4 Basics about Brownian motion

The continuous-time analogue (and limit, in an appropriate sense) of the Cox–Ross–Rubinstein binomial model is the Black–Scholes model of geometric Brownian motion. To be able to study this later in Chapter 7, we collect in this chapter some basic facts and results about *Brownian motion*. This is the stochastic process driving the Black–Scholes model; but it is of fundamental importance in many other areas as well. Very loosely, one can think of Brownian motion as a dynamic version of the normal distribution, with a comparable status as an object of central significance.

Throughout this chapter, we work on a probability space $(\Omega, \mathcal{F}, P)$ which is tacitly assumed to be big and rich enough for our purposes. In particular, Ω cannot be finite or countable. We also work with a *filtration* $\mathbb{F} = (\mathcal{F}_t)$ in continuous time; this is like in discrete time a family of σ-fields $\mathcal{F}_t \subseteq \mathcal{F}$ with $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \leq t$. The time parameter runs either through $t \in [0, T]$ with a fixed time horizon $T \in (0, \infty)$ or through $t \in [0, \infty)$. In the latter case, we define

$$\mathcal{F}_\infty := \bigvee_{t \geq 0} \mathcal{F}_t := \sigma\bigg(\bigcup_{t \geq 0} \mathcal{F}_t\bigg).$$

For technical reasons, we should also assume (or make sure, if we construct the filtration in some way) that $\mathbb{F}$ satisfies the so-called *usual conditions* of being right-continuous and P-complete, but we do not dwell on this technical mathematical issue in more detail.

4.1 Definition and first properties

Definition. A *Brownian motion with respect to* P *and a filtration* $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is a (real-valued) stochastic process $W = (W_t)_{t \geq 0}$ which is adapted to $\mathbb{F}$, starts at 0 (i.e. $W_0 = 0$ P-a.s.) and satisfies the following properties:

(BM1) For $s \leq t$, the *increment* $W_t - W_s$ is independent (under P) of $\mathcal{F}_s$ with (under P) a normal distribution $\mathcal{N}(0, t - s)$.

(BM2) W has *continuous trajectories*, meaning that for P-almost all $\omega \in \Omega$, the function $t \mapsto W_t(\omega)$ on $[0, \infty)$ is continuous.

Remarks. **1)** One can prove that Brownian motion exists, but this is a nontrivial mathematical result. See the course on "Brownian Motion and Stochastic Calculus" (in short BMSC) for more details.

2) The letter W is used in honour of Norbert Wiener who gave the first rigorous proof of the existence of Brownian motion in 1923. It is historically interesting to note, however, that Brownian motion was already introduced and used considerably earlier in both finance and physics — by Louis Bachelier in his PhD thesis in 1900 for finance and by Albert Einstein in 1905 for physics.

3) Brownian motion in $\mathbb{R}^m$ is simply an adapted $\mathbb{R}^m$-valued stochastic process null at 0 with (BM2) and such that (BM1) holds with $\mathcal{N}(0, t-s)$ replaced by $\mathcal{N}(0, (t-s)I_{m\times m})$, where $I_{m\times m}$ denotes the $m \times m$ identity matrix. This is equivalent to saying that the m components are all real-valued Brownian motions and independent (as processes). $\diamond$

There is also a definition of Brownian motion (BM for short) without any filtration $\mathbb{F}$. This is a (real-valued) stochastic process $W = (W_t)_{t\geq 0}$ which starts at 0, satisfies (BM2) and instead of (BM1) the following property:

(BM1$'$) For any $n \in \mathbb{N}$ and any times $0 = t_0 < t_1 < \cdots < t_n < \infty$, the increments $W_{t_i} - W_{t_{i-1}}$, $i = 1, \dots, n$, are independent (under P) and we have (under P) that $W_{t_i} - W_{t_{i-1}} \sim \mathcal{N}(0, t_i - t_{i-1})$, or $\sim \mathcal{N}(0, (t_i - t_{i-1})I_{m\times m})$ if W is $\mathbb{R}^m$-valued.

Instead of (BM1$'$), one also says (in words) that W has *independent stationary increments* with a (specific) normal distribution.

The two definitions of BM are equivalent if one chooses as $\mathbb{F}$ the filtration $\mathbb{F}^W$ generated by W (and made right-continuous and P-complete). This (like many other subsequent results and facts) needs a proof, which we do not give. More details can be found in the lecture notes on "Brownian Motion and Stochastic Calculus".

There are several *transformations* that produce a new Brownian motion from a given one, and this can in turn be used to prove results about BM. More precisely:

Proposition 1.1. *Suppose $W = (W_t)_{t\geq 0}$ is a BM. Then:*

1) *$W^1 := -W$ is a BM.*

2) *$W_t^2 := W_{t+T} - W_T$, $t \geq 0$, is a BM for any $T \in (0, \infty)$* (restarting at a fixed time T).

3) *$W_t^3 := cW_{\frac{t}{c^2}}$, $t \geq 0$, is a BM for any $c \in \mathbb{R}$, $c \neq 0$* (rescaling in space and time).

4) *$W_t^4 := W_{T-t} - W_T$, $0 \leq t \leq T$, is a BM on $[0, T]$ for any $T \in (0, \infty)$* (time-reversal).

5) *The process W_t^5, $t \geq 0$, defined by*

$$W_t^5 := \begin{cases} tW_{\frac{1}{t}} & \text{for } t > 0 \\ 0 & \text{for } t = 0 \end{cases}$$

is a BM (inversion of small and large times).

(Note that we always use here the definition of BM without an exogenous filtration.)

While parts 1)–4) of Proposition 1.1 are easy to prove, part 5) is a bit more tricky. However, it is also very useful because it relates the asymptotic behaviour of BM as $t \to \infty$ to the behaviour of BM close to time 0, and vice versa.

The next result gives some information about how *trajectories of BM* behave.

Proposition 1.2. *Suppose $W = (W_t)_{t \geq 0}$ is a BM. Then:*

1) Law of large numbers: $\lim_{t \to \infty} \frac{W_t}{t} = 0$ *P-a.s., i.e. BM grows more slowly than linearly as $t \to \infty$.*

2) (Global) Law of the iterated logarithm (LIL): *With $\psi_{\text{glob}}(t) := \sqrt{2t \log(\log t)}$, we have*

$$\left.\begin{array}{c} \limsup_{t \to \infty} \\ \liminf_{t \to \infty} \end{array}\right\} \frac{W_t}{\psi_{\text{glob}}(t)} = \left\{\begin{array}{c} +1 \\ -1 \end{array}\right. \qquad P\text{-a.s.},$$

i.e., for P-almost all ω, the function $t \mapsto W_t(\omega)$ for $t \to \infty$ oscillates precisely between $t \mapsto \pm\psi_{\text{glob}}(t)$.

3) (Local) Law of the iterated logarithm (LIL): *With* $\psi_{\text{loc}}(h) := \sqrt{2h \log(\log \frac{1}{h})}$, *we have for every* $t \geq 0$

$$\left.\begin{matrix} \limsup\limits_{h \searrow 0} \\ \liminf\limits_{h \searrow 0} \end{matrix}\right\} \frac{W_{t+h} - W_t}{\psi_{\text{loc}}(h)} = \begin{cases} +1 \\ -1 \end{cases} \qquad P\text{-a.s.},$$

i.e., for P*-almost all* ω*, to the right of* t*, the trajectory* $u \mapsto W_u(\omega)$ *around the level* $W_t(\omega)$ *oscillates precisely between* $h \mapsto \pm\psi_{\text{loc}}(h)$.

One immediate consequence of 2) and 3) is that BM crosses the level 0 (or, with a bit more effort for the proof, any level a) infinitely many times — and once it is at that level, it even manages to achieve infinitely many crossings in an arbitrarily short amount of time. This is already a first indication of the amazingly strong activity of BM.

We remark that part 1) of Proposition 1.2 is easily proved by using part 5) of Proposition 1.1. Moreover, part 2) follows directly from part 3) via part 5) of Proposition 1.1, and for proving part 3), it is enough to take $t = 0$, by part 2) of Proposition 1.1, and to prove the lim sup result, by part 1) of Proposition 1.1. But then the easy reductions stop and the proof becomes difficult.

The oscillation results in Proposition 1.2 already make it clear that the trajectories of BM behave rather wildly. Another result in that direction is

Proposition 1.3. *Suppose* $W = (W_t)_{t \geq 0}$ *is a BM. Then for* P*-almost all* $\omega \in \Omega$*, the function* $t \mapsto W_t(\omega)$ *from* $[0, \infty)$ *to* $I\!R$ *is continuous, but nowhere differentiable.*

The deeper reason behind the wild behaviour of Brownian trajectories, and the key to understanding stochastic calculus and Itô's formula for BM, is that Brownian trajectories are *continuous functions having* a *nonzero quadratic variation*. Heuristically, this can be seen as follows. By definition, Brownian motion increments $W_{t+h} - W_t$ have a normal distribution $\mathcal{N}(0,h)$, which implies they are symmetric around 0 with variance h so that roughly, "$W_{t+h} - W_t \approx \pm\sqrt{h}$ with probability $\frac{1}{2}$ each". In very loose and purely formal terms, this means that infinitesimal increments "$\mathrm{d}W_t = W_t - W_{t-\mathrm{d}t}$" of BM have the property that

$$\text{"}(\mathrm{d}W_t)^2 = \mathrm{d}t\text{"}.$$

While this is very helpful for an *intuitive understanding*, we emphasise that it is *purely formal* and must not be used for rigorous mathematical arguments. A more precise description is as follows.

Call a *partition* of $[0,\infty)$ any set $\Pi \subseteq [0,\infty)$ of time points with $0 \in \Pi$ and such that $\Pi \cap [0,T]$ is finite for all $T \in [0,\infty)$. This implies that Π is at most countable and can be ordered increasingly as $\Pi = \{0 = t_0 < t_1 < \cdots < t_m < \cdots < \infty\}$. The *mesh size* of Π is then defined as $|\Pi| := \sup\{t_i - t_{i-1} : t_{i-1}, t_i \in \Pi\}$, i.e. the size of the biggest time-step in Π. For any partition Π of $[0,\infty)$, any function $g : [0,\infty) \to I\!\!R$ and any $p > 0$, we first define the *p-variation of g on $[0,T]$ along Π* as

$$V_T^p(g,\Pi) := \sum_{t_i \in \Pi} |g(t_i \wedge T) - g(t_{i-1} \wedge T)|^p.$$

One can then define the *p-variation of g on $[0,T]$* as

$$V_T^p(g) := \sup_{\Pi} V_T^p(g,\Pi),$$

where the supremum is taken over all partitions Π of $[0,\infty)$. For a sequence $(\Pi_n)_{n\in I\!\!N}$ of partitions of $[0,\infty)$ with $\lim_{n\to\infty} |\Pi_n| = 0$, one can also define the *p-variation of g on $[0,T]$ along $(\Pi_n)_{n\in I\!\!N}$* as

$$\lim_{n\to\infty} V_T^p(g,\Pi_n),$$

provided that the limit exists.

With the above notations, a function g is of *finite variation* or has *finite 1-variation* if $V_T^1(g) < \infty$ for every $T \in (0,\infty)$. The interpretation is that the graph of g has finite length on any time interval. More precisely, if we define the *arc length* of (the graph of) g on the interval $[0,T]$ as

$$\sup_{\Pi} \sum_{t_i \in \Pi} \sqrt{(t_i \wedge T - t_{i-1} \wedge T)^2 + \big(g(t_i \wedge T) - g(t_{i-1} \wedge T)\big)^2},$$

with the supremum again taken over all partitions Π of $[0,\infty)$, then g has finite variation on $[0,T]$ if and only if it has finite arc length on $[0,T]$. This can be checked by using the inequality $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$ for $a,b \ge 0$.

Any monotonic (increasing or decreasing) function is clearly of finite variation, because the absolute values above disappear and we get a telescoping sum. Moreover, one can show that *any function of finite variation can be written as the difference of two increasing functions (and vice versa).*

Now let us return to Brownian motion, taking $p=2$ and as g one trajectory $W_\bullet(\omega)$. Then

$$Q_T^\Pi := \sum_{t_i \in \Pi} (W_{t_i \wedge T} - W_{t_{i-1} \wedge T})^2 = V_T^2(W_\bullet, \Pi)$$

is the *sum* up to time T *of* the *squared increments* of BM along Π. With the above formal intuition "$(\mathrm{d}W_t)^2 = \mathrm{d}t$", we then expect, at least for $|\Pi|$ very small so that time points are close together, that $(W_{t_i \wedge T} - W_{t_{i-1} \wedge T})^2 \approx t_i \wedge T - t_{i-1} \wedge T$ and hence

$$Q_T^\Pi \approx \sum_{t_i \in \Pi} (t_i \wedge T - t_{i-1} \wedge T) = T \qquad \text{for } |\Pi| \text{ small.}$$

Even if the above reasoning is only heuristic, the result surprisingly is correct:

Theorem 1.4. *Suppose $W = (W_t)_{t \ge 0}$ is a BM. For any sequence $(\Pi_n)_{n \in \mathbb{N}}$ of partitions of $[0,\infty)$ which is refining (i.e. $\Pi_n \subseteq \Pi_{n+1}$ for all n) and satisfies $\lim_{n\to\infty} |\Pi_n| = 0$, we have*

$$P\Big[\lim_{n\to\infty} Q_t^{\Pi_n} = t \text{ for every } t \ge 0\Big] = 1.$$

We express this by saying that along $(\Pi_n)_{n \in \mathbb{N}}$, the Brownian motion W has (with probability 1) quadratic variation *t on $[0,t]$ for every $t \ge 0$, and we write $\langle W \rangle_t = t$. (We sometimes also say, with a certain abuse of terminology, that P-almost all trajectories $W_\bullet(\omega) : [0,\infty) \to \mathbb{R}$ of BM have quadratic variation t on $[0,t]$, for each $t \ge 0$.)*

Remark 1.5. 1) It is a very nice and useful [$\rightarrow$ *exercise*] in analysis to prove that *every continuous function f which has nonzero quadratic variation along a sequence (Π_n) as* *above must have infinite variation*, i.e. unbounded oscillations. (This will come up again later in Section 6.1.) More generally, if $\lim_{n\to\infty} V_T^q(f,\Pi_n) > 0$ for some $q > 0$, then $\lim_{n\to\infty} V_T^p(f,\Pi_n) = +\infty$ for any p with $0 < p < q$, and if $\lim_{n\to\infty} V_T^p(f,\Pi_n) < \infty$ for some $p > 0$, then $\lim_{n\to\infty} V_T^q(f,\Pi_n) = 0$ for all $q > p$. We also recall that a classical result due to Lebesgue says that any function of finite variation is almost everywhere differentiable. So Proposition 1.3 implies that Brownian trajectories must have infinite variation, and Theorem 1.4 makes this even quantitative.

2) Caution: The comment in 1) is only true for *continuous* functions. With RCLL functions, this breaks down in general.

3) It is important in Theorem 1.4 that the partitions Π_n do not depend on the trajectory $W_\bullet(\omega)$, but are fixed a priori. One can show for P-almost all trajectories $W_\bullet(\omega)$, the (true) quadratic variation of $W_\bullet(\omega)$ is $+\infty$.

4) There is an extension of Theorem 1.4 to general local martingales M instead of Brownian motion W. But then the limit, called $[M]_t$, of the sequence $(Q_t^{\Pi_n}(M))_{n\in\mathbb{N}}$ is not t, but some ($\mathcal{F}_t$-measurable) random variable, and the convergence holds not P-almost surely, but only in probability. (Alternatively, one can obtain P-a.s. convergence along a sequence of partitions, but then this cannot be chosen, but is only shown to exist.) Moreover, $t \mapsto [M]_t(\omega)$ is then always increasing (for P-almost all ω), but only continuous if M itself has continuous trajectories. Finally, as for Brownian motion, the limit does not depend on the sequence $(\Pi_n)_{n\in\mathbb{N}}$ of partitions. ◇

4.2 Martingale properties and results

There are many martingales which are naturally associated to Brownian motion, and this is useful in many different contexts. We present here just a small sample that will be used or useful later.

As in discrete time, a *martingale* with respect to P and $\mathbb{F}$ is a (real-valued) stochastic process $M = (M_t)$ such that M is adapted to $\mathbb{F}$, M is P-integrable in the sense that each M_t is in $L^1(P)$, and the *martingale property* holds: for $s \leq t$, we have

$$E[M_t \,|\, \mathcal{F}_s] = M_s \qquad P\text{-a.s.} \tag{2.1}$$

If we have in (2.1) the inequality "$\leq$" instead of "$=$", then M is a *supermartingale*; if we have "$\geq$", then M is a *submartingale*. Of course, $\mathbb{F} = (\mathcal{F}_t)$ and $M = (M_t)$ should have the same time index set.

Remark 2.1. Because our filtration satisfies the usual conditions, a general result from the theory of stochastic processes says that any martingale has a version with nice (*RCLL*, i.e. *right-continuous with left limits*, to be precise) *trajectories*. We can and do therefore always assume that our martingales have nice trajectories in that sense, and this is important for some of the subsequent results. We shall point this out more explicitly when it is used. $\diamond$

Again exactly like in discrete time, a *stopping time* with respect to $\mathbb{F}$ is a mapping $\tau : \Omega \to [0, \infty]$ such that $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. One of the standard examples is the first time that some adapted right-continuous process X (e.g. Brownian motion W) hits an open set B (e.g. (a, ∞)), i.e.

$$\tau := \inf\{t \geq 0 : X_t \in B\} \qquad \big(= \inf\{t \geq 0 : W_t > a\}, \text{ for } X = W \text{ and } B = (a, \infty)\big).$$

We remark that checking the stopping time property above uses that the filtration is right-continuous; and we mention that τ above is still a stopping time if B is allowed to be a Borel set, but the proof of this apparently minor extension is surprisingly difficult.

One of the most useful properties of martingales is that the martingale property (2.1) and its consequences very often extend to the case where the fixed times $s \leq t$ are replaced by stopping times $\sigma \leq \tau$. "Very often" means under additional conditions, as we shall see presently. To make sense of (2.1) for σ and τ, we also first need to define, for a stopping time σ, the σ-field of *events observable up to time* σ as

$$\mathcal{F}_\sigma := \big\{A \in \mathcal{F} : A \cap \{\sigma \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\big\}.$$

(One must and can check that $\mathcal{F}_\sigma$ is a σ-field, and that one has $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$ for $\sigma \leq \tau$.) We also need to define M_τ, the value of M at the stopping time τ, by

$$(M_\tau)(\omega) := M_{\tau(\omega)}(\omega).$$

Note that this implicitly assumes that we have a random variable M_∞, because τ can take the value $+\infty$. One can then also prove that if τ is a stopping time and M is an adapted process with RC trajectories, then M_τ is $\mathcal{F}_\tau$-measurable (as one intuitively expects). Finally, we also recall the stopped process $M^\tau = (M^\tau_t)_{t\geq 0}$ which is defined by $M^\tau_t := M_{t\wedge\tau}$ for all $t \geq 0$. Again, if M is adapted with RC trajectories and τ is a stopping time, then also M^τ is adapted and has RC trajectories.

After the above preliminaries, we now have

Theorem 2.2 (**Stopping theorem**). *Suppose that $M = (M_t)_{t\geq 0}$ is a $(P, \mathbb{F})$-martingale with RC trajectories, and σ, τ are $\mathbb{F}$-stopping times with $\sigma \leq \tau$. If either τ is bounded by some $T \in (0, \infty)$ or M is uniformly integrable, then M_τ, M_σ are both in $L^1(P)$ and*

$$E[M_\tau \,|\, \mathcal{F}_\sigma] = M_\sigma \qquad P\text{-a.s.} \tag{2.2}$$

Two frequent *applications* of Theorem 2.2 are the following:

1) For any RC martingale M and any stopping time τ, we have $E[M_{\tau\wedge t} \,|\, \mathcal{F}_s] = M_{\tau\wedge s}$ for $s \leq t$, i.e., the stopped process $M^\tau = (M^\tau_t)_{t\geq 0} = (M_{t\wedge\tau})_{t\geq 0}$ is again a martingale (because we have $E[M^\tau_t \,|\, \mathcal{F}_s] = M^\tau_s$).

[Because not necessarily $s \leq \tau \wedge t$, this needs a little bit of extra work.]

2) If M is an RC martingale and τ is any stopping time, then we always have for any $t \geq 0$ that $E[M_{\tau\wedge t}] = E[M_0]$. If either τ is bounded or M is uniformly integrable, then we also obtain $E[M_\tau] = E[M_0]$.

For future use, let us also recall the notion of a local martingale null at 0, now in continuous time. An adapted process $X = (X_t)_{t \geq 0}$ null at 0 (i.e. with $X_0 = 0$) is called a *local martingale null at* 0 (with respect to P and $I\!\!F$) if there exists a sequence of stopping times $(\tau_n)_{n \in I\!\!N}$ increasing to ∞ such that for each $n \in I\!\!N$, the stopped process $X^{\tau_n} = (X_{t \wedge \tau_n})_{t \geq 0}$ is a $(P, I\!\!F)$-martingale. We then call $(\tau_n)_{n \in I\!\!N}$ a *localising sequence*. (If X is defined on $[0, T]$ for some $T \in (0, \infty)$, the requirement for a localising sequence is that (τ_n) increases to T *stationarily*, i.e. $\tau_n \nearrow T$ P-a.s. and $P[\tau_n < T] \to 0$ as $n \to \infty$.)

The next result presents a number of martingales directly related to Brownian motion.

Proposition 2.3. *Suppose $W = (W_t)_{t \geq 0}$ is a $(P, I\!\!F)$-Brownian motion. Then the following processes are all $(P, I\!\!F)$-martingales:*

1) *W itself.*

2) $W_t^2 - t$, $t \geq 0$.

3) $e^{\alpha W_t - \frac{1}{2}\alpha^2 t}$, $t \geq 0$, *for any $\alpha \in I\!\!R$.*

Proof. We do this argument (in part) because it illustrates how to work with the properties of BM. For each of the above processes, adaptedness is obvious, and integrability is also clear because each W_t has a normal distribution and hence all exponential moments. Finally, as $W_t - W_s$ is independent of $\mathcal{F}_s$ and $\sim \mathcal{N}(0, t-s)$, we get 1) from

$$E[W_t - W_s \,|\, \mathcal{F}_s] = E[W_t - W_s] = 0.$$

Using this with $W_t^2 - W_s^2 = (W_t - W_s)^2 + 2W_s(W_t - W_s)$ and $\mathcal{F}_s$-measurability of W_s then gives

$$\begin{aligned} E[W_t^2 - W_s^2 \,|\, \mathcal{F}_s] &= E[(W_t - W_s)^2 \,|\, \mathcal{F}_s] \\ &= E[(W_t - W_s)^2] = \mathrm{Var}[W_t - W_s] = t - s, \end{aligned}$$

hence 2). Finally, setting $M_t := e^{\alpha W_t - \frac{1}{2}\alpha^2 t}$ yields

$$\begin{aligned} E\left[\frac{M_t}{M_s} \,\middle|\, \mathcal{F}_s\right] &= E\big[e^{\alpha(W_t - W_s) - \frac{1}{2}\alpha^2(t-s)} \,\big|\, \mathcal{F}_s\big] \\ &= e^{-\frac{1}{2}\alpha^2(t-s)} E\big[e^{\alpha(W_t - W_s)}\big] = 1 \end{aligned}$$

because $E[e^Z] = e^{\mu+\frac{1}{2}\sigma^2}$ for $Z \sim \mathcal{N}(\mu, \sigma^2)$. So we have 3) as well. **q.e.d.**

Example. To illustrate that the conditions in Theorem 2.2 are really needed, consider a Brownian motion W and the stopping time

$$\tau := \inf\{t \geq 0 : W_t > 1\}.$$

Due to the law of the iterated logarithm in part 2) of Proposition 1.2, we have $\tau < \infty$ P-a.s., and because W has continuous trajectories, we get $W_\tau = 1$ P-a.s. For $\sigma = 0$, if (2.2) were valid for W and τ, σ, we should get by taking expectations that

$$1 = E[W_\tau] = E[W_\sigma] = E[W_0] = 0,$$

which is clearly false. So τ *cannot be bounded* by a constant (in fact, one can even show that $E[\tau] = +\infty$), and W is a martingale, but *not uniformly integrable*. Finally, we also see that (2.2) is not true in general (i.e. without assumptions on M and/or τ).

One useful application of the above martingale results is the computation of the Laplace transforms of certain *hitting times*. More precisely, let $W = (W_t)_{t\geq 0}$ be a Brownian motion and define for $a > 0$, $b > 0$ the stopping times

$$\tau_a := \inf\{t \geq 0 : W_t > a\},$$
$$\sigma_{a,b} := \inf\{t \geq 0 : W_t > a + bt\}.$$

Note that $\tau_a < \infty$ P-a.s. by the (global) law of the iterated logarithm in part 2) of Proposition 1.2, whereas $\sigma_{a,b}$ can be $+\infty$ with positive probability (see below).

Proposition 2.4. *Let W be a BM and $a > 0$, $b > 0$. Then for any $\lambda > 0$, we have*

$$E[e^{-\lambda\tau_a}] = e^{-a\sqrt{2\lambda}} \tag{2.3}$$

and

$$E[e^{-\lambda\sigma_{a,b}}] = E\big[e^{-\lambda\sigma_{a,b}} I_{\{\sigma_{a,b}<\infty\}}\big] = e^{-a(b+\sqrt{b^2+2\lambda})}. \tag{2.4}$$

Proof. We give this argument because it illustrates how to use the preceding martingale results. First of all, take $\alpha > 0$ and define $M_t := \exp(\alpha W_t - \frac{1}{2}\alpha^2 t)$, $t \geq 0$. Then M is a martingale by part 3) of Proposition 2.3, and hence so is the stopped process M^τ by (the first comment after) Theorem 2.2, for $\tau \in \{\tau_a, \sigma_{a,b}\}$. This implies (as in the second comment after Theorem 2.2) that

$$1 = E[M_0] = E[M_{\tau\wedge t}] = E\big[e^{\alpha W_{\tau\wedge t} - \frac{1}{2}\alpha^2(\tau\wedge t)}\big]$$

for all t, and we now want to let $t \to \infty$.

For $\tau = \tau_a$, we have $W_{\tau_a\wedge t} \leq a$ and therefore $M_{\tau_a\wedge t}$ is bounded uniformly in t and ω (by $e^{\alpha a}$); so dominated convergence yields for $t \to \infty$ that

$$\begin{aligned} 1 &= \lim_{t\to\infty} E\big[e^{\alpha W_{\tau_a\wedge t} - \frac{1}{2}\alpha^2(\tau_a\wedge t)}\big] \\ &= E\Big[\lim_{t\to\infty} e^{\alpha W_{\tau_a\wedge t} - \frac{1}{2}\alpha^2(\tau_a\wedge t)}\Big] \\ &= E\big[e^{\alpha W_{\tau_a} - \frac{1}{2}\alpha^2\tau_a}\big] \\ &= e^{\alpha a} E\big[e^{-\frac{1}{2}\alpha^2\tau_a}\big] \end{aligned}$$

because $\tau_a < \infty$ P-a.s., and so $\alpha := \sqrt{2\lambda}$ gives (2.3).

For $\tau = \sigma_{a,b}$, we have $W_{\sigma_{a,b}\wedge t} \leq a + b(\sigma_{a,b}\wedge t)$ so that

$$M_{\sigma_{a,b}\wedge t} \leq \exp\left(\alpha a + \left(\alpha b - \frac{1}{2}\alpha^2\right)(\sigma_{a,b}\wedge t)\right)$$

is bounded uniformly in t and ω (by $e^{\alpha a}$) for $\alpha b < \frac{1}{2}\alpha^2$, i.e. for $\alpha > 2b$. Moreover, $\alpha b - \frac{1}{2}\alpha^2 < 0$ implies that on the set $\{\sigma_{a,b} = +\infty\}$, we have both $M_{\sigma_{a,b}\wedge t} \to 0$ as $t \to \infty$ and $e^{(\alpha b - \frac{1}{2}\alpha^2)\sigma_{a,b}} = 0$. Therefore we get in the same way as above via dominated convergence that

$$1 = e^{\alpha a} E\big[e^{(\alpha b - \frac{1}{2}\alpha^2)\sigma_{a,b}} I_{\{\sigma_{a,b}<\infty\}}\big] = e^{\alpha a} E\big[e^{(\alpha b - \frac{1}{2}\alpha^2)\sigma_{a,b}}\big].$$

Then (2.4) follows for $\alpha := b + \sqrt{b^2 + 2\lambda}$, which gives by a straightforward computation that $\alpha b - \frac{1}{2}\alpha^2 = \alpha(b - \frac{1}{2}\alpha) = -\lambda < 0$. **q.e.d.**

Remark. If we let $\lambda \searrow 0$ in (2.4), we obtain $P[\sigma_{a,b} < \infty] = e^{-2ab}$ so that indeed $P[\sigma_{a,b} = +\infty] = 1 - e^{-2ab} > 0$. $\diamond$

For a general random variable $U \geq 0$, the function $\lambda \mapsto E[e^{-\lambda U}]$ for $\lambda > 0$ is called the *Laplace transform* of U. Its general importance in probability theory is that it uniquely determines the distribution of U.

In mathematical finance, both τ_a and $\sigma_{a,b}$ come up in connection with a number of so-called *exotic options*. In particular, they are important for *barrier options* whose payoff depends on whether or not a (upper or lower) level has been reached by a given time. When computing prices of such options in the Black–Scholes model, one almost immediately encounters the Laplace transforms from Proposition 2.4. For more details, see for instance Dana/Jeanblanc [3, Chapter 9].

4.3 Markovian properties

We have already seen in part 2) of Proposition 1.1 that for any fixed time $T \in (0, \infty)$, the process

$$W_{t+T} - W_T,\ t \geq 0, \qquad \text{is again a BM} \tag{3.1}$$

if $(W_t)_{t\geq 0}$ is a Brownian motion. This means that if we restart a BM from level 0 at some fixed time, it behaves exactly as if it had only just started. Moreover, one can show that the independence of increments of BM implies that

$$W_{t+T} - W_T,\ t \geq 0, \qquad \text{is independent of } \mathcal{F}_T^0, \tag{3.2}$$

where $\mathcal{F}_T^0 = \sigma(W_s, s \leq T)$ is the σ-field generated by BM up to time T. Intuitively, this means that BM at any fixed time T simply forgets its past up to time T (with the only possible exception that it remembers its current position W_T at time T), and starts afresh.

One consequence of (3.1) and (3.2) is the following. Suppose that at some fixed time T, we are interested in the behaviour of W after time T and try to predict this on the basis of the past of W up to time T, where “prediction” is done in the sense of a conditional expectation. Then we may as well forget about the past and look only at the current value W_T at time T. A bit more precisely, we can express this, for functions $g \geq 0$ applied to the part of BM after time T, as

$$E[g(W_u, u \geq T) \,|\, \sigma(W_s, s \leq T)] = E[g(W_u, u \geq T) \,|\, \sigma(W_T)]. \tag{3.3}$$

This is called the *Markov property* of BM, and it is already very useful in many situations.

Exactly as with martingales, we suspect that it might be interesting and helpful if one could in (3.3) replace the fixed time $T \in (0, \infty)$ by a stopping time τ. Note, however, that quite apart from the difficulties of writing down an analogue of (3.3) for a random time $\tau(\omega)$, it is even not clear whether this should then be true, because after all, τ itself can explicitly depend on the past behaviour of BM. Nevertheless, it turns out that such a result is true; one says that BM even has the *strong Markov property*.

Because a precise analogue of (3.3) for a stopping time becomes a bit technical, we formulate things a bit differently. If we denote almost as above by $I\!\!F^W$ the filtration generated by W (and made right-continuous, to be accurate), and if τ is a stopping time with respect to $I\!\!F^W$ and such that $\tau < \infty$ P-a.s., then

$$W_{t+\tau} - W_\tau,\ t \geq 0, \qquad \text{is again a BM and independent of } \mathcal{F}_\tau^W.$$

Of course, this includes (3.1) and (3.2) as special cases, and one can easily believe that it is even more useful than (3.3). However, the proof is too difficult to be given here.