3 Valuation and hedging in complete markets

In Chapter 2, we have characterised those financial market models in finite discrete time that are reasonable in the sense that they do not allow arbitrage. More precisely, we have studied when it is impossible to create money pumps by cleverly combining the basic traded assets (stocks and bank account).

If we now introduce into that market a *new financial instrument* (e.g. an option) and stipulate that this should not create arbitrage opportunities, what can then be said about the *price* of that new instrument? Note that "absence of arbitrage" now takes a different meaning because we consider a different market than before — the basic instruments are now the old stocks, the old bank account, and the new option. Depending on the structure of the stock price process S as well as the structure of the option under consideration, the restrictions on the possible price of the new option can be more or less severe; in the extreme, it can happen that the price of the option is uniquely determined. While this makes things nice and transparent, we should say that this is the exception rather than the rule.

Throughout this chapter, we consider as usual a (discounted) financial market in finite discrete time on $(\Omega, \mathcal{F}, P)$ with $I\!\!F = (\mathcal{F}_k)_{k=0,1,\dots,T}$, where discounted asset prices are given by $S^0 \equiv 1$ and $S = (S_k)_{k=0,1,\dots,T}$ with values in $I\!\!R^d$. Note that we again express all (discounted) quantities in terms or units of asset 0, and we think of asset 0 as representing money.

3.1 Attainable payoffs

Let us first introduce a general financial instrument of European type.

Definition. A *general European option* or *payoff* or *contingent claim* is a random variable $H \in L^0_+(\mathcal{F}_T)$.

The *interpretation* is that H describes the *net payoff* (in units of asset 0) that the owner of this instrument obtains at time T; so having $H \geq 0$ is natural and also avoids integrability issues. (A bit more generally, one could instead impose that H is bounded below P-a.s. by some constant.) As H is $\mathcal{F}_T$-measurable, the payoff can depend on the entire information up to time T; and "European" means that the time for the payoff is fixed at the end T.

Remark. We could also deal with an $\mathcal{F}_k$-measurable payoff made at time k; but as $S^0 \equiv 1$, it is financially equivalent whether such a payoff is made at k or at T, because we can use the bank account to transfer money over time without changing it or its value in any way. By using linearity, we could then also deal with *payoff streams* having a payoff at every date k (with, of course, the time k payoff being $\mathcal{F}_k$-measurable, i.e. the payoff stream being an adapted process). However, we do not consider here *American-type* payoffs where the owner of the financial instrument has some additional freedom in choosing the time of the payoff; the theory for that is a bit more complicated. $\diamond$

Example. A *European call option* on asset i with *maturity* T and *strike* K gives its owner the right, but not the obligation, to buy at time T one unit of asset i for the price K, irrespective of what the actual asset price S^i_T then is. Any rational person will make use of (*exercise*) that right if and only if $S^i_T(\omega) > K$, because it is in that, and only in that, situation that the right is more valuable than the asset itself. In that case, in purely monetary terms, the net payoff is then $S^i_T(\omega) - K$, and this is obtained by buying asset i at the low price K and immediately selling it on the market at the high price $S^i_T(\omega)$. In the other case $S^i_T(\omega) \le K$, the option is clearly worthless — it makes no monetary sense to pay K for one unit of asset i if one can get this on the market for less, namely for $S^i_T(\omega)$. So here we have for the option a net payoff, in monetary terms, of

$$H(\omega) = \max\big(0, S^i_T(\omega) - K\big) = \big(S^i_T(\omega) - K\big)^+.$$

As a random variable, this is clearly nonnegative and $\mathcal{F}_T$-measurable because S^i is adapted. Actually, H here is even simpler because it only depends on the terminal asset price S^i_T; we can write $H = h(S^i_T)$ with the function $h(x) = (x - K)^+$.

Remark. In the above example, and more generally by identifying an option with its net payoff in units of S^0, we are implicitly restricting ourselves to so-called *cash delivery* of options. However, there might be other contractual agreements. For instance, with a call option with *physical delivery*, one actually obtains at time T in case of exercise the shares or units of the specified asset and has to pay in cash the agreed amount K. If the underlying asset is some commodity like e.g. oil or grain, this distinction becomes quite important. However, we do not discuss this here any further. $\diamond$

Example. If we want to *bet on* a reasonably *stable asset price evolution*, we might be interested in a payoff of the form $H = I_B$ with

$$B = \Big\{a \le \min_{i=1,\dots,d} \min_{k=0,1,\dots,T} S^i_k < \max_{i=1,\dots,d} \max_{k=0,1,\dots,T} S^i_k \le b\Big\}.$$

In words, this option pays at time T one unit of money if and only if all stocks remain between the levels a and b up to time T. This H is also $\mathcal{F}_T$-measurable, but now depends on the asset price evolution over the whole time range $k = 0, 1, \dots, T$; it cannot be written as a function of the final stock price S_T alone.

Example. A payoff of the form

$$H = I_A\, g\bigg(\frac{1}{T}\sum_{k=1}^{T} S^i_k\bigg) \qquad \text{with } A \in \mathcal{F}_T \text{ and a function } g \ge 0$$

gives a payoff which depends on the average price (over time) of asset i, but which is only due in case that a certain event A occurs. In *insurance*, the set A could for instance be the event of the death up to time T of an insured person; then H would describe the payoff from an *index-linked insurance policy*. This is an example where H depends on more than only the basic asset prices. To get interesting examples of this type, we need the filtration $\mathbb{F}$ to be strictly larger than the filtration $\mathbb{F}^S$ generated by asset prices.

The *basic question* studied in this chapter is the following: Given a contingent claim $H \in L^0_+(\mathcal{F}_T)$, how can we assign to H a *value at* any *time* $k < T$ in such a way that this creates no arbitrage opportunities (if the claim is made available for trading at these values)? And having sold H, what can one do to *insure* oneself *against* the *risk* involved in having to pay the random, uncertain amount H at time T?

The *key idea* for answering both questions is very simple. With the help of the basic traded assets S^0 and S, we try to *construct* an *artificial product* that looks as *similar to* H as possible. The value of this product is then known because the product is constructed from the given assets; and this value should by absence of arbitrage be a good approximation for the value of H.

Let us first look at the *ideal case*. Suppose that we can find a self-financing strategy $\varphi \mathrel{\widehat{=}} (V_0, \vartheta)$ such that $V_T(\varphi) = H$ P-a.s. Then both the strategy φ and just holding H have costs of 0 at all intermediate times $k = 1, \dots, T-1$ because φ is self-financing, and both have at time T a value of H. To avoid arbitrage, the values of both structures must therefore coincide at time 0 as well, because we can otherwise buy the cheaper and sell the more expensive product to make a riskless profit. (Note that this argument crucially exploits that in finite discrete time, (NA) and (NA′) are equivalent, so that we need not worry about any admissibility condition for the "strategy", in the extended market, of combining two products.) In consequence, the value or price of H at time 0 must be V_0. An analogous argument and conclusion are valid for any time k, where the value or price of H must then be $V_k(\varphi)$.

Definition. A payoff $H \in L^0_+(\mathcal{F}_T)$ is called *attainable* if there exists an admissible self-financing strategy $\varphi \mathrel{\widehat{=}} (V_0, \vartheta)$ with $V_T(\varphi) = H$ P-a.s. The strategy φ is then said to *replicate* H and is called a *replicating strategy for* H.

Remark. Even in finite discrete time, it is important (and exploited below) that a replicating strategy should be admissible. In continuous or infinite discrete time, this becomes indispensable. ◇

The next result formalises the key idea explained just before the above definition. In addition, it also provides an efficient way of computing the resulting option price.

Theorem 1.1 (Arbitrage-free valuation of attainable payoffs). *Consider a dis-*

counted financial market in finite discrete time and suppose that S is arbitrage-free and $\mathcal{F}_0$ is trivial. Then every attainable payoff H has a unique price process $V^H = (V_k^H)_{k=0,1,\dots,T}$ which admits no arbitrage (in the extended market consisting of 1, S and V^H). It is given by

$$V_k^H = E_Q[H \mid \mathcal{F}_k] = V_k(V_0, \vartheta) \qquad \text{for } k = 0, 1, \dots, T,$$

for any equivalent martingale measure Q for S and for any replicating strategy $\varphi \mathrel{\widehat{=}} (V_0, \vartheta)$ for H.

Proof. By the DMW theorem in Theorem 2.2.1, $I\!P_{\mathrm{e}}(S)$ is nonempty because S is arbitrage-free; so there is at least one EMM Q. By assumption, H is attainable; so there is at least one replicating strategy φ. Because φ and H provide the same payoff structures, they must by absence of arbitrage in the extended market have the same value processes; so $V^H = V(\varphi)$, and this holds for any replicating φ. Because any such $\varphi \mathrel{\widehat{=}} (V_0, \vartheta)$ is admissible by definition, $V(\varphi) = V_0 + \vartheta \bullet S = V(V_0, \vartheta)$ is a Q-martingale by Theorem 1.3.3, for any $Q \in I\!P_{\mathrm{e}}(S)$, and as its final value is $V_T(\varphi) = H$ (P-a.s., hence also Q-a.s.), we get

$$V_k^H = V_k(\varphi) = E_Q[H \mid \mathcal{F}_k] \qquad \text{for all } k.$$

More precisely, V_0 is a constant because $\mathcal{F}_0$ is trivial, and φ is admissible so that $V(\varphi)$ is bounded from below. So $\vartheta \bullet S = V(\varphi) - V_0$ is also bounded from below, which justifies the use of Theorem 1.3.3. **q.e.d.**

In terms of *efficiency*, Theorem 1.1 is a substantial achievement. In a first step, we ought to check in any case whether or not the basic model we use for S is arbitrage-free, and that is most easily done by exhibiting or constructing an EMM Q for S. If we then have any attainable payoff, we very simply compute its price process by taking conditional expectations under Q, without having to spend any effort on finding a replication strategy.

However, the above statement is a bit *misleading*. First of all, for hedging purposes, we very often are interested in actually knowing and then also using a replicating strategy. But more fundamentally, how can we decide for a given payoff whether or not it is attainable, without exhibiting or constructing a replicating strategy? Is there a different and maybe simpler way to show the existence of a replicating strategy?

The next result shows how the last question can be answered by again using E(L)MMs for S.

Theorem 1.2 (Characterisation of attainable payoffs). *Consider a discounted financial market in finite discrete time and suppose that S is arbitrage-free and $\mathcal{F}_0$ is trivial. For any payoff $H \in L^0_+(\mathcal{F}_T)$, the following are equivalent:*

1) *H is attainable.*

2) $\sup_{Q \in I\!\!P_{\mathrm{e,loc}}(S)} E_Q[H] < \infty$ *is attained in some $Q^* \in I\!\!P_{\mathrm{e,loc}}(S)$, i.e. the supremum is finite and a maximum; in other words, we have* $\sup_{Q \in I\!\!P_{\mathrm{e,loc}}(S)} E_Q[H] = E_{Q^*}[H] < \infty$ *for some $Q^* \in I\!\!P_{\mathrm{e,loc}}(S)$.*

3) *The mapping $I\!\!P_{\mathrm{e}}(S) \to I\!\!R$, $Q \mapsto E_Q[H]$ is constant, i.e. H has the same and finite expectation under all EMMs Q for S.*

Proof. While some of the implications are rather straightforward, the full proof, and in particular the implication "2) $\Rightarrow$ 1)", is difficult because it relies on the so-called *optional decomposition theorem*. For the case where prices S are nonnegative, see Föllmer/Schied [9, Remark 7.17 and Theorem 5.32]. The general case is more delicate; the simplification for $S \geq 0$ is due to the fact that the sets $I\!\!P_{\mathrm{e}}(S)$ and $I\!\!P_{\mathrm{e,loc}}(S)$ then coincide. A full proof is for instance given in the lecture "Introduction to Mathematical Finance". **q.e.d.**

Remark. For models with continuous or infinite discrete time, the equivalence between 1) and 2) in Theorem 1.2 still holds (with a slightly stronger definition of attainability), but the equivalence between 2) and 3) may (surprisingly!) fail. More precisely, "3) $\Rightarrow$ 2)" remains valid if we replace $I\!\!P_{\mathrm{e}}$ by $I\!\!P_{\mathrm{e,loc}}$ in 3), but "2) $\Rightarrow$ 3)" in general only holds if H is bounded; see Delbaen/Schachermayer [4, Chapter 10] for a *counterexample*. $\diamond$

In summary, the approach to valuing and hedging a given payoff H in a financial market in finite discrete time (with $\mathcal{F}_0$ trivial) looks quite simple:

1) Check if S is arbitrage-free by finding at least one ELMM Q for S.

2) Find all ELMMs Q for S.

3) Compute $E_Q[H]$ for all ELMMs Q for S and determine the supremum of $E_Q[H]$ over Q.

4a) If the supremum is finite and a maximum, i.e. attained in some $Q^* \in I\!P_{\mathrm{e,loc}}(S)$, then H is attainable and its price process can be computed as $V_k^H = E_Q[H \mid \mathcal{F}_k]$, for any $Q \in I\!P_{\mathrm{e}}(S)$.

4b) If the supremum is not attained (or, equivalently for finite discrete time, there is a pair of EMMs Q_1, Q_2 with $E_{Q_1}[H] \neq E_{Q_2}[H]$), then H is not attainable.

In case 4a), Theorem 1.1 tells us how to *value* H; but if we also want to find a *replicating strategy*, then more work is required.

In case 4b), we are faced with a *genuine problem*: It is impossible to replicate H, so our whole conceptual approach up to here breaks down. We then have the difficult problem of *valuation and hedging* for a *non-attainable payoff*, and there are in the literature several competing approaches to that, all involving in some way the specification of *preferences* or *subjective views* of the option seller.

Remark. Because it involves no preferences, but only the assumption of absence of arbitrage, the valuation from Theorem 1.1 is often also called *risk-neutral valuation*, and an EMM Q for S is called a *risk-neutral measure*. ◇

Warning: In large parts of the literature, the terminology "risk-neutral valuation" is used for computing conditional expectations of a given payoff H under some EMM Q. This is potentially *problematic* for two reasons:

1) $V_k^{H,Q} := E_Q[H \mid \mathcal{F}_k]$ typically depends on Q if H is not attainable. So when following that approach, one should at the very least think carefully about which $Q \in I\!P_{\mathrm{e}}(S)$ one uses, and why.

2) If H is not attainable, it is at best not clear how to hedge H in any reasonably safe way, and at worst, this may be impossible to achieve.

Both of these issues are often ignored in the literature; whether this happens intentionally or through ignorance is not always clear. One area where this used to be particularly prominent is credit risk. One can of course argue that having some approach to obtain a valuation is better than nothing; but a value which has substantial arbitrariness and perhaps no clear risk management outlook should certainly be treated with care and respect.

3.2 Complete markets

As we have seen in Theorem 1.1, absence of arbitrage is already enough to value or price any attainable payoff.

Definition. A financial market model (in finite discrete time) is called *complete* if every payoff $H \in L^0_+(\mathcal{F}_T)$ is attainable. Otherwise it is called *incomplete*.

An obvious corollary of Theorem 1.1 is then

Theorem 2.1 (Valuation and hedging in complete markets). *Consider a discounted financial market model in finite discrete time and suppose that $\mathcal{F}_0$ is trivial and S is arbitrage-free and complete. Then for every payoff $H \in L^0_+(\mathcal{F}_T)$, there is a unique price process $V^H = (V^H_k)_{k=0,1,\dots,T}$ which admits no arbitrage. It is given by*

$$V^H_k = E_Q[H \,|\, \mathcal{F}_k] = V_k(V_0, \vartheta) \qquad \text{for } k = 0, 1, \dots, T$$

for any EMM Q for S and any replicating strategy $\varphi \mathrel{\widehat{=}} (V_0, \vartheta)$ for H.

While Theorem 2.1 looks very nice, it raises the important *question* of how to recognise a complete market, because completeness is a statement about *all* payoffs $H \in L^0_+(\mathcal{F}_T)$. But very fortunately, there is a very simple criterion — and it should be no surprise by now that this again involves EMMs Q.

Theorem 2.2. *Consider a discounted financial market model in finite discrete time and assume that S is arbitrage-free, $\mathcal{F}_0$ is trivial and $\mathcal{F}_T = \mathcal{F}$. Then S is complete if and only if there is a unique equivalent martingale measure for S. In brief:*

$$(NA) + \textit{completeness} \quad \Longleftrightarrow \quad \#\big(I\!\!P_e(S)\big) = 1, \textit{ i.e. } I\!\!P_e(S) \textit{ is a singleton.}$$

Proof. "$\Longleftarrow$": If $I\!\!P_e(S)$ contains only one element, then $Q \mapsto E_Q[H]$ is of course constant over $Q \in I\!\!P_e(S)$ for any $H \in L^0_+(\mathcal{F}_T)$. Hence H is attainable by Theorem 1.2.

[To be accurate and avoid the case that $Q \mapsto E_Q[H] \equiv +\infty$, one also needs to check a priori some integrability issues, namely that $E_Q[H] < \infty$ for at least one $Q \in I\!P_e(S)$; see Föllmer/Schied [9, Theorems 5.30 and 5.26] for details.]

"$\Longrightarrow$": For any $A \in \mathcal{F}_T$, the payoff $H := I_A$ is attainable; so by Theorem 1.1, we have for any pair of EMMs Q_1, Q_2 for S that

$$Q_1[A] = E_{Q_1}[H] = V_0^H = E_{Q_2}[H] = Q_2[A].$$

So Q_1 and Q_2 coincide on $\mathcal{F}_T = \mathcal{F}$, which means that there can be at most one EMM for S. By the DMW theorem in Theorem 2.2.1, there is at least one EMM because S is arbitrage-free, and so the proof is complete. **q.e.d.**

Theorem 2.2 is sometimes called the *second fundamental theorem of asset pricing*. Combining it with the first FTAP in Theorem 2.2.1, we have a very *simple and beautiful description* of discounted financial market models in finite discrete time:

– Existence of an EMM is equivalent to the market being arbitrage-free.

– Uniqueness of the EMM is equivalent to completeness of the market.

For continuous or infinite discrete time, such statements become more subtle to formulate and more difficult to prove.

Remarks. 1) We can see from the proof of Theorem 2.2 where the assumption $\mathcal{F}_T = \mathcal{F}$ is used. But it is also clear from looking at the statement why it is needed; after all, completeness is only an assertion about $\mathcal{F}_T$-measurable quantities.

2) One can show that if a financial market in finite discrete time is complete, then $\mathcal{F}_T$ must be *finite*; see Föllmer/Schied [9, Theorem 5.38]. In effect, finiteness of $\mathcal{F}_T$ means that Ω can also be taken finite. This shows that while it makes the theory nice and simple, completeness is also a very restrictive property — complete financial markets in finite discrete time are effectively given by finite tree models. $\diamond$

Example. The multinomial model with a bank account and one stock ($d = 1$) is incomplete whenever $m > 2$, i.e. as soon as there is some node in the tree which allows more than two possible stock price evolutions. This follows from Theorem 2.2 because in that situation, there are infinitely many EMMs; see Section 2.3.

Example. Consider any model with $d = 1$ (one risky asset) and i.i.d. returns $Y_1, \dots, Y_T$ under P. If Y_1 has a density (e.g. if we have lognormal returns), then S is incomplete. This is because $\mathcal{F}_1$ (and hence also $\mathcal{F}_T$) must be infinite for Y_1 to have a density. Alternatively, one can easily construct different EMMs if there is at least one. [$\rightarrow$ *Exercise*]

3.3 Example: The binomial model

In this section, we briefly illustrate how the preceding theory works out in the *binomial* or *Cox–Ross–Rubinstein model*. We recall that this model is described by parameters $p \in (0,1)$ and $u > r > d > -1$; then we have $\widetilde{S}^0_k = (1+r)^k$ and $\widetilde{S}^1_k = S^1_0 \prod_{j=1}^k Y_j$ with $S^1_0 > 0$ and $Y_1, \dots, Y_T$ i.i.d. under P taking values $1+u$ or $1+d$ with probability p or $1-p$, respectively. The filtration $\mathbb{F}$ is generated by $\widetilde{S} = (\widetilde{S}^0, \widetilde{S}^1)$ or equivalently by $\widetilde{S}^1$ or by Y. Note that $\mathcal{F}_0$ is then trivial because $\widetilde{S}^0_0 = 1$ and $\widetilde{S}^1_0 = S^1_0$ is a constant. We also take $\mathcal{F} = \mathcal{F}_T$; this is even an automatic conclusion if we construct the model on the canonical path space as in Section 1.4.

We already know from Corollary 2.2.3 that this model is *arbitrage-free* and has a *unique EMM* for $S^1 = \widetilde{S}^1/\widetilde{S}^0$. Hence S^1 is *complete* by Theorem 2.2, and so every $H \in L^0_+(\mathcal{F}_T)$ is *attainable*, with a *price process* given by

$$V^H_k = E_{Q^*}[H \,|\, \mathcal{F}_k] \qquad \text{for } k = 0, 1, \dots, T,$$

where Q^* is the unique EMM for S^1. We also recall from Corollary 2.2.3 that the Y_j are under Q^* again i.i.d., but with

$$Q^*[Y_1 = 1+u] = q^* := \frac{r-d}{u-d} \in (0,1).$$

All the above quantities S^1, H, V^H are discounted with $\widetilde{S}^0$, i.e. expressed in units of asset 0. The *undiscounted quantities* are the stock price $\widetilde{S}^1 = S^1 \widetilde{S}^0$, the payoff $\widetilde{H} := H\widetilde{S}^0_T$ and its price process $\tilde{V}^{\widetilde{H}}$ with $\tilde{V}^{\widetilde{H}}_k := V^H_k \widetilde{S}^0_k$ for $k = 0, 1, \dots, T$. Putting together all we know then yields

Corollary 3.1. *In the binomial model with $u > r > d$, the undiscounted arbitrage-free price process of any undiscounted payoff $\widetilde{H} \in L^0_+(\mathcal{F}_T)$ is given by*

$$\tilde{V}^{\widetilde{H}}_k = \widetilde{S}^0_k E_{Q^*}\left[\frac{\widetilde{H}}{\widetilde{S}^0_T} \,\middle|\, \mathcal{F}_k\right] = E_{Q^*}\left[\widetilde{H}\frac{\widetilde{S}^0_k}{\widetilde{S}^0_T} \,\middle|\, \mathcal{F}_k\right] = \frac{\widetilde{S}^0_k}{\widetilde{S}^0_T} E_{Q^*}[\widetilde{H} \,|\, \mathcal{F}_k] \qquad \text{for } k = 0, 1, \dots, T.$$

Example. For a *European call option* on $\widetilde{S}^1$ with maturity T and undiscounted strike

$\widetilde{K}$, we have

$$\widetilde{H} = (\widetilde{S}^1_T - \widetilde{K})^+ = (\widetilde{S}^1_T - \widetilde{K}) I_{\{\widetilde{S}^1_T > \widetilde{K}\}}.$$

Now

$$\{\widetilde{S}^1_T > \widetilde{K}\} = \left\{ \widetilde{S}^1_k \prod_{j=k+1}^{T} Y_j > \widetilde{K} \right\} = \left\{ \sum_{j=k+1}^{T} \log Y_j > \log(\widetilde{K}/\widetilde{S}^1_k) \right\}.$$

If we define

$$W_j := I_{\{Y_j = 1+u\}} = \begin{cases} 1 & \text{if } Y_j = 1+u, \\ 0 & \text{if } Y_j = 1+d, \end{cases}$$

then $W_1, \dots, W_T$ are under Q^* independent 0-1 experiments with success parameter q^*, so that their sum has under Q^* a *binomial distribution*. Moreover, using the fact that $\log Y_j = W_j \log(1+u) + (1-W_j)\log(1+d) = W_j \log \frac{1+u}{1+d} + \log(1+d)$ gives

$$\sum_{j=k+1}^{T} \log Y_j = W_{k,T} \log \frac{1+u}{1+d} + (T-k)\log(1+d),$$

where $W_{k,T} := \sum_{j=k+1}^{T} W_j \sim \mathrm{Bin}(T-k, q^*)$ is independent of $\mathcal{F}_k$ under Q^*. So we get

$$\{\widetilde{S}^1_T > \widetilde{K}\} = \left\{ W_{k,T} \log \frac{1+u}{1+d} > \log \frac{\widetilde{K}}{\widetilde{S}^1_k} - (T-k)\log(1+d) \right\}$$

and therefore

$$Q^*[\widetilde{S}^1_T > \widetilde{K} \mid \mathcal{F}_k] = Q^*\left[W_{k,T} > \frac{\log \frac{\widetilde{K}}{s} - (T-k)\log(1+d)}{\log \frac{1+u}{1+d}} \right]\Bigg|_{s=\widetilde{S}^1_k},$$

because $W_{k,T}$ is independent of $\mathcal{F}_k$ under Q^* and $\widetilde{S}^1_k$ is $\mathcal{F}_k$-measurable. The above probability can be computed explicitly because $W_{k,T}$ has a binomial distribution; and as

$$E_{Q^*}[\widetilde{H} \mid \mathcal{F}_k] = E_{Q^*}\left[\widetilde{S}^1_T I_{\{\widetilde{S}^1_T > \widetilde{K}\}} \,\middle|\, \mathcal{F}_k \right] - \widetilde{K} Q^*[\widetilde{S}^1_T > \widetilde{K} \mid \mathcal{F}_k],$$

we already have the second half of the so-called binomial call pricing formula.

For the first term, one can either use explicit (and lengthy) computations or more elegantly a so-called *change of numeraire* to obtain that

$$(3.1)\qquad \begin{aligned} E_{Q^*}\big[\widetilde{S}^1_T I_{\{\widetilde{S}^1_T>\widetilde{K}\}}\,\big|\,\mathcal{F}_k\big] &= \widetilde{S}^1_k \frac{\widetilde{S}^0_T}{\widetilde{S}^0_k}\frac{\widetilde{S}^0_k}{\widetilde{S}^1_k} E_{Q^*}\left[\frac{\widetilde{S}^1_T}{\widetilde{S}^0_T} I_{\{\widetilde{S}^1_T>\widetilde{K}\}}\,\middle|\,\mathcal{F}_k\right] \\ &= \widetilde{S}^1_k \frac{\widetilde{S}^0_T}{\widetilde{S}^0_k} Q^{**}[\widetilde{S}^1_T > \widetilde{K}\,|\,\mathcal{F}_k] \\ &= \widetilde{S}^1_k \frac{\widetilde{S}^0_T}{\widetilde{S}^0_k} Q^{**}\left[W_{k,T} > \frac{\log\frac{\widetilde{K}}{s} - (T-k)\log(1+d)}{\log\frac{1+u}{1+d}}\right]\Bigg|_{s=\widetilde{S}^1_k}, \end{aligned}$$

where $W_{k,T}$ under Q^{**} is $\mathrm{Bin}(T-k, q^{**})$-distributed with

$$q^{**} := q^* \frac{1+u}{1+r}, \qquad \text{hence } 1-q^{**} = (1-q^*)\frac{1+d}{1+r}.$$

Indeed, because $\widetilde{S}^1/\widetilde{S}^0 = S^1$ is under Q^* a positive martingale, one can use it to define via $\mathrm{d}Q^{**}/\,\mathrm{d}Q^* := S^1_T/S^1_0$ a probability measure $Q^{**} \approx Q^*$ on $\mathcal{F}_T$; then the Q^*-martingale S^1/S^1_0 starting at 1 is by construction the density process $Z^{Q^{**};Q^*}$ of Q^{**} with respect to Q^*, and the second equality in (3.1) is due to the Bayes formula (2.3.2) in Lemma 2.3.1. One then easily verifies [$\rightarrow$ *exercise*] that Q^{**} is the unique probability measure equivalent to P on $\mathcal{F}_T$ such that $\widetilde{S}^0/\widetilde{S}^1 = 1/S^1$ becomes a Q^{**}-martingale, and one can also check that $Y_1, \dots, Y_T$ are under Q^{**} i.i.d. with $Q^{**}[Y_1 = 1+u] = q^{**}$. Indeed, this is not really surprising — by Lemma 2.3.1, 3), the process $1/S^1$ is a Q^{**}-martingale because the product $Z^{Q^{**};Q^*}(1/S^1) = (S^1/S^1_0)(1/S^1) \equiv 1/S^1_0$ is obviously a Q^*-martingale, and $1/S^1$ has a binomial structure exactly like S^1 itself. The measure Q^{**} is sometimes called *dual martingale measure*.

So all in all, we obtain the fairly simple formula

$$(3.2)\qquad \tilde{V}^{\widetilde{H}}_k = \widetilde{S}^1_k Q^{**}[W_{k,T} > x] - \widetilde{K}\frac{\widetilde{S}^0_k}{\widetilde{S}^0_T} Q^*[W_{k,T} > x]$$

with

$$(3.3)\qquad x = \frac{\log\frac{\widetilde{K}}{s} - (T-k)\log(1+d)}{\log\frac{1+u}{1+d}}, \qquad \text{for } s = \widetilde{S}^1_k,$$

and where $W_{k,T}$ has a binomial distribution with parameter $T-k$ and with q^* under Q^*, respectively with q^{**} under Q^{**}. This *binomial call pricing formula* is the *discrete analogue of* the famous *Black–Scholes formula.*

For a *general payoff* $\widetilde{H}$, the discounted price process V^H is by its construction a Q^*-martingale with final value H, so that $V_T^H = H$ and

$$V_{k-1}^H = E_{Q^*}[V_k^H \,|\, \mathcal{F}_{k-1}] \qquad \text{for } k = 1, \dots, T.$$

This provides a very *simple recursive algorithm* by using that the filtration $I\!\!F$ in the binomial model has the structure of a *(binary) tree.* Indeed, if we fix some node (corresponding to some atom) at time $k-1$ (respectively of $\mathcal{F}_{k-1}$) and denote by v_{k-1} the value of V_{k-1}^H there (on that atom), then there are only two possible successor nodes (atoms of $\mathcal{F}_k$) and V_k^H can only take two values there, say v_k^u and v_k^d. The Q^*-martingale property then says that

$$v_{k-1} = q^* v_k^u + (1-q^*) v_k^d,$$

because the one-step transition probabilities of Q^* are the same throughout the tree and given by $q^*, 1-q^*$. In undiscounted terms, we have

$$\frac{\tilde{V}_{k-1}^{\widetilde{H}}}{\widetilde{S}_{k-1}^0} = E_{Q^*}\left[\frac{\tilde{V}_k^{\widetilde{H}}}{\widetilde{S}_k^0} \,\middle|\, \mathcal{F}_{k-1} \right]$$

or

$$\tilde{V}_{k-1}^{\widetilde{H}} = \frac{1}{1+r} E_{Q^*}[\tilde{V}_k^{\widetilde{H}} \,|\, \mathcal{F}_{k-1}],$$

which translates at the level of node values to the *recursion*

$$\tilde{v}_{k-1} = \frac{1}{1+r}\big(q^* \tilde{v}_k^u + (1-q^*)\tilde{v}_k^d\big). \tag{3.4}$$

The *terminal condition* $V_T^H = H$ or $\tilde{V}_T^{\widetilde{H}} = \widetilde{H}$ means that the values v_T or $\tilde{v}_T$ at the terminal nodes are given by the values of $\widetilde{H}$ there. Note that for a general (hence typically path-dependent) payoff $\widetilde{H}$, we have to work with the *full, non-recombining tree* and all its 2^T terminal nodes.

To work out the *replicating strategy*, also for a general payoff H, we recall from Theorem 1.1 that

$$V_k^H = V_k(V_0, \vartheta) = V_0 + \sum_{j=1}^{k} \vartheta_j \Delta S_j^1 \qquad \text{for } k = 0, 1, \dots, T.$$

For the *increments*, this means that

$$\Delta V_k^H = V_k^H - V_{k-1}^H = \vartheta_k \Delta S_k^1 = \vartheta_k (S_k^1 - S_{k-1}^1). \tag{3.5}$$

Now let us look again at some fixed node at time $k-1$ (atom of $\mathcal{F}_{k-1}$). Because ϑ is predictable, ϑ_k is $\mathcal{F}_{k-1}$-measurable and so the value of ϑ_k is already known at time $k-1$, hence in that node (on that atom), and it cannot change as we move forward to time k. If we denote as before by v_{k-1} the value of V_{k-1}^H in the chosen node (on the chosen atom) at time $k-1$ and by s_{k-1} the value of S_{k-1}^1 there, we know that v_{k-1} evolves to either v_k^u or v_k^d, and s_{k-1} evolves to $s_k^u = s_{k-1}\frac{1+u}{1+r}$ or $s_k^d = s_{k-1}\frac{1+d}{1+r}$, respectively, in the next step. But the relation (3.5) between increments must hold in all nodes (on all atoms) and at all times; so if ξ_k denotes the value of ϑ_k in the chosen node (on the chosen atom) at time $k-1$, we obtain the *two equations*

$$\begin{aligned} v_k^u - v_{k-1} &= \xi_k (s_k^u - s_{k-1}), \\ v_k^d - v_{k-1} &= \xi_k (s_k^d - s_{k-1}). \end{aligned}$$

Note that we have the same ξ_k in both equations because the value of ϑ_k cannot change as we go from time $k-1$ to time k. The above two equations are readily solved to give

$$\xi_k = \frac{v_k^u - v_k^d}{s_k^u - s_k^d} = \frac{v_k^u - v_k^d}{\frac{u-d}{1+r} s_{k-1}}. \tag{3.6}$$

Again, the right-hand side is known at time $k = T$ because we know that $V_T^H = H$. So both the price process V^H and the hedging strategy ϑ can be computed in parallel while working backward through the tree.

If the payoff $\widetilde{H}$ is like the call option of the *simple path-independent form* $\widetilde{H} = \tilde{h}(\widetilde{S}_T^1)$ for some function $\tilde{h}$, then the above formulas and computation scheme simplify considerably.

Indeed one can show by backward induction that

$$\tilde{V}_k^{\widetilde{H}} = \tilde{v}(k, \widetilde{S}_k^1) \qquad \text{for } k = 0, 1, \dots, T$$

and

$$\vartheta_k = \tilde{\xi}(k, \widetilde{S}_{k-1}^1) \qquad \text{for } k = 1, \dots, T$$

with *functions* $\tilde{v}(k,s)$ and $\tilde{\xi}(k,s)$ that are given by the recursion (compare (3.4))

$$\tilde{v}(k-1, s) = \frac{1}{1+r}\Big(q^* \tilde{v}\big(k, s(1+u)\big) + (1-q^*)\tilde{v}\big(k, s(1+d)\big)\Big)$$

with terminal condition

$$\tilde{v}(T, s) = \tilde{h}(s)$$

and, from (3.6) multiplied in both numerator and denominator by $\widetilde{S}_k^0 = (1+r)^k$, by

$$\tilde{\xi}(k, s) = \frac{\tilde{v}(k, s(1+u)) - \tilde{v}(k, s(1+d))}{(u-d)s}. \tag{3.7}$$

In particular, it is here enough to do all the computations in the *simplified, recombining tree* because neither $\tilde{V}^{\widetilde{H}}$ nor ϑ have any path-dependence, but only depend on current values of $\widetilde{S}^1$. So instead of 2^T terminal nodes for all the trajectories ω, we need here only $T+1$ terminal nodes, for all the possible values of $\widetilde{S}_T^1$. The corresponding tree is therefore also massively smaller, and so are computation times and storage requirements.
[It is a very good [→ *exercise*] to either derive the above relations for the path-independent case directly or deduce them from the preceding general results. In both cases, one uses a *backward induction* argument.]