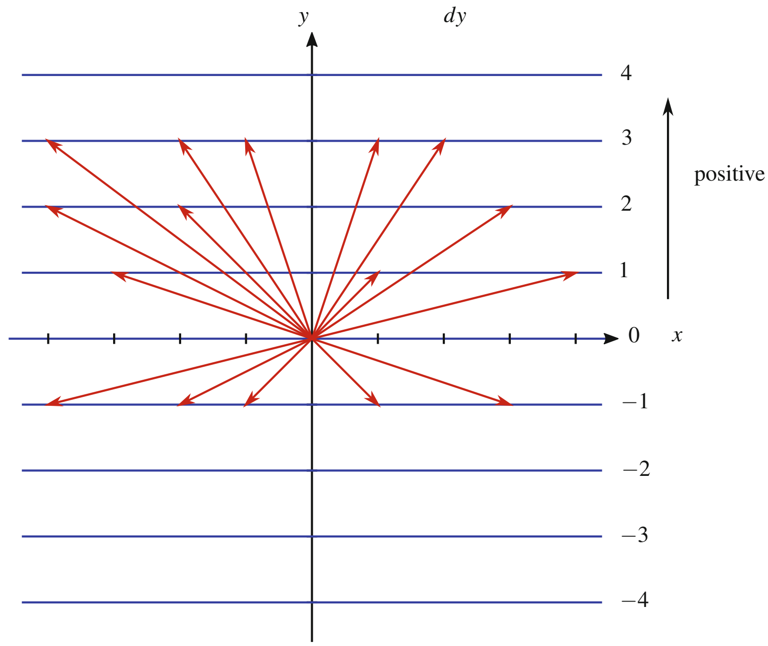

Fig. 5.2 Vectors for which the one-form dy gives outputs of $-1, 1, 2$, and 3. A positive orientation is shown

So for Fig. 5.2 we have

$$
\begin{array}{llll}
dy\left(\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right) = 3, & dy\left(\begin{bmatrix} -2 \\ 2 \end{bmatrix}\right) = 2, & dy\left(\begin{bmatrix} -3/2 \\ 1 \end{bmatrix}\right) = 1, & dy\left(\begin{bmatrix} -2 \\ -1 \end{bmatrix}\right) = -1, \\
dy\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = 3, & dy\left(\begin{bmatrix} 1/2 \\ 2 \end{bmatrix}\right) = 2, & dy\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 1, & dy\left(\begin{bmatrix} -1 \\ -1 \end{bmatrix}\right) = -1, \\
dy\left(\begin{bmatrix} 4 \\ 3 \end{bmatrix}\right) = 3, & dy\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = 2, & dy\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = 1, & dy\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = -1.
\end{array}
$$

The same comments regarding approximations apply. For example, the vector $\begin{bmatrix} 1.42 \\ 4.84 \end{bmatrix}$ pierces four lines of our picture of dy so we would have $dy\left(\begin{bmatrix} 1.42 \\ 4.84 \end{bmatrix}\right) = 4$, though the exact value would of course be 4.84.

But the differential forms dx and dy are not the only elements of $T_p^*\mathbb{R}^2$. How would the differential form $2dx_p \in T_p^*\mathbb{R}^2$ be represented in the vector space $T_p\mathbb{R}^2$? Considering the same vectors from above we have

$$
\begin{array}{llll}
2dx\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right) = 2, & 2dx\left(\begin{bmatrix} -1 \\ 4 \end{bmatrix}\right) = -2, & 2dx\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = 4, & 2dx\left(\begin{bmatrix} 3 \\ 6 \end{bmatrix}\right) = 6, \\
2dx\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = 2, & 2dx\left(\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right) = -2, & 2dx\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = 4, & 2dx\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = 6, \\
2dx\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) = 2, & 2dx\left(\begin{bmatrix} -1 \\ -2 \end{bmatrix}\right) = -2, & 2dx\left(\begin{bmatrix} 2 \\ -2 \end{bmatrix}\right) = 4, & 2dx\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right) = 6.
\end{array}
$$

So the vectors are piercing twice as many lines in the $2dx$ case as in the dx case, implying that the lines in the $2dx$ case are twice as "dense" as in the dx case. We would draw that as in Fig. 5.3.

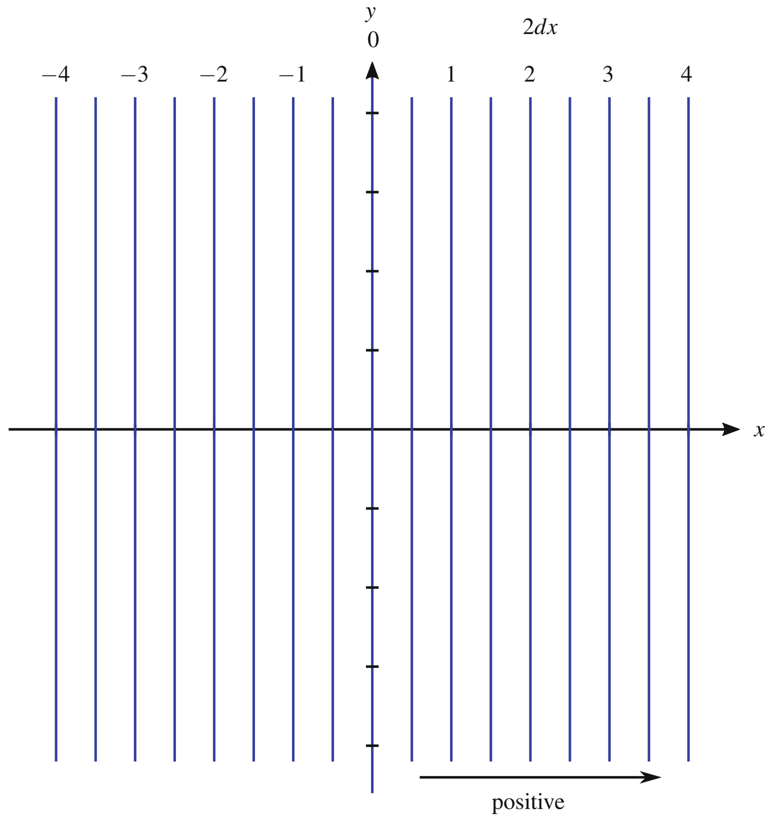

Fig. 5.3 The one-form $2dx$ shown as lines-to-be-pierced

We could perform exactly the same analysis in the $2dy$ case. Again, using the same vectors as above we would have

$$2dy\left(\begin{bmatrix}-2\\3\end{bmatrix}\right)=6,\qquad 2dy\left(\begin{bmatrix}-2\\2\end{bmatrix}\right)=4,\qquad 2dy\left(\begin{bmatrix}-3/2\\1\end{bmatrix}\right)=2,\qquad 2dy\left(\begin{bmatrix}-2\\-1\end{bmatrix}\right)=-2,$$

$$2dy\left(\begin{bmatrix}1\\3\end{bmatrix}\right)=6,\qquad 2dy\left(\begin{bmatrix}1/2\\2\end{bmatrix}\right)=4,\qquad 2dy\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=2,\qquad 2dy\left(\begin{bmatrix}-1\\-1\end{bmatrix}\right)=-2,$$

$$2dy\left(\begin{bmatrix}4\\3\end{bmatrix}\right)=6,\qquad 2dy\left(\begin{bmatrix}3\\2\end{bmatrix}\right)=4,\qquad 2dy\left(\begin{bmatrix}3\\1\end{bmatrix}\right)=2,\qquad 2dy\left(\begin{bmatrix}1\\-1\end{bmatrix}\right)=-2.$$

Again, the vectors are piercing twice as many lines in the $2dy$ case as in the dy case, so again the lines in the $2dy$ case are twice as "dense" as in the dy case. We would draw that as in Fig. 5.4.

Clearly if you performed the same analysis on $3dx$ and $3dy$ you would find the lines in the picture representations three times as "dense" as in the dx and dy case, and for ndx and ndy they would be n times as dense.

Question 5.3 Draw a picture representing the following differential forms. $3dx$, $5dy$, $\frac{1}{2}dx$, $\frac{2}{3}dx$, $\frac{1}{3}dy$, and $\frac{3}{5}dy$.

Now we turn our attention to more general elements of $T_p^*\mathbb{R}^2$. We will start off with something simple, $dx+dy$. Consider how the one-form $dx+dy$ acts on the following vectors,

$$(dx+dy)\left(\begin{bmatrix}-1\\3\end{bmatrix}\right)=-1+3=2,\qquad (dx+dy)\left(\begin{bmatrix}1\\2\end{bmatrix}\right)=1+2=3,$$

$$(dx+dy)\left(\begin{bmatrix}-1\\1\end{bmatrix}\right)=-1+1=0,\qquad (dx+dy)\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=0+1=1,$$

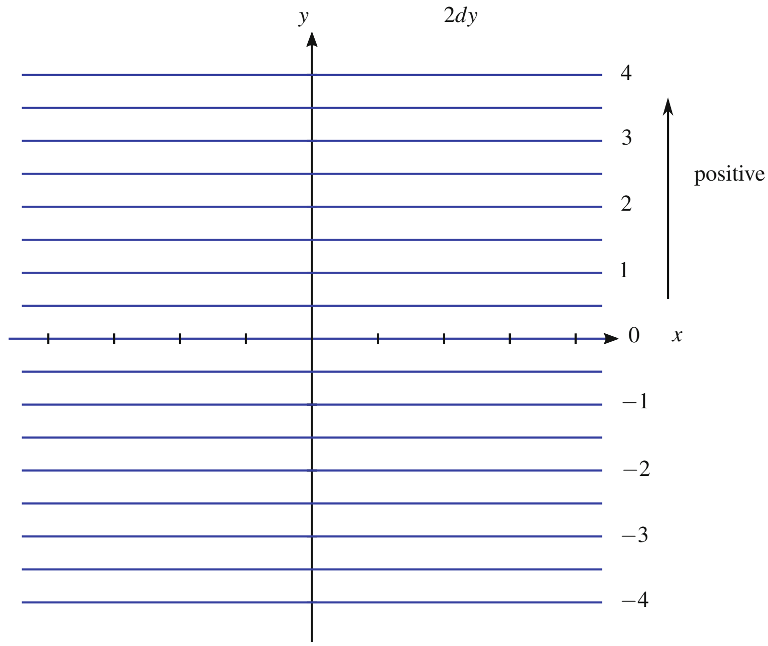

Fig. 5.4 The one-form $2dy$ shown as lines-to-be-pierced

$$(dx+dy)\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=1+1=2, \qquad (dx+dy)\left(\begin{bmatrix}-1\\0\end{bmatrix}\right)=-1+0=-1,$$

$$(dx+dy)\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=1+0=1, \qquad (dx+dy)\left(\begin{bmatrix}0\\-1\end{bmatrix}\right)=0-1=-1.$$

Figure 5.5 is a picture for $dx+dy$ that works. The fact that it works can be seen just by inspecting the vectors and counting how many lines they pierce, taking into account the orientation depicted. The picture for $dx+dy$ clearly requires slanted lines. By inspection it should be apparent that the lines are given by $y=-x+n$ for $n=0,\pm1,\pm2,\ldots$.

Now suppose we want to draw a picture for the differential one-form $2dx+dy$ or $dx-2dy$ or something more complicated still. We need a systematic way to decide what the lines are. This is actually fairly easy to do. Consider the one-form $2dx+dy$. To find the lines we need to draw to make a picture of this differential we simply start with the equation $2x+y=n$ where $n=0,\pm1,\pm2,\ldots$ and solve for y which gives $y=-2x+n$, depicted in Fig. 5.6. Consider the vector $v=\begin{bmatrix}1\\2\end{bmatrix}$ in Fig. 5.6. It "pierces" four lines in the positive direction, so according to the picture $(2dx+dy)(v)=4$. Computationally we have

$$(2dx+dy)\left(\begin{bmatrix}1\\2\end{bmatrix}\right)=2dx\left(\begin{bmatrix}1\\2\end{bmatrix}\right)+dy\left(\begin{bmatrix}1\\2\end{bmatrix}\right)=2(1)+2=4.$$

Similarly, to draw the picture for the one-form $dx-2dy$ we use the equation $x-2y=n$, which becomes $y=\frac{x}{2}-\frac{n}{2}$. See Fig. 5.7.

Question 5.4 Check that the various vectors pictured in the images for $2dx+dy$ and $dx-2dy$ give the same answers both computationally and pictorially.

Question 5.5 Explain why the above procedure for finding the lines in the pictorial representation works. To do this consider how a generic two-form $adx+bdy$ acts on a generic vector $v=\begin{bmatrix}x\\y\end{bmatrix}$. Which values of this vector give the value $(adx+bdy)(v)=n$ for any $n=0,\pm1,\pm2,\ldots$?

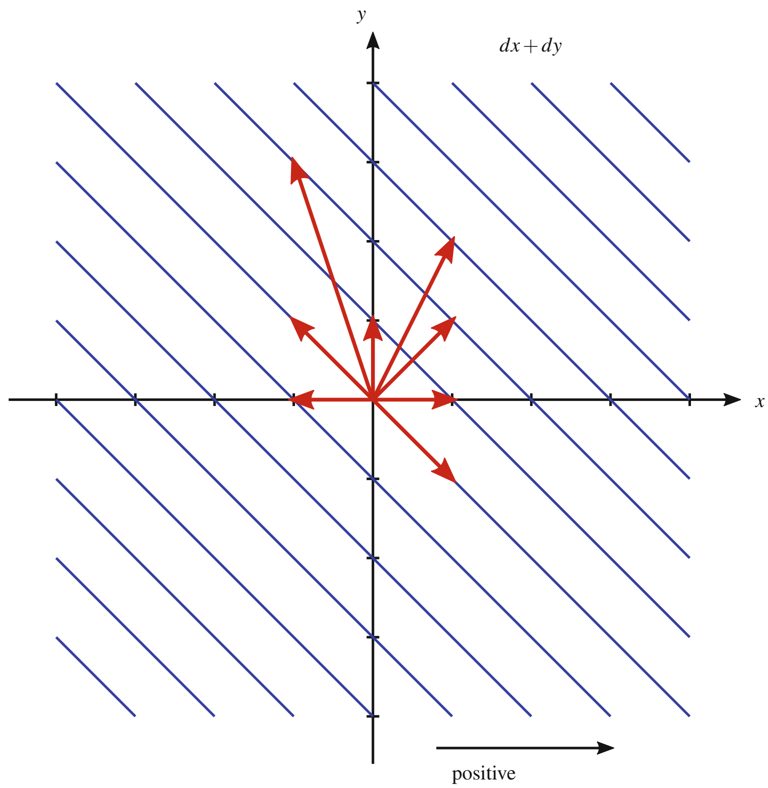

Fig. 5.5 The lines-to-be-pieced for the one-form $dx + dy$. The positive orientation is depicted

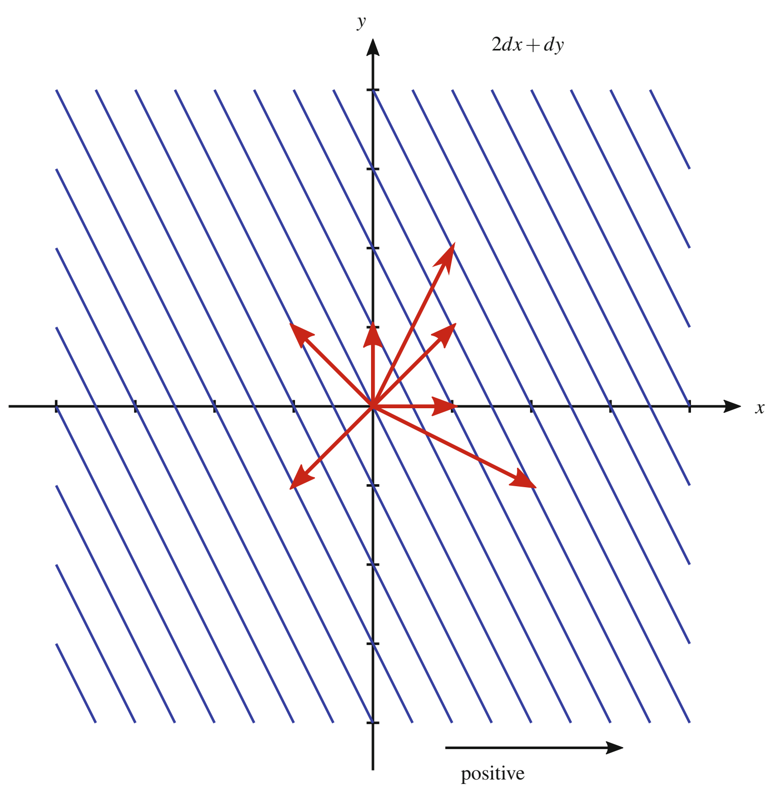

Fig. 5.6 The lines-to-be-pieced for the one-form $2dx + dy$. The positive orientation is depicted

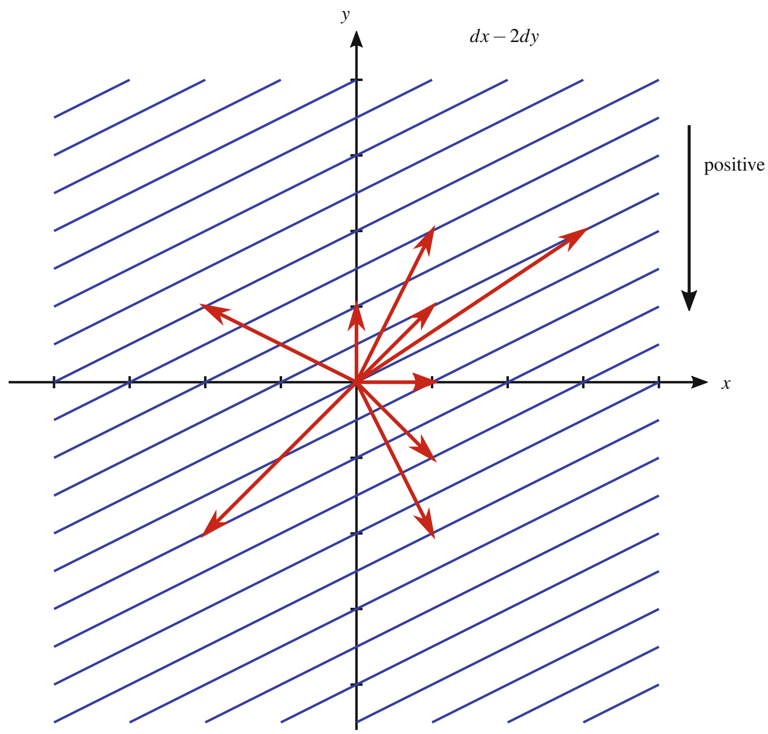

Fig. 5.7 The lines-to-be-pierced for the one-form $dx - 2dy$. The positive orientation is depicted

Question 5.6 Draw the graphical representations of the following differential forms,

(a) $3dx + dy$,
(b) $2dx - dy$,
(c) $-dx + 3dy$.

Differential one-forms in $T_p^*\mathbb{R}^2$ are not the only kind of forms that can act on vectors in $T_p\mathbb{R}^2$. Differential two-forms in $\bigwedge_p^2(\mathbb{R})$ can also act on vectors in $T_p\mathbb{R}^2$. As we know, differential two-forms act on two vectors in $T_p\mathbb{R}^2$. We begin by considering the vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and the differential two-form $dx \wedge dy$. We have

$$(dx \wedge dy)(v_1, v_2) = \begin{vmatrix} dx(v_1) & dx(v_2) \\ dy(v_1) & dy(v_2) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$$

$$(dx \wedge dy)(v_2, v_1) = \begin{vmatrix} dx(v_2) & dx(v_1) \\ dy(v_2) & dy(v_1) \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1.$$

Like before, we want to draw a picture in $T_p\mathbb{R}^2$ that represents the differential two-form $dx \wedge dy$. For differential one-forms we used a picture that consisted of lines and then we counted how many lines our vector "pierced" in order to get an approximate value for the differential form acting on the vector. Here our picture has to somehow incorporate both input vectors. We represent the differential two-form $dx \wedge dy$ in $T_p\mathbb{R}^2$ as horizontal and vertical lines along with an orientation.

Figure 5.8 shows the picture of $dx \wedge dy$, consisting of blue horizontal lines and green vertical lines, which produces a grid of squares. The parallelepiped spanned by the vectors v_1 and v_2 exactly covers one of the grid's squares, thus we say $dx \wedge dy(v_1, v_2)$ is one. Now we only have to consider orientation. Since $(dx \wedge dy)(v_1, v_2)$ is positive we use that to decide on the orientation that is needed, which is depicted in the picture as a small circular arrow that goes in the counter-clockwise direction. Going from v_1 to v_2, traversing the smallest angle between the two vectors, is counter-clockwise, so that is the

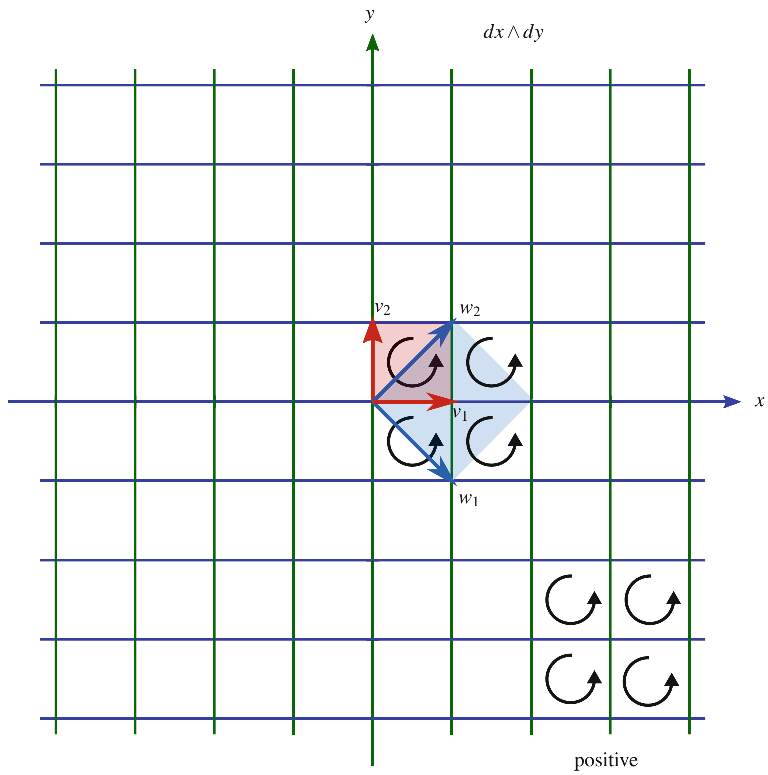

Fig. 5.8 The two-form $dx \wedge dy$ consists of both horizontal and vertical lines. The counter-clockwise circular arrows depicts a positive orientation

orientation that is chosen. Thus, if we traverse the small angle between the first and second vectors in a counter-clockwise direction we choose positive, and if we traverse the small angle between the first and second vectors in a clockwise direction we choose negative. Thus for $(dx \wedge dy)(v_2, v_1)$ we have the first vector being v_2 and the second vector being v_1. Traversing from the first vector to the second we go in a clockwise direction, so we would choose negative. The parallelepiped spanned by the vectors v_2 and v_1 covers exactly one square so the answer is -1.

Now consider the vectors

$$w_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \; w_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Computationally we have that

$$(dx \wedge dy)(w_1, w_2) = \begin{vmatrix} dx(w_1) & dx(w_2) \\ dy(w_1) & dy(w_2) \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2,$$

$$(dx \wedge dy)(w_2, w_1) = \begin{vmatrix} dx(w_2) & dx(w_1) \\ dy(w_2) & dy(w_1) \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2.$$

Using Fig. 5.8 we can see that the parallelepiped spanned by w_1 and w_2 covers a total of two grid squares and if we traverse the small angle from w_1 to w_2 it is in the counter-clockwise direction so we choose the positive sign to get $dx \wedge dy(w_1, w_2) = 2$. But if we traverse the small angle from w_2 to w_1 it is in the clockwise direction so we choose the negative sign to get $dx \wedge dy(w_2, w_1) = -2$.

Now we turn our attention to the differential two-form $2dx \wedge dy$. We have already discussed how this way of visualizing differential forms has the drawback that the answers it gives are only approximate. Now we will illustrate another drawback. There can be some ambiguity in what picture or image we use for some differential forms. Again, we will consider the vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

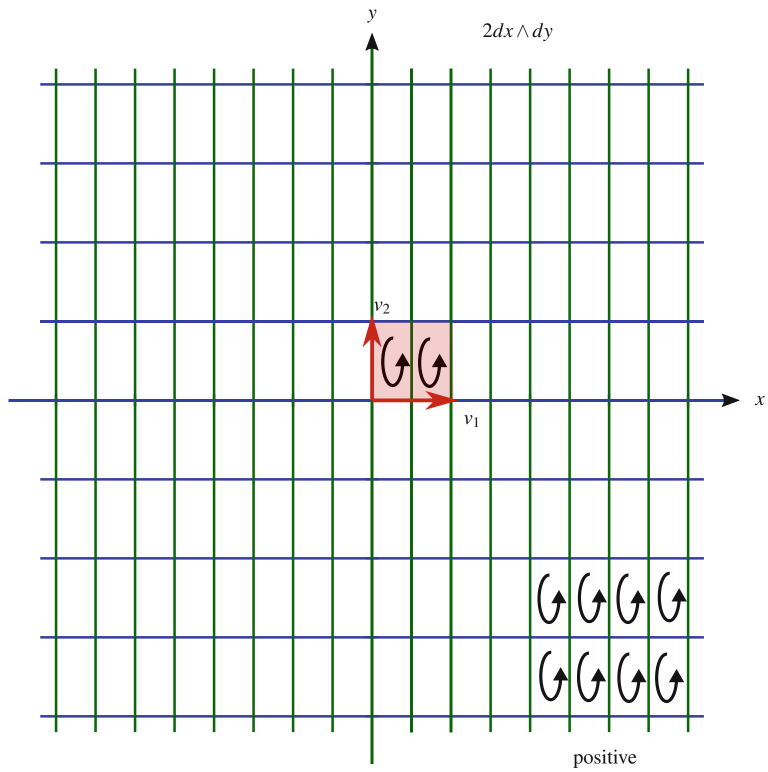

Fig. 5.9 One possible way to draw the two-form $2dx \wedge dy$. Here the lines are $x = n/2$ and $y = n$. The counter-clockwise circular arrows depicts a positive orientation

which computationally give us

$$(2dx \wedge dy)(v_1, v_2) = 2 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 2.$$

So the parallelepiped spanned by v_1 and v_2 has to cover two boxes from the grid that represents the two-form $2dx \wedge dy$. There is no unique way of doing that. For example, choosing the lines $y = n$ and $x = \frac{n}{2}$ for $n = 0, \pm 1, \pm 2, \ldots$ accomplishes this; see Fig. 5.9. Another way of accomplishing this is choosing the lines $y = \frac{n}{2}$ and $x = n$ for $n = 0, \pm 1, \pm 2, \ldots$, as in Fig. 5.10. And in fact, there are other choices as well.

Question 5.7 Find another graphical representation for the differential two-form $2dx \wedge dy$.

Question 5.8 Find three graphical representations for each of the following differential two-forms

(a) $3dx \wedge dy$,
(b) $5dx \wedge dy$,
(c) $\frac{1}{2}dx \wedge dy$.

5.2 One-Forms in $\mathbb{R}^3$

As before we will fix some point p on the manifold $\mathbb{R}^3$ and only consider one-forms in $T_p^*\mathbb{R}^3$. The "picture" of the one-forms that we will draw are actually superimposed on the vector space $T_p\mathbb{R}^3$. We will start out with the one-form $dx \in T_p^*\mathbb{R}^3$.

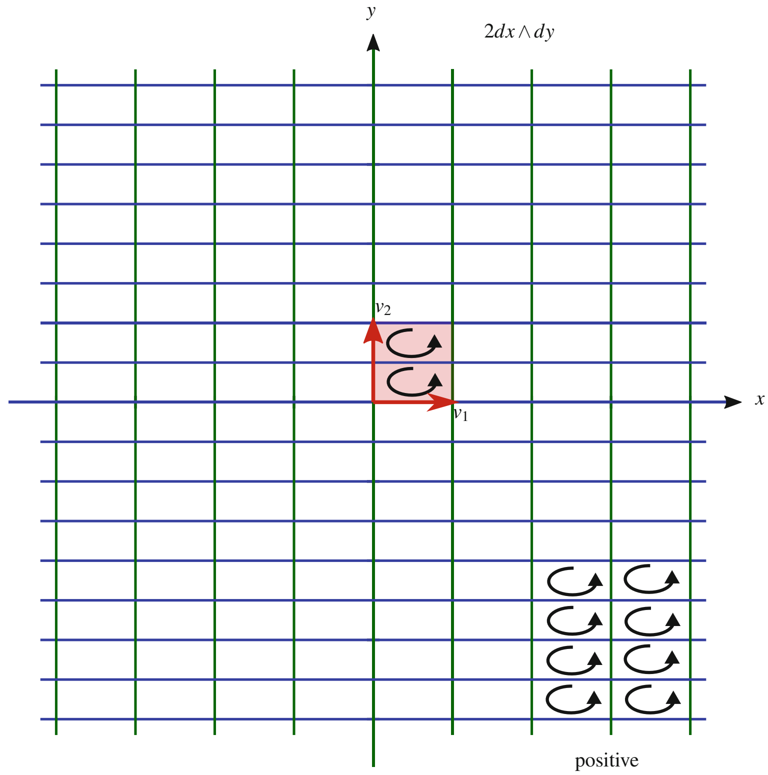

Fig. 5.10 Another possible way to draw the two-form $2dx \wedge dy$. Here the lines are $x = n$ and $y = n/2$. The counter-clockwise circular arrows depicts a positive orientation

Consider what dx does to the following vectors,

$$dx\left(\begin{bmatrix}1\\0\\5\end{bmatrix}\right) = 1, \qquad dx\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = 1, \qquad dx\left(\begin{bmatrix}1\\2\\1\end{bmatrix}\right) = 1,$$

$$dx\left(\begin{bmatrix}1\\2\\-1\end{bmatrix}\right) = 1, \qquad dx\left(\begin{bmatrix}1\\-3\\-2\end{bmatrix}\right) = 1, \qquad dx\left(\begin{bmatrix}1\\-4\\3\end{bmatrix}\right) = 1.$$

Any vector $v \in T_p\mathbb{R}^3$ that terminates anywhere on the $x = 1$ plane in $T_p\mathbb{R}^3$ has the value 1. See Fig. 5.11.

Similarly, consider what dx does to these vectors,

$$dx\left(\begin{bmatrix}3\\0\\0\end{bmatrix}\right) = 3, \qquad dx\left(\begin{bmatrix}2\\-2\\-1\end{bmatrix}\right) = 2, \qquad dx\left(\begin{bmatrix}0\\3\\4\end{bmatrix}\right) = 0,$$

$$dx\left(\begin{bmatrix}3\\-1\\2\end{bmatrix}\right) = 3, \qquad dx\left(\begin{bmatrix}2\\-2\\-4\end{bmatrix}\right) = 2, \qquad dx\left(\begin{bmatrix}0\\-3\\-1\end{bmatrix}\right) = 0,$$

$$dx\left(\begin{bmatrix}3\\2\\-5\end{bmatrix}\right) = 3, \qquad dx\left(\begin{bmatrix}2\\5\\-3\end{bmatrix}\right) = 2, \qquad dx\left(\begin{bmatrix}0\\-2\\5\end{bmatrix}\right) = 0.$$

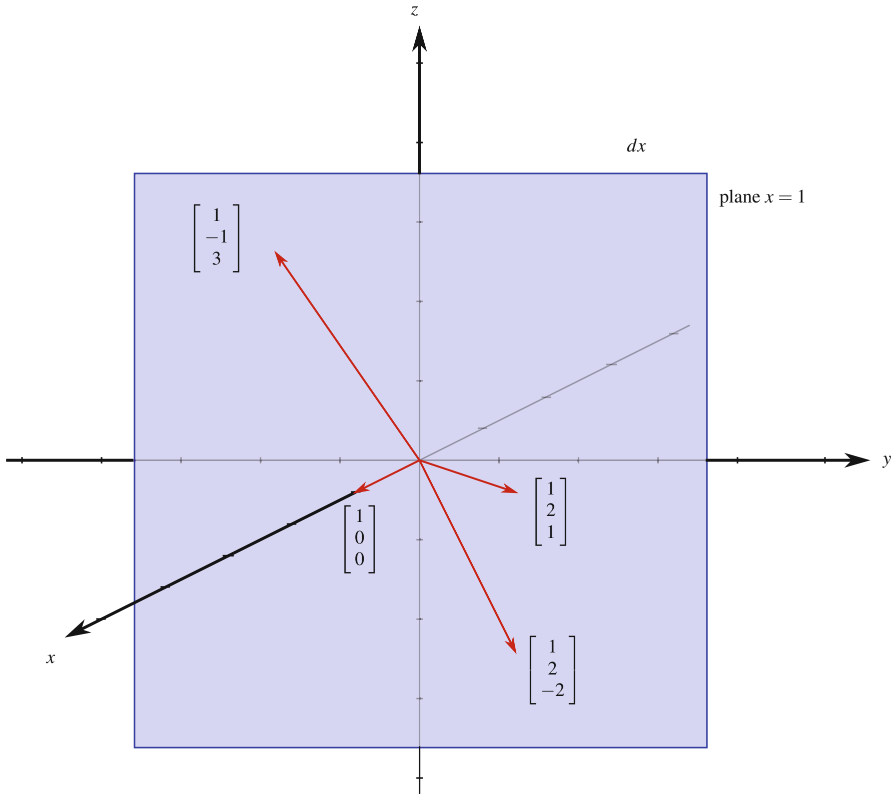

Fig. 5.11 The plane $x = 1$ in $\mathbb{R}^3$. The one-form dx sends all of these vectors that terminate on this plane to the value 1. That is, if v terminates on this plane then $dx(v) = 1$

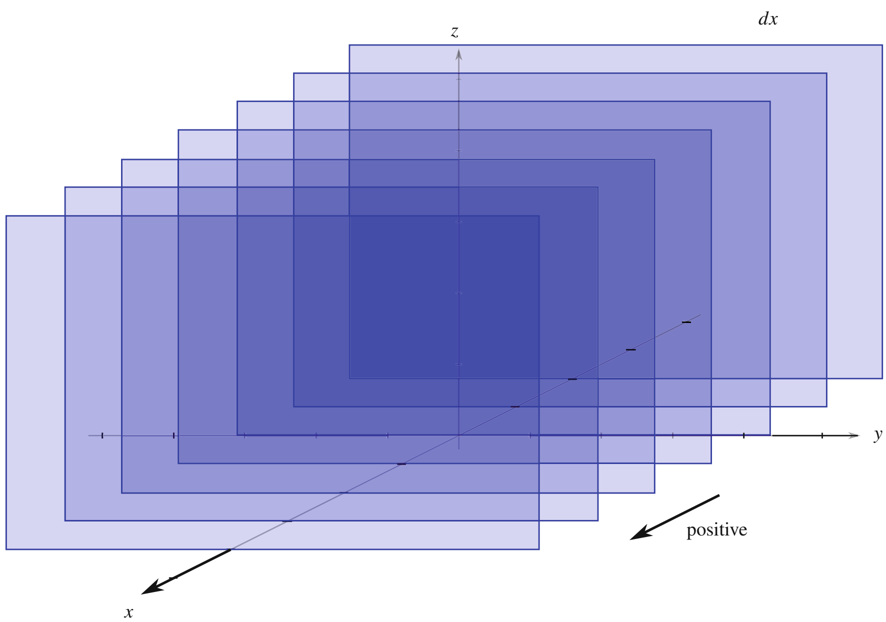

Fig. 5.12 The one-form dx depicted as planes in $\mathbb{R}^3$

The one-form sends any vector that terminates on the plane $x = 3$ to 3, any vector that terminates on the plane $x = 2$ to 2, and any vector that terminates on the plane $x = 0$ to 0. Based on this we will graphically represent the differential one-form $dx \in T_p^*\mathbb{R}^3$ as the set of planes $x = n$, where $n = 0, \pm1, \pm2, \ldots$, in the vector space $T_p\mathbb{R}^3$. In Fig. 5.12 we show dx by drawing the seven planes $x = 4$, $x = 3$, $x = 2$, $x = 1$, $x = 0$, $x = -1$, and $x = -2$.

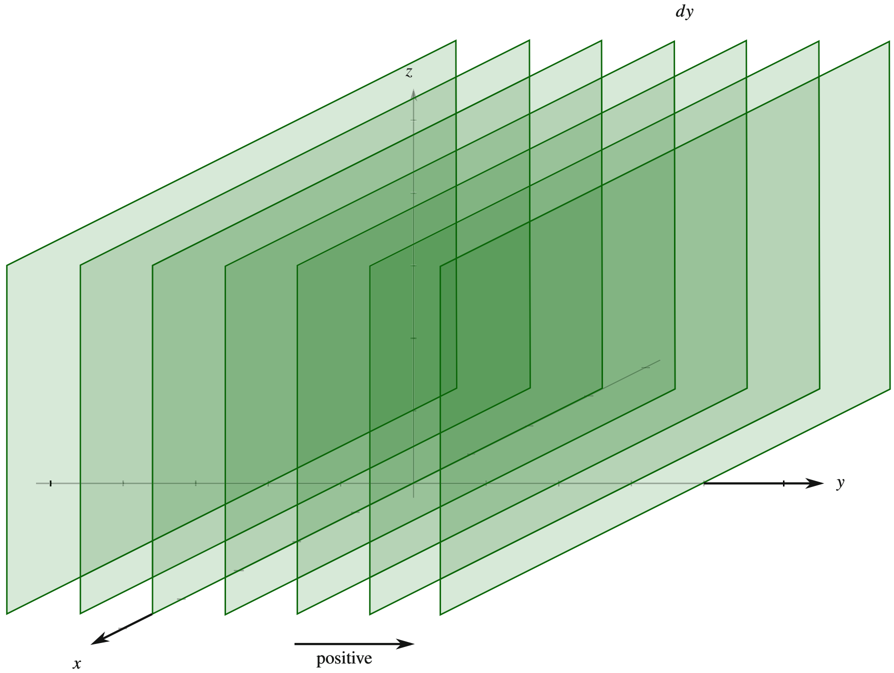

Fig. 5.13 The one-form dy depicted as planes in $\mathbb{R}^3$

Using this picture we say that dx of a vector v is the number of planes that v "pierces." And as before, the answer that this gives is only an approximate answer. For example, the vectors

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1.4 \\ 1 \\ -2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1.999 \\ -2 \\ 5 \end{bmatrix}$$

all pierce only one plane, so we would say they all have the same value, 1, though clearly the real answers are $dx(v_1) = 1$, $dx(v_2) = 1.4$, and $dx(v_3) = 1.999$. Again, an orientation needs to be assigned to the picture.

A similar line of reasoning results in dy being graphically represented as the set of planes $y = n$, where $n = 0, \pm 1, \pm 2, \ldots$, see Fig. 5.13, and dz being graphically represented as the set of planes $z = n$, where $n = 0, \pm 1, \pm 2, \ldots$, see Fig. 5.14.

Question 5.9 Show that the positive orientations depicted in Figs. 5.12, 5.13, and 5.14 are correct.

Let us take a moment to compare between the way we view a differential one-form as an element of the cotangent space $T_p^*\mathbb{R}^3$ and our way of visualizing a differential form on $T_p\mathbb{R}^3$. See Fig. 5.15. A differential one-form in $T_p^*\mathbb{R}^3$ was also called a co-vector which is drawn in the cotangent space exactly as a vector would be drawn in the tangent space. For example, the one-form dx is the unit vector along the dx-axis, the one-form dy is the unit vector along the dy-axis, and the one-form dz is the unit vector along the dz-axis. Recall that we chose to always write co-vectors as row vectors, thus in $T_p^*\mathbb{R}^3$ we have $dx = [1, 0, 0]$, $dy = [0, 1, 0]$, and $dz = [0, 0, 1]$ so the one-form $dx + dy + dz = [1, 1, 1]$. In the top of Fig. 5.15 we show each of the co-vectors dx in blue, dy in green, and dz in red.

Using the method of visualizing differential forms that we have introduced in this chapter, the differential one-form dx is regarded as an infinite number of planes in $T_p\mathbb{R}^3$ that are perpendicular to the $\frac{\partial}{\partial x}$-axis, also denoted the ∂_x-axis. Likewise, the differential one-form dy is an infinite number of planes perpendicular to the ∂_y-axis and dz is an infinite number of planes perpendicular to the ∂_z-axis. In the bottom of Fig. 5.15 we show a few planes of dx in blue, dy in green, and dz in red.

Like the one-forms dx, dy, and dz, general one-forms $adx + bdy + cdz$, where $a, b, c \in \mathbb{R}$, can also be visualized as an infinite number of planes. As an example we will consider the one-form $dx + dy$. In essence, we want to somehow add the

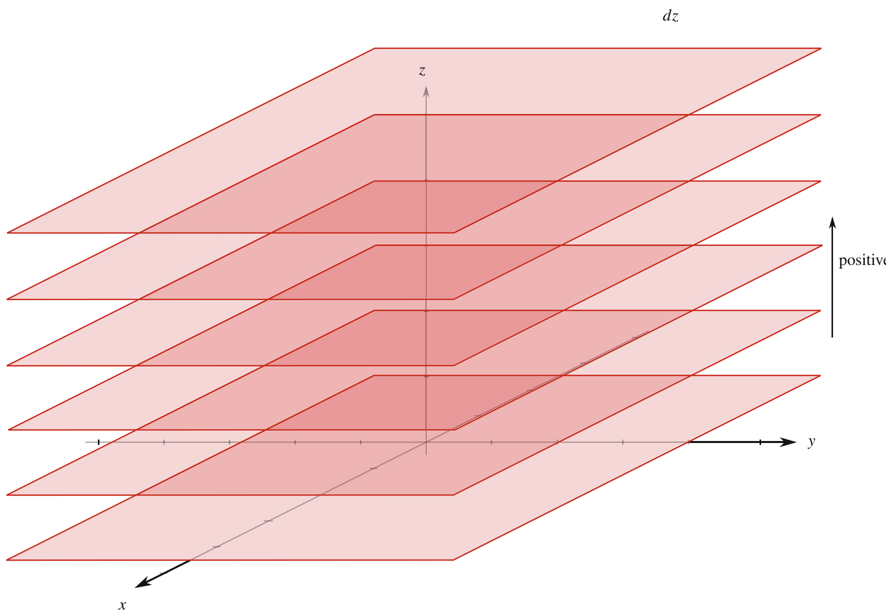

Fig. 5.14 The one-form dz depicted as planes in $\mathbb{R}^3$

dx planes with dy planes. See Fig. 5.16. In essence the planes of dx are perpendicular to the vector

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and the planes of dy are perpendicular to the vector

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

so the sum of dx and dy, or $dx + dy$ will be planes perpendicular to the sum of the vectors, that is, the planes of $dx + dy$ are perpendicular to the vector

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

The answer is shown in Fig. 5.17.

Misner, Thorne, and Wheeler's book *Gravitation* describes graphical method to find the planes associated with $dx + dy$ by choosing appropriate vectors (like v_1, v_2, w_1, w_2 in the next question), counting the number of planes they pierce, and then using the points of the vectors to determine a plane of $dx + dy$. The following questions leads you through their process.

Question 5.10 Consider the vectors

$$v_1 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, w_1 = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, w_2 = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

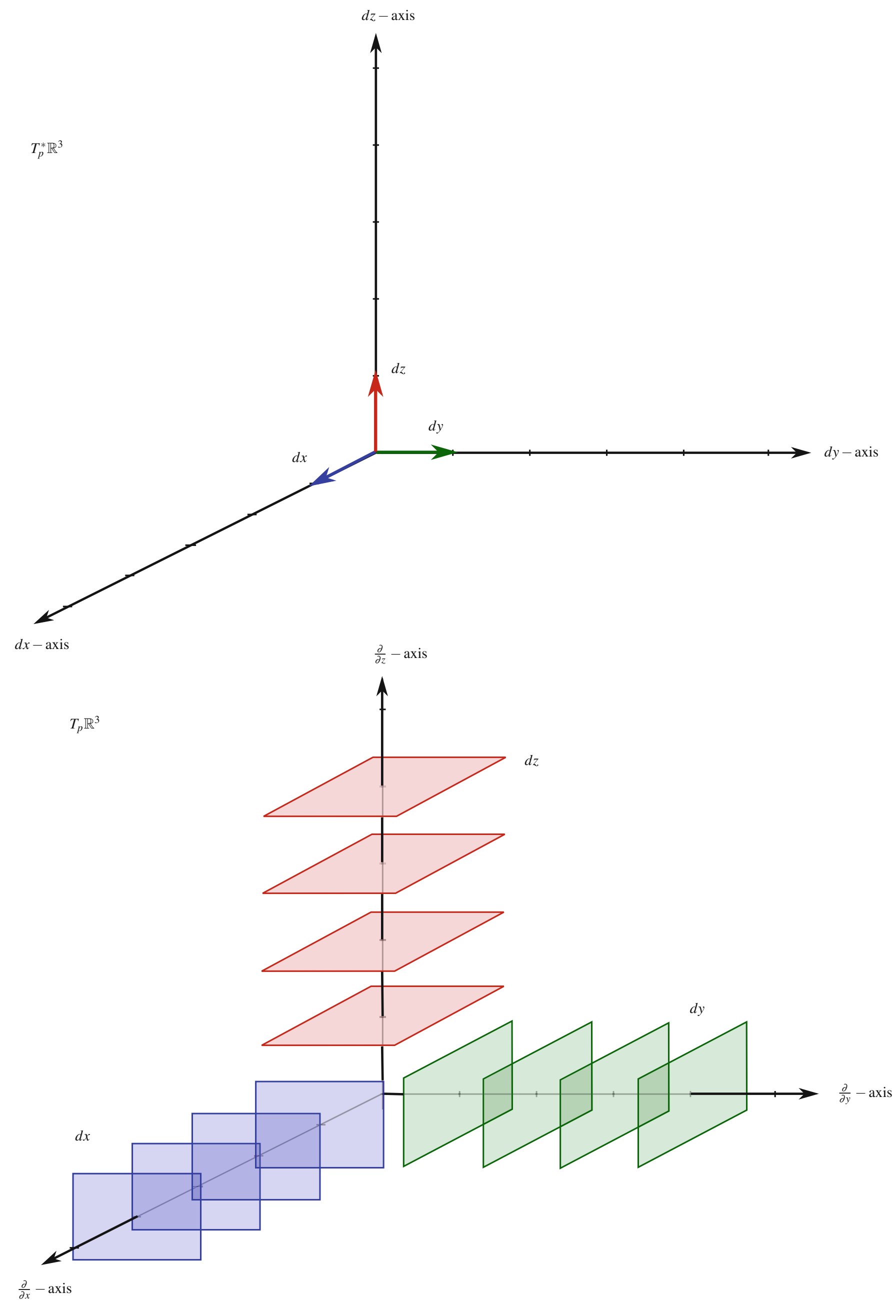

Fig. 5.15 The one-forms dx (blue), dy (green), and dz (red) pictured as co-vectors in $T_p^*\mathbb{R}^3$ (top) and as planes in $T_p\mathbb{R}^3$ (bottom)

that are shown in Fig. 5.16.

(a) Find $dx(v_1)$, $dx(v_2)$, $dx(w_1)$, and $dx(w_2)$ both computationally and graphically from the above picture.
(b) Find $dy(v_1)$, $dy(v_2)$, $dy(w_1)$, and $dy(w_2)$ both computationally and graphically from the above picture.
(c) Find $(dx + dy)(v_1)$, $(dx + dy)(v_2)$, $(dx + dy)(w_1)$, and $(dx + dy)(w_2)$ both computationally and graphically from the above picture.
(d) How many dx planes do v_1 and v_2 pierce? How many dy planes? How many dy planes do w_1 and w_2 pierce? How many dx planes? Now how many $dx + dy$ planes does each of these vectors pierce?
(e) Based on (a)–(d) determine how you would use the vectors v_1, v_2, w_1, w_2 to find the planes of $dx + dy$.

Question 5.11 Sketch the differential one-forms $dy + dz$ and $dx + dz$ on $\mathbb{R}^3$.

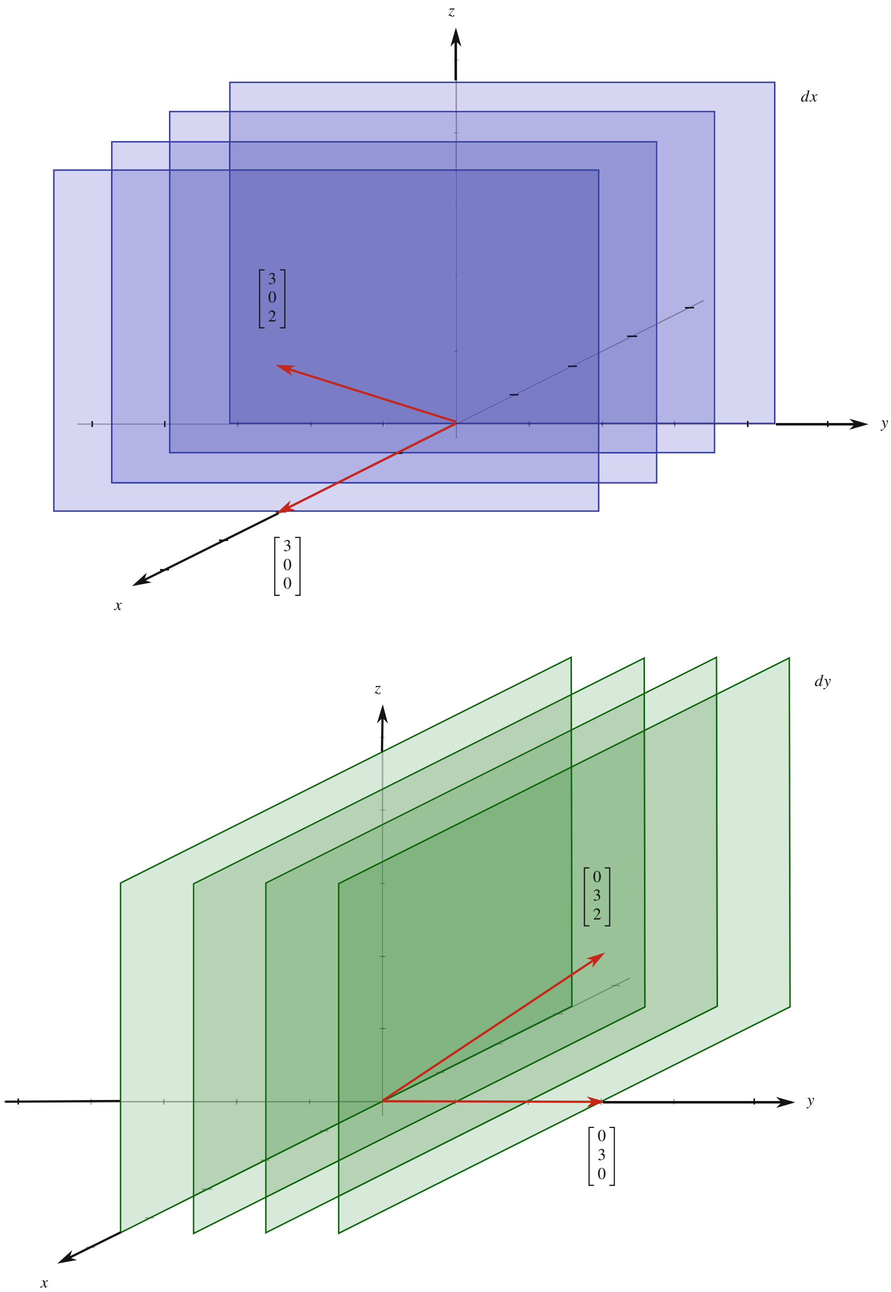

Fig. 5.16 To find the picture for $dx + dy$ we want to somehow "add" the planes for dx (top) with the planes for dy (bottom). This "addition" gives the planes depicted in Fig. 5.17

Now we have all the pieces necessary to compare the representation of $dx + dy + dx$ in $T_p^*\mathbb{R}^3$ and the way we are visualizing this same differential form in $T_p\mathbb{R}^3$, as in Fig. 5.18.

5.3 Two-Forms in $\mathbb{R}^3$

Similar to how we drew two-forms in $T_p\mathbb{R}^2$ we can draw two-forms in $T_p\mathbb{R}^3$. In fact, our graphical representation of two-forms in $T_p\mathbb{R}^3$ will essentially be a three-dimensional version of our grid-lines from the two-dimensional case. Instead of trying to walk through the development of these pictures we will simply present the pictures of the two-forms $dx \wedge dy$, $dy \wedge dz$, and $dz \wedge dx$ and illustrate how they work. We begin by a picture of the two-form $dx \wedge dy$ in Fig. 5.19.

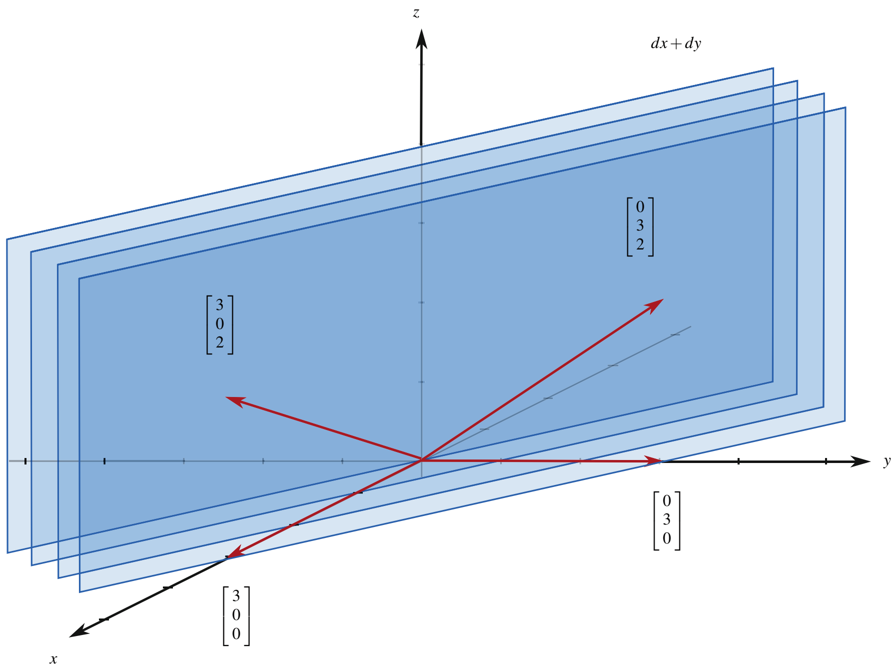

Fig. 5.17 The planes that depict the one-form $dx + dy$

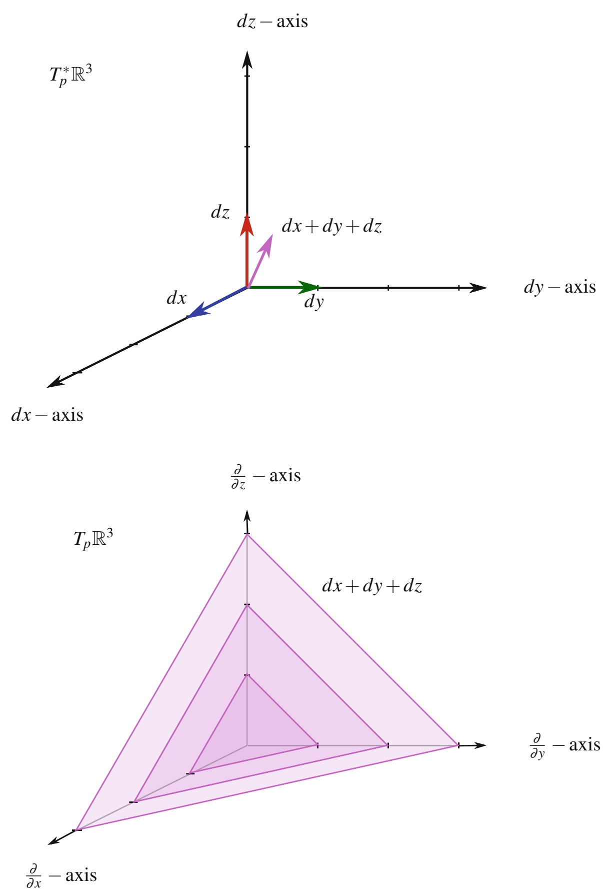

Fig. 5.18 The one-form $dx + dy + dz$ as a co-vector in $T_p^*\mathbb{R}^3$ (top) and the planes that depict $dx + dy + dz$ in $T_p\mathbb{R}^3$ (bottom)

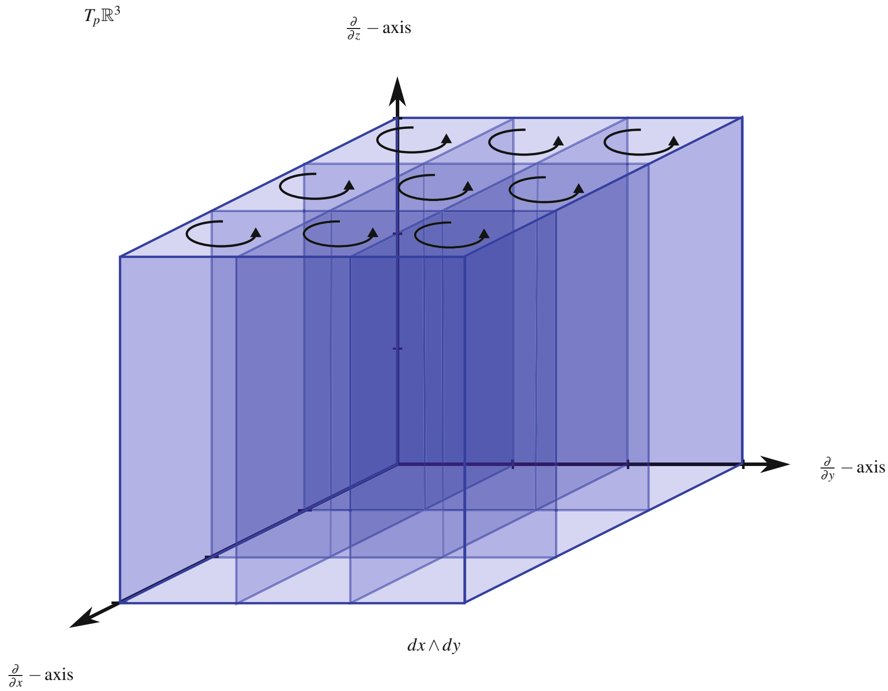

Fig. 5.19 The two-form $dx \wedge dy$ depicted in $T_p\mathbb{R}^3$ as a series of "tubes" made up of planes parallel to the ∂_x and ∂_y axes. The positive orientation is illustrated

Notice that graphically $dx \wedge dy$ is nothing more than both the dx and dy planes being drawn simultaneously which results in "tubes" that go in the ∂_z direction. The orientation is depicted by the small circular arrows in the tubes. Looking down on the picture these arrows are turning in the counter-clockwise direction. Rather than trying to prove that this is an the correct picture we will show that it works with several examples. First consider the vectors

$$u_1 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \text{ and } u_2 = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}.$$

We have

$$(dx \wedge dy)(u_1, u_2) = \begin{vmatrix} 3 & 0 \\ 0 & 3 \end{vmatrix} = 9.$$

By imagining the vectors u_1 and u_2 imposed on Fig. 5.19 you can see that the parallelepiped spanned by them has nine "tubes" of $dx \wedge dy$ passing through it. Furthermore, when traversing from u_1 to u_2 via the smallest angle between the vectors, we are moving in a direction that matches the orientation imposed on the tubes, so the nine is positive. If we traversed from u_2 to u_1 via the smallest angle between the vectors we are moving in a direction counter to the orientation imposed on the tubes, so in this case the nine is negative, just as we would expect from the computation

$$(dx \wedge dy)(u_2, u_1) = \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} = -9.$$

Now imagine the vectors

$$v_1 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

imposed on Fig. 5.19. The parallelepiped spanned by these two vectors is parallel with the tubes and so does not cut through any of the tubes, hence graphically we have $dx \wedge dy(v_1, v_2) = 0$. Computationally we get the same thing

$$(dx \wedge dy)(v_2, v_1) = \begin{vmatrix} 3 & 0 \\ 0 & 0 \end{vmatrix} = 0.$$

Finally, imagine the vectors

$$w_1 = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \text{ and } w_2 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

imposed on Fig. 5.19. Again, the parallelepiped spanned by w_1 and w_2 is parallel with the tubes and so does not cut through any of them, giving $dx \wedge dy(w_1, w_2) = 0$, which again matches the computation

$$(dx \wedge dy)(w_2, w_1) = \begin{vmatrix} 0 & 0 \\ 3 & 0 \end{vmatrix} = 0.$$

The graphical representation for $dy \wedge dz$ is very similar to that of $dx \wedge dy$ and basically consists of both the dy and dz planes resulting in tubes that are parallel with the ∂_x-axis, as in Fig. 5.20. Again we will consider how many tubes the parallelepipeds spanned by u_1 and u_2, by v_1 and v_2, and by w_1 and w_2, intersect. Computationally we have

$$(dy \wedge dz)(u_1, u_2) = \begin{vmatrix} 0 & 3 \\ 0 & 0 \end{vmatrix} = 0,$$

$$(dy \wedge dz)(v_1, v_2) = \begin{vmatrix} 0 & 0 \\ 0 & 3 \end{vmatrix} = 0,$$

$$(dy \wedge dz)(w_1, w_2) = \begin{vmatrix} 3 & 0 \\ 0 & 3 \end{vmatrix} = 9.$$

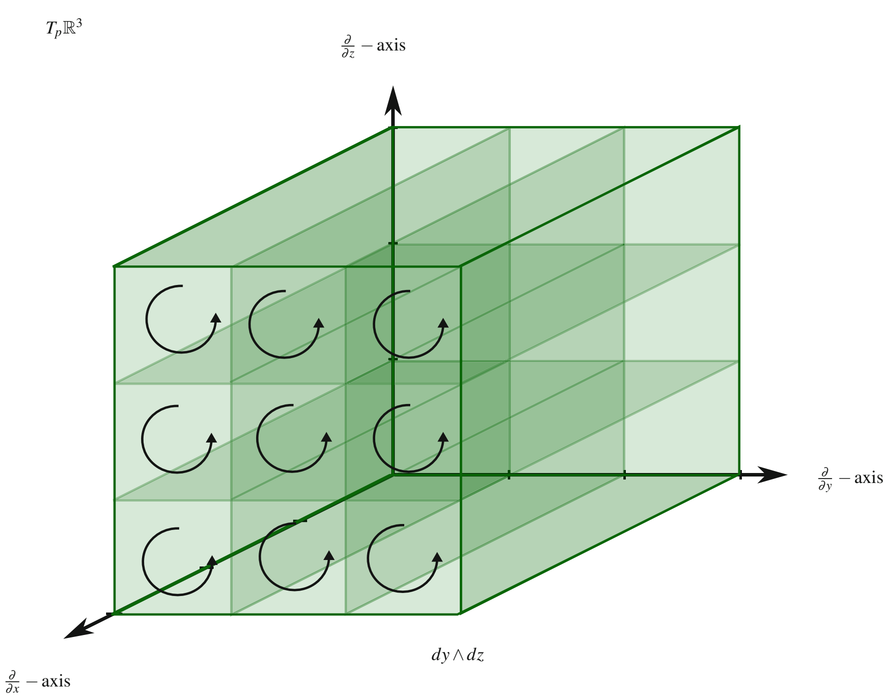

Fig. 5.20 The two-form $dy \wedge dz$ depicted in $T_p\mathbb{R}^3$ as a series of "tubes" made up of planes parallel to the ∂_y and ∂_z axes. The positive orientation is illustrated

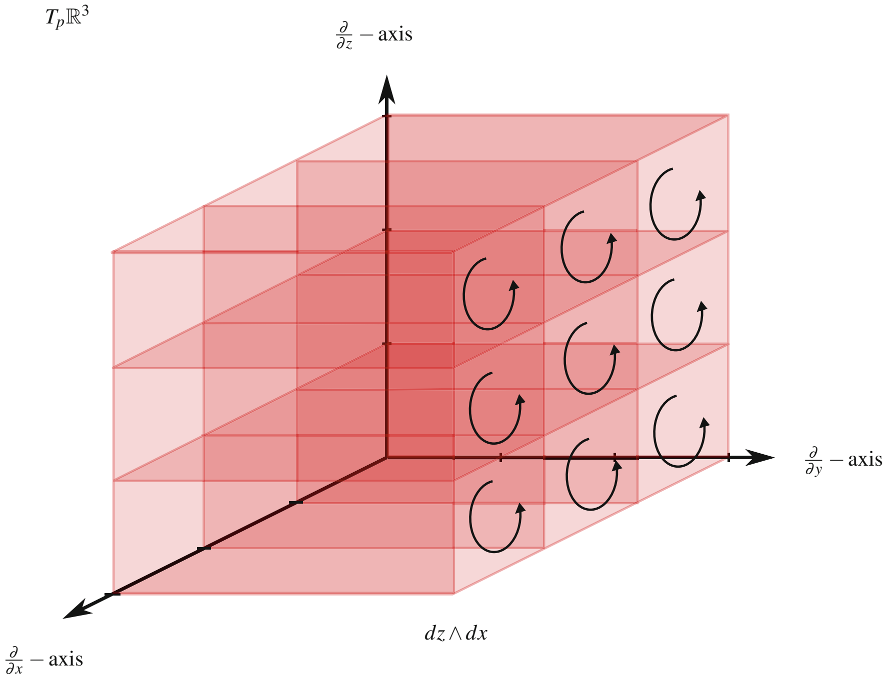

Fig. 5.21 The two-form $dz \wedge dx$ depicted in $T_p\mathbb{R}^3$ as a series of "tubes" made up of planes parallel to the ∂_z and ∂_x axes. The positive orientation is illustrated

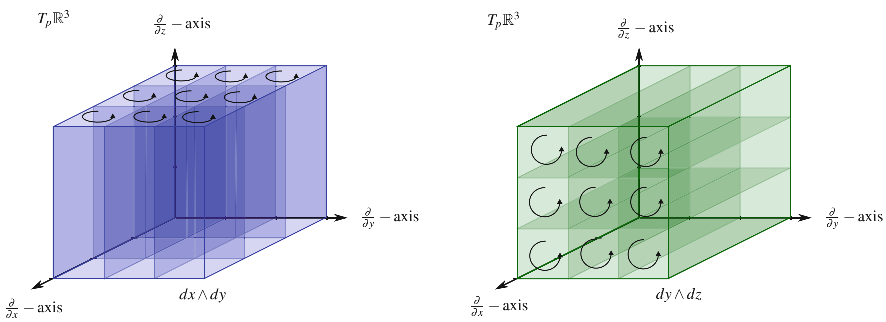

Fig. 5.22 To find the graphical representation of the two-form $dx \wedge dy + dy \wedge dz$ we have to somehow "add" the tubes of $dx \wedge dy$ (left) with those of $dy \wedge dz$ (right). The "addition" of these tubes is shown in Fig. 5.23

By imagining the vectors u_1 and u_2 imposed on the picture of $dy \wedge dz$. We can see that the parallelepiped spanned by these two vectors is parallel with the tubes and so does not cut through any of them. Similarly, the parallelepiped spanned by v_1 and v_2 does not cut through any tubes, but the parallelepiped spanned by w_1 and w_2 cuts through nine tubes.

Finally, the graphical representation for $dz \wedge dx$ consists of both the dz and dx planes resulting in tubes that are parallel with the ∂_y-axis. See Fig. 5.21. A similar analysis with the above vectors gives the expected results.

When finding a graphical representation for the two-forms $adx \wedge dy$, $bdy \wedge dz$, and $cdz \wedge dx$, where $a, b, c \in \mathbb{R}$, we have exactly the same ambiguities that we had in the-two dimensional case.

Question 5.12 Find at least three different graphical representations for the differential two-form $4dx \wedge dy$.

Now we consider general differential two-forms $adx \wedge dy + bdy \wedge dz + cdz \wedge dx$, where $a, b, c \in \mathbb{R}$. To make matters simple we first consider the two-form $dx \wedge dy + dy \wedge dz$. In essence we want to find a graphical representation that is the sum of these two pictures, see Fig. 5.22. The tubes of the differential two-form $dx \wedge dy$ are parallel to the ∂_z-axis and the tubes of the differential two-form $dy \wedge dz$ are parallel to the ∂_x-axis. Consider the unit vector in the ∂_x direction and the unit

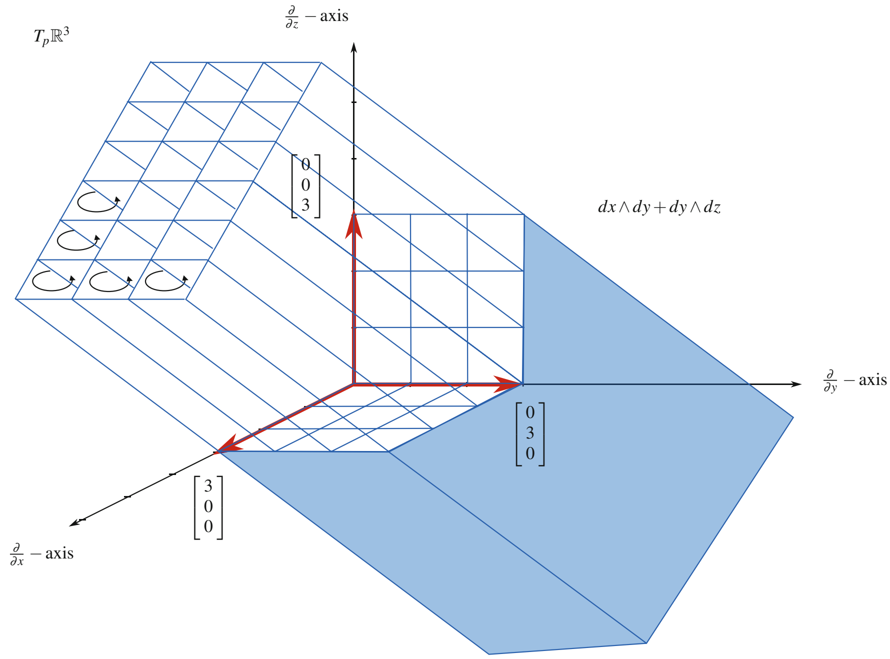

Fig. 5.23 The graphical representation of the two-form $dx \wedge dy + dy \wedge dz$ we get from "adding" the tubes of $dx \wedge dy$ and $dy \wedge dz$ shown in Fig. 5.22

vector in the ∂_z direction. When we add these two unit vectors we get

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

The tubes of $dx \wedge dy + dy \wedge dz$ are parallel to the line determined by this vector, as illustrated in Fig. 5.23.

Question 5.13 Let

$$u = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, v = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, w = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}.$$

With the aid of the Fig. 5.23 show that the answers given graphically match those given computationally for the following,

(a) $(dx \wedge dy + dy \wedge dz)(u, v)$,
(b) $(dx \wedge dy + dy \wedge dz)(v, w)$,
(c) $(dx \wedge dy + dy \wedge dz)(w, u)$.

Intuitively, the picture associated with $dx \wedge dy + dy \wedge dz$ appears to make sense. There is another way to see what is happening. Recall that the planes used to depict the one-form dx are perpendicular to the x (or ∂_x) axis and the planes used to depict dy are perpendicular to the y (or ∂_y) axis. We then found the image for $dx \wedge dy$ by simply depicting both of these sets of planes simultaneously. Our goal is to manipulate the two-form $dx \wedge dy + dy \wedge dz$ into a form which allows us to do something similar,

$$\begin{aligned} dx \wedge dy + dy \wedge dz &= -dy \wedge dx + dy \wedge dz \\ &= dy \wedge (-dx + dz) \\ &= dy \wedge d(-x + z). \end{aligned}$$

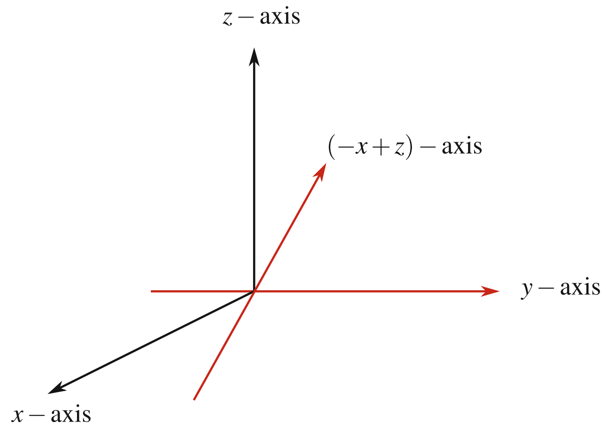

Fig. 5.24 The planes perpendicular to the y-axis and the $(-x+z)$-axis give the two-form $dx \wedge dy + dy \wedge dz$ depicted in Fig. 5.23

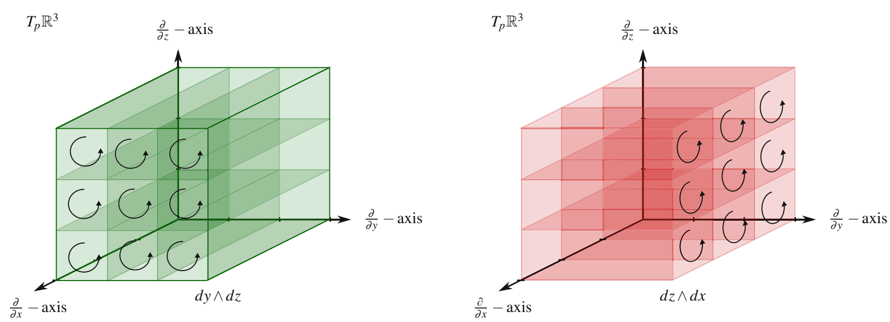

Fig. 5.25 To find the graphical representation of the two-form $dy \wedge dz + dz \wedge dx$ we need to "add" the tubes of $dy \wedge dz$ (left) and $dz \wedge dx$ (right)

So $dx \wedge dy + dy \wedge dz$ can be represented by using the planes perpendicular to the y axis and the $(-x+z)$ "axis." This last bit requires a little explanation. Consider the unit vectors in the x and z directions and add negative the first to the second to give us

$$-\begin{bmatrix}1\\0\\0\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}-1\\0\\1\end{bmatrix}.$$

The vector $\begin{bmatrix}-1\\0\\1\end{bmatrix}$ determines the axis that is used. The planes perpendicular to this axis is the second set of planes we use. In summary, for $dx \wedge dy + dy \wedge dz$ planes perpendicular to the two red axes drawn in Fig. 5.24 are used.

Now we want to find the picture for $dy \wedge dz + dz \wedge dx$, see Fig. 5.25. By inspection of the pictures for $dy \wedge dy$ and $dz \wedge dx$ we can see that the tubes of $dy \wedge dz + dz \wedge dx$ will we parallel with the vector $\begin{bmatrix}1\\1\\0\end{bmatrix}$. Or we could rewrite $dy \wedge dz + dz \wedge dx$ as before,

$$\begin{aligned}dy \wedge dz + dz \wedge dx &= -dz \wedge dy + dz \wedge dx\\&= dz \wedge (-dy + dx)\\&= dz \wedge d(-y + x),\end{aligned}$$

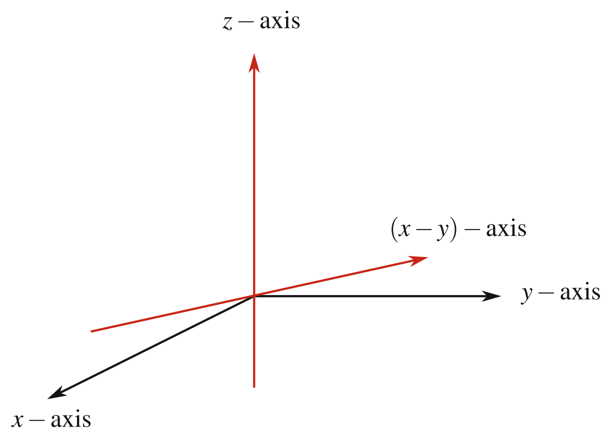

Fig. 5.26 The tubes of $dy \wedge dz + dz \wedge dx$ are made up of planes perpendicular to the x-axis and the $(x - y)$-axis

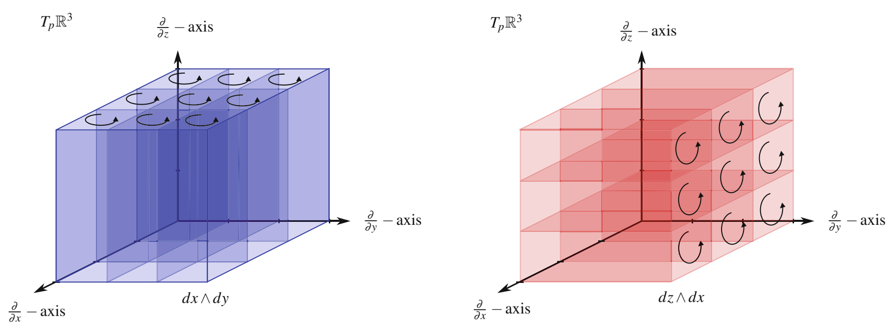

Fig. 5.27 The two-form $dx \wedge dy + dz \wedge dx$ is obtained by adding the tubes of $dx \wedge dy$ (left) with those of $dz \wedge dx$ (right)

and use planes perpendicular to the z axis and the $(-y+x)$ "axis," which is in the direction of the vector $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. These two axis are shown in red in Fig. 5.26.

Now we want to find the picture for $dx \wedge dy + dz \wedge dx$. See Fig. 5.27. By inspection we can see that the tubes of $dx \wedge dy + dz \wedge dx$ are parallel to the vector $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, or we could rewrite the two-form

$$\begin{aligned} dx \wedge dy + dz \wedge dx &= dx \wedge dy - dx \wedge dz \\ &= dx \wedge (dy - dz) \\ &= dx \wedge d(y - z) \end{aligned}$$

to get the two axes used to find the perpendicular planes. The two axes are shown in red in Fig. 5.28.

Finally we would like to find the picture for $dx \wedge dy + dy \wedge dz + dz \wedge dx$. See Fig. 5.29. We can rewrite the two-form $dx \wedge dy + dy \wedge dz + dz \wedge dx$ as follows,

$$\begin{aligned} dx \wedge dy + dy \wedge dz + dz \wedge dx &= dy \wedge d(-x+z) + dz \wedge dx \\ &= dy \wedge d(-x+z) - dz \wedge -dx + dz \wedge dz \\ &= dy \wedge d(-x+z) + dz \wedge (-dx + dz) \\ &= dy \wedge d(-x+z) + dz \wedge d(-x+z) \end{aligned}$$

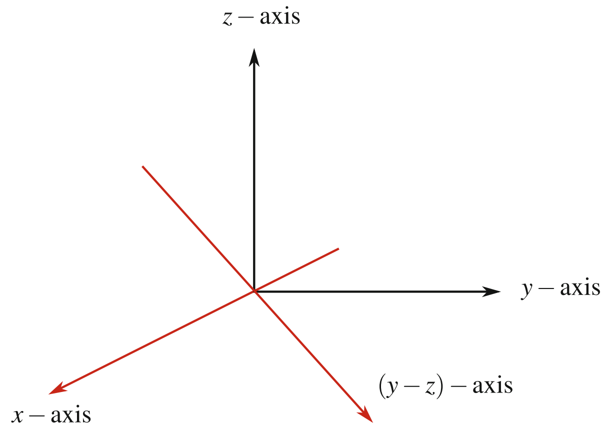

Fig. 5.28 The two-form $dx \wedge dy + dz \wedge dx$ is made up of planes perpendicular to the x-axis and the $(y - z)$-axis

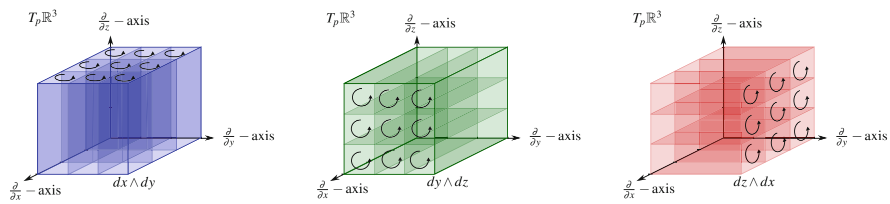

Fig. 5.29 The two-form $dx \wedge dy + dy \wedge dz + dz \wedge dx$ is obtained by "adding" the tubes for $dx + dy$ (left), $dy \wedge dz$ (middle), and $dz \wedge dx$ (right)

$$
\begin{aligned}
&= (dy + dz) \wedge d(-x + z) \\
&= d(y + z) \wedge d(-x + z),
\end{aligned}
$$

which can be used to sketch the two-form. Finally, when moving to two-forms with coefficients all the usual problems occur.

Question 5.14 Sketch the two-form $dy \wedge dz + dz \wedge dx$.

Question 5.15 Sketch the two-form $dx \wedge dy + dz \wedge dx$.

Question 5.16 Sketch the two-form $dx \wedge dy + dy \wedge dz + dz \wedge dx$.

Question 5.17 Sketch a pictures for each of the following two-forms

(a) $2dx \wedge dy + dy \wedge dz$,
(b) $dx \wedge dy + 3dy \wedge dz$,
(c) $2dx \wedge dy + 4dy \wedge dz$.

Question 5.18 Sketch a pictures for each of the following two-forms

(a) $3dy \wedge dz + 2dz \wedge dx$,
(b) $dx \wedge dy + 3dz \wedge dx$,
(c) $2dx \wedge dy + 4dy \wedge dz + 3dz \wedge dx$.

Question 5.19 Sketch the following two-forms

(a) $-dx \wedge dy + dy \wedge dz$,
(b) $dx \wedge dy - dy \wedge dz$,
(c) $-dx \wedge dy - dy \wedge dz$.

What affect do the negative signs have? Use some well chosen vectors to help you figure it out.

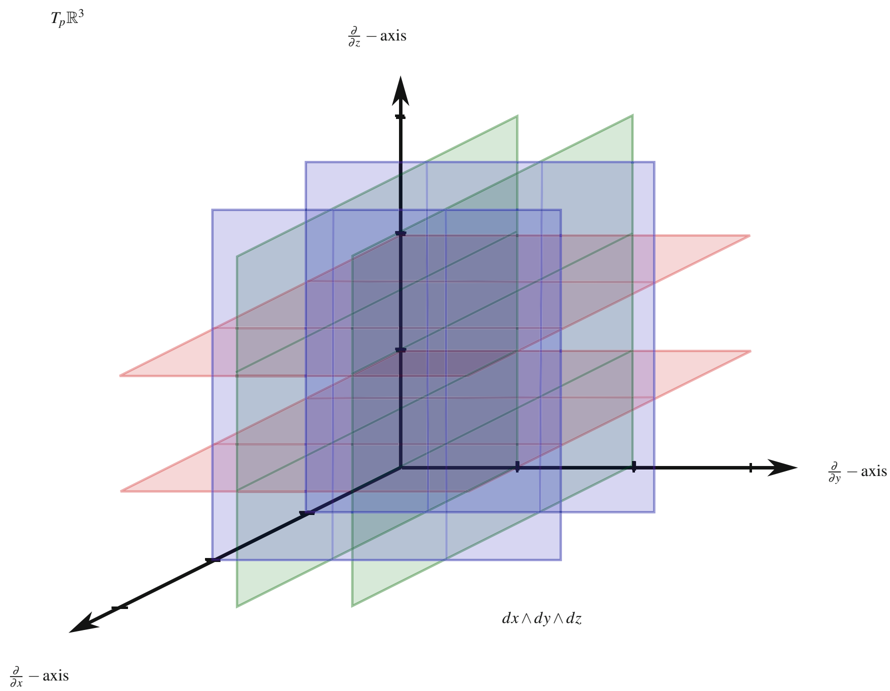

Fig. 5.30 The three-form $dx \wedge dy \wedge dz$ is made up of "cubes" whose sides are the planes perpendicular to the x-axis, the y-axis, and the z-axis

5.4 Three-Forms in $\mathbb{R}^3$

This is probably the shortest section in the entire book. And for a reason. Drawing three-forms in $T_p\mathbb{R}^3$ is in fact exceedingly simple. The space of three-forms on $\mathbb{R}$, denoted $\bigwedge^3(\mathbb{R})$, is actually of dimension one, which means that every three-form on $\mathbb{R}^3$ is of the form $c dx \wedge dy \wedge dz$ for some $c \in \mathbb{R}$. The one-form dx was given by the planes $x = n$, the one-form dy was given by the planes $y = n$, and the one-form dz was given by the planes $z = n$, where we had $n = 0, \pm 1, \pm 2, \ldots$. The three-form is simply all of these planes being drawn simultaneously, thereby filling space with unit cubes. See Fig. 5.30. For three-forms $a dx \wedge dy \wedge dz$, where $a \in \mathbb{R}$, the usual ambiguity arises.

Question 5.20 Based on Fig. 5.30 sketch three different pictures for the three-form $2dx \wedge dy \wedge dz$.

Question 5.21 Let

$$u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Evaluate $dx \wedge dy \wedge dz$ on different permutations of u, v, w and use your answers to determine what kind of orientation is $dx \wedge dy \wedge dz$ has. (Note, somehow all three directions have to be taken into account.)

Question 5.22 How might you depict the positive orientation for $dx \wedge dy \wedge dz$ in Fig. 5.30? What about the negative orientation?

5.5 Pictures of Forms on Manifolds

What we have done so far is to attempt to draw graphical representations of one-forms and two-forms in $\mathbb{R}^2$ and one-forms, two-forms, and three-forms in $\mathbb{R}^3$. In each case we looked at some form α_p at the point p and found a representation of α_p in the tangent space at the point p. For a one-form on $\mathbb{R}^2$ our picture was of lines in $T_p\mathbb{R}^2$ (Figs. 5.1–5.7), while for a two-form on $\mathbb{R}^2$ our picture was of boxes in $T_p\mathbb{R}^2$ (Figs. 5.8–5.10). For a one-form in $\mathbb{R}^3$ our picture was of planes in $T_p\mathbb{R}^3$ (Figs. 5.11–5.18), for a two-form in $\mathbb{R}^3$ our picture was of tubes in $T_p\mathbb{R}^3$ (Figs. 5.19–5.29), and for a three-form in $\mathbb{R}^3$ our pictures was of cubes in $T_p\mathbb{R}^3$ (Fig. 5.30).

Sometimes attempts are made to utilize these graphical representations of one-forms and two-forms on $\mathbb{R}^2$ and one-forms, two-forms, and three-forms on $\mathbb{R}^3$ to "draw" forms on the whole manifold, at least for manifolds that are two or three dimensional. There are two ways that this is generally done. The first method at least makes sense mathematically, though it can still end up being incomplete in some sense and can potentially misrepresent the differential form. The second method, unfortunately, has some serious mathematical shortcomings, in addition to potentially misrepresenting the differential form. But when these methods do work they can be an extremely helpful and illuminating way to think about differential forms on two and three dimensional manifolds. It is best to keep in the back of your mind that the pictures these methods give are merely cartoons at best.

First, we will present the first method of visualizing forms on manifolds. Essentially what is done here is that a lattice of points $p_{ij} = (x_i, y_j)$ on the manifold is chosen and the line-stacks that represent the one-form in $T_{p_{ij}}M$ are drawn on the manifold at each of these lattice points. Figure 5.31 shows how this is done for a series of simple one-forms on $\mathbb{R}^2$. The arrow points indicate positive orientation. Of course, we can choose lattices of points on the manifold with differing densities. Figure 5.32 shows the same one-form $xdx + ydy$ on $\mathbb{R}^2$ using two different lattices of points. A two-form would be similar, only with a small grid of boxes drawn at each point. The drawback of this method is the spacing of the lattice of points; it is entirely possible that important details of the differential form can be missed if one chooses an inappropriate lattice. Be that as it may, one generally still obtains a reasonable cartoon that can at least help one imagine the differential form.

For three dimensional manifolds the pictures are analogous. First a three-dimensional lattice of points $p_{ijk} = (x_i, y_j, z_k)$ is chosen on the manifold. A one-form would then be represented as small stacks of sheets at each point p_{ijk}, a two-form would be represented with small bundles of tubes at each point p_{ijk}, and a three-form would be represented with small sets of cubes at each point p_{ijk}.

The second method does something similar, but tries to take it one step further. Consider Fig. 5.32 again. In the second image the lattice of points was chosen close enough so that, particularly in the middle, they start to merge into each other and form concentric circles. In the second method the little pictures at the lattice points are connected up to each other to provide one continuous image on the manifold. When this is actually possible the images are very appealing and simple to think about. First we will look at how method two works when it is appropriate and then we will briefly discuss how method two fails in general.

As an example, a generic one-form α on $\mathbb{R}^3$ is shown in Fig. 5.33. Here the line-stacks at the lattice points (not shown) are connected, as well as is possible, to generate curves that cover the manifold. The value $\alpha(v)$ is (approximately) given by the number of curves the vector v pierces. The case of two-forms on the manifold $\mathbb{R}^2$ is handled identically, only instead of small stacks of lines at each lattice point one would have small patches of boxes. Again, it may be possible to connect up the boxes to give a rough image of the two-form on the manifold. Consider Fig. 5.34. Here a two-form on manifold $\mathbb{R}^2$ is depicted on the manifold $\mathbb{R}^2$. In order to find the value of the two-form α at any given point p two vectors at that point need to be given, v_1 and v_2. The value $\alpha_p(v_1, v_2)$ is then given by the number of boxes that are inside the parallelepiped spanned by v_1 and v_2.

Attempts to picture one-forms, two-forms, and three-forms on the manifold $\mathbb{R}^3$ are very similar. Again, a lattice of points $p_{ijk} = (x_i, y_j, z_k)$ is chosen and at each of these points the picture of the form in $T_{p_{ijk}}\mathbb{R}^3$ is found. For a one-form α at each point p_{ijk} one would have a stack of two-dimensional sheets, similar to Fig. 5.17. Small copies of each of these pictures would then be superimposed on the manifold $\mathbb{R}^3$ at each lattice point. One would then try to connect up these sheets, as best as one could, to arrive at a picture similar to that of Fig. 5.35. How closely the sheets are packed (in other words, how far apart the sheets are from each other) is determined by the particular one-form. To find the (approximate) value of $\alpha(v)$ one would then count the number of sheets the vector v pierces. So, the sheets of the one-form 10α are ten times as close to each other as the sheets of the one-form α.

For a two-form α at each point p_{ijk} one would have a bundle of "tubes" similar to Fig. 5.23. At each lattice point the bundle of tubes would have varying densities and be in different directions. One would then try to connect up these sheets, as best as one could, to arrive at a picture similar to that of Fig. 5.36, where tubes of varying sizes twist through $\mathbb{R}^3$. The density at which the tubes are packed depends on the exact nature of the two-form α. In order to (approximately) find $\alpha(v_p, w_p)$ one finds the parallelepiped spanned by v_p and w_p and counts the number of tubes that go through this parallelepiped.

For a three-form α at each point p_{ijk} one would have a little set of cubes, similar to Fig. 5.30. One could imagine Rubik's cubes of varying sizes and orientations at each lattice point. Then these cubes are all connected up as well as possible to fill $\mathbb{R}^3$ with little cubes. We will not attempted to show that here. Finding the (approximate) value of $\alpha(u_p, v_p, w_p)$ amounts to counting the number of cubes that are inside the parallelepiped spanned by u_p, v_p, w_p.

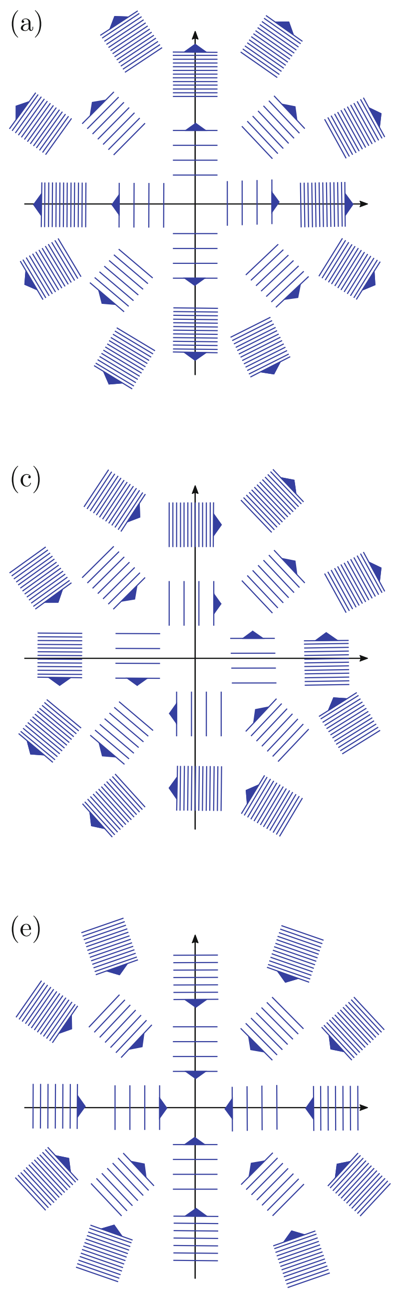

Fig. 5.31 One-forms on the manifold $\mathbb{R}^2$ represented as a grid of line-stacks on the manifold $\mathbb{R}^2$. The arrow points indicate positive direction (Images generated with Vector Field Analyzer II, Version 2, Kawski, 2001) . (**a**) $xdx + ydy$. (**b**) $-xdx + ydy$. (**c**) $ydx + xdy$. (**d**) $ydx - xdy$. (**e**) $-xdx - ydy$. (**f**) $-ydx - xdy$

There is something very appealing about this way of looking at differential forms, and in certain instances it helps provide an additional way of thinking about differential forms on two-dimensional and three-dimensional manifolds. Unfortunately, in general method two fails. In fact, one does not have to look very far to find examples of forms where it is actually impossible to visualize them using method two. One of the real issues is the "connecting" bit. Mathematically this is a complete slight-of-hand with no reasonable justification. In fact, each of the line-stacks for a one-form are in a different tangent space so "connecting" the line-stacks simply doesn't make sense. And even when we draw the pictures on the manifold as we have done, there is no reason to suppose that the line-stacks at points close to each other would be oriented in a way that would allow them to be connected, as was the case in Fig. 5.32. For a general one-form they need not be. And of course this applies to the plane-stacks and tube-bundles for three-dimensional manifolds as well.

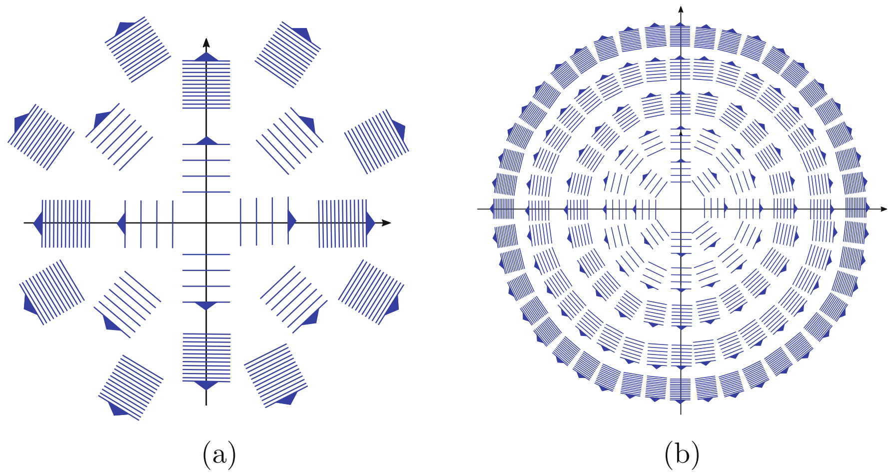

Fig. 5.32 The one-form $xdx + ydy$ on the manifold $\mathbb{R}^2$ represented as a grid of line-stacks using two different arrays of points. The arrow points indicate positive direction. Notice how in the image on the right the line-stacks in the middle start to look like they are merging into concentric circles (Images generated with Vector Field Analyzer II, Version 2, Kawski, 2001)

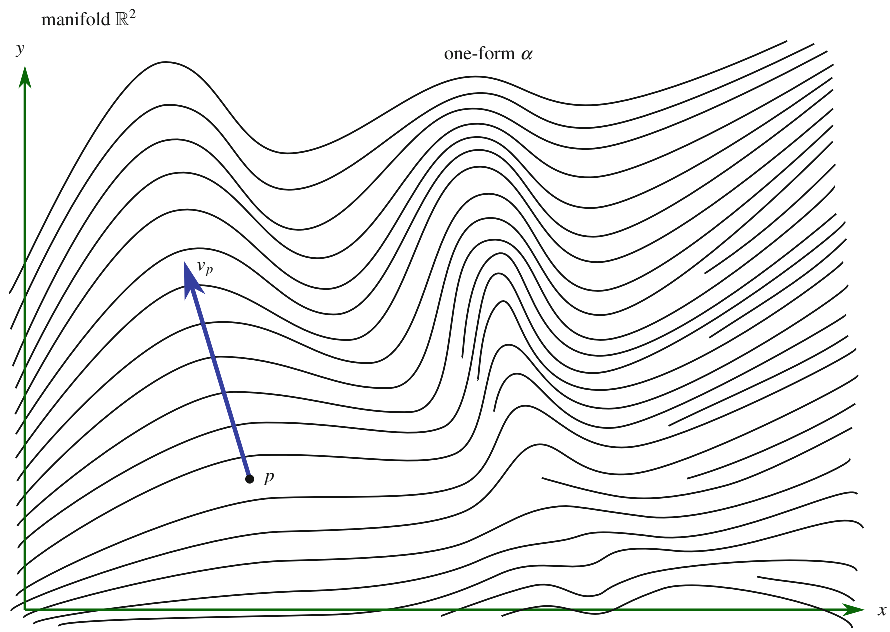

Fig. 5.33 A one-form α is shown on the manifold $\mathbb{R}^2$ with all the "line stacks" connected up smoothly as curves. The one-form applied to a vector is (approximately) equal to the number of curves "pierced" by the vector. Here, $\alpha(v_p) \approx 6$

A necessary, but not sufficient, requirement is that the distribution given by the kernel of the differential form be integrable. This would give what is called a foliation of the manifold which would allow the line-stacks, plane-stacks, or tube-bundles to line up. Exploring and explaining this is beyond the scope of this book. It is simply sufficient to realize that even a one-form on $\mathbb{R}^3$ as simple as $xdy + dz$ can not be visualized using method two.

However, this way of visualizing differential forms does appear in physics sometimes. Physicists are not so concerned about a completely general way of visualizing forms, they are interested in a way of visualizing forms that works for the physical problems and situations they are dealing with. Thus this method does have some utility in certain situations. When it is appropriate it does in fact provides a very nice way of thinking about forms. In fact, it gives a nice picture to go along with Stokes' theorem that is very appealing and will be presented in Sect. 11.6.

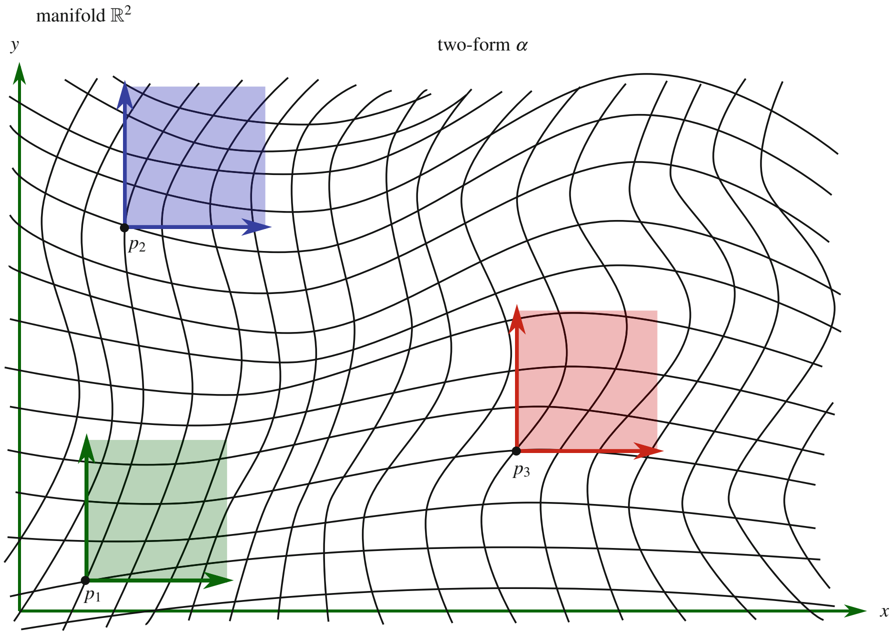

Fig. 5.34 A two-form α is shown on the manifold $\mathbb{R}^2$ with all boxes connected up smoothly. The unit vectors e_1 and e_2 are depicted at points p_1 (green), p_2 (blue), and p_3 (red). One (approximately) finds $\alpha_{p_i}(e_1, e_2)$ by counting the number of boxes that fall within the parallelepiped spanned by e_1 and e_2 at each of these points. Thus $\alpha_{p_1}(e_1, e_2) \approx 12$, $\alpha_{p_2}(e_1, e_2) \approx 16$, and $\alpha_{p_3}(e_1, e_2) \approx 9$. It is clear that while this is a nice picture the answers it gives are only very approximate

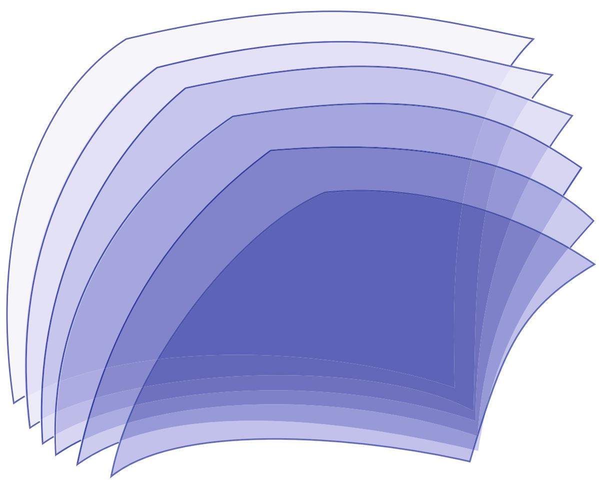

Fig. 5.35 A generic one-form in $\mathbb{R}^3$ is comprised of sheets filling $\mathbb{R}^3$

5.6 A Visual Introduction to the Hodge Star Operator

Introducing the Hodge star operator in a mathematically rigorous way at this point is a little tricky since much of the necessary mathematics has not yet been introduced, yet the geometric pictures of the Hodge star operator that go along with this chapter are really quite nice and easy to understand, so now is a natural time to do it. Thus we will try to strike a balance, providing some of the mathematics behind the pictures but not enough to overwhelm you with theoretical details. For the moment we will stick to the three dimensional case and give plausibility arguments instead of rigorous proofs.

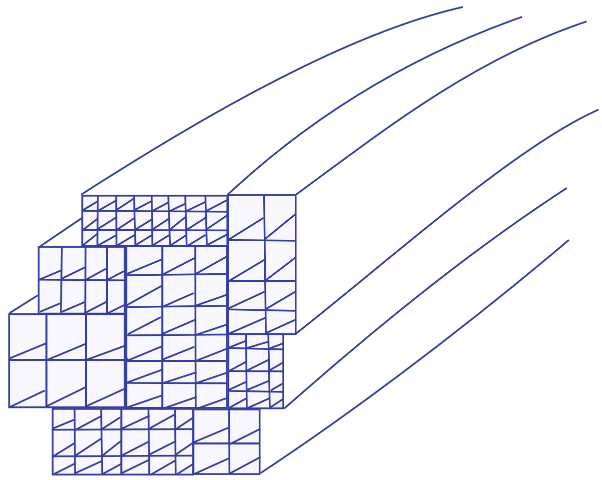

Fig. 5.36 A generic two-form in $\mathbb{R}^3$ is comprised of tubes filling $\mathbb{R}^3$

First we will consider the following vector spaces

$$\bigwedge\nolimits_p^0(\mathbb{R}^3) = \text{span}\{1\},$$
$$\bigwedge\nolimits_p^1(\mathbb{R}^3) = \text{span}\{dx, dy, dz\} = T_p^*\mathbb{R}^3,$$
$$\bigwedge\nolimits_p^2(\mathbb{R}^3) = \text{span}\{dx \wedge dy, dy \wedge dz, dz \wedge dx\},$$
$$\bigwedge\nolimits_p^3(\mathbb{R}^3) = \text{span}\{dx \wedge dy \wedge dz\},$$

which have dimension one, three, three, and one respectively. Each of these spaces has what is called an *inner product* defined on them. Without getting distracted by a rigorous definition at this point we will simply say that in essence an inner product on a vector space associates with each pair of vectors a real number. With that said, what we will do is explain the inner product on $\bigwedge_p^1(\mathbb{R}^3)$ and $\bigwedge_p^2(\mathbb{R}^3)$, the two spaces we are most interested in. We begin with the space $\bigwedge_p^1(\mathbb{R}^3)$. The inner product on $\bigwedge_p^1(\mathbb{R}^3)$ is given by the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

But how does this work? Suppose we have two one-forms elements $\alpha, \beta \in \bigwedge_p^1(\mathbb{R}^3)$. We can write these elements in terms of the basis elements as co-vectors

$$\alpha = a\,dx + b\,dy + c\,dz = [a, b, c],$$
$$\beta = r\,dx + s\,dy + t\,dz = [r, s, t].$$

Then the inner product of α and β, which is generally denoted by $\langle \alpha, \beta \rangle$, is given by

$$\begin{aligned} \langle \alpha, \beta \rangle &= \langle [a, b, c], [r, s, t] \rangle \\ &\equiv [a, b, c] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [r, s, t]^T \end{aligned}$$

$$= [a, b, c] \begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

$$= ar + bs + ct.$$

We take a moment to note that using angled brackets $\langle \cdot, \cdot \rangle$ to denote inner products is standard notation, and is quite a different thing from using angled brackets $\langle \cdot, \cdot \rangle$ to denote the canonical pairing between differential forms and vectors that we used earlier. You must pay careful attention to what is inside the brackets to know what they represent. Also, notice that we are implicitly assuming our basis elements have an order, dx first, dy second, and dz third. Only by knowing that order did we know how to write the matrix representing the inner product.

Similarly, we will define the inner product on the $\bigwedge_p^2(\mathbb{R}^3)$ by the same matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which works in exactly the same way. Thus if we have

$$\eta = adx \wedge dy + bdy \wedge dz + cdz \wedge dx = [a, b, c],$$

$$\xi = rdx \wedge dy + sdy \wedge dz + tdz \wedge dx = [r, s, t]$$

the inner product of η and ξ, which is denoted by $\langle \alpha, \beta \rangle$, is given by

$$\langle \eta, \xi \rangle = \langle [a, b, c], [r, s, t] \rangle$$

$$\equiv [a, b, c] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [r, s, t]^T$$

$$= [a, b, c] \begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

$$= ar + bs + ct.$$

Again, we implicitly assume that our basis had an order so we would know how to write the matrix.

Now we have the pieces necessary pieces to actually define the **Hodge star operator**, which is also called the Hodge star dual operator. The Hodge star operator is an isomorphism (or mapping) $* : \bigwedge_p^k(\mathbb{R}^n) \to \bigwedge_p^{n-k}(\mathbb{R}^n)$ that takes k-forms to $(n-k)$-forms. So, how is this isomorphism defined? For each k-form α there is a unique $(n-k)$-form $*\alpha$ such that for any $(n-k)$-form we have

Hodge Star Operator	$\alpha \wedge \beta = \langle *\alpha, \beta \rangle \sigma$	for all β

where the $\langle \cdot, \cdot \rangle$ represents the inner product on $\bigwedge_p^{n-k}(\mathbb{R}^n)$ and σ is the n dimensional volume form. So, $*\alpha$ is defined in terms of this fairly complicated relationship. We would like to unpack this relationship and see what it is trying to say,

$$\underbrace{\underbrace{\alpha}_{k\text{-form}} \wedge \underbrace{\beta}_{(n-k)\text{-form}}}_{\text{an } n\text{-form}} = \underbrace{\langle \underbrace{*\alpha}_{(n-k)\text{-form}}, \underbrace{\beta}_{(n-k)\text{-form}} \rangle}_{\text{a number}} \underbrace{\sigma}_{\substack{\text{volume form} \\ \text{(an } n\text{-form)}}}.$$

So we end up with an n-form on both the left and the right hand side of the equality. Recall, the volume form σ is known, the α is given, and the β can be any $(n-k)$-form at all. We then use this relationship to find out which $(n-k)$-form $*\alpha$ is. Since β can be any $(n-k)$-form at all this means that no matter which $(n-k)$-form is chosen to be β we will always find exactly the same $*\alpha$. In a way that is rather amazing. But the mapping $*$ is unique, a fact which we will not try to prove here, so there is only one $*\alpha$ no matter which $(n-k)$-form is chosen to be β.

Now, let's actually put this relationship to work in our three dimensional case. We will first look at the mapping $*: \bigwedge_p^1(\mathbb{R}^3) \to \bigwedge_p^2(\mathbb{R}^3)$. First we will tackle finding $*dx$. Since we know that $*dx$ is a two-form we know that it must have the form

$$*dx = a\ dx \wedge dy + b\ dy \wedge dz + c\ dz \wedge dx = [a, b, c]$$

for some numbers a, b, c. Our goal is to find out what a, b, and c are using this relationship

$$dx \wedge \beta = \langle *dx, \beta \rangle dx \wedge dy \wedge dz.$$

For the moment we will consider a generic two-form β given by

$$\beta = r\ dx \wedge dy + s\ dy \wedge dz + t\ dz \wedge dx = [r, s, t].$$

So our defining relationship becomes

$$\begin{aligned}
&dx \wedge \big(r\ dz \wedge dy + s\ dy \wedge dz + t\ dz \wedge dx\big) \\
&= \big\langle *dx, r\ dz \wedge dy + s\ dy \wedge dz + t\ dz \wedge dx \big\rangle dx \wedge dy \wedge dz \\
&= \big\langle [a, b, c], [r, s, t] \big\rangle dx \wedge dy \wedge dz \\
&= (ar + bs + ct) dx \wedge dy \wedge dz.
\end{aligned}$$

On the left hand side we have

$$\begin{aligned}
&dx \wedge \big(r\ dx \wedge dy + s\ dy \wedge dz + t\ dz \wedge dx\big) \\
&= r\ \underbrace{dx \wedge dx \wedge dy}_{=0} + s\ dx \wedge dy \wedge dz + t\ \underbrace{dx \wedge dz \wedge dx}_{=0} \\
&= s\ dx \wedge dy \wedge dz.
\end{aligned}$$

Combining we have

$$s\ dx \wedge dy \wedge dz = (ar + bs + ct) dx \wedge dy \wedge dz,$$

which gives us

$$s = ar + bs + ct.$$

Since our defining relationship is true no matter what β we had chosen, if we had chosen β such that $r = 1$, $s = 0$, and $t = 0$ this equality would give us that $a = 0$. If we had chosen β such that $r = 0$, $s = 1$, and $t = 0$ this equality would have given us $b = 1$. If we had chosen β such that $r = 0$, $s = 0$, and $t = 1$ this equality would have given us $c = 0$. Thus we have that

$$*dx = dy \wedge dz.$$

The relationship between dx and $*dx$ is shown pictorially in Fig. 5.37. The tubes of $*dx = dy \wedge dz$ are perpendicular to the planes of dx.

Question 5.23 Show that $dx \wedge \beta = \langle *dx, \beta \rangle dx \wedge dy \wedge dx$ for the following β,

(a) $\beta = 7dx \wedge dy - 3dy \wedge dz + 2dz \wedge dx$,
(b) $\beta = 20dx \wedge dy + 15dy \wedge dz - 10dz \wedge dx$,
(c) $\beta = -4dx \wedge dy - 6dy \wedge dz - 8dz \wedge dx$.

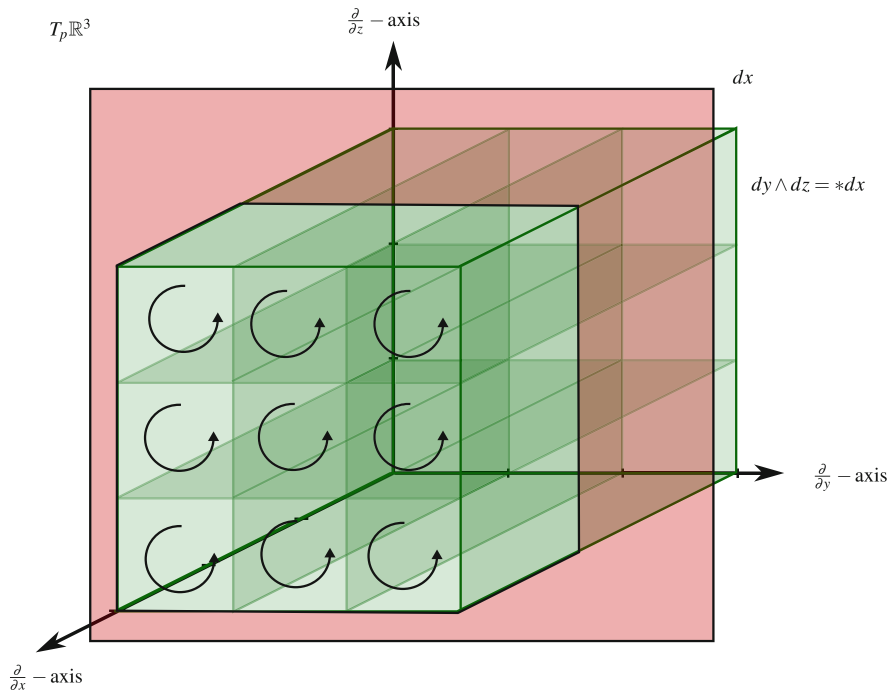

Fig. 5.37 The tubes of $*dx$ are perpendicular to the planes of dx. That is, $*dx = dy \wedge dz$

We will redo the calculation to find $*dy$. As before, since we know that $*dy$ is a two-form we know that it must have the form

$$*dy = a\ dx \wedge dy + b\ dy \wedge dz + c\ dz \wedge dx = [a, b, c]$$

for some numbers a, b, c and we wish to find out what a, b, and c are using this relationship

$$dy \wedge \beta = \langle *dy, \beta \rangle dx \wedge dy \wedge dz.$$

Again, we assume

$$\beta = r\ dx \wedge dy + s\ dy \wedge dz + t\ dz \wedge dx = [r, s, t]$$

resulting in

$$\begin{aligned} & dy \wedge \big(r\ dz \wedge dy + s\ dy \wedge dz + t\ dz \wedge dx\big) \\ &= \big\langle *dy, r\ dz \wedge dy + s\ dy \wedge dz + t\ dz \wedge dx \big\rangle dx \wedge dy \wedge dz \\ &= (ar + bs + ct) dx \wedge dy \wedge dz. \end{aligned}$$

The left hand side becomes

$$\begin{aligned} & dy \wedge \big(r\ dx \wedge dy + s\ dy \wedge dz + t\ dz \wedge dx\big) \\ &= r \underbrace{dy \wedge dx \wedge dy}_{=0} + s \underbrace{dy \wedge dy \wedge dz}_{=0} + t\ dy \wedge dz \wedge dx \\ &= -t\ dy \wedge dx \wedge dz \\ &= t\ dx \wedge dy \wedge dz. \end{aligned}$$

Combining we have

$$t\,dx \wedge dy \wedge dz = (ar + bs + ct)dx \wedge dy \wedge dz,$$

which gives us

$$t = ar + bs + ct.$$

Since our defining relationship is true no matter what β we chose, if we had chosen β such that $r = 1$, $s = 0$, and $t = 0$ we would have $a = 0$. If we had chosen it such that $r = 0$, $s = 1$, and $t = 0$ we would have $b = 0$. And if we had chosen it such that $r = 0$, $s = 0$, and $t = 1$ we would have $c = 1$. Thus we would have

$$*dy = dz \wedge dx.$$

Again, notice that the tubes of $*dy$ are perpendicular to the planes of dy. See Fig. 5.38.

Question 5.24 Show that $dy \wedge \beta = \langle *dy, \beta \rangle dx \wedge dy \wedge dx$ for the following β,

(a) $\beta = 7dx \wedge dy - 3dy \wedge dz + 2dz \wedge dx$,
(b) $\beta = 20dx \wedge dy + 15dy \wedge dz - 10dz \wedge dx$,
(c) $\beta = -4dx \wedge dy - 6dy \wedge dz - 8dz \wedge dx$.

Question 5.25 Find $*dz$ using the defining identity $dz \wedge \beta = \langle *dz, \beta \rangle dx \wedge dy \wedge dz$ for all $\beta \in \bigwedge_p^2(\mathbb{R}^3)$.

The image associated with $*dz$ shows the tubes of $*dz$ being perpendicular to the planes of dz, see Fig. 5.39.

Question 5.26 Show that $dz \wedge \beta = \langle *dz, \beta \rangle dx \wedge dy \wedge dx$ for the following β,

(a) $\beta = 7dx \wedge dy - 3dy \wedge dz + 2dz \wedge dx$,
(b) $\beta = 20dx \wedge dy + 15dy \wedge dz - 10dz \wedge dx$,
(c) $\beta = -4dx \wedge dy - 6dy \wedge dz - 8dz \wedge dx$.

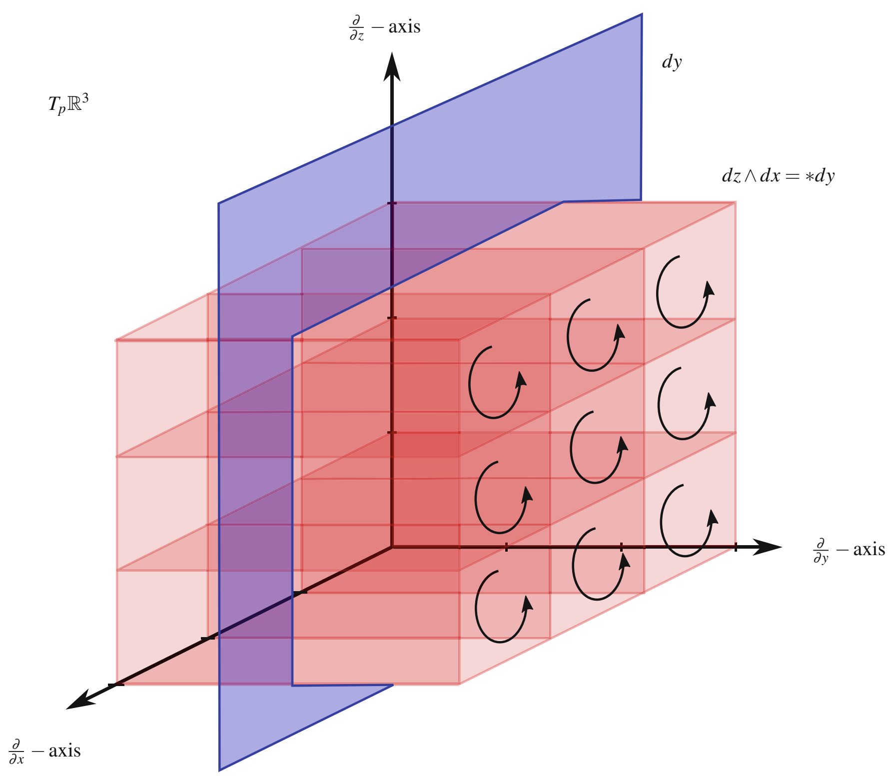

Fig. 5.38 The tubes of $*dy$ are perpendicular to the planes of dy. That is, $*dy = dz \wedge dx$

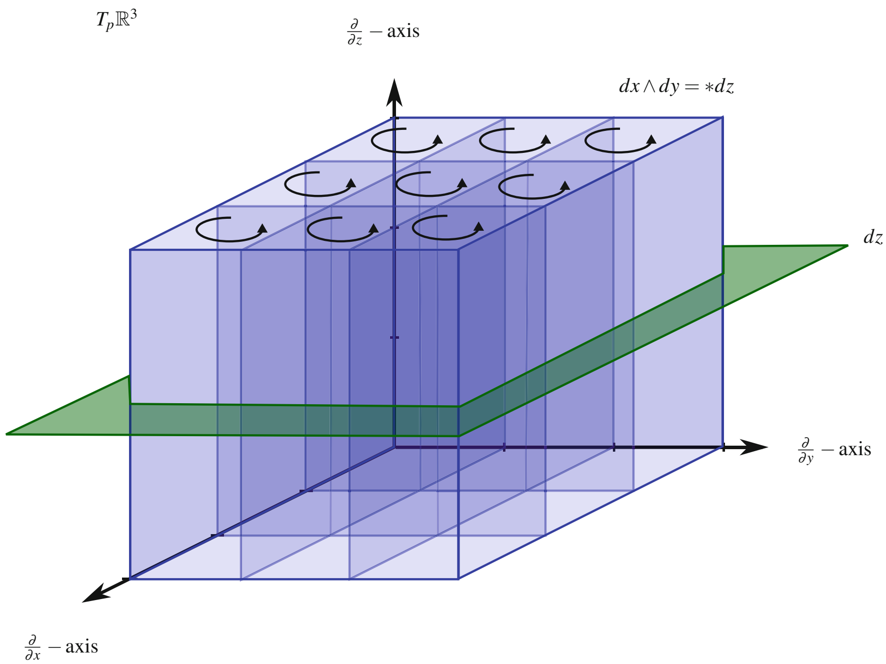

Fig. 5.39 The tubes of $*dz$ are perpendicular to the planes of dz. That is, $*dz = dx \wedge dy$

Question 5.27 Using the definition of the Hodge star operator find $*dx \wedge dy$, $*dy \wedge dz$, and $*dz \wedge dx$.

Question 5.28 You will sometimes see an alternative definition of the Hodge star operator. Since this is a somewhat easier definition to deal with we have left it for an exercise. Show that the alternative definition for the Hodge star operator

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \sigma,$$

for all α, gives the identical values for $*dx$, $*dy$, $*dz$, $*dx \wedge dy$, $*dy \wedge dz$, $*dz \wedge dx$.

We will now look at the Hodge star dual operator on forms on $\mathbb{R}^4$. In four dimensions is it much more difficult to try to draw pictures, but the mathematical computations are quite straight-forward. First we will define notation and our spaces. We will use x_1, x_2, x_3, x_4 as our Cartesian coordinate functions. Writing things cyclicly to keep things tidy worked out nicely in $\mathbb{R}^3$ but it is not so helpful in $\mathbb{R}^4$ so we will no longer do that. The convention here is to use increasing order. The spaces we will be working with are

$$\begin{aligned}
\bigwedge^0\left(\mathbb{R}^4\right) &= C\left(\mathbb{R}^4\right) = \text{ functions on } \mathbb{R}^4, \\
\bigwedge^1\left(\mathbb{R}^4\right) &= \text{Span}\left\{dx_1, dx_2, dx_3, dx_4\right\}, \\
\bigwedge^2\left(\mathbb{R}^4\right) &= \text{Span}\left\{dx_1 \wedge dx_2, dx_1 \wedge dx_3, dx_1 \wedge dx_4, dx_2 \wedge dx_3, dx_2 \wedge dx_4, dx_3 \wedge dx_4\right\}, \\
\bigwedge^3\left(\mathbb{R}^4\right) &= \text{Span}\left\{dx_1 \wedge dx_2 \wedge dx_3, dx_1 \wedge dx_2 \wedge dx_4, dx_1 \wedge dx_3 \wedge dx_4, dx_2 \wedge dx_3 \wedge dx_4\right\}, \\
\bigwedge^4\left(\mathbb{R}^4\right) &= \text{Span}\left\{dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4\right\}.
\end{aligned}$$

The inner products on these spaces, necessary for the right hand side of the definition of the Hodge star dual, are given by the identity matrix. We will remain a bit ambiguous now but we discuss the metric of a manifold, from which the inner product is derived, in greater detail in Sect. A.6.

Now we find the Hodge star dual of a one-form basis element using the definition of the Hodge star dual given in Question 5.28,

$$dx_1 \wedge *dx_1 = \underbrace{\langle dx_1, dx_1 \rangle}_{=1} dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$$
$$\Longrightarrow *dx_1 = dx_2 \wedge dx_3 \wedge dx_4.$$

Notice that by wedging dx_1 with $*dx_1$ we were able to use the nice property of the inner product on $\bigwedge^1 \left(\mathbb{R}^4\right)$ to our advantage on the right hand side of the equation where we have $\langle dx_1, dx_1 \rangle = 1$. Finding the hodge star duals of two-form basis elements is similar,

$$(dx_1 \wedge dx_2) \wedge *(dx_1 \wedge dx_2) = \underbrace{\langle dx_1 \wedge dx_2, dx_1 \wedge dx_2 \rangle}_{=1} dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$$
$$\Longrightarrow *(dx_1 \wedge dx_2) = dx_3 \wedge dx_4.$$

Question 5.29 Find $*dx_2, *dx_3, *dx_4$.

Question 5.30 Find $*(dx_1 \wedge dx_3), *(dx_1 \wedge dx_4), *(dx_2 \wedge dx_3), *(dx_2 \wedge dx_4), *(dx_3 \wedge dx_4)$.

Question 5.31 Find $*(dx_1 \wedge dx_2 \wedge dx_3), *(dx_1 \wedge dx_2 \wedge dx_4), *(dx_1 \wedge dx_3 \wedge dx_4), *(dx_2 \wedge dx_3 \wedge dx_4)$.

Question 5.32 Find $*(dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4)$.

5.7 Summary, References, and Problems

5.7.1 Summary

We now take a moment to conclude this chapter with some remarks. If you are a mathematics major you will probably never see this way of picturing or representing differential forms in any of your textbooks; I certainly never did. If, however, you are a physics major there is a good chance that may see differential forms discussed or presented in this way, either in an electromagnetics course or in a course in relativity. These images are very useful in trying to understand certain equations and picture the physical phenomenon behind those equations. We will be exploring some of these physics applications in future chapters in detail. Though these pictures can be a great aid to picturing what equations are trying to say, using them in actual computations is generally difficult and not worth the effort since they only give approximate answers. Use the computational rules of differential forms to calculate exact numerical answers and use the pictures to aid conceptual understanding.

Also, one can develop higher dimensional analogues to these images. For example, taking into account time, and viewing space-time from the pre-relativity Euclidian perspective, results in a four dimensional version of this. Including time in both the special relativity and general relativity cases also results in four dimensional versions, but in these cases since the metric (think inner product) on the manifolds is different, the Hodge star operator, which is based on the inner product, changes. However, the higher the dimension the more difficult it is to employ these pictures and the less they are used, though it is still generally possible to imagine the 1 and the $n - 1$ dimensional cases well enough.

The two equivalent definitions of the Hodge star operator are given below,

Hodge Star Operator Definition One	$\alpha \wedge \beta = \langle *\alpha, \beta \rangle \sigma$	for all β,

Hodge Star Operator Definition Two	$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \sigma$	for all α.

5.7.2 References and Further Reading

The material in this chapter is encountered almost exclusively in physics. There is a reason for this - as a rigorous way of thinking about differential forms it falls short on several fronts. See Bachman [4] for this critique of visualizing forms on manifolds and where this technique falls short. But as a fairly loose way of thinking about differential forms in various physical situations it is quite useful and can be quite illuminating from the standpoint of physical systems. The primary source for much of this material is, as mentioned at the start of the chapter, Misner, Thorne, and Wheeler [34], though Dray [16], Warnick, Selfridge, and Arnold [49], Warnick and Russer [48], and DesChapes [13] were also used. All of these sources do fall within the domain of physics. In fact, the last three references strongly argue the use of these visualization techniques for differential forms as a pedagogical tool for teaching electromagnetism, though we know of no reasonably complete attempt to actually do so. The only other genuinely mathematical source that attempts to give a geometrical explanation along the lines of the material given in this chapter is Burke [8], and his intended audience is again working physicists. The software used for generating the images of one-forms as line-stacks in $\mathbb{R}^2$ was the Vector Field Analyzer II written by Kawski [28] and available online.

5.7.3 Problems

Question 5.33 Sketch the coordinate function z on $\mathbb{R}^2$ and find the "rise" of the coordinate function y along the vectors

a) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, that is, from the point $(0, 0)$ to the point $(0, 1)$,

b) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, that is, from the point $(0, 0)$ to the point $(1, 1)$,

c) $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$, that is, from the point $(0, 0)$ to the point $(-1, 2)$,

d) $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$, that is, from the point $(0, 0)$ to the point $(2, -3)$.

Question 5.34 Draw a picture representing the one-forms $7dx$, $-3dy$, $\frac{-2}{3}dx$, $\frac{-1}{3}dy$, and $\frac{2}{5}dy$ on $\mathbb{R}^2$.

Question 5.35 Draw the graphical representations of the one-forms $\frac{1}{2}dx + 2dy$, $\frac{-1}{3}dx - \frac{1}{2}dy$, and $-dx - \frac{2}{3}dy$ on $\mathbb{R}^2$.

Question 5.36 Draw three graphical representations for each of the two-forms $5\ dx \wedge dy$, $\frac{1}{2}\ dx - \frac{1}{2}\ dy$, and $2dx + \frac{2}{3}dy$ on $\mathbb{R}^2$.

Question 5.37 Sketch the one-forms $dx - dy$, $-dx + dy$, $dy - dz$, and $-dx - dz$ on $\mathbb{R}^3$.

Question 5.38 Find two graphical representations for each of the two-forms $5dx \wedge dy$, $6dy \wedge dz$, and $\frac{1}{4}dz \wedge dx$ on $\mathbb{R}^3$.

Question 5.39 Sketch a picture for each of the two-forms $3dy \wedge dz + 2dz \wedge dx$, $dx \wedge dy + 3dz \wedge dx$, and $2dx \wedge dy + 4dy \wedge dz + 3dz \wedge dx$ on $\mathbb{R}^3$.

Question 5.40 Sketch the two-forms $-dx \wedge dy + dy \wedge dz$, $dx \wedge dy - dy \wedge dz$, and $-dx \wedge dy - dy \wedge dz$ on $\mathbb{R}^3$. What affect do the negative signs have? Use some well chosen vectors to help you figure it out.

Question 5.41 Match the below one-forms with the images depicted in Fig. 5.40

a) $\sin(x)dx + ydy$,
b) $dx - dy$,
c) $xdx + \cos(y)dy$,
d) $\sin(x)dx + xdy$.

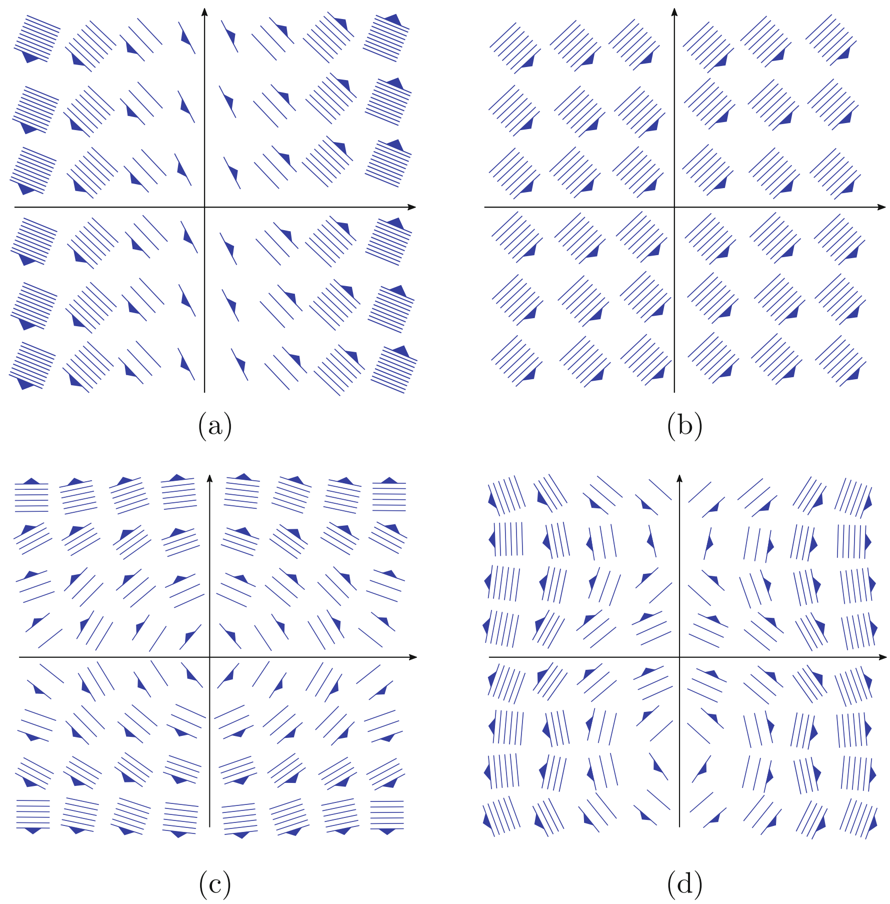

Fig. 5.40 Images for Question 5.41 (Images generated with Vector field analyzer II, version 2, Kawski, 2001)

Question 5.42 Assume the manifold is $\mathbb{R}^4$. Find $*dx_2$, $*dx_3$, $*dx_4$.

Question 5.43 Assume the manifold is $\mathbb{R}^4$. Find $*(dx_1 \wedge dx_3)$, $*(dx_1 \wedge dx_4)$, $*(dx_2 \wedge dx_3)$, $*(dx_2 \wedge dx_4)$, $*(dx_3 \wedge dx_4)$.

Question 5.44 Assume the manifold is $\mathbb{R}^4$. Find $*(dx_1 \wedge dx_2 \wedge dx_3)$, $*(dx_1 \wedge dx_2 \wedge dx_4)$, $*(dx_1 \wedge dx_3 \wedge dx_4)$, $*(dx_2 \wedge dx_3 \wedge dx_4)$.

Question 5.45 Assume the manifold is $\mathbb{R}^4$. Find $*(dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4)$.

Chapter 6
Push-Forwards and Pull-Backs

In this chapter we introduce two extremely important concepts, the push-forward of a vector and the pull-back of a differential form. In section one we take a close look at a simple change of coordinates and see what affect this change of coordinates has on the volume of the unit square. This allows us to motivate the push-forward of a vector in section two. Push-forwards of vectors allow us to move, or "push-forward," a vector from one manifold to another. In the case of coordinate changes the two manifolds are actually the same manifold, only equipped with different coordinate systems.

Using the fact that differential forms eat vectors, in section three we use the push-forwards of vectors to define the pull-back of a differential form, where we "pull-back" the differential form on one manifold to another manifold. Again, in the case of coordinate changes the two manifolds are really the same manifold, only equipped with different coordinate systems. We then look more closely at the pull-back of a volume form, which plays an essential role in the integration problems in the section three. We are able to derive a nice formula for the pull-back of a volume form that involves the Jacobian matrix of the coordinate change.

We then use push-forwards and pull-backs to look closely at some familiar examples. Polar coordinates are treated in section four and both cylindrical and spherical coordinates are considered in section five. In section six we consider the pull-back of differential forms that are not volume forms, and finally in section seven we prove three identities that are crucial for us to do computations involving pull-backs.

6.1 Coordinate Change: A Linear Example

This section will serve as our first look at coordinate changes. We aren't going to try to be at all rigorous in this section, this is really an attempt to just start to understand the big picture a little better. Before we start getting fancy with polar and spherical coordinates, let's consider something simple. A change in coordinates is nothing more than a mapping between $\mathbb{R}^n$ and $\mathbb{R}^n$ that is both one-to-one and onto. Since $\mathbb{R}^n$ is the same as $\mathbb{R}^n$ we often think of it as a mapping from $\mathbb{R}^n$ to itself. Consider the two mappings $u, v : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$u(x, y) = x + y,$$
$$v(x, y) = x - y.$$

Figure 6.1 shows this change of coordinates. On the left is the $\mathbb{R}^2$ in xy-coordinates, which we will also call the xy-plane or $\mathbb{R}^2_{xy}$, and on the right is $\mathbb{R}^2$ in uv-coordinates, which we will also call the uv-plane or $\mathbb{R}^2_{uv}$. So, even though $\mathbb{R}^2$ is $\mathbb{R}^2$, we are making a distinction between the two copies based on what coordinates we are using. On the top left the x and y grid lines from the xy-plane are mapped to lines on the uv-plane on the right, and on the bottom right the u and v grid lines are mapped to lines on the xy-plane on the left.

Question 6.1 Find the image of the following lines under the mapping $u = x + y$ and $v = x - y$ and compare what you find with the mapping shown at the top of Fig. 6.1.

$a)\ x = 0$	$b)\ x = 1$	$c)\ x = 4$
$d)\ y = 0$	$e)\ y = 1$	$f)\ y = -3$

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https://doi.org/10.1007/978-3-319-96992-3_6

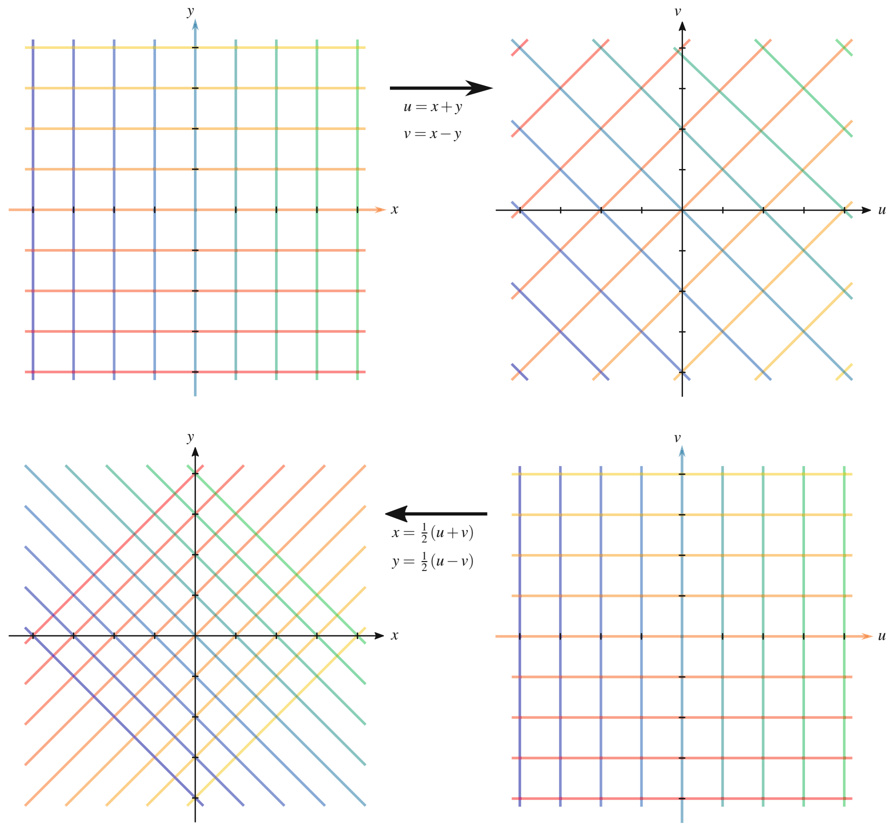

Fig. 6.1 The planes $\mathbb{R}^2_{xy}$ (left) and $\mathbb{R}^2_{uv}$ (right). The change of coordinates given by $u(x, y) = x + y$ and $v(x, y) = x - y$ is on the top, left to right. The inverse change of coordinates $x(u, v) = 0.5(u + v)$ and $y(u, v) = 0.5(u - v)$ is on the bottom, right to left. The color coding shows how various lines are mapped

Question 6.2 Find the image of the following lines under the inverse mapping $x = \frac{1}{2}(u + v)$ and $y = \frac{1}{2}(u - v)$ and compare what you find with the mapping shown at the bottom of Fig. 6.1.

$a)\ u = 0$	$b)\ u = 1$	$c)\ u = 4$
$d)\ v = 0$	$e)\ v = 1$	$f)\ v = -3$

We use the coordinate functions $u = x + y$ and $v = x - y$ to map points in the plane $\mathbb{R}^2_{xy}$ to points in $\mathbb{R}^2_{uv}$ as follows:

$$\begin{aligned}
\underline{(x, y)} &\longrightarrow \underline{(x + y, x - y) = (u, v)} \\
(0, 0) &\longrightarrow (0 + 0, 0 - 0) = (0, 0) \\
(1, 0) &\longrightarrow (1 + 0, 1 - 0) = (1, 1) \\
(1, 1) &\longrightarrow (1 + 1, 1 - 1) = (2, 0) \\
(0, 1) &\longrightarrow (0 + 1, 0 - 1) = (1, -1)
\end{aligned}$$

As we move around the unit square in $\mathbb{R}^2_{xy}$ we can follow what happen in $\mathbb{R}^2_{uv}$.

$$\begin{aligned}
(x, y):&\quad (0, 0) \longrightarrow (1, 0) \longrightarrow (1, 1) \longrightarrow (0, 1) \longrightarrow (0, 0) \\
(u, v):&\quad (0, 0) \longrightarrow (1, 1) \longrightarrow (2, 0) \longrightarrow (1, -1) \longrightarrow (0, 0)
\end{aligned}$$

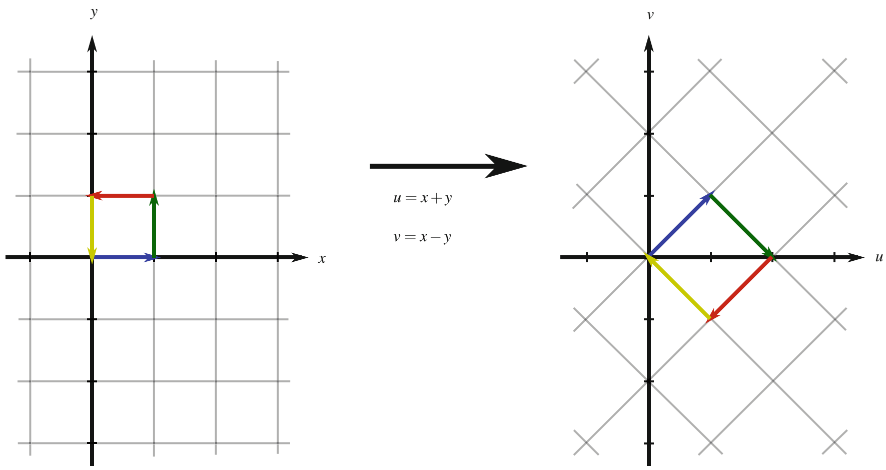

Fig. 6.2 A unit square in $\mathbb{R}^2_{xy}$ (left) is mapped to the diamond in $\mathbb{R}^2_{uv}$ (right). Notice the orientation switches from counter-clockwise to clockwise and the area increases from one to two

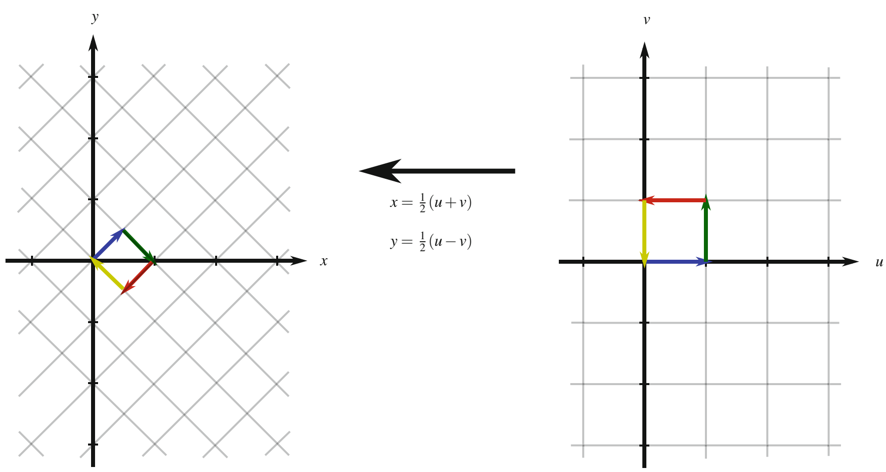

Fig. 6.3 A unit square in $\mathbb{R}^2_{uv}$ (right) is mapped to the diamond in $\mathbb{R}^2_{xy}$ (left). Notice the orientation switches from counter-clockwise to clockwise and the area decreases from one to a half

With this we can see, as shown in Fig. 6.2, that the unit square from the xy-coordinates gets mapped to a diamond in uv-coordinates. Notice that as you move around the unit square in xy-coordinates in a counter-clockwise direction, shown on the left of Fig. 6.2, you move around the diamond in the uv-coordinates in a clockwise direction, shown on the right of Fig. 6.2. There is also another interesting difference. Consider the area of the unit square in the xy-plane; it has an area of one, while the area of the diamond in the uv-plane, which is the image of the unit square, has an area of two.

Question 6.3 Show what happens to the points (0, 0), (1, 0), (1, 1), and (0, 1) under the inverse mapping $x = \frac{1}{2}(u + v)$ and $y = \frac{1}{2}(u - v)$.

Similarly, if we used the inverse mapping to take a unit square from the uv-coordinate plane, shown on the right of Fig. 6.3, to the xy-coordinate plane we get a diamond in the xy-coordinate plane, shown on the left of Fig. 6.3, with half the area and the direction reversed. Next notice that

$$dx \wedge dy \underbrace{\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{(0,0)}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{(0,0)} \right)}_{\text{vectors in } xy\text{-plane}} = 1,$$