7 The Black–Scholes formula

Our goal in this final chapter is to combine the modelling and financial ideas from the discrete-time setting with the continuous-time techniques from stochastic calculus. We introduce and study a simple continuous-time financial market model and show how this allows us to derive the celebrated Black–Scholes formula together with the underlying methodology. We emphasise that the latter is much more important than the formula itself, for obvious reasons.

7.1 The Black–Scholes model

The *Black–Scholes model* or *Samuelson model* is the continuous-time analogue of the Cox–Ross–Rubinstein binomial model we have seen at length in earlier chapters. Like the latter, it is too simple to be realistic, but still very popular because it allows many explicit calculations and results. It also serves as a basic starting point or reference model.

To set up the model, we start with a fixed time horizon $T \in (0, \infty)$ and a probability space $(\Omega, \mathcal{F}, P)$ on which there is a Brownian motion $W = (W_t)_{0 \le t \le T}$. We take as *filtration* $I\!\!F = (\mathcal{F}_t)_{0 \le t \le T}$ the one generated by W and augmented as in Section 6.3 by the P-nullsets from $\mathcal{F}^0_T := \sigma(W_s, s \le T)$ so that $I\!\!F = I\!\!F^W$ satisfies the usual conditions under P. We shall see soon that this choice of filtration is important.

The *financial market model* has two basic traded assets: a *bank account* with constant continuously compounded *interest rate* $r \in I\!\!R$, and a *risky asset* (usually called *stock*) having two parameters $\mu \in I\!\!R$ and $\sigma > 0$. Undiscounted prices are given by

$$\widetilde{S}^0_t = e^{rt}, \tag{1.1}$$

$$\widetilde{S}^1_t = S^1_0 \exp\left(\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right)t\right) \tag{1.2}$$

with a constant $S^1_0 > 0$. Applying Itô's formula easily yields

$$\mathrm{d}\widetilde{S}^0_t = \widetilde{S}^0_t r \,\mathrm{d}t, \tag{1.3}$$

$$\mathrm{d}\widetilde{S}^1_t = \widetilde{S}^1_t \mu \,\mathrm{d}t + \widetilde{S}^1_t \sigma \,\mathrm{d}W_t, \tag{1.4}$$

which can be rewritten as

$$\frac{\mathrm{d}\widetilde{S}^0_t}{\widetilde{S}^0_t} = r\,\mathrm{d}t, \tag{1.5}$$

$$\frac{\mathrm{d}\widetilde{S}^1_t}{\widetilde{S}^1_t} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t. \tag{1.6}$$

This means that the bank account has a *relative price change* $(\widetilde{S}^0_t - \widetilde{S}^0_{t-\mathrm{d}t})/\widetilde{S}^0_{t-\mathrm{d}t}$ over a short time period $(t-\mathrm{d}t, t]$ of $r\,\mathrm{d}t$; so r is the growth rate of the bank account. In the same way, the relative price change of the stock has a part $\mu\,\mathrm{d}t$ giving a growth at rate μ, and a second part $\sigma\,\mathrm{d}W_t$ "with mean 0 and variance $\sigma^2\,\mathrm{d}t$" that causes random fluctuations. We call μ the *drift* (rate) and σ the (instantaneous) *volatility* of $\widetilde{S}^1$. The formulation (1.5), (1.6) also makes it clear why this model is the continuous-time analogue of the CRR binomial model; see Example 6.1.3 in Section 6.1 for a more detailed discussion. (Because $\widetilde{S}^0$ and $\widetilde{S}^1$ are both continuous, we can replace $\widetilde{S}^0_{t-\mathrm{d}t}$ and $\widetilde{S}^1_{t-\mathrm{d}t}$ in the denominators above by $\widetilde{S}^0_t$ and $\widetilde{S}^1_t$, respectively.)

As usual, we pass to quantities *discounted* with $\widetilde{S}^0$; so we have $S^0 = \widetilde{S}^0/\widetilde{S}^0 \equiv 1$, and $S^1 = \widetilde{S}^1/\widetilde{S}^0$ is by (1.1) and (1.2) given by

$$S^1_t = S^1_0 \exp\left(\sigma W_t + \left(\mu - r - \frac{1}{2}\sigma^2\right)t\right). \tag{1.7}$$

Either from (1.7) or from (1.3), (1.4), we obtain via Itô's formula that S^1 solves the SDE

$$\mathrm{d}S^1_t = S^1_t\big((\mu - r)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t\big). \tag{1.8}$$

For later use, we observe that this gives

$$\mathrm{d}\langle S^1\rangle_t = (S^1_t)^2\sigma^2\,\mathrm{d}\langle W\rangle_t = (S^1_t)^2\sigma^2\,\mathrm{d}t \tag{1.9}$$

for the *quadratic variation* of S^1, because $\langle W\rangle_t = t$.

Remark 1.1. Because the cofficients μ, r, σ are all constant and $\sigma > 0$, the undiscounted prices $(\widetilde{S}^0, \widetilde{S}^1)$, the discounted prices (S^0, S^1), the discounted stock price S^1 alone, and the Brownian motion W all generate the same filtration. This means that there is here no compromise between mathematical convenience (the filtration $\mathbb{F}$ is generated by W) and financial modelling (the filtration is generated by information about prices). $\diamond$

As in discrete time, we should like to have an *equivalent martingale measure* for the discounted stock price process S^1. To get an idea how to find this, we rewrite (1.8) as

$$\mathrm{d}S_t^1 = S_t^1\sigma\left(\mathrm{d}W_t + \frac{\mu - r}{\sigma}\,\mathrm{d}t\right) = S_t^1\sigma\,\mathrm{d}W_t^*, \tag{1.10}$$

with $W^* = (W_t^*)_{0\le t\le T}$ defined by

$$W_t^* := W_t + \frac{\mu - r}{\sigma}t = W_t + \int_0^t \lambda\,\mathrm{d}s \qquad \text{for } 0 \le t \le T.$$

The quantity

$$\lambda := \frac{\mu - r}{\sigma}$$

is often called the instantaneous *market price of risk* or infinitesimal *Sharpe ratio* of S^1. By looking at Girsanov's theorem in the form of Theorem 6.2.3, we see that W^* is a Brownian motion on $[0, T]$ under the probability measure Q^* given by

$$\frac{\mathrm{d}Q^*}{\mathrm{d}P} := \mathcal{E}\left(-\int \lambda\,\mathrm{d}W\right)_T = \exp\left(-\lambda W_T - \frac{1}{2}\lambda^2 T\right) \qquad \text{on } \mathcal{F}_T,$$

whose density process with respect to P is

$$Z_t^{Q^*;P} = Z_t^* = \mathcal{E}\left(-\int \lambda\,\mathrm{d}W\right)_t = \exp\left(-\lambda W_t - \frac{1}{2}\lambda^2 t\right) \qquad \text{for } 0 \le t \le T.$$

By (1.10), the stochastic integral process

$$S_t^1 = S_0^1 + \int_0^t S_u^1\sigma\,\mathrm{d}W_u^*$$

is then a continuous local Q^*-martingale like W^*; it is even a Q^*-martingale because we have the explicit expression

$$S_t^1 = S_1^0\,\mathcal{E}(\sigma W^*)_t = S_0^1 \exp\left(\sigma W_t^* - \frac{1}{2}\sigma^2 t\right) \tag{1.11}$$

from (1.10) by Itô's formula, and so we can use Proposition 4.2.3 under Q^*.

All in all, then, S^1 admits an equivalent martingale measure, explicitly given by Q^*, and so we expect that S^1 should be "arbitrage-free" in any reasonable sense. However, we cannot make this precise here before defining more carefully what "trading strategy", "self-financing", "arbitrage opportunity" etc. should mean in this context.

Remark. Suppose Q is any probability measure equivalent to P on $\mathcal{F}_T$ and denote its P-density process by $Z^{Q;P} = Z = (Z_t)_{0\le t\le T}$. Then we can write $Z = Z_0\,\mathcal{E}(L)$ as in Section 6.2, where L is a local $(P, I\!\!F)$-martingale null at 0. But $I\!\!F$ is generated by W; so Itô's representation theorem in Corollary 6.3.2 says that

$$L = \int \nu_s \,\mathrm{d}W_s \qquad \text{for some } \nu \in L^2_{\mathrm{loc}}(W)$$

and therefore $\mathrm{d}Z_t = Z_{t-}\,\mathrm{d}L_t = Z_t\nu_t\,\mathrm{d}W_t$ (as Z is automatically continuous like L).

Now suppose in addition that S^1 is a local Q-martingale, i.e. Q is an ELMM for S^1. By the Bayes rule in Lemma 6.2.1, this implies that ZS^1 is a local P-martingale. But the product rule, (1.8) and the rules for computing covariations of stochastic integrals give

$$\begin{aligned}
\mathrm{d}(Z_tS^1_t) &= Z_t\,\mathrm{d}S^1_t + S^1_t\,\mathrm{d}Z_t + \mathrm{d}\langle Z, S^1\rangle_t\\
&= Z_tS^1_t(\mu - r)\,\mathrm{d}t + Z_tS^1_t\sigma\,\mathrm{d}W_t + S^1_tZ_t\nu_t\,\mathrm{d}W_t + Z_t\nu_tS^1_t\sigma\,\mathrm{d}\langle W, W\rangle_t\\
&= Z_tS^1_t(\sigma + \nu_t)\,\mathrm{d}W_t + Z_tS^1_t\sigma(\lambda + \nu_t)\,\mathrm{d}t,
\end{aligned}$$

using that $\mu - r = \sigma\lambda$. On the left-hand side, we have by assumption a local P-martingale, and on the right-hand side, the $\mathrm{d}W$-integral is also a local P-martingale. Therefore the last term,

$$A_t := \int_0^t Z_sS^1_s\sigma(\lambda + \nu_s)\,\mathrm{d}s \qquad \text{for } 0 \le t \le T,$$

must also be a local P-martingale. But A is adapted and continuous (hence predictable) and of finite variation; so it has quadratic variation 0, hence must be constant, and so its integrand must be 0. This implies that $\nu_s \equiv -\lambda$, because Z, S^1, σ are all > 0, and therefore we get

$$Z = Z_0\,\mathcal{E}(L) = Z_0\,\mathcal{E}\left(\int \nu\,\mathrm{d}W\right) = Z_0\,\mathcal{E}\left(-\int \lambda\,\mathrm{d}W\right).$$

Finally, Z_0 has $E_P[Z_0] = E_P[Z_T] = Q[\Omega] = 1$ and is measurable with respect to $\mathcal{F}_0 = \mathcal{F}_0^W$ which is P-trivial (because W_0 is constant P-a.s.); so $Z_0 = E_P[Z_0] = 1$ and therefore

$$Z = \mathcal{E}\left(-\int \lambda \, \mathrm{d}W\right) = Z^*, \qquad \text{or } Q = Q^*.$$

Thus we have shown that in the Black–Scholes model, there is a *unique equivalent martingale measure*, which is given explicitly by Q^*. So we expect that the Black–Scholes model is not only "arbitrage-free", but also "complete" in a suitable sense. Note that the latter point (as well as the above proof of uniqueness) depends via Itô's representation theorem in a crucial way on the assumption that the filtration $I\!\!F$ is generated by W. $\diamond$

Now take any $H \in L^0_+(\mathcal{F}_T)$ and view H as a random *payoff* (in discounted units) due at time T. Recall that $I\!\!F$ is generated by W and that $W^*_t = W_t + \lambda t$, $0 \le t \le T$, is a Q^*-Brownian motion. Because λ is deterministic, W and W^* generate the same filtration, and so we can also apply Itô's representation theorem with Q^* and W^* instead of P and W. So if H is also in $L^1(Q^*)$, the Q^*-martingale $V^*_t := E_{Q^*}[H \mid \mathcal{F}_t]$, $0 \le t \le T$, can be represented as

$$V^*_t = E_{Q^*}[H] + \int_0^t \psi^H_s \, \mathrm{d}W^*_s \qquad \text{for } 0 \le t \le T,$$

with some unique $\psi^H \in L^2_{\mathrm{loc}}(W^*)$ such that $\int \psi^H \, \mathrm{d}W^*$ is a Q^*-martingale. Recall from (1.10) that

$$\mathrm{d}S^1_t = S^1_t \sigma \, \mathrm{d}W^*_t.$$

So if we define for $0 \le t \le T$

$$\vartheta^H_t := \frac{\psi^H_t}{S^1_t \sigma},$$
$$\eta^H_t := V^*_t - \vartheta^H_t S^1_t$$

(which are both predictable because ψ^H is and S^1, V^* are both adapted and continuous), then we can interpret $\varphi^H = (\vartheta^H, \eta^H)$ as a *trading strategy* whose discounted value process is given by

$$V_t(\varphi^H) = \vartheta^H_t S^1_t + \eta^H_t S^0_t = V^*_t \qquad \text{for } 0 \le t \le T,$$

and which is *self-financing* in the (usual) sense that

$$V_t(\varphi^H) = V_t^* = V_0^* + \int_0^t \psi_u^H \, \mathrm{d}W_u^* = V_0(\varphi^H) + \int_0^t \vartheta_u^H \, \mathrm{d}S_u^1 \qquad \text{for } 0 \le t \le T. \tag{1.12}$$

Moreover,

$$V_T(\varphi^H) = V_T^* = H \qquad \text{a.s.}$$

shows that the strategy φ^H *replicates* H, and

$$\int \vartheta^H \, \mathrm{d}S^1 = V(\varphi^H) - V_0(\varphi^H) = V^* - E_{Q^*}[H] \ge -E_{Q^*}[H]$$

(because $V^* \ge 0$, as $H \ge 0$) shows that ϑ^H is admissible (for S^1) in the usual sense.

In summary, then, every $H \in L^1_+(\mathcal{F}_T, Q^*)$ is attainable in the sense that it can be replicated by a dynamic strategy trading in the stock and the bank account in such a way that the strategy is self-financing and admissible, and its value process is a Q^*-martingale. In that sense, we can say that the Black–Scholes model is complete. By analogous arguments as in discrete time, we then also obtain the arbitrage-free value at time t of any payoff $H \in L^1_+(\mathcal{F}_T, Q^*)$ as its conditional expectation

$$V_t^H = V_t^* = E_{Q^*}[H \mid \mathcal{F}_t]$$

under the unique equivalent martingale measure Q^* for S^1. This is in perfect parallel to the results we have seen for the CRR binomial model; see Section 3.3.

Remarks. 1) All the above computations and results are in *discounted* units. Of course, we could also go back to undiscounted units.

2) Itô's representation theorem gives the *existence* of a strategy, but does not tell us how it looks. To get more explicit results, additional structure (for the payoff H) and more work is needed. [→ *Exercise*]

3) The SDE (1.8) for discounted prices is

$$\frac{\mathrm{d}S_t^1}{S_t^1} = (\mu - r)\, \mathrm{d}t + \sigma \, \mathrm{d}W_t,$$

and this is rather restrictive as μ, r, σ are all constant. An obvious *extension* is to allow the coefficients μ, r, σ to be (suitably integrable) predictable processes, or possibly functionals of S or $\widetilde{S}$. This brings up several issues:

a) If μ, r, σ are specified as functionals of S, it is no longer clear whether there exists a solution of the resulting SDE. This needs a more careful and usually case-based analysis.

b) If μ, r, σ are stochastic processes that depend on extra randomness apart from W, we have to work in a larger filtration and a result like Itô's representation theorem is perhaps no longer available. Typical examples are *stochastic volatility* models where σ usually depends on a second Brownian motion as well, or *credit risk* models where the default of an asset often involves the jump of some process.

c) Even if μ, r, σ are predictable with respect to the filtration $\mathbb{F}$ generated by W, the process $W^* = W + \int \lambda_s \, \mathrm{d}s$ in general does not generate $\mathbb{F}$, but only a smaller filtration. Fortunately, there is still a representation result with respect to W^* and Q^*, but one must work a little to prove this.

4) From the point of view of finance, the *natural filtration* to work with would be the one generated by S or $\widetilde{S}$, i.e. by prices, not by W. From the explicit formulae (1.1), (1.2), one can see that $\widetilde{S}$ and W generate the same filtrations when the coefficients μ, r, σ are deterministic. (This has already been pointed out in Remark 1.1.) But in general (i.e. for more general coefficients), working with the price filtration is rather difficult because it is hard to describe.

5) A closer look at the no-arbitrage argument for valuing H shows that in continuous time, we can only say that the arbitrage-free *seller price process* for the payoff H is given by $V^H = V^*$. The reason is that the strategy φ^H is admissible, but $-\varphi^H$ is not, in general, unless H is in addition bounded from above. In finite discrete time, this phenomenon does not appear because absence of arbitrage for admissible or for general self-financing strategies is the same there. ◇

7.2 Markovian payoffs and PDEs

The presentation in Section 7.1 is often called the *martingale approach* to valuing options, for obvious reasons. If one has more structure for the payoff H (and, in more general models, also for S), an alternative method involves the use of partial differential equations (PDEs) and is thus called the *PDE approach*. We briefly outline some aspects of this here.

Suppose that the (discounted) payoff is of the form $H = h(S^1_T)$ for some measurable function $h \geq 0$ on $I\!R_+$. We also suppose that H is in $L^1(Q^*)$. One example discussed in detail in the next section is the European call option on $\widetilde{S}^1$ with maturity T and undiscounted strike $\widetilde{K}$; here, $H = (\widetilde{S}^1_T - \widetilde{K})^+/\widetilde{S}^0_T = (S^1_T - \widetilde{K}e^{-rT})^+$ so that the payoff function is $h(x) = (x - \widetilde{K}e^{-rT})^+ =: (x - K)^+$. Our goal, for general h, is to compute the value process V^* and the strategy ϑ^H more explicitly.

We start with the *value process*. Because we have $V^*_t = E_{Q^*}[H \,|\, \mathcal{F}_t] = E_{Q^*}[h(S^1_T) \,|\, \mathcal{F}_t]$, we look at the explicit expression for S^1 in (1.11) and write

$$S^1_T = S^1_t \frac{S^1_T}{S^1_t} = S^1_t \exp\left(\sigma(W^*_T - W^*_t) - \frac{1}{2}\sigma^2(T-t)\right).$$

In the last term, the first factor S^1_t is obviously $\mathcal{F}_t$-measurable. Moreover, W^* is a Q^*-Brownian motion with respect to $I\!F$, and so in the second factor, $W^*_T - W^*_t$ is under Q^* independent of $\mathcal{F}_t$ and has an $\mathcal{N}(0, T-t)$-distribution. Therefore we get

$$V^*_t = E_{Q^*}[h(S^1_T) \,|\, \mathcal{F}_t] = v(t, S^1_t) \tag{2.1}$$

with the function $v(t,x)$ given, for $Y \sim \mathcal{N}(0,1)$ under Q^*, by

$$\begin{aligned} v(t,x) &= E_{Q^*}\left[h\left(x \exp\left(\sigma(W^*_T - W^*_t) - \frac{1}{2}\sigma^2(T-t)\right)\right)\right] \\ &= E_{Q^*}\left[h\left(x e^{\sigma\sqrt{T-t}\,Y - \frac{1}{2}\sigma^2(T-t)}\right)\right] \\ &= \int_{-\infty}^{\infty} h\left(x e^{\sigma\sqrt{T-t}\,y - \frac{1}{2}\sigma^2(T-t)}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\,\mathrm{d}y. \end{aligned} \tag{2.2}$$

This already gives a fairly precise structural description of V^*_t as a function of (t and) S^1_t, instead of a general $\mathcal{F}_t$-measurable random variable.

Because we have an explicit formula for the function v as essentially the convolution of h with a very smooth function (the density of a lognormally distributed random variable), one can prove that the function v is *sufficiently smooth* to allow the use of Itô's formula. This gives, writing subscripts in the function v for partial derivatives and using (1.10) and (1.9),

$$\begin{aligned}
\mathrm{d}V_t^* &= \mathrm{d}v(t, S_t^1) \\
&= v_t(t, S_t^1)\,\mathrm{d}t + v_x(t, S_t^1)\,\mathrm{d}S_t^1 + \frac{1}{2}v_{xx}(t, S_t^1)\,\mathrm{d}\langle S^1\rangle_t \\
&= v_x(t, S_t^1)\sigma S_t^1\,\mathrm{d}W_t^* + \left(v_t(t, S_t^1) + \frac{1}{2}v_{xx}(t, S_t^1)\sigma^2 (S_t^1)^2\right)\mathrm{d}t.
\end{aligned} \tag{2.3}$$

But V^* is a local (even a true) Q^*-martingale, by its definition, and so is the integrated $\mathrm{d}W^*$-term on the right-hand side above. Therefore the integrated $\mathrm{d}t$-term on the right-hand side of (2.3) is at the same time continuous and adapted and of finite variation, and a local Q^*-martingale. Hence it must vanish, and so (2.3) and (1.12) yield

$$v_x(t, S_t^1)\,\mathrm{d}S_t^1 = dV_t^* = \vartheta_t^H\,\mathrm{d}S_t^1.$$

In consequence, we obtain the *strategy* explicitly as

$$\vartheta_t^H = \frac{\partial v}{\partial x}(t, S_t^1), \tag{2.4}$$

i.e., as the spatial derivative of v, evaluated along the trajectories of S^1. This is parallel to the result in Section 3.3 for the CRR binomial model; see (3.3.6) or (3.3.7).

A closer look at the above argument also allows us to extract some information about the function v. This is similar to our arguments in Example 6.1.4 for the representation of the random variable $h(W_T)$ as a stochastic integral of W. Indeed, the fact that the $\mathrm{d}t$-term vanishes means that the function $v_t(t,x)+\frac{1}{2}v_{xx}(t,x)\sigma^2 x^2$ must vanish along the trajectories of the space-time process $(t, S_t^1)_{0<t<T}$. But by the explicit expression in (1.11), each S_t^1 is lognormally distributed and hence has all of $(0,\infty)$ in its support. So the support of the space-time process contains $(0,T)\times(0,\infty)$, and so $v(t,x)$ must satisfy the (linear, second-order) *partial differential equation (PDE)*

$$0 = \frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} \qquad \text{on } (0,T)\times(0,\infty). \tag{2.5}$$

Moreover, the definition of v via (2.1) gives the *boundary condition*

$$v(T, \cdot\,) = h(\,\cdot\,) \qquad \text{on } (0, \infty), \tag{2.6}$$

because $v(T, S_T^1) = V_T^* = H = h(S_T^1)$ and the support of the distribution of S_T^1 contains $(0, \infty)$. So even if we cannot compute the integral in (2.2) explicitly, we can at least obtain $v(t, x)$ *numerically* by solving the PDE (2.5), (2.6).

Remarks. 1) Instead of using the above probabilistic argument, one can also derive the PDE (2.5) *analytically*. Using in (2.2) the substitution $u = x \exp(\sigma\sqrt{T-t}\,y - \frac{1}{2}\sigma^2(T-t))$ gives $y = (\log\frac{u}{x} + \frac{1}{2}\sigma^2(T-t))/\sigma\sqrt{T-t}$, hence $\mathrm{d}y = \frac{1}{u\sigma\sqrt{T-t}}\,\mathrm{d}u$, and then

$$v(t,x) = \int_0^\infty h(u) \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \frac{1}{u} \exp\left(-\frac{(\log\frac{u}{x} + \frac{1}{2}\sigma^2(T-t))^2}{2\sigma^2(T-t)} \right) \mathrm{d}u.$$

One can now first check, by using that $h(S_T^1)$ is in $L^1(Q^*)$, that v may be differentiated by differentiating under the integral sign, and by *brute force computations*, one can then check in this way that v indeed satisfies the PDE (2.5). The deeper reason behind this is the fact that the density function $\varphi(t,z) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{z^2}{2t}}$ of an $\mathcal{N}(0,t)$-distribution satisfies the heat equation $\varphi_t = \frac{1}{2}\varphi_{zz}$.

2) The above approach works not only for the Black–Scholes model, but more generally in a *Markovian setting*, because conditional expectations given $\mathcal{F}_t$ can there typically be written as functions of the state variables at time t. The martingale property then essentially translates into saying that the generator of the driving Markov process applied to the above functions must vanish. For diffusion state variables, the generator is a second-order differential operator and so this leads to PDEs; for Lévy state variables, one has additional integral terms coming from the jumps of the driving Lévy process, and so one obtains PIDEs (partial integro-differential equations). However, there are a number of substantial technical issues; for instance, regularity or existence of smooth solutions to the resulting equations is often not clear, and one must also be careful whether or not one has uniqueness of solutions. Not all the literature is equally rigorous and precise about these issues. $\diamond$

When comparing the PDE (2.5), (2.6) to some of those found in the literature, one might be puzzled by the simple form of (2.5). This is because we have expressed everything in discounted units. If the *undiscounted* payoff is $\widetilde{H} = \tilde{h}(\widetilde{S}^1_T)$ and the undiscounted value at time t is written as $\tilde{v}(t, \widetilde{S}^1_t)$, we have the relations

$$\tilde{h}(\widetilde{S}^1_T) = \tilde{h}(e^{rT} S^1_T) = \widetilde{H} = e^{rT} H = e^{rT} h(S^1_T)$$

and

$$\tilde{v}(t, \widetilde{S}^1_t) = e^{rt} v(t, S^1_t)$$

so that

$$\begin{aligned} v(t,x) &= e^{-rt}\tilde{v}(t, xe^{rt}),\\ \tilde{v}(t,\tilde{x}) &= e^{rt} v(t, \tilde{x}e^{-rt}). \end{aligned}$$

For the function $\tilde{v}$, we can then compute the partial derivatives

$$\begin{aligned} \frac{\partial \tilde{v}}{\partial t}(t,\tilde{x}) &= r\tilde{v}(t,\tilde{x}) + e^{rt}\frac{\partial v}{\partial t}(t,\tilde{x}e^{-rt}) - e^{rt}\frac{\partial v}{\partial x}(t,\tilde{x}e^{-rt})\tilde{x}re^{-rt},\\ \frac{\partial \tilde{v}}{\partial \tilde{x}}(t,\tilde{x}) &= e^{rt}\frac{\partial v}{\partial x}(t,\tilde{x}e^{-rt})e^{-rt} = \frac{\partial v}{\partial x}(t,\tilde{x}e^{-rt}),\\ \frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2}(t,\tilde{x}) &= \frac{\partial^2 v}{\partial x^2}(t,\tilde{x}e^{-rt})e^{-rt}, \end{aligned}$$

and by plugging in, we obtain from (2.5) the PDE

$$0 = \frac{\partial \tilde{v}}{\partial t} + r\tilde{x}\frac{\partial \tilde{v}}{\partial \tilde{x}} + \frac{1}{2}\sigma^2\tilde{x}^2\frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} - r\tilde{v} \qquad \text{on } (0,T)\times(0,\infty)$$

with the boundary condition

$$\tilde{v}(T,\,\cdot\,) = \tilde{h}(\,\cdot\,).$$

[It is a nice [$\rightarrow$ *exercise*] to convince oneself that this is correct. Possible ways include straightforward but tedious calculus, or alternatively again a martingale argument.]

7.3 The Black–Scholes formula

In the special case of a *European call option*, the value process and the corresponding strategy can be computed explicitly, and this has found widespread use in the financial industry. Suppose the undiscounted strike price is $\widetilde{K}$ so that the undiscounted payoff is

$$\widetilde{H} = (\widetilde{S}^1_T - \widetilde{K})^+.$$

Then $H = \widetilde{H}/\widetilde{S}^0_T = (S^1_T - \widetilde{K}e^{-rT})^+ =: (S^1_T - K)^+$, and we obtain from (2.2) that the discounted value of H at time t is

$$V^H_t = V^*_t = E_{Q^*}[H \,|\, \mathcal{F}_t] = E_{Q^*}[(S^1_T - K)^+ \,|\, \mathcal{F}_t] = E_{Q^*}\Big[\Big(xe^{\sigma\sqrt{T-t}\,Y - \frac{1}{2}\sigma^2(T-t)} - K\Big)^+\Big]\Big|_{x=S^1_t},$$

with $Y \sim \mathcal{N}(0,1)$ under Q^*. An elementary computation with normal distributions yields for $x > 0$, $a > 0$ and $b \geq 0$ that

$$E_{Q^*}\big[\big(xe^{aY - \frac{1}{2}a^2} - b\big)^+\big] = x\Phi\left(\frac{\log\frac{x}{b} + \frac{1}{2}a^2}{a}\right) - b\Phi\left(\frac{\log\frac{x}{b} - \frac{1}{2}a^2}{a}\right),$$

where

$$\Phi(y) = Q^*[Y \leq y] = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\,\mathrm{d}z$$

is the cumulative distribution function of the standard normal distribution $\mathcal{N}(0,1)$. Plugging in $x = S^1_t$, $a = \sigma\sqrt{T-t}$, $b = K$ and then passing to undiscounted quantities via $S^1_t = \widetilde{S}^1_t e^{-rt}$, $K = \widetilde{K}e^{-rT}$ therefore yields the famous *Black–Scholes formula* in the form

$$\tilde{V}^{\widetilde{H}}_t = \tilde{v}(t, \widetilde{S}^1_t) = \widetilde{S}^1_t \Phi(d_1) - \widetilde{K}e^{-r(T-t)}\Phi(d_2) \tag{3.1}$$

with

$$d_{1,2} = \frac{\log(\widetilde{S}^1_t/\widetilde{K}) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}. \tag{3.2}$$

Note that the drift μ of the stock does not appear here; this is analogous to the result that the probability p of an up move in the CRR binomial model does not appear in the binomial option pricing formula (3.2), (3.3) in Section 3.3. What does appear is the volatility σ, in analogy to the difference $\log(1+u) - \log(1+d)$ which gives an indication of the spread between future stock prices from one time point to the next.

To compute the *replicating strategy*, we recall from (2.4) that the stock price holdings at time t are given by

$$\vartheta_t^H = \frac{\partial v}{\partial x}(t, S_t^1).$$

Moreover, $v(t,x) = e^{-rt}\tilde{v}(t, xe^{rt})$ so that

$$\frac{\partial v}{\partial x}(t,x) = e^{-rt}\frac{\partial \tilde{v}}{\partial x}(t, xe^{rt}) = e^{-rt}\frac{\partial \tilde{v}}{\partial \tilde{x}}(t, xe^{rt})e^{rt} = \frac{\partial \tilde{v}}{\partial \tilde{x}}(t, \tilde{x}).$$

Computing the above derivative explicitly [→ *exercise*] gives

$$\vartheta_t^H = \frac{\partial \tilde{v}}{\partial \tilde{x}}(t, \widetilde{S}_t^1) = \Phi(d_1) = \Phi\left(\frac{\log(\widetilde{S}_t^1/\widetilde{K}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right), \tag{3.3}$$

which always lies between 0 and 1.

One very useful feature of the above results is that the explicit formula (3.1), (3.2) allows to compute all partial derivatives of the option price with respect to the various parameters. These sensitivities are usually called *Greeks* and denoted by (genuine or invented) Greek letters. Examples are

- *Delta*: the partial derivative with respect to the asset price $\widetilde{S}_t^1$, computed in (3.3), also called *hedge ratio*.
- *Gamma*: the second partial derivative with respect to $\widetilde{S}_t^1$; it measures the reaction of Delta to a stock price change.
- *Rho*: the partial derivative with respect to the interest rate r.
- *Vega*: the partial derivative with respect to the volatility σ.
- *Theta*: the partial derivative with respect to $T-t$, the time to maturity.

- *Vanna*: the partial derivative of Delta with respect to σ, or the second partial derivative of the option price, once with respect to $\widetilde{S}_t^1$ and once with respect to σ.
- *Vomma*: the second partial derivative of the option price with respect to σ.
- *Charm*: the partial derivative of Delta with respect to $T - t$, the time to maturity.
- *Volga*: another term for Vomma.

Of course, the above definitions per se make sense for any model; but in the Black–Scholes model, one has even explicit expressions for them.

Remark. One can find in the literature many different derivations for the Black–Scholes *formula*. One especially popular approach is to first derive the binomial call pricing formula in the CRR model via arbitrage arguments, as we have done in Section 3.3, and to then pass to the limit by appropriately rescaling the parameters. More precisely, one considers for each $n \in \mathbb{N}$ a binomial model with time step T/n so that letting n increase corresponds to more and more frequent trading. It is intuitively plausible that the CRR models should then converge in some sense to the BS model, and one can make this mathematically precise via Donsker's theorem. Obtaining the Black–Scholes formula as a limit is similar but simpler; it is essentially an application of the central limit theorem.

The above limiting "derivation" of the Black–Scholes formula is mathematically much simpler; but it is also far less satisfactory, especially at the conceptual level. Most importantly, it does not give the key insight of the *methodology behind the formula*, namely that the price is the initial capital for a self-financing replication strategy in the continuous-time model. We do have the corresponding insight for each binomial model; but the elementary analysis usually done in the literature does not study whether that important structural property is preserved when passing to the limit. To obtain that insight (and to develop it further in other applications or maybe generalisations), stochastic calculus in continuous time is indispensable.

It is interesting to note that the above view was also shared by the Nobel Prize Committee; when it awarded the 1997 Nobel Prize in Economics to Robert C. Merton and Myron Scholes (Fischer Black had died in 1995), the award was given "for a new method to determine the value of derivatives". The emphasis here is clearly on "method", as opposed to "formula". $\diamond$