Compare and contrast these two procedures. They are essentially the same, except for how areas are found. In step two of the first procedure we simply find the numbers Δx_i and Δy_j and in the second procedure we find the vectors $V^1_{i,j}$ and $V^2_{i,j}$. Then, in step three of the first procedure we found the area of the rectangle in the manifold by simply finding the product $\Delta x_i \Delta y_j$ whereas in the second procedure we found the area by using the area form $dx \wedge dy$ and the vectors $V^1_{i,j}, V^2_{i,j}$, namely, we found $dx \wedge dy(V^1_{i,j}, V^2_{i,j})$. These are, in essence, the only differences between the two procedures. But notice, in the first procedure we defined the integral $\int\int_R f(x, y)dxdy$ whereas in the second procedure we defined the integral $\int\int_R f(x, y)dx \wedge dy$. It is because of this that one will sometimes see the integral of the differential form $f(x, y)dx \wedge dy$ to simply be defined as

$$\int\int_R f(x, y)dx \wedge dy \equiv \int\int_R f(x, y)dxdy$$

instead of being defined in terms of the second procedure using Riemann sums. The only problem here is that the first procedure and its associated integral $\int\int_R f(x, y)dxdy$ does not keep track of orientation whereas the second procedure and its associated integral $\int\int_R f(x, y)dx \wedge dy$ does.

Question 7.1 Using the five steps above as a guide, define the integral of a differential form $\alpha = f(x_1, x_2, \ldots, x_n)dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ on a region $R \subset \mathbb{R}^n$ using Riemann sums.

Now, suppose we find that taking the integral $\int_R f(x_1, \ldots, x_n)dx_1 \wedge \cdots \wedge dx_n$, or equivalently the integral $\int_R f(x_1, \ldots, x_n)dx_1 \cdots dx_n$, is difficult but we notice that it would become easier by doing an appropriate change of coordinates

$$\begin{aligned} \mathbb{R}^n_{x_1x_2\cdots x_n} &\xrightarrow{\phi} \mathbb{R}^n_{\phi_1\phi_2\cdots\phi_n} \\ (x_1, x_2, \ldots, x_n) &\longmapsto (\phi_1, \phi_2, \ldots, \phi_n). \end{aligned}$$

We want to develop this change of coordinates formula from first principles, that is, from the Riemann sum definition, in the context of differential forms. In other words, we are going to show that

$$\int_R f\ dx_1 \wedge \cdots \wedge dx_n = \int_{\phi(R)} f \circ \phi^{-1}\ \ T^*\phi^{-1} \cdot \big(dx_1 \wedge \cdots \wedge dx_n\big).$$

By doing the change of coordinates ϕ we map $R \subset \mathbb{R}^n_{x_1x_2\cdots x_n}$ to $\phi(R) \subset \mathbb{R}^n_{\phi_1\phi_2\cdots\phi_n}$. We also need to assume that ϕ has an inverse ϕ^{-1}.

(1) Choose a lattice of points $\big\{(x_{i_1}, \ldots, x_{i_n})\big\}$ on $\mathbb{R}^n_{x_1x_2\cdots x_n}$. This in turn gives a lattice of points

$$\Big\{\big(\phi_1(x_{i_1}, \ldots, x_{i_n}), \ldots, \phi_n(x_{i_1}, \ldots, x_{i_n})\big)\Big\} = \big\{(\phi_1, \ldots, \phi_n)\big\}$$

in the new coordinate system on $\mathbb{R}^n_{\phi_1\phi_2\cdots\phi_n}$, see Fig. 7.5.

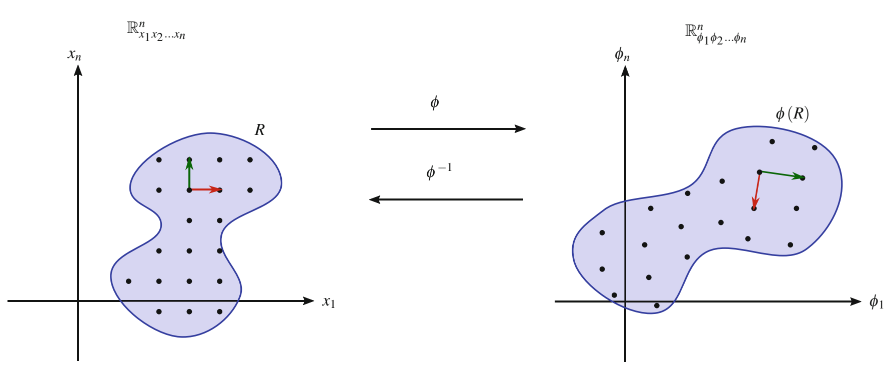

Fig. 7.5 The lattice $\{(x_{i_1}, \ldots, x_{i_n})\}$ in $\mathbb{R}^n_{x_1x_2\cdots x_n}$ produces the lattice $\{(\phi_1, \ldots, \phi_n)\}$ in $\mathbb{R}^n_{\phi_1\phi_2\cdots\phi_n}$. The push-forwards of two vectors are shown

(2) For each $i_1, \ldots, i_n$ define the vectors

$$V^1_{i_1 \cdots i_n} = \begin{bmatrix} x_{i_1+1} - x_{i_1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{(x_{i_1}, x_{i_2}, \ldots, x_{i_n})},$$

$$\vdots$$

$$V^n_{i_1 \cdots i_n} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x_{i_n+1} - x_{i_n} \end{bmatrix}_{(x_{i_1}, x_{i_2}, \ldots, x_{i_n})}.$$

Note that since the manifold is a vector space which is isomorphic to the tangent space we can claim that the vectors $V^1_{i_1 \cdots i_n}, \ldots, V^n_{i_1 \cdots i_n} \in T_{(x_{i_1}, \ldots, x_{i_n})} \mathbb{R}^n_{x_1 \cdots x_n}$. The volume of the parallelepiped spanned by the vectors $V^1_{i_1 \cdots i_n}, \ldots, V^n_{i_1 \cdots i_n}$ is given by

$$dx_1 \wedge \cdots \wedge dx_n (V^1_{i_1 \cdots i_n}, \ldots, V^n_{i_1 \cdots i_n}).$$

(3) The $(n + 1)$-dimensional volume over the n-dimensional parallelepiped and under the graph of f is approximated by

$$f(x_{i_1}, \ldots x_{i_n}) \, dx_1 \wedge \cdots \wedge dx_n (V^1_{i_1 \cdots i_n}, \ldots, V^n_{i_1 \cdots i_n}).$$

Next, at each lattice point $p = \{(x_{i_1}, \ldots, x_{i_n})\} \in \mathbb{R}^n_{x_1 \cdots x_n}$ the push-forwards of the vectors $V^1_{i_1 \cdots i_n}, \ldots, V^n_{i_1 \cdots i_n}$ at that point are

$$T_p\phi \cdot V^1_{i_1 \cdots i_n}, \ldots, T_p\phi \cdot V^n_{i_1 \cdots i_n} \in T_{\phi(p)} \mathbb{R}^n_{\phi_1 \phi_2 \cdots \phi_n}.$$

Also, note that for each i, $T\phi^{-1} \cdot T\phi \cdot V^i_{i_1 \cdots i_n} = V^i_{i_1 \cdots i_n}$ so, by definition, we have

$$\begin{aligned} &T^*\phi^{-1} \cdot \big(dx_1 \wedge \cdots \wedge dx_n\big)\Big(T\phi \cdot V^1_{i_1 \cdots i_n}, \ldots, T\phi \cdot V^n_{i_1 \cdots i_n}\Big) \\ &= dx_1 \wedge \cdots \wedge dx_n \Big(T\phi^{-1} \cdot T\phi \cdot V^1_{i_1 \cdots i_n}, \ldots, T\phi^{-1} \cdot T\phi \cdot V^n_{i_1 \cdots i_n}\Big) \\ &= dx_1 \wedge \cdots \wedge dx_n \Big(V^1_{i_1 \cdots i_n}, \ldots, V^n_{i_1 \cdots i_n}\Big). \end{aligned}$$

Next, we notice that $f(x_{i_1}, \ldots x_{i_n}) = f \circ \phi^{-1} \circ \phi(x_{i_1}, \ldots x_{i_n})$. Combining this we have that

$$\begin{aligned} &f(x_{i_1}, \ldots x_{i_n}) \, dx_1 \wedge \cdots \wedge dx_n \Big(V^1_{i_1 \cdots i_n}, \ldots, V^n_{i_1 \cdots i_n}\Big) \\ &= f \circ \phi^{-1} \circ \phi(x_{i_1}, \ldots x_{i_n}) \, T^*\phi^{-1} \cdot \big(dx_1 \wedge \cdots \wedge dx_n\big)\Big(T\phi \cdot V^1_{i_1 \cdots i_n}, \ldots, T\phi \cdot V^n_{i_1 \cdots i_n}\Big) \\ &= f \circ \phi^{-1}(\phi_1, \ldots, \phi_n) \, T^*\phi^{-1} \cdot \big(dx_1 \wedge \cdots \wedge dx_n\big)\Big(T\phi \cdot V^1_{i_1 \cdots i_n}, \ldots, T\phi \cdot V^n_{i_1 \cdots i_n}\Big). \end{aligned}$$

Notice that in the last equality above we wrote

$$\phi(x_{i_1}, \ldots x_{i_n}) = \big(\phi_1(x_{i_1}, \ldots x_{i_n}), \ldots, \phi_n(x_{i_1}, \ldots x_{i_n})\big) = (\phi_1, \ldots, \phi_n).$$

(4) Now we sum over all $i_1, \ldots, i_n$ to get

$$\begin{aligned}&\sum_{i_1,\ldots,i_n} f(x_{i_1}, \ldots x_{i_n})\, dx_1 \wedge \cdots \wedge dx_n\Big(V^1_{i_1\cdots i_n}, \ldots, V^n_{i_1\cdots i_n}\Big)\\ &= \sum_{i_1,\ldots,i_n} f \circ \phi^{-1}(\phi_1, \ldots, \phi_n)\, T^*\phi^{-1} \cdot \big(dx_1 \wedge \cdots \wedge dx_n\big)\Big(T\phi \cdot V^1_{i_1\cdots i_n}, \ldots, T\phi \cdot V^n_{i_1\cdots i_n}\Big).\end{aligned}$$

(5) Taking the limit as $\left|x_{i_j+1} - x_{i_j}\right| \to 0$ for $j = 1, \ldots, n$ we define

$$\begin{aligned}&\int_R f(x_1, \ldots, x_n)\, dx_1 \wedge \cdots \wedge dx_n\\ &\quad= \lim \sum_{i_1,\ldots,i_n} f(x_{i_1}, \ldots x_{i_n})\, dx_1 \wedge \cdots \wedge dx_n\Big(V^1_{i_1\cdots i_n}, \ldots, V^n_{i_1\cdots i_n}\Big)\end{aligned}$$

and

$$\begin{aligned}&\int_{\phi(R)} f \circ \phi^{-1}(\phi_1, \ldots, \phi_n)\;\; T^*\phi^{-1} \cdot \big(dx_1 \wedge \cdots \wedge dx_n\big)\\ &= \lim \sum_{i_1,\ldots,i_n} f \circ \phi^{-1}(\phi_1, \ldots, \phi_n)\;\; T^*\phi^{-1} \cdot \big(dx_1 \wedge \cdots \wedge dx_n\big)\Big(T\phi \cdot V^1_{i_1\cdots i_n}, \ldots, T\phi \cdot V^n_{i_1\cdots i_n}\Big).\end{aligned}$$

Combining step (4) with these definitions gives us the following equality:

Change of coordinates formula	$\int_R f(x_1, \ldots, x_n)\;\; dx_1 \wedge \ldots \wedge dx_n = \int_{\phi(R)} f \circ \phi^{-1}(\phi_1, \ldots, \phi_n)\;\; T^*\phi^{-1} \cdot (dx_1 \wedge \ldots \wedge dx_n).$

Let us take a closer look at this equality. The left hand side is fairly straight forward; the integral takes place in $x_1 \cdots x_n$-coordinates and we are integrating the function $f(x_1, \ldots, x_n)$ over the region R using the volume form $dx_1 \wedge \cdots \wedge dx_n$ associated with the $x_1 \cdots x_n$-coordinates. Now, let's unpack the right hand side of this equality. First, we have the coordinate transformation $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ given by

$$\begin{aligned}\mathbb{R}^n_{x_1x_2\cdots x_n} &\xrightarrow{\phi} \mathbb{R}^n_{\phi_1\phi_2\cdots\phi_n}\\ (x_1, x_2, \ldots, x_n) &\longmapsto (\phi_1, \phi_2, \ldots, \phi_n)\end{aligned}$$

which gives the picture in Fig. 7.6. The region we are integrating over in $\mathbb{R}^n_{x_1x_2\cdots x_n}$ is R and the region we are integrating over in $\mathbb{R}^n_{\phi_1\phi_2\cdots\phi_n}$ is its image $\phi(R)$. The function $f : \mathbb{R}^n \to \mathbb{R}$ is a function of the variables $x_1, x_2, \cdots x_n$ whereas the function $f \circ \phi^{-1} : \mathbb{R}^n \to \mathbb{R}$ is a function of the variables $\phi_1, \phi_2, \cdots \phi_n$. Finally, $dx_1 \wedge \cdots \wedge dx_n$ is the area form on $\mathbb{R}^n_{x_1x_2\cdots x_n}$. But notice that the form on the right hand side is $T^*\phi^{-1} \cdot \big(dx_1 \wedge \cdots \wedge dx_n\big)$, the pull-back of the area form $dx_1 \wedge \cdots \wedge dx_n$ by ϕ^{-1} and NOT the area form $d\phi_1 \wedge \cdots \wedge d\phi_n$. This is an essential point when doing changes of variables.

There is now one final issue we need to mention briefly. Whether or not you have taken an introductory analysis course and have seen the proof of Fubini's theorem, or have even heard of Fubini's theorem before, you are doubtless aware of its consequences. Suppose we have a rectangular region $R = \{a \le x \le b, c \le y \le d\}$ and a continuous function $f : R \to \mathbb{R}$, then Fubini's theorem states

$$\int_a^b \int_c^d \;\; f(x, y)\;\; dy\, dx = \int_c^d \int_a^b \;\; f(x, y)\;\; dx\, dy.$$

Essentially it says we can change the order of integration. This actually requires our function f be somehow "nice." We do not want to get into the technical details here (we leave that for your analysis course) but suffice it to say that most function you will be dealing with are in fact "nice." However, in the context of differential forms we of course have $dx \wedge dy = -dy \wedge dx$

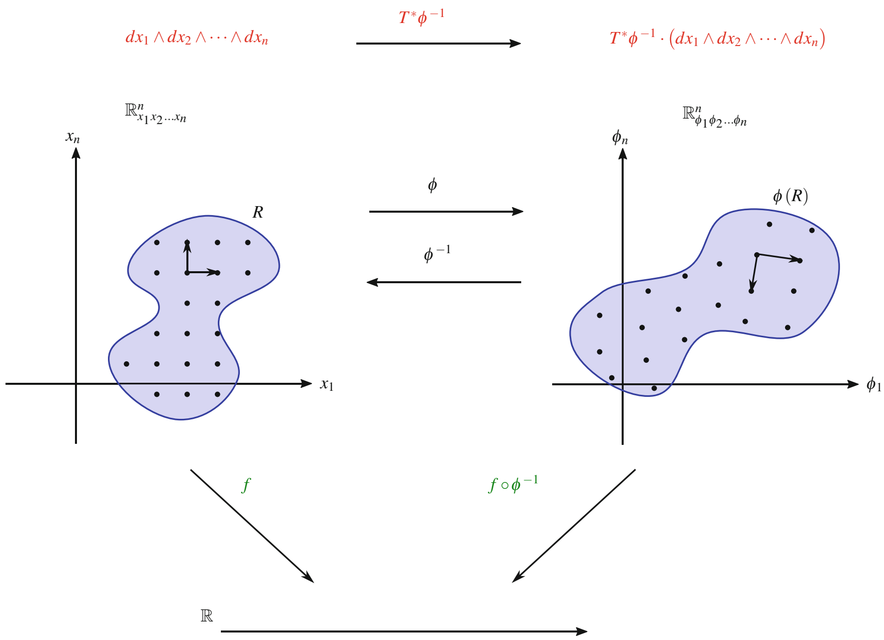

Fig. 7.6 When we change variables we have to change three things in our integral, the region (blue) the function (green), and the volume form (red). The vectors on the left are pushed forward by $T\phi$ to the vectors on the right

so we would have

$$\int_a^b \int_c^d f(x, y)\ dy \wedge dx = -\int_c^d \int_a^b f(x, y)\ dx \wedge dy,$$

unless of course you also change the orientation of the rectangle R as well. This issue will come up again later, but we will not spend a great deal of time on it. In general, integrating with differential forms takes into account orientation, which was not done in calculus. Thus, your answer may be different by a sign.

7.2 A Simple Example

We now apply what we have just done by talking a look at the role that volume forms play in integration during changes of variable by looking at a simple example that uses the same coordinate change that we encountered in Sect. 6.1. We will begin by integrating a function over a given region in $\mathbb{R}^2_{xy}$. Then we will apply the change-of-variables formula and perform the integration on $\mathbb{R}^2_{uv}$. We will also make implicit use of the following identification,

$$\int\int_R f(x, y)\, dx\, dy \equiv \int\int_R f(x, y)\, dx \wedge dy.$$

Suppose we want to integrate the function $f(x, y) = x$ over the region in $\mathbb{R}^2_{xy}$ bounded by the lines $x = 0$, $y = 0$, and $y = x - 2$, shown in Fig. 7.7. We will first integrate with respect to x and then with respect to y,

$$\begin{aligned}\int_{-2}^{0}\int_{y+2}^{0} x\, dx\, dy &= \int_{-2}^{0} \left[\frac{x^2}{2}\right]_{y+2}^{0} dy \\ &= \int_{-2}^{0} \frac{-(y+2)^2}{2}\, dy\end{aligned}$$

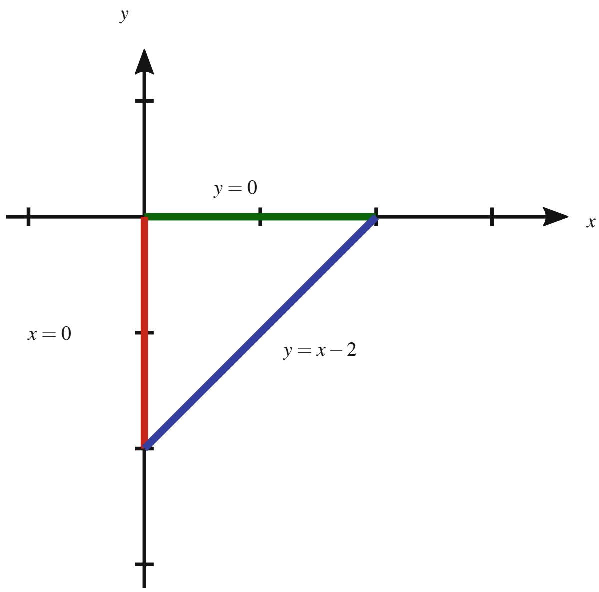

Fig. 7.7 The region in $\mathbb{R}^2$ bounded by the lines $x = 0$, $y = 0$, and $y = x - 2$

$$
\begin{aligned}
&= \int_{-2}^{0} -\left(\frac{y^2 + 4y + 4}{2}\right) dy \\
&= -\left[\frac{y^3}{6} + y^2 + 2y\right]_{-2}^{0} \\
&= \frac{-4}{3}.
\end{aligned}
$$

Now we want to do the same integral, only in different coordinates. To do this we must make use of the change of coordinates formula that we derived in Sect. 7.1,

$$\int_R f(x_1, \ldots, x_n) \; dx_1 \wedge \ldots \wedge dx_n = \int_{\phi(R)} f \circ \phi^{-1}(\phi_1, \ldots, \phi_n) \; T^*\phi^{-1} \cdot (dx_1 \wedge \ldots \wedge dx_n).$$

For our particular example the general change of coordinates formula simplifies to

$$\int\int_R f(x, y) \; dx \wedge dy = \int\int_{\phi(R)} f \circ \phi^{-1}(u, v) \; T^*\phi^{-1} \cdot (dx \wedge dy).$$

All of this comes down to the following: when we change variables we have to change three things in our integral, the region (blue), the function (green), and the volume form (red). See Fig. 7.8.

Returning to our example, we will change coordinates to the uv-coordinate systems defined by $u = x + y$ and $v = x - y$. In other words, we will use the coordinate transformation $\phi : \mathbb{R}^2_{xy} \to \mathbb{R}^2_{uv}$ given by $\phi(x, y) = \big(u(x, y), v(x, y)\big) = (x + y, x - y)$. We consider each of the three things we need to change. First we will consider the region we are integrating over in $\mathbb{R}^2_{uv}$. The coordinate transformations $u = x + y$ and $v = x - y$ can be rewritten as $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$ which means the under the transformations the equations that define the borders become

$$
\begin{aligned}
x = 0 &\Longrightarrow u = -v, \\
y = 0 &\Longrightarrow u = v, \\
y = x - 2 &\Longrightarrow v = 2.
\end{aligned}
$$

Thus the image of our region in $\mathbb{R}^2_{uv}$ is shown in Fig. 7.9.

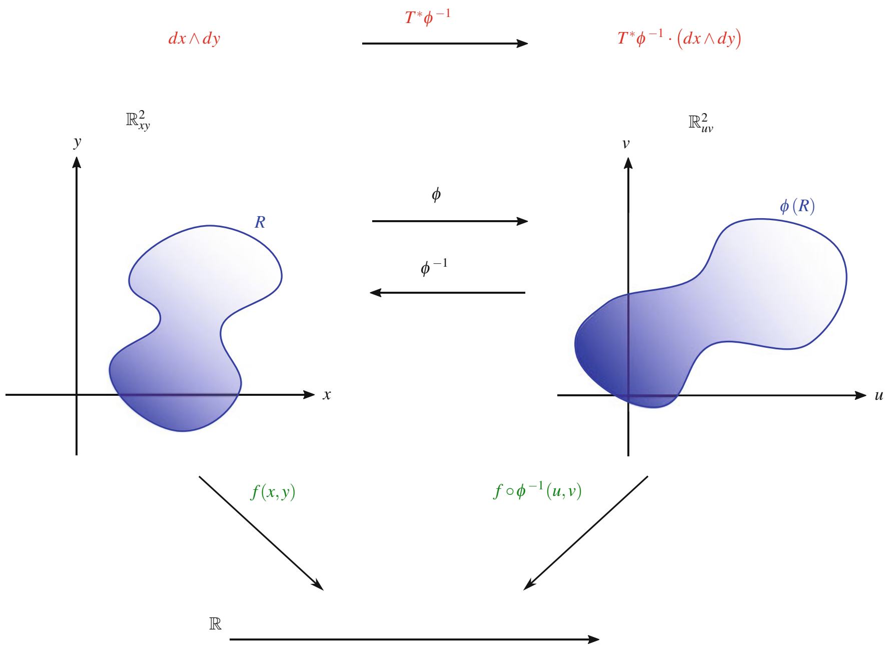

Fig. 7.8 When we change variables we have to change three things in our integral, the region (blue) the function (green), and the volume form (red)

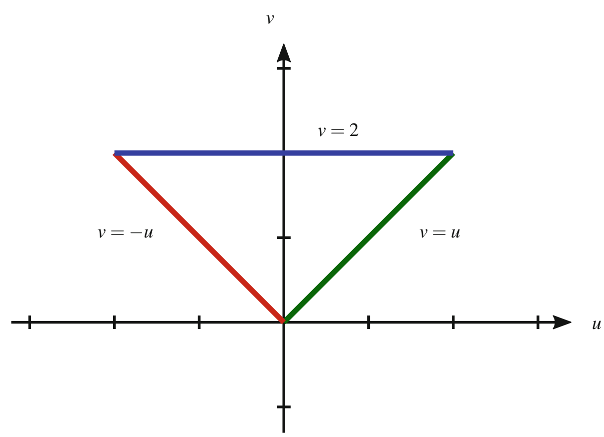

Fig. 7.9 The image in $\mathbb{R}^2_{uv}$ of the region shown in Fig. 7.7 under the transformation $\phi(x, y) = \big(u(x, y), v(x, y)\big) = (x + y, x - y)$

Second, we will consider the function. The function $f(x, y) = x$ that we want to integrate gets transformed into $f \circ \phi^{-1}(u, v) = \frac{u+v}{2}$. Third, we need to consider how the volume form changes. Recall that the mapping ϕ^{-1} induces the pullback mapping $T^*\phi^{-1}$,

$$\begin{aligned} \mathbb{R}^2_{uv} &\xrightarrow{\phi^{-1}} \mathbb{R}^2_{xy} \\ (u, v) &\longmapsto (x, y) \\ \bigwedge\left(\mathbb{R}^2_{uv}\right) &\xleftarrow{T^*\phi^{-1}} \bigwedge\left(\mathbb{R}^2_{xy}\right) \\ T^*\phi^{-1} \cdot \big(dx \wedge dy\big) &\longleftarrow dx \wedge dy. \end{aligned}$$

In Chap. 6 we found $T^*\phi^{-1} \cdot \big(dx \wedge dy\big) = \frac{-1}{2} du \wedge dv$.

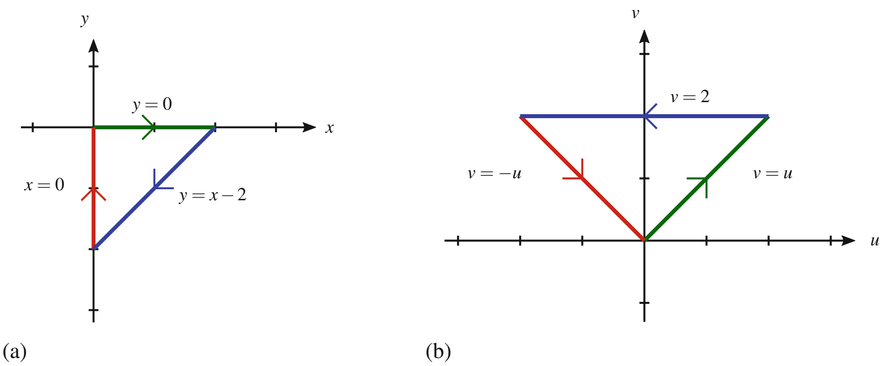

Fig. 7.10 The affect of the mapping $\phi : \mathbb{R}^2_{xy} \longrightarrow \mathbb{R}^2_{uv}$ given by $\phi(x, y) = \big(u(x, y), v(x, y)\big) = (x + y, x - y)$ on the region we are integrating over . (**a**) Enclosed area in $\mathbb{R}^2_{xy}$ is two and orientation is clockwise. (**b**) Enclosed area in $\mathbb{R}^2_{uv}$ is four and orientation is counter-clockwise

Under the transformation ϕ the corners of the region are mapped as follows,

$$\begin{aligned} \mathbb{R}^2_{xy} &\xrightarrow{\phi} \mathbb{R}^2_{uv} \\ (x, y) &\longmapsto (x + y, x - y) \equiv (u, v) \\ (0, 0) &\longmapsto (0, 0) \\ (2, 0) &\longmapsto (2, 2) \\ (0, -2) &\longmapsto (-2, 2). \end{aligned}$$

Thus we can see that as we trace around the regions the orientation in $\mathbb{R}^2_{uv}$ is different from the orientation in $\mathbb{R}^2_{xy}$. Also, compare the areas of the two regions in Fig. 7.10. This pictures makes it clear why we need to use the pull-back of the area form. Under the change of coordinates both the area and the orientation of the region we are integrating over changes. By using the pull-back of the area form we compensate for these changes. Differential forms, volume forms in particular, carry with them the information we need in order to do changes of coordinates for integration problems.

In summary, $\int\int_R x\, dx\, dy$ should really be thought of as $\int\int_R x\, dx \wedge dy$. When we change the coordinates the region $R = \big\{(x, y)\big|0 \le x \le 2, x - 2 \le y \le 0\big\}$ becomes $\tilde{R} = \big\{(u, v)\big|0 \le v \le 2, -v \le u \le v\big\}$, $f(x, y) = x$ becomes $\tilde{f}(u, v) = \frac{u+v}{2}$, and $dx\ dy$, which is really $dx \wedge dy$, becomes $\frac{-1}{2} du \wedge dv$, or $\frac{-1}{2} du\ dv$. In other words, we have the following

$$\int\int_R f(x, y)\ dx\ dy = \int\int_{\tilde{R}} \tilde{f}(u, v)\ \frac{-1}{2} du\ dv$$

which is really

$$\int\int_R f(x, y)\ dx \wedge dy = \int\int_{\tilde{R}} \tilde{f}(u, v)\ \frac{-1}{2} du \wedge dv.$$

Let us now integrate the transformed function over the transformed area,

$$\begin{aligned} \int_0^2 \int_{-v}^{v} \frac{u+v}{2} \frac{-1}{2}\, du\, dv &= \frac{-1}{2} \int_0^2 \left[\frac{u^2}{4} + \frac{uv}{2}\right]_{-v}^{v} dv \\ &= \frac{-1}{2} \int_0^2 v^2\, dv \end{aligned}$$

$$
\begin{aligned}
&= \frac{-1}{2}\left[\frac{v^3}{3}\right]_0^2 \\
&= \frac{-1}{2}\cdot\frac{8}{3} \\
&= -\frac{4}{3}.
\end{aligned}
$$

Often changing variables (also called changing basis) makes the function we are we are trying to integrate much simpler thereby making the integration easier. To see an example suppose we want to integrate $f(x, y) = e^{\frac{x+y}{x-y}}$ over the same region as before, $R = \left\{(x, y)\middle| 0 \le x \le 2, x-2 \le y \le 0\right\}$. Wanting to simplify this function, and thereby simplify the integration, would be the motivation of choosing the new coordinates $u = x + y$ and $v = x - y$,

$$
\begin{aligned}
\int_{-2}^{0}\int_{y+2}^{0} e^{\frac{x+y}{x-y}}\,dx\,dy &= \int_{-2}^{0}\int_{y+2}^{0} e^{\frac{x+y}{x-y}}\,dx\wedge dy \\
&= \int_0^2\int_{-v}^{v} e^{\frac{u}{v}}\left(\frac{-1}{2}\right)du\wedge dv \\
&= \int_0^2\int_{-v}^{v} e^{\frac{u}{v}}\left(\frac{-1}{2}\right)du\,dv \\
&= \frac{-1}{2}\int_0^2\left[ve^{\frac{u}{v}}\right]_{-v}^{v}dv \\
&= \frac{-1}{2}\int_0^2 v\left(e^{\frac{v}{v}} - e^{\frac{-v}{v}}\right)dv \\
&= \frac{-1}{2}\left(e^1 - e^{-1}\right)\int_0^2 v\,dv \\
&= \frac{-1}{2}\left(e^1 - e^{-1}\right)\left[\frac{v^2}{2}\right]_0^2 \\
&= -\left(e^1 - e^{-1}\right) \\
&\approx -2.35.
\end{aligned}
$$

Clearly, $e^{\frac{x+y}{x-y}} > 0$ for all x and y, so you are integrating a function that is positive over an area, which you generally think of as positive, so you would rather naturally expect to get a positive volume instead of a negative one. Of course, in thinking of it this way we are not taking into account the "orientation" of space. But the fact that one would naturally expect volumes to be positive is why in classical calculus classes the absolute value of the extra term, the Jacobian, which is $\frac{-1}{2}$ here, is taken. In essence this amounts to the convention that we always assume that our coordinate axis follows the "right-hand rule" regardless of what happens during the coordinate transformation.

However, as we start to deal with more advanced and abstract mathematics, we want to be able to keep track of the orientations of our manifold, which is something that differential forms do. Recall, the fact that differential forms keep track of orientations all follows from the fact that our wedgeproduct is defined in terms of the determinant, which itself, by the very way we developed it, keeps track of orientation. The negative sign shows up in our example because when we push-forward two vectors from the xy-plane to the uv-plane using our transformations the image vectors change "handedness" so positively oriented areas become negatively oriented areas, and visa-versa.

7.3 Polar, Cylindrical, and Spherical Coordinates

Now we will look at a variety of examples involving polar, cylindrical, and spherical coordinates. Though you have seen these coordinate systems in calculus the presentation here will relate what you already know from the perspective of differential forms.

7.3.1 *Polar Coordinates Example*

Now, let's get our hands dirty with an actual example involving polar coordinates. Suppose we want to find the volume of the 3-dimensional solid bounded by the $z = 0$ plane and the paraboloid $z = 3 - x^2 - y^2$, Fig. 7.11. First we find the intersection of paraboloid with the $z = 0$ plane

$$z = 0 \;\Rightarrow\; 0 = 3 - x^2 - y^2 \;\Rightarrow\; x^2 + y^2 = 3,$$

which gives the region $R = \{x^2 + y^2 \leq 3\}$ in the xy-plane. Thus the integral we want to compute is

$$\int_R 3 - x^2 - y^2 \, dx \wedge dy$$

or, in more traditional calculus notation,

$$\int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-y^2}}^{\sqrt{3-y^2}} 3 - x^2 - y^2 \, dx dy.$$

There is no doubt the computation would be messy.

Question 7.2 Attempt to do this integration with the techniques you remember from calculus class without doing a change of coordinates and see how far you can get.

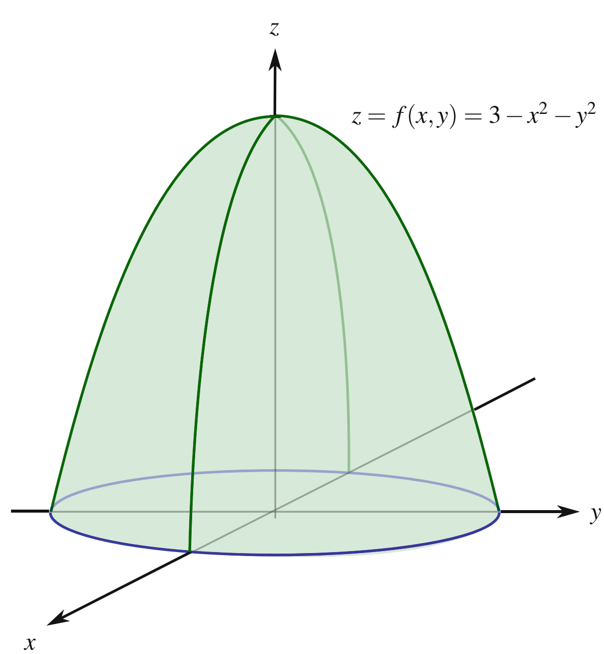

Fig. 7.11 The 3-dimensional solid bounded by the $z = 0$ plane and the paraboloid $z = 3 - x^2 - y^2$

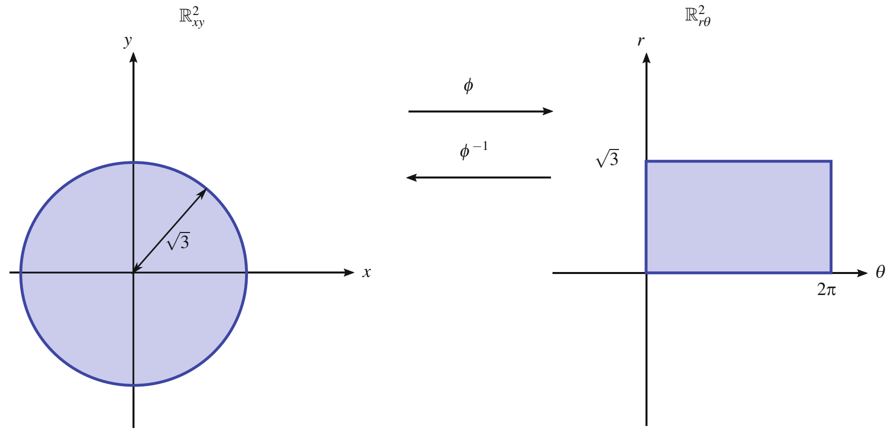

Fig. 7.12 The image of the circular region $\{x^2 + y^2 \leq 3\}$ under a polar coordinate change is the rectangular region $\{0 \leq r \leq \sqrt{3}, 0 \leq \theta \leq 2\pi\}$

When we actually compute integrals it is clearly much simpler to integrate over rectangular regions then regions of other shapes. Since this region is a disk in the xy-plane using a polar coordinate change results in a rectangular region in the $r\theta$-plane. That is, given the polar coordinate transformation

$$x = r\cos\theta,$$
$$y = r\sin\theta$$

the image of this region under a polar coordinate change is

$$\phi(R) = \{0 \leq r \leq \sqrt{3}, 0 \leq \theta \leq 2\pi\}$$

as is shown in Fig. 7.12.

Question 7.3 Show that the region $\phi(R)$ is indeed $\{0 \leq r \leq \sqrt{3}, 0 \leq \theta \leq 2\pi\}$.

Thus we want to use the coordinate transformation to simplify our integral by making use of the equality derived from first principles in the last section,

$$\int_R f\, dx_1 \wedge \cdots \wedge dx_n = \int_{\phi(R)} f \circ \phi^{-1} \;\; T^*\phi^{-1} \cdot \big(dx_1 \wedge \cdots \wedge dx_n\big).$$

Here we take a few moments to make some points about both the notation we are using and the comparison between the xy-plane and the $r\theta$-plane. First is that the polar transformation that is usually presented in calculus textbooks, $x = r\cos\theta$ and $y = r\sin\theta$, is the transformation from the $r\theta$-plane to the xy-plane and not visa-versa. The transformation from the xy-plane to the $r\theta$-plane is given by the substantially more complex

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \arctan\left(\frac{y}{x}\right).$$

We would also need to restrict ourselves to $0 < \theta < 2\pi$ in the $r\theta$-plane in order for arctan to be defined. We will play a little fast and loose with this, but for our integration problems doing so will not present a problem. Also, which direction we label ϕ and which we label ϕ^{-1} is of course completely arbitrary, but in order to remain consistent with the notation in the previous section we label the transformation from the xy-plane to the $r\theta$-plane ϕ and the reverse transformation is labeled ϕ^{-1}. It is a little surprising when you think about it, but actually having ϕ^{-1}, that is, x and y in terms of r and θ, allows us to find $\phi(R)$ and $f \circ \phi^{-1}$ easily,

$$\begin{aligned} f \circ \phi^{-1} &= 3 - (r\cos\theta)^2 - (r\sin\theta)^2 \\ &= 3 - r^2(\sin^2\theta + \cos^2\theta) \\ &= 3 - r^2. \end{aligned}$$

Finally, we find $T^*\phi^{-1} \cdot \big(dx \wedge dy\big)$. To do this we first note that $T^*\phi^{-1} \cdot \big(dx \wedge dy\big)$ has the form $g(r,\theta)d\theta \wedge dr$ for some function g. Next we notice that

$$\begin{aligned} g(r,\theta) &= \Big(g d\theta \wedge dr\Big)\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right) \\ &= \Big(T^*\phi^{-1} \cdot (dx \wedge dy)\Big)\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right) \\ &= dx \wedge dy \left(T\phi^{-1} \cdot \begin{bmatrix}1\\0\end{bmatrix}, T\phi^{-1} \cdot \begin{bmatrix}0\\1\end{bmatrix}\right). \end{aligned}$$

In order to proceed further we now need to find $T\phi^{-1}$,

$$T\phi^{-1} = \begin{bmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial r} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial r} \end{bmatrix} = \begin{bmatrix} -r\sin\theta & \cos\theta \\ r\cos\theta & \sin\theta \end{bmatrix}$$

so we can then compute

$$\begin{aligned} T\phi^{-1} \cdot \begin{bmatrix}1\\0\end{bmatrix} &= \begin{bmatrix} -r\sin\theta & \cos\theta \\ r\cos\theta & \sin\theta \end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix} -r\sin\theta \\ r\cos\theta \end{bmatrix}, \\ T\phi^{-1} \cdot \begin{bmatrix}0\\1\end{bmatrix} &= \begin{bmatrix} -r\sin\theta & \cos\theta \\ r\cos\theta & \sin\theta \end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \end{aligned}$$

which in turn gives us

$$\begin{aligned} & dx \wedge dy \left(T\phi^{-1} \cdot \begin{bmatrix}1\\0\end{bmatrix}, T\phi^{-1} \cdot \begin{bmatrix}0\\1\end{bmatrix}\right) \\ &= dx \wedge dy \left(\begin{bmatrix} -r\sin\theta \\ r\cos\theta \end{bmatrix}, \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}\right) \\ &= \begin{vmatrix} -r\sin\theta & \cos\theta \\ r\cos\theta & \sin\theta \end{vmatrix} \\ &= -r\sin^2\theta - r\cos^2\theta \\ &= -r \end{aligned}$$

which means that $g(r,\theta) = -r$. Hence

$$T^*\phi^{-1} \cdot \big(dx \wedge dy\big) = -r d\theta \wedge dr.$$

We had of course already found in the previous chapter.

Now we have all the ingredients we need to rewrite the integral in polar coordinates. Of course, we have gone through a lot of trouble to show all the steps in excruciating detail. Once you have done this a few times the end results are available and you can simply write down the integral in polar coordinates,

$$
\begin{aligned}
\int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-y^2}}^{\sqrt{3-y^2}} 3 - x^2 - y^2 \, dxdy &= \int_R f \, dx \wedge dy \\
&= \int_{\phi(R)} f \circ \phi^{-1} \; T^*\phi^{-1} \cdot (dx \wedge dy) \\
&= \int_0^{\sqrt{3}} \int_0^{2\pi} (3 - r^2) \; (-r) d\theta \wedge dr \\
&= \int_0^{\sqrt{3}} \int_0^{2\pi} -3r + r^3 \, d\theta dr.
\end{aligned}
$$

This is an altogether easier integral to compute than the one we started with. The only caveat is the negative sign in front of the r in the pull-back of the area form $dx \wedge dy$, which reflects the change in orientation.

Question 7.4 Do this integration. Is the answer the same as in Question 7.2, if you managed to complete that integration?

7.3.2 Cylindrical Coordinates Example

Now we are going to do an integration problem where a change of variables to cylindrical coordinates makes the integral easier. Suppose we want to integrate the function $f(x, y) = x^2 + y^2$ in the region bounded by the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 2$. In other words, we want to find the integral

$$
\int_R (x^2 + y^2) \; dx \, dy \, dz.
$$

In order to draw a picture representing this integral accurately we would need four dimensions, so we will not try to do that. Simply imagine a cartoon along the lines of Fig. 7.1. One way of writing the region R in Cartesian coordinates would be

$$
R = \Big\{ -2 \leq x \leq 2, \;\; -\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2}, \;\; \sqrt{x^2 + y^2} \leq z \leq 2 \Big\},
$$

which results in the integral

$$
\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2 + y^2) \; dz \, dy \, dx.
$$

However, like before, we can tell that doing this integral would be quite complicated.

Question 7.5 Show that the region bounded by the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 2$ can indeed be written as $\{-2 \leq x \leq 2, \;\; -\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2}, \;\; \sqrt{x^2 + y^2} \leq z \leq 2\}$.

Question 7.6 Attempt this calculation with the techniques you remember from calculus class without performing a change of variable and see how far you get.

But when we consider the cylindrical coordinate change

$$
\begin{aligned}
\mathbb{R}^3_{r\theta z} &\xrightarrow{\phi^{-1}} \mathbb{R}^3_{xyz} \\
(r, \theta, z) &\longmapsto (r\cos\theta, \;\; r\sin\theta, \;\; z)
\end{aligned}
$$

it is fairly obvious that the region has a much simpler description in cylindrical coordinates,

$$
\phi(R) = \Big\{ 0 \leq \theta \leq 2\pi, \;\; 0 \leq r \leq 2, \;\; r \leq z \leq 2 \Big\}.
$$

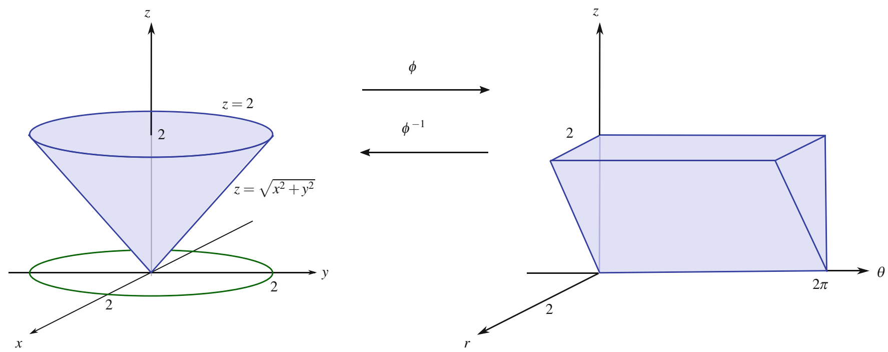

Fig. 7.13 The affect of the mapping $\phi : \mathbb{R}^3_{xyz} \longrightarrow \mathbb{R}^2_{r\theta z}$ on region we are integrating over

Question 7.7 Show that the region $\phi(R)$ is indeed $\{0 \le \theta \le 2\pi, \quad 0 \le r \le 2, \quad r \le z \le 2\}$.

We point out that to keep our notation in this section consistent with the last section, we are denoting the mapping from $\mathbb{R}^3_{r\theta z}$ to $\mathbb{R}^3_{xyz}$ as ϕ^{-1} instead of ϕ as we would have in the last chapter. As long as we restrict our mappings to the appropriate domain of $\mathbb{R}^3_{r\theta z}$ so our inverse is well-defined then this should hopefully not cause any confusion. See Fig. 7.13. We also have that $f \circ \phi^{-1} = r^2$ and $T^*\phi^{-1} \cdot (dx \wedge dy \wedge dz) = -r d\theta \wedge dr \wedge dz$.

Question 7.8 Find both $f \circ \phi^{-1}$ and $T^*\phi^{-1} \cdot (dx \wedge dy \wedge dz)$.

Finally, using the identity developed in the last section we get

$$\begin{aligned}
\int_R f \, dx \wedge dy \wedge dz &= \int_{\phi(R)} f \circ \phi^{-1} \;\; T^*\phi^{-1} \cdot \big(dx \wedge dy \wedge dz\big) \\
&= \int_{\phi(R)} (r^2) \;\; (-r) d\theta \wedge dr \wedge dz \\
&= \int_{\phi(R)} (r^2) \;\; r d\theta \wedge dz \wedge dr \\
&= \int_0^{2\pi} \int_0^2 \int_r^2 r^3 \;\; d\theta \, dz \, dr.
\end{aligned}$$

Question 7.9 Complete this integration.

7.3.3 Spherical Coordinates Example

We finish off this section with an example of a coordinate change using spherical coordinates. Suppose we wanted to integrate the function $f(x, y, z) = e^{(x^2+y^2+z^2)^{3/2}}$ over the unit ball

$$R = \Big\{x^2 + y^2 + z^2 \le 1\Big\}.$$

One integral that would do this is

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} e^{(x^2+y^2+z^2)^{3/2}} \;\; dz \, dy \, dx,$$

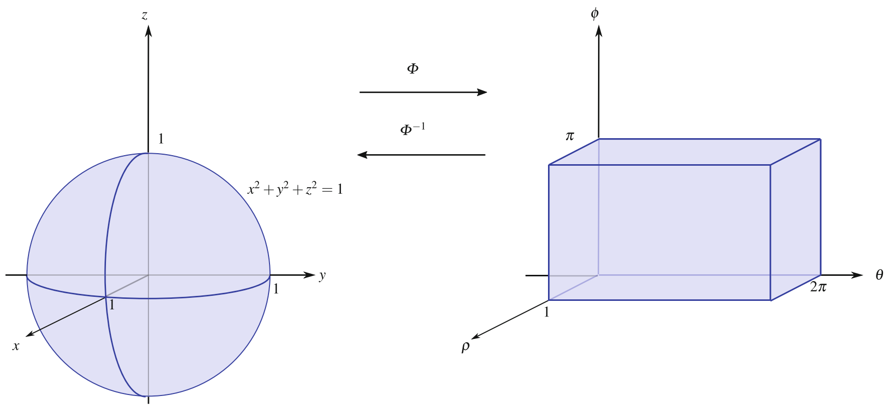

Fig. 7.14 The effect of the mapping $\Phi : \mathbb{R}^3_{xyz} \longrightarrow \mathbb{R}^2_{\rho\theta\phi}$ on region we are integrating over

which would clearly be difficult to do at best. Under the spherical coordinate transformation

$$\mathbb{R}^3_{\rho\theta\phi} \xrightarrow{\Phi^{-1}} \mathbb{R}^3_{xyz}$$
$$(\rho, \theta, \phi) \longmapsto (\rho \sin\phi \cos\theta, \quad \rho \sin\phi \sin\theta, \quad \rho \cos\phi)$$

our region becomes

$$\Phi(R) = \Big\{ 0 \le \rho \le 1, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi \Big\}.$$

Since ϕ is one of the standard variables used in spherical coordinates, we have denoted our mapping by the upper-case Φ. See Fig. 7.14.

Question 7.10 Show that the region $\Phi(R)$ is indeed $\{0 \le \rho \le 1, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi\}$.

We also get $f \circ \Phi^{-1} = e^{(\rho^2)^{3/2}} = e^{\rho^3}$ and $T^*\Phi^{-1} \cdot (dx \wedge dy \wedge dz) = -\rho^2 \sin\phi \, d\rho \wedge d\theta \wedge d\phi$. Again, using the identity developed in the last section we have

$$\begin{aligned}
\int_R f \, dx \wedge dy \wedge dz &= \int_{\Phi(R)} f \circ \Phi^{-1} \quad T^*\Phi^{-1} \cdot \big(dx \wedge dy \wedge dz\big) \\
&= \int_{\phi(R)} \left(e^{\rho^3}\right) \ \left(-\rho^2 \sin\phi\right) \, d\rho \wedge d\theta \wedge d\phi \\
&= \int_0^{\pi} \int_0^{2\pi} \int_0^1 -\rho^2 e^{\rho^3} \sin\phi \ \ d\rho \, d\theta \, d\phi.
\end{aligned}$$

Question 7.11 Complete this integration.

7.4 Integration of Differential Forms on Parameterized Surfaces

So far we have looked at a specific kind of integration problem, problems where the region we want to integrate over has a much simpler representation in some other coordinate system. There are, however, other kinds of integration problems we can do using differential forms.

As long as we have an n-form on an n-dimensional manifold we can integrate it. For example, we can integrate one-forms on curves, two-forms on surfaces, three-forms on three-dimensional spaces, et cetera. As long as we have a parametrization of the manifold $\Sigma_k \subset \mathbb{R}^n$, $k < n$, that is, an invertible mapping $\phi^{-1} : U \subset \mathbb{R}^k \longrightarrow \Sigma_k \subset \mathbb{R}^n$, we can integrate an k-form on Σ_k by pulling the k-form back to $U \subset \mathbb{R}^k$ using $T^*\phi^{-1}$ and integrating there. It is a little unusual to name a parametrisation with ϕ^{-1} instead of ϕ but we want to keep notation consistent with the last chapter. The rest of this section will be about integrating forms over submanifolds of $\mathbb{R}^n$ that are given as parameterized surfaces Σ_k. The equality we had developed earlier was

$$\int_R f\, dx_1 \wedge \cdots \wedge dx_n = \int_{\phi(R)} f \circ \phi^{-1}\; T^*\phi^{-1} \cdot \big(dx_1 \wedge \cdots \wedge dx_n\big).$$

The version of this equality we generally want to use is

$$\boxed{\begin{array}{c}\text{Integral of}\\ k\text{–Form on}\\ k\text{–Dimensional}\\ \text{Parameterized}\\ \text{Surface } \Sigma_k\end{array} \qquad \int_{\Sigma_k} \alpha = \int_{\phi(\Sigma_k)} T^*\phi^{-1} \cdot \alpha}$$

When $k = 1$ then we have a curve and we we often write $\Sigma_1 = C$.

Question 7.12 Define this formula using an argument involving Riemann sums similar to the argument in Sect. 7.1.

But before doing that we also want to reiterate, you can, and often do, integrate all sorts of things that are not differential forms. For example, to find the surface area of a 2-dimensional surface in $\mathbb{R}^3$ you would need to integrate

$$\sqrt{(dx \wedge dy)^2 + (dy \wedge dz)^2 + (dz \wedge dx)^2}$$

which is not a differential form since it is not multi-linear, even though it is composed of the differential forms $dx \wedge dy$, $dy \wedge dz$, and $dz \wedge dx$. However, for simple surfaces sometimes integrals like this can still be done by hand. Otherwise a numerical answer can generally be found using numerical methods. One of the things about forms that makes them so useful is the generalized Stokes' theorem. However, the generalized Stokes' theorem does not apply to integrands that are not forms.

7.4.1 Line Integrals

Here we will look at some examples that involve integrating one-forms on a one-dimensional manifold. We will make use of the identity

$$\int_C \alpha = \int_{\phi(C)} T^*\phi^{-1} \cdot \alpha.$$

Example One
We begin with an example in $\mathbb{R}^2$. We will integrate the one-form $y^2 dx + x dy$ along two different curves. The first curve C_1 will be the line segment from $(-5, -3)$ to $(0, 2)$. The second curve C_2 will be the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(0, 2)$. We begin with the integral along C_1. In order to do this we first of all parameterize the path C_1 by $\phi^{-1}(t) = \big(x(t), y(t)\big) = (5t - 5, 5t - 3)$. Letting $R = [0, 1] \subset \mathbb{R}$ we have $\phi^{-1}(R) = \big\{(x, y) \mid x = 5t - 5,\ y = 5t - 3,\ \ 0 \le t \le 1\big\} = C_1$. See Fig. 7.15. Using the mapping ϕ^{-1} we find $T\phi^{-1}$,

$$T\phi^{-1} = \begin{bmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

so we have the induced tangent and cotangent mappings,

$$\begin{array}{rcl} T_p(\mathbb{R}) & \xrightarrow{T\phi^{-1}} & T_{\phi^{-1}(p)}\left(\mathbb{R}^2\right) \\ [1] \in T_p(\mathbb{R}) & \longmapsto & T\phi^{-1} \cdot [1] = \begin{bmatrix} 5 \\ 5 \end{bmatrix} [1] = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \in T_{\phi^{-1}(p)}\left(\mathbb{R}^2\right) \end{array}$$

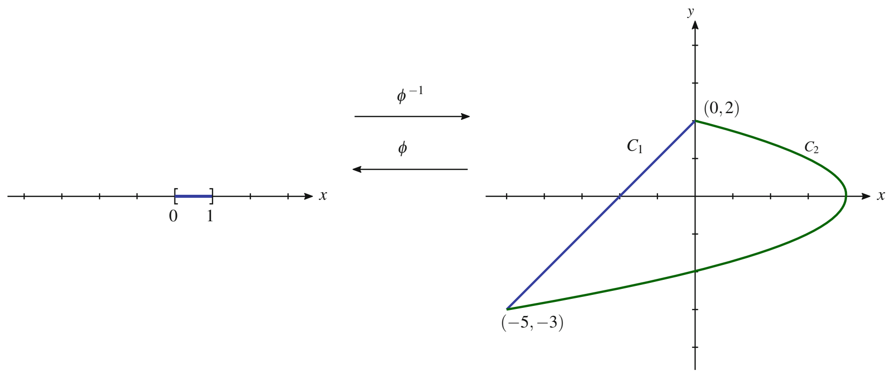

Fig. 7.15 The mapping ϕ^{-1} sends $[0, 1]$ to curve C_1

$$\begin{array}{ccc} T_p^*(\mathbb{R}) & \xleftarrow{T^*\phi^{-1}} & T_{\phi^{-1}(p)}\left(\mathbb{R}^2\right) \\ f(t)dt = T^*\phi^{-1}\cdot(y^2dx + xdy) & \longleftarrow & y^2dx + xdy \end{array}$$

for some $f(t)$. This function of t is what we want to find. Notice that $f(t)dt([1]) = f(t)$ so we have

$$\begin{aligned} f(t) &= f(t)dt([1]) \\ &= \left(T^*\phi^{-1}\cdot\left(y^2dx + xdy\right)\right)([1]) \\ &= \left(y^2dx + xdy\right)\left(T\phi^{-1}\cdot[1]\right) \\ &= \left(y^2dx + xdy\right)\left(\begin{bmatrix}5\\5\end{bmatrix}\right) \\ &= y^2dx\left(\begin{bmatrix}5\\5\end{bmatrix}\right) + xdy\left(\begin{bmatrix}5\\5\end{bmatrix}\right) \\ &= y^2(5) + x(5) \\ &= (5t-3)^2(5) + (5t-5)(5) \\ &= 125t^2 - 125t + 20 \end{aligned}$$

so we have

$$T^*\phi^{-1}\cdot\left(y^2dx + xdy\right) = (125t^2 - 125t + 20)dt.$$

Using the following equality

$$\int_C \alpha = \int_{\phi(C)} T^*\phi^{-1}\cdot\alpha$$

our integral becomes

$$\begin{aligned} \int_{C_1}\left(y^2dx + xdy\right) &= \int_{[0,1]}\left(125t^2 - 125t + 20\right)\,dt \\ &= \int_0^1\left(125t^2 - 125t + 20\right)\,dt \\ &= \frac{-5}{6}. \end{aligned}$$

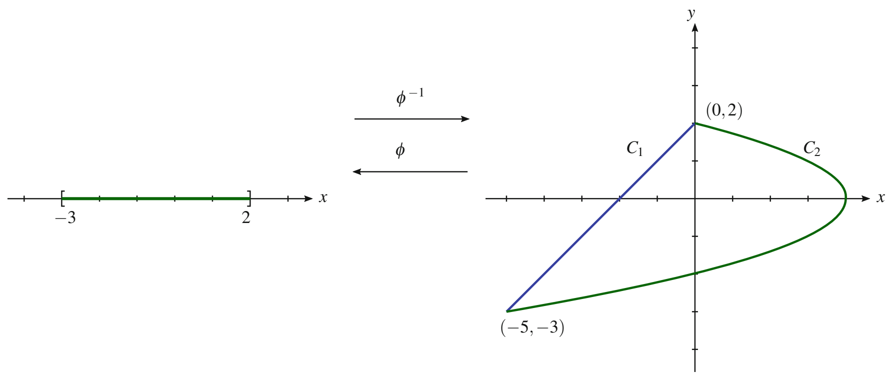

Fig. 7.16 The mapping ϕ^{-1} sends $[-3, 2]$ to curve C_2

Now we will find the integral of the one-form $y^2dx + xdy$ along the second path C_2.In order to do this we first of all parameterize the path C_2 by $\phi^{-1}(t) = \big(x(t), y(t)\big) = (4-t^2, t)$. Letting $R = [-3, 2] \subset \mathbb{R}$ we have $\phi^{-1}(R) = \{\ (x, y) \mid x = 4 - t^2,\ \ y = t,\ \ -3 \le t \le 2\} = C_2$. See Fig. 7.16. Using the mapping ϕ^{-1} we find $T\phi^{-1}$,

$$T_p\phi^{-1} = \begin{bmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \end{bmatrix}_p = \begin{bmatrix} -2t \\ 1 \end{bmatrix}_p$$

which is actually the mapping

$$\begin{array}{ccc} T_p(\mathbb{R}) & \xrightarrow{T\phi^{-1}} & T_{\phi^{-1}(p)}\left(\mathbb{R}^2\right) \\ [1] \in T_p(\mathbb{R}) & \longmapsto & T_p\phi^{-1} \cdot [1] = \begin{bmatrix} -2t \\ 1 \end{bmatrix}[1] = \begin{bmatrix} -2t \\ 1 \end{bmatrix} \in T_{\phi^{-1}(p)}\left(\mathbb{R}^2\right). \end{array}$$

From this we can clearly see the mapping $T_p\phi^{-1}$ depends on the point our vector is at. To make this more concrete we do a few examples showing what the vector [1] based at differing points gets pushed to.

$$[1] \in T_{-3}(\mathbb{R}) \implies T_{-3}\phi^{-1} \cdot [1] = \begin{bmatrix} -2(-3) \\ 1 \end{bmatrix}_{-3} \cdot [1] = \begin{bmatrix} 6 \\ 1 \end{bmatrix} \in T_{(-5,-3)}\left(\mathbb{R}^2\right),$$

$$[1] \in T_{-2}(\mathbb{R}) \implies T_{-2}\phi^{-1} \cdot [1] = \begin{bmatrix} -2(-2) \\ 1 \end{bmatrix}_{-2} \cdot [1] = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \in T_{(0,-2)}\left(\mathbb{R}^2\right),$$

$$[1] \in T_{-1}(\mathbb{R}) \implies T_{-1}\phi^{-1} \cdot [1] = \begin{bmatrix} -2(-1) \\ 1 \end{bmatrix}_{-1} \cdot [1] = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \in T_{(3,-1)}\left(\mathbb{R}^2\right),$$

et cetera. We also have the cotangent mapping

$$\begin{array}{ccc} T_p^*(\mathbb{R}) & \xleftarrow{T^*\phi^{-1}} & T_{\phi^{-1}(p)}\left(\mathbb{R}^2\right) \\ f(t)dt = T^*\phi^{-1} \cdot (y^2dx + xdy) & \longleftarrow & y^2dx + xdy \end{array}$$

for some $f(t)$. This $f(t)$ is what we want to find. Similar to before we have

$$\begin{aligned} f(t) &= f(t)dt([1]) \\ &= \left(T^*\phi^{-1} \cdot \left(y^2dx + xdy\right)\right)([1]) \end{aligned}$$

$$
\begin{aligned}
&= \left(y^2dx + xdy\right)\left(T\phi^{-1}\cdot[1]\right)\\
&= \left(y^2dx + xdy\right)\left(\begin{bmatrix} -2t \\ 1 \end{bmatrix}\right)\\
&= y^2dx\left(\begin{bmatrix} -2t \\ 1 \end{bmatrix}\right) + xdy\left(\begin{bmatrix} -2t \\ 1 \end{bmatrix}\right)\\
&= y^2(-2t) + x(1)\\
&= (t)^2(-2t) + (4-t^2)(1)\\
&= -2t^3 - t^2 + 4
\end{aligned}
$$

which means that

$$T^*\phi^{-1}\cdot\left(y^2dx + xdy\right) = \left(-2t^3 - t^2 + 4\right)dt.$$

Again, using the following equality

$$\int_C \alpha = \int_{\phi(C)} T^*\phi^{-1}\cdot\alpha$$

our integral becomes

$$
\begin{aligned}
\int_{C_2}\left(y^2dx + xdy\right) &= \int_{[-3,2]}\left(-2t^3 - t^2 + 4\right)\,dt\\
&= \int_{-3}^{2}\left(-2t^3 - t^2 + 4\right)\,dt\\
&= 40\frac{5}{6}.
\end{aligned}
$$

Notice that our answers are not the same. Even though the start point and the end point for paths C_1 and C_2 are the same the integral of the one-form $y^2dx + xdy$ depends on the path taken.

Example Two

In this example we will do something similar to what we did in example one. Recall that in example one we integrated a one-form along two different curves C_1 and C_2 that had the same starting point and ending point and discovered that the two integrals were different, that is, the integral depended on the path taken. We will repeat this using a different one-form. The one-form we integrate this time will be a special kind of one-form, it will be the exterior derivative of a zero-form.

We will choose the zero-form (function) on $\mathbb{R}^2$, $f(x,y) = x^2y$. Taking the exterior derivative of this zero-form gives us the one-form

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 2xy\,dx + x^2\,dy.$$

This is the one-form that we will integrate. We will use the same paths and parameterizations as in example one. The procedure is essentially the same as the previous examples. We know that $T^*\phi^{-1}\cdot\left(2xydx + x^2dy\right)$ is a one-form on $\mathbb{R}$ and so it must take the form $f(t)dt$ for some function $f(t)$. For curve C_1 we find

$$
\begin{aligned}
f(t) &= f(t)dt([1])\\
&= \left(T^*\phi^{-1}\cdot\left(2xydx + x^2dy\right)\right)([1])\\
&= \left(2xydx + x^2dy\right)\left(\begin{bmatrix} 5 \\ 5 \end{bmatrix}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= 10xy + 5x^2 \\
&= 10(5t-5)(5t-3) + 5(5t-5)^2 \\
&= 5(75t^2 - 130t + 55)
\end{aligned}
$$

giving us $T^*\phi^{-1} \cdot \big(2xydx + x^2dy\big) = 5(75t^2 - 130t + 55)\ dt$. We use the identity $\int_{C_1} \alpha = \int_{\phi(C_1)} T^*\phi^{-1} \cdot \alpha$ to get

$$
\begin{aligned}
\int_{C_1} \Big(2xydx + x^2dy\Big) &= \int_{\phi(C_1)} T^*\phi^{-1} \cdot \Big(2xydx + x^2dy\Big) \\
&= \int_{[0,1]} 5(75t^2 - 130t + 55)\ dt \\
&= 75.
\end{aligned}
$$

Now we take the integral along path C_2. Similarly we find

$$
\begin{aligned}
f(t) &= f(t)dt([1]) \\
&= \Big(T^*\phi^{-1} \cdot \Big(2xydx + x^2dy\Big)\Big)([1]) \\
&= \Big(2xydx + x^2dy\Big)\left(\begin{bmatrix} -2t \\ 1 \end{bmatrix}\right) \\
&= 2xy(-2t) + x^2(1) \\
&= 2(4-t^2)(t)(-2t) + (4-t^2)^2 \\
&= 5t^4 - 24t^2 + 16
\end{aligned}
$$

giving us $T^*\phi^{-1} \cdot \big(2xydx + x^2dy\big) = (5t^4 - 24t^2 + 16)\ dt$. The integral is

$$
\begin{aligned}
\int_{C_2} \Big(2xydx + x^2dy\Big) &= \int_{\phi(C_2)} T^*\phi^{-1} \cdot \Big(2xydx + x^2dy\Big) \\
&= \int_{[-3,2]} (5t^4 - 24t^2 + 16)\ dt \\
&= 75.
\end{aligned}
$$

Something interesting has just happened. When our one-form was the exterior derivative of a zero-form then the result of our integration does not seem to depend on the path taken. In fact this is true in general, not just in this example. We will look at this in more detail later on.

Example Three
Our next example involves integrating the one-form $ydx + zdy + xdz$ along the straight path C in $\mathbb{R}^3$ from the point $(2, 0, 0)$ to the point $(3, 4, 5)$. We find a parameterizations $\phi^{-1} : \mathbb{R} \to \mathbb{R}^3$ of this path given by $\phi^{-1}(t) = \big(x(t), y(t), z(t)\big) = (2+t, 4t, 5t)$. Letting $R = [0, 1] \subset \mathbb{R}$ we have $\phi^{-1}(R) = \Big\{ (x, y, z) \ \Big|\ x = 2+t, y = 4t, z = 5t,\ \ 0 \le t \le 1 \Big\} = C$. This gives

$$
T_p\phi^{-1} = \begin{bmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial t} \end{bmatrix}_p = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}.
$$

Clearly, $T_p\phi^{-1}$ does not depend on the base point $p \in R$. As before we know $T^*\phi^{-1} \cdot (ydx + zdy + xdz) = f(t)dt$ for some $f(t)$, which we wish to find. The computation proceeds just like before

$$\begin{aligned}
f(t) &= f(t)dt([1]) \\
&= \left(T^*\phi^{-1} \cdot (ydx + zdy + xdz)\right)([1]) \\
&= (ydx + zdy + xdz)\left(T\phi^{-1} \cdot [1]\right) \\
&= (ydx + zdy + xdz)\left(\begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} \cdot [1]\right) \\
&= ydx\left(\begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}\right) + zdy\left(\begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}\right) + xdz\left(\begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}\right) \\
&= y(1) + z(4) + x(5) \\
&= (4t) + 4(5t) + 5(2 + t) \\
&= 29t + 10
\end{aligned}$$

giving us $T^*\phi^{-1} \cdot (ydx + zdy + xdz) = (29t + 10)dt$. Using $\int_C \alpha = \int_{\phi(C)} T^*\phi^{-1} \cdot \alpha$ we have the integral

$$\begin{aligned}
\int_C (ydx + zdy + xdz) &= \int_{\phi(C)} T^*\phi^{-1} \cdot (ydx + zdy + xdz) \\
&= \int_{[0,1]} (29t + 10)\, dt \\
&= 24\frac{1}{2}.
\end{aligned}$$

7.4.2 Surface Integrals

Now we will do some examples where we integrate a two-form over a parameterized two-dimensional surface.

Example One
We will integrate the two-form $z^2 dx \wedge dy$ over the top half of the unit sphere. We will use the parametrization given by

$$\begin{aligned}
\phi^{-1}(r,\theta) &= \big(x(r,\theta), y(r,\theta), z(r,\theta)\big) \\
&= \big(r\cos\theta, r\sin\theta, \sqrt{1-r^2}\big).
\end{aligned}$$

Letting $R = \left\{ (r,\theta) \;\middle|\; 0 \le r \le 1, 0 \le \theta \le 2\pi \right\}$ we have that $\phi^{-1}(R)$ is the top half of the unit sphere. See Fig. 7.17. First we find $T\phi^{-1}$,

$$T\phi^{-1} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} \end{bmatrix}$$

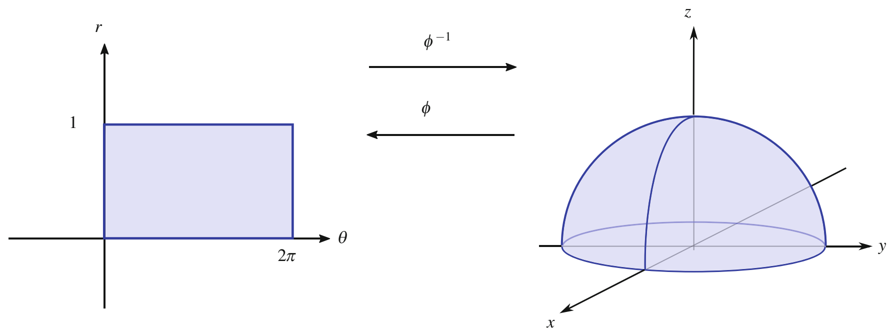

Fig. 7.17 The mapping ϕ^{-1} sends $R = \{ (r,\theta) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi \}$ to the top of the unit sphere in $\mathbb{R}^3$

$$= \begin{bmatrix} \frac{\partial(r\cos\theta)}{\partial r} & \frac{\partial(r\cos\theta)}{\partial \theta} \\ \frac{\partial(r\sin\theta)}{\partial r} & \frac{\partial(r\sin\theta)}{\partial \theta} \\ \frac{\partial(\sqrt{1-r^2})}{\partial r} & \frac{\partial(\sqrt{1-r^2})}{\partial \theta} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \\ \frac{-r}{\sqrt{1-r^2}} & 0 \end{bmatrix}.$$

Now we want to find $T^*\phi^{-1} \cdot \left(z^2 dx \wedge dy\right)$. We know this is a two-form on $\mathbb{R}^2$ so it must have the form $g(r,\theta) dr \wedge d\theta$ for some function $g(r,\theta)$. Our goal is to find that function. Notice that

$$g(r,\theta) dr \wedge d\theta \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = g(r,\theta) \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = g(r,\theta).$$

Therefore,

$$\begin{aligned} g(r,\theta) &= g(r,\theta) dr \wedge d\theta \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \\ &= T^*\phi^{-1} \cdot \left(z^2 dx \wedge dy\right) \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \\ &= z^2 dx \wedge dy \left(T\phi^{-1} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}, T\phi^{-1} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \\ &= z^2 dx \wedge dy \left(\begin{bmatrix} \cos\theta \\ \sin\theta \\ \frac{-r}{\sqrt{1-r^2}} \end{bmatrix}, \begin{bmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{bmatrix} \right) \\ &= z^2 \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \\ &= z^2\, r \\ &= \left(\sqrt{1-r^2}\right)^2\, r \\ &= r - r^3 \\ \implies \quad T^*\phi^{-1} \cdot \left(z^2 dx \wedge dy\right) &= (r - r^3)\, dr \wedge d\theta. \end{aligned}$$

Using the identity $\int_R \alpha = \int_{\phi(R)} T^*\phi^{-1} \cdot \alpha$ we have

$$\begin{aligned}
\int_R z^2\, dx \wedge dy &= \int_{\phi(R)} (r - r^3)\, dr \wedge d\theta \\
&= \int_0^{2\pi} \int_0^1 (r - r^3)\, dr d\theta \\
&= \int_0^{2\pi} \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 d\theta \\
&= \int_0^{2\pi} \frac{1}{4}\, d\theta \\
&= \frac{\pi}{2}.
\end{aligned}$$

Example Two

Now we will integrate the two-form $\alpha = \frac{1}{x} dy \wedge dz - \frac{1}{y} dx \wedge dz$ on the top half of the unit sphere using the following three parameterizations. We will do the first parametrization and leave the other two as an exercise.

(a) $(r, \theta) \mapsto \left(r\cos\theta, r\sin\theta, \sqrt{1 - r^2} \right), \quad 0 \le \theta \le 2\pi,\ 0 \le r \le 1$
(b) $(\theta, \phi) \mapsto (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\theta), \quad 0 \le \theta \le 2\pi,\ 0 \le \phi \le \frac{\pi}{2}$
(c) $(x, y) \mapsto \left(x, y, \sqrt{1 - x^2 - y^2} \right), \quad \sqrt{x^2 + y^2} \le 1$

We should expect that we will get the same answer regardless of the parameterizations. We now proceed with the first parametrization which follows example one, except that we have

$$g(r,\theta) dr \wedge d\theta = T^*\phi^{-1} \cdot \left(\frac{1}{x} dy \wedge dz - \frac{1}{y} dx \wedge dz \right) \xleftarrow{T^*\phi^{-1}} \frac{1}{x} dy \wedge dz - \frac{1}{y} dx \wedge dz.$$

Wanting to find $g(r, \theta)$ we let e_1 and e_2 be the unit vectors in the r and θ directions respectively, so we have

$$\begin{aligned}
g(r,\theta) &= g(r,\theta)\, dr \wedge d\theta\, (e_1, e_2) \\
&= T^*\phi^{-1} \cdot \left(\frac{1}{x} dy \wedge dz - \frac{1}{y} dx \wedge dz \right) (e_1, e_2) \\
&= \left(\frac{1}{x} dy \wedge dz - \frac{1}{y} dx \wedge dz \right) (T\phi \cdot e_1, T\phi \cdot e_2) \\
&= \left(\frac{1}{x} dy \wedge dz - \frac{1}{y} dx \wedge dz \right) \left(\begin{bmatrix} \cos\theta \\ \sin\theta \\ \frac{-r}{\sqrt{1-r^2}} \end{bmatrix}, \begin{bmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{bmatrix} \right) \\
&= \frac{1}{x} dy \wedge dz \left(\begin{bmatrix} \cos\theta \\ \sin\theta \\ \frac{-r}{\sqrt{1-r^2}} \end{bmatrix}, \begin{bmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{bmatrix} \right) - \frac{1}{y} dx \wedge dz \left(\begin{bmatrix} \cos\theta \\ \sin\theta \\ \frac{-r}{\sqrt{1-r^2}} \end{bmatrix}, \begin{bmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{bmatrix} \right) \\
&= \frac{1}{x} \begin{vmatrix} \sin\theta & r\cos\theta \\ \frac{-r}{\sqrt{1-r^2}} & 0 \end{vmatrix} - \frac{1}{y} \begin{vmatrix} \cos\theta & -r\sin\theta \\ \frac{-r}{\sqrt{1-r^2}} & 0 \end{vmatrix} \\
&= \frac{1}{x} \left(\frac{r^2\cos\theta}{\sqrt{1-r^2}} \right) - \frac{1}{y} \left(\frac{-r^2\sin\theta}{\sqrt{1-r^2}} \right) \\
&= \frac{1}{r\cos\theta} \left(\frac{r^2\cos\theta}{\sqrt{1-r^2}} \right) - \frac{1}{r\sin\theta} \left(\frac{-r^2\sin\theta}{\sqrt{1-r^2}} \right)
\end{aligned}$$

$$= \frac{2r}{\sqrt{1-r^2}}$$

$$\implies T^*\phi^{-1} \cdot \left(\frac{1}{x} dy \wedge dz - \frac{1}{y} dx \wedge dz \right) = \frac{2r}{\sqrt{1-r^2}} dr \wedge d\theta.$$

Now we use the identity $\int_R \alpha = \int_{\phi(R)} T^*\phi^{-1} \cdot \alpha$ to take the integral,

$$\begin{aligned}
&\int_R \left(\frac{1}{x} dy \wedge dz - \frac{1}{y} dx \wedge dz \right) \\
&= \int_{\phi(R)} \frac{2r}{\sqrt{1-r^2}} \, dr \wedge d\theta \\
&= \int_0^{2\pi} \int_0^1 \frac{2r}{\sqrt{1-r^2}} \, dr d\theta \\
&= 2 \left(\int_0^{2\pi} d\theta \right) \cdot \left(\int_0^1 \frac{2r}{\sqrt{1-r^2}} \, dr \right) \\
&\qquad \text{substitution } u = 1 - r^2, \; dr = \frac{du}{-2r} \\
&= 2 \cdot 2\pi \cdot \left(\frac{-1}{2} \int \frac{1}{\sqrt{u}} \, du \right) \\
&= 4\pi \cdot \left(-\sqrt{u} \right) \\
&= 4\pi \cdot \left[-\sqrt{1-r^2} \right]_0^1 \\
&= 4\pi.
\end{aligned}$$

Question 7.13 Complete this example by finding the integral using the second and third parameterizations.

7.5 Summary, References, and Problems

7.5.1 Summary

If finding the integral $\int_R f(x_1, x_2 \ldots, x_n) dx_1 dx_2 \cdots dx_n$ is difficult but it would become easier by doing the change of coordinates

$$\begin{aligned}
\mathbb{R}^n_{x_1 x_2 \cdots x_n} &\xrightarrow{\phi} \mathbb{R}^n_{\phi_1 \phi_2 \cdots \phi_n} \\
(x_1, x_2, \ldots, x_n) &\longmapsto (\phi_1, \phi_2, \ldots, \phi_n)
\end{aligned}$$

then the change of coordinates formula is used

Change of coordinates formula	$\int_R f(x_1, \ldots, x_n) \; dx_1 \wedge \ldots \wedge dx_n = \int_{\phi(R)} f \circ \phi^{-1}(\phi_1, \ldots, \phi_n) \; T^*\phi^{-1} \cdot (dx_1 \wedge \ldots \wedge dx_n)$

The left hand side takes place in $x_1 \cdots x_n$-coordinates and we are integrating the function $f(x_1, \ldots, x_n)$ over the region R using the volume form $dx_1 \wedge \cdots \wedge dx_n$ associated with the $x_1 \cdots x_n$-coordinates. On the right hand side the function $f \circ \phi^{-1} : \mathbb{R}^n \to \mathbb{R}$ is a function of the variables $\phi_1, \phi_2, \cdots \phi_n$. The region we are integrating over in $\mathbb{R}^n_{\phi_1 \phi_2 \cdots \phi_n}$ is the image

$\phi(R)$. Finally, the form on the right hand side is $T^*\phi^{-1} \cdot \big(dx_1 \wedge \cdots \wedge dx_n\big)$, the pull-back of the area form $dx_1 \wedge \cdots \wedge dx_n$ by ϕ^{-1} and NOT the area form $d\phi_1 \wedge \cdots \wedge d\phi_n$. This gives us exactly what you used in multivariable calculus when you used polar, cylindrical, and spherical changes of coordinates.

We can integrate one-forms on curves, two-forms on surfaces, three-forms on three-dimensional spaces, et cetera. As long as we have an n-form on an n-dimensional manifold we can integrate it. Given a parameterizations of the manifold $\Sigma_k \subset \mathbb{R}^n$, $k < n$, that is, a mapping $\phi^{-1} : U \subset \mathbb{R}^k \longrightarrow \Sigma_k \subset \mathbb{R}^n$, where ϕ^{-1} is invertible we can integrate an k-form on Σ_k by pulling the k-form back to $U \subset \mathbb{R}^k$ and integrating there. In this case the version of the change of coordinate formula we want to use is

Integral of $k-$Form on $k-$Dimensional Parameterized Surface Σ_k	$\displaystyle\int_{\Sigma_k} \alpha = \int_{\phi(\Sigma_k)} T^*\phi^{-1} \cdot \alpha$

When $k = 1$ then we have a curve and often write $\Sigma_1 = C$, when $k = 2$ then we have a surface and often write $\Sigma_2 = S$, and when $k = 3$ then we have a volume and often write $\Sigma_3 = V$.

7.5.2 *References and Further Reading*

Both Riemann sums and changes of variables are introduced and discussed, often at length, in virtually every book on calculus or introductory analysis, see for example Steward [43], Hubbard and Hubbard [27], or Marsden and Hoffman [31]. However, in this chapter we have generally followed the exposition of Bachman [4] at it relates directly to the differential forms case. Also see Watschap [47] or Renteln [37] for somewhat more theoretical introductions to the same material.

7.5.3 *Problems*

Question 7.14 Let C be an oriented curve in $\mathbb{R}^3$ parameterized by $\gamma(t) = \big(2t+1, t^2, t^3\big)$ for $1 \le t \le 3$ and let $\alpha = (3x-1)^2\, dx + 5\, dy + 2\, dz$ be a one-form on $\mathbb{R}^3$. Find $\int_C \alpha$.

Question 7.15 Let C be an oriented curve in $\mathbb{R}^3$ parameterized by $\gamma(t) = \big(1 - t^2 t^3 - t, 0\big)$ for $0 \le t \le 2$. Notice that the curve C lies in $\mathbb{R}^2 \subset \mathbb{R}^3$. Let $\alpha = -5xy\, dx + dz$ be a one-form on $\mathbb{R}^3$. Find $\int_C \alpha$.

Question 7.16 Let C be a curve in $\mathbb{R}^2$ parameterized by $\gamma(t) = \big(t^2, 2t+1\big)$ for $1 \le t \le 2$ and let $\alpha = 4x\, dx + ydy$ be a one-form on $\mathbb{R}^2$. Find $\int_C \alpha$.

Question 7.17 Let C be the curve in the plane $\mathbb{R}^2$ which is the graph of the function $y = 1 + x^2$ for $-1 \le x \le 1$, oriented in the direction of increasing x, and let $\alpha = y\, dx + x\, dy$ be a one-form on the plane. Find $\int_C \alpha$. (While not necessary, one could parameterize the curve C as $\gamma(t) = (t, 1 + t^2)$ for $-1 \le t \le 1$.)

Question 7.18 Let C_1 and C_2 be curves in the plane $\mathbb{R}^2$ parameterized by $\gamma_1(t) = \big(t-1, 2t-1\big)$ for $1 \le t \le 2$ and $\gamma_2(u) = \big(2-u, 5-2u\big)$ for $1 \le u \le 2$, respectively. Let $\alpha = y\, dx$ be a one-form on $\mathbb{R}^2$. Find $\int_{C_1} \alpha$ and $\int_{C_2} \alpha$. Compare the two answers and explain their relationship with each other.

Question 7.19 Let C be the unit circle in $\mathbb{R}^2$ parameterized by $\gamma(\theta) = \big(\cos(\theta), \sin(\theta)\big)$ for $0 \le \theta \le 2\pi$. Let $\alpha_1 = \frac{-y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dz$ and let $\alpha_2 = \frac{x}{x^2+y^2}\, dx + \frac{y}{x^2+y^2}\, dz$ be one-forms on $\mathbb{R}^2$. Find $\int_C \alpha_1$ and $\int_C \alpha_2$. Notice that one can write $\alpha_1 = -\sin(\theta)\, dx + \cos(\theta)\, dy$ and $\alpha_2 = \cos(\theta)\, dx + \sin(\theta)\, dy$.

Question 7.20 Let C be a curve in $\mathbb{R}^2$. The curve C can be parameterized by either $\gamma_1(t) = \big(t, \sqrt{1+t^2}\big)$ where $-1 \le t \le 1$ or $\gamma_1(\theta) = \big(\tan(\theta), \sec(\theta)\big)$ where $\frac{-\pi}{4} \le \theta \le \frac{\pi}{4}$. Let $\alpha = y^2\, dx$ be a one-form on $\mathbb{R}^2$. Find $\int_C \alpha$ using each parametrization of C.

Question 7.21 Let C be a curve in $\mathbb{R}^2$. The curve C can be parameterized as $\gamma_1(t) = \big(t, t^2\big),\ -2 \le t \le 2$ or as $\gamma_2(u) = \big(u^3 + u, u^6 + 2u^4 + u^2\big),\ -1 \le u \le 1$ or as $\gamma_3(\theta) = \big(4\sin(\theta), 16\sin^2(\theta)\big),\ \frac{-\pi}{6} \le \theta \le \frac{\pi}{6}$. If $\alpha = 5xy\, dy$ is a one-form on $\mathbb{R}^2$ find $\int_C \alpha$ using all three parameterizations.

Question 7.22 Consider the surface S in $\mathbb{R}^3$ parameterized by $\phi(x, y) = \big(x, y, x^3 + xy^2\big)$ for $1 \le x \le 3$, $1 \le y \le 2$ and the two-form $\beta = 2xyz\, dx \wedge dy + 6(2x^3 - y^3)\, dy \wedge dz + z\, dz \wedge dx$ on $\mathbb{R}^3$. Find $\int_S \beta$.

Question 7.23 Let U be the top half of the unit disk, $U = \big\{(x, y)\big| -1 \le x \le 1,\ 0 \le y \le \sqrt{1 - x^2}\big\}$. Let the surface S in $\mathbb{R}^3$ be parameterized by $\phi(x, y) = \big(x, y, x^2 + y^2\big)$, where $(x, y) \in U$ and let $\beta = 6y\, dx \wedge dy$ be a two-form on $\mathbb{R}^3$. Find $\int_S \beta$.

Question 7.24 Let U be the triangular region $U = \big\{(u, v)\big|0 \le u \le 1,\ 0 \le v \le 1 - u\big\}$ in $\mathbb{R}^2$. Let the surface S in $\mathbb{R}^3$ be parameterized by $\phi(u, v) = \big(u - v, uv, u^2 + (v + 1)^2\big)$ for $(u, v) \in U$ and let $\beta = -(1 + x^2)\, dx \wedge dy + dy \wedge dz$. Find $\int_S \beta$.

Question 7.25 Let $U = \big\{(u, v)\big| -1 \le u \le 1,\ -2 \le v \le 2\big\} \subset \mathbb{R}^2$ and let the surface S in $\mathbb{R}^3$ be parameterized by $\phi(u, v) = \big(u, v, u^3 + 3uv^2\big)$ where $(u, v) \in U$. Let $\beta = -dy \wedge dz + dz \wedge dx$. Find $\int_S \beta$.

Question 7.26 Let $U = \big\{(u, v)\big|0 \le u \le 1,\ 0 \le v \le 2\big\} \subset \mathbb{R}^2$ and let the surface S in $\mathbb{R}^3$ be parameterized by $\phi(u, v) = \big(u, v, u^2 + v^3\big)$ where $(u, v) \in U$. Let $\beta = 31x^2y^3\, dx \wedge dy + 5xz\, dy \wedge dz + 7yz\, dz \wedge dx$. Find $\int_S \beta$.

Question 7.27 Let $U = \big\{(u, v)\big|u^2 + v^2 \le 9\big\} \subset \mathbb{R}^2$ and let the surface S in $\mathbb{R}^3$ be parameterized by $\phi(u, v) = \big(u, v, 7 + (u^2 + v^2)\big)$ where $(u, v) \in U$. Let $\beta = z^2\, dx \wedge dy$. Find $\int_S \beta$.

Question 7.28 Let $U = \big\{(u, v)\big|u^2 + v^2 \le 9\big\} \subset \mathbb{R}^2$ and let the surface S in $\mathbb{R}^3$ be parameterized by $\phi(u, v) = \big(u, v, 25 - (u^2 + v^2)\big)$ where $(u, v) \in U$. Let $\beta = z^2\, dx \wedge dy$. Find $\int_S \beta$.

Question 7.29 Let $U = \big\{(u, v)\big|0 \le u \le 1,\ 0 \le v \le 2\big\} \subset \mathbb{R}^2$ and let the surface S in $\mathbb{R}^3$ be parameterized by $\phi(u, v) = \big(u + v, u - v^2, uv\big)$ where $(u, v) \in U$. Let $\beta = 6z\, dz \wedge dx$. Find $\int_S \beta$.

Question 7.30 Let $U = \big\{(u, v)\big|1 \le u \le 2,\ 0 \le v \le 2\pi\big\} \subset \mathbb{R}^2$ and let the surface S in $\mathbb{R}^3$ be parameterized by $\phi(u, v) = \big(u^3\cos(v), u^3\sin(v), u^2\big)$ where $(u, v) \in U$. Let $\beta = 2z\, dx \wedge dy - x\, dy \wedge dz - y\, dz \wedge dx$. Find $\int_S \beta$.

Question 7.31 Let $\alpha = xy\, dx \wedge dy$ be a two-form on $\mathbb{R}^2$ and let D be the disk centered at the origin with radius 3. Find $\int_D \alpha$ using a polar change of coordinates.

Question 7.32 Let $\alpha = (3x + 4y^2)\, dx \wedge dy$ be a two-form on $\mathbb{R}^2$ and let R be the region in the upper half plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$. Find $\int_R \alpha$ using a polar change of coordinates.

Question 7.33 Let $\alpha = (x + y)\, dx \wedge dy$ be a two-form on $\mathbb{R}^2$ and let R be the region to the left of the y-axis and bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$. Find $\int_R \alpha$ using a polar change of coordinates.

Question 7.34 Let $\alpha = \cos(x^2 + y^2)\, dx \wedge dy$ be a two-form on $\mathbb{R}^2$ and let R be the region above the x-axis and inside the circle $x^2 + y^2 = 4$. Find $\int_R \alpha$ using a polar change of coordinates.

Question 7.35 Let $V \subset \mathbb{R}^3$ be the region that lies within the cylinder $x^2 + y^2 = 1$ and between the planes $z = 0$ and $z = 5$ and let $\beta = x\, dx \wedge dy \wedge dz$ be a three-form on $\mathbb{R}^3$. Find $\int_V \beta$ using a cylindrical change of coordinates.

Question 7.36 Let $V \subset \mathbb{R}^3$ be the region that lies within the cylinder $x^2 + y^2 = 25$ and between the planes $z = -5$ and $z = 5$ and let $\beta = \sqrt{x^2 + y^2}\, dx \wedge dy \wedge dz$ be a three-form on $\mathbb{R}^3$. Find $\int_V \beta$ using a cylindrical change of coordinates.

Question 7.37 Let $V = \big\{(x, y, z)\big| -2 \le x \le 2,\ -\sqrt{4 - x^2} \le y \le \sqrt{4 - x^2},\ \sqrt{x^2 + y^2} \le z \le 2\big\} \subset \mathbb{R}^3$ and let $\beta = (x^2 + y^2)\, dx \wedge dy \wedge dz$ be a three-form on $\mathbb{R}^3$. Find $\int_V \beta$ using a cylindrical change of coordinates.

Question 7.38 Let $V \subset \mathbb{R}^3$ be the region that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y + 2 = 4$ and between the planes $z = 0$ and $z = x + 2$ and let $\beta = y\, dx \wedge dy \wedge dz$ be a three-form on $\mathbb{R}^3$. Find $\int_V \beta$ using a cylindrical change of coordinates.

Question 7.39 Let $V = \big\{(x, y, z)\big|x^2 + y^2 + z^2 \le 1\big\}$, the unit ball in $\mathbb{R}^3$, and let $\beta = e^{(x^2+y^2+z^2)^{3/2}}\, dx \wedge dy \wedge dz$ be a three-form on $\mathbb{R}^3$. Find $\int_V \beta$ using a spherical change of coordinates.

Question 7.40 Let $V = \big\{(x, y, z)\big|x^2 + y^2 + z^2 \le 1\big\}$, the unit ball in $\mathbb{R}^3$, and let $\beta = e^{(x^2+y^2+z^2)^{3/2}}\, dx \wedge dy \wedge dz$ be a three-form on $\mathbb{R}^3$. Find $\int_V \beta$ using a spherical change of coordinates.

Question 7.41 Let $V = \{(x, y, z) | x^2 + y^2 + z^2 \leq 5\}$, the ball in $\mathbb{R}^3$ with the origin as the center and radius 5, and let $\beta = (x^2 + y^2 + z^2)^2 \, dx \wedge dy \wedge dz$ be a three-form on $\mathbb{R}^3$. Find $\int_V \beta$ using a spherical change of coordinates.

Question 7.42 Find the volume of the region in $\mathbb{R}^3$ above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$. This means the three-form we want to integrate is the volume form $dx \wedge dy \wedge dz$. Use a spherical change of coordinates.

Chapter 8
Poincaré Lemma

Briefly put, the Poincaré lemma is as follows.

Theorem 8.1 (Poincaré Lemma) *Every closed form on* $\mathbb{R}^n$ *is exact.*

Section one gives a detailed introduction to the Poincaré lemma and what this means. Strictly speaking, the whole space $\mathbb{R}^n$ is not necessary. The proof works on any star-shaped region that is contractible to a point, but as this plays no essential role in the proof, to keep everything as clear as possible we will simply assume the manifolds are all of $\mathbb{R}^n$, $n \geq 0$.

The proof of the Poncareé lemma is based on induction. In case you have not yet had a class in mathematical proofs we will proceed slowly and carefully, trying to fully explain each step. While this is not a class on proof techniques, hopefully when you are done with this chapter you will have a good idea of what proofs by induction are like. Section two covers what is called the base case of the induction proof while section three covers the general case of the proof.

The Poincaré lemma plays an important role in a branch of mathematics called de Rham cohomology which is briefly introduced in Appendix B.

8.1 Introduction to the Poincaré Lemma

The Poincaré lemma states that *every closed form on* $\mathbb{R}^n$ *is exact*. A differential form α is called **closed** if $d\alpha = 0$. A differential form α is called **exact** if there is another differential form β such that $\alpha = d\beta$. Obviously, if α is an exact k-form then β must be a $(k-1)$-form. So, another way of phrasing the Poincaré lemma is to say that *if* α *is a* k*-form on* $\mathbb{R}^n$ *such that* $d\alpha = 0$, *then there exists some* $(k-1)$*-form* β *such that* $\alpha = d\beta$.

We begin by considering the exact forms on some manifold M. A k-form is called exact if it is equal to the exterior derivative of some $(k-1)$-form. Thus the set of all exact k-forms is exactly the set of all $d\alpha$ where α is a $(k-1)$-form. Another way of saying this is that the set of exact forms is the image under d of $\bigwedge^{k-1}(M)$. Figure 8.1 gives a Venn-like diagram of the mapping $d : \bigwedge^{k-1}(M) \to \bigwedge^{k}(M)$ to give a picture of the exact k-forms.

A k-form on M is called closed if the exterior derivative of that k-form is zero; that is, the zero $(k+1)$-form. The zero $(k+1)$-form is the form that sends every set of $k+1$ vectors to zero and should not be confused with the zero-forms, which are functions. We generally denote the zero k-form, at any point p, simply with 0. Thus the set of all closed k-forms are those k-forms α where $d\alpha = 0$. Another way of saying this is that the set of closed k-forms is the pre-image of the zero $(k+1)$-form under the mapping $d : \bigwedge^{k}(M) \to \bigwedge^{k+1}(M)$. Figure 8.2 uses a Venn-like diagram to give a picture of the closed k-forms.

Now the question becomes, how do closed k-forms and exact k-forms relate to each other. A first guess may be go along the lines of Fig. 8.3 where a number of different possibilities are shown:

- The exact forms are a subset of the closed forms.
- The exact forms are the same as the closed forms.
- The closed forms are a subset of the exact forms.
- The closed and exact forms are mutually exclusive.
- The closed and exact forms are not subsets of each other but are not mutually exclusive.

J. P. Fortney, *A Visual Introduction to Differential Forms and Calculus on Manifolds*,
https://doi.org/10.1007/978-3-319-96992-3_8

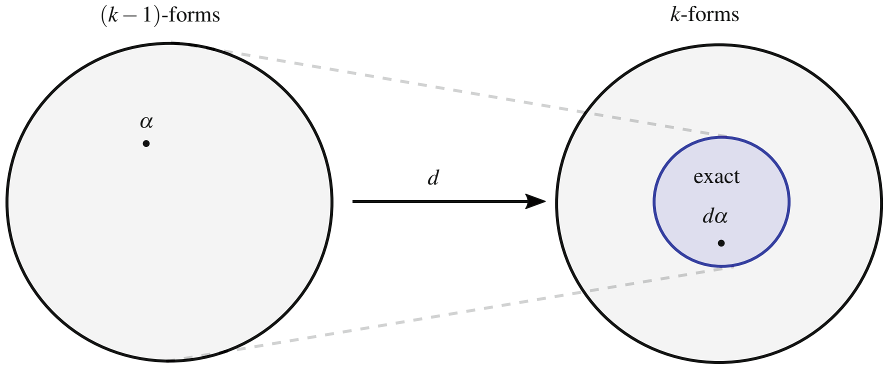

Fig. 8.1 The mapping $d : \bigwedge^{k-1}(M) \to \bigwedge^{k}(M)$ illustrated using a Venn-like diagram. The exact k-forms are the image, under d, of $\bigwedge^{k-1}(M)$

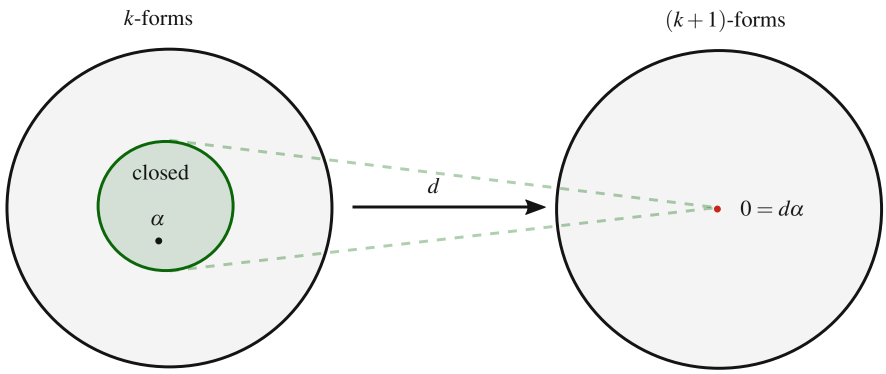

Fig. 8.2 The mapping $d : \bigwedge^{k}(M) \to \bigwedge^{k+1}(M)$ illustrated using a Venn-like diagram. The closed k-forms are the pre-image, under d, of the zero $(k+1)$-form 0

However, a little reflection will almost immediately eliminate the last three possibilities. All we have to do is recognize two facts. First, every exact k-form is $d\alpha$ for some $(k-1)$-form α, and second that no matter what the form is, we have $dd\alpha = 0$. This means that every exact form is also a closed form, thereby eliminating the last three possibilities of Fig. 8.3. That leaves the first two possibilities; Fig. 8.4.

So, which of the two possibilities in Fig. 8.4 is it? Are exact forms a subset of closed forms or are exact forms the same as closed forms? It turns out that the answer to this question depends on what manifold M the forms are on. The Poincaré lemma tells us that exact forms are the same as closed forms when the manifold is $\mathbb{R}^n$ for any number n. But if the manifold is something different from $\mathbb{R}^n$ then this may not be the case. The branch of mathematics called de Rham cohomology looks at how different the set of closed forms on M is from the set of exact forms on M and uses this information to figure out certain properties of the underlying manifold M.

Figure 8.5 is a commutative diagram showing all the spaces of forms on $\mathbb{R}^n$, which extends to infinity to the bottom and to the right. Clearly, exterior differentiation gives the sequence of maps

$$\bigwedge^0(\mathbb{R}^n) \xrightarrow{d} \bigwedge^1(\mathbb{R}^n) \xrightarrow{d} \bigwedge^2(\mathbb{R}^n) \xrightarrow{d} \bigwedge^3(\mathbb{R}^n) \xrightarrow{d} \cdots$$

for the manifold $\mathbb{R}^n$, for $n = 0, 1, 2, \ldots$. The map $\mathscr{K}$ is needed to prove the Poincaré lemma and will be explained in the next section. This gives the columns in the commutative diagram. The mappings

$$\bigwedge^k(\mathbb{R}^n) \underset{\mathscr{C}}{\overset{\mathscr{L}}{\rightleftarrows}} \bigwedge^k\left(\mathbb{R}^{n+1}\right),$$

where $k = 1, 2, \ldots$ and $n = 0, 1, 2, \ldots$ are mappings that we need to use to prove the Poincaré lemma and which will also be introduced in the next section. As we move through the proof of the Poincaré lemma we will refer back to this commutative diagram. Our goal is to show that every closed k-form α, for $k > 0$, on $\mathbb{R}^n$, for $n > 0$, is exact. In other words, we want to show that for every $\alpha \in \bigwedge^k(\mathbb{R}^n)$, where $d\alpha = 0$, there is a $\beta \in \bigwedge^{k-1}(\mathbb{R}^n)$ such that $\alpha = d\beta$. Actually, we will not be finding this β explicitly, instead we will be using induction on n. That is, if we know it is true for k-forms on $\mathbb{R}^n$ then we

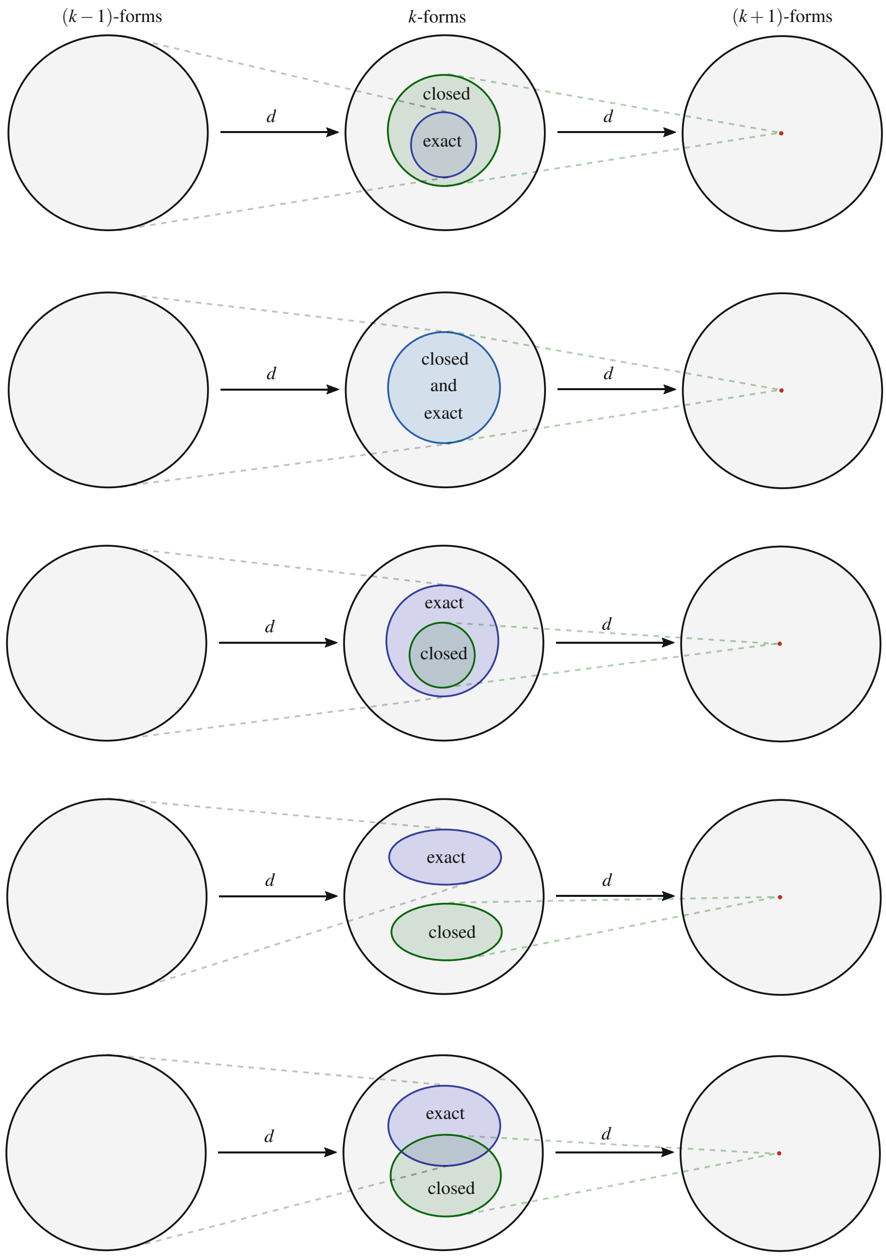

Fig. 8.3 Various possibilities on how closed and exact k-forms may be related to each other

will be able to prove that it is true for $\mathbb{R}^{n+1}$. In Fig. 8.6 we give a Venn-like diagram representation of the columns of the commutative diagram in Fig. 8.5.

8.2 The Base Case and a Simple Example Case

First of all we recall that zero-forms α on $\mathbb{R}^n$ are exactly the real-valued functions on $\mathbb{R}^n$. The zero-forms on $\mathbb{R}^n$ are denoted by $\bigwedge^0(\mathbb{R}^n)$, which is the top row of Fig. 8.5. While it is certainly possible to have closed zero forms on $\mathbb{R}^n$ it is not possible to have an exact zero form on $\mathbb{R}^n$ since there are no such things as (-1)-forms. Hence the Poincaré lemma does not apply to zero forms but only to k-forms where $k > 0$.

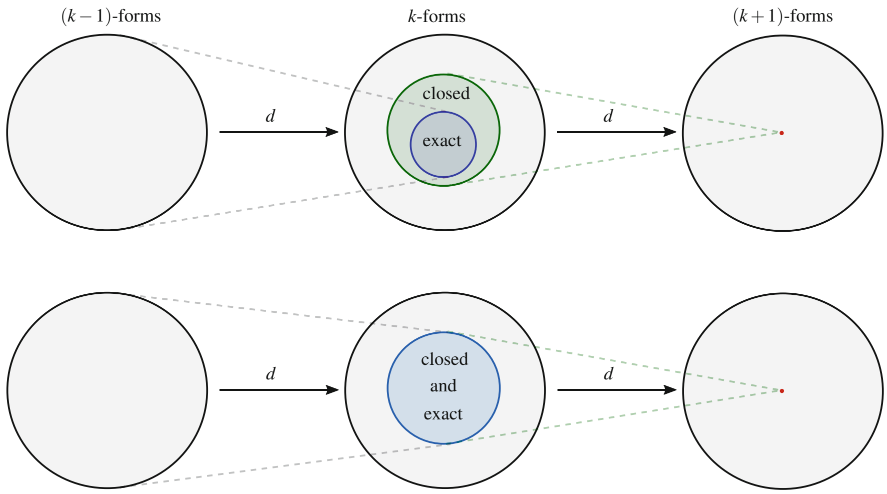

Fig. 8.4 Are exact forms a subset of closed forms (above) or are exact forms the same as closed forms (below)? The answer depends on the manifold M

$$
\begin{tikzcd}
\Lambda^0(\mathbb{R}^0) \arrow[d, shift left, "d"] & \Lambda^0(\mathbb{R}^1) \arrow[d, shift left, "d"] & \Lambda^0(\mathbb{R}^2) \arrow[d, shift left, "d"] & \Lambda^0(\mathbb{R}^3) \arrow[d, shift left, "d"] & \\
\Lambda^1(\mathbb{R}^0) \arrow[u, shift left, "\mathscr{K}"] \arrow[d, shift left, "d"] \arrow[r, shift left, "\mathscr{C}"] & \Lambda^1(\mathbb{R}^1) \arrow[l, shift left, "\mathscr{Z}"] \arrow[u, shift left, "\mathscr{K}"] \arrow[d, shift left, "d"] \arrow[r, shift left, "\mathscr{C}"] & \Lambda^1(\mathbb{R}^2) \arrow[l, shift left, "\mathscr{Z}"] \arrow[u, shift left, "\mathscr{K}"] \arrow[d, shift left, "d"] \arrow[r, shift left, "\mathscr{C}"] & \Lambda^1(\mathbb{R}^3) \arrow[l, shift left, "\mathscr{Z}"] \arrow[u, shift left, "\mathscr{K}"] \arrow[d, shift left, "d"] \arrow[r, shift left, "\mathscr{C}"] & \cdots \arrow[l, shift left, "\mathscr{Z}"] \\
\Lambda^2(\mathbb{R}^0) \arrow[u, shift left, "\mathscr{K}"] \arrow[d, shift left, "d"] \arrow[r, shift left, "\mathscr{C}"] & \Lambda^2(\mathbb{R}^1) \arrow[l, shift left, "\mathscr{Z}"] \arrow[u, shift left, "\mathscr{K}"] \arrow[d, shift left, "d"] \arrow[r, shift left, "\mathscr{C}"] & \Lambda^2(\mathbb{R}^2) \arrow[l, shift left, "\mathscr{Z}"] \arrow[u, shift left, "\mathscr{K}"] \arrow[d, shift left, "d"] \arrow[r, shift left, "\mathscr{C}"] & \Lambda^2(\mathbb{R}^3) \arrow[l, shift left, "\mathscr{Z}"] \arrow[u, shift left, "\mathscr{K}"] \arrow[d, shift left, "d"] \arrow[r, shift left, "\mathscr{C}"] & \cdots \arrow[l, shift left, "\mathscr{Z}"] \\
\Lambda^3(\mathbb{R}^0) \arrow[u, shift left, "\mathscr{K}"] \arrow[d, shift left, "d"] \arrow[r, shift left, "\mathscr{C}"] & \Lambda^3(\mathbb{R}^1) \arrow[l, shift left, "\mathscr{Z}"] \arrow[u, shift left, "\mathscr{K}"] \arrow[d, shift left, "d"] \arrow[r, shift left, "\mathscr{C}"] & \Lambda^3(\mathbb{R}^2) \arrow[l, shift left, "\mathscr{Z}"] \arrow[u, shift left, "\mathscr{K}"] \arrow[d, shift left, "d"] \arrow[r, shift left, "\mathscr{C}"] & \Lambda^3(\mathbb{R}^3) \arrow[l, shift left, "\mathscr{Z}"] \arrow[u, shift left, "\mathscr{K}"] \arrow[d, shift left, "d"] \arrow[r, shift left, "\mathscr{C}"] & \cdots \arrow[l, shift left, "\mathscr{Z}"] \\
\Lambda^4(\mathbb{R}^0) \arrow[u, shift left, "\mathscr{K}"] \arrow[d, shift left, "d"] \arrow[r, shift left, "\mathscr{C}"] & \Lambda^4(\mathbb{R}^1) \arrow[l, shift left, "\mathscr{Z}"] \arrow[u, shift left, "\mathscr{K}"] \arrow[d, shift left, "d"] \arrow[r, shift left, "\mathscr{C}"] & \Lambda^4(\mathbb{R}^2) \arrow[l, shift left, "\mathscr{Z}"] \arrow[u, shift left, "\mathscr{K}"] \arrow[d, shift left, "d"] \arrow[r, shift left, "\mathscr{C}"] & \Lambda^4(\mathbb{R}^3) \arrow[l, shift left, "\mathscr{Z}"] \arrow[u, shift left, "\mathscr{K}"] \arrow[d, shift left, "d"] \arrow[r, shift left, "\mathscr{C}"] & \cdots \arrow[l, shift left, "\mathscr{Z}"] \\
\vdots \arrow[u, shift left, "\mathscr{K}"] & \vdots \arrow[u, shift left, "\mathscr{K}"] & \vdots \arrow[u, shift left, "\mathscr{K}"] & \vdots \arrow[u, shift left, "\mathscr{K}"] &
\end{tikzcd}
$$

Fig. 8.5 Commutative diagram that will be helpful in understanding the proof of the Poincaré lemma

Next we will prove our base case. Every proof by induction requires you to first prove a base case. Usually, as will be the case here, the base case is very easy to prove. That is part of the beauty of proofs by induction. Here our base case is the first column of Fig. 8.5, which is also shown as the first row in Fig. 8.6 using Venn-like diagrams. $\mathbb{R}^0$ is a set of exactly one point, $\mathbb{R}^0 = \{0\}$, so the space $\bigwedge^0(\mathbb{R}^0)$ is just the set of real-valued functions on a point, $f : \{0\} \to \mathbb{R}$. This is shown as the first space in the upper left hand corner of Fig. 8.5.

We also have previously seen that $\bigwedge^k(\mathbb{R}^n) = \{0\}$ for $k > n$; that is, all k-forms on $\mathbb{R}^n$, where $k > n$, are the zero k-form. What we mean is that the zero k-form sends every set of k vectors to zero. Thus, for $k > n$ if $\alpha \in \bigwedge^k(\mathbb{R}^n)$ then $\alpha(v_1, \ldots v_k) = 0$ for every set of vectors $v_1, \ldots, v_k$ and hence we write $\alpha \equiv 0$. Thus $\bigwedge^k(\mathbb{R}^0) = \{0\}$ for $k > 0$. These spaces are all shown as a single point in Fig. 8.6.

Now we proceed to prove our base case. The proof of the base case is essentially trivial but still happens in two simple steps:

1. If $\alpha \in \bigwedge^1(\mathbb{R}^0)$ is closed show that it is exact.
2. If $\alpha \in \bigwedge^k(\mathbb{R}^0)$ for $k > 1$ is closed show that it is exact.