CHAPTER 11

VAR Mapping

> The second [principle], to divide each of the difficulties under examination into as many parts as possible, and as might be necessary for its adequate solution.
>
> —*René Descartes*

Whichever value-at-risk (VAR) method is used, the risk measurement process needs to simplify the portfolio by *mapping* the positions on the selected risk factors. Mapping is the process by which the current values of the portfolio positions are replaced by exposures on the risk factors.

Mapping arises because of the fundamental nature of VAR, which is portfolio measurement at the highest level. As a result, this is usually a very large-scale aggregation problem. It would be too complex and time-consuming to model all positions individually as risk factors. Furthermore, this is unnecessary because many positions are driven by the same set of risk factors and can be aggregated into a small set of exposures without loss of risk information. Once a portfolio has been mapped on the risk factors, any of the three VAR methods can be used to build the distribution of profits and losses.

This chapter illustrates the mapping process for major financial instruments. Section 11.1 first reviews the basic principles behind mapping for VAR. We then proceed to illustrate cases where instruments are broken down into their constituent components. We will see that the mapping process is instructive because it reveals useful insights into the risk drivers of derivatives. Section 11.2 deals with fixed-income securities, and Section 11.3 with linear derivatives. We cover the most important instruments, forward contracts, forward rate agreements, and interest-rate swaps. Section 11.4 then describes nonlinear derivatives, or options.

11.1 MAPPING FOR RISK MEASUREMENT

11.1.1 Why Mapping?

The essence of VAR is aggregation at the highest level. This generally involves a very large number of positions, including bonds, stocks, currencies, commodities, and their derivatives. As a result, it would be impractical to consider each position separately (see Box 11-1). Too many computations would be required, and the time needed to measure risk would slow to a crawl.

Fortunately, mapping provides a shortcut. Many positions can be simplified to a smaller number of positions on an set of elementary, or *primitive*, risk factors. Consider, for instance, a trader's desk with thousands of open dollar/euro forward contracts. The positions may differ owing to different maturities and delivery prices. It is unnecessary, however, to model all these positions individually. Basically, the positions are exposed to a single major risk factor, which is the dollar/euro spot exchange rate. Thus they could be summarized by a single aggregate exposure on this risk factor. Such aggregation, of course, is not appropriate for the pricing of the portfolio. For risk measurement purposes, however, it is perfectly acceptable. This is why risk management methods can differ from pricing methods.

Mapping is also the only solution when the characteristics of the instrument change over time. The risk profile of bonds, for instance, changes as they age. One cannot use the history of prices on a bond directly. Instead, the bond must be mapped on yields that best represent its current profile. Similarly, the risk profile of options changes very quickly. Options must be mapped on their primary risk factors. Mapping provides a way to tackle these practical problems.

11.1.2 Mapping as a Solution to Data Problems

Mapping is also required in many common situations. Often a complete history of all securities may not exist or may not be relevant. Consider a

BOX 11-1

WHY MAPPING?

"J.P. Morgan Chase's VAR calculation is highly granular, comprising more than 2.1 million positions and 240,000 pricing series (e.g., securities prices, interest rates, foreign exchange rates)." (Annual report, 2004) mutual fund with a strategy of investing in *initial public offerings* (IPOs) of common stock. By definition, these stocks have no history. They certainly cannot be ignored in the risk system, however. The risk manager would have to replace these positions by exposures on similar risk factors already in the system.

Another common problem with global markets is the time at which prices are recorded. Consider, for instance, a portfolio or mutual funds invested in international stocks. As much as 15 hours can elapse from the time the market closes in Tokyo at 1:00 A.M. EST (3:00 P.M. in Japan) to the time it closes in the United States at 4:00 P.M. As a result, prices from the Tokyo close ignore intervening information and are said to be *stale*. This led to the mutual-fund scandal of 2003, which is described in Box 11-2.

BOX 11-2

MARKET TIMING AND STALE PRICES

In September 2003, New York Attorney General Eliot Spitzer accused a number of investment companies of allowing *market timing* into their funds. Market timing is a short-term trading strategy of buying and selling the same funds.

Consider, for example, our portfolio of Japanese and U.S. stocks, for which prices are set in different time zones. The problem is that U.S. investors can trade up to the close of the U.S. market. *Market timers* could take advantage of this discrepancy by rapid trading. For instance, if the U.S. market moves up following good news, it is likely the Japanese market will move up as well the following day. Market timers would buy the fund at the stale price and resell it the next day.

Such trading, however, creates transactions costs that are borne by the other investors in the fund. As a result, fund companies usually state in their prospectus that this practice is not allowed. In practice, Eliot Spitzer found out that many mutual-fund companies had encouraged market timers, which he argued was fraudulent. Eventually, a number of funds settled by paying more than $2 billion.

This practice can be stopped in a number of ways. Many mutual funds now impose short-term redemption fees, which make market timing uneconomical. Alternatively, the cutoff time for placing trades can be moved earlier.

For risk managers, stale prices cause problems. Because returns are not synchronous, daily correlations across markets are too low, which will affect the measurement of portfolio risk.

One possible solution is mapping. For instance, prices at the close of the U.S. market can be estimated from a regression of Japanese returns on U.S. returns and using the forecast value conditional on the latest U.S. information. Alternatively, correlations can be measured from returns taken over longer time intervals, such as weekly. In practice, the risk manager needs to make sure that the data-collection process will lead to meaningful risk estimates.

11.1.3 The Mapping Process

Figure 11-1 illustrates a simple mapping process, where six instruments are mapped on three risk factors. The first step in the analysis is marking all positions to market in current dollars or whatever reference currency is used. The market value for each instrument then is allocated to the three risk factors.

FIGURE 11-1

Mapping instruments on risk factors.

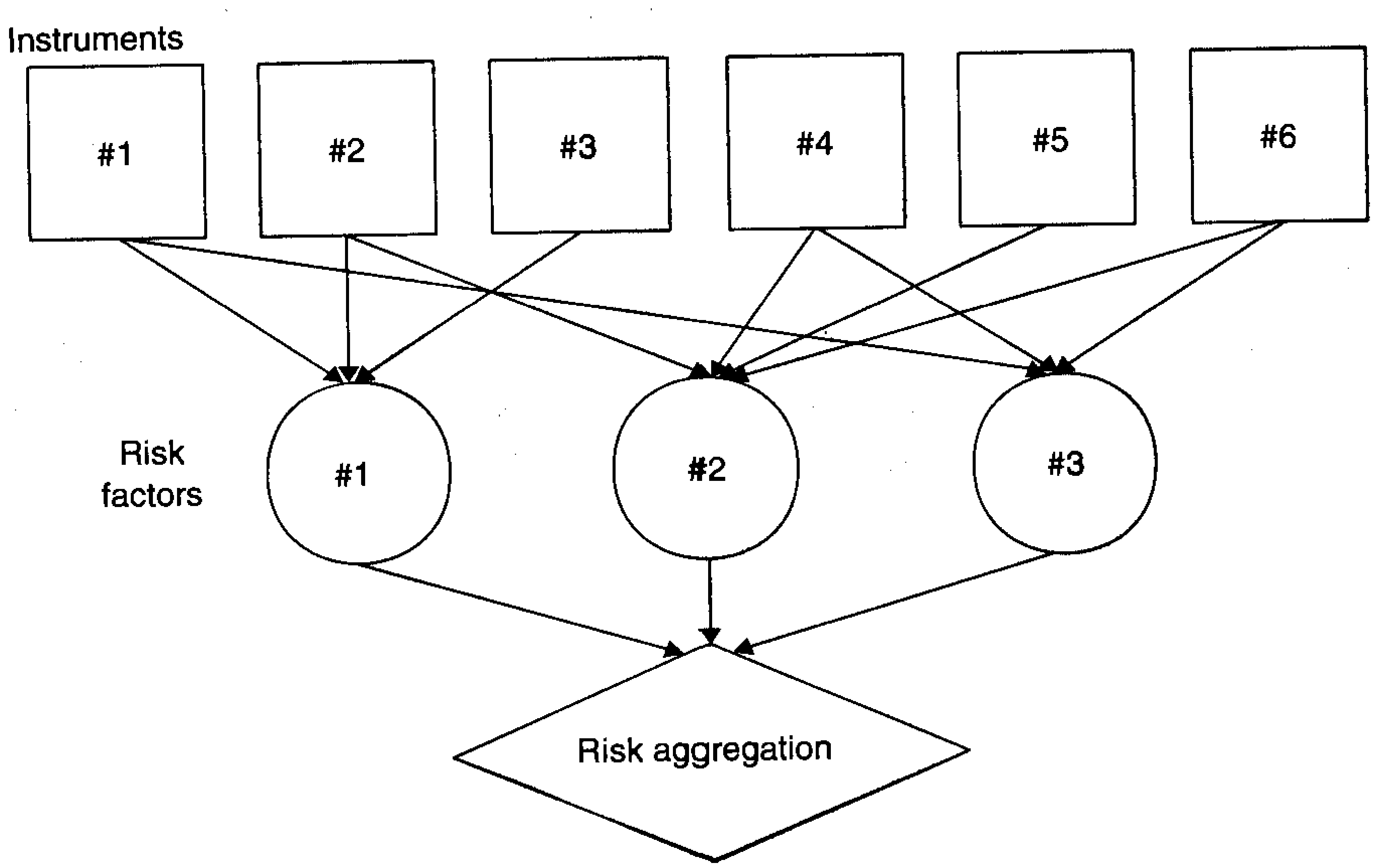


TABLE 11-1

Mapping Exposures

	Market Value	Exposure on Risk Factor 1	2	3
Instrument 1	V_1	x_{11}	x_{12}	x_{13}
Instrument 2	V_2	x_{21}	x_{22}	x_{23}
⋮	⋮	⋮	⋮	⋮
Instrument 6	V_6	x_{61}	x_{62}	x_{63}
Total portfolio	V	$x_1=\Sigma_{i=1}^{6}x_{i1}$	$x_2=\Sigma_{i=1}^{6}x_{i2}$	$x_3=\Sigma_{i=1}^{6}x_{i3}$

Table 11-1 shows that the first instrument has a market value of V_1, which is allocated to three exposures, x_{11}, x_{12}, and x_{13}. If the current market value is not fully allocated to the risk factors, it must mean that the remainder is allocated to cash, which is not a risk factor because it has no risk.

Next, the system allocates the position for instrument 2 and so on. At the end of the process, positions are summed for each risk factor. For the first risk factor, the dollar exposure is $x_1=\Sigma_{i=1}^{6}x_{i1}$. This creates a vector x of three exposures that can be fed into the risk measurement system.

Mapping can be of two kinds. The first provides an exact allocation of exposures on the risk factors. This is obtained for derivatives, for instance, when the price is an exact function of the risk factors. As we shall see in the rest of this chapter, the partial derivatives of the price function generate *analytical* measures of exposures on the risk factors.

Alternatively, exposures may have to be *estimated*. This occurs, for instance, when a stock is replaced by a position in the stock index. The exposure then is estimated by the slope coefficient from a regression of the stock return on the index return.

11.1.4 General and Specific Risk

This brings us to the issue of the choice of the set of primitive risk factors. This choice should reflect the tradeoff between better quality of the approximation and faster processing. More factors lead to tighter risk measurement but also require more time devoted to the modeling process and risk computation.

The choice of primitive risk factors also influences the size of specific risks. *Specific risk* can be defined as risk that is due to issuer-specific price movements, after accounting for general market factors. Hence the definition of specific risk depends on that of general market risk. The Basel rules have a separate charge for specific risk.[1]

To illustrate this decomposition, consider a portfolio of N stocks. We are mapping each stock on a position in the stock market index, which is our primitive risk factor. The return on a stock R_i is regressed on the return on the stock market index R_m, that is,

$$R_i = \alpha_i + \beta_i R_m + \epsilon_i \tag{11.1}$$

which gives the exposure β_i. In what follows, ignore α, which does not contribute to risk. We assume that the specific risk owing to ϵ is not correlated across stocks or with the market. The relative weight of each stock in the portfolio is given by w_i. Thus the portfolio return is

$$R_p = \sum_{i=1}^{N} w_i R_i = \sum_{i=1}^{N} w_i \beta_i R_m + \sum_{i=1}^{N} w_i \epsilon_i \tag{11.2}$$

These exposures are aggregated across all the stocks in the portfolio. This gives

$$\beta_p = \sum_{i=1}^{N} w_i \beta_i \tag{11.3}$$

If the portfolio value is W, the mapping on the index is $x = W\beta_p$.

Next, we decompose the variance of R_p in Equation (11.2) and find

$$V(R_p) = (\beta_p^2) V(R_m) + \sum_{i=1}^{N} w_i^2 \sigma_{\epsilon i}^2 \tag{11.4}$$

The first component is the general market risk. The second component is the aggregate of specific risk for the entire portfolio. This decomposition shows that with more detail on the primitive or general-market risk factors, there will be less specific risk for a fixed amount of total risk $V(R_p)$.

As another example, consider a corporate bond portfolio. Bond positions describe the distribution of money flows over time by their amount, timing, and credit quality of the issuer. This creates a continuum of risk factors, going from overnight to long maturities for various credit risks.

[1] Typically, the charge is 4 percent of the position value for equities and unrated debt, assuming that the banks' models do not incorporate specific risks. See Chapter 3.

In practice, we have to restrict the number of risk factors to a small set. For some portfolios, one risk factor may be sufficient. For others, 15 maturities may be necessary. For portfolios with options, we need to model movements not only in yields but also in their implied volatilities.

Our primitive risk factors could be movements in a set of J government bond yields z_j and in a set of K credit spreads s_k sorted by credit rating. We model the movement in each corporate bond yield dy_i by a movement in z at the closest maturity and in s for the same credit rating. The remaining component is ϵ_i.

The movement in value W then is

$$dW = \sum_{i=1}^{N} \text{DVBP}_i dy_i = \sum_{j=1}^{J} \text{DVBP}_j dz_j + \sum_{k=1}^{K} \text{DVBP}_k ds_k + \sum_{i=1}^{N} \text{DVBP}_i d\epsilon_i \quad (11.5)$$

where DVBP is the total dollar value of a basis point for the associated risk factor. The values for DVBP_j then represent the summation of the DVBP across all individual bonds for each maturity.

This leads to a total risk decomposition of

$$V(dW) = \text{ general risk} + \sum_{i=1}^{N} \text{DVBP}_i^2 V(d\epsilon_i) \quad (11.6)$$

A greater number of general risk factors should create less residual risk. Even so, we need to ascertain the size of the second, specific risk term. In practice, there may not be sufficient history to measure the specific risk of individual bonds, which is why it is often assumed that all issuers within the same risk class have the same risk.

11.2 MAPPING FIXED-INCOME PORTFOLIOS

11.2.1 Mapping Approaches

Once the risk factors have been selected, the question is how to map the portfolio positions into exposures on these risk factors. We can distinguish three mapping systems for fixed-income portfolios: principal, duration, and cash flows. With *principal mapping*, one risk factor is chosen that corresponds to the average portfolio maturity. With *duration mapping*, one risk factor is chosen that corresponds to the portfolio duration. With *cash-flow mapping*, the portfolio cash flows are grouped into maturity buckets. Mapping should preserve the market value of the position. Ideally, it also should preserve its market risk.

TABLE 11-2

Mapping for a Bond Portfolio ($ Millions)

	Cash Flows			Mapping (PV)		
Term (Year)	**5-Year**	**1-Year**	**Spot Rate**	**Principal**	**Duration**	**Cash Flow**
1	$6	$104	4.000%	0.00	0.00	$105.77
2	$6	0	4.618%	0.00	0.00	$5.48
2.733	—	—		—	$200.00	—
3	$6	0	5.192%	$200.00	0.00	$5.15
4	$6	0	5.716%	0.00	0.00	$4.80
5	$106	0	6.112%	0.00	0.00	$78.79
Total				$200.00	$200.00	$200.00

As an example, Table 11-2 describes a two-bond portfolio consisting of a $100 million 5-year 6 percent issue and a $100 million 1-year 4 percent issue. Both issues are selling at par, implying a market value of $200 million. The portfolio has an average maturity of 3 years and a duration of 2.733 years. The table lays out the present value of all portfolio cash flows discounted at the appropriate zero-coupon rate.

Principal mapping considers the timing of redemption payments only. Since the average maturity of this portfolio is 3 years, the VAR can be found from the risk of a 3-year maturity, which is 1.484 percent from Table 11-3. VAR then is $\$200 \times 1.484/100 = \$2.97$ million. The only positive aspect of this method is its simplicity. This approach overstates the true risk because it ignores intervening coupon payments.

The next step in precision is duration mapping. We replace the portfolio by a zero-coupon bond with maturity equal to the duration of the portfolio, which is 2.733 years. We discuss in Appendix 11.A how to allocate the portfolio to the adjoining 2- and 3-year vertices. Table 11-3 shows VARs of 0.987 and 1.484 for these maturities, respectively. Using a linear interpolation, we find a risk of $0.987 + (1.484-0.987) \times (2.733-2) = 1.351$ percent for this hypothetical zero. With a $200 million portfolio, the duration-based VAR is $\$200 \times 1.351/100 = \$2.70$ million, slightly less than before.

Finally, the cash-flow mapping method consists of grouping all cash flows on term-structure "vertices" that correspond to maturities for which volatilities are provided. Each cash flow is represented by the present value of the cash payment, discounted at the appropriate zero-coupon rate.

The diversified VAR is computed as

$$\text{VAR} = \alpha\sqrt{x'\Sigma x} = \sqrt{(x \times V)'R(x \times V)} \tag{11.7}$$

where $V = \alpha\sigma$ is the vector of VAR for zero-coupon bond returns, and R is the correlation matrix.

Table 11-4 shows how to compute the portfolio VAR using cash-flow mapping. The second column reports the cash flows x from Table 11-2. Note that the current value of $200 million is fully allocated to the five risk factors. The third column presents the product of these cash flows with the risk of each vertex $x \times V$, which represents the individual VARs.

With perfect correlation across all zeroes, the VAR of the portfolio is

$$\text{Undiversified VAR} = \sum_{i=1}^{N} |x_i| V_i$$

which is $2.63 million. This number is close to the VAR obtained from the duration approximation, which was $2.70 million.

The right side of the table presents the correlation matrix of zeroes for maturities ranging from 1 to 5 years. To obtain the portfolio VAR, we premultiply and postmultiply the matrix by the dollar amounts (xV) at each vertex. Taking the square root, we find a diversified VAR measure of $2.57 million.

Note that this is slightly less than the duration VAR of $2.70 million. This difference is due to two factors. First, risk measures are not perfectly linear with maturity, as we have seen in a previous section. Second, cor-

TABLE 11-3

Computing VAR from Change in Prices of Zeroes

Term (Year)	Cash Flows	Old Zero Value	Old PV of Flows	Risk (%)	New Zero Value	New PV of Flows
1	$110	0.9615	$105.77	0.4696	0.9570	$105.27
2	$6	0.9136	$5.48	0.9868	0.9046	$5.43
3	$6	0.8591	$5.15	1.4841	0.8463	$5.08
4	$6	0.8006	$4.80	1.9714	0.7848	$4.71
5	$106	0.7433	$78.79	2.4261	0.7252	$76.88
Total			$200.00			$197.37
Loss						$2.63

relations are below unity, which reduces risk even further. Thus, of the \$130,000 difference in these measures, (\$2.70−\$2.57 million), \$70,000 is due to differences in yield volatility, and (\$2.70−\$2.63 million), \$60,000 is due to imperfect correlations. The last column presents the component VAR using computations as explained in Chapter 7.

11.2.2 Stress Test

Table 11-3 presents another approach to VAR that is directly derived from movements in the value of zeroes. This is an example of stress testing. Assume that all zeroes are perfectly correlated. Then we could decrease all zeroes' values by their VAR. For instance, the 1-year zero is worth 0.9615. Given the VAR in Table 11-3 of 0.4696, a 95 percent probability move would be for the zero to fall to $0.9615 \times (1-0.4696/100)=0.9570$. If all zeroes are perfectly correlated, they should all fall by their respective VAR. This generates a new distribution of present-value factors that can be used to price the portfolio. Table 11-3 shows that the new value is \$197.37 million, which is exactly \$2.63 million below the original value. This number is exactly the same as the undiversified VAR just computed.

The two approaches illustrate the link between computing VAR through matrix multiplication and through movements in underlying prices. Computing VAR through matrix multiplication is much more direct,

TABLE 11-4

Computing the VAR of a \$200 Million Bond Portfolio (Monthly VAR at 95 Percent Level)

	PV Cash Flows	Individual VAR	Correlation Matrix R					Component VAR
Term (Year)	x	$x \times V$	1Y	2Y	3Y	4Y	5Y	$x\Delta$VAR
1	\$105.77	0.4966	1					\$0.45
2	\$5.48	0.0540	0.897	1				\$0.05
3	\$5.15	0.0765	0.886	0.991	1			\$0.08
4	\$4.80	0.0947	0.866	0.976	0.994	1		\$0.09
5	\$78.79	1.9115	0.855	0.966	0.988	0.998	1	\$1.90
Total	\$200.00	2.6335						
Undiversified VAR		\$2.63						
Diversified VAR								\$2.57

however, and more appropriate because it allows nonperfect correlations across different sectors of the yield curve.

11.2.3 Benchmarking a Portfolio

Next, we provide a practical fixed-income example by showing how to compute VAR in relative terms, that is, relative to a performance benchmark. Table 11-5 presents the cash-flow decomposition of the J.P. Morgan U.S. bond index, which has a duration of 4.62 years. Assume that we are trying to benchmark a portfolio of $100 million. Over a monthly horizon, the VAR of the index at the 95 percent confidence level is $1.99 million. This is about equivalent to the risk of a 4-year note.

TABLE 11-5

Benchmarking a $100 Million Bond Index (Monthly Tracking Error VAR at 95 Percent Level)

			Position: Portfolio				
Vertex	Risk (%)	Position: Index ($)	1 ($)	2 ($)	3 ($)	4 ($)	5 ($)
≤1m	0.022	1.05	0.0	0.0	0.0	0.0	84.8
3m	0.065	1.35	0.0	0.0	0.0	0.0	0.0
6m	0.163	2.49	0.0	0.0	0.0	0.0	0.0
1Y	0.470	13.96	0.0	0.0	0.0	59.8	0.0
2Y	0.987	24.83	0.0	0.0	62.6	0.0	0.0
3Y	1.484	15.40	0.0	59.5	0.0	0.0	0.0
4Y	1.971	11.57	38.0	0.0	0.0	0.0	0.0
5Y	2.426	7.62	62.0	0.0	0.0	0.0	0.0
7Y	3.192	6.43	0.0	40.5	0.0	0.0	0.0
9Y	3.913	4.51	0.0	0.0	37.4	0.0	0.0
10Y	4.250	3.34	0.0	0.0	0.0	40.2	0.0
15Y	6.234	3.00	0.0	0.0	0.0	0.0	0.0
20Y	8.146	3.15	0.0	0.0	0.0	0.0	0.0
30Y	11.119	1.31	0.0	0.0	0.0	0.0	15.2
Total		100.00	100.0	100.0	100.0	100.0	100.0
Duration		4.62	4.62	4.62	4.62	4.62	4.62
Absolute VAR		$1.99	$2.25	$2.16	$2.04	$1.94	$1.71
Tracking error VAR		$0.00	$0.43	$0.29	$0.16	$0.20	$0.81

Next, we try to match the index with two bonds. The rightmost columns in the table display the positions of two-bond portfolios with duration matched to that of the index. Since no zero-coupon has a maturity of exactly 4.62 years, the closest portfolio consists of two positions, each in a 4- and a 5-year zero. The respective weights for this portfolio are \$38 million and \$62 million.

Define the new vector of positions for this portfolio as x and for the index as x_0. The VAR of the deviation relative to the benchmark is

$$\text{Tracking error VAR} = \alpha\sqrt{(x - x_0)'\Sigma(x - x_0)} \tag{11.8}$$

After performing the necessary calculations, we find that the *tracking error* VAR (TE-VAR) of this duration-hedged portfolio is \$0.43 million. Thus the maximum deviation between the index and the portfolio is at most \$0.43 million under normal market conditions. This potential shortfall is much less than the \$1.99 million absolute risk of the index. The remaining tracking error is due to nonparallel moves in the term structure.

Relative to the original index, the tracking error can be measured in terms of variance reduction, similar to an R^2 in a regression. The variance improvement is

$$1 - \left(\frac{0.43}{1.99}\right)^2 = 95.4\,\text{percent}$$

which is in line with the explanatory power of the first factor in the variance decomposition detailed in Chapter 8.

Next, we explore the effect of altering the composition of the tracking portfolio. Portfolio 2 widens the bracket of cash flows in years 3 and 7. The TE-VAR is \$0.29 million, which is an improvement over the previous number. Next, portfolio 3 has positions in years 2 and 9. This comes the closest to approximating the cash-flow positions in the index, which has the greatest weight on the 2-year vertex. The TE-VAR is reduced further to \$0.16 million. Portfolio 4 has positions in years 1 and 10. Now the TE-VAR increases to \$0.20 million. This mistracking is even more pronounced for a portfolio consisting of 1-month bills and 30-year zeroes, for which the TE-VAR increases to \$0.81 million.

Among the portfolios considered here, the lowest tracking error is obtained with portfolio 3. Note that the absolute risk of these portfolios is lowest for portfolio 5. As correlations decrease for more distant maturities, we should expect that a duration-matched portfolio should have the lowest absolute risk for the combination of most distant maturities, such as a *barbell* portfolio of cash and a 30-year zero. However, minimizing absolute market risk is not the same as minimizing relative market risk.

This example demonstrates that duration hedging only provides a first approximation to interest-rate risk management. If the goal is to minimize tracking error relative to an index, it is essential to use a fine decomposition of the index by maturity.

11.3 MAPPING LINEAR DERIVATIVES

11.3.1 Forward Contracts

Forward and futures contracts are the simplest types of derivatives. Since their value is linear in the underlying spot rates, their risk can be constructed easily from basic building blocks. Assume, for instance, that we are dealing with a forward contract on a foreign currency. The basic valuation formula can be derived from an arbitrage argument.

To establish notations, define

S_t = spot price of one unit of the underlying cash asset
K = contracted forward price
r = domestic risk-free rate
y = income flow on the asset
τ = time to maturity.

When the asset is a foreign currency, y represents the foreign risk-free rate r^*. We will use these two notations interchangeably. For convenience, we assume that all rates are compounded continuously.

We seek to find the current value of a forward contract f_t to buy one unit of foreign currency at K after time τ. To do this, we consider the fact that investors have two alternatives that are economically equivalent: (1) Buy $e^{-y\tau}$ units of the asset at the price S_t and hold for one period, or (2) enter a forward contract to buy one unit of the asset in one period. Under alternative 1, the investment will grow, with reinvestment of dividend, to exactly one unit of the asset after one period. Under alternative 2, the contract costs f_t upfront, and we need to set aside enough cash to pay K in the future, which is $Ke^{-r\tau}$. After 1 year, the two alternatives lead to the same position, one unit of the asset. Therefore, their initial cost must be identical. This leads to the following valuation formula for outstanding forward contracts:

$$f_t = S_t e^{-y\tau} - Ke^{-r\tau} \tag{11.9}$$

Note that we can repeat the preceding reasoning to find the current forward rate F_t that would set the value of the contract to zero. Setting $K = F_t$ and $f_t = 0$ in Equation (11.9), we have

$$F_t = (S_t e^{-y\tau})e^{r\tau} \tag{11.10}$$

This allows us to rewrite Equation (11.9) as

$$f_t = F_t e^{-r\tau} - Ke^{-r\tau} = (F_t - K)e^{-r\tau} \tag{11.11}$$

In other words, the current value of the forward contract is the present value of the difference between the current forward rate and the locked-in delivery rate. If we are long a forward contract with contracted rate K, we can liquidate the contract by entering a new contract to sell at the current rate F_t. This will lock in a profit of $(F_t - K)$, which we need to discount to the present time to find f_t.

Let us examine the risk of a 1-year forward contract to purchase 100 million euros in exchange for \$130.086 million. Table 11-6 displays pricing information for the contract (current spot, forward, and interest rates), risk, and correlations. The first step is to find the market value of the contract. We can use Equation (11.9), accounting for the fact that the quoted interest rates are discretely compounded, as

$$f_t = \$1.2877\frac{1}{(1+2.2810/100)} - \$1.3009\frac{1}{(1+3.3304/100)} = \$1.2589 - \$1.2589 = 0$$

TABLE 11-6

Risk and Correlations for Forward Contract Risk Factors (Monthly VAR at 95 Percent Level)

			Correlations		
Risk Factor	**Price or Rate**	**VAR (%)**	**EUR Spot**	**EUR 1Y**	**USD 1Y**
EUR spot	\$1.2877	4.5381	1	0.1289	0.0400
Long EUR bill	2.2810%	0.1396	0.1289	1	−0.0583
Short USD bill	3.3304%	0.2121	0.0400	−0.0583	1
EUR forward	\$1.3009				

Thus the initial value of the contract is zero. This value, however, may change, creating market risk.

Among the three sources of risk, the volatility of the spot contract is the highest by far, with a 4.54 percent VAR (corresponding to 1.65 standard deviations over a month for a 95 percent confidence level). This is much greater than the 0.14 percent VAR for the EUR 1-year bill or even the 0.21 percent VAR for the USD bill. Thus most of the risk of the forward contract is driven by the cash EUR position.

But risk is also affected by correlations. The positive correlation of 0.13 between the EUR spot and bill positions indicates that when the EUR goes up in value against the dollar, the value of a 1-year EUR investment is likely to appreciate. Therefore, higher values of the EUR are associated with lower EUR interest rates.

This positive correlation increases the risk of the combined position. On the other hand, the position is also short a 1-year USD bill, which is correlated with the other two legs of the transaction. The issue is, what will be the net effect on the risk of the forward contract?

VAR provides an exact answer to this question, which is displayed in Table 11-7. But first we have to compute the positions x on each of the three building blocks of the contract. By taking the partial derivative of Equation (11.9) with respect to the risk factors, we have

$$df = \frac{\partial f}{\partial S} dS + \frac{\partial f}{\partial r^*} dr^* + \frac{\partial f}{\partial r} dr = e^{-r^*\tau} dS - Se^{-r^*\tau} \tau dr^* + Ke^{-r\tau} \tau dr \qquad (11.12)$$

Here, the building blocks consist of the spot rate and interest rates. Alternatively, we can replace interest rates by the price of bills. Define these

TABLE 11-7

Computing VAR for a EUR 100 Million Forward Contract (Monthly VAR at 95 Percent Level)

Position	Present-Value Factor	Cash Flows (CF)	PV of Flows, x	Individual VAR, $\|x\|V$	Component VAR, $x\Delta$VAR
EUR spot			\$125.89	\$5.713	\$5.704
Long EUR bill	0.977698	EUR100.00	\$125.89	\$0.176	\$0.029
Short USD bill	0.967769	−\$130.09	−\$125.89	\$0.267	\$0.002
Undiversified VAR				\$6.156	
Diversified VAR					\$5.735

as $P = e^{-r\tau}$ and $P^*=e^{-r^*\tau}$. We then replace dr with dP using $dP = (-\tau)e^{-r\tau}$ dr and $dP^*=(-\tau)e^{-r^*\tau}$ dr^*. The risk of the forward contract becomes

$$df = (Se^{-r^*\tau})\frac{dS}{S} + (Se^{-r^*\tau})\frac{dP^*}{P^*} - (Ke^{-r\tau})\frac{dP}{P} \tag{11.13}$$

This shows that the forward position can be separated into three cash flows: (1) a long spot position in EUR, worth EUR 100 million = \$130.09 million in a year, or $(Se^{-r^*\tau})$ = \$125.89 million now, (2) a long position in a EUR investment, also worth \$125.89 million now, and (3) a short position in a USD investment, worth \$130.09 million in a year, or $(Ke^{-r\tau})$ = \$125.89 million now. Thus a position in the forward contract has three building blocks:

> Long forward contract = long foreign currency spot + long foreign currency bill + short U.S. dollar bill

Considering only the spot position, the VAR is \$125.89 million times the risk of 4.538 percent, which is \$5.713 million. To compute the diversified VAR, we use the risk matrix from the data in Table 11-6 and pre- and postmultiply by the vector of positions (PV of flows column in the table). The total VAR for the forward contract is \$5.735 million. This number is about the same size as that of the spot contract because exchange-rate volatility dominates the volatility of 1-year bonds.

More generally, the same methodology can be used for long-term currency swaps, which are equivalent to portfolios of forward contracts. For instance, a 10-year contract to pay dollars and receive euros is equivalent to a series of 10 forward contracts to exchange a set amount of dollars into marks. To compute the VAR, the contract must be broken down into a currency-risk component and a string of USD and EUR fixed-income components. As before, the total VAR will be driven primarily by the currency component.

11.3.2 Commodity Forwards

The valuation of forward or futures contracts on commodities is substantially more complex than for financial assets such as currencies, bonds, or stock indices. Such financial assets have a well-defined income flow y, which is the foreign interest rate, the coupon payment, or the dividend yield, respectively.

Things are not so simple for commodities, such as metals, agricultural products, or energy products. Most products do not make monetary payments but instead are consumed, thus creating an implied benefit. This flow of benefit, net of storage cost, is loosely called *convenience yield* to represent the benefit from holding the cash product. This convenience yield, however, is not tied to another financial variable, such as the foreign interest rate for currency futures. It is also highly variable, creating its own source of risk.

As a result, the risk measurement of commodity futures uses Equation (11.11) directly, where the main driver of the value of the contract is the current forward price for this commodity. Table 11-8 illustrates the term structure of volatilities for selected energy products and base metals. First, we note that monthly VAR measures are very high, reaching 29 percent for near contracts. In contrast, currency and equity market

TABLE 11-8

Risk of Commodity Contracts (Monthly VAR at 95 Percent Level)

	Energy Products			
Maturity	**Natural Gas**	**Heating Oil**	**Unleaded Gasoline**	**Crude Oil–WTI**
1 month	28.77	22.07	20.17	19.20
3 months	22.79	20.60	18.29	17.46
6 months	16.01	16.67	16.26	15.87
12 months	12.68	14.61	—	14.05
	Base Metals			
Maturity	**Aluminum**	**Copper**	**Nickel**	**Zinc**
Cash	11.34	13.09	18.97	13.49
3 months	11.01	12.34	18.41	13.18
15 months	8.99	10.51	15.44	11.95
27 months	7.27	9.57	—	11.59
	Precious Metals			
Maturity	**Gold**	**Silver**	**Platinum**	
Cash	6.18	14.97	7.70	

VARs are typically around 6 percent. Thus commodities are much more volatile than typical financial assets.

Second, we observe that volatilities decrease with maturity. The effect is strongest for less storable products such as energy products and less so for base metals. It is actually imperceptible for precious metals, which have low storage costs and no convenience yield. For financial assets, volatilities are driven primarily by spot prices, which implies basically constant volatilities across contract maturities.

Let us now say that we wish to compute the VAR for a 12-month forward position on 1 million barrels of oil priced at \$45.2 per barrel. Using a present-value factor of 0.967769, this translates into a current position of \$43,743,000.

Differentiating Equation (11.11), we have

$$df = \frac{\partial f}{\partial F} dF = e^{-r\tau} dF = (e^{-r\tau} F) \frac{dF}{F} \tag{11.14}$$

The term between parentheses therefore represents the exposure. The contract VAR is

$$\text{VAR} = \$43{,}743{,}000 \times 14.05/100 = \$6{,}146{,}000$$

In general, the contract cash flows will fall between the maturities of the risk factors, and present values must be apportioned accordingly.

11.3.3 Forward Rate Agreements

Forward rate agreements (FRAs) are forward contracts that allow users to lock in an interest rate at some future date. The buyer of an FRA locks in a borrowing rate; the seller locks in a lending rate. In other words, the "long" receives a payment if the spot rate is above the forward rate.

Define the timing of the short leg as τ_1 and of the long leg as τ_2, both expressed in years. Assume linear compounding for simplicity. The forward rate can be defined as the implied rate that equalizes the return on a τ_2-period investment with a τ_1-period investment rolled over, that is,

$$(1 + R_2\tau_2) = (1 + R_1\tau_1)\,[1 + F_{1,2}(\tau_2 - \tau_1)] \tag{11.15}$$

For instance, suppose that you sold a 6 × 12 FRA on \$100 million. This is equivalent to borrowing \$100 million for 6 months and investing the proceeds for 12 months. When the FRA expires in 6 months, assume that the prevailing 6-month spot rate is higher than the locked-in forward rate. The seller then pays the buyer the difference between the spot and forward rates applied to the principal. In effect, this payment offsets the higher return that the investor otherwise would receive, thus guaranteeing a return equal to the forward rate. Therefore, an FRA can be decomposed into two zero-coupon building blocks.

Long 6 × 12 FRA = long 6-month bill + short 12-month bill

Table 11-9 provides a worked-out example. If the 360-day spot rate is 5.8125 percent and the 180-day rate is 5.6250 percent, the forward rate must be such that

$$(1+F_{1,2}/2)=\frac{(1+5.8125/100)}{(1+5.6250/200)}$$

or F = 5.836 percent. The present value of the notional \$100 million in 6 months is x = \$100/(1+5.625/200)= \$97.264 million. This amount is invested for 12 months. In the meantime, what is the risk of this FRA?

Table 11-9 displays the computation of VAR for the FRA. The VARs of 6- and 12-month zeroes are 0.1629 and 0.4696, respectively, with a correlation of 0.8738. Applied to the principal of \$97.26 million, the individual VARs are \$0.158 million and \$0.457 million, which gives an undiversified VAR of \$0.615 million. Fortunately, the correlation substantially lowers the FRA risk. The largest amount the position can lose over a month at the 95 percent level is \$0.327 million.

TABLE 11-9

Computing the VAR of a \$100 Million FRA (Monthly VAR at 95 Percent Level)

Position	PV of Flows, x	Risk (%), V	Correlation Matrix, R		Individual VAR, $\|x\|V$	Component VAR, $x\Delta$VAR
180 days	−\$97.264	0.1629	1	0.8738	\$0.158	−\$0.116
360 days	\$97.264	0.4696	0.8738	1	\$0.457	\$0.444
Undiversified VAR					\$0.615	
Diversified VAR						\$0.327

11.3.4 Interest-Rate Swaps

Interest-rate swaps are the most actively used derivatives. They create exchanges of interest-rate flows from fixed to floating or vice versa. Swaps can be decomposed into two legs, a fixed leg and a floating leg. The fixed leg can be priced as a coupon-paying bond; the floating leg is equivalent to a floating-rate note (FRN).

To illustrate, let us compute the VAR of a $100 million 5-year interest-rate swap. We enter a dollar swap that pays 6.195 percent annually for 5 years in exchange for floating-rate payments indexed to London Interbank Offer Rate (LIBOR). Initially, we consider a situation where the floating-rate note is about to be reset. Just before the reset period, we know that the coupon will be set at the prevailing market rate. Therefore, the note carries no market risk, and its value can be mapped on cash only. Right after the reset, however, the note becomes similar to a bill with maturity equal to the next reset period.

Interest-rate swaps can be viewed in two different ways: as (1) a combined position in a fixed-rate bond and in a floating-rate bond or (2) a portfolio of forward contracts. We first value the swap as a position in two bonds using risk data from Table 11-4. The analysis is detailed in Table 11-10.

TABLE 11-10

Computing the VAR of a $100 Million Interest-Rate Swap (Monthly VAR at 95 Percent Level)

	Cash Flows					
Term (Year)	Fixed	Float	Spot Rate	PV of Net Cash Flows	Individual VAR	Component VAR
0	$0	+$100		+$100.000	$0	$0
1	−$6.195	$0	5.813%	−$5.855	$0.027	$0.024
2	−$6.195	$0	5.929%	−$5.521	$0.054	$0.053
3	−$6.195	$0	6.034%	−$5.196	$0.077	$0.075
4	−$6.195	$0	6.130%	−$4.883	$0.096	$0.096
5	−$106.195	$0	6.217%	−$78.546	$1.905	$1.905
Total				$0.000		
Undiversified VAR					$2.160	
Diversified VAR						$2.152

The second and third columns lay out the payments on both legs. Assuming that this is an at-the-market swap, that is, that its coupon is equal to prevailing swap rates, the short position in the fixed-rate bond is worth \$100 million. Just before reset, the long position in the FRN is also worth \$100 million, so the market value of the swap is zero. To clarify the allocation of current values, the FRN is allocated to cash, with a zero maturity. This has no risk.

The next column lists the zero-coupon swap rates for maturities going from 1 to 5 years. The fifth column reports the present value of the net cash flows, fixed minus floating. The last column presents the component VAR, which adds up to a total diversified VAR of \$2.152 million. The undiversified VAR is obtained from summing all individual VARs. As usual, the \$2.160 million value somewhat overestimates risk.

This swap can be viewed as the sum of five forward contracts, as shown in Table 11-11. The 1-year contract promises payment of \$100 million plus the coupon of 6.195 percent; discounted at the spot rate of 5.813 percent, this yields a present value of −\$100.36 million. This is in exchange for \$100 million now, which has no risk.

The next contract is a 1×2 forward contract that promises to pay the principal plus the fixed coupon in 2 years, or −\$106.195 million; discounted at the 2-year spot rate, this yields −\$94.64 million. This is in exchange for \$100 million in 1 year, which is also \$94.50 million when discounted at the 1-year spot rate. And so on until the fifth contract, a 4×5 forward contract.

Table 11-11 shows the VAR of each contract. The undiversified VAR of \$2.401 million is the result of a simple summation of the five VARs. The fully diversified VAR is \$2.152 million, exactly the same as in the preceding table. This demonstrates the equivalence of the two approaches.

Finally, we examine the change in risk after the first payment has just been set on the floating-rate leg. The FRN then becomes a 1-year bond initially valued at par but subject to fluctuations in rates. The only change in the pattern of cash flows in Table 11-10 is to add \$100 million to the position on year 1 (from −\$5.855 to \$94.145). The resulting VAR then decreases from \$2.152 million to \$1.763 million. More generally, the swap's VAR will converge to zero as the swap matures, dipping each time a coupon is set.

TABLE 11-11

An Interest-Rate Swap Viewed as Forward Contracts (Monthly VAR at 95 Percent Level)

	PV of Flows: Contract					
Term (Year)	**1**	**1 × 2**	**2 × 3**	**3 × 4**	**4 × 5**	**VAR**
1	−\$100.36	\$94.50				
2		−\$94.64	\$89.11			
3			−\$89.08	\$83.88		
4				−\$83.70	\$78.82	
5					−\$78.55	
VAR	\$0.471	\$0.571	\$0.488	\$0.446	\$0.425	
Undiversified VAR						\$2.401
Diversified VAR						\$2.152

11.4 MAPPING OPTIONS

We now consider the mapping process for nonlinear derivatives, or options. Obviously, this nonlinearity may create problems for risk measurement systems based on the delta-normal approach, which is fundamentally linear.

To simplify, consider the Black-Scholes (BS) model for European options.[2] The model assumes, in addition to perfect capital markets, that the underlying spot price follows a continuous *geometric brownian motion* with constant volatility $\sigma(dS/S)$. Based on these assumptions, the Black-Scholes (1973) model, as expanded by Merton (1973), gives the value of a European call as

$$c = c(S,K,\tau,r,r^*,\sigma) = Se^{-r^*\tau}N(d_1) - Ke^{-r\tau}N(d_2) \qquad (11.16)$$

where $N(d)$ is the cumulative normal distribution function described in Chapter 5 with arguments

$$d_1 = \frac{\ln(Se^{-r^*\tau} / Ke^{-r\tau})}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2}, \qquad d_2 = d_1 - \sigma\sqrt{\tau}$$

where K is now the *exercise price* at which the option holder can, but is not obligated to, buy the asset.

[2] For a systematic approach to pricing derivatives, see the excellent book by Hull (2005).

Changes in the value of the option can be approximated by taking partial derivatives, that is,

$$\begin{aligned} dc &= \frac{\partial c}{\partial S}dS + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}dS^2 + \frac{\partial c}{\partial r^*}dr^* + \frac{\partial c}{\partial r}dr + \frac{\partial c}{\partial \sigma}d\sigma + \frac{\partial c}{\partial t}dt \\ &= \Delta dS + \tfrac{1}{2}\Gamma dS^2 + \rho^* dr^* + \rho dr + \Lambda d\sigma + \Theta dt \end{aligned} \tag{11.17}$$

The advantage of the BS model is that it leads to closed-form solutions for all these partial derivatives. Table 11-12 gives typical values for 3-month European call options with various exercise prices.

The first partial derivative, or *delta*, is particularly important. For a European call, this is

$$\Delta = e^{-r^*\tau}N(d_1) \tag{11.18}$$

This is related to the cumulative normal density function covered in Chapter 5. Figure 11-2 displays its behavior as a function of the underlying spot price and for various maturities.

The figure shows that delta is not a constant, which may make linear methods inappropriate for measuring the risk of options. Delta increases with the underlying spot price. The relationship becomes more nonlinear

TABLE 11-12

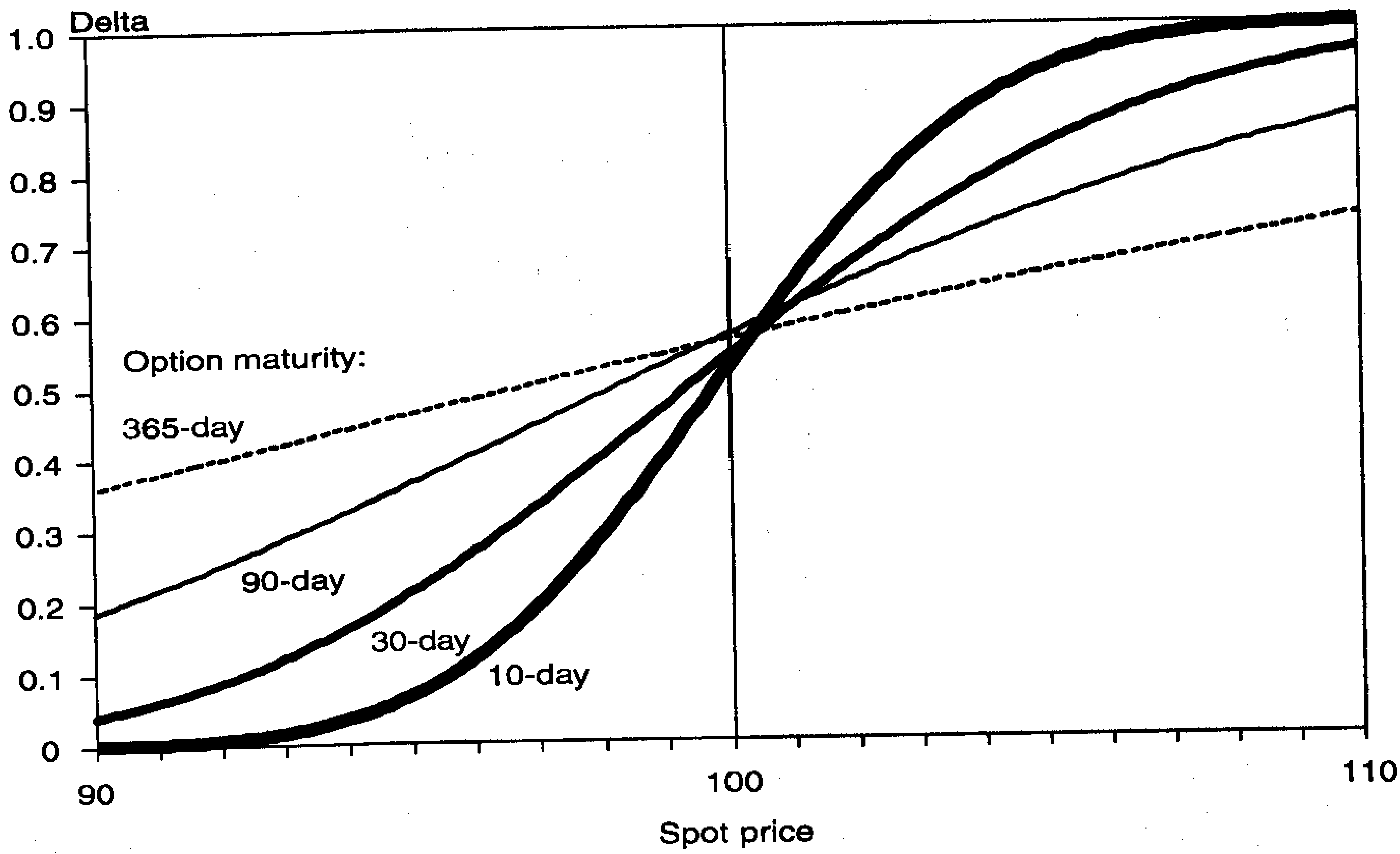
Derivatives for a European Call

Parameters: S = \$100, σ = 20%, r = 5%, r^* = 3%, τ = 3 months

			Exercise Price		
	Variable	**Unit**	**K = 90**	**K = 100**	**K = 110**
c		*Dollars*	11.01	4.20	1.04
		Change per			
Δ	Spot price	Dollar	0.869	0.536	0.195
Γ	Spot price	Dollar	0.020	0.039	0.028
Λ	Volatility	(% pa)	0.102	0.197	0.138
ρ	Interest rate	(% pa)	0.190	0.123	0.046
ρ^*	Asset yield	(% pa)	−0.217	−0.133	−0.049
θ	Time	Day	−0.014	−0.024	−0.016

FIGURE 11-2

Delta as a function of the risk factor.



for short-term options, for example, with an option maturity of 10 days. Linear methods approximate delta by a constant value over the risk horizon. The quality of this approximation depends on parameter values.

For instance, if the risk horizon is 1 day, the worst down move in the spot price is $-\alpha S\sigma\sqrt{T} = -1.645 \times \$100 \times 0.20\sqrt{1/252} = -\$2.08$, leading to a worst price of \$97.92. With a 90-day option, delta changes from 0.536 to 0.452 only. With such a small change, the linear effect will dominate the nonlinear effect. Thus linear approximations may be acceptable for options with long maturities when the risk horizon is short.

It is instructive to consider only the linear effects of the spot rate and two interest rates, that is,

$$\begin{aligned}
dc &= \Delta dS + \rho^* dr^* + \rho dr \\
&= [e^{-r^*\tau} N(d_1)]dS + [-Se^{-r^*\tau}\tau N(d_1)]dr^* + [Ke^{-r\tau}\tau N(d_2)]dr \\
&= [Se^{-r^*\tau} N(d_1)]\frac{dS}{S} + [Se^{-r^*\tau} N(d_1)]\frac{dP^*}{P^*} - [Ke^{-r\tau} N(d_2)]\frac{dP}{P} \qquad (11.19) \\
&= x_1\frac{dS}{S} + x_2\frac{dP^*}{P^*} + x_3\frac{dP}{P}
\end{aligned}$$

This formula bears a striking resemblance to that for foreign currency forwards, as in Equation (11.13). The only difference is that the position on the spot foreign currency and on the foreign currency bill $x_1 = x_2$ now involves $N(d_1)$, and the position on the dollar bill x_3 involves $N(d_2)$. In the extreme case, where the option is deep in the money, both $N(d_1)$ and $N(d_2)$ are equal to unity, and the option behaves exactly like a position in a forward contract. In this case, the BS model reduces to $c = Se^{-r^*\tau} - Ke^{-r\tau}$, which is indeed the valuation formula for a forward contract, as in Equation (11.9).

Also note that the position on the dollar bill $Ke^{-r\tau}N(d_2)$ is equivalent to $Se^{-r^*\tau}N(d_1) - c = S\Delta - c$. This shows that the call option is equivalent to a position of Δ in the underlying asset plus a short position of $(\Delta S - c)$ in a dollar bill, that is

Long option = long Δ asset + short $(\Delta S - c)$ bill

For instance, assume that the delta for an at-the-money call option on an asset worth \$100 is $\Delta = 0.536$. The option itself is worth \$4.20. This option is equivalent to a $\Delta S =$ \$53.60 position in the underlying asset financed by a loan of $\Delta S - c =$ \$53.60 − \$4.20 = \$49.40.

The next step in the risk measurement process is the aggregation of exposures across the portfolio. Thus all options on the same underlying risk factor are decomposed into their delta equivalents, which are summed across the portfolio. This generalizes to movements in the implied volatility, if necessary. The option portfolio would be characterized by its net *vega*, or Λ. This decomposition also can take into account second-order derivatives using the net *gamma*, or Γ. These exposures can be combined with simulations of the underlying risk factors to generate a risk distribution.

11.5 CONCLUSIONS

Risk measurement at financial institutions is a top-level aggregation problem involving too many positions to be modeled individually. As a result, instruments have to be mapped on a smaller set of primitive risk factors.

Choosing the appropriate set of risk factors, however, is part of the art of risk management. Too many risk factors would be unnecessary, slow, and wasteful. Too few risk factors, in contrast, could create blind spots in the risk measurement system. These issues will be discussed in Chapter 21, where we will discuss limitations of VAR.

The mapping process consists of replacing the current values of all instruments by their exposures on these risk factors. Next, exposures are aggregated across the portfolio to create a net exposure to each risk factor. The risk engine then combines these exposures with the distribution of risk factors to generate a distribution of portfolio values.

For some instruments, the allocation into general-market risk factors is exhaustive. In other words, there is no specific risk left. This is typically the case with derivatives, which are tightly priced in relation to their underlying risk factor. For others positions, such as individual stocks or corporate bonds, there remains some risk, called *specific risk*. In large, well-diversified portfolios, this remaining risk tends to wash away. Otherwise, specific risk needs to be taken into account.