CHAPTER 10

VAR Methods

In practice, this works, but how about in theory?

—*Attributed to a French mathematician*

Value at risk (VAR) has become an essential tool for risk managers because it provides a quantitative measure of downside risk based on current positions. In practice, the objective should be to provide a reasonably accurate estimate of risk at a reasonable cost. This involves choosing from among the various industry standards a method that is most appropriate for the portfolio at hand. To help with this selection, this chapter presents and critically evaluates various approaches to VAR.

The potential for losses results from exposures to the risk factors, as well as the distributions of these risk factors. This dichotomy finds its way into the structure of risk management systems, which can be classified into models for exposure and models for the distributions of risk factors.

Models for exposure can be classified into two groups. The first group uses local valuation. *Local-valuation methods* measure risk by valuing the portfolio once, at the initial position, and using local derivatives to infer possible movements. Within this class, the *delta-normal method* uses linear, or delta, exposures and assumes normal distributions. This is sometimes called the *variance-covariance method.* For portfolios exposed to a small number of risk factors, second-order derivatives sometimes are used. The second group uses full valuation. *Full-valuation methods* measure risk by fully repricing the portfolio over a range of scenarios.

Models for risk factors include parametric approaches, such as the normal distribution, and nonparametric approaches based on historical data.

Section 10.1 gives an overview of VAR systems. The local- and full-valuation approaches are discussed in Section 10.2. Initially, we consider a simple portfolio that is driven by one risk factor only. This chapter then turns to VAR methods for large portfolios. The delta-normal method is explained in Section 10.3. The historical simulation and Monte Carlo (MC) simulation methods are discussed next in Sections 10.4 and 10.5. All these methods require mapping, which is developed in Chapter 11.

This classification reflects a fundamental tradeoff between speed and accuracy. Speed is important for large portfolios exposed to many risk factors that involve a large number of correlations. These are handled most easily in the delta-normal approach. Accuracy may be more important, however, when the portfolio has substantial nonlinear components. Section 10.6 presents some empirical comparisons of the VAR approaches. Finally, Section 10.7 summarizes the pros and cons of each of the three main methods.

10.1 VAR SYSTEMS

The potential for gains and losses can be attributed to two sources. On the one hand are the exposures, which represent active choices by the trader or portfolio manager. On the other hand are the movements in the risk factors, which are outside their control.

This dichotomy is reflected in the structure of risk management systems, which is described in Figure 10-1. The left-hand side describes the portfolio *positions*, which have as input trades from the front office. The right-hand side describes the *risk factors*, which have as input data feeds with current market prices. Positions and risk-factor distributions are brought together in the *risk engine*, which generates a distribution of portfolio values that can be summarized, for instance, by its VAR.

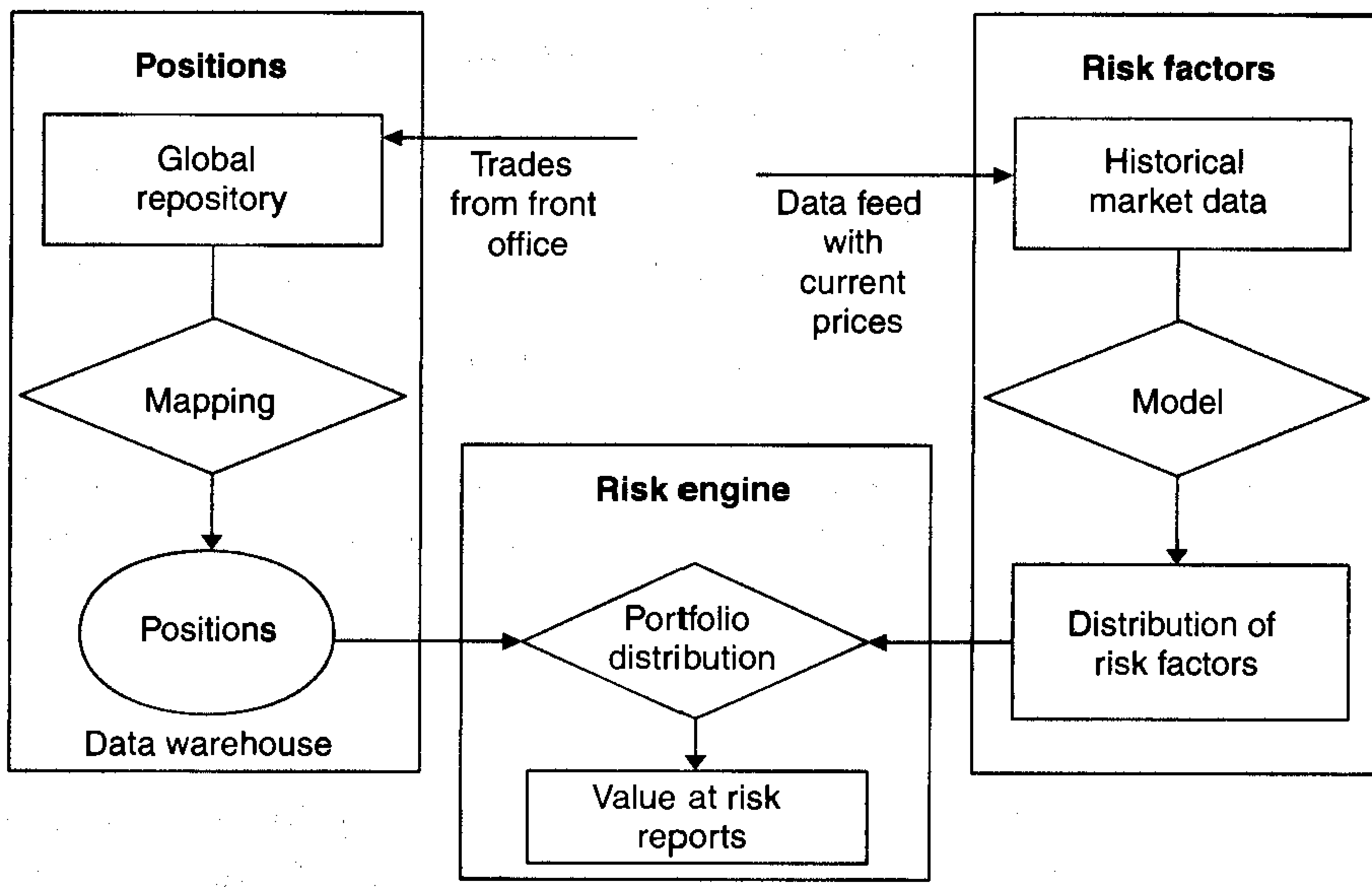
Different VAR methods make different assumptions for the modeling of positions and risk factors. The positions can be replaced by their linear exposures to the risk factors, or by the quadratic exposures, or by using full repricing. The distributions of risk factors can be modeled using a normal distribution, or the historical data, or Monte Carlo simulations.

Modern risk measurement methods are applied at the highest level of the portfolio. This generally involves a very large number of instruments and risk factors. It would be impractical to model all these positions individually. Realistically, simplifications are required.

The first step in the implementation of a risk measurement system involves choosing an appropriate number of risk factors. Positions then are simplified by *mapping* each and every one on the risk factors. This

FIGURE 10-1

VAR systems.



replaces the dollar value of positions in each instrument by a set of dollar exposures on the risk factors. These exposures then are aggregated across the whole portfolio to create net positions that are matched to the risk factors. This mapping process will be detailed further in Chapter 11. In this chapter we focus on integration of exposures with the risk factors.

10.2 LOCAL VERSUS FULL VALUATION

10.2.1 Delta-Normal Valuation

Local-valuation methods measure exposures with partial derivatives. To illustrate the approach, take an instrument whose value depends on a single underlying risk factor S. The first step consists of valuing the asset at the initial point, that is,

$$V_0 = V(S_0) \tag{10.1}$$

along with analytical or numerical derivatives. Define delta (Δ_0) as the first partial derivative, or the asset sensitivity to changes in prices, evaluated at the current position V_0. This would be called *delta* for a derivative or *modified duration* for a fixed-income portfolio. For instance, with an at-the-money call, $\Delta = 0.5$, and a long position in one option is simply replaced by a 50 percent position in one unit of the underlying asset. Thus this is a linear exposure to the risk factor.

The potential loss in value of an option dV then is computed as

$$dV = \frac{\partial V}{\partial S}\Big|_0 dS = \Delta_0 \times dS = (\Delta_0 S)\frac{dS}{S} \tag{10.2}$$

which involves the potential change in prices dS. Here, the dollar exposure is given by $x = \Delta_0 S$.

Because this is a linear relationship, the worst loss for V is attained for an extreme value of S. If the distribution is normal, the portfolio VAR can be derived from the product of the exposure and the VAR of the underlying variable, that is,

$$\text{VAR} = |\Delta_0| \times \text{VAR}_S = |\Delta_0| \times (\alpha\sigma S_0) \tag{10.3}$$

where α is the standard normal deviate corresponding to the specified confidence level, for example, 1.645 for a 95 percent level. Here, we take $\sigma(dS/S)$ as the standard deviation of *rates* of changes in the price.

This approach is called the *delta-normal method.* Because VAR is obtained as a closed-form solution, this is an *analytical* method. Note that VAR was measured by computing the portfolio value only once, at the current position V_0.[1]

For a fixed-income portfolio, the risk factor is the yield y, and the price-yield relationship is

$$dV = (-D^*V)dy \tag{10.4}$$

where D^* is the *modified duration.* Here, the dollar exposure is given by $x = -D^*V$. In this case, the portfolio VAR is

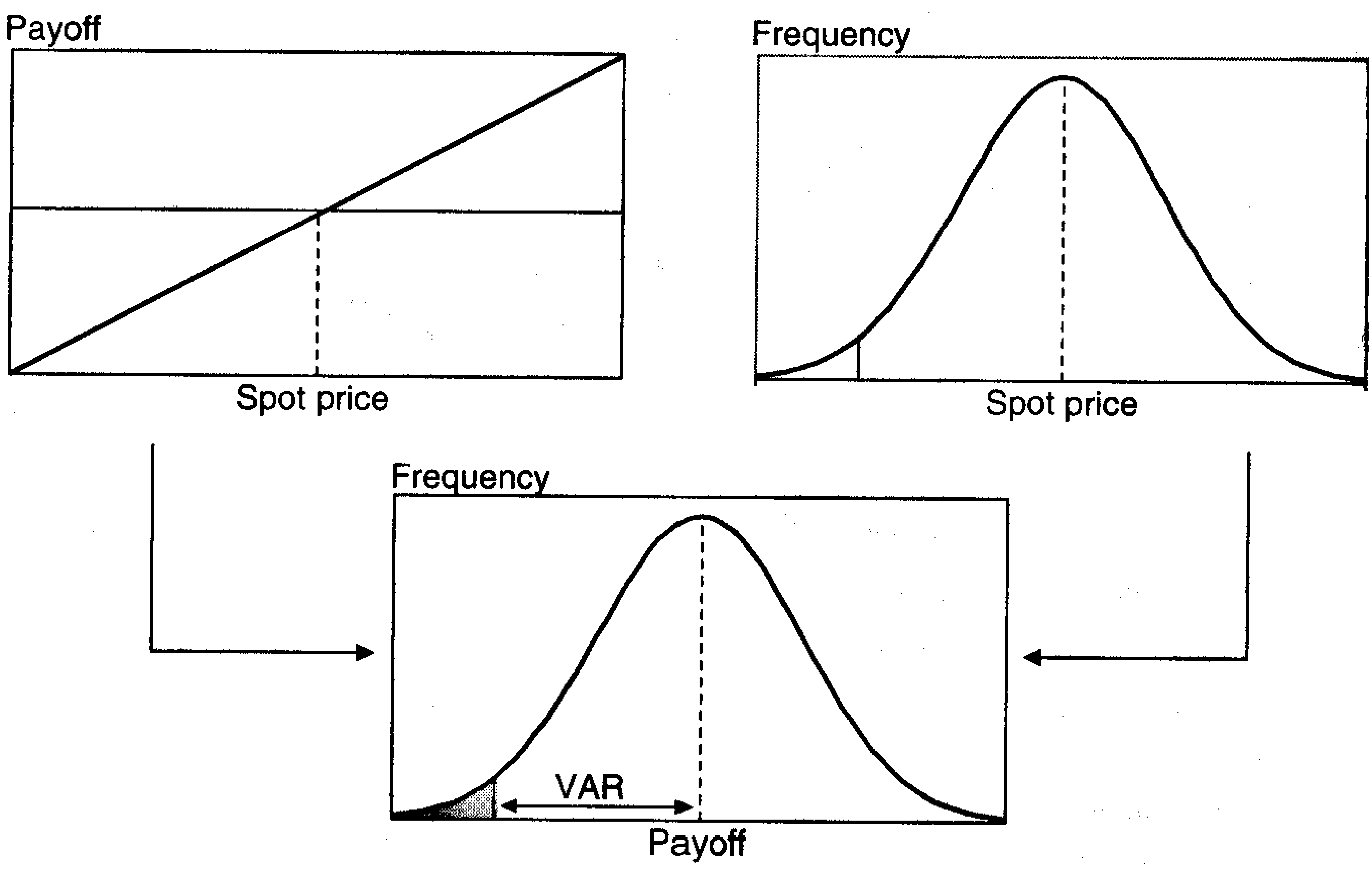
$$\text{VAR} = |D^*V| \times (\alpha\sigma) \tag{10.5}$$

where $\sigma(dy)$ is now the volatility of changes in the *level* of yield. The assumption here is that changes in yields are normally distributed, although this is ultimately an empirical issue.

[1] Usually, delta can be computed easily. If the instrument is an option valued using a binomial tree, for instance, delta can be computed from the up and down values of the option at the first step divided by the changes in asset values.

FIGURE 10-2

Distribution with linear exposures.



This method is illustrated in Figure 10-2, where the profit payoff V is a linear function of the underlying spot price S and is displayed in the upper left panel. As shown in the right panel, the price is normally distributed. As a result, the profit itself is normally distributed, as shown at the bottom of the figure. This implies that the VAR for the profit can be derived from the exposure and the VAR for the underlying price. There is a one-to-one mapping between the two VAR measures.

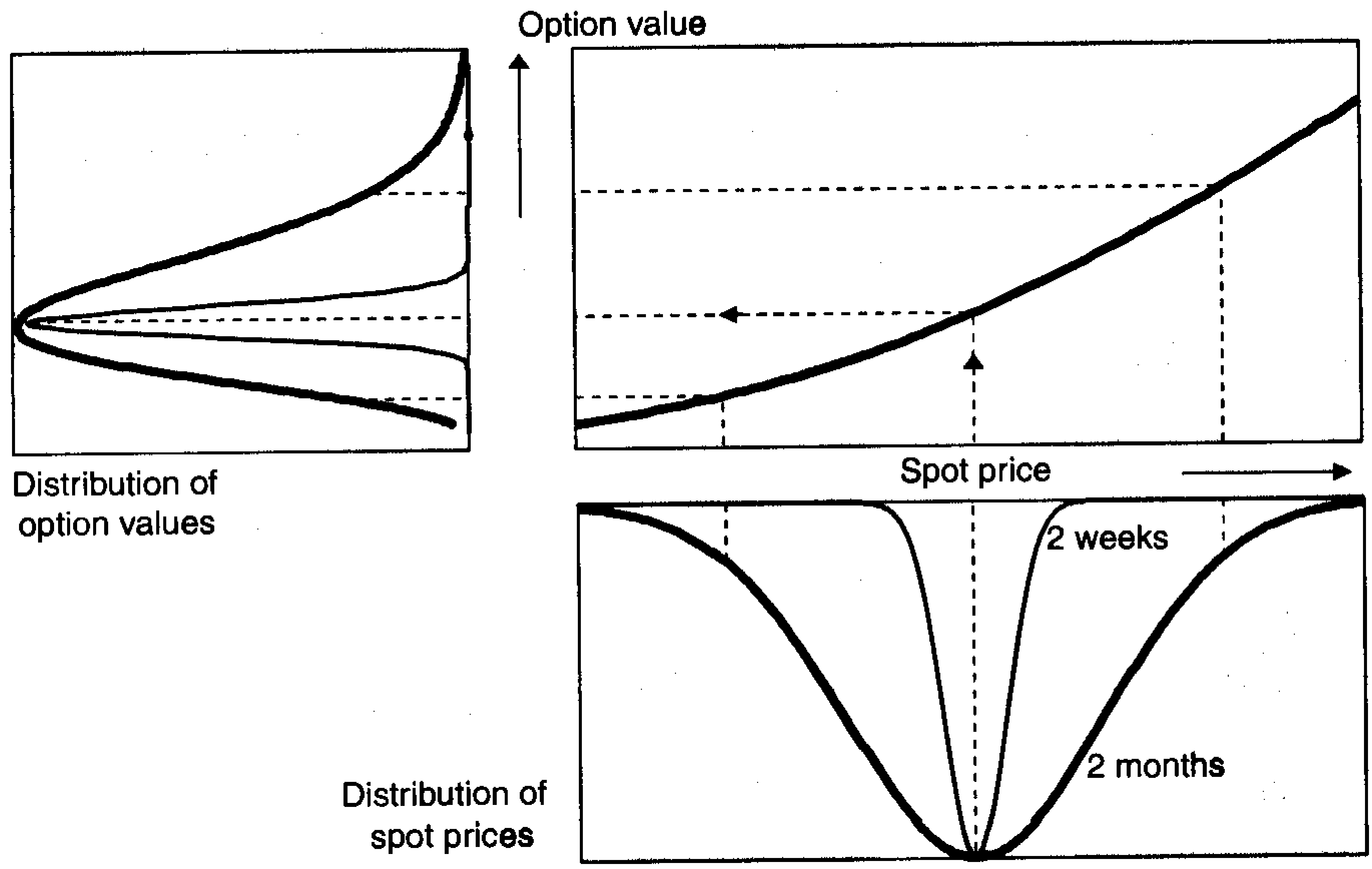
10.2.2 Full Valuation

In some situations, the delta-normal approach is totally inadequate. This is the case, for instance, with combinations of options that have very nonlinear payoffs. Sometimes the worst loss may not be obtained for extreme realizations of the underlying spot rate.

Consider, for instance, a simple example of a long position in a call option. In this case, we can describe the distribution of option values easily. This is so because there is a one-to-one relationship between V and S. In other words, given the pricing function, any value for S can be translated into a value for V, and vice versa.

FIGURE 10-3

Transformation of distributions.



This is illustrated in Figure 10-3, which shows how the distribution of the spot price is translated into a distribution for the option value (in the left panel). Note that the option distribution has a long right tail owing to the upside potential, whereas the downside is limited to the option premium. This shift is due to the nonlinear payoff on the option. Note how the positive skewness translates into a shorter left tail or lower VAR than otherwise.

Here, the cth quantile for V is simply the function evaluated at the cth quantile of S. For the long call option, the worst loss at a given confidence level is achieved at $S^* = S_0 - \alpha\sigma S_0$, and

$$\text{VAR} = V(S_0) - V(S_0 - \alpha\sigma S_0) \tag{10.6}$$

Because this is a *monotonic* transformation, the quantiles can be translated from S to V directly. This result, unfortunately, does not translate to general payoff functions.

The nonlinearity effect is not obvious, though. It also depends on the maturity of the option and on the range of spot prices over the horizon. The option illustrated here is a call option with 3 months to expiration. To obtain a visible shift in the shape of the option distribution, the volatility was set at 20 percent per annum and the VAR horizon at 2 months, which is rather long.

The figure also shows thinner distributions that correspond to a VAR horizon of 2 weeks. Here, the option distribution is indistinguishable from the normal. In other words, the mere presence of options does not necessarily invalidate the delta-normal approach. The quality of the approximation depends on the extent of nonlinearities, which is a function of the type of options, of their maturities, as well as of the volatility of risk factors and VAR horizon. The shorter the VAR horizon, the better is the delta-normal approximation.

Equation (10.6) is a convenient transformation of quantiles but does not apply with more complex, nonmonotonic payoffs. An example is that of a long *straddle*, which involves the purchase of a call and a put. The worst payoff, which is the sum of the premiums, will be realized if the spot rate does not move at all. In general, it is not sufficient to evaluate the portfolio at the two extremes. All intermediate values must be checked.

The *full-valuation approach* considers the portfolio value for a wide range of price levels, that is,

$$dV = V(S_1) - V(S_0) \tag{10.7}$$

The new values S_1 can be generated by simulation methods. The *Monte Carlo simulation approach* relies on parametric distributions. For instance, the realizations can be drawn from a normal distribution, that is,

$$\frac{dS}{S} \approx N(0,\sigma^2) \tag{10.8}$$

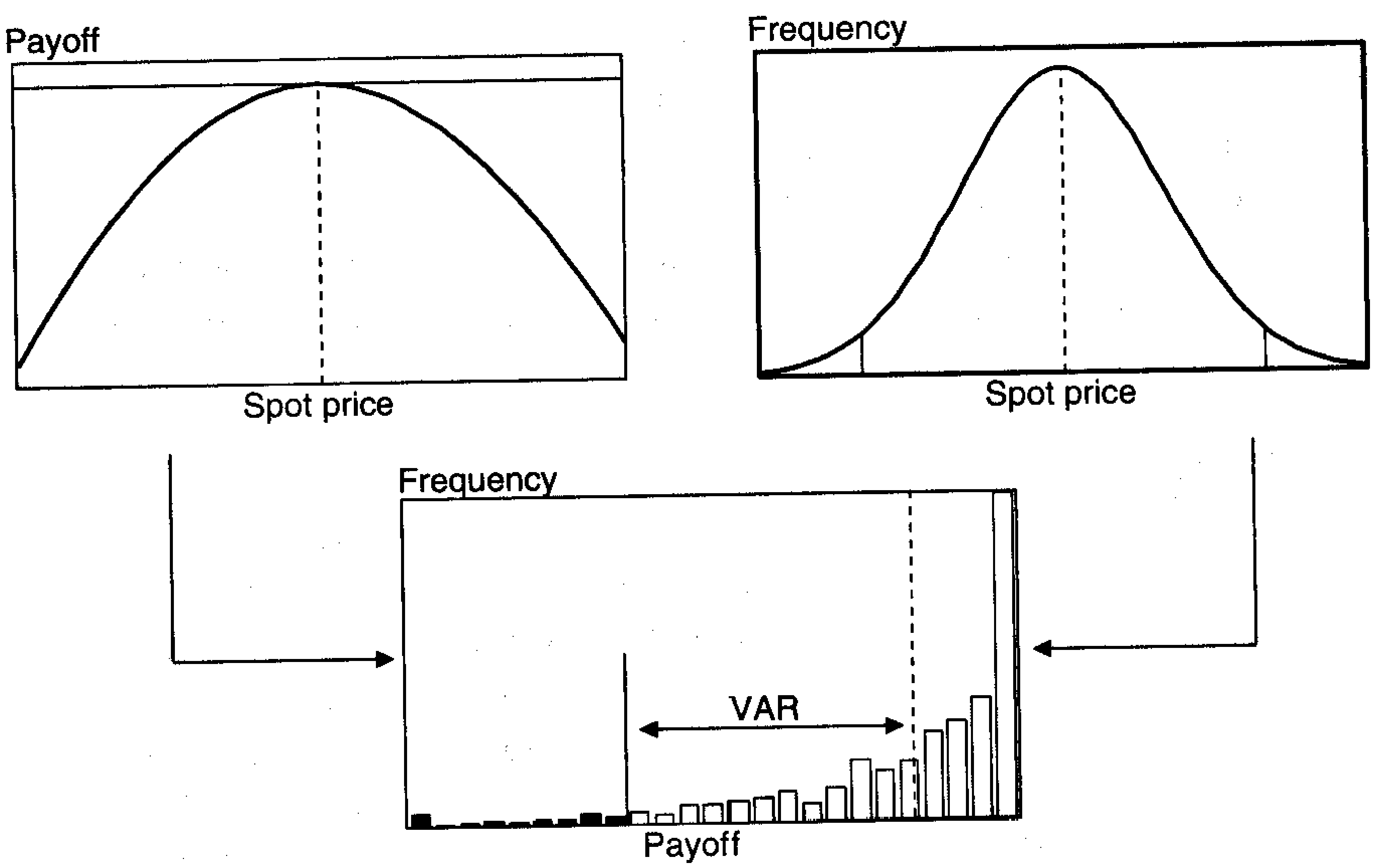
Alternatively, the *historical simulation approach* simply samples from recent historical data.

For each of these draws, the portfolio is priced on the target date using a full-valuation method. This method is potentially the most accurate because it accounts for nonlinearities, income payments, and even time-decay effects that are ignored in the delta-normal approach. VAR then is calculated from the percentiles of the full distribution of payoffs.

To illustrate the result of nonlinear exposures, Figure 10-4 displays the payoff function for a short straddle that is highly nonlinear. The resulting distribution is severely skewed to the left. Further, there is no direct way to relate the VAR of the portfolio to that of the underlying asset.

FIGURE 10-4

Distribution with nonlinear exposures.



Computationally, this approach is quite demanding because it requires marking to market the whole portfolio over a large number of realizations of underlying random variables. As a result, methods have been developed to speed up the computations. In general, these approaches try to break the link between the number of Monte Carlo draws and the number of times the portfolio is repriced.

One example is the *grid Monte Carlo approach*, which starts by an exact valuation of the portfolio over a limited number of grid points.[2] For each simulation, the portfolio value then is approximated using a linear interpolation from the exact values at the adjoining grid points. This approach is especially efficient if exact valuation of the instrument is complex. Take, for instance, a portfolio with one risk factor for which we require 1000 values $V(S_1)$. With the grid Monte Carlo method, 10 full valuations at the grid points may be sufficient. In contrast, the full Monte Carlo method would require 1000 full valuations.

[2] Picoult (1997) describes this method in more detail.

10.2.3 Delta-Gamma Approximations (The "Greeks")

It may be possible to extend the analytical tractability of the delta-normal method with higher-order terms. Because the method uses partial derivatives defined using Greek letters, it is sometimes called the *Greeks*.

We can improve the quality of the linear approximation by adding terms in the Taylor expansion of the valuation function, that is,

$$dV = \frac{\partial V}{\partial S} dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} dS^2 + \frac{\partial V}{\partial t} dt + \cdots = \Delta dS + \frac{1}{2}\Gamma dS^2 + \Theta dt + \cdots \tag{10.9}$$

where Γ is now the second derivative of the portfolio value, and Θ is the time drift, which is deterministic. For a fixed-income portfolio, the instantaneous price-yield relationship is now

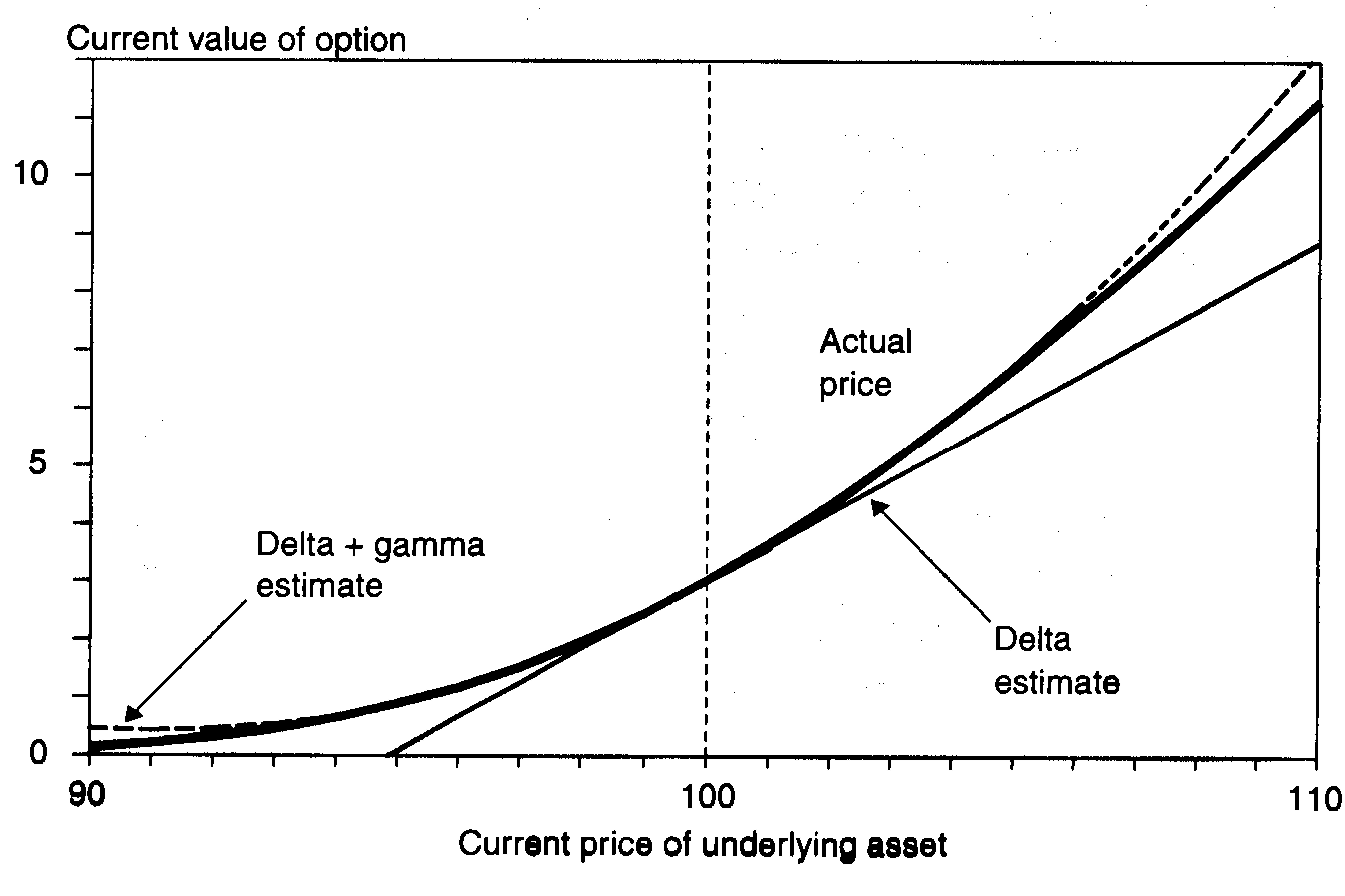
$$dV = -(D^*V)dy + \tfrac{1}{2}(CV)dy^2 + \cdots \tag{10.10}$$

where the second-order coefficient C is called *convexity* and is akin to Γ.

Figure 10-5 describes the approximation for a simple position, a long position in a European call option. The actual price is represented by the

FIGURE 10-5

Delta-gamma approximation for a long call.



thick line. The delta estimate is the straight line below. The delta plus gamma estimate is the dashed line. Because Γ is positive, the term ΓdS^2 must be positive, and the quadratic estimate must lie above the linear estimate. The graph shows that the linear estimate is only valid for small movements around the initial value. For larger movements, the delta-gamma estimate creates a better fit.

The figure also shows that delta is not constant but rather changes as a function of the spot price; gamma gives the rate of change in delta. Delta also changes with the passage of time. This has implications for the extrapolation of risk across horizons. With linear models, as we have seen in Chapter 4, daily VAR can be adjusted easily to other periods by scaling by a square-root-of-time factor. This adjustment assumes that the position is fixed and that daily returns are independent and identically distributed. Such adjustment, however, is not appropriate for options, even when the positions are fixed. This is so because the option delta changes dynamically over time. Hence the square-root-of-time adjustment may not be valid for options.

We now turn to the computation of VAR for the long-call option position. Using the Taylor expansion in Equation (10.6) gives

$$\begin{aligned}\text{VAR} &= V(S_0) - V(S_0 - \alpha\sigma S_0)\\ &= V(S_0) - [V(S_0) + \Delta(-\alpha\sigma S) + \tfrac{1}{2}\Gamma(-\alpha\sigma S)^2] \qquad (10.11)\\ &= |\Delta|(\alpha\sigma S) - \tfrac{1}{2}\Gamma(\alpha\sigma S)^2\end{aligned}$$

This formula is valid for long and short positions in calls and puts and, more generally, for portfolios whose payoff is monotonic in S. If Γ is positive, which corresponds to a net long position in options, the second term will decrease the linear VAR. Indeed, Figure 10-5 shows that the downside risk for the option is less than that given by the delta approximation. If Γ is negative, which corresponds to a net short position in options, VAR is increased.

This closed-form solution does not apply, unfortunately, to more complex functions $V(S)$. Appendix 10.A lists some analytical approximations, including the *delta-gamma-delta.* Generally, however, quadratic approximations are not used at the highest level of aggregation for large portfolios. Full implementation would require knowledge of all gammas and cross-gammas, that is, second derivatives with respect to other risk factors.

On the other hand, quadratic approximations are very useful to speed up computations with simulations. An example is the *delta-gamma-Monte-Carlo approach*, which creates random simulations of the risk factor S and then uses the Taylor approximation to create simulated movements in the option value. This method is also known as a *partial-simulation approach*. Note that this is still a local-valuation method because the asset is fully valued at the initial point V_0 only. The portfolio can be valued by adding the approximated option positions to all others.

10.2.4 Comparison of Methods

To summarize, Table 10-1 classifies the various VAR methods. Overall, each of these methods is best adapted to a different environment:

- For large portfolios where optionality is not a dominant factor, the delta-normal method provides a fast and efficient method for measuring VAR.
- For fast approximations of option values, mixed methods such as delta-gamma-Monte-Carlo or grid Monte Carlo are efficient.
- For portfolios with substantial option components (such as mortgages) or longer horizons, a full-valuation method may be required.

10.2.5 An Example: Leeson's Straddle

The Barings story provides a good illustration of these various methods. In addition to the long futures positions described in Chapter 7, Leeson also sold options, about 35,000 calls and puts each on Nikkei futures. This

TABLE 10-1

Comparison of VAR Methods

	Valuation Method	
Risk Factor Distribution	**Local Valuation**	**Full Valuation**
Analytical	Delta-normal Delta-gamma-delta	Not used
Simulated	Delta-gamma-Monte-Carlo	Monte Carlo Grid Monte Carlo Historical

position is known as a *short straddle* and is about delta-neutral because the positive delta from the call is offset by a negative delta from the put, assuming that most of the options are at the money.

Leeson did not deal in small amounts. With a multiplier of 500 yen for the options contract and a 100 yen/$ exchange rate, the dollar exposure of the call options to the Nikkei was delta times $0.175 million. Initially, the market value of the position was zero. The position was designed to turn in a profit if the Nikkei remained stable. Unfortunately, it also had an unlimited potential for large losses.

Figure 10-6 displays the payoffs from the straddle, using a Black-Scholes model with a 20 percent annual volatility. We assume that the options have a maturity of 3 months. At the current index value of 19,000, the delta VAR for this position is close to zero. Of course, reporting a zero delta-normal VAR is highly misleading. Any move up or down has the potential to create a large loss. A drop in the index to 17,000, for instance, would lead to an immediate loss of about $150 million. The graph also shows that the delta-gamma approximation provides increased accuracy over the delta method. How do we compute the potential loss over a horizon of, say, 1 month?

FIGURE 10-6

Leeson's straddle.

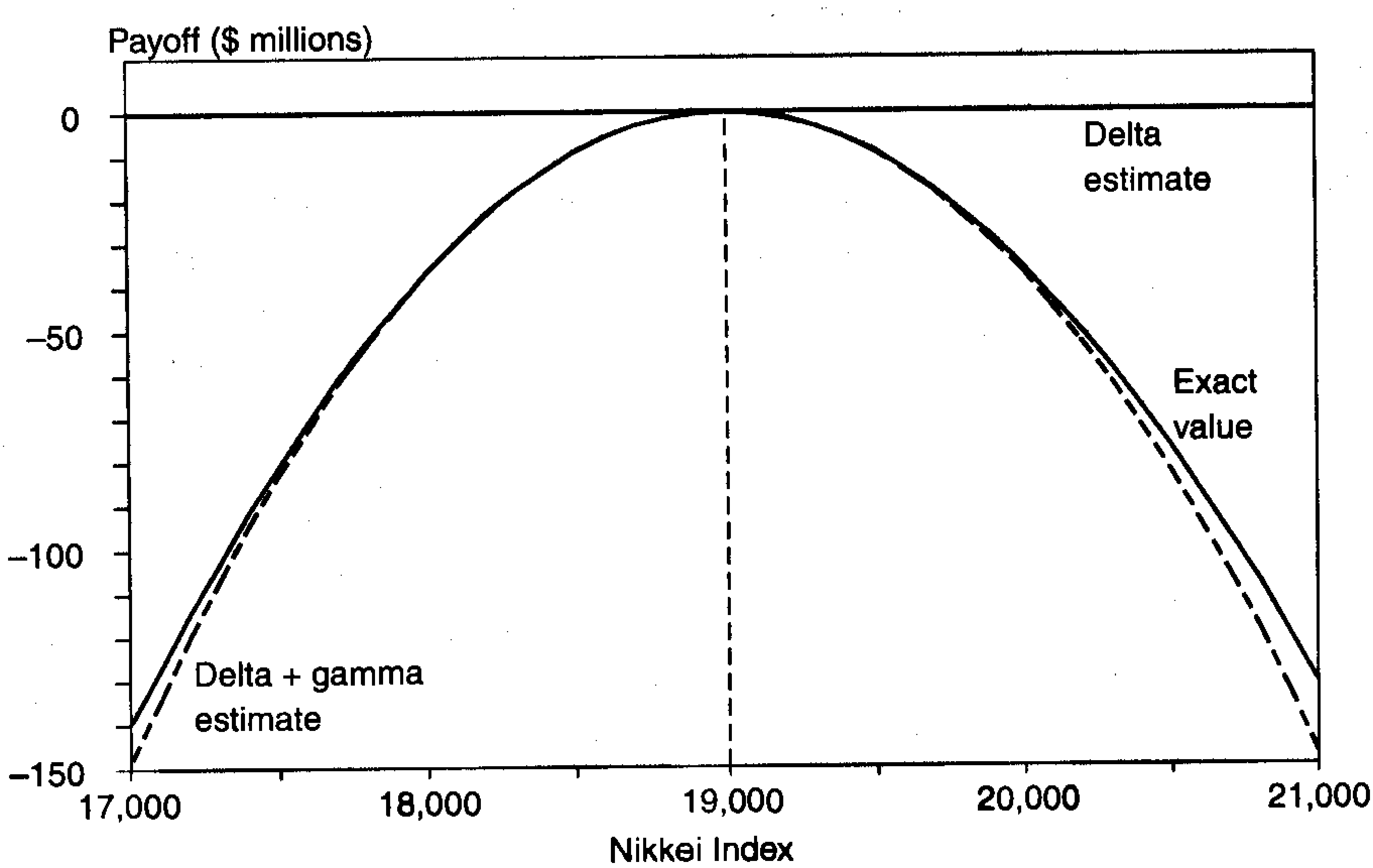
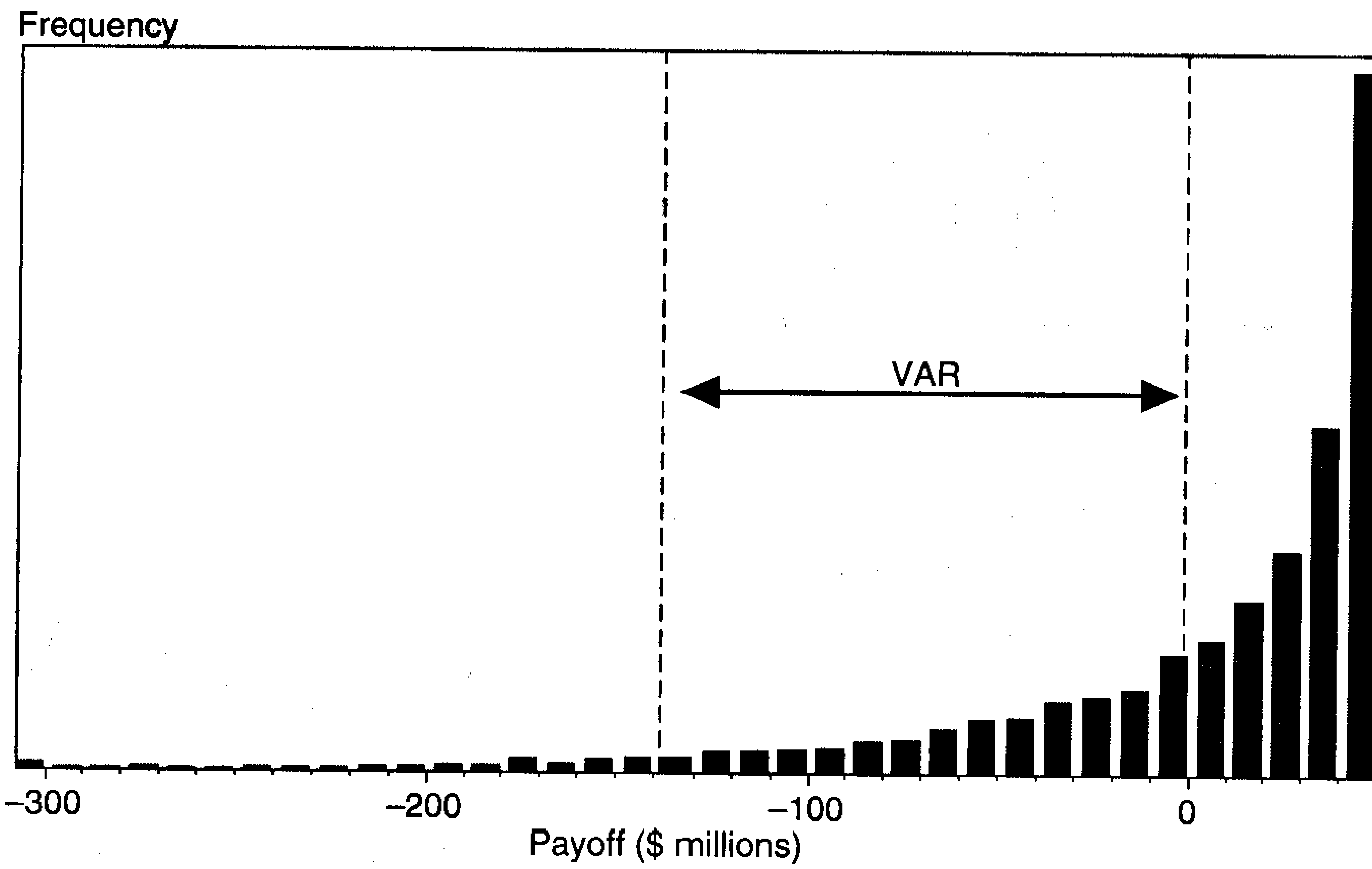


FIGURE 10-7

Distribution of 1-month payoff for straddle.



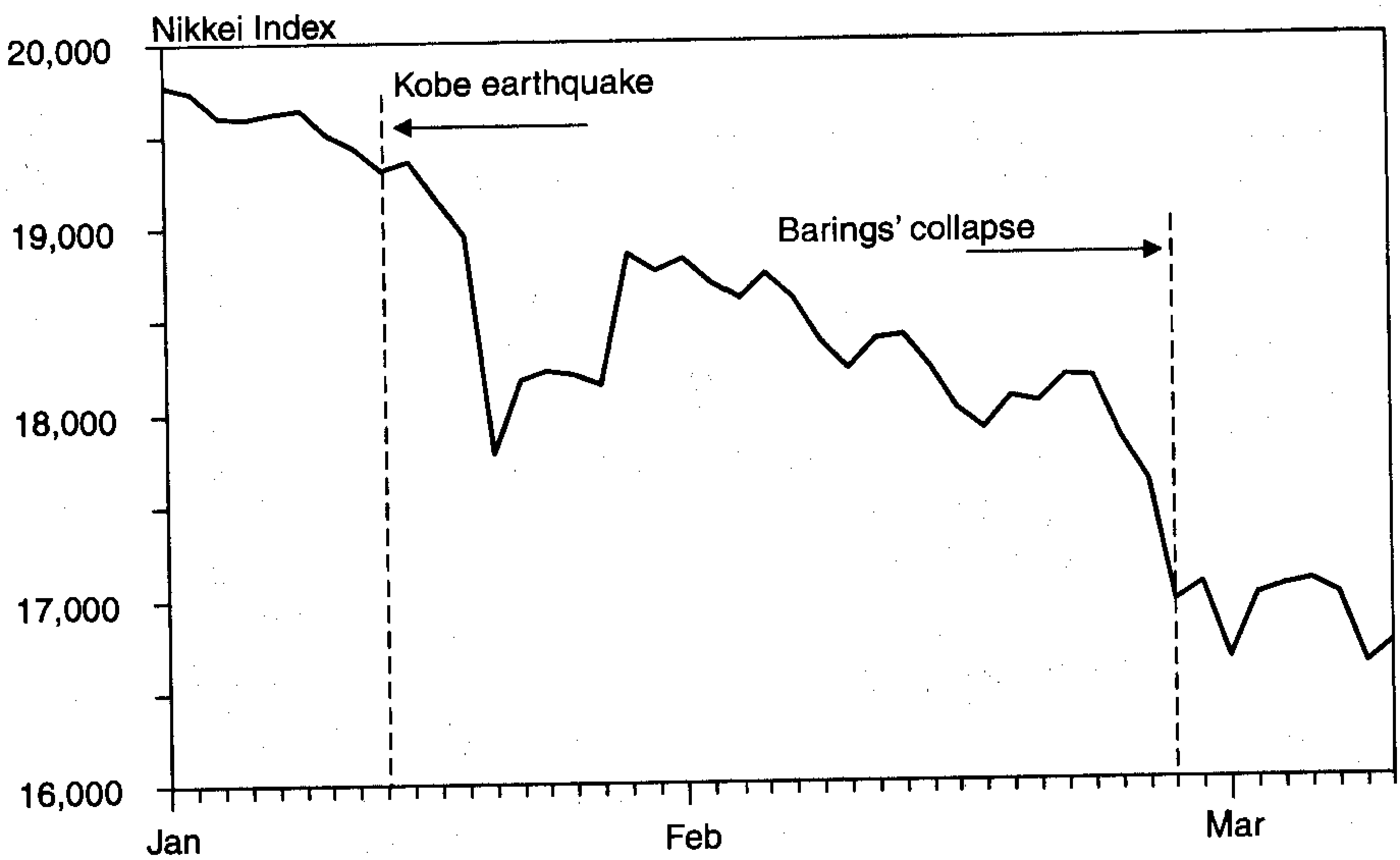
The risks involved are described in Figure 10-7, which plots the frequency distribution of payoffs on the straddle using a *full Monte Carlo simulation* with 10,000 replications. This distribution is obtained from a revaluation of the portfolio after a month over a range of values for the Nikkei. Each replication uses full valuation with a remaining maturity of 2 months (the 3-month original maturity minus the 1-month VAR horizon). The distribution looks highly skewed to the left. Its mean is −$1 million, and the 95th percentile is −$139 million. Hence the 1-month 95 percent VAR is $138 million.

Next, we can use the delta-gamma-Monte-Carlo approach, which consists of using the simulations of S but valuing the portfolio on the target date using only the partial derivatives. This yields a VAR of $128 million, not too far from the true value.

And indeed, the option position contributed to Barings' fall. As January 1995 began, the historical volatility on the Japanese market was very low, around 10 percent. At the time, the Nikkei was hovering around 19,000. The option position would have been profitable if the market had been stable. Unfortunately, this was not so. The Kobe earthquake struck Japan on January 17 and led to a drop in the Nikkei to 18,000, shown in Figure 10-8. To make

FIGURE 10-8

The Nikkei's fall.



things worse, options became more expensive as market volatility increased. Both the long futures and the straddle positions lost money. As losses ballooned, Leeson increased his exposure in a desperate attempt to recoup the losses, but to no avail. On February 27, the Nikkei dropped further to 17,000. Unable to meet the mounting margin calls, Barings went bust.

10.3 DELTA-NORMAL METHOD

10.3.1 Implementation

When the risk factors are jointly normally distributed and the positions can be represented by their delta exposures, the measurement of VAR is considerably simplified. We have N risk factors. Define $x_{i,t}$ as the exposures aggregated across all instruments for each risk factor i and measured in currency units. Equivalently, we could divide these by the current portfolio value W to obtain the portfolio weights $w_{i,t}$.

The portfolio *rate of return* is

$$R_{p,t+1} = \sum_{i=1}^{N} w_{i,t} R_{i,t+1} \tag{10.12}$$

where the weights $w_{i,t}$ are indexed by time to indicate that this is the current portfolio. This method allows easy aggregation of risks for large portfolios because of the invariance property of normal variables: Portfolios of jointly normal variables are themselves normally distributed. The portfolio normality assumption is also justified by the *central limit theorem*, which states that the average of independent random variables converges to a normal distribution. For portfolios diversified across a number of risk factors that have modest correlations, these conditions could be approximately met.

Using matrix notations, as in Chapter 7, the portfolio variance is given by

$$\sigma^2(R_{p,t+1}) = w_t' \Sigma_{t+1} w_t \tag{10.13}$$

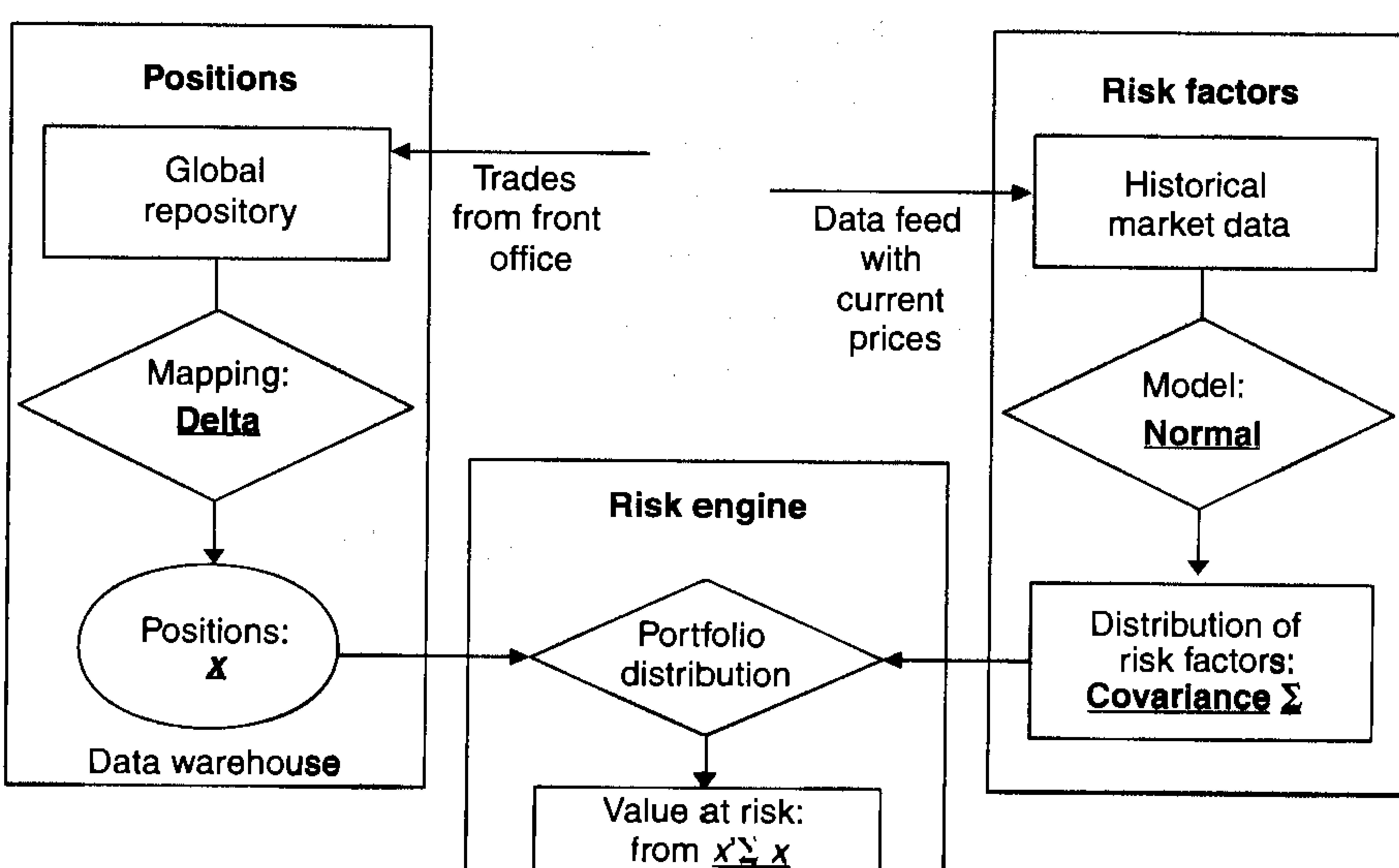
where Σ_{t+1} is the forecast of the covariance matrix over the VAR horizon, perhaps using models developed in Chapter 9. The portfolio VAR then is

$$\text{VAR} = \alpha\sqrt{x_t' \Sigma_{t+1} x_t} = \alpha W \sqrt{w_t' \Sigma_{t+1} w_t} \tag{10.14}$$

where α is the deviate corresponding to the confidence level for the normal distribution or for another parametric distribution. Figure 10-9 details the steps involved in this approach.

FIGURE 10-9

Delta-normal method.



10.3.2 Advantages

The delta-normal method is particularly *easy* to implement because it involves a simple matrix multiplication. It is also *computationally fast*, even with a very large number of assets, because it replaces each position by its linear exposure. As a result, it can be run in *real time*, or during the day as positions change.

As a parametric approach, VAR is easily *amenable to analysis* because measures of marginal and incremental risk are a by-product of the VAR computation. This is useful to manage the portfolio risk.

10.3.3 Drawbacks

The delta-normal method has a number of drawbacks, however. A first problem is the existence of *fat tails* in the distribution of returns on most financial assets. These fat tails are particularly worrisome precisely because VAR attempts to capture the behavior of the portfolio return in the left tail. In this situation, a model based on a normal distribution would underestimate the proportion of outliers and hence the true VAR. A simple ad hoc adjustment consists of increasing the parameter α to compensate.

This problem depends on the choice of the confidence level. Typically, there is not much bias from using a normal distribution at the 95 percent confidence level. The underestimation increases, however, for higher confidence levels.

Another problem is that the method is inadequate for *nonlinear instruments,* such as options and mortgages. As we have seen in the preceding section, asymmetries in the distribution of options are not captured by the delta-normal VAR.

For simple portfolios, however, the delta-normal method may be adequate. At the highest level of financial institutions, asymmetries tend to wash away, as predicted by the central limit theorem. For more complex portfolios, however, the delta-normal method generally is not sufficient.

10.4 HISTORICAL SIMULATION METHOD

10.4.1 Implementation

The historical simulation (HS) approach is a nonparametric method that makes no specific assumption about the distribution of risk factors. It consists of going back in time and replaying the tape of history on the current positions. Positions can be priced using full or local valuation.

In the most simple case, this method applies current weights to a time series of historical asset returns, that is,

$$R_{p,k} = \sum_{i=1}^{N} w_{i,t} R_{i,k} \quad k = 1, \ldots, t \tag{10.15}$$

Note that the weights w_t are kept at their current values. This return does not represent an actual portfolio but rather reconstructs the history of a hypothetical portfolio using the current position. The approach is sometimes called *bootstrapping* because it uses the actual distribution of recent historical data without replacement. Each scenario k is drawn from the history of t observations.

More generally, the method can use *full valuation*, employing hypothetical values for the risk factors, which are obtained from applying historical changes in prices to the current level of prices, that is,

$$S_{i,k}^{*} = S_{i,0} + \Delta S_{i,k} \quad i = 1, \ldots, N \tag{10.16}$$

A new portfolio value $V_{p,k}^{*}$ then is computed from the full set of hypothetical prices, perhaps incorporating nonlinear relationships $V_k^{*} = V(S_{i,k}^{*})$. Note that to capture *vega risk*, owing to changing volatilities, the set of risk factors can incorporate implied volatility measures. This creates the hypothetical return corresponding to simulation k, that is,

$$R_{p,k} = \frac{V_k^{*} - V_0}{V_0} \tag{10.17}$$

VAR then is obtained from the entire distribution of hypothetical returns, where each historical scenario is assigned the same weight of $(1/t)$. Figure 10-10 details the process. Because the approach does not assume a parametric distribution for the risk factors, it is called *nonparametric*.

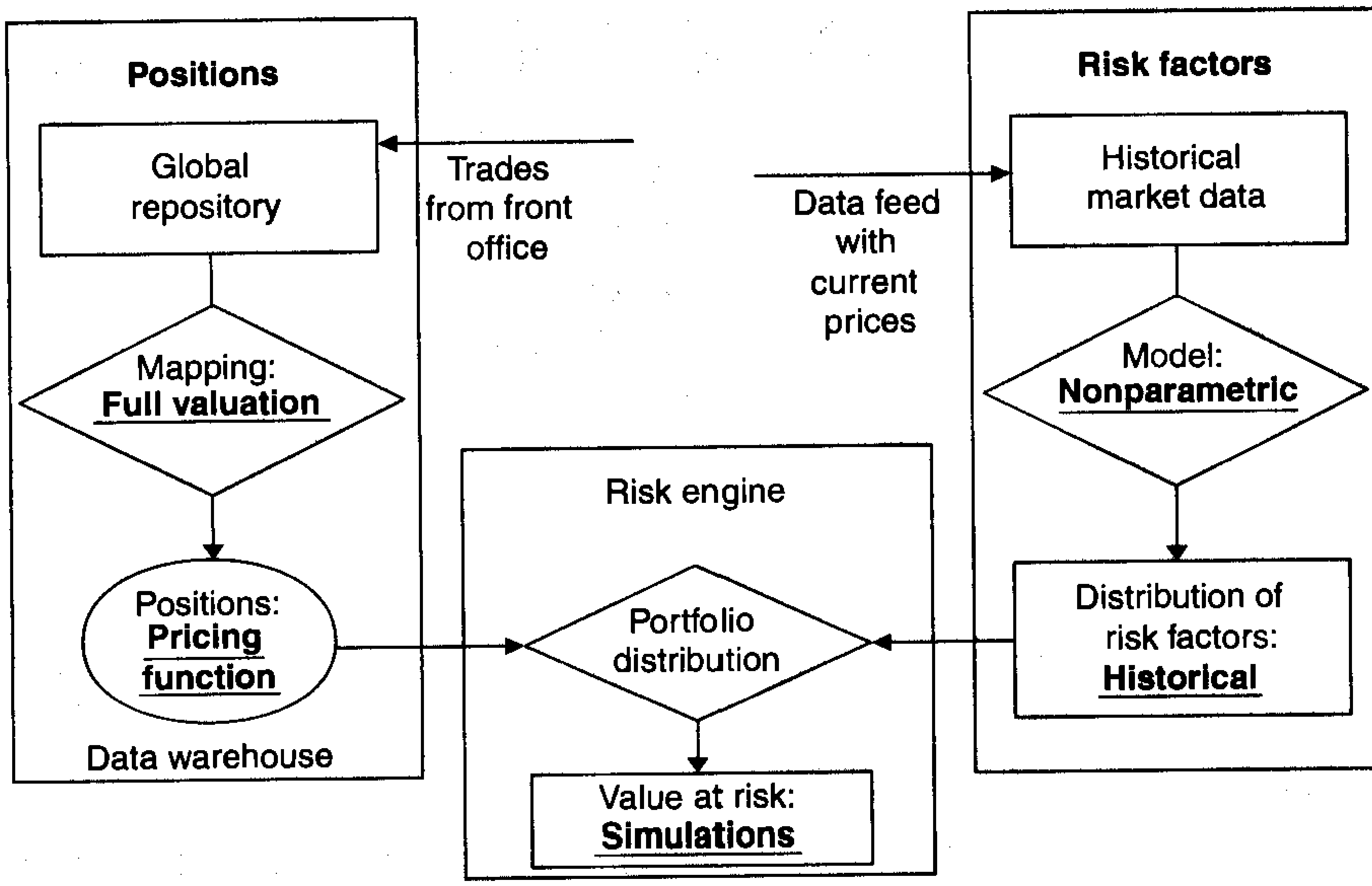
10.4.2 Advantages

This method is relatively *simple to implement* if historical data on risk factors have been collected in-house for daily marking to market. The same data can be stored for later reuse in estimating VAR.

Historical simulation also short circuits the need to estimate a covariance matrix. This simplifies the computations in cases of portfolios with a large number of assets and short sample periods. All that is needed is the time series of the aggregate portfolio value.

FIGURE 10-10

Historical simulation method.



Perhaps most important, historical simulation accounts for *fat tails* to the extent that they are present in the historical data. The method does not require distributional assumptions and therefore is *robust*. Historical simulation can be implemented using *full valuation*. Thus the method can capture gamma and vega risk.

The method also deals directly with the *choice of horizon* for measuring VAR. Returns simply are measured over intervals that correspond to the length of the horizon. For instance, to obtain a monthly VAR, the user would reconstruct historical monthly portfolio returns over, say, the last 5 years.

Historical simulation is also *intuitive*. VAR corresponds to a large loss sustained over a recent period. Hence users can go back in time and explain the circumstances behind the VAR measure.

10.4.3 Drawbacks

On the other hand, the historical simulation method has a number of drawbacks. Only *one sample path* is used. The assumption is that the past represents the immediate future fairly. If the window omits important events, the tails will not be well represented. Vice versa, the sample may contain events that will not reappear in the future.

Next, the *sampling variation* of the historical simulation VAR is greater than for a parametric method. As was pointed out in Chapter 5, there is substantial estimation error in the sample quantile, especially with short sample sizes and high confidence levels. For instance, a 99 percent daily VAR estimated over a window of 100 days only produces one observation in the tail on average, which necessarily leads to an imprecise VAR measure. Thus long sample paths are required to obtain meaningful quantiles. The dilemma is that this may involve observations that are no longer relevant. In practice, most banks use periods between 250 and 750 days, which is taken as a reasonable tradeoff between precision and nonstationarity.

Finally, the method assumes that the distribution is stationary over the selected window. In practice, there may be significant and predictable time variation in risk. This can be taken into account with the following steps. First, we fit a time-series model for the volatility of the series R_t; assume that the volatility forecast is σ_t for each day. The residual then is measured as $\epsilon_t = R_t/\sigma_t$. Second, we bootstrap the scaled residuals from the selected window. Third, we apply these residuals to tomorrow's volatility forecast σ_{t+1}. This is essentially a historical simulation on the ϵ's, which then are multiplied by the current volatility forecast. This method is called *filtered simulation*.[3]

10.5 MONTE CARLO SIMULATION METHOD

10.5.1 Implementation

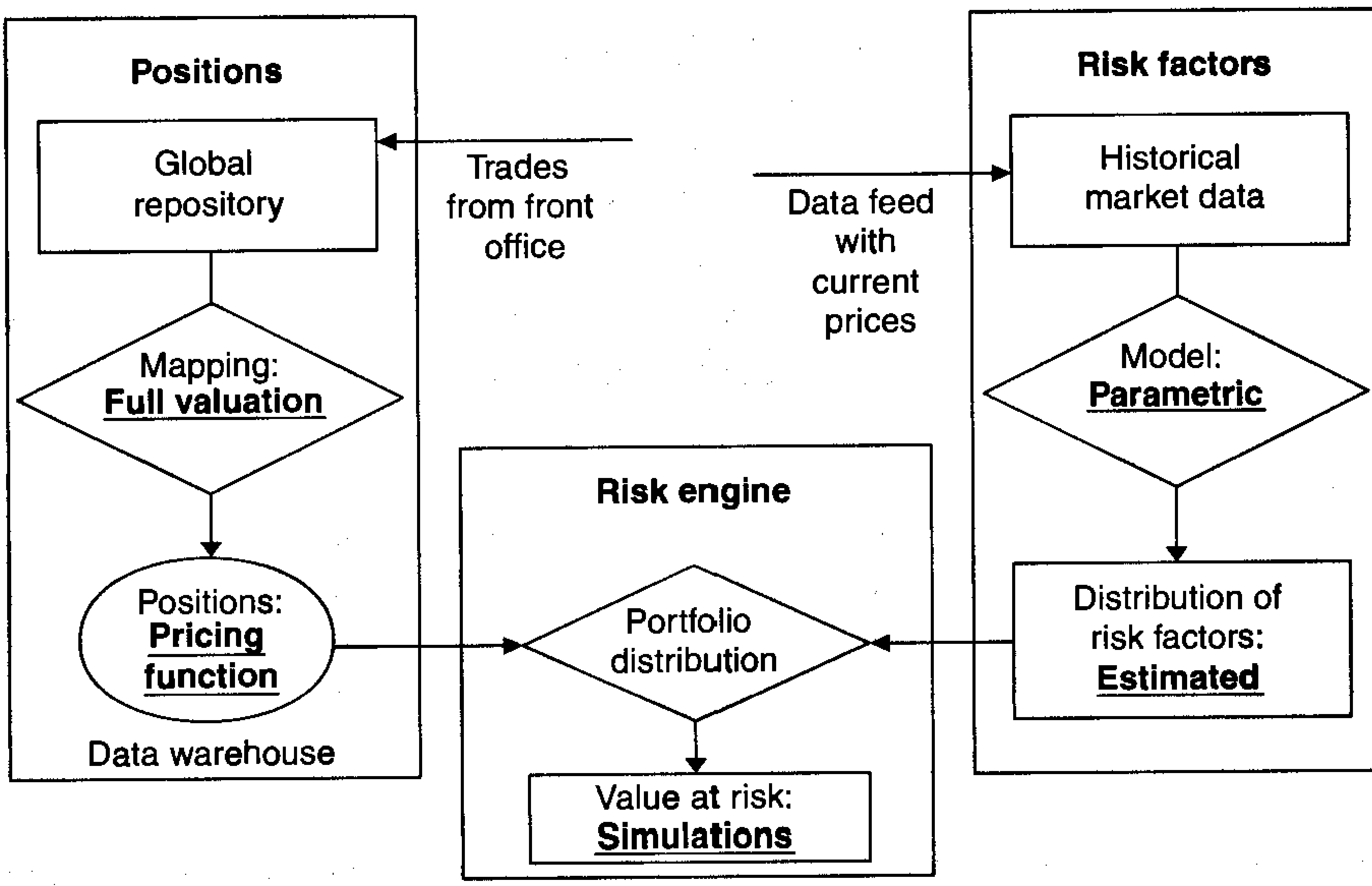
The Monte Carlo (MC) simulation approach is a parametric method that generates random movements in risk factors from estimated parametric distributions. Positions can be priced using full valuation.

The methodology behind MC simulation will be developed in more detail in Chapter 12. In brief, the method proceeds in two steps. First, the risk manager specifies a parametric stochastic process for all risk factors.

[3] For applications, see Hull and White (1998). Another issue is that the method puts the same weight on all observations in the window, including old data points. To alleviate this problem, Boudoukh et al. (1998) propose a scheme whereby each observation R_k is assigned a weight w_k that declines as it ages. The distribution then is obtained from ranking the R_k observations and cumulating the associated weights to find the selected confidence level.

FIGURE 10-11

Monte-Carlo method.



Parameters such as risk and correlations can be derived from historical or options data. Second, fictitious price paths are simulated for all the risk factors. At each horizon considered, the portfolio is marked to market using full valuation as in the historical simulation method, that is, $V_k^* = V(S_{i,k}^*)$. Each of these "pseudo" realizations then is used to compile a distribution of returns, from which a VAR figure can be measured. The method is summarized in Figure 10-11.

The Monte Carlo method thus is similar to the historical simulation method, except that the hypothetical changes in prices ΔS_i for asset i in Equation (10.16) are created by random draws from a prespecified stochastic process instead of sampled from historical data.

10.5.2 Advantages

Monte Carlo analysis is by far the most *powerful method* to compute VAR. For the risk factors, it is flexible enough to incorporate time variation in volatility or in expected returns, *fat tails*, and extreme scenarios. For the instruments in the portfolios, it can account for *nonlinear price exposure*, vega risk, and complex pricing models.

MC simulation can incorporate the *passage of time*, which will create structural changes in the portfolio. This includes the time decay of options; the daily settlement of fixed, floating, or contractually specified cash flows; and the effect of prespecified trading or hedging strategies. These effects are especially important as the time horizon lengthens, which is the case for the measurement of credit risk.

10.5.3 Drawbacks

The biggest drawback to this method is its *computational time*. If 1000 sample paths are generated with a portfolio of 1000 assets, the total number of valuations amounts to 1 million. In addition, if the valuation of assets on the target date involves a simulation, the method requires a "simulation within a simulation." This quickly becomes too onerous to implement on a frequent basis.

This method is the most *expensive to implement* in terms of systems infrastructure and especially intellectual development. MC simulation needs powerful computer systems. It also requires substantial investment in human capital if developed from scratch. Perhaps, then, it should be purchased from outside vendors. On the other hand, when the institution already has in place a system to model complex structures using simulations, implementing MC simulation is less costly because the required expertise is in place. Also, these are situations where proper risk management of complex positions is absolutely necessary.

Another potential weakness of the method is *model risk*. MC relies on specific stochastic processes for the underlying risk factors, which could be wrong. To check if the results are robust to changes in the model, simulation results should be complemented by some sensitivity analysis. Otherwise, the approach is like a black box that provides no intuition for the results.

Finally, VAR estimates from MC simulation are subject to *sampling variation*, which is due to the limited number of replications. Consider, for instance, a case where the risk factors are jointly normal and all payoffs linear. The delta-normal method then will provide the correct measure of VAR in one easy step. MC simulations based on the same covariance matrix will only give an approximation, albeit increasingly good as the number of replications increases.

Overall, this method probably is the most comprehensive approach to measuring market risk if the modeling is done correctly. This is the only method that can handle credit risks. A full chapter will be devoted to the implementation of Monte Carlo simulation methods (see Chapter 12).

10.6 EMPIRICAL COMPARISONS

It is instructive to compare the VAR numbers obtained from these three methods. Hendricks (1996), for instance, calculated 1-day VARs for randomly selected foreign-currency portfolios using a delta-normal method based on fixed windows of equal weights and exponential weights as well as a historical simulation method.

Table 10-2 summarizes the results, which are compared in terms of percentage of outcomes falling within the VAR forecast. This is also one minus the fraction of exceptions. At the 95 percent confidence level, all methods give a coverage that is very close to the ideal number. At the 99 percent confidence level, however, the delta-normal methods seem to underestimate VAR slightly. The historical-simulation method with windows of 1 year or more seem well calibrated.

TABLE 10-2

Empirical Comparison of VAR Methods: Fraction of Outcomes Covered

Method	95% VAR	99% VAR
Delta-normal		
Equal weights over		
50 days	95.1%	98.4%
250 days	95.3%	98.4%
1250 days	95.4%	98.5%
Delta-normal		
Exponential weights		
$\lambda = 0.94$	94.7%	98.2%
$\lambda = 0.97$	95.0%	98.4%
$\lambda = 0.99$	95.4%	98.5%
Historical simulation		
Equal weights over		
125 days	94.4%	98.3%
250 days	94.9%	98.8%
1250 days	95.1%	99.0%

Hendricks also indicates that the delta-normal VAR measures should be increased by about 9 to 15 percent to achieve correct coverage. In other words, the fat tails in the data could be modeled by choosing a distribution with a greater α parameter. A student t distribution with 4 to 6 degrees of freedom, for example, would be appropriate.

This empirical analysis, however, examined positions with linear risk profiles. The delta-normal methods could prove less accurate with options positions, although it should be much faster. Pritsker (1997) examines the tradeoff between speed and accuracy for a portfolio of options.

Table 10-3 reports the accuracy of various methods, measured as the mean absolute percentage error in VAR, as well as their computational times. The table shows that the delta method, as expected, has the highest average absolute error, at 5.34 percent of the true VAR. It is also by far the fastest method, with an execution time of 0.08 second. At the other end, the most accurate method is the full Monte Carlo, which comes arbitrarily close to the true VAR, but with an average run time of 66 seconds. In between, the delta-gamma-delta, delta-gamma-Monte-Carlo, and grid Monte Carlo methods offer a tradeoff between accuracy and speed.

An interesting but still unresolved issue is, how would these approximation work in the context of large, diversified bank portfolios? There is very little evidence on this point. The industry initially seemed to prefer the analytical covariance approach owing to its simplicity. With the rapidly decreasing cost of computing power, however, there is now a marked trend toward the generalized use of historical simulation methods.

TABLE 10-3

Accuracy and Speed VAR Methods: 99 Percent VAR for Option Portfolios

Method	Accuracy, Mean Absolute Error in VAR, %	Speed, Computation Time, seconds
Delta	5.34	0.08
Delta-gamma-delta	4.72	1.17
Delta-gamma-MC	3.08	3.88
Grid Monte Carlo	3.07	32.19
Full Monte Carlo	0	66.27

10.7 SUMMARY

A number of different methods are available to measure VAR. At the most fundamental level, they separate into local valuation and full valuation. This separation reflects a tradeoff between speed of computation and accuracy of valuation.

Among local-valuation models, delta-normal models use a combination of the delta or linear exposures with the covariance matrix. Among full-valuation models, historical simulation is the easiest to implement. It uses the recent history of the risk factors to generate hypothetical scenarios, to which full valuation is applied. Finally, the most complete model but also the most difficult to implement is the Monte Carlo simulation approach. This imposes a particular stochastic process for the risk factors, from which various sample paths are simulated. Full valuation for each sample path generates a distribution of portfolio values.

Table 10-4 describes the pros and cons of each method. The choice of the method largely depends on the composition of the portfolio. For

TABLE 10-4

Comparison of Approaches to VAR

Features	Delta-Normal Simulation	Historical Simulation	Monte Carlo Simulation
Positions			
Valuation	Linear	Full	Full
Distribution			
Shape	Normal	Actual	General
Time varying	Yes	Possible	Yes
Implied data	Possible	No	Possible
Extreme events	Low probability	In recent data	Possible
Use correlations	Yes	Yes	Yes
VAR precision	Excellent	Poor with short window	Good with many iterations
Implementation			
Ease of computation	Yes	Yes	No
Pricing accuracy	Depends on portfolio	Yes	Yes
Communicability	Easy	Easy	Difficult
VAR analysis	Easy	More difficult	More difficult
Major pitfalls	Nonlinearities, fat tails	Time variation in risk, unusual events	Model risk

portfolios with no options and whose distributions are close to the normal, the delta-normal method may well be the best choice. VAR will be relatively easy to compute, fast, and accurate. In addition, it is not too prone to model risk owing to faulty assumptions or computations. The resulting VAR is easy to explain to management and to the public. Because the method is analytical, it provides tools for decomposing VAR into marginal and component measures. For portfolios with options positions, however, the method may not be appropriate. Instead, users should turn to a full-valuation method.

The second method, historical simulation, is also relatively easy to implement and uses full valuation of all securities. However, the method relies on a narrow window only and creates substantial imprecision in VAR numbers.

In theory, the Monte Carlo approach can alleviate all these difficulties. It can incorporate nonlinear positions, nonnormal distributions, and even user-defined scenarios. The price to pay for this flexibility, however, is heavy. Computer and data requirements are a quantum step above the other two approaches, model risk looms large, and VAR loses its intuitive appeal. As the price of computing power continues to fall, however, this method is bound to take on increasing importance.

In practice, all these methods are used. Initially, banks used the delta-normal method because of its simplicity. By now, many institutions are using historical simulation over a window of 1 to 4 years, duly supplemented by stress tests to help minimize the possibility of blinds spots in the risk management system.