = b+s$), the contribution is $+k(b+s)s$. Therefore,

$$\text{circulation} = k(b+s)s - kbs = ks^2.$$

The area of the square is s^2, and the vorticity vector is $\operatorname{curl}\mathbb{F} = (0,0,k)$, so we find once again that the circulation per unit area equals the magnitude of the vorticity vector.

Staying with Example 2, let us determine circulation per unit area when the path is a circle instead of a square. We take the same oriented circle $\vec{C}$ we used for Example 1,

$$(x,y,z) = (a + r\cos t, b + r\sin t, c), \quad 0 \le t \le 2\pi.$$

With the flow field $\mathbb{F} = (-ky, 0, 0)$, we have

$$\text{circulation} = \oint_{\vec{C}} -ky\,dx = \int_0^{2\pi} (krb\sin t + kr^2\sin^2 t)\,dt = k\pi r^2.$$

For these paths, at least, circulation per unit area also equals k.

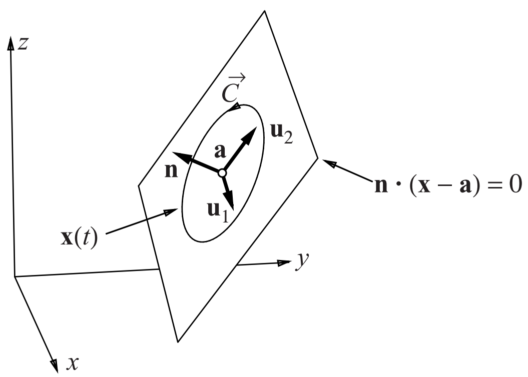
Circulation of a flow

Definition 11.3 *The **circulation** of the flow $\mathbb{F}$ around the oriented closed loop $\vec{C}$ is the path integral*

$$\textit{circulation of } \mathbb{F} \textit{ around } \vec{C} = \oint_{\vec{C}} \mathbb{F} \cdot d\mathbf{x}.$$

Circulation around other curves

In our two examples, vorticity was constant in both magnitude and direction, and we calculated the circulation only around curves lying in planes perpendicular to the vorticity vector $\operatorname{curl}\mathbb{F}$. Suppose we take an arbitrary plane; how does the circulation around a curve in that plane depend on the orientation of the plane?



Circles with arbitrary orientation

We can parametrize the oriented circle $\vec{C}$ of radius r that is centered at the point $\mathbf{a}$ and lies in the plane with unit normal $\mathbf{n}$ by choosing two perpendicular unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ for which $\mathbf{u}_1 \times \mathbf{u}_2 = \mathbf{n}$ and setting

$$\mathbf{x}(t) = \mathbf{a} + (r\cos t)\mathbf{u}_1 + (r\sin t)\mathbf{u}_2, \quad 0 \le t \le 2\pi.$$

Let $\mathbf{a} = (a,b,c)$, $\mathbf{u}_1 = (\alpha_1, \beta_1, \gamma_1)$, and $\mathbf{u}_2 = (\alpha_2, \beta_2, \gamma_2)$; then the circulation of $\mathbb{F} = (-ky, 0, 0)$ (Example 2) around $\vec{C}$ is

$$\begin{aligned}\oint_{\vec{C}} -ky\,dx &= \int_0^{2\pi} -k(b + r\beta_1\cos t + r\beta_2\sin t)(-r\alpha_1\sin t + r\alpha_2\cos t)\,dt \\ &= kr^2\int_0^{2\pi} (\alpha_1\beta_2\sin^2 t - \beta_1\alpha_2\cos^2 t)\,dt \\ &= kr^2(\alpha_1\beta_2\,\pi - \beta_1\alpha_2\,\pi) = \pi r^2 k \begin{vmatrix}\alpha_1 & \beta_1 \\ \alpha_2 & \beta_2\end{vmatrix}.\end{aligned}$$

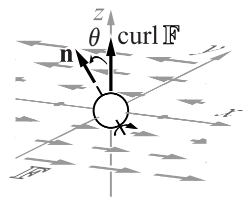
(Of the six terms in the second integral, only the two that make a nonzero contribution have been carried over to the third integral.) Because

$$\mathbf{n} = (\alpha_1, \beta_1, \gamma_1) \times (\alpha_2, \beta_2, \gamma_2) = \left(\begin{vmatrix}\beta_1 & \gamma_1 \\ \beta_2 & \gamma_2\end{vmatrix}, \begin{vmatrix}\gamma_1 & \alpha_1 \\ \gamma_2 & \alpha_2\end{vmatrix}, \begin{vmatrix}\alpha_1 & \beta_1 \\ \alpha_2 & \beta_2\end{vmatrix}\right)$$

and $\operatorname{curl}\mathbb{F} = (0,0,k)$, we can express the circulation of $\mathbb{F}$ around $\vec{C}$ as

$$(\text{area inside } \vec{C})\ \operatorname{curl}\mathbb{F}\cdot\mathbf{n}.$$

For this example, at least, we see how the orientation of the plane containing $\vec{C}$ affects the value of the circulation. It says that if the unit normal $\mathbf{n}$ makes an angle of θ radians with $\operatorname{curl}\mathbb{F}$, then



$$\frac{\text{circulation of } \mathbb{F} \text{ around } \vec{C}}{\text{area inside } \vec{C}} = \|\operatorname{curl}\mathbb{F}\|\cos\theta.$$

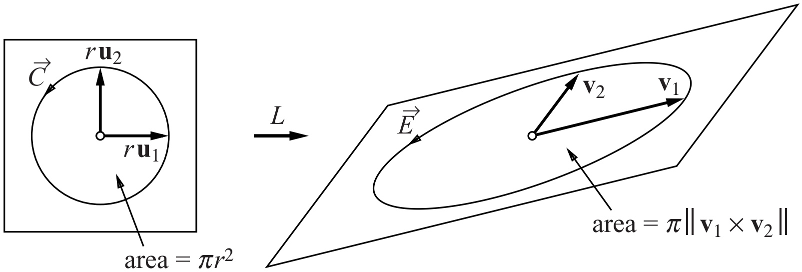
For example, suppose the test ball in Example 2 is constrained to rotate around an axis parallel to $\mathbf{n}$. If $\mathbf{n}$ is vertical (i.e., $\theta = 0$), the ball will spin as before. However, as we increase θ and tilt $\mathbf{n}$ away from the vertical, the shearing effect of the nearby fluid—and with it the ball's rate of spin—will decrease. As the axis becomes horizontal, the rate of spin will approach zero.

Parametrizing an ellipse in any plane

The connection between vorticity and circulation that we have noted in the examples holds, in fact, quite generally. To generalize, let us first replace the orthogonal vectors $r\mathbf{u}_1$ and $r\mathbf{u}_2$ in the parametrization of the circle $\vec{C}$ above by arbitrary linearly independent vectors $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\mathbf{x}(t) = \mathbf{a} + (\cos t)\mathbf{v}_1 + (\sin t)\mathbf{v}_2, \quad 0 \le t \le 2\pi.$$

The image is an oriented ellipse $\vec{E}$ whose area is $\pi\|\mathbf{v}_1 \times \mathbf{v}_2\|$. To see this, consider the linear map L of the plane containing $\vec{C}$ to the plane containing $\vec{E}$ given by $L(r\mathbf{u}_i) = \mathbf{v}_i$, $i = 1,2$.



Then $\vec{E}$ is an ellipse because it is the image of a circle under a linear map. Furthermore, because L maps the square $r\mathbf{u}_1 \wedge r\mathbf{u}_2$ to the parallelogram $\mathbf{v}_1 \wedge \mathbf{v}_2$,

$$|\text{area magnification factor for } L| = \left|\frac{\text{area}(\mathbf{v}_1 \wedge \mathbf{v}_2)}{\text{area}(r\mathbf{u}_1 \wedge r\mathbf{u}_2)}\right| = \frac{\|\mathbf{v}_1 \times \mathbf{v}_2\|}{r^2}.$$

Finally, because $\vec{E} = L(\vec{C})$ and $\vec{C}$ has area πr^2, the area of $\vec{E}$ is $\pi\|\mathbf{v}_1 \times \mathbf{v}_2\|$.

Flows and maps

To continue with our generalization of the connection between vorticity and circulation, we note that a flow field $\mathbb{F} = (A,B,C)$ defined on an open region Ω in $\mathbb{R}^3$ can be considered a map $\mathbb{F} : \Omega \to \mathbb{R}^3$,

$$\mathbb{F} : \begin{cases} u = A(x,y,z), \\ v = B(x,y,z), \\ w = C(x,y,z). \end{cases}$$

In particular, if the map $\mathbb{F}$ is differentiable on Ω, then we have

$$\mathbb{F}(\mathbf{a} + \Delta\mathbf{x}) = \mathbb{F}(\mathbf{a}) + d\mathbb{F}_{\mathbf{a}}(\Delta\mathbf{x}) + \boldsymbol{o}(\Delta\mathbf{x}) \quad \text{as } \Delta\mathbf{x} \to \mathbf{0},$$

for any point $\mathbf{a}$ in Ω (Definition 4.6, p. 129). The following theorem now considers the circulation of $\mathbb{F}$ around the family of ellipses

$$\vec{E}_\varepsilon : \mathbf{x}(t) = \mathbf{a} + (\varepsilon\cos t)\mathbf{v}_1 + (\varepsilon\sin t)\mathbf{v}_2, \quad 0 \le t \le 2\pi.$$

Note that area $\vec{E}_\varepsilon = \pi\varepsilon^2\|\mathbf{v}_1 \times \mathbf{v}_2\|$. In our examples, the ratio of circulation to area had a constant value, namely, $\operatorname{curl}\mathbb{F} \cdot \mathbf{n}$. Here the ratio is no longer constant; nevertheless, its limiting value equals $\operatorname{curl}\mathbb{F} \cdot \mathbf{n}$ as the area approaches zero.

Limiting value of circulation/area

Theorem 11.8. *If $\mathbb{F} : \Omega \to \mathbb{R}^3$ is a differentiable flow field, then*

$$\lim_{\varepsilon \to 0} \frac{\textit{circulation of } \mathbb{F} \textit{ around } \vec{E}_\varepsilon}{\textit{area inside } \vec{E}_\varepsilon} = \operatorname{curl}\mathbb{F} \cdot \mathbf{n},$$

where $\mathbf{n}$ is the unit normal in the plane containing the ellipses $\vec{E}_\varepsilon$.

Proof. The proof follows from a sequence of lemmas.

Lemma 11.1. $\displaystyle\oint_{\vec{E}_\varepsilon} \mathbb{F} \cdot d\mathbf{x} = \pi\varepsilon^2\left[d\mathbb{F}_{\mathbf{a}}(\mathbf{v}_1) \cdot \mathbf{v}_2 - d\mathbb{F}_{\mathbf{a}}(\mathbf{v}_2) \cdot \mathbf{v}_1\right] + o(\varepsilon^2)$ *as* $\varepsilon \to 0$.

Proof. The circulation of $\mathbb{F}$ around $\vec{E}_\varepsilon$ is the integral

$$\oint_{\vec{E}_\varepsilon} \mathbb{F}\cdot d\mathbf{x} = \int_0^{2\pi} \mathbb{F}(\mathbf{a}+\varepsilon\cos t\,\mathbf{v}_1+\varepsilon\sin t\,\mathbf{v}_2)\cdot(-\varepsilon\sin t\,\mathbf{v}_1+\varepsilon\cos t\,\mathbf{v}_2)\,dt.$$

Set $\Delta\mathbf{x}=\varepsilon\cos t\,\mathbf{v}_1+\varepsilon\sin t\,\mathbf{v}_2$. Because $\Delta\mathbf{x}\to\mathbf{0}$ as $\varepsilon\to 0$, we have $\boldsymbol{o}(\Delta\mathbf{x})=o(\varepsilon)$. Therefore, using the differentiblity of $\mathbb{F}$, we can write the integral as

$$\int_0^{2\pi} (\mathbb{F}(\mathbf{a})+\varepsilon\cos t\,d\mathbb{F}_{\mathbf{a}}(\mathbf{v}_1)+\varepsilon\sin t\,d\mathbb{F}_{\mathbf{a}}(\mathbf{v}_2)+o(\varepsilon))\cdot(-\varepsilon\sin t\,\mathbf{v}_1+\varepsilon\cos t\,\mathbf{v}_2)\,dt.$$

When we expand the dot product, we get eight terms. Four have average value zero and therefore vanish when integrated; the others combine to give

$$\begin{aligned}\oint_{\vec{E}_\varepsilon} \mathbb{F}\cdot d\mathbf{x} &= \int_o^{2\pi} (\varepsilon^2\cos^2 t\,d\mathbb{F}_{\mathbf{a}}(\mathbf{v}_1)\cdot\mathbf{v}_2-\varepsilon^2\sin^2 t\,d\mathbb{F}_{\mathbf{a}}(\mathbf{v}_2)\cdot\mathbf{v}_1+o(\varepsilon^2))\,dt\\ &= \pi\varepsilon^2\left[d\mathbb{F}_{\mathbf{a}}(\mathbf{v}_1)\cdot\mathbf{v}_2-d\mathbb{F}_{\mathbf{a}}(\mathbf{v}_2)\cdot\mathbf{v}_1\right]+o(\varepsilon^2)\quad \text{as } \varepsilon\to 0.\end{aligned}$$ □

We find that the expression $\psi(\mathbf{v}_1,\mathbf{v}_2)=d\mathbb{F}_{\mathbf{a}}(\mathbf{v}_1)\cdot\mathbf{v}_2-d\mathbb{F}_{\mathbf{a}}(\mathbf{v}_2)\cdot\mathbf{v}_1$ is the key to relating the circulation of $\mathbb{F}$ to curl $\mathbb{F}$. $\psi(\mathbf{v}_1,\mathbf{v}_2)$

Lemma 11.2. *There is a unique vector* $\mathbf{q}$ *for which*

$$\psi(\mathbf{v}_1,\mathbf{v}_2)=d\mathbb{F}_{\mathbf{a}}(\mathbf{v}_1)\cdot\mathbf{v}_2-d\mathbb{F}_{\mathbf{a}}(\mathbf{v}_2)\cdot\mathbf{v}_1=\mathbf{q}\cdot(\mathbf{v}_1\times\mathbf{v}_2),$$

for all vectors $\mathbf{v}_1$, $\mathbf{v}_2$ *in* $\mathbb{R}^3$.

Proof. For any vector $\mathbf{q}$, define the function

$$\tau_{\mathbf{q}}(\mathbf{v}_1,\mathbf{v}_2)=\mathbf{q}\cdot(\mathbf{v}_1\times\mathbf{v}_2).$$

Then ψ and $\tau_{\mathbf{q}}$ are both bilinear and antisymmetric (cf. p. 62); that is, they are linear functions of each of their inputs and, furthrmore,

$$\psi(\mathbf{v}_2,\mathbf{v}_1)=-\psi(\mathbf{v}_1,\mathbf{v}_2),\quad \tau_{\mathbf{q}}(\mathbf{v}_2,\mathbf{v}_1)=-\tau_{\mathbf{q}}(\mathbf{v}_1,\mathbf{v}_2),$$

for all pairs $\mathbf{v}_1,\mathbf{v}_2$. Therefore, if ψ and $\tau_{\mathbf{q}}$ agree on a basis for $\mathbb{R}^3$, they must agree everywhere (cf. Exercise 11.11).

Set $\mathbf{q}=(q_1,q_2,q_3)$ where

$$q_1=\psi(\mathbf{e}_2,\mathbf{e}_3),\quad q_2=\psi(\mathbf{e}_3,\mathbf{e}_1),\quad q_3=\psi(\mathbf{e}_1,\mathbf{e}_2),$$

and $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ is the standard basis for $\mathbb{R}^3$. Then

$$\begin{aligned}\tau_{\mathbf{q}}(\mathbf{e}_2,\mathbf{e}_3)&=\mathbf{q}\cdot(\mathbf{e}_2\times\mathbf{e}_3)=\mathbf{q}\cdot\mathbf{e}_1=q_1=\psi(\mathbf{e}_2,\mathbf{e}_3),\\ \tau_{\mathbf{q}}(\mathbf{e}_3,\mathbf{e}_1)&=\mathbf{q}\cdot(\mathbf{e}_3\times\mathbf{e}_1)=\mathbf{q}\cdot\mathbf{e}_2=q_2=\psi(\mathbf{e}_3,\mathbf{e}_1),\\ \tau_{\mathbf{q}}(\mathbf{e}_1,\mathbf{e}_2)&=\mathbf{q}\cdot(\mathbf{e}_1\times\mathbf{e}_2)=\mathbf{q}\cdot\mathbf{e}_3=q_3=\psi(\mathbf{e}_1,\mathbf{e}_2);\end{aligned}$$

by antisymmetry, we find $\tau_{\mathbf{q}}(\mathbf{e}_i,\mathbf{e}_j)=\psi(\mathbf{e}_i,\mathbf{e}_j)$ for all $i,j=1,2,3$. $\square$

Any square matrix M can be written uniquely as the sum of a symmetric and an antisymmetric matrix. Set

$$\mathcal{S}=\tfrac{1}{2}(M+M^\dagger),\quad \mathcal{A}=\tfrac{1}{2}(M-M^\dagger),$$

where $M^\dagger$ is the transpose of M. Then $\mathcal{S}^\dagger=\mathcal{S}$, $\mathcal{A}^\dagger=-\mathcal{A}$, and $M=\mathcal{S}+\mathcal{A}$.

$\psi_M=\psi_{\mathcal{A}}$

Lemma 11.3. *For any* 3×3 *matrix* M*,*

$$\psi_M=M\mathbf{v}_1\cdot\mathbf{v}_2-M\mathbf{v}_2\cdot\mathbf{v}_1=\mathcal{A}\mathbf{v}_1\cdot\mathbf{v}_2-\mathcal{A}\mathbf{v}_2\cdot\mathbf{v}_1=\psi_{\mathcal{A}},$$

where $\mathcal{A}$ *is the antisymmetric part of* M*:* $\mathcal{A}=\frac{1}{2}(M-M^\dagger)$.

Proof. Write $M=\mathcal{S}+\mathcal{A}$, where $\mathcal{S}=\frac{1}{2}(M+M^\dagger)$; then it is easy to see that $\psi_M=\psi_{\mathcal{S}}+\psi_{\mathcal{A}}$. Now write the dot products as matrix multiplications using column vectors and their transposes:

$$\psi_{\mathcal{S}}=(\mathcal{S}\mathbf{v}_1)^\dagger\,\mathbf{v}_2-(\mathcal{S}\mathbf{v}_2)^\dagger\,\mathbf{v}_1=\mathbf{v}_1^\dagger\mathcal{S}^\dagger\,\mathbf{v}_2-\mathbf{v}_2^\dagger\mathcal{S}^\dagger\,\mathbf{v}_1=\mathbf{v}_1^\dagger\mathcal{S}^\dagger\,\mathbf{v}_2-\mathbf{v}_2^\dagger\mathcal{S}\mathbf{v}_1.$$

In the last term we used the symmetry of $\mathcal{S}$. Each term is a scalar (i.e., a 1×1 matrix), so is equal to its own transpose. Thus we can write

$$\mathbf{v}_1^\dagger\mathcal{S}^\dagger\,\mathbf{v}_2=(\mathbf{v}_1^\dagger\mathcal{S}^\dagger\,\mathbf{v}_2)^\dagger=\mathbf{v}_2^\dagger\mathcal{S}\,\mathbf{v}_1,$$

from which it follows that $\psi_{\mathcal{S}}\equiv 0$ and hence $\psi_M\equiv\psi_{\mathcal{A}}$. $\square$

Antisymmetric part of $d\mathbb{F}_{\mathbf{a}}$

For the map $\mathbb{F}$ with components (A,B,C), the derivative and its antisymmetric part are

$$d\mathbb{F}_{\mathbf{a}}=\begin{pmatrix}A_x & A_y & A_z\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{pmatrix},\quad \mathcal{A}=\frac{1}{2}\begin{pmatrix}0 & -\gamma & \beta\\ \gamma & 0 & -\alpha\\ -\beta & \alpha & 0\end{pmatrix},$$

where

$$\alpha=C_y-B_z,\quad \beta=A_z-C_x,\quad \gamma=B_x-A_y$$

and all components are evaluated at $\mathbf{x}=\mathbf{a}$. Labels are chosen for the components of $\mathcal{A}$ so that $(\alpha,\beta,\gamma)=\operatorname{curl}\mathbb{F}(\mathbf{a})$. With $\mathcal{A}$ we can now determine the vorticity vector $\mathbf{q}$ that is provided by Lemma 11.2.

Lemma 11.4. $d\mathbb{F}_{\mathbf{a}}(\mathbf{v}_1)\cdot\mathbf{v}_2-d\mathbb{F}_{\mathbf{a}}(\mathbf{v}_2)\cdot\mathbf{v}_1=\operatorname{curl}\mathbb{F}(\mathbf{a})\cdot(\mathbf{v}_1\times\mathbf{v}_2)$.

Proof. Because $\psi_{d\mathbb{F}_{\mathbf{a}}}=\psi_{\mathcal{A}}$, we have

$$q_1=\psi_{\mathcal{A}}(\mathbf{e}_2,\mathbf{e}_3)=\mathcal{A}\mathbf{e}_2\cdot\mathbf{e}_3-\mathcal{A}\mathbf{e}_3\cdot\mathbf{e}_2=\tfrac{1}{2}\begin{pmatrix}-\gamma\\0\\\alpha\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}-\tfrac{1}{2}\begin{pmatrix}\beta\\-\alpha\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}=\alpha.$$

In a similar way, you can show $q_2=\beta$, $q_3=\gamma$. $\square$

To complete the proof of Theorem 11.8, use Lemma 11.1 and Lemma 11.4 to write

$$\begin{aligned}\frac{\text{circulation of } \mathbb{F} \text{ around } \vec{E}_\varepsilon}{\text{area inside } \vec{E}_\varepsilon} &= \frac{\pi\varepsilon^2\left[d\mathbb{F}_{\mathbf{a}}(\mathbf{v}_1)\cdot\mathbf{v}_2 - d\mathbb{F}_{\mathbf{a}}(\mathbf{v}_2)\cdot\mathbf{v}_1\right] + o(\varepsilon^2)}{\pi\varepsilon^2\|\mathbf{v}_1\times\mathbf{v}_2\|} \\ &= \operatorname{curl}\mathbb{F}\cdot\frac{\mathbf{v}_1\times\mathbf{v}_2}{\|\mathbf{v}_1\times\mathbf{v}_2\|} + \frac{o(\varepsilon^2)}{\varepsilon^2} \\ &= \operatorname{curl}\mathbb{F}\cdot\mathbf{n} + \frac{o(\varepsilon^2)}{\varepsilon^2}.\end{aligned}$$

In the second term, the divisor $\pi\|\mathbf{v}_1\times\mathbf{v}_2\|$ has been absorbed into $o(\varepsilon^2)$. The theorem then follows because, by definition, $o(\varepsilon^2)/\varepsilon^2 \to 0$ as $\varepsilon \to 0$. □

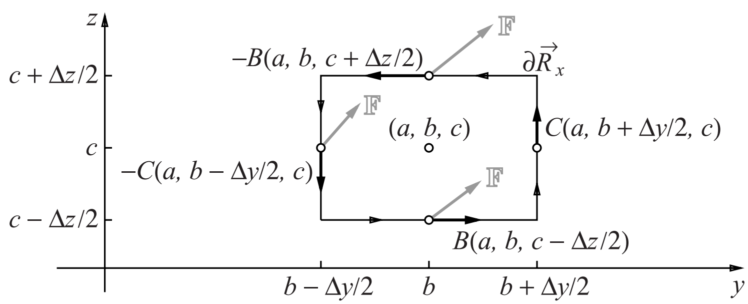
Reconstructing the vorticity vector $\mathbf{q}$

In the proof of Lemma 11.2, there is no motivation (other than hindsight) for the formula

$$\mathbf{q} = (\psi(\mathbf{e}_2,\mathbf{e}_3), \psi(\mathbf{e}_3,\mathbf{e}_1), \psi(\mathbf{e}_1,\mathbf{e}_2))$$

that expresses the components of the vorticity vector $\mathbf{q}$ in terms of particular circulation calculations. According to this equation, the x-component of vorticity comes from circulation in a plane normal to the x-axis (i.e., a plane parallel to the vectors $\mathbf{e}_2$ and $\mathbf{e}_3$ that determine the (y,z)-plane). Similarly, the y-component of vorticity $\mathbf{q}$ uses a plane normal to the y-axis, and the z-component of $\mathbf{q}$ uses a plane normal to the z-axis. Let us now reconstruct $\mathbf{q}$ by calculating anew the circulation in those planes. The work will look similar to the initial work (pp. 449–451) that led to our identifying the divergence of a field.

It is convenient to calculate the circulation around the boundary of a rectangle—as we did in Example 2—rather than around an ellipse. On a rectangle whose sides are parallel to the axes, the tangential component of $\mathbb{F}$ on a side is just one of the Cartesian components of $\mathbb{F}$. We begin with the oriented rectangle $\vec{R}_x$ centered at the point $\mathbf{a} = (a,b,c)$ and lying in the plane $x = a$ (and thus parallel to the (y,z)-plane). Let the lengths of its sides be denoted Δy and Δz, and orient it counterclockwise when viewed from the side where $x > a$. Its orientation normal is then $\mathbf{n} = (1,0,0)$.



On the bottom edge (where $z = c - \Delta z/2$), the tangential component of $\mathbb{F} = (A,B,C)$ is $B(a,y,c-\Delta z/2)$. If we approximate B everywhere on this edge by its value at the center, $(a,b,c-\Delta z/2)$, then the contribution this edge makes to the circulation of $\mathbb{F}$ around $\partial\vec{R}_x$ is approximately

$$B(a,b,c-\Delta z/2)\,\Delta y.$$

Along the top edge, the tangential component of $\mathbb{F}$ is $-B(a,y,c+\Delta z/2)$. The minus sign is needed because the unit tangent to $\partial\vec{R}_x$ on this edge is $\mathbf{t}=(0,-1,0)$. If we approximate $-B$ everywhere by its value $-B(a,b,c+\Delta z/2)$ at the center, the approximate contribution this edge makes to the circulation is

$$-B(a,b,c+\Delta z/2)\,\Delta y.$$

If we put these together and use the microscope equation, we have

$$\begin{aligned}-B(a,b,c+\Delta z/2)\,\Delta y &+ B(a,b,c-\Delta z/2)\,\Delta y\\ &= -\frac{B(a,b,c+\Delta z/2)-B(a,b,c-\Delta z/2)}{\Delta z}\,\Delta z\,\Delta y\\ &\approx -B_z(a,b,c)\,\Delta y\,\Delta z.\end{aligned}$$

For the right and left edges there is a similar result: together they contribute approximately

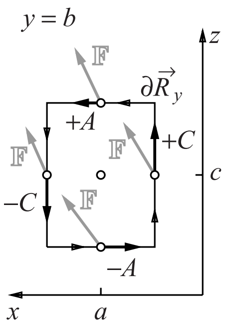
$$\begin{aligned}C(a,b+\Delta y/2,c)\,\Delta z &- C(a,b-\Delta y/2)\,\Delta z\\ &= \frac{C(a,b+\Delta y/2,c)-C(a,b-\Delta y/2,c)}{\Delta y}\,\Delta y\,\Delta z\\ &\approx C_y(a,b,c)\,\Delta y\,\Delta z.\end{aligned}$$

Thus we can write

$$\text{circulation of } \mathbb{F} \text{ around } \partial\vec{R}_x \approx (C_y(\mathbf{a})-B_z(\mathbf{a}))\ \text{area}\,\vec{R}_x.$$

The factor $C_y - B_z$ is indeed the x-component of curl$\,\mathbb{F}$.

For a similar rectangle $\vec{R}_y$ in the plane $y=b$ parallel to the (z,x)-plane, the tangential components of $\mathbb{F}$ at the centers of the sides parallel to the z-axis are $C(a-\Delta x/2,b,c)$ and $-C(a+\Delta x/2,b,c)$. Their contribution to the circulation is approximately



$$\begin{aligned}-C(a+\Delta x/2,b,c)\,\Delta z &+ C(a-\Delta x/2,b,c)\,\Delta z\\ &= -\frac{C(a+\Delta x/2,b,c)-C(a-\Delta x/2,b,c)}{\Delta x}\,\Delta x\,\Delta z\\ &\approx -C_x(a,b,c)\,\Delta z\,\Delta x.\end{aligned}$$

The tangential components of $\mathbb{F}$ at the centers of the remaining two sides are $A(a,b,c+\Delta z/2)$ and $-A(a,b,c-\Delta z/2)$, making together the approximate contribution

$$A(a,b,c+\Delta z/2)\,\Delta x - A(a,b,c-\Delta z/2)\,\Delta x$$
$$= \frac{A(a,b,c+\Delta z/2) - A(a,b,c-\Delta z/2)}{\Delta z}\,\Delta z\,\Delta x \approx A_z(a,b,c)\;\Delta z\,\Delta x$$

to the circulation around $\partial\vec{R}_y$. Thus

$$\text{circulation of } \mathbb{F} \text{ around } \partial\vec{R}_y \approx (A_z(\mathbf{a}) - C_x(\mathbf{a}))\;\text{area}\,\vec{R}_y;$$

the factor $A_z - C_x$ is the y-component of curl$\,\mathbb{F}$. A similar analysis of an oriented rectangle $\vec{R}_z$ in the plane $z = c$ leads to the result

$$\text{circulation of } \mathbb{F} \text{ around } \partial\vec{R}_z \approx (B_x(\mathbf{a}) - A_y(\mathbf{a}))\;\text{area}\,\vec{R}_z;$$

the factor $B_x - A_y$ gives the final component of curl$\,\mathbb{F}$.

Circulation and vorticity

Thus, the circulation around any one of the rectangles $\vec{R}$ is approximately curl$\,\mathbb{F}(\mathbf{a})\cdot\mathbf{n}$ area$\,\vec{R}$. But this expression is itself an approximation to the flux of the vector field curl$\,\mathbb{F}$ through the small rectangle $\vec{R}$ (Definition 10.1, p. 388). Hence,

$$\text{circulation of } \mathbb{F} \text{ around } \partial\vec{R} \approx \text{flux of curl}\,\mathbb{F} \text{ through } \vec{R}.$$

Even more is true. In the next section, we show that, for an oriented surface $\vec{S}$ with boundary $\partial\vec{S}$, the circulation of the vector field $\mathbb{F}$ around $\partial\vec{S}$ is equal to the flux of its vorticity field curl$\,\mathbb{F}$ through $\vec{S}$:

$$\text{circulation of } \mathbb{F} \text{ around } \partial\vec{S} = \text{flux of curl}\,\mathbb{F} \text{ through } \vec{S}$$
$$\oint_{\partial S} \mathbb{F}\cdot\mathbf{t}\,ds = \iint_S \text{curl}\,\mathbb{F}\cdot\mathbf{n}\,dA.$$

This is the physical content of Stokes' theorem. It also gives us a new way to look at vorticity. Instead of having to rely on the physical image of a little ball set spinning by the shear action of a fluid, we can use the mathematical notion of the circulation of that fluid in various planes. Note that the circulation/flux identity involves two distinct flows: $\mathbb{F}$ and its *vorticity* curl$\,\mathbb{F}$.

The vorticity flow of a flow

Definition 11.4 *Let the continuously differentiable vector field* $\mathbb{F}$ *represent a steady fluid flow field; then the* ***vorticity flow field****, or the* ***vortex flow field****, of* $\mathbb{F}$ *is represented by the vector field* $\mathbb{V} = \text{curl}\,\mathbb{F}$.

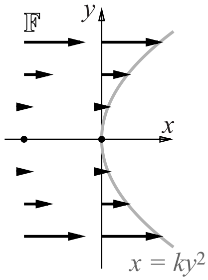
The relative motions of nearby fluid particles

We attribute the vorticity of a fluid flow—as given by the curl—to shearing forces induced by the flow. We attribute these forces, in turn, to the relative motions of nearby fluid particles. Let us now analyze the relative motions; as we show in Theorem 11.10, they explain the divergence as well as the curl.

Example 3: $\mathbb{F} = (ky^2, 0, 0)$

The velocity field $\mathbb{F}$ determines the overall motion of the fluid; it must, therefore, determine the relative motion as well. To explore this connection and to clarify the distinction between the two kinds of motion, we work through a third example:

$$\mathbb{F} = (ky^2, 0, 0), \quad k > 0.$$



Think of this as a nonlinear modification of Example 2; the nonlinearity will help us see the relative motion more clearly. The fluid flows in straight lines parallel to the x-axis. In the (x,y)-plane, the speed of a particle is proportional to the square of its distance from the x-axis; in any parallel plane, the flow looks the same.

As in the previous examples, the position $\mathbf{x}(t) = (x(t), y(t), z(t))$ of a particle at time t is determined by the differential equations

$$\mathbf{x}'(t) = \mathbb{F}(\mathbf{x}(t)), \text{ that is, } x' = ky^2, \quad y' = 0, \quad z' = 0.$$

The solutions are

$$x(t) = a + kb^2 t, \quad y(t) = b, \quad z(t) = c;$$

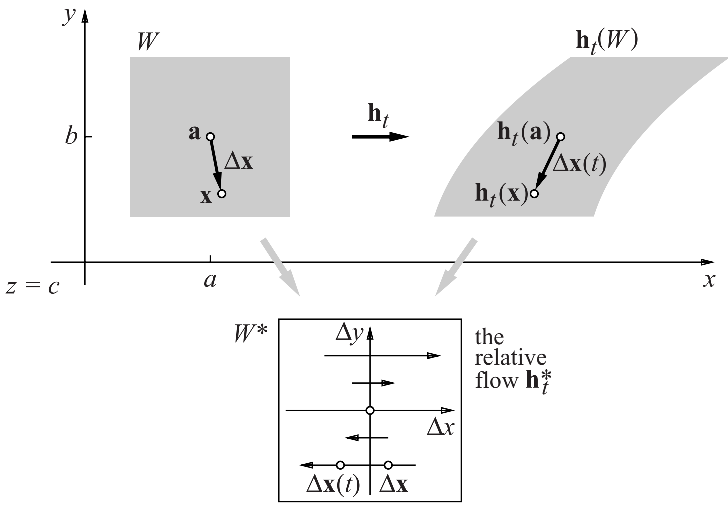
a, b, and c are arbitrary constants of integration. These equations parametrize the path of the particle that is initially (i.e., when $t = 0$) at the point $\mathbf{a} = (a, b, c)$. The particle does not move if $b = 0$; therefore we assume $b \neq 0$ for the rest of the discussion.

The flow maps $\mathbf{h}_t$

More generally, then, the equations say that the particle at the point $\mathbf{x} = (x, y, z)$ at time $t = 0$ is at the point $\mathbf{h}_t(\mathbf{x}) = (x + ky^2 t, y, z)$ at an arbitrary (earlier or later) time t. In other words, the flow defines, and is defined by, the family of maps $\mathbf{h}_t : \mathbb{R}^3 \to \mathbb{R}^3$:

$$\mathbf{h}_t : \begin{cases} u = x + ky^2 t, \\ v = y, \\ w = z; \end{cases} \qquad d(\mathbf{h}_t)_{\mathbf{x}} = \begin{pmatrix} 1 & 2kyt & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $\mathbf{h}_s \circ \mathbf{h}_t = \mathbf{h}_{s+t}$ for every s, t. This implies each $\mathbf{h}_t$ is invertible, because $\mathbf{h}_{-t} \circ \mathbf{h}_t = \mathbf{h}_0 =$ identity.



Describing the relative flow

To describe the relative motion of fluid particles near the particle that is intially at the point $\mathbf{a}$, choose a small cube W (a "window") centered at $\mathbf{a}$, and let $\mathbf{h}_t(W)$ denote the image of W under the flow $\mathbf{h}_t$ (t can take negative as well as positive values). If

$\mathbf{x}$ is a point in W, and $\Delta x = \mathbf{x} - \mathbf{a}$ indicates the position of $\mathbf{x}$ in relation to $\mathbf{a}$, then the vector

$$\Delta\mathbf{x}(t) = \mathbf{h}_t(\mathbf{x}) - \mathbf{h}_t(\mathbf{a})$$

in $\mathbf{h}_t(W)$ indicates how the relative position $\Delta\mathbf{x}$ varies over time. When the vectors $\Delta\mathbf{x}(t)$ (for t near 0) are translated to a common point (at the origin of a window W^* with local coordinates $\Delta\mathbf{x} = (\Delta x, \Delta y, \Delta z)$), they exhibit the relative flow we seek to describe. With these local coordinates in mind, we rewrite $\mathbf{h}_t(\mathbf{x})$ as

$$\mathbf{h}_t(\mathbf{a} + \Delta\mathbf{x}) = (a + \Delta x + k(b + \Delta y)^2 t,\ b + \Delta y, c + \Delta z),$$

which shows how $\Delta\mathbf{x}(t) = \mathbf{h}_t(\mathbf{x}) - \mathbf{h}_t(\mathbf{a})$ can be described by the equations

$$\mathbf{h}_t^* : \begin{cases} \Delta x(t) = \Delta x + t\,(2kb\,\Delta y + k(\Delta y)^2), \\ \Delta y(t) = \Delta y, \\ \Delta z(t) = \Delta z. \end{cases}$$

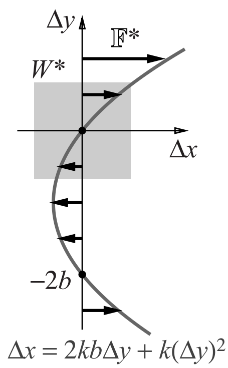
The relative-flow maps $\mathbf{h}_t^*$

From one point of view, these equations parametrize a straight line that is parallel to the Δx-axis and passes through the arbitrary, but fixed, point $\Delta\mathbf{x} = (\Delta x, \Delta y, \Delta z)$ in W^*. This straight line is one of the relative flow lines we see in W^*, above. From a second point of view (in which $\Delta\mathbf{x}$ is variable, not fixed), these equations define the family of *relative-flow maps* $\mathbf{h}_t^* : W^* \to \mathbb{R}^3$. When the center of the original window W lies off the (z,x)-plane (i.e., when $b \neq 0$), the relative flow described by $\mathbf{h}_t^*$ is along paths that head in opposite directions on opposite sides of the Δx-axis.

The relative-flow field $\mathbb{F}^*$

The relative flow has its own velocity field; let us call it $\mathbb{F}^*$. The field vector $\mathbb{F}^*$ at the point $\Delta\mathbf{x} = (\Delta x, \Delta y, \Delta z)$ is, by definition, the velocity of the path $\Delta\mathbf{x}(t)$ at its intitial point:

$$\mathbb{F}^*(\Delta x, \Delta y, \Delta z) = \frac{d}{dt}\Delta\mathbf{x}(t)\bigg|_{t=0} = (2kb\Delta y + k(\Delta y)^2, 0, 0)$$



Alternatively, the relative flow field is the map $\mathbb{F}^* : W^* \to \mathbb{R}^3$,

$$\mathbb{F}^* : \begin{cases} \Delta u = 2kb\,\Delta y + k(\Delta y)^2, \\ \Delta v = 0, \\ \Delta w = 0. \end{cases}$$

Because we assume W^* is small, $k(\Delta y)^2$ is negligible in comparison to $2kb\,\Delta y$ (recall that $b \neq 0$), so $\mathbb{F}^*$ is well approximated by its linear part,

$$\begin{pmatrix} 0 & 2kb & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix},$$

which is just $d\mathbb{F}_{\mathbf{a}}(\Delta\mathbf{x})$. This gives us the following links between the relative flow $\mathbb{F}^*$ in W^* and the overall flow $\mathbb{F}$.

$$\mathbb{F}^*(\Delta\mathbf{x}) = d\mathbb{F}_{\mathbf{a}}(\Delta\mathbf{x}) + \boldsymbol{o}(\Delta\mathbf{x}),$$
$$\mathbf{h}_t^*(\Delta\mathbf{x}) = \Delta\mathbf{x} + t\,d\mathbb{F}_{\mathbf{a}}(\Delta\mathbf{x}) + t\,\boldsymbol{o}(\Delta\mathbf{x}).$$

Because we can make W^* arbitrarily small, we can neglect the higher-order terms $\boldsymbol{o}(\Delta\mathbf{x})$ in analyzing the relative flow, so that

$$\mathbf{h}_t^* \approx d(\mathbf{h}_t^*)_{\mathbf{0}} = I + t\,d\mathbb{F}_{\mathbf{a}}.$$

Splitting $d\mathbb{F}_{\mathbf{a}}$ to describe its action

Hence, we look to the action of $d\mathbb{F}_{\mathbf{a}}$ to explain the relative flow. First split $d\mathbb{F}_{\mathbf{a}}$ into its symmetric and antisymmetric parts (cf. p. 470); thus, $d\mathbb{F}_{\mathbf{a}} = \mathcal{S} + \mathcal{A}$, where

$$\mathcal{S} = \begin{pmatrix} 0 & kb & 0 \\ kb & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} 0 & kb & 0 \\ -kb & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Because $\mathcal{S}$ is symmetric, it is a pure strain (cf. Theorem 2.6, p. 40): it has three mutually perpendicular eigenvectors with real eigenvalues, as follows:

$$\lambda_1 = kb, \qquad \lambda_2 = -kb, \qquad \lambda_3 = 0,$$
$$\mathbf{b}_1 = \begin{pmatrix} \varepsilon/\sqrt{2} \\ \varepsilon/\sqrt{2} \\ 0 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} -\varepsilon/\sqrt{2} \\ \varepsilon/\sqrt{2} \\ 0 \end{pmatrix}, \quad \mathbf{b}_3 = \begin{pmatrix} 0 \\ 0 \\ \varepsilon \end{pmatrix}.$$

We take $\varepsilon > 0$, so each eigenvector has length ε. As it happens, $\mathbf{b}_3$ is also an eigenvector for $\mathcal{A}$ (with the same eigenvalue, 0). Furthermore, $\mathcal{A}$ maps the plane determined by the other two eigenvectors to itself, because

$$\mathcal{A}\mathbf{b}_1 = -kb\,\mathbf{b}_2 \quad \text{and} \quad \mathcal{A}\mathbf{b}_2 = kb\,\mathbf{b}_1.$$

Therefore, the map $d(\mathbf{h}_t^*)_{\mathbf{0}} = I + t(\mathcal{S} + \mathcal{A})$ acts in a simple way on the basis $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ for $\mathbb{R}^3$:

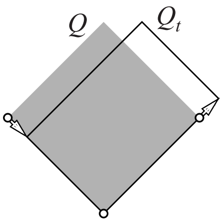
$$\mathbf{b}_1 \to \mathbf{b}_1 + kbt\,\mathbf{b}_1 - kbt\,\mathbf{b}_2,$$
$$\mathbf{b}_2 \to \mathbf{b}_2 - kbt\,\mathbf{b}_2 + kbt\,\mathbf{b}_1,$$
$$\mathbf{b}_3 \to \mathbf{b}_3.$$

Let us see how the cube $K = \mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \mathbf{b}_3$ "flows" under this transformation.

The action of $I + t\mathcal{S}$

First remove $\mathcal{A}$ in order to isolate the action of the symmetric component $\mathcal{S}$:

$$\mathbf{b}_1 \to \mathbf{b}_1 + kbt\,\mathbf{b}_1, \quad \mathbf{b}_2 \to \mathbf{b}_2 - kbt\,\mathbf{b}_2, \quad \mathbf{b}_3 \to \mathbf{b}_3.$$



The vertical edge $\mathbf{b}_3$ is left unchanged, so we concentrate on what happens to the base square $Q = \mathbf{b}_1 \wedge \mathbf{b}_2$ in the $(\Delta x, \Delta y)$-plane. When $t = 0$, we have the original square Q; as t changes, one edge of Q grows longer and the other grows shorter at the same rate, yielding a rectangle Q_t whose sides are parallel to the square. (The white arrows in the figure are $kbt\,\mathbf{b}_1$ and $-kbt\,\mathbf{b}_2$.) Because we are interested in $t \to 0$, we can assume $|kbt| \ll 1$, so the deformation $Q \to Q_t$ is small. The original cube K becomes a rectangular parallelepiped K_t (a "brick") whose base is the rectangle Q_t. In effect, $I + t\mathcal{S}$ is a *strain* that changes the shape of the cube, and also its volume:

$$\operatorname{vol} K_t = \varepsilon(1+kbt) \times \varepsilon(1-kbt) \times \varepsilon = \varepsilon^3(1-(kbt)^2).$$

To first order, t has no effect on the volume. More precisely, the change in volume is $\Delta \operatorname{vol} K = \operatorname{vol} K_t - \operatorname{vol} K = -(kbt)^2\varepsilon^3$, so the *relative* change (i.e., the change in volume as a fraction of the volume itself) is the function

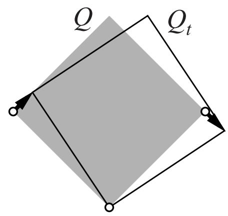
$$V(t) = \frac{\Delta \operatorname{vol} K}{\operatorname{vol} K} = -(kbt)^2, \quad \text{for which} \quad V'(0) = 0.$$

It turns out that $V'(0) = 0$ is a consequence of the particular nature of $\mathcal{S}$ and thus, ultimately, of $\mathbb{F}$. We show that, for a general flow $\mathbb{F}$, the relative growth rate $V'(0)$ of volume is $\operatorname{div} \mathbb{F}$. Note that, in our example, $\operatorname{div} \mathbb{F} = 0$.

the action of $I + t\mathcal{A}$

Now remove $\mathcal{S}$ from $I + t(\mathcal{S} + \mathcal{A})$ in order to isolate the action of the antisymmetric component $\mathcal{A}$:

$$\mathbf{b}_1 \to \mathbf{b}_1 - kbt\,\mathbf{b}_2, \quad \mathbf{b}_2 \to \mathbf{b}_2 + kbt\,\mathbf{b}_1, \quad \mathbf{b}_3 \to \mathbf{b}_3.$$



Again it is sufficient to see what happens to the base square Q in the $(\Delta x, \Delta y)$-plane. This time the small changes (the black arrows in the figure) are perpendicular to the sides; the effect is to turn the square, rather than to strain it. Nevertheless, Q_t is slightly larger than Q when $t \neq 0$, but t again has only a second-order effect on the volume of K_t:

$$\operatorname{vol} K_t = \varepsilon\sqrt{1+(kbt)^2} \times \varepsilon\sqrt{1+(kbt)^2} \times \varepsilon = \varepsilon^3 + O(t^2) \text{ as } t \to 0.$$

We show that, in the general case, $I + t\mathcal{A}$ will continue to have only a second-order effect on volume. To first order, $I + t\mathcal{A}$ is a uniform rotation.

Lemma 11.5. *To first order in t, the flow*

$$I + t\mathcal{A} = \begin{pmatrix} 1 & kbt & 0 \\ -kbt & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a uniform rotation with angular velocity $\boldsymbol{\omega} = (0,0,-kb)$, that is, a uniform rotation around the positive Δz-axis with angular speed $-kb$.

Proof. In (x,y,z)-space, uniform rotation with angular speed ω around the positive z-axis is given by the matrix function

$$R_{\omega t} = \begin{pmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The Taylor approximations

$$\cos \omega t = 1 + O(t^2) \text{ and } \sin \omega t = \omega t + O(t^3) \text{ as } t \to 0$$

imply

$$R_{\omega t} = \begin{pmatrix} 1 & -\omega t & 0 \\ \omega t & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \boldsymbol{O}(t^2) \text{ as } t \to 0. \qquad \square$$

Combining the rotation and the strain

To first order in t, $t\mathcal{S}$ induces no rotation and $t\mathcal{A}$ induces no strain. Thus, to first order in t and in a sufficiently small window W^*, the relative flow

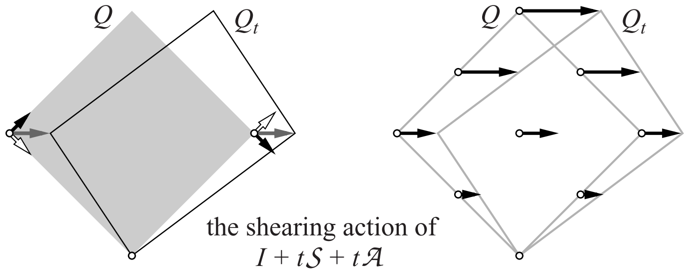
$$\mathbf{h}_t^* \approx I + t\mathcal{S} + t\mathcal{A}$$

rotates the cube K around the positive Δz-axis with angular speed $-kb$ while altering the lengths of its sides at the rates kb, $-kb$, and 0.

The curl describes local rotation

For the flow $\mathbb{F} = (ky^2, 0, 0)$ of our example, $\operatorname{curl}\mathbb{F}(\mathbf{a}) = (0, 0, -2kb) = 2\boldsymbol{\omega}$. We originally interpreted the curl as describing the tendency of a flow to spin a small object carried along with it. Example 3 suggests a new interpretation, in which the curl directly describes the local rotation of a small blob of the fluid itself under the action of the relative flow $\mathbf{h}_t^*$.

With the decomposition $d\mathbb{F}_{\mathbf{a}} = \mathcal{S} + \mathcal{A}$, we were able to focus separately on the two distinct aspects of the relative flow: strain and rotation. In fact, these aspects act together, and together they should produce the shear flow in W^* that we observed at the outset. The following figure confirms this. Each gray arrow, as the sum of a white and a black arrow, expresses the action of $I + t\mathcal{S} + t\mathcal{A}$.



the shearing action of
$I + t\mathcal{S} + t\mathcal{A}$

The relative flow for an arbitrary $\mathbb{F}$

Using Example 3 as a guide, we now take up the general case of an arbitrary continuously differentiable velocity field $\mathbb{F} = (A, B, C)$ defined on an open set Ω in $\mathbb{R}^3$. The corresponding flow $\mathbf{h}_t : \Omega \to \mathbb{R}^3$ is defined by $\mathbf{h}_t(\mathbf{x}) = \mathbf{x}(t)$, where $\mathbf{x}(t)$ is the unique solution of the initial-value problem

$$\mathbf{x}'(t) = \mathbb{F}(\mathbf{x}(t)), \quad \mathbf{x}(0) = \mathbf{x}.$$

To describe the relative flow of fluid particles near the particle that is initially at the point $\mathbf{a}$, let W be a small cube centered at $\mathbf{a}$, and let $\mathbf{x}$ be an arbitrary point in W. Then $\Delta\mathbf{x} = \mathbf{x} - \mathbf{a}$ gives the position of $\mathbf{x}$ in relation to $\mathbf{a}$, and the vector

$$\Delta\mathbf{x}(t) = \mathbf{h}_t(\mathbf{x}) - \mathbf{h}_t(\mathbf{a}) = \mathbf{h}_t(\mathbf{a}+\Delta\mathbf{x}) - \mathbf{h}_t(\mathbf{a}) = \mathbf{h}_t^*(\Delta\mathbf{x})$$

describes how that relative position varies over time. To get a formula that ties $\mathbf{h}_t^*$ back to $\mathbb{F}$, we first use Taylor's theorem to write

$$\mathbf{h}_t(\mathbf{x}) = \mathbf{x}(t) = \mathbf{x}(0) + t\,\mathbf{x}'(0) + \boldsymbol{O}(t^2) = \mathbf{x} + t\,\mathbb{F}(\mathbf{x}) + \boldsymbol{O}(t^2) \ \text{ as } t \to 0.$$

Note that $\mathbf{x}'(0) = \mathbb{F}(\mathbf{x}(0)) = \mathbb{F}(\mathbf{x})$ and that $\mathbf{x}(t)$ has the continuous second derivative required by Taylor's theorem, because $\mathbf{x}'(t) = \mathbb{F}(\mathbf{x}(t))$ and $\mathbb{F}$ is continuously differentiable. Because $\mathbf{x} = \mathbf{a} + \Delta\mathbf{x}$, it follows that

$$\begin{aligned} \mathbf{h}_t(\mathbf{a}+\Delta\mathbf{x}) &= \mathbf{a} + \Delta\mathbf{x} + t\,\mathbb{F}(\mathbf{a}+\Delta\mathbf{x}) + \boldsymbol{O}(t^2) \\ \text{and } \ \mathbf{h}_t(\mathbf{a}) &= \mathbf{a} + t\,\mathbb{F}(\mathbf{a}) + \boldsymbol{O}(t^2) \end{aligned}$$

as $t \to 0$, and hence (using the differentiability of $\mathbb{F}$ at $\mathbf{a}$) that

$$\begin{aligned} \mathbf{h}_t^*(\Delta\mathbf{x}) = \mathbf{h}_t(\mathbf{a}+\Delta\mathbf{x}) - \mathbf{h}_t(\mathbf{a}) &= \Delta\mathbf{x} + t\,(\mathbb{F}(\mathbf{a}+\Delta\mathbf{x}) - \mathbb{F}(\mathbf{a})) + \boldsymbol{O}(t^2) \\ &= \Delta\mathbf{x} + t\,(d\mathbb{F}_{\mathbf{a}}(\Delta\mathbf{x}) + \boldsymbol{o}(\Delta\mathbf{x})) + \boldsymbol{O}(t^2) \end{aligned}$$

as $\Delta\mathbf{x} \to \mathbf{0}$ and $t \to 0$.

The relative flow and $d\mathbb{F}_{\mathbf{a}}$

As we did in Example 3, we consider that this formula for the relative flow defines a family of maps

$$\mathbf{h}_t^* : W^* \to \mathbb{R}^3$$

of a window W^* centered at the origin of local coordinates $\Delta\mathbf{x}$. Compare the general formula for $\mathbf{h}_t^*$ that we have just obtained with the earlier one in Example 3 (p. 475); the only new ingredient is the higher-order term $\boldsymbol{O}(t^2)$. The derviative $d\mathbb{F}_{\mathbf{a}}$ is still the key to the relative flow. The velocity field $\mathbb{F}^*$ of the relative flow is

$$\mathbb{F}^*(\Delta\mathbf{x}) = \left.\frac{d}{dt}\mathbf{h}_t^*(\Delta\mathbf{x})\right|_{t=0} = d\mathbb{F}_{\mathbf{a}}(\Delta\mathbf{x}) + \boldsymbol{o}(\Delta\mathbf{x}) \ \text{ as } \Delta\mathbf{x} \to \mathbf{0}.$$

Furthermore, when W^* is small enough for us to ignore higher-order terms in $\Delta\mathbf{x}$, we can approximate $\mathbf{h}_t^*$ by its derivative at the origin $\Delta\mathbf{x} = \mathbf{0}$:

$$\mathbf{h}_t^* \approx d(\mathbf{h}_t^*)_{\mathbf{0}} = I + t\,d\mathbb{F}_{\mathbf{a}} + \boldsymbol{O}(t^2) \ \text{ as } t \to 0.$$

How $d(\mathbf{h}_t^*)_{\mathbf{0}}$ affects volumes

Let us now use the approximation $d(\mathbf{h}_t^*)_{\mathbf{0}}$ to get an idea how $\mathbf{h}_t^*$ affects volumes near $\Delta\mathbf{x} = \mathbf{0}$. We expect that the volume of a small cube of fluid may change as the fluid flows; we want to measure that change. As before, it is helpful to split $d\mathbb{F}_{\mathbf{a}}$ into its symmetric and antisymmetric parts: $d\mathbb{F}_{\mathbf{a}} = \mathcal{S} + \mathcal{A}$. The symmetric matrix $\mathcal{S}$ has mutually orthogonal eigenvectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ of length ε, with eigenvalues $\lambda_1, \lambda_2, \lambda_3$, respectively. Choose an order for the eigenvectors so that the cube $K = \mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \mathbf{b}_3$ has positive volume ε^3. Its image $K_t = (I + t\mathcal{S})(K)$ is the rectangular parallelepiped with

$$\begin{aligned} \operatorname{vol} K_t &= \varepsilon(1+t\lambda_1) \times \varepsilon(1+t\lambda_2) \times \varepsilon(1+t\lambda_3) \\ &= \varepsilon^3(1+t(\lambda_1+\lambda_2+\lambda_3)+O(t^2)) \quad \text{as } t \to 0. \end{aligned}$$

Because the sum of the eigenvalues of a matrix equals the trace of that matrix, and because the diagonal elements of an antisymmetric matrix are all 0, we have

$$\lambda_1+\lambda_2+\lambda_3 = \operatorname{tr}\mathcal{S} = \operatorname{tr}(\mathcal{S}+\mathcal{A}) = \operatorname{tr} d\mathbb{F}_{\mathbf{a}}.$$

Furthermore, because $\mathbb{F} = (A,B,C)$,

$$\operatorname{tr} d\mathbb{F}_{\mathbf{a}} = \frac{\partial A}{\partial x}(\mathbf{a}) + \frac{\partial B}{\partial y}(\mathbf{a}) + \frac{\partial C}{\partial z}(\mathbf{a}) = \operatorname{div} F(\mathbf{a}),$$

finally linking volume change to divergence:

$$\operatorname{vol} K_t = \operatorname{vol} K(1+t\operatorname{div}\mathbb{F}(\mathbf{a})+O(t^2)) \quad \text{as } t \to 0.$$

Thus, the relative (or percentage) change in volume (p. 477) is

$$V(t) = \frac{\operatorname{vol} K_t - \operatorname{vol} K}{\operatorname{vol} K} = t\operatorname{div}\mathbb{F}(\mathbf{a})+O(t^2) \quad \text{as } t \to 0;$$

consequently, $V'(0) = \operatorname{div}\mathbb{F}(\mathbf{a})$. Note that we obtained this result under the assumption we could replace the relative flow by its linear approximation and we could restrict ourselves to cubes whose edges were the eigenvectors of the symmetric part of the linear approximation.

How $\mathbf{h}_t^*$ itself affects volume

We now remove the restrictive assumptions and determine $V'(0)$ using the original nonlinear map $\mathbf{h}_t^*$ itself. First, let K range over closed sets with volume that contain the point $\mathbf{a}$ (so $\Delta\mathbf{x} = \mathbf{0}$). Let $K_t = \mathbf{h}_t^*(K)$; its volume is

$$\operatorname{vol} \mathbf{h}_t^*(K) = \iiint_K J_{\mathbf{h}_t^*}\, dV$$

($J_{\mathbf{h}_t^*}$ is the Jacobian of $\mathbf{h}_t^*$). Because $\mathbf{h}_t^*$ is nonlinear, the ratio

$$\frac{\operatorname{vol} \mathbf{h}_t^*(K) - \operatorname{vol} K}{\operatorname{vol} K}$$

is no longer independent of the diameter δK of K (Definition 8.14, p. 291). However, to determine percentage growth of volume, it is sufficient to see what happens to this ratio as $\delta K \to 0$. Because $\operatorname{vol} \mathbf{h}_t^*(K)$ is a set function of integral type (cf. pp. 310–312), we can calculate its derivative as K shrinks down to the point $\mathbf{a}$; by Theorem 8.39, (p. 312), we find

$$\lim_{\delta K \to 0} \frac{\operatorname{vol} \mathbf{h}_t^*(K)}{\operatorname{vol} K} = J_{\mathbf{h}_t^*}(\mathbf{0}).$$

This allows us to define the percentage change of volume as the limit

$$V(t) = \lim_{\delta K \to 0} \frac{\operatorname{vol} \mathbf{h}_t^*(K) - \operatorname{vol} K}{\operatorname{vol} K} = J_{\mathbf{h}_t^*}(\mathbf{0}) - 1.$$

Because

$$J_{\mathbf{h}_t^*}(\mathbf{0}) = \det d(\mathbf{h}_t^*)_{\mathbf{0}} = 1 + t \operatorname{div} \mathbb{F}(\mathbf{a}) + O(t^2) \quad \text{as } t \to 0,$$

we again have

$$V(t) = t \operatorname{div} \mathbb{F}(\mathbf{a}) + O(t^2) \quad \text{as } t \to 0,$$

and $V'(0) = \operatorname{div} \mathbb{F}(\mathbf{a})$.

Action of $I + t\mathcal{A}$

We now consider the action of the flow $I + t\mathcal{A}$, where $\mathcal{A}$ is the antisymmetric part of $d\mathbb{F}_{\mathbf{a}}$. As we noted on page 470, when $\mathbb{F} = (A, B, C)$, then

$$\mathcal{A} = \frac{1}{2} \begin{pmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{pmatrix}, \quad \text{where } Z = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} C_y - B_z \\ A_z - C_x \\ B_x - A_y \end{pmatrix},$$

and all functions are evaluated at $\mathbf{x} = \mathbf{a}$.

Lemma 11.6. *The vector Z is an eigenvector of $\mathcal{A}$ with eigenvalue 0.* □

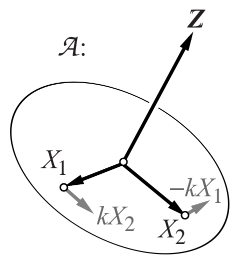
The following theorem is the analogue, for a general flow, of Lemma 11.5 for the flow of Example 3. Note that $Z = \operatorname{curl} \mathbb{F}(\mathbf{a})$.

Theorem 11.9. *To first order in t, the flow $I + t\mathcal{A}$ is a uniform rotation with angular velocity $\frac{1}{2}Z$.*

Proof. By definition, a rotation matrix R is an orthogonal matrix (i.e., one whose transpose equals its inverse: $R^\dagger R = I$) with positive determinant. Let $R_t = I + t\mathcal{A}$. Then $R_t^\dagger = I - t\mathcal{A}$ and $(R_t)^\dagger R_t = I - t^2\mathcal{A}$; thus, to first order in t, R_t is orthogonal. Becaue Z is an eigenvector of $\mathcal{A}$ with eigenvalue 0, $R_t Z = Z$. This implies Z is the rotation axis of each R_t.

To determine the angular speed and show that it is constant, we need more detailed information about $\mathcal{A}$. The vectors

$$X_1 = \frac{1}{\sqrt{\alpha^2 + \beta^2}} \begin{pmatrix} -\beta \\ \alpha \\ 0 \end{pmatrix}, \quad X_2 = \frac{Z \times X_1}{\|Z\|} = \frac{1}{\|Z\| \sqrt{\alpha^2 + \beta^2}} \begin{pmatrix} -\alpha\gamma \\ -\beta\gamma \\ \alpha^2 + \beta^2 \end{pmatrix}$$



will provide this information. They are orthogonal unit vectors that span a plane orthogonal to Z; we claim the rotation leaves that plane invariant. This follows from quick calculations that show

$$\mathcal{A}X_1 = kX_2 \quad \text{and} \quad \mathcal{A}X_2 = -kX_1,$$

where $k = \frac{1}{2}\|Z\|$. In terms of the basis $\{X_1, X_2, Z\}$, the matrix for $\mathcal{A}$ is

$$\begin{pmatrix} 0 & -k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the matrix for the flow $I + t\mathcal{A}$ is

$$\begin{pmatrix} 1 & -kt & 0 \\ kt & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It follows from Lemma 11.5 that, to first order in t, $I + t\mathcal{A}$ is a uniform rotation with angular velocity $k = \frac{1}{2}Z$. □

Theorem 11.10. *Suppose a steady fluid flow is governed by the velocity field $\mathbb{F}$, and W^* is a frame that is translated in space so as to remain centered on the fluid particle initially at the point $\mathbf{a}$. Then a vanishingly small ball of fluid centered at the origin of W^* does the following.*

- *Rotates with instantaneous angular velocity* $\frac{1}{2}\operatorname{curl}\mathbb{F}(\mathbf{a})$
- *Changes its relative volume at the instantaneous rate* $\operatorname{div}\mathbb{F}(\mathbf{a})$ □

11.3 Stokes' theorem

Stokes' theorem is our final setting for the assertion that the boundary operator and the exterior derivative are adjoints in the symbolic integral pairing: $\langle \partial\vec{S}, \omega\rangle = \langle \vec{S}, d\omega\rangle$. In this setting, $\vec{S}$ is a piecewise-smooth oriented surface in (x,y,z)-space, and $\omega = \omega(x,y,z)$ is a differential 1-form. In physical terms, Stokes' theorem equates the circulation of a flow around the boundary of a surface to the flux of the vorticity field (Definition 11.4, p. 473) of that flow through the surface.

Linking Stokes' and Green's theorems

To begin the process of proving Stokes' theorem, we first note how similar it is to Green's theorem. Both assert that

$$\oint_{\partial\vec{S}} \omega = \iint_{\vec{S}} d\omega,$$

where ω is a 1-form and $\vec{S}$ is an oriented 2-dimensional region. The only difference is the dimension of the ambient space: Green's theorem is set in $\mathbb{R}^2$ (forcing $\vec{S}$ to be planar), whereas Stokes' theorem is set in $\mathbb{R}^3$ (allowing $\vec{S}$ to be curved.) But because a surface patch in space is parametrized by a plane region, we are able to use Green's theorem to prove Stokes'.

Ingredients of the proof

The proof follows quickly once the ingredients are assembled. Recall that a *piecewise-smooth oriented surface* (Definitions 10.7, p. 412, and 10.12, p. 420) is a finite sum of oriented surface patches whose common boundary segments have opposite orientations. An *oriented surface patch* $\vec{S}$ (Definition 10.2, p. 392) is the image $\vec{S} = \mathbf{f}(\vec{U})$ of a closed bounded, positively oriented set $\vec{U} \subset \Omega$ with area, where the *parametrization*

$$\mathbf{f} : \Omega \to \mathbb{R}^3$$

is a continuously differentiable 1–1 immersion on the open set $\Omega \subseteq \mathbb{R}^2$. The map $\mathbf{f}$ is an *immersion* at the point $\mathbf{a}$ if the derivative $d\mathbf{f}_{\mathbf{a}} : \mathbb{R}^2 \to \mathbb{R}^3$ is 1–1.

The *surface integral* of a 2-form α defined everywhere on the surface patch $\vec{S}$ is given by the pullback $\mathbf{f}^*$:

$$\iint_{\vec{S}} \alpha = \iint_{\vec{U}} \mathbf{f}^* \alpha$$

(Definition 10.4, as reformulated on p. 432). The value of the surface integral is independent of the parametrization used to represent $\vec{S}$ (Corollary 10.4, p. 401). The surface integral of a 2-form over a piecewise-smooth oriented surface is the sum of the integrals over its smooth oriented pieces (Definition 10.13, p. 420). We also use the facts that the pullback $\mathbf{f}^*$ commutes with the exterior derivative (Theorem 10.16, p. 437), and that $\mathbf{f}$ and $\mathbf{f}^*$ are adjoints on piecewise-smooth curves $\vec{C}$ (Exercise 4.37, p. 149). Because these facts were first established in slightly different circumstances, we reconstruct them here, taking the target of $\mathbf{f}$ to be $\mathbb{R}^3$ instead of $\mathbb{R}^2$.

Lemma 11.7. *For any continuously differentiable map* $\mathbf{f}:\Omega \to \mathbb{R}^3$ *and k-form* $\alpha(x,y,z)$, $\mathbf{f}^*(d\alpha) = d(\mathbf{f}^*\alpha)$.

$\mathbf{f}^*$ and d commute

Proof. All aspects of the proof of Theorem 10.16 for $k \neq 2$ carry over, *mutatis mutandis*. The only difference occurs for $k = 2$, where the 3-form $d\alpha$ may be nonzero. However, the pullbacks $\mathbf{f}^*(d\alpha)$ and $d(\mathbf{f}^*\alpha)$ are 3-forms in two variables, so they are both zero and $\mathbf{f}^*(d\alpha) = d(\mathbf{f}^*\alpha)$ for all k. □

The proof of the second lemma exploits, in addition, the fact that $\mathbf{f}$ is a 1–1 immersion.

Lemma 11.8. *For any piecewise-smooth oriented curve* $\vec{C}$ *and* 1*-form* β *defined everywhere on* $\mathbf{f}(\vec{C})$,

$\mathbf{f}$ and $\mathbf{f}^*$ are adjoints

$$\langle \mathbf{f}(\vec{C}), \beta \rangle = \int_{\mathbf{f}(\vec{C})} \beta = \int_{\vec{C}} \mathbf{f}^*\beta = \langle \vec{C}, \mathbf{f}^*(\beta) \rangle.$$

Proof. Let $\vec{C} = \vec{C}_1 + \cdots + \vec{C}_m$ be a decomposition into smooth oriented curves, each of which is either simple or is a simple closed curve. Because $\mathbf{f}$ is a 1–1 immersion, each $\mathbf{f}(\vec{C}_i)$ is likewise either simple, or a simple closed, smooth oriented curve, providing a decomposition

$$\mathbf{f}(\vec{C}) = \mathbf{f}(\vec{C}_1) + \cdots + \mathbf{f}(\vec{C}_m).$$

Let $\mathbf{u}_i(t) = (u_i(t), v_i(t))$, $a_i \leq t \leq b_i$ parametrize $\vec{C}_i$; then

$$\begin{gathered} \mathbf{x}_i(t) = \mathbf{f}(\mathbf{u}_i(t)), \\ (x_i(t), y_i(t), z_i(t)) = (x(u_i(t), v_i(t)), y(u_i(t), v_i(t)), z(u_i(t), v_i(t))), \end{gathered}$$

parametrizes $\mathbf{f}(\vec{C}_i)$ with the same domain $a_i \leq t \leq b_i$. Suppose, for simplicity, that $\beta = P\,dx$; then

$$\mathbf{f}^*\beta = P(\mathbf{f}(\mathbf{u}))\,\mathbf{f}^*(dx) = P(\mathbf{f}(\mathbf{u}))(x_u\,du + x_v\,dv)$$

and

$$\int_{\vec{C}_i} \mathbf{f}^* \beta = \int_{a_i}^{b_i} P(\mathbf{f}(\mathbf{u}_i(t)))\,(x_u u_i' + x_v v_i')\,dt.$$

On the other hand,

$$\int_{\mathbf{f}(\vec{C}_i)} \beta = \int_{a_i}^{b_i} P(\mathbf{f}(\mathbf{u}_i(t)))\,x_i'\,dt = \int_{a_i}^{b_i} P(\mathbf{f}(\mathbf{u}_i(t)))\,(x_u u_i' + x_v v_i')\,dt = \int_{\vec{C}_i} \mathbf{f}^* \beta.$$

You can treat the 1-forms $\beta = Q\,dy$ and $\beta = R\,dz$ the same way. $\square$

Stokes' theorem for a surface patch

Theorem 11.11. *Let* $\mathbf{f}:\Omega \to \mathbb{R}^3$ *be a continuously differentiable* 1–1 *immersion on an open set* $\Omega \subseteq \mathbb{R}^2$. *Let* $\vec{U} \subset \Omega$ *be a closed, bounded, positively oriented set with area on which Green's theorem holds. If* ω *is a continuously differentiable* 1*-form defined on the oriented surface patch* $\vec{S} = \mathbf{f}(\vec{U})$, *then*

$$\oint_{\partial \vec{S}} \omega = \iint_{\vec{S}} d\omega.$$

Proof. We have the following sequence of equalities:

$$\begin{aligned}
\iint_{\vec{S}} d\omega = \iint_{\mathbf{f}(\vec{U})} d\omega = \iint_{\vec{U}} \mathbf{f}^* d\omega & \qquad \text{definition of surface integral}\\
= \iint_{\vec{U}} d(\mathbf{f}^*\omega) & \qquad d \text{ and } \mathbf{f}^* \text{ commute (Lemma 11.7)}\\
= \oint_{\partial\vec{U}} \mathbf{f}^*\omega & \qquad \text{Green's theorem for } \vec{U}\\
= \oint_{\mathbf{f}(\partial\vec{U})} \omega & \qquad \mathbf{f} \text{ and } \mathbf{f}^* \text{ are adjoints (Lemma 11.8)}\\
= \oint_{\partial\vec{S}} \omega. &
\end{aligned}$$

$\square$

Stokes' theorem

Corollary 11.12 (Stokes' theorem) *Suppose* $\vec{S} = \vec{S}_1 + \cdots + \vec{S}_m$ *is a piecewise-smooth oriented surface, and suppose the theorem holds on each of the surface patches* $\vec{S}_i$, $i = 1,\ldots,m$; *then*

$$\oint_{\partial \vec{S}} \omega = \iint_{\vec{S}} d\omega.$$

Proof. Because the common segments of the various $\partial\vec{S}_i$ have opposite orientation, path integrals over those segments cancel in pairs; only the segments of $\partial\vec{S}_i$ that lie in $\partial\vec{S}$ make a nonzero contribution to the path integral. Therefore,

$$\oint_{\partial\vec{S}} \omega = \sum_{i=1}^{m} \oint_{\partial\vec{S}_i} \omega = \sum_{i=1}^{m} \iint_{\vec{S}_i} d\omega = \iint_{\vec{S}} d\omega.$$

The final equality is just the definition of a surface integral over $\vec{S}$. $\square$

From differential forms to vector fields

If $\omega = A\,dx + B\,dy + C\,dz$, then

$$d\omega = (C_y - B_z)\,dy\,dz + (A_z - C_x)\,dz\,dx + (B_x - A_y)\,dx\,dy,$$

and Stokes' theorem states that

$$\oint_{\partial\vec{S}} A\,dx + B\,dy + C\,dz = \iint_{\vec{S}} (C_y - B_z)\,dy\,dz + (A_z - C_x)\,dz\,dx + (B_x - A_y)\,dx\,dy.$$

Our discussion of the connection between differential forms and scalar and vector fields (pp. 460–462) makes it easy to convert Stokes' theorem into a statement about the integrals of vector fields. If $\omega = A\,dx + B\,dy + C\,dz$, then

$$\omega = \omega^1_{\mathbb{F}} \leftrightarrow \mathbb{F} = (A, B, C)$$

and

$$d\omega = d\big(\omega^1_{\mathbb{F}}\big) = \omega^2_{\operatorname{curl}\mathbb{F}} \leftrightarrow \operatorname{curl}\mathbb{F} = (C_y - B_z, A_z - C_x, B_x - A_y).$$

If $\mathbf{t}$ is the positively oriented unit tangent vector on $\partial\vec{S}$, then (cf. p. 19)

$$\oint_{\partial\vec{S}} A\,dx + B\,dy + C\,dz = \oint_{\partial S} \mathbb{F}\cdot\mathbf{t}\,ds = \text{circulation of } \mathbb{F} \text{ around } \partial\vec{S}.$$

If $\mathbf{n}$ is the unit normal that determines the orientation of $\vec{S}$, then (cf. pp. 403–404)

$$\iint_{\vec{S}} (C_y - B_z)\,dy\,dz + (A_z - C_x)\,dz\,dx + (B_x - A_y)\,dx\,dy$$
$$= \iint_S \operatorname{curl}\mathbb{F}\cdot\mathbf{n}\,dA = \text{total flux of } \operatorname{curl}\mathbb{F} \text{ through } \vec{S}.$$

With these connections, we can restate Stokes' theorem for vector fields.

Theorem 11.13 (Physical form of Stokes' theorem). *If $\mathbb{F}$ is a continuously differentiable flow field defined on a piecewise-smooth oriented surface $\vec{S}$, and* curl$\mathbb{F}$ *is its vorticity field, then*

$$\oint_{\partial S} \mathbb{F}\cdot\mathbf{t}\,ds = \iint_S \operatorname{curl}\mathbb{F}\cdot\mathbf{n}\,dA,$$

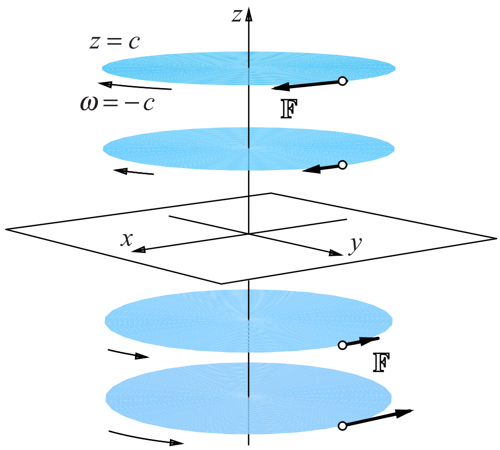
circulation of $\mathbb{F}$ around $\partial\vec{S}$ = total flux of curl$\mathbb{F}$ *through $\vec{S}$.* □

Example 3: $\mathbb{F} = (yz, -xz, 0)$

To illustrate the theorem in its physical form, let us add an extended example to the two we considered in the last section. We take the flow field and its vorticity field (Definition 11.4, p. 473) to be

$$\mathbb{F} = (yz, -xz, 0), \quad \operatorname{curl}\mathbb{F} = (x, y, -2z).$$

This flow is similar to the flow of Example 1, where $\mathbb{F} = (-\omega y, \omega x, 0)$ (pp. 462–463). In that case, the entire fluid rotated rigidly (i.e., without the particles changing their relative positions over time) with constant angular speed ω around the z-axis. The flow in Example 3 is only slightly more complicated: the variable $-z$ simply replaces the constant ω. Thus the fluid at each level $z = c$ rotates rigidly around the z-axis with its own constant angular speed $\omega = -c$.



Shearing actions of the fluid

Imagine the fluid is separated into parallel disks. The disks above the (x,y)-plane rotate clockwise (when viewed from above); those below, counterclockwise. The farther a disk is from the (x,y)-plane, the faster it spins. This difference introduces a new shearing action within the fluid that we did not see in Example 1. There, the only shearing action was caused by the greater speed of particles farther from the z-axis. That is present here as well, and accounts for the component $-2z$ in the vorticity vector $\operatorname{curl}\mathbb{F} = (x, y, -2z)$. The new shearing action, between disks, accounts for the other components.

Vortex lines of $\mathbb{F}$

Let us examine all this in more detail. To connect the circulation of a field $\mathbb{F}$ to the flux of its vorticity field $\operatorname{curl}\mathbb{F}$, we naturally think of the second field, $\operatorname{curl}\mathbb{F}$, as a flow of "particles." The paths of these particles are called the **vortex lines** of $\mathbb{F}$. In the present case, the vortex lines are paths $\mathbf{x}(t) = (x(t), y(t), z(t))$ that satisfy the differential equations (cf. p. 462)

$$\mathbf{x}'(t) = \operatorname{curl}\mathbb{F}(\mathbf{x}(t)), \text{ or } x' = x, \quad y' = y, \quad z' = -2z.$$

The general solution here is the three-parameter family

$$\mathbf{x}(t) = (a\mathrm{e}^t, b\mathrm{e}^t, c\mathrm{e}^{-2t}).$$

This describes the motion of the particle that is initially at the point $\mathbf{x}(0) = (a, b, c)$, which can thus be anywhere in space. In particular, if the initial point lies in the vertical plane $y = mx$ (so that $b = ma$), then the entire path is in the same plane, because

$$y(t) = b\mathrm{e}^t = ma\,\mathrm{e}^t = m\,x(t).$$

In fact, this equation shows that we obtain all paths by rotating the paths that lie in a single vertical plane (e.g., in the plane $y = 0$) around the z-axis. The flow of $\operatorname{curl}\mathbb{F}$ has rotational symmetry around the z-axis.

The vortex lines have a saddle at the origin

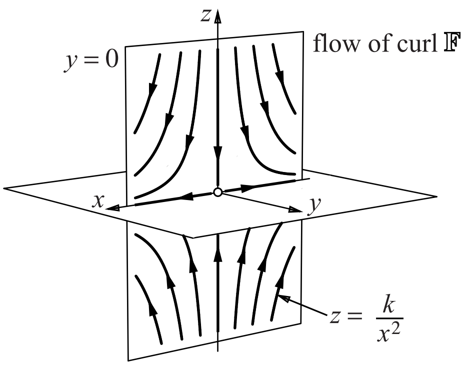
The solutions on the plane $y = 0$ (i.e., where $b = 0$) are

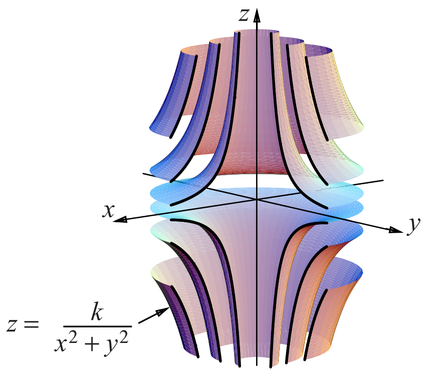
$$x(t) = a\mathrm{e}^t, \quad y(t) = 0, \quad z(t) = c\mathrm{e}^{-2t}.$$

For a given $a \neq 0$ and c, this vortex line lies on the graph of the function

$$z = k/x^2, \quad k = ca^2,$$

in the (x,z)-plane. Particles on these trajectories flow simultaneously away from the z-axis and toward the (x,y)-plane. Particles on the z-axis (where $a = 0$) flow directly toward the origin. Particles in the (x,y)-plane ($c = 0$) flow radially away from the origin on straight lines. The origin is said to be a *saddle point* of the flow.



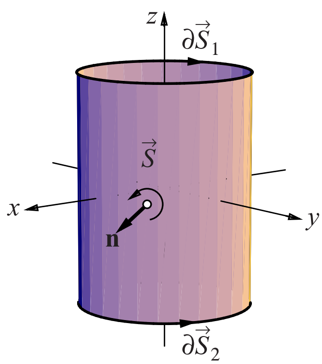


How the flows intersect $z = k/(x^2+y^2)$

When the graph $z = k/x^2$ is rotated around the z-axis, it sweeps out the horn-shaped surface that is the graph of $z = k/(x^2+y^2)$. (In the figure on the right, above, the portion of each surface that lies in the first quadrant has been cut away for better visibility.) The horn-shaped surfaces make it is easy to visualize the two flows and the way they are related: the flow lines of curl $\mathbb{F}$ (the vortex lines) are intersections of those surfaces with the vertical planes $y = mx$ that are "hinged" on the z-axis; the flow lines of $\mathbb{F}$ itself are their intersections with the horizontal planes $z = c$.

The surface $\vec{S}$

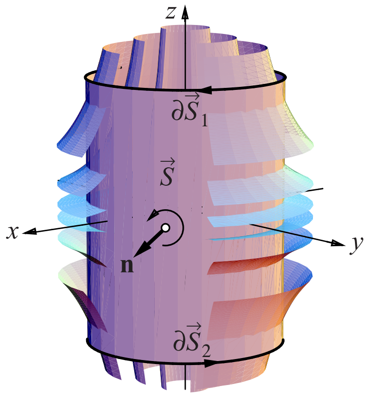
To examine the link between the two fields, we need an oriented surface $\vec{S}$. We take $\vec{S}$ to be a cylinder centered at the origin; let its axis be the z-axis, and let its orientation normal $\mathbf{n}$ be outward-pointing. Let the radius be R and the height $2H$. The boundary of $\vec{S}$ is a pair of circles. The orientation induced by $\vec{S}$ on the upper one, $\partial\vec{S}_1$, is clockwise when viewed from above; on the lower one, $\partial\vec{S}_2$, it is counterclockwise. We want to compare the circulation of $\mathbb{F}$ around $\partial\vec{S} = \partial\vec{S}_1 + \partial\vec{S}_2$ with the total flux of curl $\mathbb{F}$ through $\vec{S}$.



Circulation of $\mathbb{F}$ around $\partial\vec{S}$

The flow $\mathbb{F}$ is everywhere tangent to $\partial\vec{S}$, so the circulation is quite simple to calculate. The upper circle, $\partial\vec{S}_1$, is in the horizontal plane $z = H$, wherein the fluid governed by $\mathbb{F}$ rotates with constant angular speed $\omega = H$ in the clockwise direction, the direction of $\partial\vec{S}_1$. Because $\partial\vec{S}_1$ is a circle of radius R, the fluid on it moves with speed HR. The circulation of $\mathbb{F}$ on $\partial\vec{S}_1$ is therefore just the product of this speed and the length of $\partial\vec{S}_1$, namely the positive quantity $2\pi R^2 H$. Taking orientations properly into account, we see that the circulation around the bottom of the cylinder, $\partial\vec{S}_2$, is the same; thus,

$$\text{circulation of } \mathbb{F} \text{ around } \partial\vec{S} = 4\pi R^2 H.$$



Total flux of curl $\mathbb{F}$ through $\vec{S}$

Now consider the vortex field curl $\mathbb{F}$ on $\vec{S}$. Although curl $\mathbb{F}$ is neither constant in magnitude nor perpendicular to $\vec{S}$, we now show that its projection onto the orienting normal $\mathbf{n}$ is constant, so total flux Φ is also simple to calculate. First, write the coordinates of a point on $\vec{S}$ in the form

$$(x,y,z) = (R\cos\theta, R\sin\theta, z);$$

(R,θ,z) are the *cylindrical coordinates* of the point. At this point, the vectors $\mathbf{n}$ and curl $\mathbb{F}$ have the form

$$\mathbf{n} = (\cos\theta, \sin\theta, 0) \quad \text{and} \quad \operatorname{curl}\mathbb{F} = (R\cos\theta, R\sin\theta, -2z),$$

from which it follows that

$$\operatorname{curl}\mathbb{F}\cdot\mathbf{n} = R$$

everywhere on $\vec{S}$. In principle (Definition 10.1, p. 389), Φ is the product of this projection length and the area of $\vec{S}$. When the projection length varies, the product needs to be rendered as a surface integral, but that is unnecessary here. Thus we have

$$\Phi = \operatorname{curl}\mathbb{F}\cdot\mathbf{n}\ \text{area}\,\vec{S} = R \times (2\pi R \times 2H) = 4\pi R^2 H;$$

hence

$$\text{circulation of } \mathbb{F} \text{ around } \partial\vec{S} = \text{total flux of curl}\,\mathbb{F} \text{ through } \vec{S},$$

as we wished to show.

We now confirm this relation between $\mathbb{F}$ and $\operatorname{curl}\mathbb{F}$ on a second surface. Let $\vec{\Sigma}_1$ be the flat disk whose boundary is $\partial\vec{S}_1$. To make $\partial\vec{\Sigma}_1 = \partial\vec{S}_1$, that is, to make the orientations match, the orienting normal for $\vec{\Sigma}_1$ must point downward: $\mathbf{n}_1 = (0,0,-1)$. Let $\vec{\Sigma}_2$ be the disk whose boundary is $\partial\vec{S}_2$; here the orienting normal is $\mathbf{n}_2 = (0,0,+1)$. Finally, let $\vec{\Sigma} = \vec{\Sigma}_1 + \vec{\Sigma}_2$. Then $\partial\vec{\Sigma} = \partial\vec{S}$, so

$$\text{circulation of } \mathbb{F} \text{ around } \partial\vec{\Sigma} = 4\pi R^2 H,$$

as before. The total flux of $\operatorname{curl}\mathbb{F}$ through $\vec{\Sigma}$ is once again a simple calculation. On $\vec{\Sigma}_1$, a point has coordinates (x,y,H), so

$$\operatorname{curl}\mathbb{F}\cdot\mathbf{n}_1 = (x,y,-2H)\cdot(0,0,-1) = +2H$$

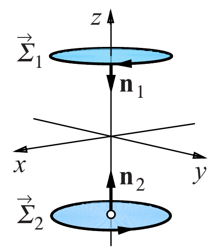
there. Because this is a constant, the flux Φ_1 through $\vec{\Sigma}_1$ is just

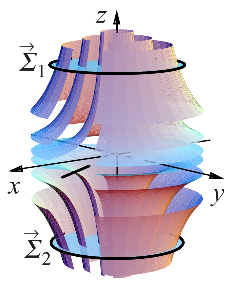
$$\Phi_1 = \operatorname{curl}\mathbb{F}\cdot\mathbf{n}_1 \ \text{area}\,\vec{\Sigma}_1 = 2H \times \pi R^2 = 2\pi R^2 H.$$

On $\vec{\Sigma}_2$, $\operatorname{curl}\mathbb{F}\cdot\mathbf{n}_2 = (x,y,2H)\cdot(0,0,1) = 2H$ once again, and the flux is $\Phi_2 = 2\pi R^2 H$. Hence,

$$\text{circulation of } \mathbb{F} \text{ around } \partial\vec{\Sigma} = \text{total flux of } \operatorname{curl}\mathbb{F} \text{ through } \vec{\Sigma}.$$

The surface Σ

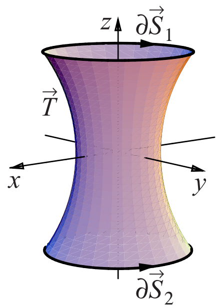




With $\vec{S}$ and $\vec{\Sigma}$, we were able to determine total flux without calculating an integral. Here is a third surface, $\vec{T}$, for which the integral is necessary. We define $\vec{T}$ by a parametrization $\mathbf{f}: \vec{U} \to \vec{T}$ (with α to be determined):

$$\mathbf{f}: \begin{cases} x = \alpha\cos u\cosh v, \\ y = \alpha\sin u\cosh v, \\ z = v, \end{cases} \qquad \vec{U}: \begin{array}{l} -\pi \le u \le \pi, \\ -H \le v \le H. \end{array}$$

The surface $\vec{T}$



This is a surface of revolution around the z-axis; if $\alpha = 1$, it is called a *catenoid*. However, we want to choose α so that $\partial\vec{T} = \partial\vec{S}_1 + \partial\vec{S}_2$. Thus, in the first quadrant in the (x,z)-plane (where $\cos u = 1$), we want $x = R$ when $z = H$. Consequently $R = \alpha\cosh H$, so $\alpha = R/\cosh H$. Note: $\mathbf{f}$ is not 1–1 on $\vec{U}$, so $\vec{T}$ is not a surface patch, strictly speaking. We should break up $\vec{T}$ into two separate pieces. We can accomplish that by breaking up $\vec{U}$ into two pieces (e.g., with $-\pi \le u \le 0$ and $0 \le u \le \pi$) and using the same formula $\mathbf{f}$ for each. But nothing essential is lost by treating these two pieces together, as we do; see a similar comment (p. 396) about parametrizing a sphere.

To determine the total flux of $\operatorname{curl}\mathbb{F}$ through $\vec{T}$, we must pull back

$$d\omega = x\,dy\,dz + y\,dz\,dx - 2z\,dx\,dy$$

to $\vec{U}$ using $\mathbf{f}^*$. We have

$$\mathbf{f}^*(dy\,dz) = \alpha\cos u\cosh v\,du\,dv, \qquad \mathbf{f}^*(dz\,dx) = \alpha\sin u\cosh v\,du\,dv,$$
$$\mathbf{f}^*(dx\,dy) = -\alpha^2\sinh v\cosh v\,du\,dv,$$

from which it follows that

$$\mathbf{f}^*(d\omega) = \alpha^2(\cosh^2 v + 2v\sinh v\cosh v)\,du\,dv.$$

Therefore, the total flux is

$$\iint_{\vec{T}} d\omega = \iint_{\vec{U}} \mathbf{f}^*(d\omega) = \alpha^2\int_{-\pi}^{\pi} du \int_{-H}^{H}(\cosh^2 v + 2v\sinh v\cosh v)\,dv$$
$$= \alpha^2 \times 2\pi \times 2H\cosh^2 H.$$

(Note that $\cosh^2 v + 2v\sinh v\cosh v = (v\cosh^2 v)'$.) Because $\alpha^2 = R^2/\cosh^2 H$, we find

$$\text{total flux of } \operatorname{curl}\mathbb{F} \text{ through } \vec{T} = 4\pi R^2 H,$$

agreeing with the values for $\vec{S}$ and $\vec{\Sigma}$.

Oriented surfaces with the same boundary

Of course it is not an accident that the total flux of $\operatorname{curl}\mathbb{F}$ through one of these surfaces has the same value as through any other. It is a simple consequence of Stokes' theorem and the fact that the three surfaces have the same boundary, including orientation.

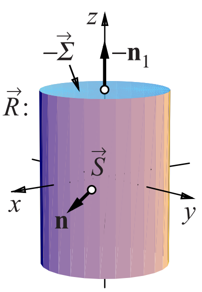
Theorem 11.14. *Suppose Stokes' theorem holds for the piecewise-smooth oriented surfaces $\vec{S}$ and $\vec{\Sigma}$, and $\partial\vec{S} = \partial\vec{\Sigma}$. If ω is any 1-form defined on a region containing $\vec{S}$ and $\vec{\Sigma}$, then*

$$\iint_{\vec{S}} d\omega = \iint_{\vec{\Sigma}} d\omega.$$

Proof. The proof involves two applications of Stokes' theorem:

$$\iint_{\vec{S}} d\omega = \oint_{\partial\vec{S}} \omega = \oint_{\partial\vec{\Sigma}} \omega = \iint_{\vec{\Sigma}} d\omega. \qquad \square$$

Implications of the divergence theorem



The divergence theorem gives us another way to show that the total flux of $\operatorname{curl}\mathbb{F}$ through any two of the surfaces in our Example 3 must be equal. The key is that any two of the surfaces, properly reoriented, form the total boundary of a 3-dimensional region. For example, $\vec{S}$ and $-\vec{\Sigma}$ make up the boundary of the positively oriented cylindrical region

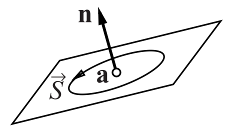
$$\vec{R}: \begin{array}{l} x^2+y^2 \le R^2, \\ -H \le z \le H. \end{array}$$

Now apply the divergence theorem to the region $\vec{R}$ and the 2-form $d\omega$ on $\partial\vec{R}$; because $d(d\omega) = 0$ on $\vec{R}$ (because $d^2 = 0$ always), we find

$$0 = \iiint_{\vec{R}} d(d\omega) = \iint_{\partial\vec{R}} d\omega = \iint_{\vec{S}-\vec{\Sigma}} d\omega = \iint_{\vec{S}} d\omega - \iint_{\vec{\Sigma}} d\omega.$$

Of course, this argument using the divergence theorem works equally well for any pair of piecewise-smooth oriented surfaces that have a common boundary and that together form the complete boundary (when properly reoriented) of a 3-dimensional region.

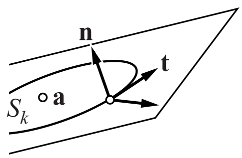
In Theorem 11.8 and also in the discussion on pages 471–473, we established that the fundamental link between circulation and curl has the form



$$\lim_{\delta(\vec{S})\to 0} \frac{\text{circulation of } \mathbb{F} \text{ around } \partial\vec{S}}{\text{area of } \vec{S}} = \operatorname{curl}\mathbb{F}(\mathbf{a})\cdot\mathbf{n},$$

where $\vec{S}$ is centered at $\mathbf{a}$ and lies in the plane with normal $\mathbf{n}$ passing through $\mathbf{a}$; $\delta(\vec{S})$ is the *diameter* of $\vec{S}$ (Definition 8.14, p. 291). However, to carry out the computations, we needed $\vec{S}$ to be either an ellipse or a rectangle. Stokes' theorem is a more powerful computational tool; with it, we can now remove this restriction.

Theorem 11.15. *Suppose the flow field* $\mathbb{F}$ *is defined on an open set* $\Omega \subseteq \mathbb{R}^3$. *Let* $S_k \subset \Omega$ *be a sequence of closed surfaces with area that pass through a common point* $\mathbf{a}$ *and lie in a common plane with unit normal* $\mathbf{n}$. *Suppose that the diameter* $\delta(S_k) \to 0$ *as* $k \to \infty$ *and the boundary of each* S_k *is a piecewise-smooth simple closed curve. Then*



$$\lim_{k\to\infty} \frac{\textit{circulation of } \mathbb{F} \textit{ around } \partial S_k}{\textit{area of } S_k} = \operatorname{curl}\mathbb{F}(\mathbf{a})\cdot\mathbf{n},$$

where circulation is computed in the direction $\mathbf{t}$ *along* ∂S_k *for which the ordered triple of vectors* $\{$*outward normal to* S_k *in the plane*$, \mathbf{t}, \mathbf{n}\}$ *has the same orientation as the coordinate axes.*

Proof. By Stokes' theorem, the circulation of $\mathbb{F}$ around ∂S_k is

$$\oint_{\partial S_k} \mathbb{F}\cdot\mathbf{t}\,ds = \iint_{S_k} \operatorname{curl}\mathbb{F}\cdot\mathbf{n}\,dA.$$

By an adaptation of the law of the mean for double integrals (Theorem 3.7, p. 76),

$$\iint_{S_k} \operatorname{curl}\mathbb{F}\cdot\mathbf{n}\,dA = \operatorname{curl}\mathbb{F}(\mathbf{a}_k)\cdot\mathbf{n}\ \operatorname{area} S_k,$$

where $\mathbf{a}_k$ is a point in S_k; note that $\mathbf{n}$ is constant in the integral. Now let $k \to \infty$; then $\delta(S_k) \to 0$, so $\mathbf{a}_k \to \mathbf{a}$. □

The theorem calls our attention to the quantity

Circulation per unit area

$$q(\mathbf{a},\mathbf{n}) = \lim_{k\to\infty} \frac{\text{circulation of } \mathbb{F} \text{ around } \partial S_k}{\text{area of } S_k}$$

that depends on the point $\mathbf{a}$ and the unit normal $\mathbf{n}$, but not on the particular sets S_k used to define it; let us call it the **circulation of** $\mathbb{F}$ **per unit area** at the point $\mathbf{a}$ in the direction $\mathbf{n}$. Now let

$$q_i(\mathbf{a}) = q(\mathbf{a}, \mathbf{e}_i), \quad i = 1, 2, 3,$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the standard basis in $\mathbb{R}^3$.

Corollary 11.16 $\operatorname{curl} \mathbb{F}(\mathbf{a}) = (q_1(\mathbf{a}), q_2(\mathbf{a}), q_3(\mathbf{a}))$. □

This corollary serves, as Lemmas 11.2–11.4 did, to give us insight into the way the vorticity vector $\operatorname{curl} \mathbb{F}$ is linked to $\mathbb{F}$ itself. It provides us with an alternative to the physical imagery of a little ball set spinning by the shearing action of a fluid flow represented by $\mathbb{F}$.

11.4 Closed and exact forms

Using differential forms

In this section, we use differential forms to define and to solve ordinary and partial differential equations. When the differential forms have certain special properties (e.g., when they are *closed* or *exact*), the solutions can be particularly simple and elegant. In a different vein, we also show how those special forms give us information about the geometry of the domains on which they are defined.

Solving a differential equation by integrating

Analytic methods for solving differential equations frequently involve integration, or *quadrature*, as it is traditionally called in this context. For example, the solution of the basic equation $dy/dx = f(x)$ is a quadrature:

$$y = \int f(x)\, dx.$$

The integral represents the infinite collection of functions $F(x) + c$, where F is a specific antiderivative of f (i.e., $F'(x) = f(x)$), and c is an arbitrary constant. We say the solutions form a one-parameter family; c is the parameter.

For the more complicated equation $dy/dx = f(x)/g(y)$, a solution is a function $y = \varphi(x)$ for which

$$\varphi'(x) = \frac{f(x)}{g(\varphi(x))} \quad \text{for all } x \text{ in some nonempty interval.}$$

To find φ, first rewrite the differential equation as an "equation with differentials" in such a way that the two variables are separated:

$$g(y)\, dy - f(x)\, dx = 0.$$

This implies $G(y) - F(x) = c$, where c is an arbitrary constant, and G and F are specific antiderivatives of g and f, respectively. For each c that we specify, the equation $G(y) - F(x) = c$ is a relation between x and y that defines y implicitly as a function of x (cf. Chapter 6.1) if there is a "seed point" $(x, y) = (a, b)$ for which

$$G(b) - F(a) = c \quad \text{and} \quad G'(b) \neq 0.$$

The implicit function $y = \varphi_c(x)$ that is supplied by Theorem 6.1, page 189 (and by the specified c), has $\varphi_c(a) = b$ and satisfies the conditions

$$G(\varphi_c(x)) - F(x) = c \quad \text{and} \quad \varphi_c'(x) = -\frac{-F'(x)}{G'(\varphi_c(x))} = \frac{f(x)}{g(\varphi_c(x))}$$

at all points x on some open interval including $x = a$. Thus $\varphi_c(x)$ is indeed a solution to the differential equation. As in the simpler case, c serves to parametrize an infinite family of such solutions.

Primitives and first integrals

The function $G(y) - F(x)$ is called a **primitive**, or **first integral**, of the differential equation. The first name is suggested by the fact that solutions emerge from it (as implicitly defined functions). The second name is suggested by the fact that we can write it as an integral:

$$G(y) - F(x) = \int g(y)\,dy - \int f(x)\,dx.$$

Simply put, we solve the differential equation by integrating it to obtain a first integral/primitive that defines solutions implicitly.

Exact differential equations

There is a larger class of differential equations, called *exact*, that can be integrated the same way. An **exact differential equation** has the form

$$\Phi_x(x,y)\,dx + \Phi_y(x,y)\,dy = 0;$$

it has this name because the left-hand side is exactly equal to the differential (i.e., the exterior derivative) of the function $\Phi(x,y)$: $d\Phi = \Phi_x\,dx + \Phi_y\,dy$. A solution is a function $y = \varphi(x)$ for which

$$\Phi_x(x,\varphi(x)) + \Phi_y(x,\varphi(x))\,\varphi'(x) = 0$$

for all x in some nonempty interval. (In other words, the given differential equation is satisfied when we substitute $\varphi(x)$ for y and $\varphi'(x)\,dx$ for dy.) Because we can write the differential equation as $d\Phi = 0$, we have

$$\Phi(x,y) = \int d\Phi = c,$$

implying that Φ is a *first integral* for the differential equation. In other words, if we fix c and find a “seed point” $(x,y) = (a,b)$ for which

$$\Phi(a,b) = c \quad \text{and} \quad \Phi_y(a,b) \neq 0,$$

then the implicit function theorem provides a function $y = \varphi_c(x)$ with $\varphi_c(a) = b$ and for which

$$\Phi(x,\varphi_c(x)) = c \quad \text{and} \quad \varphi_c'(x) = \frac{-\Phi_x(x,\varphi_c(x))}{\Phi_y(x,\varphi_c(x))}.$$

for all x on some open interval containing $x = a$. Thus φ_c is a solution to the differential equation.

Examples

The differential equation $y\,dx + x\,dy = 0$ is exact. One first integral is $\Phi = xy$, and the solutions are the functions $\varphi_c(x) = c/x$. In other cases, it may be more difficult to see whether the differential equation is exact. For example,

$$\frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = 0, \quad (x,y) \neq (0,0),$$

is exact; its first integral is

$$\theta(x,y) = \arctan\left(\frac{y}{x}\right),$$

as can be verified immediately (cf. pp. 429–431). The solution implicitly defined by the equation $\theta = c$ is the linear function $\varphi_c(x) = (\tan c)x$, $x \neq 0$. These solutions form a one-parameter family whose graphs are the straight lines that radiate from the origin. We cannot expect the solutions to be defined at the origin because the differential equation itself is not defined there. The example has an even more remarkable feature: the first integral $\theta(x,y)$ must be multiple-valued if it is to avoid discontinuities on the punctured plane. See the graph of $z = \theta(x,y)$ on page 430.

Each differential equation we have been considering can be written in the form $\omega = 0$ when ω is the general 1-form $P(x,y)\,dx + Q(x,y)\,dy$. In terms of ω, an *exact* differential equation is one for which $\omega = d\Phi$ for some 0-form Φ. In this case, we now say ω itself is exact, and then extend this definition to general k-forms.

Exact forms

Definition 11.5 *A differential k-form ω in n variables is said to be* ***exact*** *if there is a $(k-1)$-form α for which $\omega = d\alpha$.*

When $\omega = d\alpha$ is exact, then $d\omega = d^2\alpha = 0$ (because $d^2 = 0$ by Theorem 10.17, p. 439). Recall that the exterior derivative "d" and the boundary operator "∂" are paired as *adjoints* (cf. p. 428). We use the term *closed* for a curve or surface S that has zero boundary, $\partial S = \emptyset$; therefore the pairing suggests the same term, *closed*, for a differential form ω that has zero exterior derivative, $d\omega = 0$.

Closed forms

Definition 11.6 *We say the k-form ω is* ***closed*** *if $d\omega = 0$.*

These definitions lead to the following conclusion.

Corollary 11.17 *Every exact form is closed.* □

An integrability condition

The corollary gives us a necessary condition for a differential equation of the more general form

$$\omega = P(x,y)\,dx + Q(x,y)\,dy = 0$$

to be exact: we must have $Q_x = P_y$ (because $d\omega = (Q_x - P_y)\,dx\,dy$). This also follows from the equality of mixed partial derivatives, for if ω is an exact 1-form, with $\omega = d\Phi = \Phi_x\,dx + \Phi_y\,dy$, then $P = \Phi_x$, $Q = \Phi_y$, and

$$Q_x = (\Phi_y)_x = (\Phi_x)_y = P_y.$$

Because an exact differential equation is "integrable" (i.e., solvable by integrations), we think of $Q_x = P_y$ as an **integrability condition**, in this case, a *necessary* condition. As we show below, $Q_x = P_y$ is also a *sufficient* condition for the integrability of $\omega = 0$, at least locally.

Integrating factors

The integrability condition shows that, for example, $x\,dy - y\,dx = 0$ cannot be exact ($Q_x - P_y = 2$). Nevertheless, when we rewrite this differential equation as

$$\frac{x\,dy - y\,dx}{x^2 + y^2} = 0,$$

it becomes exact. It has the first integral $\arctan(y/x)$, as noted above. Because the added factor $1/(x^2 + y^2)$ makes the differential equation integrable, we call it an **integrating factor**. Another integrating factor is $1/x^2$, because

$$\frac{x\,dy - y\,dx}{x^2} = \frac{-y}{x^2}\,dx + \frac{1}{x}\,dy = d\left(\frac{y}{x}\right).$$

In this case the first integral is y/x, not $\arctan(y/x)$, but the solution graphs are unchanged; they are the same straight lines that radiate from the origin. With yet another integrating factor, $1/xy$, the differential equation even becomes separable:

$$\frac{x\,dy - y\,dx}{xy} = \frac{dy}{y} - \frac{dx}{x} = d(\ln|y| - \ln|x|).$$

Here the first integral is $\ln|y/x|$; it leads once again to the same solution graphs. Of course most differential equations fail to be exact and fail to have integrating factors that make them exact. We leave further discussion of the art of finding integrating factors to texts on differential equations.

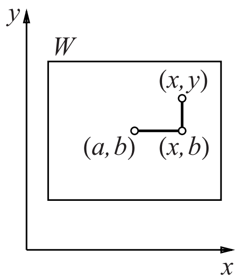
Is a closed form exact?

Every exact form is closed; is every closed form exact? The closed form

$$\omega = \frac{-y\,dx + x\,dy}{x^2 + y^2}$$

reveals a difficulty. The form is defined everywhere in the *punctured plane* $\mathcal{P} = \mathbb{R}^2 \setminus (0,0)$, but there is no continuously differentiable, single-valued function $\Phi(x,y)$ on $\mathcal{P}$ for which $d\Phi = \omega$ (see Exercise 11.12). As pointed out above (and on pp. 429–431), the angle function $\theta(x,y) = \arctan(y/x)$ does have $d\theta = \omega$ everywhere on $\mathcal{P}$, but it is multiple-valued. There is no way to assign a unique angle to every point in $\mathcal{P}$ without having discontinuities. (There is a similar difficulty with the earth's time zones: time increases steadily in the eastward direction, until the International Date Line, where it drops back 24 hours.) The obstruction to continuitity disappears if we restrict ω to a domain that has no closed path encircling the origin. For example, we can use a disk or a rectangle that excludes the origin. More generally, we have the following result for closed 1-forms in two variables.

Theorem 11.18. *Suppose the* 1*-form* $\omega = P(x,y)\,dx + Q(x,y)\,dy$ *is defined and* closed *on a rectangular window W centered at a point (a,b) in $\mathbb{R}^2$. Then $\omega(x,y) = d\Phi(x,y)$ for every (x,y) in W, where*



$$\Phi(x,y) = \int_a^x P(t,b)\,dt + \int_b^y Q(x,t)\,dt.$$

Proof. It suffices to show that $\Phi_x = P$, $\Phi_y = Q$ in W. Because y does not appear in the first integral, and appears only in the upper limit of integration in the second, the equality $\Phi_y = Q$ is immediate. To verify the other equality, we use the integrability condition $Q_x = P_y$ to write

$$\begin{aligned}\Phi_x(x,y) &= P(x,b) + \int_b^y Q_x(x,t)\,dt = P(x,b) + \int_b^y P_y(x,t)\,dt \\ &= P(x,b) + P(x,t)\Big|_b^y = P(x,b) + P(x,y) - P(x,b) = P(x,y).\end{aligned}$$ □

Local exactness

Our goal is to prove this result in full generality: to show that a k-form in n variables that is closed inside a rectangular parallelepiped (a "window") is exact there. We say that such a closed form is **locally exact**.

From *partial* to *ordinary* differential equations

Consider how Theorem 11.18 accomplished the goal. The function $\Phi(x,y)$ for which

$$d\Phi = \Phi_x\,dx + \Phi_y\,dy = P\,dx + Q\,dy = \omega,$$

is a solution to the pair of partial differential equations

$$\Phi_x = P, \quad \Phi_y = Q,$$

where $P(x,y)$ and $Q(x,y)$ are given functions that satisfy the integrability condition $Q_x = P_y$. But the theorem presents $\Phi(x,y)$ as $F_b(x) + G_x(y)$, where F_b and G_x are the particular solutions of the ordinary differential equations

$$F_b'(t) = P(t,b), \quad G_x'(t) = Q(x,t),$$

that satisfy the initial conditions

$$F_b(a) = 0, \quad G_x(b) = 0.$$

In the first function, b is a parameter; in the second, x is.

In general, showing that a closed k-form in n variables is locally exact reduces to solving a set of partial differential equations in the presence of certain integrability conditions. For example, take $k = 2$, $n = 3$, and suppose

$$\omega = P(x,y,z)\,dy\,dz + Q(x,y,z)\,dz\,dx + R(x,y,z)\,dx\,dy$$

is closed; this implies $P_x + Q_y + R_z = 0$. If ω is to be exact, we need a 1-form

$$\alpha = A(x,y,z)\,dx + B(x,y,z)\,dy + C(x,y,z)\,dz$$

with $d\alpha = \omega$; this implies

$$C_y - B_z = P, \quad A_z - C_x = Q, \quad B_x - A_y = R.$$

These are three partial differential equations for the three unknown functions A, B, and C, together with one integrability condition $P_x + Q_y + R_z = 0$ imposed on the known functions P, Q, and R. More generally, local exactness of a closed k-form in n variables involves $\binom{n}{k}$ partial differential equations for $\binom{n}{k-1}$ unknown functions together with $\binom{n}{k+1}$ integrability conditions (see Exercise 11.25). To prove local exactness for a general k-form by the approach of Theorem 11.18, we must first reduce the partial differential equations with their integrability conditions into ordinary differential equations whose solutions (expressed as ordinary integrals) supply the coefficients of the needed $(k-1)$-form.

Solving the case $k = 2$, $n = 3$

In "The Poincaré Lemma and an Elementary Construction of Vector Potentials" [22], Shirley Llamado Yap introduces an algorithm for carrying out this approach. The algorithm constructs a solution for every k and $n \geq k$, using induction on n. Because the argument involves a flurry of subscripts, we first step through the (subscript-free) example with $k = 2$ and $n = 3$ we have just introduced. Thus we are given three functions $P(x,y,z)$, $Q(x,y,z)$, and $R(x,y,z)$ that are defined in a window centered at $(x,y,z) = (a,b,c)$. They satisfy the integrability condition $P_x + Q_y + R_z = 0$. We seek three functions $A(x,y,z)$, $B(x,y,z)$, and $C(x,y,z)$ that satisfy the three partial differential equations

$$C_y - B_z = P, \quad A_z - C_x = Q, \quad B_x - A_y = R,$$

in that window.

Step 1

A system of partial differential equations typically has many solutions. We seek a solution in which $C(x,y,z) \equiv 0$. In that case the first two equations reduce to differentiation with respect to z alone: $A_z = Q$, $B_z = -P$. By treating x and y as parameters, we can think of these as *ordinary* differential equations in z, whose solutions are then given by integration:

$$\begin{aligned} A(x,y,z) &= A(x,y,c) + \int_c^z Q(x,y,t)\,dt, \\ B(x,y,z) &= B(x,y,c) - \int_c^z P(x,y,t)\,dt. \end{aligned}$$

The first equation expresses the values of A off the plane $z = c$ in terms of its values on that plane (and on the values of Q in the window). But we have not yet determined the values of A on the plane.

The situation is similar for B, but, following the approach we took with C, we seek a solution in which $B(x,y,c) \equiv 0$. In that case, the equation for B reduces to

$$B(x,y,z) = -\int_c^z P(x,y,t)\,dt.$$

Step 2

Now consider the third partial differential equation, $B_x - A_y = R$. For the moment, we look for a solution only on the plane $z = c$; in Step 3, we remove this restriction. (The move from the plane to 3-space becomes the induction step in the general algorithm.) On $z = c$, we have $B = 0$ by Step 1, so $B_x - A_y = R$ reduces to

$$A_y(x,y,c) = -R(x,y,c).$$

We can treat this as an ordinary differential equation in y (with x and c as parameters); its solution is the integral

$$A(x,y,c) = A(x,b,c) - \int_b^y R(x,t,c)\,dt.$$

By analogy with what we have done with C and B, we seek a solution for which $A(x,b,c) = 0$; then

$$A(x,y,c) = -\int_b^y R(x,t,c)\,dt.$$

Step 3 Combining the results from Steps 1 and 2, we obtain the following formulas for A and B that are defined in the entire window:

$$\begin{aligned} A(x,y,z) &= -\int_b^y R(x,t,c)\,dt + \int_c^z Q(x,y,t)\,dt, \\ B(x,y,z) &= -\int_c^z P(x,y,t)\,dt. \end{aligned}$$

But we have not yet verified that A and B, as defined by these formulas, satisfy the third partial differential equation everywhere in the window. In Step 2, we constructed A and B to satisfy that third equation *only on the plane* $z = c$.

To show that A and B also satisfy the third equation when $z \neq c$, we take the following approach. We can write the third equation as $E = 0$, where

$$E(x,y,z) = B_x(x,y,z) - A_y(x,y,z) - R(x,y,z).$$

By Step 2, E equals zero when $z = c$ (and (x,y,z) lies in the window); we must show that E remains equal to zero for all z in some open neighborhood of $z = c$. We claim that the derivative of E with respect to z is zero; it will then follow that the value of E does not change—and will thus remain equal to zero—as z moves away from c. To prove the claim, we invoke the integrability condition $P_x + Q_y + R_z = 0$:

$$\frac{\partial E}{\partial z} = B_{xz} - A_{yz} - R_z = (B_z)_x - (A_z)_y - R_z = -P_x - Q_y - R_z = 0.$$

This completes the construction of the 1-form α for which $d\alpha = \omega$, and thus proves the Poincaré lemma in this case.

Theorem 11.19. *If $\omega = P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy$ is closed in a window centered at $(x,y,z) = (a,b,c)$, then $\omega = d\alpha$, where $\alpha = A\,dx + B\,dy + C\,dz$ and*

$$\begin{aligned} A(x,y,z) &= -\int_b^y R(x,t,c)\,dt + \int_c^z Q(x,y,t)\,dt, \\ B(x,y,z) &= -\int_c^z P(x,y,t)\,dt, \\ C(x,y,z) &= 0. \end{aligned}$$

□

The 1-form α that makes ω locally exact is not unique. If β also makes ω locally exact (i.e., $d\beta = \omega$), then

$\omega = d(\alpha + d\Phi)$ for any Φ

$$d(\beta - \alpha) = \omega - \omega = 0.$$

Hence $\beta - \alpha$ is a *closed* 1-form, so by the Poincaré lemma for 1-forms (Theorem 11.18, as extended to higher dimensions in Exercises 11.23 and 11.24), $\beta - \alpha$ is itself locally exact: $\beta - \alpha = d\Phi$ for some 0-form Φ. The most general 1-form that makes ω locally exact is thus $\alpha + d\Phi$, where Φ is an arbitrary 0-form.

The general algorithm

We move on now to Yap's general algorithm for constructing a $(k-1)$-form α that makes a *closed* k-form ω locally exact: $\omega = d\alpha$. The construction proceeds inductively on the dimension n of the space on which the forms are defined. For simplicity, we assume that ω is defined on a window (rectangular parallelepiped) centered at the origin in $\mathbb{R}^n$.

The base, with $n = k$

Throughout the argument, k is fixed. The induction on n begins with $n = k$. In this case, a k-form ω has only a single term,

$$\omega = P(x_1, \ldots, x_k)\, dx_1 \cdots dx_k,$$

and ω is automatically closed. Let us take

$$\alpha = A(x_1, \ldots, x_k)\, dx_1 \cdots dx_{k-1};$$

then

$$d\alpha = \frac{\partial A}{\partial x_k}\, dx_k\, dx_1 \cdots dx_{k-1} = (-1)^{k-1} \frac{\partial A}{\partial x_k}\, dx_1 \cdots dx_{k-1}\, dx_k.$$

Thus $\omega = d\alpha$ if A satisfies the partial differential equation

$$P = (-1)^{k-1} \frac{\partial A}{\partial x_k}.$$

We can solve this differential equation immediately by integration:

$$A(x_1, \ldots, x_k) = (-1)^{k-1} \int_0^{x_k} P(x_1, \ldots, x_{k-1}, t)\, dt.$$

This completes the construction of α when $n = k$.

The induction, with $n > k$

Now take $n > k$ and use induction. That is, assume the algorithm works for k-forms in $\mathbb{R}^{n-1}$ and then show that it works for k-forms in $\mathbb{R}^n$. The arguments make extensive use of multi-indices; see pages 439–443.

We are given a closed k-form

$$\omega = \sum_I P_I(x_1, \ldots, x_n)\, d\mathbf{x}_I, \quad I = (i_1, \ldots, i_k),$$

with $1 \le i_1 < \cdots < i_k \le n$. We want to find a $(k-1)$-form

$$\alpha = \sum_{\widehat{I}_s} A_{\widehat{I}_s}(x_1,\ldots,x_n)\, d\mathbf{x}_{\widehat{I}_s}, \quad \widehat{I}_s = (1_i,\ldots,\widehat{i_s},\ldots,i_k),$$

for which $\omega = d\alpha$ in an open neighborhood of $\mathbf{x} = \mathbf{0}$. Because

$$d\alpha = \sum_I \left(\sum_{s=1}^{k} (-1)^{s-1} \frac{\partial A_{\widehat{I}_s}}{\partial x_{i_s}} \right) d\mathbf{x}_I$$

(Theorem 10.19, p. 441), the condition $\omega = d\alpha$ yields the following $\binom{n}{k}$ partial differential equations

$$P_I = \sum_{s=1}^{k} (-1)^{s-1} \frac{\partial A_{\widehat{I}_s}}{\partial x_{i_s}}$$

for the $\binom{n}{k-1}$ unknown functions $A_{\widehat{I}_s}$. To obtain these functions, we follow the same steps as in the example above (pp. 497–499).

Step 1

First, restrict the multi-index I to the case where $i_k = n$. Consider the new multi-index $J = (i_1,\ldots,i_{k-1})$ with $i_{k-1} \le n-1$. If we define $\widehat{J}_s$ by analogy with $\widehat{I}_s$, then the restriction $i_k = n$ means that $I = J,n$ and

$$\widehat{I}_s = \widehat{J}_s, n \ \text{ if } s < k, \quad \widehat{I}_k = J.$$

Now consider the partial differential equation for which $I = J,n$:

$$P_{J,n} = \frac{\partial A_{\widehat{J}_1,n}}{\partial x_{i_1}} - \frac{\partial A_{\widehat{J}_2,n}}{\partial x_{i_2}} + \cdots + (-1)^{k-1} \frac{\partial A_J}{\partial x_n}.$$

Following the example, we begin the process of determining the functions $A_{\widehat{I}_s}$ by setting

$$A_{\widehat{J}_1,n}(x_1,\ldots,x_n) = \cdots = A_{\widehat{J}_{k-1},n}(x_1,\ldots,x_n) \equiv 0.$$

Because the multi-index $\widehat{J}_s$ selects $k-2$ distinct elements from the first $n-1$ positive integers, these equations for $A_{\widehat{J}_s,n}$ determine $\binom{n-1}{k-2}$ of the functions we seek.

Each partial differential equation for which $I = J,n$ thus reduces to a single term on the right and involves only one unknown function, A_J. We write this equation in the form

$$\frac{\partial A_J}{\partial x_n} = (-1)^{k-1} P_{J,n}.$$

Integration yields

$$A_J(x_1,\ldots,x_n) = A_J(x_1,\ldots,x_{n-1},0) + (-1)^{k-1} \int_0^{x_n} P_{J,n}(x_1,\ldots,x_{n-1},t)\, dt.$$

This determines A_J in terms of its values on the hyperplane $x_n = 0$ (and the values of the known function $P_{J,n}$ in the window), but we have not yet determined the values of A_J on that hyperplane.

There are $\binom{n-1}{k-1}$ functions A_J defined by these integrals; together with the $\binom{n-1}{k-2}$ functions already set equal to zero, we have identified all the $\binom{n}{k-1}$ unknown functions, because (cf. Exercise 11.26)

$$\binom{n-1}{k-1}+\binom{n-1}{k-2}=\binom{n}{k-1}.$$

We now determine the functions A_J on the hyperplane $x_n = 0$. In Step 1 we exhausted the possibility that $i_k = n$, so from this point on we take $i_k < n$. This means that dx_n no longer appears in any basic differential $d\mathbf{x}_I$. Because x_n itself no longer appears as a variable on the hyperplane $x_n = 0$, we have reduced the setting to differential forms on $\mathbb{R}^{n-1}$. Therefore, if we can pull back ω and α to forms ω^* and α^* on $\mathbb{R}^{n-1}$ in such a way that $d\omega^* = 0$ and $d\alpha^* = \omega^*$ on $\mathbb{R}^{n-1}$, we can use the induction hypothesis to obtain the functions A_J. Step 2

Consider the map $\mathbf{f} : \mathbb{R}^{n-1} \to \mathbb{R}^n : \mathbf{y} \to \mathbf{x}$ into the hyperplane $x_n = 0$:

$$\mathbf{f}: \begin{cases} x_1 = y_1, \\ \quad\vdots \\ x_{n-1} = y_{n-1}, \\ x_n = 0. \end{cases}$$

By Theorem 10.20, page 443, and the discussion preceding it, the pullback of $\mathbf{f}$ on a basic k-form is

$$\mathbf{f}^* d\mathbf{x}_I = \begin{cases} d\mathbf{y}_I & \text{if } i_k < n, \\ 0 & \text{if } i_k = n. \end{cases}$$

This suggests we define a new multi-index $I^* = I$ with the restriction that $i_k \le n-1$. The pullback of ω is then

$$\omega^*(y_1,\ldots,y_{n-1}) = \mathbf{f}^*\omega = \sum_{I^*} P_{I^*}(y_1,\ldots,y_{n-1},0)\, d\mathbf{y}_{I^*}.$$

Because d and $\mathbf{f}^*$ commute, ω^* is closed:

$$d\omega^* = d\mathbf{f}^*\omega = \mathbf{f}^* d\omega = \mathbf{f}^* 0 = 0.$$

For α we have

$$\alpha^*(y_1,\ldots,y_{n-1}) = \mathbf{f}^*\alpha = \sum_{\widehat{I^*}_s} A_{\widehat{I^*}_s}(y_1,\ldots,y_{n-1},0)\, d\mathbf{y}_{\widehat{I^*}_s},$$

$$d\alpha^*(y_1,\ldots,y_{n-1}) = d\mathbf{f}^*\alpha = \mathbf{f}^* d\alpha = \sum_{I^*}\left(\sum_{s=1}^{k}(-1)^{s-1}\frac{\partial A_{\widehat{I^*}_s}}{\partial x_{i_s}}\right) d\mathbf{y}_{I^*},$$

and the condition $\omega = d\alpha$ implies

$$\omega^* = \mathbf{f}^*\omega = \mathbf{f}^* d\alpha = d\mathbf{f}^*\alpha = d\alpha^*.$$

The equation $\omega^* = d\alpha^*$ then implies the partial differential equations

$$P_{I^*}(y_1,\dots,y_{n-1},0) = \sum_{s=1}^{k}(-1)^{s-1}\frac{\partial A_{\widehat{I^*}_s}}{\partial x_{i_s}}(y_1,\dots,y_{n-1},0).$$

By the induction hypothesis, the algorithm supplies solutions

$$A_{\widehat{I^*}_s}(y_1,\dots,y_{n-1},0)$$

to these differential equations. The multi-index $\widehat{I^*}_s$ is an increasing sequence of $k-1$ positive integers between 1 and $n-1$, so it equals a unique multi-index heretofore written as J. Conversely, suppose J is an arbitrary $(k-1)$-multi-index. Because $k-1 \le n-2$, at least one integer j in the range from 1 to $n-1$ is missing from J. Let I^* be the k-multi-index constructed by augmenting J by inserting j in the proper place. If j is in the ℓth place in I^*, then $\widehat{I^*}_\ell = J$. Thus, every J equals some $\widehat{I^*}_s$, and we have determined all the functions

$$A_J(x_1,\dots,x_{n-1},0)$$

that remained to be found at the end of Step 1.

Step 3

Steps 1 and 2 together give us all the functions $A_J(x_1,\dots,x_n)$ defined in a window centered at $\mathbf{0}$ in $\mathbb{R}^n$, but as yet we know only that those functions satisfy the partial differential equations when $x_n = 0$. We can put the matter this way (cf. Step 3 of the example). The partial differential equation indexed by I^* can be written as $E_{I^*} = 0$, where

$$E_{I^*}(x_1,\dots,x_n) = \sum_{s=1}^{k}(-1)^{s-1}\frac{\partial A_{\widehat{I^*}_s}}{\partial x_{i_s}}(x_1,\dots,x_n) - P_{I^*}(x_1,\dots,x_n).$$

By Step 2, E_{I^*} equals zero when $x_n = 0$ (and when $(x_1,\dots,x_n)$ is in the window). We claim E_{I^*} remains equal to zero for all x_n in an open interval centered at 0. To prove the claim, it is enough to show $\partial E_{I^*}/\partial x_n = 0$.

In the example, we invoked the integrability conditions to prove the claim. For the same reason, we invoke them here. The integrability conditions are the coefficients of the $(k+1)$-form $d\omega$ set equal to zero; therefore we begin by expressing ω in a way that allows us to read off those coefficients of $d\omega$. To index $(k+1)$-forms, let $L = (i_1,\dots,i_{k+1})$, $1 \le i_i < \cdots < i_{k+1} \le n$, and write

$$\omega = \sum_{\widehat{L}_s} P_{\widehat{L}_s}(x_1,\dots,x_n)\, d\mathbf{x}_{\widehat{L}_s},$$

where $s = 1,\dots,k+1$. Then

$$d\omega = \sum_L \left(\sum_{s=1}^{k+1} (-1)^{s-1} \frac{\partial P_{\widehat{L}_s}}{\partial x_{i_s}} \right) d\mathbf{x}_L,$$

so the integrability conditions are

$$\sum_{s=1}^{k+1} (-1)^{s-1} \frac{\partial P_{\widehat{L}_s}}{\partial x_{i_s}} = 0.$$

There are $\binom{n}{k+1}$ such conditions, one for each multi-index $L = (i_1, \ldots, i_{k+1})$. We focus on those multi-indices L for which $i_{k+1} = n$. Then $L = I^*, n$ and

$$\widehat{L}_s = \widehat{I^*}_s, n \ \text{ if } s < k+1, \ \text{ and } \ \widehat{L}_{k+1} = I^*.$$

The integrability condition indexed by this L is

$$\sum_{s=1}^{k} (-1)^{s-1} \frac{\partial P_{\widehat{I^*}_s, n}}{\partial x_{i_s}} + (-1)^k \frac{\partial P_{I^*}}{\partial x_n} = 0.$$

Now let us determine $\partial E_{I^*} / \partial x_n$. For each multi-index I^*, we have

$$\frac{\partial E_{I^*}}{\partial x_n} = \sum_{s=1}^{k} (-1)^{s-1} \frac{\partial}{\partial x_n} \left(\frac{\partial A_{\widehat{I^*}_s}}{\partial x_{i_s}} \right) - \frac{\partial P_{I^*}}{\partial x_n} = \sum_{s=1}^{k} (-1)^{s-1} \frac{\partial}{\partial x_{i_s}} \left(\frac{\partial A_{\widehat{I^*}_s}}{\partial x_n} \right) - \frac{\partial P_{I^*}}{\partial x_n}$$

We know that each $\widehat{I^*}_s = J$ for a suitable multi-index J; thus we can write, using the partial differential equation for A_J from Step 1,

$$\frac{\partial A_{\widehat{I^*}_s}}{\partial x_n} = \frac{\partial A_J}{\partial x_n} = (-1)^{k-1} P_{J,n} = (-1)^{k-1} P_{\widehat{I^*}_s, n}.$$

Hence

$$\frac{\partial E_{I^*}}{\partial x_n} = (-1)^{k-1} \sum_{s=1}^{k} (-1)^{s-1} \frac{\partial P_{\widehat{I^*}_s, n}}{\partial x_{i_s}} - \frac{\partial P_{I^*}}{\partial x_n},$$

so

$$(-1)^{k-1} \frac{\partial E_{I^*}}{\partial x_n} = \sum_{s=1}^{k} (-1)^{s-1} \frac{\partial P_{\widehat{I^*}_s, n}}{\partial x_{i_s}} + (-1)^k \frac{\partial P_{I^*}}{\partial x_n} = 0$$

by the integrability condition. This proves the claim, and thus establishes the algorithm.

Theorem 11.20 (Poincaré lemma). *Suppose ω is a closed k-form defined in a window centered at the origin in $\mathbb{R}^n$. Then there is a $(k-1)$-form α for which $\omega = d\alpha$ in that window. The coefficients of α can be obtained from the coefficients of ω by integration (quadrature).* □

The $(k-1)$-form α in the Poincaré lemma is not unique: if γ is any $(k-2)$-form, then $\omega = d(\alpha + d\gamma)$. In fact, it follows from the Poincaré lemma that *all* $(k-1)$-forms β with $\omega = d\beta$ can be expressed this way.

Corollary 11.21 *If $\omega = d\alpha = d\beta$, then locally $\beta = \alpha + d\gamma$ for some properly chosen $(k-2)$-form γ.*

Proof. Note that $\alpha - \beta$ is a closed $(k-1)$-form; therefore, by the Poincaré lemma it is locally exact. □

The effect of the domain

The Poincaré lemma says that a closed form will be exact if its domain is sufficiently simple. The closed 1-form

$$\omega = \frac{-y\,dx + x\,dy}{x^2+y^2},$$

whose domain is the punctured plane $\mathcal{P} = \mathbb{R}^2 \setminus (0,0)$, shows that exactness may be lost if the domain is even slightly complicated. This example is not isolated; there is an analogue of ω in every dimension. We explore them now to get a better idea how the shape of a domain can become an obstruction to the exactness of a closed form.

We take the domain to be punctured 3-space $\mathcal{Q} = \mathbb{R}^3 \setminus (0,0,0)$, and define the 2-form

$$\beta = X\,dy\,dz + Y\,dz\,dx + Z\,dx\,dy$$

by

$$X = \frac{x}{(x^2+y^2+z^2)^{3/2}}, \quad Y = \frac{y}{(x^2+y^2+z^2)^{3/2}}, \quad Z = \frac{z}{(x^2+y^2+z^2)^{3/2}}.$$

Because

$$d\beta = \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) dx\,dy\,dz$$

and

$$\frac{\partial X}{\partial x} = \frac{(x^2+y^2+z^2)^{3/2} - x \cdot 3x(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3} = \frac{x^2+y^2+z^2-3x^2}{(x^2+y^2+z^2)^{5/2}},$$

$$\frac{\partial Y}{\partial y} = \frac{x^2+y^2+z^2-3y^2}{(x^2+y^2+z^2)^{5/2}}, \quad \frac{\partial Z}{\partial z} = \frac{x^2+y^2+z^2-3z^2}{(x^2+y^2+z^2)^{5/2}},$$

β is closed, but...

we see $d\beta = 0$ everywhere on $\mathcal{Q}$.

If β were exact, so that $\beta = d\alpha$ for some 1-form α defined on $\mathcal{Q}$, then we would have

$$\iint_{\vec{S}^2} \beta = \iint_{\vec{S}^2} d\alpha = \int_{\partial \vec{S}^2} \alpha = \int_{\emptyset} \alpha = 0,$$

where $\vec{S}^2$ is the outwardly oriented unit sphere in $\mathbb{R}^3$. The path integral equals zero because $\partial\vec{S}^2$ is empty. However, because $x^2+y^2+z^2 = 1$ on S^2, β reduces to radial flow out of the sphere (cf. p. 396), so

$$\iint_{\vec{S}2} \beta = \iint_{\vec{S}^2} x\,dy\,dz + y\,dz\,dx + z\,dx\,dy = 4\pi \neq 0.$$

Consequently, β is not exact on Q.

... β is not exact

An analogue of β in each dimension

The 2-form β is the prototype for the following sequence of examples, one in each dimension. Let $Q_n = \mathbb{R}^n \setminus \mathbf{0}$, and let

$$\beta_{n-1}(\mathbf{x}) = \sum_{s=1}^{n} (-1)^{s-1} \frac{x_s}{r^n}\, d\mathbf{x}_{\widehat{N}_s},$$

where $\mathbf{x} = (x_i, \ldots, x_n)$, $r = \|\mathbf{x}\|$, $N = (1, \ldots, n)$, and $\widehat{N}_s = (1, \ldots, \widehat{s}, \ldots, n)$. Note that

$$\beta_1 = \frac{x_2\; dx_1 - x_1\, dx_2}{x_1^2 + x_2^2} \quad \text{and} \quad \beta_2 = \frac{x_1\, dx_2\, dx_3 - x_2\, dx_1\, dx_3 + x_3\, dx_1\, dx_2}{(x_1^2 + x_2^2 + x_3^2)^{3/2}},$$

so the $(n-1)$-form β_{n-1} generalizes ω and β. We have

$$d\beta_{n-1} = \sum_{s=1}^{n} \frac{\partial}{\partial x_s} \left(\frac{x_s}{r^n} \right) (-1)^{s-1}\, dx_s\, d\mathbf{x}_{\widehat{N}_s} = \sum_{s=1}^{n} \frac{r^2 - n x_s^2}{r^{n+2}}\, d\mathbf{x}_N = 0,$$

so β_{n-1} is closed. If β_{n-1} were exact, then we would have $\beta_{n-1} = d\alpha_{n-2}$ for some $(n-2)$-form defined on Q_n. Let $\vec{S}^{n-1}$ be the unit $(n-1)$-sphere in $\mathbb{R}^n$, oriented by its outward normal; as a set, S^{n-1} consists of all points $\mathbf{x}$ in $\mathbb{R}^n$ for which $r = 1$. Then we would have

$$\iint \cdots \int_{\vec{S}^{n-1}} \beta_{n-1} = \iint \cdots \int_{\vec{S}^{n-1}} d\alpha_{n-2} = \int \cdots \int_{\partial \vec{S}^{n-1}} \alpha_{n-2} = 0,$$

because $\partial \vec{S}^{n-1} = \emptyset$. Nevertheless, β is not exact, because

$$\iint \cdots \int_{\vec{S}^{n-1}} \beta_{n-1} \neq 0.$$

This follows immediately from the general n-dimensional Stokes' theorem (which we do not prove). In dimension $n = 4$, however, you can prove by a direct computation (cf. Exercise 11.13) that

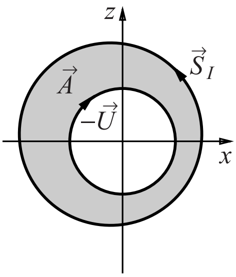
$$\iiint_{\vec{S}^3} \beta_3 = 2\pi^2.$$

The two kinds of spheres in Q

Let us resume our analysis of $Q = \mathbb{R}^3 \setminus (0,0,0)$. We distinguish between two kinds of 2-spheres in Q: those that enclose the origin, and those that do not. Each oriented sphere $\vec{S}_I$ of the first kind is the boundary of an oriented ball $\vec{B}_I$ that contains the origin. But the origin is not in Q, so, although $\vec{S}_I$ is the boundary of a ball in $\mathbb{R}^3$, it is not the boundary of a ball in Q. By contrast, each oriented sphere $\vec{S}_{II}$ of the second kind is the boundary of an oriented ball $\vec{B}_{II}$ that lies entirely in Q. For spheres of the first kind, we have the following result.

Theorem 11.22. *Suppose the origin lies in the interior of the positively oriented ball $\vec{B}_I$ in $\mathbb{R}^3$. Let $\vec{S}_I = \partial\vec{B}_I$ in $\mathbb{R}^3$; then*

$$E(\vec{S}_I) = \frac{1}{4\pi}\iint_{\vec{S}_I}\beta = +1.$$

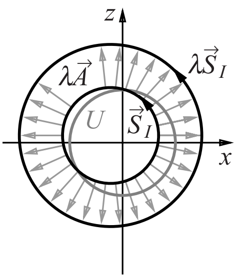


Proof. We show that $E(\vec{S}_I) = E(\vec{U})$, where $\vec{U}$ is the unit sphere (centered at the origin) with its outward orientation; we have already established (p. 504) that $E(\vec{U}) = +1$.

Suppose the given sphere, $\vec{S}_I$, lies everywhere outside $\vec{U}$, and suppose that $\vec{A}$ is the 3-dimensional positively oriented shell that lies between the two spheres; then $\partial\vec{A} = \vec{S}_I - \vec{U}$. The divergence theorem applies to β on $\vec{A}$; we thus find

$$E(\vec{S}_I) - E(\vec{U}) = \frac{1}{4\pi}\iint_{\vec{S}_I}\beta - \frac{1}{4\pi}\iint_{\vec{U}}\beta = \frac{1}{4\pi}\iint_{\partial\vec{A}}\beta = \frac{1}{4\pi}\iiint_{\vec{A}}d\beta = 0,$$

because β is closed on Q.



Even if $\vec{S}_I$ does not lie entirely outside the unit sphere U, some concentric enlargement $\lambda\vec{S}_I$ does. If $\lambda\vec{A}$ is the positively oriented 3-dimensional shell that lies between these concentric spheres, then $\partial(\lambda\vec{A}) = \lambda\vec{S}_I - \vec{S}_I$. Just as in the previous case, the divergence theorem applies, giving $E(\lambda\vec{S}_I) - E(\vec{S}_I) = 0$. Thus, in all cases, $E(\vec{S}_I) = E(\vec{U}) = +1$. □

Corollary 11.23 *If the sphere $\vec{S}_I$ encloses the origin and has inward orientation, then $E(\vec{S}_I) = -1$.* □

Theorem 11.24. *If $\vec{B}_{II}$ is an oriented ball that lies entirely in Q and $\vec{S}_{II} = \partial\vec{B}_{II}$, then $E(\vec{S}_{II}) = 0$.*

Proof. Because β is defined everywhere in an open neighborhood of $\vec{B}_{II}$, the divergence theorem applies, and

$$E(\vec{S}_{II}) = \frac{1}{4\pi}\iint_{\vec{S}_{II}}\beta = \frac{1}{4\pi}\iint_{\partial\vec{B}_{II}}\beta = \frac{1}{4\pi}\iiint_{\vec{B}_{II}}d\beta = 0. \qquad \square$$

Differential forms that detect holes

We see that the function $E(\vec{S})$ plays the same role for oriented spheres in Q that the winding number

$$W(\vec{C}) = \frac{1}{2\pi}\oint_{\vec{C}}\beta_1, \quad \beta_1 = \frac{x\,dy - y\,dx}{x^2+y^2},$$

(p. 430) plays for oriented circles $\vec{C}$ in the punctured plane. That is, each domain has a hole, and a nonzero value of the function indicates that the sphere or circle encloses that hole. The function is, in each case, determined by the differential form; thus we can just as well say it is the differential form that detects the hole.

Every closed 1-form on Q is exact

In the case of the 3-dimensional region Q, it is a 2-form that detects the hole. Is there a 1-form on Q that does the same thing? In other words, is there a 1-form α that is closed on Q but fails to be exact on Q?

Theorem 11.25. *Every closed* 1*-form* α *on* Q *is exact.*

Proof. We construct a function $\Phi(x,y,z)$ for which $d\Phi = \alpha$ on Q. Select a smooth path $\vec{C}$ in Q that starts at a fixed point (a,b,c) in Q and ends at an arbitrary point (x,y,z) in Q. Define

$$\Phi(x,y,z) = \int_{\vec{C}} \alpha;$$

for this to be meaningful, we must show the integral is path-independent.

Let $\vec{C}_1$ be any other smooth path in Q with the same starting and ending points as $\vec{C}$. Then $\vec{C}_1 - \vec{C}$ is a closed piecewise-smooth oriented path in Q, and is therefore the boundary of an oriented surface $\vec{\Sigma}$ that can be chosen to avoid the origin. In other words, $\vec{\Sigma}$ lies entirely in Q, so Stokes' theorem applies. Thus

$$\int_{\vec{C}_1} \alpha - \int_{\vec{C}} \alpha = \int_{\vec{C}_1 - \vec{C}} \alpha = \int_{\partial \vec{\Sigma}} \alpha = \iint_{\vec{\Sigma}} d\alpha = 0,$$

because $d\alpha = 0$ by hypothesis, so the integral is path-independent. □

Because the exterior derivative on forms corresponds to the classical operations of the gradient, divergence, and curl on scalar and vector fields, we can translate the relations between closed and exact forms into relations between these operations. On pages 460–462, we made the following correspondence between fields and forms in $\mathbb{R}^3$.

$$\begin{aligned} f &\leftrightarrow \omega_f^0 = f, \\ \mathbb{F} = (A,B,C) &\leftrightarrow \omega_{\mathbb{F}}^1 = A\,dx + B\,dy + C\,dz, \\ \mathbb{V} = (P,Q,R) &\leftrightarrow \omega_{\mathbb{V}}^2 = P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy, \\ H &\leftrightarrow \omega_H^3 = H\,dx\,dy\,dz. \end{aligned}$$

Using the differential operator ∇ (i.e., *nabla*, Definition 3.3, p. 93) to express the classical operators,

$$\operatorname{grad} f = \nabla f, \quad \operatorname{curl} \mathbb{F} = \nabla \times \mathbb{F}, \quad \operatorname{div} \mathbb{V} = \nabla \cdot \mathbb{V},$$

we can express the correspondences between those operators and the exterior derivative in the following way.

$$\begin{aligned} \operatorname{grad} f: \quad & \nabla f = (f_x, f_y, f_z) \;\leftrightarrow\; d\big(\omega_f^0\big) = f_x\,dx + f_y\,dy + f_z\,dz \\ \operatorname{curl} \mathbb{F}: \quad & \nabla \times (A,B,C) \;\leftrightarrow\; d\big(\omega_{\mathbb{F}}^1\big) = (C_y - B_z)\,dy\,dz + (A_z - C_x)\,dz\,dx \\ & \qquad\qquad + (B_x - A_y)\,dx\,dy \\ \operatorname{div} \mathbb{V}: \quad & \nabla \cdot (P,Q,R) \;\leftrightarrow\; d\big(\omega_{\mathbb{V}}^2\big) = (P_x + Q_y + R_z)\,dx\,dy\,dz \end{aligned}$$

As already noted (pp. 460–462), these correspondences define the forms

$$\omega^1_{\mathrm{grad}\, f} = d\left(\omega^0_f\right) = f_x\, dx + f_y\, dy + f_z\, dz$$
$$\omega^2_{\mathrm{curl}\,\mathbb{F}} = d\left(\omega^1_{\mathbb{F}}\right) = (C_y - B_z)\, dy\, dz + (A_z - C_x)\, dz\, dx + (B_x - A_y)\, dx\, dy$$
$$\omega^3_{\mathrm{div}\,\mathbb{V}} = d\left(\omega^2_{\mathbb{V}}\right) = (P_x + Q_y + R_z)\, dx\, dy\, dz.$$

Theorem 11.26. *Suppose f is a scalar field, and $\mathbb{F}$ a vector field, and each is defined on an open set in $\mathbb{R}^3$; then*

$$\nabla \times \nabla f = \mathrm{curl}\,\mathrm{grad}\, f = \mathbf{0} \;\; \textit{and} \;\; \nabla \cdot \nabla \times \mathbb{F} = \mathrm{div}\,\mathrm{curl}\,\mathbb{F} = 0.$$

Proof. These are just translations of $d^2 = 0$. □

The Poincaré lemma itself translates into the following pair of theorems.

Theorem 11.27. *Suppose $\mathbb{F}$ is a vector field and* $\mathrm{curl}\,\mathbb{F} = \mathbf{0}$ *in a neighborhood of some point in $\mathbb{R}^3$; then there is a scalar field Φ for which $\mathbb{F} = \nabla\Phi = \mathrm{grad}\,\Phi$ in a window centered at that point.* □

Theorem 11.28. *Suppose $\mathbb{V}$ is a vector field and* $\mathrm{div}\,\mathbb{V} = 0$ *in a neighborhood of some point in $\mathbb{R}^3$; then there is another vector field $\mathbb{P}$ for which*

$$\mathbb{V} = \nabla \times \mathbb{P} = \mathrm{curl}\,\mathbb{P}$$

in a window centered at that point. □

Vector and scalar potentials

The function Φ in Theorem 11.27 is a *potential function* of the vector field $\mathbb{F}$ (Definition 1.3, p. 25). Because the vector field $\mathbb{P}$ in Theorem 11.28 stands in the same relation to the field $\mathbb{V}$, we call $\mathbb{P}$ a **vector potential** for $\mathbb{V}$. (For the sake of clarity, we now refer to the function f as a **scalar potential** for the field $\mathbb{F}$ of Theorem 11.27.) By extension, we call α a *vector potential* for any k-form ω whenever $d\alpha = \omega$. The Poincaré lemma thus asserts the existence of a local vector potential for any closed k-form; indeed, Yap's result [22] is expressed in this language. By Corollary 11.21, a vector potential is not unique (when it exists).

Irrotational and incompressible flows

Suppose a fluid flow is represented by a continuously differentiable vector field $\mathbb{V}$. We say the flow is **irrotational** if $\mathrm{curl}\,\mathbb{V} = \nabla \times \mathbb{V} = \mathbf{0}$; we say it is **incompressible** if $\mathrm{div}\,\mathbb{V} = \nabla \cdot \mathbb{V} = 0$. In these terms the previous theorems say the following.

- A gradient flow is irrotational.
- A vortex flow (Definition 11.4, p. 473) is incompressible.
- An irrotational flow is locally a gradient.
- An incompressible flow is locally a vortex flow.

The Laplacian

There are meaningful ways to compose a pair of classical operators that do not correspond to the composition d^2. One is $\mathrm{div}\,\mathrm{grad}\, f = \nabla \cdot \nabla f$, where $f = f(x,y,z)$ is a scalar field. We have

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \text{ and } \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

The second-order differential operator

$$\nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

that appears here is called the **Laplacian**; it is also denoted as Δ:

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

The Laplacian operates on a scalar field (i.e., a function) to produce another scalar field. We can extend the Laplacian to vector fields by operating on each component. If $\mathbb{F} = (A, B, C)$, just set

$$\Delta \mathbb{F} = (\Delta A, \Delta B, \Delta C),$$

another vector field.

Two more classical composites that are similarly unrelated to d^2 are

$$\operatorname{grad}\operatorname{div}\mathbb{V} = \nabla(\nabla \cdot \mathbb{V}) \text{ and } \operatorname{curl}(\operatorname{curl}\mathbb{F}) = \nabla \times (\nabla \times \mathbb{F}).$$

Each composite operates on a vector field to produce another vector field; the two are connected by the following identity (see Exercise 11.28):

$$\begin{aligned} \operatorname{curl}(\operatorname{curl}\mathbb{F}) &= \operatorname{grad}(\operatorname{div}\mathbb{F}) - \operatorname{div}(\operatorname{grad}\mathbb{F}) \\ \nabla \times (\nabla \times \mathbb{F}) &= \nabla(\nabla \cdot \mathbb{F}) - \Delta \mathbb{F}. \end{aligned}$$

Exercises

11.1. Calculate $\operatorname{div}\mathbb{V} = \nabla \cdot \mathbb{V}$ when:

a. $\mathbb{V} = (x\cos y, x\sin y, 0)$.
b. $\mathbb{V} = (y+z, z+x, x+y)$.
c. $\mathbb{V} = (x/yz, y/zx, z/xy)$.
d. $\mathbb{V} = \operatorname{grad} f$, $f(x,y,z) = ax+by+cz$.
e. $\mathbb{V} = \operatorname{grad} f$, $f(x,y,z)$ arbitrary.

11.2. Calculate $\nabla \times \mathbb{F} = \operatorname{curl}\mathbb{F}$ when

a. $\mathbb{F} = (yz, zx, xy)$.
b. $\mathbb{F} = (y+z, z+x, x+y)$.
c. $\mathbb{F} = \operatorname{grad} f$, $f(x,y,z) = ax+by+cz$.
d. $\mathbb{F} = \operatorname{grad}\varphi$, $\varphi(x,y,z) = ax^2+2bxy+cy^2+2dyz+ez^2+2fzx$.