2

Model Predictive Control—Regulation

2.1 Introduction

In Chapter 1 we investigated a special, but useful, form of model predictive control (MPC); an important feature of this form of MPC is that, *if* the terminal cost is chosen to be the value function of infinite horizon *unconstrained* optimal control problem, there exists a set of initial states for which MPC is actually optimal for the *infinite horizon constrained* optimal control problem and therefore inherits its associated advantages. Just as there are many methods other than infinite horizon linear quadratic control for stabilizing linear systems, there are alternative forms of MPC that can stabilize linear and even nonlinear systems. We explore these alternatives in the remainder of this chapter. But first we place MPC in a more general setting to facilitate comparison with other control methods.

MPC is, as we have seen earlier, a form of control in which the control action is obtained by solving *online*, at each sampling instant, a *finite horizon* optimal control problem in which the initial state is the current state of the plant. Optimization yields a finite control sequence, and the first control action in this sequence is applied to the plant. MPC differs, therefore, from conventional control in which the control law is precomputed offline. But this is not an essential difference; MPC implicitly implements a control law that can, in principle, be computed offline as we shall soon see. Specifically, if the current state of the system being controlled is x, MPC obtains, by solving an open-loop optimal control problem for this initial state, a specific control action u to apply to the plant.

Dynamic programming (DP) may be used to solve a feedback version of the same optimal control problem, however, yielding a receding horizon control *law* $\kappa(\cdot)$. The important fact is that if x is the current state, the optimal control u obtained by MPC (by solving an open-loop optimal control problem) satisfies $u = \kappa(x)$; MPC computes the *value* $\kappa(x)$ of the optimal receding horizon control law for the current state x, while DP yields the control *law* $\kappa(\cdot)$ that can be used for *any* state. DP would appear to be preferable since it provides a control law that can be implemented simply (as a look-up table). However, obtaining a DP solution is difficult, if not impossible, for most optimal control problems if the state dimension is reasonably high — unless the system is linear, the cost quadratic and there are no control or state constraints. The great advantage of MPC is that open-loop optimal control problems often can be solved rapidly enough, using standard mathematical programming algorithms, to permit the use of MPC even though the system being controlled is nonlinear, and hard constraints on states and controls must be satisfied. Thus MPC permits the application of a DP solution, even though explicit determination of the optimal control law is intractable. MPC is an effective *implementation* of the DP solution.

In this chapter we study MPC for the case when the state is known. This case is particularly important, even though it rarely arises in practice, because important properties, such as stability and performance, may be relatively easily established. The relative simplicity of this case arises from the fact that if the state is known and if there are no disturbances or model error, the problem is *deterministic*, i.e., there is no uncertainty making feedback unnecessary in principle. As we pointed out previously, for deterministic systems the MPC action for a given state is identical to the receding horizon control law, determined using DP, and evaluated at the given state. When the state is *not* known, it has to be estimated and state estimation error, together with model error and disturbances, makes the system uncertain in that future trajectories cannot be precisely predicted. The simple connection between MPC and the DP solution is lost because there does not exist an open-loop optimal control problem whose solution yields a control action that is the same as that obtained by the DP solution. A practical consequence is that special techniques are required to ensure robustness against these various forms of uncertainty. So the results of this chapter hold when there is no uncertainty. We prove, in particular, that the optimal control problem that defines the model predictive control can always be solved if the initial optimal control problem can be solved (recursive feasibility), and that the optimal cost can always be reduced allowing us to prove asymptotic or exponential stability of the target state. We refer to stability in the absence of uncertainty as *nominal or inherent stability*.

When uncertainty is present, however, neither of these two assertions is necessarily true; uncertainty may cause the state to wander outside the region where the optimal control problem can be solved and may lead to instability. Procedures for overcoming the problems arising from uncertainty are presented in Chapters 3 and 5. In most of the control algorithms presented in this chapter, the decrease in the optimal cost, on which the proof of stability is founded, is based on the assumption that the next state is exactly as predicted and that the global solution to the optimal control problem can be computed. In the suboptimal control algorithm presented in Chapter 6, where global optimality is not required, the decrease in the optimal cost is still based on the assumption that the current state is exactly the state as predicted at the previous time.

2.2 Model Predictive Control

As discussed briefly in Chapter 1, most nonlinear system descriptions derived from physical arguments are continuous time models in the form of nonlinear differential equations

$$\frac{dx}{dt} = f(x, u)$$

For this class of systems, the control law with the best closed-loop properties is the solution to the following infinite horizon, constrained optimal control problem. The cost is defined to be

$$V_\infty(x, u(\cdot)) = \int_0^\infty \ell(x(t), u(t))dt$$

in which $x(t)$ and $u(t)$ satisfy $\dot{x} = f(x, u)$. The optimal control problem $\mathbb{P}_\infty(x)$ is defined by

$$\min_{u(\cdot)} V_\infty(x, u(\cdot))$$

subject to

$$\begin{gathered}\dot{x} = f(x, u) \quad x(0) = x_0 \\ (x(t), u(t)) \in \mathbb{Z} \text{ for all } t \in \mathbb{R}_{\geq 0}\end{gathered}$$

If $\ell(\cdot)$ is positive definite, the goal of the regulator is to steer the state of the system to the origin.

We denote the solution to this problem (when it exists) by $u^0_\infty(\cdot;x)$ and the resultant optimal value function by $V^0_\infty(x)$. The closed-loop system under this optimal control law evolves as

$$\frac{dx(t)}{dt} = f(x(t), u^0_\infty(t;x))$$

If $f(\cdot)$, $\ell(\cdot)$ and $V_f(\cdot)$ satisfy certain differentiability and growth assumptions, and if the class of admissible controls is sufficiently rich, then a solution to $\mathbb{P}_\infty(x)$ exists for all x and satisfies

$$\dot{V}^0_\infty(x) = -\ell(x, u^0_\infty(0;x))$$

Using this and upper and lower bounds on $V^0_\infty(\cdot)$ enables global asymptotic stability of the origin to be established.

Although the control law $u^0_\infty(0;\cdot)$ provides excellent closed-loop properties, there are several impediments to its use. A feedback, rather than an open-loop, solution of the optimal control problem is desirable because of uncertainty; solution of the optimal control problem $\mathbb{P}_\infty(x)$ yields the optimal control sequence $u^0_\infty(0;x)$ for the state x but does not provide a control law. Dynamic programming may, in principle, be employed, but is generally impractical if the state dimension and the horizon are not small.

If we turn instead to an MPC approach in which we generate online only the value of optimal control sequence $u^0_\infty(\cdot;x)$ for the currently measured value of x, rather than for all x, the problem remains formidable for the following reasons. First, we are optimizing a time *function*, $u(\cdot)$, and functions are infinite dimensional. Secondly, the time interval of interest, $[0,\infty)$, is a semi-infinite interval, which poses other numerical challenges. Finally, the cost function $V(x, u(\cdot))$ is usually not a convex function of $u(\cdot)$, which presents significant optimization difficulties, especially in an online setting. Even proving existence of the optimal control in this general setting is a challenge. However, see Pannocchia, Rawlings, Mayne, and Mancuso (2015) in which it is shown how an *infinite horizon* optimal control may be solved online if the system is linear, the cost quadratic and the control but not the state is constrained.

Our task in this chapter may therefore be viewed as restricting the system and control parameterization to make problem $\mathbb{P}_\infty(x)$ more easily computable. We show how to pose various problems for which we can establish existence of the optimal solution and asymptotic closed-loop stability of the resulting controller. For these problems, we almost always replace the continuous time differential equation with a discrete time difference equation. We often replace the semi-infinite time interval with a finite time interval and append a terminal region so that we can approximate the cost to go for the semi-infinite interval once the system enters the terminal region. Although the solution of problem $\mathbb{P}_\infty(x)$ in its full generality is out of reach with today's computational methods, its value lies in distinguishing what is *desirable* in the control problem formulation and what is *achievable* with available computing technology.

We develop here MPC for the control of constrained nonlinear time-invariant systems. The nonlinear system is described by the nonlinear difference equation

$$x^+ = f(x, u) \tag{2.1}$$

in which $x \in \mathbb{R}^n$ is the current state, u is the current control, and x^+ the successor state; $x^+ = f(x, u)$ is the discrete time analog of the continuous time differential equation $\dot{x} = f(x, u)$. The function $f(\cdot)$ is assumed to be continuous and to satisfy $f(0, 0) = 0$; $(0, 0)$ is the desired equilibrium pair. The subsequent analysis is easily extended to the case when the desired equilibrium pair is (x_s, u_s) satisfying $x_s = f(x_s, u_s)$.

We introduce here some notation that we employ in the sequel. The set $\mathbb{I}$ denotes the set of integers, $\mathbb{I}_{\geq 0} := \{0, 1, 2, \ldots\}$ and, for any two integers m and n satisfying $m \leq n$, $\mathbb{I}_{m:n} := \{m, m+1, \ldots, n\}$. We refer to the pair (x, i) as an event; an event (x, i) denotes that the state at time i is x. We use $\mathbf{u}$ to denote the possibly infinite control sequence $(u(k))_{k \in \mathbb{I}_{\geq 0}} = (u(0), u(1), u(2), \ldots)$. In the context of MPC, $\mathbf{u}$ frequently denotes the finite sequence $\mathbf{u}_{\mathbb{I}_{0:N-1}} = (u(0), u(1), \ldots, u(N-1))$ in which N is the control *horizon*. For any integer $j \in \mathbb{I}_{\geq 0}$, we sometimes employ $\mathbf{u}_j$ to denote the finite sequence $\big(u(0), u(1), \ldots, u(j-1)\big)$. Similarly $\mathbf{x}$ denotes the possibly infinite state sequence $(x(0), x(1), x(2), \ldots)$ and $\mathbf{x}_j$ the finite sequence $\big(x(0), x(1), \ldots, x(j)\big)$. When no confusion can arise we often employ, for simplicity in notation, $\mathbf{u}$ in place of $\mathbf{u}_N$ and $\mathbf{x}$ in place of $\mathbf{x}_N$. Also for simplicity in notation, $\mathbf{u}$, when used in algebraic expressions, denotes the column vector $(u(0)', u(1)', \ldots, u(N-1)')'$; similarly $\mathbf{x}$ in algebraic expressions denotes the column vector $(x(0)', x(1)', \ldots, x(N)')'$.

The solution of (2.1) at time k, if the initial state at time zero is x and the control sequence is $\mathbf{u}$, is denoted by $\phi(k; x, \mathbf{u})$; the solution at time k depends only on $u(0), u(1), \ldots, u(k-1)$. Similarly, the solution of the system (2.1) at time k, if the initial state at time i is x and the control sequence is $\mathbf{u}$, is denoted by $\phi(k; (x, i), \mathbf{u})$. Because the system is time invariant, the solution does not depend on the initial time; if the initial state is x at time i, the solution at time $j \geq i$ is $\phi(j - i; x, \mathbf{u})$. Thus the solution at time k if the initial event is (x, i) is identical to the solution at time $k - i$ if the initial event is $(x, 0)$. For each k, the function $(x, \mathbf{u}) \mapsto \phi(k; x, \mathbf{u})$ is continuous as we show next.

Proposition 2.1 (Continuity of system solution)**.** *Suppose the function $f(\cdot)$ is continuous. Then, for each integer $k \in \mathbb{I}$, the function $(x, \mathbf{u}) \mapsto \phi(k; x, \mathbf{u})$ is continuous.*

Proof.
Since $\phi(1; x, u(0)) = f(x, u(0))$, the function $(x, u(0)) \mapsto \phi(1; x, u(0))$ is continuous. Suppose the function $(x, \mathbf{u}_{j-1}) \mapsto \phi(j; x, \mathbf{u}_{j-1})$ is continuous and consider the function $(x, \mathbf{u}_j) \mapsto \phi(j+1; x, \mathbf{u}_j)$. Since

$$\phi(j + 1; x, \mathbf{u}_j) = f(\phi(j; x, \mathbf{u}_{j-1}), u(j))$$

in which $f(\cdot)$ and $\phi(j; \cdot)$ are continuous and since $\phi(j + 1; \cdot)$ is the composition of two continuous functions $f(\cdot)$ and $\phi(j; \cdot)$, it follows that $\phi(j + 1; \cdot)$ is continuous. By induction $\phi(k; \cdot)$ is continuous for any positive integer k. ■

The system (2.1) is subject to hard constraints which may take the form

$$(x(k), u(k)) \in \mathbb{Z} \qquad \text{for all } k \in \mathbb{I}_{\geq 0} \tag{2.2}$$

in which $\mathbb{Z}$ is generally polyhedral, i.e., $\mathbb{Z} = \{(x, u) \mid Fx + Eu \leq e\}$ for some F, E, e. For example, many problems have a rate constraint $|u(k) - u(k - 1)| \leq c$ on the control. This constraint may equivalently be expressed as $|u(k) - z(k)| \leq c$ in which z is an extra state satisfying $z^+ = u$ so that $z(k) = u(k - 1)$. The constraint $(x, u) \in \mathbb{Z}$ implies the control constraint is possibly state-dependent, i.e., $(x, u) \in \mathbb{Z}$ implies that

$$u \in \mathbb{U}(x) := \{u \in \mathbb{R}^m \mid (x, u) \in \mathbb{Z}\}$$

It also implies that the state must satisfy the constraint

$$x \in \mathbb{X} := \{x \in \mathbb{R}^n \mid \mathbb{U}(x) \neq \emptyset\}$$

If there are no mixed constraints, then $\mathbb{Z} = \mathbb{X} \times \mathbb{U}$ so the system constraints become $x(k) \in \mathbb{X}$ and $u(k) \in \mathbb{U}$.

We assume in this chapter that the state x is known; if the state x is estimated, uncertainty (state estimation error) is introduced and *robust* MPC, discussed in Chapter 3, is required.

The next ingredient of the optimal control problem is the cost function. Practical considerations normally require that the cost be defined over a finite horizon N to ensure the resultant optimal control problem can be solved sufficiently rapidly to permit effective control. We consider initially the regulation problem in which the target state is the origin. If x is the current state and i the current time, then the optimal control problem may be posed as minimizing a cost defined over the interval from time i to time $N+i$. The optimal control problem $\mathbb{P}_N(x,i)$ at event (x,i) is the problem of minimizing the cost

$$\sum_{k=i}^{i+N-1} \ell(x(k), u(k)) + V_f(x(N+i))$$

with respect to the sequences $\mathbf{x} := (x(i), x(i+1), \ldots, x(i+N))$ and $\mathbf{u} := (u(i), u(i+1), \ldots, u(i+N-1))$ subject to the constraints that $\mathbf{x}$ and $\mathbf{u}$ satisfy the difference equation (2.1), the initial condition $x(i) = x$, and the state and control constraints (2.2). We assume that $\ell(\cdot)$ is continuous and that $\ell(0,0) = 0$. The optimal control and state sequences, obtained by solving $\mathbb{P}_N(x,i)$, are functions of the initial event (x,i)

$$\mathbf{u}^0(x,i) = \big(u^0(i;(x,i)), u^0(i+1;(x,i)), \ldots, u^0(i+N-1;(x,i))\big)$$
$$\mathbf{x}^0(x,i) = \big(x^0(i;(x,i)), x^0(i+1;(x,i)), \ldots, x^0(i+N;(x,i))\big)$$

with $x^0(i;(x,i)) = x$. In MPC, the first control action $u^0(i;(x,i))$ in the optimal control sequence $\mathbf{u}^0(x,i)$ is applied to the plant, i.e., $u(i) = u^0(i;(x,i))$. Because the system $x^+ = f(x,u)$, the stage cost $\ell(\cdot)$, and the terminal cost $V_f(\cdot)$ are all time invariant, however, the solution of $\mathbb{P}_N(x,i)$, for any time $i \in \mathbb{I}_{\geq 0}$, is identical to the solution of $\mathbb{P}_N(x,0)$ so that

$$\mathbf{u}^0(x,i) = \mathbf{u}^0(x,0)$$
$$\mathbf{x}^0(x,i) = \mathbf{x}^0(x,0)$$

In particular, $u^0(i;(x,i)) = u^0(0;(x,0))$, i.e., the control $u^0(i;(x,i))$ applied to the plant is equal to $u^0(0;(x,0))$, the first element in the sequence $\mathbf{u}^0(x,0)$. Hence we may as well merely consider problem $\mathbb{P}_N(x,0)$ which, since the initial time is irrelevant, we call $\mathbb{P}_N(x)$. Similarly, for simplicity in notation, we replace $\mathbf{u}^0(x,0)$ and $\mathbf{x}^0(x,0)$ by, respectively, $\mathbf{u}^0(x)$ and $\mathbf{x}^0(x)$.

The optimal control problem $\mathbb{P}_N(x)$ may then be expressed as minimization of

$$\sum_{k=0}^{N-1} \ell(x(k), u(k)) + V_f(x(N))$$

with respect to the *decision variables* $(\mathbf{x}, \mathbf{u})$ subject to the constraints that the state and control sequences $\mathbf{x}$ and $\mathbf{u}$ satisfy the difference equation (2.1), the initial condition $x(0) = x$, and the state, control constraints (2.2). Here $\mathbf{u}$ denotes the control sequence $(u(0), u(1), \ldots, u(N-1))$ and $\mathbf{x}$ the state sequence $(x(0), x(1), \ldots, x(N))$. Retaining the state sequence in the set of decision variables is discussed in Chapters 6 and 8. For the purpose of analysis, however, it is preferable to constrain the state sequence $\mathbf{x}$ *a priori* to be a solution of $x^+ = f(x, u)$ enabling us to express the problem in the equivalent form of minimizing, with respect to the decision variable $\mathbf{u}$, a cost that is purely a function of the initial state x and the control sequence $\mathbf{u}$. This formulation is possible since the state sequence $\mathbf{x}$ may be expressed, via the difference equation $x^+ = f(x, u)$, as a function of $(x, \mathbf{u})$. The cost becomes $V_N(x, \mathbf{u})$ defined by

$$V_N(x, \mathbf{u}) := \sum_{k=0}^{N-1} \ell(x(k), u(k)) + V_f(x(N)) \tag{2.3}$$

in which $x(k) := \phi(k; x, \mathbf{u})$ for all $k \in \mathbb{I}_{0:N}$. Similarly the constraints (2.2), together with an additional terminal constraint

$$x(N) \in \mathbb{X}_f \subseteq \mathbb{X}$$

impose an implicit constraint on the control sequence of the form

$$\mathbf{u} \in \mathcal{U}_N(x) \tag{2.4}$$

The control constraint set $\mathcal{U}_N(x)$ is the set of control sequences $\mathbf{u} := (u(0), u(1), \ldots, u(N-1))$ satisfying the state and control constraints. It is therefore defined by

$$\mathcal{U}_N(x) := \{\mathbf{u} \mid (x, \mathbf{u}) \in \mathbb{Z}_N\} \tag{2.5}$$

in which the set $\mathbb{Z}_N \subset \mathbb{R}^n \times \mathbb{R}^{Nm}$ is defined by

$$\begin{aligned}\mathbb{Z}_N := \big\{(x, \mathbf{u}) \mid (\phi(k; x, \mathbf{u}), u(k)) \in \mathbb{Z}, \forall k \in \mathbb{I}_{0:N-1},& \\ \phi(N; x, \mathbf{u}) \in \mathbb{X}_f\big\}&\end{aligned} \tag{2.6}$$

The optimal control problem $\mathbb{P}_N(x)$, is, therefore

$$\mathbb{P}_N(x): \qquad V_N^0(x) := \min_{\mathbf{u}}\{V_N(x,\mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_N(x)\} \tag{2.7}$$

Problem $\mathbb{P}_N(x)$ is a *parametric* optimization problem in which the decision variable is $\mathbf{u}$, and both the cost and the constraint set depend on the *parameter* x. The set $\mathbb{Z}_N$ is the set of admissible $(x,\mathbf{u})$, i.e., the set of $(x,\mathbf{u})$ for which the constraints of $\mathbb{P}_N(x)$ are satisfied. Let $\mathcal{X}_N$ be the set of states in $\mathbb{X}$ for which $\mathbb{P}_N(x)$ has a solution

$$\mathcal{X}_N := \{x \in \mathbb{X} \mid \mathcal{U}_N(x) \neq \emptyset\} \tag{2.8}$$

It follows from (2.7) and (2.8) that

$$\mathcal{X}_N = \{x \in \mathbb{R}^n \mid \exists \mathbf{u} \in \mathbb{R}^{Nm} \text{ such that } (x,\mathbf{u}) \in \mathbb{Z}_N\}$$

which is the orthogonal projection of $\mathbb{Z}_N \subset \mathbb{R}^n \times \mathbb{R}^{Nm}$ onto $\mathbb{R}^n$. The domain of $V_N^0(\cdot)$, i.e., the set of states in $\mathbb{X}$ for which $\mathbb{P}_N(x)$ has a solution, is $\mathcal{X}_N$.

Not every optimization problem has a solution. For example, the problem $\min\{x \mid x \in (0,1)\}$ does not have a solution; $\inf\{x \mid x \in (0, 1)\} = 0$ but $x = 0$ does not lie in the constraint set $(0,1)$. By Weierstrass's theorem, however, an optimization problem does have a solution if the cost is continuous (in the decision variable) and the constraint set compact (see Proposition A.7). This is the case for our problem as shown subsequently in Proposition 2.4. We assume, without further comment, that the following two standing conditions are satisfied in the sequel.

Assumption 2.2 (Continuity of system and cost)**.** The functions $f : \mathbb{Z} \to \mathbb{R}^n$, $\ell : \mathbb{Z} \to \mathbb{R}_{\geq 0}$ and $V_f : \mathbb{X}_f \to \mathbb{R}_{\geq 0}$ are continuous, $f(0,0) = 0$, $\ell(0,0) = 0$ and $V_f(0) = 0$.

In by far the majority of applications the control is constrained. Nevertheless, it is of theoretical interest to consider the case when the optimal control problem has no constraints on the control. To analyze this case we employ an implicit control constraint set $\bar{\mathcal{U}}_N^c(x)$ defined as follows. Let $c := V_N(x,\mathbf{u}^*)$ in which $\mathbf{u}^*$ is an arbitrary control sequence in $\mathcal{U}_N(x)$. Then

$$\bar{\mathcal{U}}_N^c(x) := \{\mathbf{u} \mid V_N(x,\mathbf{u}) \leq c\}$$

We also define the feasible set $\bar{\mathcal{X}}_N^c$ for the optimal control problem with no constraints on the control by

$$\bar{\mathcal{X}}_N^c := \{x \mid \bar{\mathcal{U}}_N^c(x) \neq \emptyset\}$$

Assumption 2.3 (Properties of constraint sets)**.** The set $\mathbb{Z}$ is closed. If there are control constraints, the set $\mathbb{U}(x)$ is compact and is uniformly bounded in $\mathbb{X}$. The set $\mathbb{X}_f \subseteq \mathbb{X}$ is compact. Each set contains the origin. If there are no control constraints ($\mathbb{Z} = \mathbb{X} \times \mathbb{R}^m$), the function $\mathbf{u} \mapsto V_N(x, \mathbf{u})$ is coercive, i.e., $V_N(x, \mathbf{u}) \to \infty$ as $|\mathbf{u}| \to \infty$ for all $x \in \mathbb{X}$).

It is implicitly assumed in these assumptions that the desired equilibrium pair is $(\bar{x}, \bar{u}) = (0, 0)$ because the first problem we tackle is regulation to the origin.

Proposition 2.4 (Existence of solution to optimal control problem)**.** *Suppose Assumptions 2.2 and 2.3 hold. Then*

(a) The function $V_N(\cdot)$ is continuous in $\mathbb{Z}_N$.

(b) For each $x \in \mathcal{X}_N$ (for each $x \in \bar{\mathcal{X}}_N^c$, each $c \in \mathbb{R}_{>0}$), the control constraint set $\mathcal{U}_N(x)$ ($\bar{\mathcal{U}}_N^c(x)$) is compact.

(c) For each $x \in \mathcal{X}_N$ (for each $\bar{\mathcal{X}}_N^c$, each $c \in \mathbb{R}_{>0}$) a solution to $\mathbb{P}_N(x)$ exists.

Proof.

(a) That $(x, \mathbf{u}) \mapsto V_N(x, \mathbf{u})$ is continuous follows from continuity of $\ell(\cdot)$ and $V_f(\cdot)$ in Assumption 2.2, and the continuity of $(x, \mathbf{u}) \mapsto \phi(j; x, \mathbf{u})$ for each $j \in \mathbb{I}_{0:N-1}$, established in Proposition 2.1.

(b) If there are control constraints, $\mathcal{U}_N(x)$ is defined by a finite set of inequalities each of which has the form $\eta(x, \mathbf{u}) \leq 0$ in which $\eta(\cdot)$ is continuous. It follows that $\mathcal{U}_N(x)$ is closed. Boundedness of $\mathcal{U}_N(x)$ follows from Assumption 2.3. Hence $\mathcal{U}_N(x)$ is compact for all $x \in \mathcal{X}_N$. If instead there are no control constraints, that $\mathcal{U}_N^c(x)$ is closed follows from the fact that $V_N(\cdot)$ is continuous. To prove $\mathcal{U}_N^c(x)$ is bounded for all c, suppose the contrary: there exists a c such that $\mathcal{U}_N^c(x)$ is unbounded. Then there exists a sequence $(\mathbf{u}_i)_{i \in \mathbb{I}_{\geq 0}}$ in $\mathcal{U}_N^c(x)$ such that $\mathbf{u}_i \to \infty$ as $i \to \infty$. Because $V_N(\cdot)$ is coercive, $V_N(x, \mathbf{u}_i) \to \infty$ as $i \to \infty$, a contradiction. Hence $\mathcal{U}_N^c(x)$ is closed and bounded and, hence, compact.

(c) Since $V_N(x, \cdot)$ is continuous and $\mathcal{U}_N(x)$ ($\bar{\mathcal{U}}_N(x)$) is compact, it follows from Weierstrass's theorem (Proposition A.7) a solution to $\mathbb{P}_N(x)$ exists for each $x \in \mathcal{X}_N$ ($\bar{\mathcal{X}}_N^c$). ■

Although the function $(x, \mathbf{u}) \mapsto V_N(x, \mathbf{u})$ is continuous, the function $x \mapsto V_N^0(x)$ is not necessarily continuous; we discuss this possibility and its implications later. For each $x \in \mathcal{X}_N$, the solution of $\mathbb{P}_N(x)$ is

$$\mathbf{u}^0(x) = \arg\min_{\mathbf{u}} \{V_N(x, \mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_N(x)\}$$

If $\mathbf{u}^0(x) = (u^0(0;x), u^0(1;x), \ldots, u^0(N-1;x))$ is unique for each $x \in \mathcal{X}_N$, then $\mathbf{u}^0 : \mathbb{R}^n \to \mathbb{R}^{Nm}$ is a function; otherwise it is a set-valued function.[1] In MPC, the control applied to the plant is the first element $u^0(0;x)$ of the optimal control sequence. At the next sampling instant, the procedure is repeated for the successor state. Although MPC computes $\mathbf{u}^0(x)$ only for specific values of the state x, it could, in principle, be used to compute $\mathbf{u}^0(x)$ and, hence, $u^0(0;x)$ for every x for which $\mathbb{P}_N(x)$ is feasible, yielding the MPC control law $\kappa_N(\cdot)$ defined by

$$\kappa_N(x) := u^0(0;x), \qquad x \in \mathcal{X}_N$$

MPC does *not* require determination of the control law $\kappa_N(\cdot)$, a task that is usually intractable when constraints or nonlinearities are present and the state dimension is large; it is this fact that makes MPC so useful.

If, at a given state x, the solution of $\mathbb{P}_N(x)$ is not unique, then $\kappa_N(\cdot) = u^0(0;\cdot)$ is set valued and the model predictive controller selects one element from the set $\kappa_N(x)$.

Example 2.5: Linear quadratic MPC

Suppose the system is described by

$$x^+ = f(x,u) := x + u$$

with initial state x. The stage cost and terminal cost are

$$\ell(x,u) := (1/2)(x^2 + u^2) \qquad V_f(x) := (1/2)x^2$$

The control constraint is

$$u \in [-1, 1]$$

and there are no state or terminal constraints. Suppose the horizon is $N = 2$. Under the first approach, the decision variables are $\mathbf{u}$ and $\mathbf{x}$, and the optimal control problem is minimization of

$$V_N(x(0), x(1), x(2), u(0), u(1)) = \\ (1/2)\left(x(0)^2 + x(1)^2 + x(2)^2 + u(0)^2 + u(1)^2\right)$$

[1]A set-valued function $\phi(\cdot)$ is a function whose value $\phi(x)$ for each x in its domain is a set.

with respect to $(x(0), x(1), x(2))$, and $(u(0), u(1))$ subject to the following constraints

$$\begin{array}{lll} x(0) = x & x(1) = x(0) + u(0) & x(2) = x(1) + u(1) \\ u(0) \in [-1, 1] & u(1) \in [-1, 1] & \end{array}$$

The constraint $u \in [-1, 1]$ is equivalent to two inequality constraints, $u \leq 1$ and $-u \leq 1$. The first three constraints are equality constraints enforcing satisfaction of the difference equation.

In the second approach, the decision variable is merely $\mathbf{u}$ because the first three constraints are automatically enforced by requiring $\mathbf{x}$ to be a solution of the difference equation. Hence, the optimal control problem becomes minimization with respect to $\mathbf{u} = (u(0), u(1))$ of

$$\begin{aligned} V_N(x, \mathbf{u}) &= (1/2)\big(x^2 + (x + u(0))^2 + (x + u(0) + u(1))^2 + \\ &\qquad u(0)^2 + u(1)^2\big) \\ &= (3/2)x^2 + \begin{bmatrix} 2x & x \end{bmatrix} \mathbf{u} + (1/2)\mathbf{u}' H \mathbf{u} \end{aligned}$$

in which

$$H = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

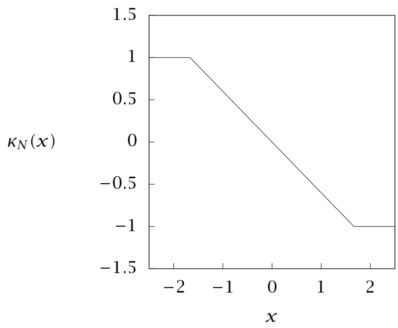
subject to the constraint $\mathbf{u} \in \mathcal{U}_N(x)$ where

$$\mathcal{U}_N(x) = \{\mathbf{u} \mid |u(k)| \leq 1 \;\; k = 0, 1\}$$

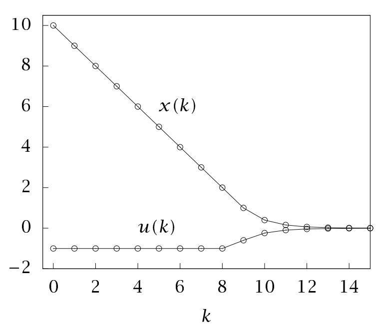
Because there are no state or terminal constraints, the set $\mathcal{U}_N(x) = \mathcal{U}_N$ for this example does not depend on the parameter x; often it does. Both optimal control problems are quadratic programs.[2] The solution for $x = 10$ is $u^0(1; 10) = u^0(2; 10) = -1$ so the optimal state trajectory is $x^0(0; 10) = 10$, $x^0(1; 10) = 9$ and $x^0(2; 10) = 8$. The value $V_N^0(10) = 124$. By solving $\mathbb{P}_N(x)$ for every $x \in [-10, 10]$, the optimal control law $\kappa_N(\cdot)$ on this set can be determined, and is shown in Figure 2.1(a). The implicit MPC control law is *time invariant* since the system being controlled, the cost, and the constraints are all time invariant. For our example, the controlled system (the system with MPC) satisfies the difference equation

$$x^+ = x + \kappa_N(x) \qquad \kappa_N(x) = -\mathrm{sat}(3x/5)$$

[2]A quadratic program is an optimization problem in which the cost is quadratic and the constraint set is polyhedral, i.e., defined by linear inequalities.



(a) Implicit MPC control law.



(b) Trajectories of controlled system.

Figure 2.1: Example of MPC.

and the state and control trajectories for an initial state of $x = 10$ are shown in Figure 2.1(b). It turns out that the origin is exponentially stable for this simple case; often, however, the terminal cost and terminal constraint set have to be carefully chosen to ensure stability. □

Example 2.6: Closer inspection of linear quadratic MPC

We revisit the MPC problem discussed in Example 2.5. The objective function is

$$V_N(x, \mathbf{u}) = (1/2)\mathbf{u}'H\mathbf{u} + c(x)'\mathbf{u} + d(x)$$

where $c(x)' = [2\ 1]x$ and $d(x) = (3/2)x^2$. The objective function may be written in the form

$$V_N(x, \mathbf{u}) = (1/2)(\mathbf{u} - a(x))'H(\mathbf{u} - a(x)) + e(x)$$

Expanding the second form shows the two forms are equal if

$$a(x) = -H^{-1}c(x) = K_1 x \qquad K_1 = -(1/5)\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

and

$$e(x) + (1/2)a(x)'Ha(x) = d(x)$$

Since H is positive definite, $a(x)$ is the unconstrained minimizer of the objective function; indeed $\nabla_{\mathbf{u}} V_N(x, a(x)) = 0$ since

$$\nabla_{\mathbf{u}} V_N(x, \mathbf{u}) = H\mathbf{u} + c(x)$$

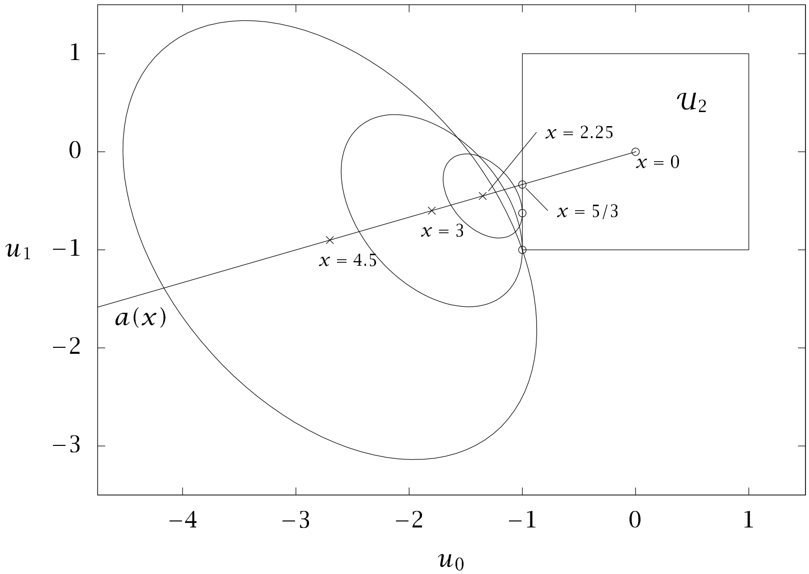


Figure 2.2: Feasible region $\mathcal{U}_2$, elliptical cost contours and ellipse center $a(x)$, and constrained minimizers for different values of x.

The locus of $a(x)$ for $x \geq 0$ is shown in Figure 2.2. Clearly the unconstrained minimizer $a(x) = K_1 x$ is equal to the constrained minimizer $\mathbf{u}^0(x)$ for all x such that $a(x) \in \mathcal{U}_2$ where $\mathcal{U}_2$ is the unit square illustrated in Figure 2.2; since $a(x) = K_1 x$, $a(x) \in \mathcal{U}_2$ for all $x \in \mathbb{X}_1 = [0, x_{c1}]$ where $x_{c1} = 5/3$. For $x > x_{c1}$, the unconstrained minimizer lies outside $\mathcal{U}_2$ as shown in Figure 2.2 for $x = 2.25$, $x = 3$ and $x = 5$. For such x, the constrained minimizer $\mathbf{u}^0(x)$ is a point that lies on the intersection of a level set of the objective function (which is an ellipse) and the boundary of $\mathcal{U}_2$. For $x \in [x_{c1}, x_{c2})$, $\mathbf{u}^0(x)$ lies on the left face of the box $\mathcal{U}_2$ and for $x \geq x_{c2} = 3$, $\mathbf{u}^0(x)$ remains at $(-1, -1)$, the bottom left vertex of $\mathcal{U}_2$.

When $u^0(x)$ lies on the left face of $\mathcal{U}_2$, the gradient $\nabla_{\mathbf{u}} V_N(x, \mathbf{u}^0(x))$ of the objective function is normal to the left face of $\mathcal{U}_2$, i.e., the level set of $V_N^0(\cdot)$ passing through $\mathbf{u}^0(x)$ is tangential to the left face of $\mathcal{U}_2$. The outward normal to $\mathcal{U}_2$ at a point on the left face is $-e_1 = (-1, 0)$

so that at $\mathbf{u} = \mathbf{u}^0(x)$

$$\nabla_{\mathbf{u}} V(x, \mathbf{u}^0(x)) + \lambda(-e_1) = 0$$

for some $\lambda > 0$; this is a standard condition of optimality. Since $\mathbf{u} = [-1 \;\; v]'$ for some $v \in [-1, 1]$ and since $\nabla_{\mathbf{u}} V(x, \mathbf{u}) = H(\mathbf{u} - a(x)) = H\mathbf{u} + c(x)$, the condition of optimality is

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ v \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} x - \begin{bmatrix} \lambda \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$\begin{aligned} -3 + v + 2x - \lambda &= 0 \\ -1 + 2v + x &= 0 \end{aligned}$$

which, when solved, yields $v = (1/2) - (1/2)x$ and $\lambda = -(5/2) + (3/2)x$. Hence

$$\mathbf{u}^0(x) = b_2 + K_2 x \qquad b_2 = \begin{bmatrix} -1 \\ (1/2) \end{bmatrix} \qquad K_2 = \begin{bmatrix} 0 \\ -(1/2) \end{bmatrix}$$

for all $x \in \mathbb{X}_2 = [x_{c1}, x_{c2}]$ where $x_{c2} = 3$ since $\mathbf{u}^0(x) \in \mathcal{U}_2$ for all x in this range. For all $x \in \mathbb{X}_3 = [x_{c_2}, \infty)$, $\mathbf{u}^0(x) = (-1, -1)'$. Summarizing

$$\begin{aligned} x \in [0, (5/3)] &\implies \mathbf{u}^0(x) = K_1 x \\ x \in [(5/3), 3] &\implies \mathbf{u}^0(x) = K_2 x + b_2 \\ x \in [3, \infty) &\implies \mathbf{u}^0(x) = b_3 \end{aligned}$$

in which

$$K_1 = \begin{bmatrix} -(3/5) \\ -(1/5) \end{bmatrix} \qquad K_2 = \begin{bmatrix} 0 \\ -(1/2) \end{bmatrix} \qquad b_2 = \begin{bmatrix} -1 \\ (1/2) \end{bmatrix} \qquad b_3 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

The optimal control for $x \leq 0$ may be obtained by symmetry; $\mathbf{u}^0(-x) = -\mathbf{u}^0(x)$ for all $x \geq 0$ so that

$$\begin{aligned} x \in [0, -(5/3)] &\implies \mathbf{u}^0(x) = -K_1 x \\ x \in [-(5/3), -3] &\implies \mathbf{u}^0(x) = -K_2 x - b_2 \\ x \in [-3, -\infty) &\implies \mathbf{u}^0(x) = -b_3 \end{aligned}$$

It is easily checked that $\mathbf{u}^0(\cdot)$ is continuous and satisfies the constraint for all $x \in \mathbb{R}$. The MPC control law $\kappa_N(\cdot)$ is the first component of $\mathbf{u}^0(\cdot)$

and, therefore, is defined by

$$\begin{array}{ll} \kappa_N(x) = 1 & x \in [-(5/3), -\infty) \\ \kappa_N(x) = -(3/5)x & x \in [-(5/3), (5/3)] \\ \kappa_N(x) = -1 & x \in [(5/3), \infty) \end{array}$$

i.e., $\kappa_N(x) = -\mathrm{sat}(3x/5)$ which is the saturating control law depicted in Figure 2.1(a). The control law is piecewise affine and the value function piecewise quadratic. The structure of the solution to constrained linear quadratic optimal control problems is explored more fully in Chapter 7. □

As we show in Chapter 3, continuity of the value function is desirable. Unfortunately, this is not true in general; the major difficulty is in establishing that the set-valued function $\mathcal{U}_N(\cdot)$ has certain continuity properties. Continuity of the value function $V_N^0(\cdot)$ and of the implicit control law $\kappa_N(\cdot)$ may be established for a few important cases, however, as is shown by the next result, which assumes satisfaction of our standing assumptions: 2.2 and 2.3 so that the cost function $V_N(\cdot)$ is continuous in $(x, \mathbf{u})$.

Theorem 2.7 (Continuity of value function and control law)**.** *Suppose that Assumptions 2.2 and 2.3 hold.*

(a) Suppose that there are no state constraints so that $\mathbb{Z} = \mathbb{X} \times \mathbb{U}$ *in which* $\mathbb{X} = \mathbb{X}_f = \mathbb{R}^n$. *Then the value function* $V_N^0 : \mathcal{X}_N \to \mathbb{R}$ *is continuous and* $\mathcal{X}_N = \mathbb{R}^n$.

(b) Suppose $f(\cdot)$ *is linear (*$x^+ = Ax + Bu$*) and that the state-control constraint set* $\mathbb{Z}$ *is polyhedral.*[3] *Then the value function* $V_N^0 : \mathcal{X}_N \to \mathbb{R}$ *is continuous.*

(c) If, in addition, the solution $\mathbf{u}^0(x)$ *of* $\mathbb{P}_N(x)$ *is unique at each* $x \in \mathcal{X}_N$, *then the implicit MPC control law* $\kappa_N(\cdot)$ *is continuous.*

The proof of this theorem is given in Section C.3 of Appendix C. The following example, due to Meadows, Henson, Eaton, and Rawlings (1995), shows that there exist nonlinear examples where the value function and implicit control law are not continuous.

[3] A set $\mathbb{Z}$ is polyhedral if it may be defined as set of linear inequalities, i.e., if it may be expressed in the form $\mathbb{Z} = \{z \mid Mz \leq m\}$.

Example 2.8: Discontinuous MPC control law

Consider the nonlinear system defined by

$$\begin{aligned} x_1^+ &= x_1 + u \\ x_2^+ &= x_2 + u^3 \end{aligned}$$

The control horizon is $N = 3$ and the cost function $V_3(\cdot)$ is defined by

$$V_3(x, \mathbf{u}) := \sum_{k=0}^{2} \ell(x(k), u(k))$$

and the stage cost $\ell(\cdot)$ is defined by

$$\ell(x, u) := |x|^2 + u^2$$

The constraint sets are $\mathbb{X} = \mathbb{R}^2$, $\mathbb{U} = \mathbb{R}$, and $\mathbb{X}_f := \{0\}$, i.e., there are no state and control constraints, and the terminal state must satisfy the constraint $x(3) = 0$. Hence, although there are three control actions, $u(0)$, $u(1)$, and $u(2)$, two must be employed to satisfy the terminal constraint, leaving only one degree of freedom. Choosing $u(0)$ to be the free decision variable automatically constrains $u(1)$ and $u(2)$ to be functions of the initial state x and the first control action $u(0)$. Solving the equation

$$\begin{aligned} x_1(3) &= x_1 + u(0) + u(1) + u(2) &&= 0 \\ x_2(3) &= x_2 + u(0)^3 + u(1)^3 + u(2)^3 &&= 0 \end{aligned}$$

for $u(1)$ and $u(2)$ yields

$$\begin{aligned} u(1) &= -x_1/2 - u(0)/2 \pm \sqrt{b} \\ u(2) &= -x_1/2 - u(0)/2 \mp \sqrt{b} \end{aligned}$$

in which

$$b = \frac{3u(0)^3 - 3u(0)^2 x_1 - 3u(0)x_1^2 - x_1^3 + 4x_2}{12(u(0) + x_1)}$$

Clearly a real solution exists only if b is positive, i.e., if both the numerator and denominator in the expression for b have the same sign. The optimal control problem $\mathbb{P}_3(x)$ is defined by

$$V_3^0(x) = \min_{\mathbf{u}} \{V_3(x, \mathbf{u}) \mid \phi(3; x, \mathbf{u}) = 0\}$$

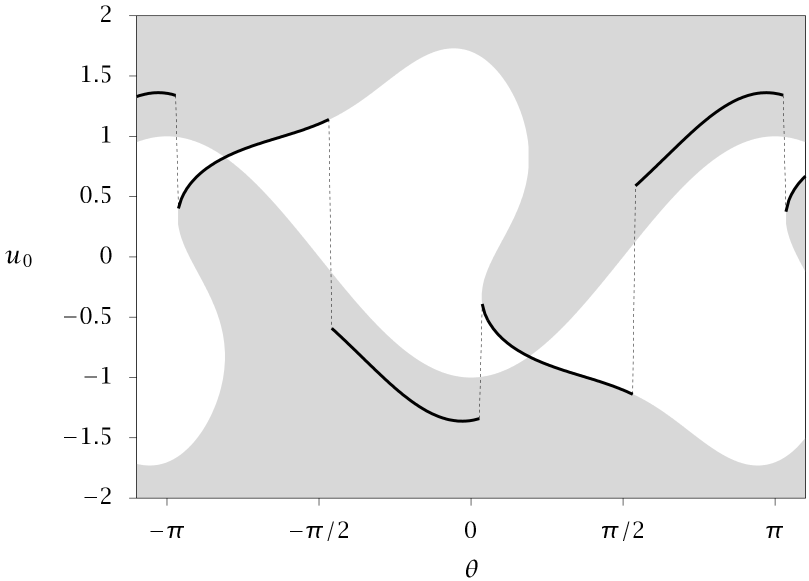


Figure 2.3: First element of control constraint set $\mathcal{U}_3(x)$ (shaded) and control law $\kappa_3(x)$ (line) versus $x = (\cos(\theta), \sin(\theta))$, $\theta \in [-\pi, \pi]$ on the unit circle for a nonlinear system with terminal constraint.

and the implicit MPC control law is $\kappa_3(\cdot)$ where $\kappa_3(x) = u^0(0; x)$, the first element in the minimizing sequence $\mathbf{u}^0(x)$. It can be shown, using analysis presented later in this chapter, that the origin is asymptotically stable for the controlled system $x^+ = f(x, \kappa_N(x))$. That this control law is necessarily discontinuous may be shown as follows. If the control is strictly positive, any trajectory originating in the first quadrant (x_1, $x_2 > 0$) moves away from the origin. If the control is strictly negative, any control originating in the third quadrant ($x_1, x_2 < 0$) also moves away from the origin. But the control cannot be zero at any nonzero point lying in the domain of attraction. If it were, this point would be a fixed point for the controlled system, contradicting the fact that it lies in the domain of attraction.

In fact, both the value function $V_3^0(\cdot)$ and the MPC control law $\kappa_3(\cdot)$ are discontinuous. Figures 2.3 and 2.4 show how $\mathcal{U}_3(x)$, $\kappa_3(x)$, and $V_3^0(x)$ vary as $x = (\cos(\theta), \sin(\theta))$ ranges over the unit circle. A further conclusion that can be drawn from this example is that it is possible

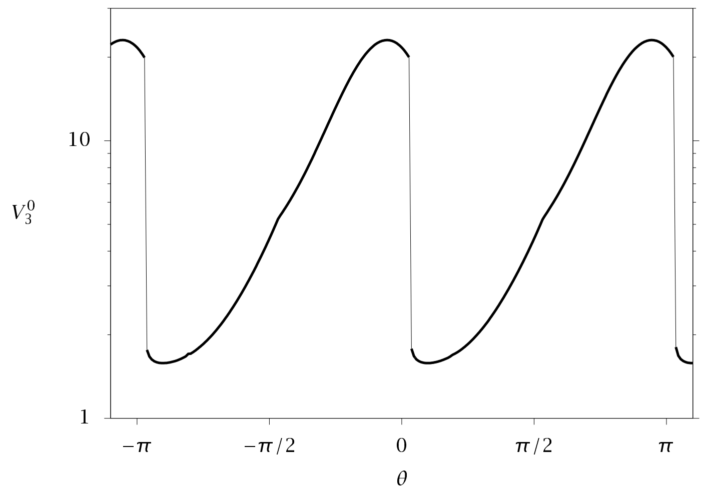


Figure 2.4: Optimal cost $V_3^0(x)$ versus $x = (\cos(\theta), \sin(\theta))$, $\theta \in [-\pi, \pi]$ on the unit circle; the discontinuity in V_3^0 is caused by the discontinuity in $\mathcal{U}_3$ as θ crosses the dashed line in Figure 2.3.

for the MPC control law to be discontinuous at points where the value function is continuous. □

2.3 Dynamic Programming Solution

We examine next the DP solution of the optimal control problem $\mathbb{P}_N(x)$, not because it provides a practical procedure but because of the insight it provides. DP can rarely be used for constrained and/or nonlinear control problems unless the state dimension n is small. MPC is best regarded as a practical means of implementing the DP solution; for a given state x it provides $V_N^0(x)$ and $\kappa_N(x)$, the value, respectively, of the value function and control law at a *point* x. DP, on the other hand, yields the value function $V_N^0(\cdot)$ and the implicit MPC control law $\kappa_N(\cdot)$.

The optimal control problem $\mathbb{P}_N(x)$ is defined, as before, by (2.7) with the cost function $V_N(\cdot)$ defined by (2.3) and the constraints by (2.4). DP yields an optimal policy $\boldsymbol{\mu}^0 = \left(\mu_0^0(\cdot), \mu_1^0(\cdot), \ldots, \mu_{N-1}^0(\cdot)\right)$, i.e., a sequence of control laws $\mu_i : \mathcal{X}_i \to \mathbb{U}$, $i = 0, 1, \ldots, N-1$. The domain $\mathcal{X}_i$ of each control law will be defined later. The optimal controlled system is time varying and satisfies

$$x^+ = f(x, \mu_i^0(x)), \ i = 0, 1, \ldots, N-1$$

in contrast with the system using MPC, which is time invariant and satisfies

$$x^+ = f(x, \kappa_N(x)), \ i = 0, 1, \ldots, N-1$$

with $\kappa_N(\cdot) = \mu_0^0(\cdot)$. The optimal control law at time i is $\mu_i^0(\cdot)$, but receding horizon control (RHC) uses the time-invariant control law $\kappa_N(\cdot) = \mu_0(\cdot)$ obtained by assuming that at each time t, the terminal time or *horizon* is $t + N$ so that the horizon $t + N$ recedes as t increases. One consequence is that the time-invariant control law $\kappa_N(\cdot)$ is *not* optimal for the problem of controlling $x^+ = f(x, u)$ over the fixed interval $[0, T]$ in such a way as to minimize V_N and satisfy the constraints.

For all $j \in \mathbb{I}_{0:N-1}$, let $V_j(x, \mathbf{u})$, $\mathcal{U}_j(x)$, $\mathbb{P}_j(x)$, and $V_j^0(x)$ be defined, respectively, by (2.3), (2.4), (2.5), and (2.6), with N replaced by j. As shown in Section C.1 of Appendix C, DP solves not only $\mathbb{P}_N(x)$ for all $x \in \mathcal{X}_N$, the domain of $V_N^0(\cdot)$, but also $\mathbb{P}_j(x)$ for all $x \in \mathcal{X}_j$, the domain of $V_j^0(\cdot)$, all $j \in \mathbb{I}_{0:N-1}$. The DP equations are, for all $x \in \mathcal{X}_j$

$$V_j^0(x) = \min_{u \in \mathbb{U}(x)} \{\ell(x, u) + V_{j-1}^0(f(x, u)) \mid f(x, u) \in \mathcal{X}_{j-1}\} \tag{2.9}$$

$$\kappa_j(x) = \arg \min_{u \in \mathbb{U}(x)} \{\ell(x, u) + V_{j-1}^0(f(x, u)) \mid f(x, u) \in \mathcal{X}_{j-1}\} \tag{2.10}$$

with

$$\mathcal{X}_j = \{x \in \mathbb{X} \mid \exists u \in \mathbb{U}(x) \text{ such that } f(x, u) \in \mathcal{X}_{j-1}\} \tag{2.11}$$

for $j = 1, 2, \ldots, N$ (j is *time to go*), with terminal conditions

$$V_0^0(x) = V_f(x) \ \forall x \in \mathcal{X}_0 \qquad \mathcal{X}_0 = \mathbb{X}_f$$

For each j, $V_j^0(x)$ is the optimal cost for problem $\mathbb{P}_j(x)$ if the current state is x, current time is $N - j$, and the terminal time is N; $\mathcal{X}_j$ is the domain of $V_j^0(x)$ and is also the set of states in $\mathbb{X}$ that can be steered to the terminal set $\mathbb{X}_f$ in j steps by an *admissible* control sequence, i.e., a control sequence that satisfies the control, state, and terminal constraints. Hence, for each j

$$\mathcal{X}_j = \{x \in \mathbb{X} \mid \mathcal{U}_j(x) \neq \emptyset\}$$

DP yields much more than an optimal control sequence for a given initial state; it yields an optimal feedback *policy* $\boldsymbol{\mu}^0$ or sequence of control laws where

$$\boldsymbol{\mu}^0 := (\mu_0(\cdot), \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot)) = (\kappa_N(\cdot), \kappa_{N-1}(\cdot), \ldots, \kappa_1(\cdot))$$

At event (x, i), i.e., at state x at time i, the time to go is $N - i$ and the optimal control is

$$\mu_i^0(x) = \kappa_{N-i}(x)$$

i.e., $\mu_i^0(\cdot)$ is the optimal control law at time i. Consider an initial *event* $(x, 0)$, i.e., state x at time zero. If the terminal time (horizon) is N, the optimal control for $(x, 0)$ is $\kappa_N(x)$. The successor state, at time 1, is

$$x^+ = f(x, \kappa_N(x))$$

At event $(x^+, 1)$, the time to go to the terminal set $\mathbb{X}_f$ is $N - 1$ and the optimal control is $\kappa_{N-1}(x^+) = \kappa_{N-1}(f(x, \kappa_N(x)))$. For a given initial event $(x, 0)$, the optimal policy generates the optimal state and control trajectories $\mathbf{x}^0(x)$ and $\mathbf{u}^0(x)$ that satisfy the difference equations

$$x(0) = x \qquad u(0) = \kappa_N(x) = \mu_0(x) \tag{2.12}$$

$$x(i+1) = f(x(i), u(i)) \qquad u(i) = \kappa_{N-i}(x(i)) = \mu_i(x(i)) \tag{2.13}$$

for $i = 0, 1, \ldots, N - 1$. These state and control trajectories are identical to those obtained, as in MPC, by solving $\mathbb{P}_N(x)$ directly for the particular initial event $(x, 0)$ using a mathematical programming algorithm. Dynamic programming, however, provides a solution for *any* event (x, i) such that $i \in \mathbb{I}_{0:N-1}$ and $x \in \mathcal{X}_i$.

Optimal control, in the classic sense of determining a control that minimizes a cost over the interval $[0, N]$ (so that the cost for $k > N$ is irrelevant), is generally time varying (at event (x, i), $i \in \mathbb{I}_{0:N}$, the optimal control is $\mu_i(x) = \kappa_{N-i}(x)$). Under fairly general conditions, $\mu_i(\cdot) \to \kappa_\infty(\cdot)$ as $N \to \infty$ where $\kappa_\infty(\cdot)$ is the stationary infinite horizon optimal control law. MPC and RHC, on the other hand, employ the time-invariant control $\kappa_N(x)$ for all $i \in \mathbb{I}_{\geq 0}$. Thus the state and control trajectories $\mathbf{x}_{\mathrm{mpc}}(x)$ and $\mathbf{u}_{\mathrm{mpc}}(x)$ generated by MPC for an initial event $(x, 0)$ satisfy the difference equations

$$x(0) = x \qquad u(0) = \kappa_N(x)$$

$$x(i+1) = f(x(i), u(i)) \qquad u(i) = \kappa_N(x(i))$$

and can be seen to differ in general from $\mathbf{x}^0(x)$ and $\mathbf{u}^0(x)$, which satisfy (2.12) and (2.13).

Before leaving this section, we obtain some properties of the solution to each partial problem $\mathbb{P}_j(x)$. For this, we require a few definitions (Blanchini and Miani, 2008).

Definition 2.9 (Positive and control invariant sets).

(a) A set $X \subseteq \mathbb{R}^n$ is positive invariant for $x^+ = f(x)$ if $x \in X$ implies $f(x) \in X$.

(b) A set $X \subseteq \mathbb{R}^n$ is control invariant for $x^+ = f(x, u)$, $u \in \mathbb{U}$, if, for all $x \in X$, there exists a $u \in \mathbb{U}$ such that $f(x, u) \in X$.

We recall from our standing assumptions 2.2 and 2.3 that $f(\cdot)$, $\ell(\cdot)$ and $V_f(\cdot)$ are continuous, that $\mathbb{X}$ and $\mathbb{X}_f$ are closed, $\mathbb{U}$ is compact and that each of these sets contains the origin.

Proposition 2.10 (Existence of solutions to DP recursion)**.** *Suppose Assumptions 2.2 and 2.3 hold. Then*

(a) For all $j \geq 0$, the cost function $V_j(\cdot)$ is continuous in $\mathcal{Z}_j$, and, for each $x \in \mathcal{X}_j$, the control constraint set $\mathcal{U}_j(x)$ is compact and a solution $\mathbf{u}^0(x) \in \mathcal{U}_j(x)$ to $\mathbb{P}_j(x)$ exists.

(b) If $\mathcal{X}_0 := \mathbb{X}_f$ is control invariant for $x^+ = f(x, u)$, $u \in \mathbb{U}(x)$ and $0 \in \mathbb{X}_f$, then, for each $j \in \mathbb{I}_{\geq 0}$, the set $\mathcal{X}_j$ is also control invariant, $\mathcal{X}_j \supseteq \mathcal{X}_{j-1}$, and $0 \in \mathcal{X}_j$. In addition, the set $\mathcal{X}_N$ is positive invariant for $x^+ = f(x, \kappa_N(x))$.

(c) For all $j \in \mathbb{I}_{\geq 0}$, the set $\mathcal{X}_j$ is closed.

(d) If, in addition, $\mathbb{X}_f$ is compact and the function $f^{-1}(\cdot)$[4] is bounded on bounded sets ($f^{-1}(S)$ is bounded for every bounded set $S \in \mathbb{R}^n$), then, for all $j \in \mathbb{I}_{\geq 0}$, $\mathcal{X}_j$ is compact.

Proof.

(a) This proof is almost identical to the proof of Proposition 2.4.

(b) By assumption, $\mathcal{X}_0 = \mathbb{X}_f \subseteq \mathbb{X}$ is control invariant. By (2.11)

$$\mathcal{X}_1 = \{x \in \mathbb{X} \mid \exists u \in \mathbb{U}(x) \text{ such that } f(x, u) \in \mathcal{X}_0\}$$

Since $\mathcal{X}_0$ is control invariant for $x^+ = f(x, u), u \in \mathbb{U}$, for every $x \in \mathcal{X}_0$ there exist a $u \in \mathbb{U}$ such that $f(x, u) \in \mathcal{X}_0$ so that $x \in \mathcal{X}_1$. Hence $\mathcal{X}_1 \supseteq \mathcal{X}_0$. Since for every $x \in \mathcal{X}_1$, there exists a $u \in \mathbb{U}$ such that $f(x, u) \in \mathcal{X}_0 \subseteq \mathcal{X}_1$, it follows that $\mathcal{X}_1$ is control invariant for $x^+ = f(x, u)$, $u \in \mathbb{U}(x)$. If for some integer $j \in \mathbb{I}_{\geq 0}$, $\mathcal{X}_{j-1}$ is control invariant for $x^+ = f(x, u)$, it follows by similar reasoning that $\mathcal{X}_j \supseteq \mathcal{X}_{j-1}$ and that $\mathcal{X}_j$ is control invariant. By induction $\mathcal{X}_j$ is control invariant and $\mathcal{X}_j \supseteq \mathcal{X}_{j-1}$ for all $j > 0$. Hence $0 \in \mathcal{X}_j$ for all $j \in \mathbb{I}_{\geq 0}$. That $\mathcal{X}_N$ is positive invariant for $x^+ = f(x, \kappa_N(x))$ follows from (2.10), which shows that $\kappa_N(\cdot)$ steers every $x \in \mathcal{X}_N$ into $\mathcal{X}_{N-1} \subseteq \mathcal{X}_N$.

[4] For any $S \subset \mathbb{R}^n$, $f^{-1}(S) := \{z \in \mathbb{Z} \mid f(z) \in S\}$

(c) By Assumption 2.3, $\mathcal{X}_0 = \mathbb{X}_f$ is closed. Suppose, for some $j \in \mathbb{I}_{\geq 1}$, that $\mathcal{X}_{j-1}$ is closed. Then $\mathcal{Z}_j := \{(x, u) \in \mathbb{Z} \mid f(x, u) \in \mathcal{X}_{j-1}\}$ is closed since $f(\cdot)$ is continuous. To prove that $\mathcal{X}_j$ is closed, take any sequence $(x_i)_{i \in \mathbb{I}_{\geq 0}}$ in $\mathcal{X}_j$ that converges to, say, $\bar{x}$. For each i, select a $u_i \in \mathbb{U}(x_i)$ such that $z_i = (x_i, u_i) \in \mathcal{Z}_j$; this is possible since $x_i \in \mathcal{X}_j$ implies $x_i \in \mathbb{X} := \{x \in \mathbb{R}^n \mid \mathbb{U}(x) \neq \emptyset\}$. Since $\mathcal{Z}_j$ is closed, there exists a subsequence, indexed by $\mathbb{I}$, such that $z_i = (x_i, u_i) \to \bar{z} = (\bar{x}, \bar{u}) \in \mathcal{Z}_j$ as $i \to \infty$, $i \in \mathbb{I}$. But $\mathcal{X}_j = \{x \in \mathbb{X} \mid \exists\ u \in U(x)$ such that $f(x, u) \in \mathcal{X}_{j-1}\} = \{x \in \mathbb{X} \mid \exists\ u \in \mathbb{U}(x)$ such that $(x, u) \in \mathcal{Z}_j\}$ (see (2.11)). Hence $\bar{x} \in \mathcal{X}_j$ so that $\mathcal{X}_j$ is closed. By induction $\mathcal{X}_j$ is closed for all $j \in \mathbb{I}_{\geq 0}$.

(d) Since $\mathbb{X}_f$ and $\mathbb{U}$ are bounded, so is $\mathcal{Z}_1 \subset f^{-1}(X_f) := \{(x, u) \in \mathbb{Z} \mid f(x, u) \in \mathbb{X}_f\}$ and its projection $\mathcal{X}_1$ onto $\mathbb{R}^n$. Assume then, for some $j \in \mathbb{I}_{\geq 0}$ that $\mathcal{Z}_{j-1}$ is bounded; its projection $\mathcal{X}_{j-1}$ is also bounded. Consequently, $\mathcal{Z}_j \subset f^{-1}(\mathcal{X}_{j-1})$ is also bounded and so is its projection $\mathcal{X}_j$. By induction, $\mathcal{X}_j$ is bounded, and hence, compact, for all $j \in \mathbb{I}_{\geq 0}$.

■

Part (d) of Proposition 2.10 requires that the function $f^{-1}(\cdot)$ is bounded on bounded sets. This is a mild requirement if $f(\cdot)$ is the discrete time version of a continuous system as is almost always the case in process control. If the continuous time system satisfies $\dot{x} = f_c(x, u)$ and if the sample time is Δ, then

$$f(x, u) = x + \int_0^{\Delta} f_c(x(s; x), u)ds$$

in which $x(s; x)$ is the solution of $\dot{x} = f_c(x, u)$ at time s if $x(0) = x$ and u is the constant input in the interval $[0, \Delta]$. It is easily shown that $f^{-1}(\cdot)$ is bounded on bounded sets if $f(x, u) = Ax + Bu$ and A is nonsingular or if $f(\cdot)$ is Lipschitz in x (see Exercise 2.2).

The fact that $\mathcal{X}_N$ is positive invariant for $x^+ = f(x, \kappa_N(x))$ can also be established by observing that $\mathcal{X}_N$ is the set of states x in $\mathbb{X}$ for which there exists a $\mathbf{u}$ that is feasible for $\mathbb{P}_N(x)$, i.e., for which there exists a control $\mathbf{u}$ satisfying the control, state and terminal constraints. It is shown in the next section that for every $x \in \mathcal{X}_N$, there exists a feasible control sequence $\widetilde{\mathbf{u}}$ for $\mathbb{P}_N(x^+)$ ($x^+ = f(x, \kappa_N(x))$ is the successor state) provided that $\mathbb{X}_f$ is control invariant, i.e., $\mathcal{X}_N$ is positive invariant for $x^+ = f(x, \kappa_N(x))$ if $\mathbb{X}_f$ is control invariant. An important practical consequence is that if $\mathbb{P}_N(x(0))$ can be solved for the initial state $x(0)$, then $\mathbb{P}_N(x(i))$ can be solved for any subsequent state $x(i)$ of the controlled system $x^+ = f(x, \kappa_N(x))$, a property that is sometimes called recursive feasibility. Uncertainty, in the form of additive disturbances, model error or state estimation error, may destroy this important property; techniques to restore this property when uncertainty is present are discussed in Chapter 3.

2.4 Stability

2.4.1 Introduction

The classical definition of stability was employed in the first edition of this text. This states the origin in $\mathbb{R}^n$ is globally asymptotically stable (GAS) for $x^+ = f(x)$ if the origin is *locally stable* and if the origin is *globally attractive*. The origin is *locally stable* if, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $|x| < \delta$ implies $|\phi(k; x)| < \varepsilon$ for all $k \in \mathbb{I}_{\geq 0}$ (small perturbations of the initial state from the origin cause subsequent perturbations to be small). The origin is *globally attractive* for $x^+ = f(x)$ if $|\phi(k; x)| \to 0$ as $k \to \infty$ for all $x \in \mathbb{R}^n$. This definition of stability has been widely used and is equivalent to the recently defined stronger definition given below if $f(\cdot)$ is continuous but has some disadvantages; there exist examples of systems that are asymptotically stable (AS) in the classical sense in which small perturbations in the initial state from its initial value, not the origin, can cause subsequent perturbations to be arbitrarily large. Hence we employ in this section, as discussed more fully in Appendix B, a stronger definition of asymptotic stability that avoids this undesirable behavior.

To establish stability we make use of Lyapunov theorems that are defined in terms of the function classes $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$. A function belongs to class $\mathcal{K}$ if it is continuous, zero at zero, and strictly increasing; a function belongs to class $\mathcal{K}_\infty$ if it is in class $\mathcal{K}$ and unbounded; a function $\beta(\cdot)$ belongs to class $\mathcal{KL}$ if it is continuous and if, for each $k \geq 0$, $\beta(\cdot, k)$ is a class $\mathcal{K}$ function and for each $s \geq 0$, $\beta(s, \cdot)$ is nonincreasing and $\beta(s, i)$ converges to zero as $i \to \infty$. We can now state the stronger definition of stability.

Definition 2.11 (Asymptotically stable and GAS)**.** Suppose $\mathbb{X}$ is positive invariant for $x^+ = f(x)$. The origin is AS for $x^+ = f(x)$ in $\mathbb{X}$ if there exists a $\mathcal{KL}$ function $\beta(\cdot)$ such that, for each $x \in \mathbb{X}$

$$\phi(i; x) \leq \beta(|x|, i) \qquad \forall i \in \mathbb{I}_{\geq 0}$$

If $\mathbb{X} = \mathbb{R}^n$, the origin is GAS for $x^+ = f(x)$.

The set $\mathbb{X}$ is called a region of attraction. Energy in a passive electrical or mechanical system provides a useful analogy to Lyapunov stability theory. In a lumped mechanical system, the total stored energy, the sum of the potential and kinetic energy, is dissipated by friction and decays to zero at which point the dynamic system is in equilibrium. Lyapunov theory follows a similar path; if a real-valued function (a Lyapunov function) can be found that is positive and decreasing if the state is not the origin, then the state converges to the origin.

Definition 2.12 (Lyapunov function)**.** Suppose that $\mathbb{X}$ is positive invariant for $x^+ = f(x)$. A function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be a Lyapunov function in $\mathbb{X}$ for $x^+ = f(x)$ if there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a continuous, positive definite function α_3 such that for any $x \in \mathbb{X}$

$$V(x) \geq \alpha_1(|x|) \tag{2.14}$$

$$V(x) \leq \alpha_2(|x|) \tag{2.15}$$

$$V(f(x)) - V(x) \leq -\alpha_3(|x|) \tag{2.16}$$

We now employ the following stability theorem.

Theorem 2.13 (Lyapunov stability theorem)**.** *Suppose* $\mathbb{X} \subset \mathbb{R}^n$ *is positive invariant for* $x^+ = f(x)$*. If there exists a Lyapunov function in* $\mathbb{X}$ *for the system* $x^+ = f(x)$*, then the origin is asymptotically stable in* $\mathbb{X}$ *for* $x^+ = f(x)$*. If* $\mathbb{X} = \mathbb{R}^n$*, then the origin is globally asymptotically stable. If* $\alpha_i(|x|) = c_i |x|^a$*,* $a, c_i \in \mathbb{R}_{>0}$*,* $i = 1, 2, 3$*, then the origin is exponentially stable.*

A standard approach to establish stability is to employ the value function of an infinite horizon optimal control problem as a Lyapunov function. This suggests the use of $V_N^0(\cdot)$, the value function for the finite horizon optimal control problem whose solution yields the model predictive controller, as a Lyapunov function. It is simple to show, under mild assumptions on $\ell(\cdot)$, that $V_N^0(\cdot)$ has property (2.14) for all $x \in \mathcal{X}_N$. The value function $V_\infty(\cdot)$ for infinite horizon optimal control problems does satisfy, under mild conditions, $V_\infty^0(f(x, \kappa_\infty(x))) = V_\infty^0(x) - \ell(x, \kappa_\infty(x))$ thereby ensuring satisfaction of property (2.16). Since, as is often pointed out, optimality does not imply stability, this property does not usually hold when the horizon is finite. One of the main tasks of this chapter is show that if the ingredients $V_f(\cdot)$, $\ell(\cdot)$, and $\mathbb{X}_f$ of the finite horizon optimal control problem are chosen appropriately, then $V_N^0(f(x, \kappa_N(x))) \leq V_N^0(x) - \ell(x, \kappa_N(x))$ for all x in $\mathcal{X}_N$ enabling property (2.16) to be obtained. Property (2.15), an upper bound on the value function, is more difficult to establish but we also show that appropriate ingredients that ensures satisfaction of property (2.16) also ensures satisfaction of property (2.15).

We now address a point that we have glossed over. The solution to an optimization problem is not necessarily unique. Thus $\mathbf{u}^0(x)$ and $\kappa_N(x)$ may be set valued; any point in the set $\mathbf{u}^0(x)$ is a solution of $\mathbb{P}_N(x)$. Similarly $\mathbf{x}^0(x)$ is set valued. Uniqueness may be obtained by choosing that element in the set $\mathbf{u}^0(x)$ that has least norm; and if the minimum-norm solution is not unique, applying an arbitrary selection map in the set of minimum-norm solutions. To avoid expressions such as "let $\mathbf{u}$ be any element of the minimizing set $\mathbf{u}^0(x)$," we shall, in the sequel, use $\mathbf{u}^0(x)$ to denote any sequence in the set of minimizing sequences and use $\kappa_N(x)$ to denote $u^0(0;x)$, the first element of this sequence.

2.4.2 Stabilizing Conditions

To show that the value function $V_N^0(\cdot)$ is a valid Lyapunov function for the closed-loop system $x^+ = f(x, \kappa_N(x))$ we have to show that it satisfies (2.14), (2.15), and (2.16). We show below that $V_N^0(\cdot)$ is a valid Lyapunov function if, in addition to Assumptions 2.2 and 2.3, the following assumption is satisfied.

Assumption 2.14 (Basic stability assumption)**.** $V_f(\cdot)$, $\mathbb{X}_f$ and $\ell(\cdot)$ have the following properties:

(a) For all $x \in \mathbb{X}_f$, there exists a u (such that $(x,u) \in \mathbb{Z}$) satisfying

$$\begin{aligned} &f(x,u) \in \mathbb{X}_f \\ &V_f(f(x,u)) - V_f(x) \leq -\ell(x,u)r \end{aligned}$$

(b) There exist $\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$ and $\alpha_f(\cdot)$ satisfying

$$\begin{aligned} \ell(x,u) &\geq \alpha_1(|x|) && \forall x \in \mathcal{X}_N, \forall u \text{ such that } (x,u) \in \mathbb{Z} \\ V_f(x) &\leq \alpha_f(|x|) && \forall x \in \mathbb{X}_f \end{aligned}$$

We now show that $V_N^0(\cdot)$ is a Lyapunov function satisfying (2.14), (2.15), and (2.16) if Assumptions 2.2, 2.3, and 2.14 hold.

Lower bound for $V_N^0(\cdot)$. The lower bound property (2.14) is easily obtained. Since $V_N^0(x) \geq \ell(x, \kappa_N(x))$ for all $x \in \mathcal{X}_N$, the lower bound (2.14) follows from Assumption 2.14(b) in which it is assumed that there exists a $\mathcal{K}_\infty$ function $\alpha_1(\cdot)$ such that $\ell(x, u) \geq \alpha_1(|x|)$ for all $x \in \mathcal{X}_N$, for all u such that $(x, u) \in \mathbb{Z}$. This assumption is satisfied by the usual choice $\ell(x, u) = (1/2)(x'Qx + u'Ru)$ with Q and R positive definite. Condition (2.14) is satisfied.

Upper bound for $V_N^0(\cdot)$. If $\mathbb{X}_f$ contains the origin in its interior, the upper bound property (2.15) can be established as follows. We show below in Proposition 2.18 that, under Assumption 2.14, $V_j^0(x) \leq V_f(x)$ for all $x \in \mathbb{X}_f$, all $j \in \mathbb{I}_{\geq 0}$. Also, under the same Assumption, there exists a $\mathcal{K}_\infty$ function $\alpha_f(\cdot)$ such that $V_f(x) \leq \alpha_f(|x|)$ for all $x \in \mathbb{X}_f$. It follows that $V_N^0(\cdot)$ has the same upper bound $\alpha_f(|x|)$ in $\mathbb{X}_f$. We now have to show that this bound on $V_N^0(\cdot)$ in $\mathbb{X}_f$ can be extended to a similar bound on $V_N^0(\cdot)$ in $\mathcal{X}_N$. We do this through two propositions. The first proposition proves that the value function $V_N^0(\cdot)$ is locally bounded.

Proposition 2.15 (The value function $V_N^0(\cdot)$ is locally bounded)**.** *Suppose Assumptions 2.2 and 2.3 hold. Then $V_N^0(\cdot)$ is locally bounded on $\mathcal{X}_N$.*

Proof. Let X be an arbitrary compact subset of $\mathcal{X}_N$. The function $V_N : \mathbb{R}^n \times \mathbb{R}^{Nm} \to \mathbb{R}_{\geq 0}$ is continuous and therefore has an upper bound on the compact set $X \times \mathbb{U}^N$. Since $\mathcal{U}_N(x) \subset \mathbb{U}^N$ for all $x \in \mathcal{X}_N$, $V_N^0 : \mathcal{X}_N \to \mathbb{R}_{\geq 0}$ has the same upper bound on X. Since X is arbitrary, V_N^0 is locally bounded on $\mathcal{X}_N$. ■

The second proposition shows the upper bound of $V_N^0(\cdot)$ in $\mathcal{X}_f$ implies the existence of a similar upper bound in the larger set $\mathcal{X}_N$.

Proposition 2.16 (Extension of upper bound to $\mathcal{X}_N$)**.** *Suppose Assumptions 2.2 and 2.3 hold and that $\mathbb{X}_f \subseteq \mathbb{X}$ is control invariant for $x^+ = f(x, u)$, $u \in \mathbb{U}(x)$ and contains the origin in its interior. Suppose also that there exists a $\mathcal{K}_\infty$ function $\alpha(\cdot)$ such that $V_f(x) \leq \alpha(|x|)$ for all $x \in \mathbb{X}_f$. Then there exists a $\mathcal{K}_\infty$ function $\alpha_2(\cdot)$ such that*

$$V_N^0(x) \leq \alpha_2(|x|) \ \forall x \in \mathcal{X}_N$$

Proof. We have that $0 \leq V_N^0(x) \leq V_f(x) \leq \alpha(|x|)$ for $x \in \mathbb{X}_f$ which contains a neighborhood of zero. Therefore $V_N^0(\cdot)$ is continuous at zero. The set $\mathcal{X}_N$ is closed, and V_N^0 is locally bounded on $\mathcal{X}_N$. Therefore Proposition B.25 of Appendix B applies, and the result is established. ■

In situations where $\mathbb{X}_f$ does not have an interior, such as when $\mathbb{X}_f = \{0\}$, we cannot establish an upper bound for $V_N^0(\cdot)$ and resort to the following assumption.

Assumption 2.17 (Weak controllability)**.** There exists a $\mathcal{K}_\infty$ function $\alpha(\cdot)$ such that

$$V_N^0(x) \leq \alpha(|x|) \ \forall x \in \mathcal{X}_N$$

Assumption 2.17 is weaker than a controllability assumption. It confines attention to those states that can be steered to $\mathbb{X}_f$ in N steps and merely requires that the cost of doing so is not excessive.

Descent property for $V_N^0(\cdot)$. Let x be any state in $\mathcal{X}_N$ at time zero. Then

$$V_N^0(x) = V_N(x, \mathbf{u}^0(x))$$

in which

$$\mathbf{u}^0(x) = \left(u^0(0;x), u^0(1;x), \ldots, u^0(N-1;x)\right)$$

is any minimizing control sequence. The resultant optimal state sequence is

$$\mathbf{x}^0(x) = \left(x^0(0;x), x^0(1;x), \ldots, x^0(N;x)\right)$$

in which $x^0(0;x) = x$ and $x^0(1;x) = x^+$. The successor state to x at time zero is $x^+ = f(x, \kappa_N(x)) = x^0(1;x)$ at time 1 where $\kappa_N(x) = u^0(0;x)$, and

$$V_N^0(x^+) = V_N(x^+, \mathbf{u}^0(x^+))$$

in which

$$\mathbf{u}^0(x^+) = \left(u^0(0;x^+), u^0(1;x^+), \ldots, u^0(N-1;x^+)\right)$$

It is difficult to compare $V_N^0(x)$ and $V_N^0(x^+)$ directly, but

$$V_N^0(x^+) = V_N(x^+, \mathbf{u}^0(x^+)) \leq V_N(x^+, \tilde{\mathbf{u}})$$

where $\tilde{\mathbf{u}}$ is any feasible control sequence for $\mathbb{P}_N(x^+)$, i.e., any control sequence in $\mathbb{U}^N$. To facilitate comparison of $V_N(x^+, \tilde{\mathbf{u}})$ with $V_N^0(x) = V_N(x, \mathbf{u}^0(x))$, we choose

$$\tilde{\mathbf{u}} = \left(u^0(1;x), \ldots, u^0(N-1;x), u\right)$$

in which $u \in \mathbb{U}$ still has to be chosen. Comparing $\tilde{\mathbf{u}}$ with $\mathbf{u}^0(x)$ shows that $\tilde{\mathbf{x}}$, the state sequence due to control sequence $\tilde{\mathbf{u}}$, is

$$\tilde{\mathbf{x}} = \Big(x^0(1;x), x^0(2;x), \ldots, x^0(N;x), f(x^0(N;x), u)\Big)$$

in which $x^0(1;x) = x^+ = f(x, \kappa_N(x))$. Since there are no state or terminal constraints, the state sequence $\tilde{\mathbf{x}}$ is clearly feasible if $u \in \mathbb{U}$. Because $\mathbf{x}^0$ coincides with $\tilde{\mathbf{x}}$ and $\mathbf{u}(\cdot)$ coincides with $\tilde{\mathbf{u}}$ for $i = 1, 2, \ldots, N-1$ (but not for $i = N$), a simple calculation yields

$$V_N(x^+, \tilde{\mathbf{u}}) = \sum_{j=1}^{N-1} \ell(x^0(j;x), u^0(j;x)) + \ell(x^0(N;x)) + V_f(f(x^0(N;x), u))$$

But

$$\begin{aligned} V_N^0(x) &= V_N(x, \mathbf{u}^0(x)) \\ &= \ell(x, \kappa_N(x)) + \sum_{j=1}^{N-1} \ell(x^0(j;x), u^0(j;x)) + V_f(x^0(N;x)) \end{aligned}$$

so that

$$\sum_{j=1}^{N-1} \ell(x^0(j;x), u^0(j;x)) = V_N^0(x) - \ell(x, \kappa_N(x)) - V_f(x^0(N;x))$$

Hence

$$\begin{aligned} V_N^0(x) \leq V_N(x^+, \tilde{\mathbf{u}}) = V_N^0(x) - \ell(x, \kappa_N(x)) - V_f(x^0(N;x)) + \\ \ell(x^0(N;x), u) + V_f(f(x^0(N;x), u)) \end{aligned}$$

It follows that

$$V_N^0(f(x, \kappa_N(x))) \leq V_N^0(x) - \ell(x, \kappa_N(x)) \tag{2.17}$$

for all $x \in \mathbb{R}^n$ *if* the function $V_f(\cdot)$ and the set $\mathbb{X}_f$ have the property that, for all $x \in \mathbb{X}_f$, there exists a $u \in \mathbb{R}^m$ such that if, for all $x \in \mathbb{X}_f$, there exists a u such that

$$(x, u) \in \mathbb{Z}, \ V_f(f(x, u)) \leq V_f(x) - \ell(x, u), \text{ and } f(x, u) \in \mathbb{X}_f \tag{2.18}$$

But this condition is satisfied by the stabilizing condition, Assumption 2.14. Since $\ell(x, \kappa_N(x)) \geq \alpha_1(|x|)$ for all $x \in \mathbb{X}$, $V_N^0(\cdot)$ has the desired descent property (2.16).

To complete the proof that the value function satisfies (2.14), (2.15), and (2.16), we have to prove the assertion, made in obtaining the upper bound for $V_N^0(\cdot)$, that $V_j^0(x) \leq V_f(x)$ for all $x \in \mathbb{X}_f$, all $j \in \mathbb{I}_{\geq 0}$. This assertion follows from the monotonicity property of the value function $V_N^0(\cdot)$. This interesting result was first obtained for the unconstrained linear quadratic optimal control problem.

Proposition 2.18 (Monotonicity of the value function)**.** *Suppose that Assumptions 2.2, 2.3, and 2.14 hold. Then*

$$V_{j+1}^0(x) \leq V_j^0(x) \;\; \forall x \in \mathcal{X}_j, \;\; \forall j \in \mathbb{I}_{\geq 0}$$

and

$$V_j^0(x) \leq V_f(x) \;\; \forall x \in \mathcal{X}_f, \;\; \forall j \in \mathbb{I}_{\geq 0}$$

Proof. From the DP recursion (2.9)

$$V_1^0(x) = \min_{u \in \mathbb{U}(x)} \{\ell(x,u) + V_0^0(f(x,u)) \mid f(x,u) \in \mathcal{X}_0\}$$

But $V_0^0(\cdot) := V_f(\cdot)$ and $\mathcal{X}_0 := \mathbb{X}_f$. Also, by Assumption 2.14

$$\min_{u \in \mathbb{U}(x)} \{\ell(x,u) + V_f(f(x,u)) \mid f(x,u) \in \mathbb{X}_f\} \leq V_f(x) \qquad \forall x \in \mathbb{X}_f$$

so that

$$V_1^0(x) \leq V_0^0(x) \qquad \forall x \in \mathcal{X}_0 = \mathbb{X}_f$$

Next, suppose that for some $j \geq 1$

$$V_j^0(x) \leq V_{j-1}^0(x) \qquad \forall x \in \mathcal{X}_{j-1}$$

Then, using the DP equation (2.9)

$$\begin{aligned} V_{j+1}^0(x) - V_j^0(x) = \ell(x, \kappa_{j+1}(x)) + V_j^0(f(x, \kappa_{j+1}(x))) \\ - \ell(x, \kappa_j(x)) - V_{j-1}^0(f(x, \kappa_j(x))) \qquad \forall x \in \mathcal{X}_j \subseteq \mathcal{X}_{j+1} \end{aligned}$$

Since $\kappa_j(x)$ may *not* be optimal for $\mathbb{P}_{j+1}(x)$ for all $x \in \mathcal{X}_j \subseteq \mathcal{X}_{j+1}$, we have

$$\begin{aligned} V_{j+1}^0(x) - V_j^0(x) \leq \ell(x, \kappa_j(x)) + V_j^0(f(x, \kappa_j(x))) \\ - \ell(x, \kappa_j(x)) - V_{j-1}^0(f(x, \kappa_j(x))) \qquad \forall x \in \mathcal{X}_j \end{aligned}$$

Also, from (2.11), $x \in \mathcal{X}_j$ implies $f(x, \kappa_j(x)) \in \mathcal{X}_{j-1}$ so that, by assumption, $V_j^0(f(x, \kappa_j(x))) \leq V_{j-1}^0(f(x, \kappa_j(x)))$ for all $x \in \mathcal{X}_j$. Hence

$$V_{j+1}^0(x) \leq V_j^0(x) \qquad \forall x \in \mathcal{X}_j$$

By induction

$$V_{j+1}^0(x) \leq V_j^0(x) \qquad \forall x \in \mathcal{X}_j, \ \forall j \in \mathbb{I}_{\geq 0}$$

Since the set sequence $(\mathcal{X}_j)_{\mathbb{I}_{\geq 0}}$ has the nested property $\mathcal{X}_j \subset \mathcal{X}_{j+1}$ for all $j \in \mathbb{I}_{\geq 0}$, it follows that $V_j^0(x) \leq V_f(x)$ for all $x \in \mathbb{X}_f$, all $j \in \mathbb{I}_{\geq 0}$. ∎

The monotonicity property Proposition 2.18 also holds even if $\mathbb{U}(x)$ is not compact provided that the minimizer in the DP recursion always exists; this is the case for the linear-quadratic problem.

The monotonicity property can be used to establish the descent property of $V_N^0(\cdot)$ proved in Theorem 2.19 by noting that

$$\begin{aligned} V_N^0(x) &= \ell(x, \kappa_N(x)) + V_{N-1}^0(f(x, \kappa_N(x))) \\ &= \ell(x, \kappa_N(x)) + V_N^0(f(x, \kappa_N(x))) + \\ &\quad [V_{N-1}^0(f(x, \kappa_N(x))) - V_N^0(f(x, \kappa_N(x)))] \end{aligned}$$

so that using the monotonicity property

$$\begin{aligned} V_N^0(f(x, \kappa_N(x))) &= V_N^0(x) - \ell(x, \kappa_N(x)) + \\ &\qquad [V_N^0(f(x, \kappa_N(x))) - V_{N-1}^0(f(x, \kappa_N(x)))] \\ &\leq V_N^0(x) - \ell(x, \kappa_N(x)) \qquad \forall x \in \mathcal{X}_N \end{aligned}$$

which is the desired descent property.

Since inequalities (2.14), (2.15), and (2.16) are all satisfied we have proved

Theorem 2.19 (Asymptotic stability of the origin). *Suppose Assumptions 2.2, 2.3, 2.14, and 2.17 are satisfied. Then*

(a) There exists $\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ such that

$$\begin{aligned} \alpha_1(|x|) \leq V_N^0(x) &\leq \alpha_2(|x|) \\ V_N^0(f(x, \kappa_N(x))) - V_N^0(x) &\leq -\alpha_1(|x|) \end{aligned}$$

for all $x \in \mathcal{X}_N$.

(b) The origin is asymptotically stable in $\mathcal{X}_N$ for $x^+ = f(x, \kappa_N(x))$.

As discussed above, Assumption 2.17 is immediate if the origin lies in the interior of $\mathbb{X}_f$. In other cases, e.g., when the stabilizing ingredient is the terminal equality constraint $x(N) = 0$ ($\mathbb{X}_f = \{0\}$), Assumption 2.17 is taken directly. See Proposition 2.38 for some additional circumstances in which Assumption 2.17 is satisfied.

2.4.3 Exponential Stability

Exponential stability is defined as follows.

Definition 2.20 (Exponential stability)**.** Suppose $X \subseteq \mathbb{R}^n$ is positive invariant for $x^+ = f(x)$. The origin is exponentially stable for $x^+ = f(x)$ in X if there exist $c \in \mathbb{R}_{>0}$ and $\gamma \in (0, 1)$ such that

$$\left|\phi(i; x)\right| \leq c\,|x|\,\gamma^i$$

for all $x \in X$, all $i \in \mathbb{I}_{\geq 0}$.

Theorem 2.21 (Lyapunov function and exponential stability)**.** *Suppose $X \subset \mathbb{R}^n$ is positive invariant for $x^+ = f(x)$. If there exists a Lyapunov function in X for the system $x^+ = f(x)$ with $\alpha_i(\cdot) = c_i\,|\cdot|^a$ in which a, $c_i \in \mathbb{R}_{>0}$ $i = 1, 2, 3$, then the origin is exponentially stable for $x^+ = f(x)$ in X.*

The proof of this result is left as an exercise.

2.4.4 Controllability and Observability

We have not yet made any assumptions on controllability (stabilizability) or observability (detectability) of the system (2.1) being controlled, which may be puzzling since such assumptions are commonly required in optimal control to, for example, establish existence of a solution to the optimal control problem. The reasons for this omission are that such assumptions are implicitly required, at least locally, for the basic stability Assumption 2.14, and that we restrict attention to $\mathcal{X}_N$, the set of states that can be steered to $\mathbb{X}_f$ in N steps satisfying all constraints.

Stage cost $\ell(\cdot)$ not positive definite. In the previous stability analysis we assume that the function $(x, u) \mapsto \ell(x, u)$ is positive definite; more precisely, we assume that there exists a $\mathcal{K}_\infty$ function $\alpha_1(\cdot)$ such that $\ell(x, u) \geq \alpha_1(|x|)$ for all (x, u). Often we assume that $\ell(\cdot)$ is quadratic, satisfying $\ell(x, u) = (1/2)(x'Qx + u'Ru)$ where Q and R

are positive definite. In this section we consider the case where the stage cost is $\ell(y, u)$ where $y = h(x)$ and the function $h(\cdot)$ is not necessarily invertible. An example is the quadratic stage cost $\ell(y, u) = (1/2)(y' Q_y y + u' R u)$ where Q_y and R are positive definite, $y = Cx$, and C is not invertible; hence the stage cost is $(1/2)(x' Q x + u' R u)$ where $Q = C' Q_y C$ is merely positive semidefinite. Since now $\ell(\cdot)$ does not satisfy $\ell(x, u) \geq \alpha_1(|x|)$ for all $(x, u) \in \mathbb{Z}$ and some $\mathcal{K}_\infty$ function $\alpha_1(\cdot)$, we have to make an additional assumption in order to establish asymptotic stability of the origin for the closed-loop system. An appropriate assumption is input/output-to-state-stability (IOSS), which ensures the state goes to zero as the input and output go to zero. We recall Definition B.51, restated here.

Definition 2.22 (Input/output-to-state stable (IOSS))**.** The system $x^+ = f(x, u)$, $y = h(x)$ is IOSS if there exist functions $\beta(\cdot) \in \mathcal{KL}$ and $\gamma_1(\cdot)$, $\gamma_2(\cdot) \in \mathcal{K}$ such that for every initial state $x \in \mathbb{R}^n$, every control sequence $\mathbf{u}$, and all $i \geq 0$

$$|x(i)| \leq \max\{\beta(|x|, i), \gamma_1(\|\mathbf{u}\|_{0:i-1}), \gamma_2(\|\mathbf{y}\|_{0:i})\}$$

in which $x(i) := \phi(i; x, \mathbf{u})$ is the solution of $x^+ = f(x, u)$ at time i if the initial state is x and the input sequence is $\mathbf{u}$; $y(i) := h(x(i))$ is the output, and $\|\mathbf{d}\|_{a:b} := \max_{a \leq j \leq b} |d(j)|$ denotes the max norm of a sequence.

Note that for linear systems, IOSS is equivalent to detectability of (A, C) (see Exercise 4.5).

We assume as usual that Assumptions 2.2 and 2.3 are satisfied, but we replace Assumption 2.14 by the following.

Assumption 2.23 (Modified basic stability assumption)**.** $V_f(\cdot)$, $\mathbb{X}_f$ and $\ell(\cdot)$ have the following properties.

(a) For all $x \in \mathbb{X}_f$, there exists a u (such that $(x, u) \in \mathbb{Z}$) satisfying

$$V_f(f(x, u)) - V_f(x) \leq -\ell(h(x), u), \quad f(x, u) \in \mathbb{X}_f$$

(b) There exist $\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$ and $\alpha_f(\cdot)$ satisfying

$$\begin{aligned} \ell(y, u) &\geq \alpha_1(|(y, u)|) && \forall (y, u) \in \mathbb{R}^p \times \mathbb{R}^m \\ V_f(x) &\leq \alpha_f(|x|) && \forall x \in \mathbb{X}_f \end{aligned}$$

Note that in the modification of Assumption 2.14 we have changed only the lower bound inequality for stage cost $\ell(y, u)$. With these assumptions we can then establish asymptotic stability of the origin.

Theorem 2.24 (Asymptotic stability with stage cost $\ell(y, u)$)**.** *Suppose Assumptions 2.2, 2.3, 2.17 and 2.23 are satisfied, and the system $x^+ = f(x, u), y = h(x)$ is IOSS. Then there exists a Lyapunov function in $\mathcal{X}_N$ for the closed-loop system $x^+ = f(x, \kappa_N(x))$, and the origin is asymptotically stable in $\mathcal{X}_N$.*

Proof. Assumptions 2.2, 2.3, and 2.23(a) guarantee the existence of the optimal solution of the MPC problem and the positive invariance of $\mathcal{X}_N$ for $x^+ = f(x, \kappa_N(x))$, but the nonpositive definite stage cost gives the following modified inequalities

$$\ell(h(x), u) \leq V_N^0(x) \leq \alpha_2(|x|)$$
$$V_N^0(f(x, \kappa_N(x))) - V_N^0(x) \leq -\ell(h(x), u)$$

so $V_N^0(\cdot)$ is no longer a Lyapunov function for the closed-loop system. Because the system is IOSS and $\ell(y, u) \geq \alpha_1(\left|(y, u)\right|)$, however, Theorem B.53 in Appendix B provides that for *any* $\gamma(\cdot) \in \mathcal{K}_\infty$ there exists an IOSS-Lyapunov function $\Lambda(\cdot)$ for which the following holds for all $(x, u) \in \mathbb{Z}$ for which $f(x, u) \in \mathbb{X}$

$$\gamma_1(|x|) \leq \Lambda(x) \leq \gamma_2(|x|)$$
$$\Lambda(f(x, u)) - \Lambda(x) \leq -\rho(|x|) + \gamma(\ell(h(x), u))$$

with $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ and *continuous* function $\rho \in \mathcal{PD}$. Note that these inequalities certainly apply for $u = \kappa_N(x)$ since $(x, \kappa_N(x)) \in \mathbb{Z}$ and $f(x, \kappa_N(x)) \in \mathcal{X}_N \subseteq \mathbb{X}$. Therefore we choose the linear $\mathcal{K}_\infty$ function $\gamma(\cdot) = (\cdot)$, take $V(\cdot) = V_N^0(\cdot) + \Lambda(\cdot)$ as our candidate Lyapunov function, and obtain for all $x \in \mathcal{X}_N$

$$\overline{\alpha}_1(|x|) \leq V(x) \leq \overline{\alpha}_2(|x|)$$
$$V(f(x, \kappa_N(x))) - V(x) \leq -\rho(|x|)$$

with $\mathcal{K}_\infty$ functions $\overline{\alpha}_1(\cdot) := \gamma_1(\cdot)$ and $\overline{\alpha}_2(\cdot) := \alpha_2(\cdot) + \gamma_2(\cdot)$. From Definition 2.12, $V(\cdot)$ is a Lyapunov function in $\mathcal{X}_N$ for the system $x^+ = f(x, \kappa_N(x))$. Therefore the origin is asymptotically stable in $\mathcal{X}_N$ from Theorem 2.13. ■

Note that we have here the appearance of a Lyapunov function that is *not* the optimal value function of the MPC regulation problem. In earlier MPC literature, observability rather than detectability was often employed as the extra assumption required to establish asymptotic stability. Exercise 2.14 discusses that approach.

2.4.5 Time-Varying Systems

Most of the control problems discussed in this book are time invariant. Time-varying problems do arise in practice, however, even if the system being controlled is time invariant. One example occurs when an observer or filter is used to estimate the state of the system being controlled since bounds on the state estimation error are often time varying. In the deterministic case, for example, state estimation error decays exponentially to zero. Another example occurs when the desired equilibrium is not a state-control pair (x_s, u_s) but a periodic trajectory. In this section, which may be omitted in the first reading, we show how MPC may be employed for a class of time-varying systems.

The problem. The time-varying nonlinear system is described by

$$x^+ = f(x, u, i)$$

where x is the current state at time i, u the current control, and x^+ the successor state at time $i + 1$. For each integer i, the function $f(\cdot, i)$ is assumed to be continuous. The solution of this system at time $k \geq i$ given that the initial state is x at time i is denoted by $\phi(k; x, \mathbf{u}, i)$; the solution now depends on both the time i and current time k rather than merely on the difference $k - i$ as in the time-invariant case. The cost $V_N(x, \mathbf{u}, i)$ also depends on time i and is defined by

$$V_N(x, \mathbf{u}, i) := \sum_{k=i}^{i+N-1} \ell(x(k), u(k), k) + V_f(x(i+N), i+N)$$

in which $x(k) := \phi(k; x, \mathbf{u}, i)$, $\mathbf{u} = (u(i), u(i+1), \ldots, u(i+N-1))$, and the stage cost $\ell(\cdot)$ and terminal cost $V_f(\cdot)$ are time varying. The state and control constraints are also time varying

$$x(i) \in \mathbb{X}(k) \qquad u(k) \in \mathbb{U}(i)$$

for all i. In addition, there is a time-varying terminal constraint

$$x(i+N) \in \mathbb{X}_f(i+N)$$

in which i is the current time. The time-varying optimal control problem at event (x, i) is $\mathbb{P}_N(x, i)$ defined by

$$\mathbb{P}_N(x, i): \quad V_N^0(x, i) = \min\{V_N(x, \mathbf{u}, i) \mid \mathbf{u} \in \mathcal{U}_N(x, i)\}$$

in which $\mathcal{U}_N(x,i)$ is the set of control sequences $\mathbf{u} = \big((u(i), u(i+1), \ldots, u(i+N-1)\big)$ satisfying the state, control and terminal constraints, i.e.,

$$\mathcal{U}_N(x,i) := \{\mathbf{u} \mid (x,\mathbf{u}) \in \mathbb{Z}_N(i)\}$$

in which, for each i, $\mathbb{Z}_N(i) \subset \mathbb{R}^n \times \mathbb{R}^{Nm}$ is defined by

$$\mathbb{Z}_N(i) := \big\{(x,\mathbf{u}) \mid u(k) \in \mathbb{U}(k), \quad \phi(k;x,\mathbf{u},i) \in \mathbb{X}(k), \forall k \in \mathbb{I}_{i:i+N-1}, \\ \phi(i+N;x,\mathbf{u},i) \in \mathbb{X}_f(i+N)\big\}$$

For each time i, the domain of $V_N^0(\cdot,i)$ is $\mathcal{X}_N(i)$ where

$$\begin{aligned}\mathcal{X}_N(i) &:= \{x \in \mathbb{X}(i) \mid \mathcal{U}_N(x,i) \neq \emptyset\} \\ &= \{x \in \mathbb{R}^n \mid \exists \mathbf{u} \text{ such that } (x,\mathbf{u}) \in \mathbb{Z}_N(i)\}\end{aligned}$$

which is the projection of $\mathbb{Z}_N(i)$ onto $\mathbb{R}^n$. Our standing assumptions (2.2 and 2.3) are replaced, in the time-varying case, by

Assumption 2.25 (Continuity of system and cost; time-varying case)**.** The functions $(x,u) \mapsto f(x,u,i)$, $(x,u) \mapsto \ell(x,u,i)$ and $(x,u) \mapsto V_f(x,u,i)$ are continuous for all $i \in \mathbb{I}_{\geq 0}$. Also, for all $i \in \mathbb{I}_{\geq 0}$, $f(0,0,i) = 0$, $\ell(0,0,i) = 0$ and $V_f(0,i) = 0$.

Assumption 2.26 (Properties of constraint sets; time-varying case)**.** For each $i \in \mathbb{I}_{\geq 0}$, $\mathbb{X}(i)$ and $\mathbb{X}_f(i)$ are closed, $\mathbb{X}_f(i) \subset \mathbb{X}(i)$ and $\mathbb{U}(i)$ are compact; the sets $\mathbb{U}(i), i \in \mathbb{I}_{\geq 0}$ are uniformly bounded by the compact set $\bar{\mathbb{U}}$. Each set contains the origin.

In making these assumptions we are implicitly assuming, as before, that the desired setpoint has been shifted to the origin, but in this case, it need not be constant in time. For example, letting $\bar{x}$ and $\bar{u}$ be the original positional variables, we can consider a time-varying reference trajectory $(\bar{x}_r(i), \bar{u}_r(i))$ by defining $x(i) := \bar{x}(i) - \bar{x}_r(i)$ and $u(i) := \bar{u}(i) - \bar{u}_r(i)$. Depending on the application, $\bar{x}_r(i)$ and $\bar{u}_r(i)$ could be constant, periodic, or generally time varying. In any case, because of the time-varying nature of the problem, we need to extend our definitions of invariance and control invariance.

Definition 2.27 (Sequential positive invariance and sequential control invariance)**.**

(a) A sequence of sets $(X(i))_{i\geq 0}$ is *sequentially positive invariant* for the system $x^+ = f(x,i)$ if for any $i \geq 0$, $x \in X(i)$ implies $f(x,i) \in X(i+1)$.

(b) A sequence of sets $(X(i))_{i\geq 0}$ is *sequentially control invariant* for the system $x^+ = f(x,u,i)$ if for any $i \geq 0$ and $x \in X(i)$, there exists a $u \in \mathbb{U}(i)$ such that $x^+ = f(x,u,i) \in X(i+1)$.

Let $(\mathcal{X}(i))_{i\geq 0}$ be sequentially positive invariant. If $x \in \mathcal{X}(i_0)$ for some $i_0 \geq 0$, then $\phi(i;x,i_0) \in \mathcal{X}(i)$ for all $i \geq i_0$.

The following results, which are analogs of the results for time-invariant systems given previously, are stated without proof.

Proposition 2.28 (Continuous system solution; time-varying case)**.** *Suppose Assumptions 2.25 and 2.26 are satisfied. For each initial time $i_0 \geq 0$ and final time $i \geq i_0$, the function $(x,\mathbf{u}) \mapsto \phi(i;x,\mathbf{u},i_0)$ is continuous.*

Proposition 2.29 (Existence of solution to optimal control problem; time-varying case)**.** *Suppose Assumptions 2.25 and 2.26 are satisfied. Then for each time $i \in \mathbb{I}_{\geq 0}$*

(a) The function $(x,\mathbf{u}) \mapsto V_N(x,\mathbf{u},i)$ is continuous in $\mathbb{Z}_N(i)$.

(b) For each $x \in \mathcal{X}_N(i)$, the control constraint set $\mathcal{U}_N(x,i)$ is compact.

(c) For each $x \in \mathcal{X}_N(i)$, a solution to $\mathbb{P}_N(x,i)$ exists.

(d) $\mathcal{X}_N(i)$ is closed and $x = 0 \in \mathcal{X}_N(i)$.

(e) If $\left(\mathbb{X}_f(i)\right)_{i\in\mathbb{I}_{\geq 0}}$ is sequentially control invariant for $x^+ = f(x,u,i)$, then $(\mathcal{X}_N(i))_{i\in\mathbb{I}_{\geq 0}}$ is sequentially control invariant for $x^+ = f(x,u,i)$ and sequentially positive invariant for $x^+ = f(x,\kappa_N(x,i),i)$.

Stability. Our definitions of AS (asymptotic stability) and GAS (global asymptotic stability) also require slight modification for the time-varying case.

Definition 2.30 (Asymptotically stable and GAS for time-varying systems)**.** Suppose that the sequence $(X(i))_{i\geq 0}$ is sequentially positive invariant for $x^+ = f(x,i)$. The origin is *asymptotically stable* in the sequence $(X(i))_{i\geq 0}$ for $x^+ = f(x,i)$ if the following holds for all $i \geq i_0 \geq 0$, and $x \in X(i_0)$

$$\left|\phi(i;x,i_0)\right| \leq \beta(|x|, i - i_0) \tag{2.19}$$

in which $\beta \in \mathcal{KL}$ and $\phi(i;x,i_0)$ is the solution to $x^+ = f(x,i)$ at time $i \geq i_0$ with initial condition x at time $i_0 \geq 0$. If $X(i) = \mathbb{R}^n$, the origin is *globally asymptotically stable* (GAS).

This definition is somewhat restrictive because $\left|\phi(i,x,i_0)\right|$ depends on $i - i_0$ rather than on i.

Definition 2.31 (Lyapunov function: time-varying, constrained case)**.** Let the sequence $(X(i))_{i\geq 0}$ be sequentially positive invariant, and let $V(\cdot, i) : X(i) \to \mathbb{R}_{\geq 0}$ satisfy for all $x \in X(i)$, $i \in \mathbb{I}_{\geq 0}$

$$\begin{aligned} \alpha_1(|x|_{\mathcal{A}}) \leq V(x,i) &\leq \quad \alpha_2(|x|_{\mathcal{A}}) \\ \Delta V(x,i) &\leq -\alpha_3(|x|_{\mathcal{A}}) \end{aligned}$$

with $\Delta V(x,i) := V(f(x,i), i+1) - V(x,i)$, $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$. Then the function $V(\cdot,\cdot)$ is a time-varying Lyapunov function in the sequence $(X(i))_{i\geq 0}$ for $x^+ = f(x,i)$.

This definition requires a single, time-invariant bound for each $\alpha_j(\cdot)$, $j \in \{1,2,3\}$, which is not overly restrictive. For example, supposing there is a sequence of lower bounds $\left(\alpha_1^i(\cdot)\right)_{\mathbb{I}_{\geq 0}}$, it is necessary only that the infimum

$$\alpha_1(\cdot) := \inf_{i \in \mathbb{I}_{\geq 0}} \alpha_1^i(\cdot)$$

is class $\mathcal{K}_\infty$. If the system is time invariant or periodic, this property is satisfied (as the inf becomes a min over a finite set), but it does preclude bounds that become arbitrarily flat, such as $\alpha_1^i(s) = \frac{1}{i+1}s^2$. A similar argument holds for $j \in \{2,3\}$ (using sup instead of inf for $j = 2$). We can now state a stability definition that we employ in this chapter

Theorem 2.32 (Lyapunov theorem for asymptotic stability (time-varying, constrained))**.** *Let the sequence $(X(i))_{\geq 0}$ be sequentially positive invariant for the system $x^+ = f(x,i)$, and $V(\cdot,\cdot)$ be a time-varying Lyapunov function in the sequence $(X(i))_{\geq 0}$ for $x^+ = f(x,i)$. Then the origin is asymptotically stable in $X(i)$ at each time $i \geq 0$ for $x^+ = f(x,i)$.*

The proof of this theorem is given in Appendix B (see Theorem B.24).

Model predictive control of time-varying systems. As before, the receding horizon control law $\kappa_N(\cdot)$, which is now time varying, is not necessarily optimal or stabilizing. By choosing the time-varying ingredients $V_f(\cdot)$ and $\mathbb{X}_f$ in the optimal control problem appropriately, however, stability can be ensured, as we now show. We replace the basic stability assumption 2.14 by its time-varying extension.

Assumption 2.33 (Basic stability assumption; time-varying case)**.**

(a) For all $i \in \mathbb{I}_{\geq 0}$, all $x \in \mathbb{X}_f(i)$, there exists a $u \in \mathbb{U}(i)$ such that

$$\begin{aligned} &f(x,u,i) \in \mathbb{X}_f(i+1) \\ &V_f(f(x,u,i), i+1) - V_f(x,i) \leq -\ell(x,u,i) \end{aligned}$$

(b) There exist $\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$ and $\alpha_f(\cdot)$ satisfying

$$\ell(x,u,i) \geq \alpha_1(|x|)\ \forall x \in \mathcal{X}_N(i), \forall u \text{ such that } (x,u) \in \mathbb{Z}_N(i), \forall i \in \mathbb{I}_{\geq 0}$$
$$V_f(x,i) \leq \alpha_f(|x|),\ \forall x \in \mathbb{X}_f(i), \forall i \in \mathbb{I}_{\geq 0}$$

As in the case of the time-varying Lyapunov function, requiring time-invariant bounds is typically not restrictive. A direct consequence of Assumption 2.33 is the descent property given in the following proposition.

Proposition 2.34 (Optimal cost decrease; time-varying case)**.** *Suppose Assumptions 2.25, 2.26, and 2.33 hold. Then*

$$V_N^0(f(x,\kappa_N(x,i),i),i+1) \leq V_N^0(x,i) - \ell(x,\kappa_N(x,i),i) \tag{2.20}$$

for all $x \in \mathcal{X}_N(i)$, all $i \in \mathbb{I}_{\geq 0}$.

Proposition 2.35 (MPC cost is less than terminal cost)**.** *Suppose Assumptions 2.25, 2.26, and 2.33 hold. Then*

$$V_N^0(x,i) \leq V_f(x,i) \qquad \forall x \in \mathbb{X}_f(i), \quad \forall i \in \mathbb{I}_{\geq 0}$$

The proofs of Propositions 2.34 and 2.35 are left as Exercises 2.9 and 2.10.

Proposition 2.36 (Optimal value function properties; time-varying case)**.** *Suppose Assumptions 2.25, 2.26, and 2.33 are satisfied. Then there exist two $\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ such that, for all $i \in \mathbb{I}_{\geq 0}$*

$$\begin{aligned} V_N^0(x,i) &\geq \alpha_1(|x|) && \forall x \in \mathcal{X}_N(i) \\ V_N^0(x,i) &\leq \alpha_2(|x|) && \forall x \in \mathbb{X}_f(i) \\ V_N^0(f(x,\kappa_N(x,i),i+1)) - V_N^0(x,i) &\leq -\alpha_1(|x|) && \forall x \in \mathcal{X}_N(i) \end{aligned}$$

We can deal with the obstacle posed by the fact that the upper bound on $V_N^0(\cdot)$ holds only in $\mathbb{X}_f(i)$ in much the same way as we did previously for the time-invariant case. In general, we invoke the following assumption.

Assumption 2.37 (Uniform weak controllability)**.** There exists a $\mathcal{K}_\infty$ function $\alpha(\cdot)$ such that

$$V_N^0(x,i) \leq \alpha(|x|) \qquad \forall x \in \mathcal{X}_N(i),\ \forall i \in \mathbb{I}_{\geq 0}$$

It can be shown that Assumption 2.37 holds in a variety of other circumstances as described in the following proposition.

Proposition 2.38 (Conditions for uniform weak controllability)**.** *Suppose the functions $f(\cdot)$, $\ell(\cdot)$, and $V_f(\cdot)$ are uniformly bounded for all $i \in \mathbb{I}_{\geq 0}$, i.e., on any compact set $Z \subset \mathbb{R}^n \times \bar{\mathbb{U}}$, the set*

$$\{(f(x,u,i), \ell(x,u,i), V_f(x,i)) \mid (x,u) \in Z, i \in \mathbb{I}_{\geq 0}\}$$

is bounded. Assumption 2.37 is satisfied if any of the following conditions holds:

(a) There exists a neighborhood of the origin X such that $X \subseteq \mathbb{X}_f(i)$ for each $i \in \mathbb{I}_{\geq 0}$

(b) For $i \in \mathbb{I}_{\geq 0}$, the optimal value function $V_N^0(x,i)$ is uniformly continuous in x at $x = 0$

(c) There exists a neighborhood of the origin X and a $\mathcal{K}$ function $\alpha(\cdot)$ such that $V_N^0(x,i) \leq \alpha(|x|)$ for all $i \in \mathbb{I}_{\geq 0}$ and $x \in X \cap \mathcal{X}_N(i)$

(d) The functions $f(\cdot)$ and $\ell(\cdot)$ are uniformly continuous at the origin $(x,u) = (0,0)$ for all $i \in \mathbb{I}_{\geq 0}$, and the system is stabilizable with small inputs, i.e., there exists a $\mathcal{K}_\infty$ function $\gamma(\cdot)$ such that for all $i \in \mathbb{I}_{\geq 0}$ and $x \in \mathcal{X}_N(i)$, there exists $\mathbf{u} \in \mathcal{U}_N(x,i)$ with $|\mathbf{u}| \leq \gamma(|x|)$.

Proof.

(a) Similar to Proposition 2.16, one can show that the optimal cost

$$V_N^0(x,i) \leq V_f(x,i) \leq \alpha_2(|x|) \qquad \text{for all } x \in X$$

Thus, condition (c) is implied.

(b) From uniform continuity, we know that for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x| \leq \delta \quad \text{implies} \quad V_N^0(x,i) \leq \varepsilon \quad \text{for all } i \in \mathbb{I}_{\geq 0}$$

recalling that $V_N^0(\cdot)$ is nonnegative and zero at the origin. By Rawlings and Risbeck (2015, Proposition 13), this is equivalent to the existence of a $\mathcal{K}$ function $\gamma(\cdot)$ defined on $[0,b]$ (with $b > 0$) such that

$$V_N^0(x,i) \leq \gamma(|x|) \quad \text{for all } x \in X$$

with $X := \{x \in \mathbb{R}^n \mid |x| \leq b\}$ a neighborhood of the origin. Thus, condition (c) is also implied.

(c) First, we know that $V_N(\cdot)$ is uniformly bounded because it is the finite sum and composition of the uniformly bounded functions $f(\cdot)$, $\ell(\cdot)$, and $V_f(\cdot)$. Thus, $V_N^0(\cdot)$ is also uniformly bounded, because

$$0 \leq V_N^0(x,i) \leq V_N(x,\mathbf{u},i) \quad \text{for all } \mathbf{u} \in \mathcal{U}_N(x,i)$$

and $V_N(\cdot)$ is uniformly bounded. Next, because X is a neighborhood of the origin, there exists $b_0 > 0$ such that $V_N^0(x,i) \leq \alpha(|x|)$ whenever $x \in \mathcal{X}_N(i)$ and $|x| \leq b_0$. Following Rawlings and Risbeck (2015, Proposition 14), we choose any strictly increasing and unbounded sequence $(b_k)_{k=0}^{\infty}$ and define

$$B_k(i) := \{x \in \mathcal{X}_N(i) \mid |x| \leq b_k\}$$

We then compute a new sequence $(\beta_k)_{k=0}^{\infty}$ as

$$\beta_k := k + \sup_{\substack{i \in \mathbb{I}_{\geq 0} \\ x \in B_k(i)}} V_N^0(x,i)$$

We know that this sequence is well-defined because $V_N^0(x,i)$ is uniformly bounded for $i \in \mathbb{I}_{\geq 0}$ on $\bigcup_{i \in \mathbb{I}_{\geq 0}} B_k(i)$. We then define

$$\alpha(s) := \begin{cases} \dfrac{\beta_1}{\gamma(b_0)}\gamma(s) & s \in [0, b_0) \\ \beta_{k+1} + (\beta_{k+2} - \beta_{k+1})\dfrac{s - b_i}{b_{i+1} - b_i} & s \in [b_k, b_{k+1}) \text{ for all } k \in \mathbb{I}_{\geq 0} \end{cases}$$

which is a $\mathcal{K}_\infty$ function that satisfies

$$V_N^0(x,i) \leq \alpha(|x|) \quad \text{for all } i \in \mathbb{I}_{\geq 0}$$

as desired.

(d) See Exercise 2.22. Note that the uniform continuity of $f(\cdot)$ and $\ell(\cdot)$ implies the existence of $\mathcal{K}$ function upper bounds of the form

$$\begin{aligned} \left|f(x,u,i)\right| &\leq \alpha_{fx}(|x|) + \alpha_{fu}(|u|) \\ \ell(x,u,i) &\leq \alpha_{\ell x}(|x|) + \alpha_{\ell u}(|u|) \end{aligned}$$

for all $i \in \mathbb{I}_{\geq 0}$. ■

Hence, if Assumptions 2.25, 2.26, 2.33, and 2.37 hold it follows from Proposition 2.36 that, for all $i \in \mathbb{I}_{\geq 0}$, all $x \in \mathcal{X}_N(i)$

$$\begin{aligned} \alpha_1(|x|) \leq V_N^0(x,i) &\leq \alpha_2(|x|) \\ V_N^0(f(x,\kappa_N(x,i),i+1)) - V_N^0(x,i) &\leq -\alpha_1(|x|) \end{aligned} \tag{2.21}$$

Assumption	Title	Page
2.2	Continuity of system and cost	97
2.3	Properties of constraint sets	98
2.14	Basic stability assumption	114
2.17	Weak controllability	116

Table 2.1: Stability assumptions; time-invariant case.

so that, by Definition 2.31, $V_N^0(\cdot)$ is a time-varying Lyapunov function in the sequence $(X(i))_{i\geq 0}$ for $x^+ = f(x, \kappa_N(x,i), i)$. It can be shown, by a slight extension of the arguments employed in the time-invariant case, that problem $\mathbb{P}_N(\cdot)$ is recursively feasible and that $(\mathcal{X}_N(i))_{i\in\mathbb{I}_{\geq 0}}$ is sequentially positive invariant for the system $x^+ = f(x, \kappa_N(x,i))$. The sequence $(\mathcal{X}_N(i))_{i\geq 0}$ in the time-varying case replaces the set $\mathcal{X}_N$ in the time-invariant case.

Theorem 2.39 (Asymptotic stability of the origin: time-varying MPC)**.** *Suppose Assumptions 2.25, 2.26, 2.33, and 2.37 holds. Then,*

(a) There exist $\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ such that, for all $i \in \mathbb{I}_{\geq 0}$ and all $x \in \mathcal{X}_{\mathcal{N}}(i)$, inequalities (2.21) *are satisfied.*

(b) The origin is asymptotically stable in $\mathcal{X}_N(i)$ at each time $i \geq 0$ for the time-varying system $x+ = f(x, \kappa_N(x,i), i)$.

Proof.

(a) It follows from Assumptions 2.25, 2.26, 2.33, and 2.37 and Proposition 2.36 that $V_N^0(\cdot)$ satisfies the inequalities (2.21).

(b) It follows from (a) and definition 2.31 that $V_N^0(\cdot)$ is a time-varying Lyapunov function. It follows from Theorem 2.32 that the origin is asymptotically stable in $\mathcal{X}_N(i)$ at each time $i \geq 0$ for the time-varying system $x^+ = f(x, \kappa_N(x,i), i)$.

■

2.5 Examples of MPC

We already have discussed the general principles underlying the design of stabilizing model predictive controllers. The stabilizing conditions on $\mathbb{X}_f$, $\ell(\cdot)$, and $V_f(\cdot)$ that guarantee stability can be implemented

Assumption	Title	Page
2.25	Continuity of system and cost	124
2.26	Properties of constraint sets	124
2.33	Basic stability assumption	126
2.37	Uniform weak controllability	127

Table 2.2: Stability assumptions; time-varying case.

in a variety of ways so that MPC can take many different forms. We present the most useful forms of MPC for applications. These examples also display the roles of the three main assumptions used to guarantee closed-loop asymptotic stability. These assumptions are summarized in Table 2.1 for the time-invariant case, and Table 2.2 for the time-varying case. Referring back to these tables may prove helpful while reading this section and comparing the various forms of MPC.

One question that is often asked is whether or not the terminal constraint is necessary. Since the conditions given previously are sufficient, necessity cannot be claimed. We discuss this further later. It is evident that the constraint arises because one often has a local, rather than a global, control Lyapunov function (CLF) for the system being controlled. In a few situations, a global CLF *is* available, in which case a terminal constraint is not necessary.

All model predictive controllers determine the control action to be applied to the system being controlled by solving, at each state, an optimal control problem that is usually constrained. If the constraints in the optimal control problem include hard state constraints, then the feasible region $\mathcal{X}_N$ is a subset of $\mathbb{R}^n$. The analysis given previously shows that if the initial state $x(0)$ lies in $\mathcal{X}_N$, so do all subsequent states, a property known as *recursive feasibility*. It is always possible, however, for unanticipated events to cause the state to become infeasible. In this case, the optimal control problem, as stated, cannot be solved, and the controller fails. It is therefore desirable, if this does not conflict with design aims, to employ soft state constraints in place of hard constraints. Otherwise, any implementation of the algorithms described subsequently should be modified to include a feature that enables recovery from faults that cause infeasibility. One remedy is to replace the hard constraints by soft constraints when the current state is infeasible, thereby restoring feasibility, and to revert back to the hard constraints as soon as they can be satisfied at the current state.

In establishing stability of the examples of MPC presented below, we make use of Theorem 2.19 (or Theorem 2.24) for time-invariant systems and Theorem 2.39 for time-varying systems. We must therefore establish that Assumptions 2.2, 2.3, and 2.14 are satisfied in the time-invariant case and that Assumptions 2.25, 2.26, and 2.33 are satisfied in the time-varying case. We normally assume that 2.2, 2.3, and 2.14(b) or 2.25, 2.26, and 2.33(b) are satisfied, so our main task below in each example is establishing satisfaction of the basic stability assumption (cost decrease) 2.14(a) or 2.33(a).

2.5.1 The Unconstrained Linear Quadratic Regulator

Consider the linear, time-invariant model $x^+ = Ax + Bu$, $y = Cx$ with quadratic penalties on output and state $\ell(y,u) = (1/2)(y'Q_y y + u'Ru)$ for both the finite and infinite horizon cases. We first consider what the assumptions of Theorem 2.24 imply in this case, and compare these assumptions to the standard LQR assumptions (listed in Exercise 1.20(b)).

Assumptions 2.2 is satisfied for $f(x,u) = Ax + Bu$ and $\ell(x,u) = (1/2)(C'Q_y C + u'Ru)$ for all A, B, C, Q_y, R. Assumption 2.3 is satisfied with $\mathbb{Z} = \mathbb{R}^n \times \mathbb{R}^m$ and $R > 0$. Assumption 2.23 implies that $Q_y > 0$ as well. The system being IOSS implies that (A,C) is detectable (see Exercise 4.5). We can choose $\mathbb{X}_f$ to be the stabilizable subspace of (A,B) and Assumption 2.23(a) is satisfied. The set $\mathcal{X}_N$ contains the system controllability information. The set $\mathcal{X}_N$ is the stabilizable subspace of (A,B), and we can satisfy Assumption 2.23(b) by choosing $V_f(x) = (1/2)x'\Pi x$ in which Π is the solution to the steady-state Riccati equation for the stabilizable modes of (A,B).

In particular, if (A,B) is stabilizable, then $V_f(\cdot)$ can be chosen to be $V_f(x) = (1/2)x'\Pi x$ in which Π is the solution to the steady-state Riccati equation (1.18), which is positive definite. The terminal set can be taken as any (arbitrarily large) sublevel set of the terminal penalty, $\mathbb{X}_f = \text{lev}_a V_f$, $a > 0$, so that any point in $\mathbb{R}^n$ is in $\mathbb{X}_f$ for large enough a. We then have $\mathcal{X}_N = \mathbb{R}^n$ for all $N \in \mathbb{I}_{0:\infty}$. The horizon N can be finite or infinite with this choice of $V_f(\cdot)$ and the control law is invariant with respect to the horizon length, $\kappa_N(x) = Kx$ in which K is the steady-state linear quadratic regulator gain given in (1.18). Theorem 2.24 then gives that the origin of the closed-loop system $x^+ = f(x, \kappa_N(x)) = (A + BK)x$ is globally, asymptotically stable. This can be strengthened to globally, *exponentially* stable because of the choice of quadratic stage cost and form of the resulting Lyapunov function in Theorem 2.24.

The standard assumptions for the LQR with stage cost $l(y,u) = (1/2)(y'Q_y y + u'Ru)$ are

$$Q_y > 0 \quad R > 0 \quad (A, C) \text{ detectable} \quad (A, B) \text{ stabilizable}$$

and we see that LQ theory establishes that the standard steady-state LQR is covered by Theorem 2.24. Summarizing we have

> Given the standard LQR problem, Assumptions 2.2, 2.3, and 2.23 are satisfied and $\mathcal{X}_N = \mathbb{X}_f = \mathbb{R}^n$. It follows from Theorem 2.24 and 2.21 that the origin is globally, exponentially stable for the controlled system $x^+ = Ax + B\kappa_N(x) = (A + BK)x$.

2.5.2 Unconstrained Linear Periodic Systems

In the special case where the system is time varying but periodic, a global CLF can be determined as in the LQR case. Suppose the objective function is

$$\ell(x,u,i) := \frac{1}{2}\left(x'Q(i)x + u'R(i)u\right)$$

with each $Q(i)$ and $R(i)$ positive definite. To start, choose a sequence of linear control laws

$$\kappa_f(x,i) := K(i)x$$

and let

$$A_K(i) := A(i) + B(i)K(i)$$
$$Q_K(i) := Q(i) + K(i)'R(i)K(i)$$

For integers m and n satisfying $m \geq n \geq 0$, let

$$\mathcal{A}(m,n) := A_K(m-1)A_K(m-2)\cdots A_K(n+1)A_K(n)$$

Given these matrices, the closed-loop evolution of the system under the terminal control law is

$$x(m) = \mathcal{A}(m,n)x(n)$$

for $f(x,u,i) = f(x,\kappa_f(u,i),i) = A_K(i)x$.

Suppose the periodic system $(A(i), B(i))$ is stabilizable. It follows that the control laws $K(i)$ can be chosen so that each $\mathcal{A}(i+T, i)$ is stable. Such control laws can be found, e.g., by iterating the periodic discrete algebraic Riccati equation or by solving the Riccati equation for a larger, time-invariant system (see Exercise 2.23).

For a terminal cost, we require matrices $P(i)$ that satisfy

$$A_K(i)'P(i+1)A_K(i) + Q_K(i) = P(i)$$

Summing this relationship for $i \in \mathbb{I}_{0:T-1}$ gives

$$\mathcal{A}(T,0)'P(T)\mathcal{A}(T,0) + \sum_{i=0}^{T-1} \mathcal{A}(k,0)'Q(k)\mathcal{A}(k,0) = P(0)$$

and by periodicity, $P(T) = P(0)$. Noting that $\mathcal{A}(T,0)$ is stable and the summation is positive definite (recall that the first term $|\mathcal{A}(0,0)|^2_{Q(0)} = Q(0)$ is positive definite), there exists a unique solution to this equation, and the remaining $P(i)$ are determined by the recurrence relationship. Thus, taking

$$V_f(x,i) = \frac{1}{2}x'P(i)x$$

we have, for $u = \kappa_f(x,i) = K(i)x$

$$\begin{aligned} V_f(f(x,u,i), i+1) + \ell(x,u,i) &= \frac{1}{2}x'A_K(i)'P(i+1)A_K(i)x + \\ &\frac{1}{2}x'Q_K(i)x = \frac{1}{2}x'P(i)x \leq V_f(x,i) \end{aligned}$$

as required. The terminal region can then be taken as $\mathbb{X}_f(i) = \mathbb{R}^n$. Summarizing we have

> If the periodic system is stabilizable, there exists a periodic sequence of controller gains and terminal penalties such that $\mathcal{X}_N(i) = \mathbb{X}_f(i) = \mathbb{R}^n$ for all $i \geq 0$. The origin is globally asymptotically stable by Theorem 2.39, which can be strengthened to globally exponentially stable due to the quadratic stage cost. The function $V_f(\cdot, i)$ is a global, time-varying CLF.

2.5.3 Stable Linear Systems with Control Constraints

Usually, when constraints and/or nonlinearities are present, it is impossible to obtain a *global* CLF to serve as the terminal cost function $V_f(\cdot)$. There are, however, a few special cases where this is possible, such as the stable linear system.

The system to be controlled is $x^+ = Ax + Bu$ where A is stable (its eigenvalues lie strictly inside the unit circle) and the control u is subject to the constraint $u \in \mathbb{U}$ where $\mathbb{U}$ is compact and contains the origin in its interior. The stage cost is $\ell(x,u) = (1/2)(x'Qx + u'Ru)$ where Q and R are positive definite. To establish stability of the systems under MPC, we wish to obtain a *global* CLF to serve as the terminal cost function $V_f(\cdot)$. This is usually difficult because any linear control law $u = Kx$, say, transgresses the control constraint for x sufficiently large. In other words, it is usually impossible to find a $V_f(\cdot)$ such that there exists a $u \in \mathbb{U}$ satisfying $V_f(Ax + Bu) \leq V_f(x) - \ell(x,u)$ for all x in $\mathbb{R}^n$. Since A is stable, however, it is possible to obtain a Lyapunov function for the autonomous system $x^+ = Ax$ that is a suitable candidate for $V_f(\cdot)$; in fact, for all $Q > 0$, there exists a $P > 0$ such that

$$A'PA + Q = P$$

Let $V_f(\cdot)$ be defined by

$$V_f(x) = (1/2)x'Px$$

With $f(\cdot)$, $\ell(\cdot)$, and $V_f(\cdot)$ defined thus, $\mathbb{P}_N(x)$ is a parametric quadratic problem if the constraint set $\mathbb{U}$ is polyhedral and global solutions may be computed online. The terminal cost function $V_f(\cdot)$ satisfies

$$V_f(Ax) + (1/2)x'Qx - V_f(x) = (1/2)x'(A'PA + Q - P)x = 0$$

for all $x \in \mathbb{X}_f := \mathbb{R}^n$. We see that for all $x \in \mathbb{X}_f$, there exists a u, namely $u = 0$, such that $V_f(Ax + Bu) \leq V_f(x) - \ell(x,u)$; $\ell(x,u) = (1/2)x'Qx$ when $u = 0$. Since there are no state or terminal constraints, $\mathcal{X}_N = \mathbb{R}^n$. It follows that there exist positive constants c_1 and c_2 such that

$$c_1 |x|^2 \leq V_N^0(x) \leq c_2 |x|^2$$
$$V_N^0(f(x, \kappa_N(x))) - V_N^0(x) \leq -c_1 |x|^2$$

for all $x \in \mathcal{X}_N = \mathbb{R}^n$. Summarizing, we have

> Assumptions 2.2, 2.3, and 2.14 are satisfied and $\mathcal{X}_N = \mathbb{X}_f = \mathbb{R}^n$. It follows from Theorems 2.19 and 2.21 that the origin is globally, exponentially stable for the controlled system $x^+ = Ax + B\kappa_N(x)$.

An extension of this approach for unstable A is used in Chapter 6.

2.5.4 Linear Systems with Control and State Constraints

We turn now to the consideration of systems with control and state constraints. In this situation determination of a global CLF is usually difficult if not impossible. Hence we show how local CLFs may be determined together with an invariant region in which they are valid.

The system to be controlled is $x^+ = Ax + Bu$ where A is not necessarily stable, the control u is subject to the constraint $u \in \mathbb{U}$ where $\mathbb{U}$ is compact and contains the origin in its interior, and the state x is subject to the constraint $x \in \mathbb{X}$ where $\mathbb{X}$ is closed and contains the origin in its interior. The stage cost is $\ell(x,u) = (1/2)(x'Qx + u'Ru)$ where Q and R are positive definite. Because of the constraints, it is difficult to obtain a global CLF. Hence we restrict ourselves to the more modest goal of obtaining a local CLF and proceed as follows. If (A,B) is stabilizable, the solution to the infinite horizon *unconstrained* optimal control problem $\mathbb{P}_\infty^{\text{uc}}(x)$ is known. The value function for this problem is $V_\infty^{\text{uc}}(x) = (1/2)x'Px$ where P is the unique (in the class of positive semidefinite matrices) solution to the discrete algebraic Riccati equation

$$P = A_K' P A_K + Q_K$$

in which $A_K := A + BK$, $Q_K := Q + K'RK$, and $u = Kx$, in which K is defined by

$$K := -(B'PB + R)^{-1}B'PA'$$

is the optimal controller. The value function $V_\infty^{\text{uc}}(\cdot)$ for the infinite horizon unconstrained optimal control problem $\mathbb{P}_\infty^{\text{uc}}(x)$ satisfies

$$V_\infty^{\text{uc}}(x) = \min_u \{\ell(x,u) + V_\infty^{\text{uc}}(Ax + Bu)\} = \ell(x, Kx) + V_\infty^{\text{uc}}(A_K x)$$

It is known that P is positive definite. We define the terminal cost $V_f(\cdot)$ by

$$V_f(x) := V_\infty^{\text{uc}}(x) = (1/2)x'Px$$

If $\mathbb{X}$ and $\mathbb{U}$ are polyhedral, problem $\mathbb{P}_N(x)$ is a parametric quadratic program that may be solved online using standard software. The terminal cost function $V_f(\cdot)$ satisfies

$$V_f(A_K x) + (1/2)x'Q_K x - V_f(x) \leq 0 \;\; \forall x \in \mathbb{R}^n$$

The controller $u = Kx$ does not necessarily satisfy the control and state constraints, however. The terminal constraint set $\mathbb{X}_f$ must be chosen with this requirement in mind. We may choose $\mathbb{X}_f$ to be the maximal invariant constraint admissible set for $x^+ = A_K x$; this is the largest set W with respect to inclusion[5] satisfying: (a) $W \subseteq \{x \in \mathbb{X} \mid Kx \in \mathbb{U}\}$, and (b) $x \in W$ implies $x(i) = A_K^i x \in W$ for all $i \geq 0$. Thus $\mathbb{X}_f$, defined this way, is control invariant for $x^+ = Ax + Bu$, $u \in \mathbb{U}$. If the initial state x of the system is in $\mathbb{X}_f$, the controller $u = Kx$ maintains the state in $\mathbb{X}_f$ and satisfies the state and control constraints for all future time ($x(i) = A_K^i x \in \mathbb{X}_f \subset \mathbb{X}$ and $u(i) = Kx(i) \in \mathbb{U}$ for all $i \geq 0$). Hence, with $V_f(\cdot)$, $\mathbb{X}_f$, and $\ell(\cdot)$ as defined previously, Assumptions 2.2, 2.3, and 2.14 are satisfied. Summarizing, we have

> Assumptions 2.2, 2.3, and 2.14 are satisfied, and $\mathbb{X}_f$ contains the origin in its interior. Hence, by Theorems 2.19 and 2.21, the origin is exponentially stable in $\mathcal{X}_N$.

It is, of course, not necessary to choose K and $V_f(\cdot)$ as above. Any K such that $A_K = A + BK$ is stable may be chosen, and P may be obtained by solving the Lyapunov equation $A_K' P A_K + Q_K = P$. With $V_f(x) := (1/2)x'Px$ and $\mathbb{X}_f$ the maximal constraint admissible set for $x^+ = A_K x$, the origin may be shown, as above, to be asymptotically stable with a region of attraction $\mathcal{X}_N$ for $x^+ = Ax + B\kappa_N(x)$, and exponentially stable with a region of attraction any sublevel set of $V_N^0(\cdot)$. The optimal control problem is, again, a quadratic program. The terminal set $\mathbb{X}_f$ may be chosen, as above, to be the maximal invariant constraint admissible set for $x^+ = A_K x$, or it may be chosen to be a suitably small sublevel set of $V_f(\cdot)$; by suitably small, we mean small enough to ensure $\mathbb{X}_f \subseteq \mathbb{X}$ and $K\mathbb{X}_f \subseteq \mathbb{U}$. The set $\mathbb{X}_f$, if chosen this way, is ellipsoidal, a subset of the maximal constraint admissible set, and is positive invariant for $x^+ = A_K x$. The disadvantage of this choice is that $\mathbb{P}_N(x)$ is no longer a quadratic program, though it remains a convex program for which software exists.

The choice $V_f(\cdot) = V_\infty^{\mathrm{uc}}(\cdot)$ results in an interesting property of the closed-loop system $x^+ = Ax + B\kappa_N(x)$. Generally, the terminal constraint set $\mathbb{X}_f$ is *not* positive invariant for the controlled system $x^+ = Ax + B\kappa_N(x)$. Thus, in solving $\mathbb{P}_N(x)$ for an initial state $x \in \mathbb{X}_f$, the "predicted" state sequence $\mathbf{x}^0(x) = (x^0(0;x), x^0(1;x), \ldots, x^0(N;x))$ starts and ends in $\mathbb{X}_f$ but does not necessarily remain in $\mathbb{X}_f$. Thus $x^0(0;x) = x \in \mathbb{X}_f$ and $x^0(N;x) \in \mathbb{X}_f$, because of the terminal constraint in the optimal control problem, but, for any $i \in \mathbb{I}_{1:N-1}$, $x^0(i;x)$ may lie outside of $\mathbb{X}_f$. In particular, $x^+ = Ax + B\kappa_N(x) = x^0(1;x)$

[5] $W \in \mathcal{W}$ is the largest set in $\mathcal{W}$ with respect to inclusion if $W' \subseteq W$ for any $W' \in \mathcal{W}$.

may lie outside of $\mathbb{X}_f$; $\mathbb{X}_f$ is *not* necessarily positive invariant for the controlled system $x^+ = Ax + B\kappa_N(x)$.

Consider now the problem $\mathbb{P}_N^{\text{uc}}(x)$ defined in the same way as $\mathbb{P}_N(x)$ except that *all* constraints are omitted so that $\mathcal{U}_N(x) = \mathbb{R}^{Nm}$

$$\mathbb{P}_N^{\text{uc}}(x): \quad V_N^{\text{uc}}(x) = \min_{\mathbf{u}} V_N(x, \mathbf{u})$$

in which $V_N(\cdot)$ is defined as previously by

$$V_N(x, \mathbf{u}) := \sum_{i=0}^{N-1} \ell(x(i), u(i)) + V_f(x(N))$$

with $V_f(\cdot)$ the value function for the infinite horizon unconstrained optimal control problem, i.e., $V_f(x) := V_\infty^{\text{uc}}(x) = (1/2)x'Px$. With these definitions, it follows that

$$V_N^{\text{uc}}(x) = V_\infty^{\text{uc}}(x) = V_f(x) = (1/2)x'Px$$
$$\kappa_N^{\text{uc}}(x) = Kx, \quad K = -(B'PB + R)^{-1}B'PA$$

for all $x \in \mathbb{R}^n$; $u = Kx$ is the optimal controller for the unconstrained infinite horizon problem. But $\mathbb{X}_f$ *is* positive invariant for $x^+ = A_K x$.

We now claim that with $V_f(\cdot)$ chosen to equal to $V_\infty^{\text{uc}}(\cdot)$, the terminal constraint set $\mathbb{X}_f$ *is* positive invariant for $x^+ = Ax + B\kappa_N(x)$. We do this by showing that $V_N^0(x) = V_N^{\text{uc}}(x) = V_\infty^{\text{uc}}(x)$ for all $x \in \mathbb{X}_f$, so that the associated control laws are the same, i.e., $\kappa_N(x) = Kx$. First, because $\mathbb{P}_N^{\text{uc}}(x)$ is identical with $\mathbb{P}_N(x)$ except for the absence of all constraints, we have

$$V_N^{\text{uc}}(x) = V_f(x) \leq V_N^0(x) \quad \forall x \in \mathcal{X}_N \supseteq \mathbb{X}_f$$

Second, from Lemma 2.18

$$V_N^0(x) \leq V_f(x) \quad \forall x \in \mathbb{X}_f$$

Hence $V_N^0(x) = V_N^{\text{uc}}(x) = V_f(x)$ for all $x \in \mathbb{X}_f$. That $\kappa_N(x) = Kx$ for all $x \in \mathbb{X}_f$ follows from the uniqueness of the solutions to the problems $\mathbb{P}_N(x)$ and $\mathbb{P}_N^{\text{uc}}(x)$. Summarizing, we have

> If $V_f(\cdot)$ is chosen to be the value function for the unconstrained infinite horizon optimal control problem, if $u = Kx$ is the associated controller, and if $\mathbb{X}_f$ is invariant for $x^+ = A_K x$, then $\mathbb{X}_f$ is also positive invariant for the controlled system $x^+ = Ax + B\kappa_N(x)$. Also $\kappa_N(x) = Kx$ for all $x \in \mathbb{X}_f$.

2.5.5 Constrained Nonlinear Systems

The system to be controlled is

$$x^+ = f(x, u)$$

in which $f(\cdot)$ is assumed to be twice continuously differentiable. The system is subject to state and control constraints

$$x \in \mathbb{X} \qquad u \in \mathbb{U}$$

in which $\mathbb{X}$ is closed and $\mathbb{U}$ is compact; each set contains the origin in its interior. The cost function is defined by

$$V_N(x, \mathbf{u}) = \sum_{i=0}^{N-1} \ell(x(i), u(i)) + V_f(x(N))$$

in which, for each i, $x(i) := \phi(i; x, \mathbf{u})$, the solution of $x^+ = f(x, u)$ at time i if the initial state is x at time zero and the control is $\mathbf{u}$. The stage cost $\ell(\cdot)$ is defined by

$$\ell(x, u) := (1/2)(|x|_Q^2 + |u|_R^2)$$

in which Q and R are positive definite. The optimal control problem $\mathbb{P}_N(x)$ is defined by

$$\mathbb{P}_N(x): \quad V_N^0(x) = \min_{\mathbf{u}}\{V_N(x, \mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_N(x)\}$$

in which $\mathcal{U}_N(x)$ is defined by (2.5) and includes the terminal constraint $x(N) = \phi(N; x, \mathbf{u}) \in \mathbb{X}_f$ (in addition to the state and control constraints).

Our first task is to choose the ingredients $V_f(\cdot)$ and $\mathbb{X}_f$ of the optimal control problem to ensure asymptotic stability of the origin for the controlled system. We obtain a terminal cost function $V_f(\cdot)$ and a terminal constraint set $\mathbb{X}_f$ by linearization of the nonlinear system $x^+ = f(x, u)$ at the origin. Hence we assume $f(\cdot)$ and $\ell(\cdot)$ are twice continuously differentiable so that Assumption 2.2 is satisfied . Suppose then that the linearized system is

$$x^+ = Ax + Bu$$

where $A := f_x(0, 0)$ and $B := f_u(0, 0)$. We assume that (A, B) is stabilizable and we choose any controller $u = Kx$ such that the origin is globally exponentially stable for the system $x^+ = A_K x$, $A_K := A + BK$, i.e., such that A_K is stable. Suppose also that the stage cost $\ell(\cdot)$ is defined by $\ell(x,u) := (1/2)(|x|_Q^2 + |u|_R^2)$ where Q and R are positive definite; hence $\ell(x,Kx) = (1/2)x'Q_K x$ where $Q_K := (Q + K'RK)$. Let P be defined by the Lyapunov equation

$$A_K' P A_K + \mu Q_K = P$$

for some $\mu > 1$ The reason for the factor μ will become apparent soon. Since Q_K is positive definite and A_K is stable, P is positive definite. Let the terminal cost function $V_f(\cdot)$ be defined by

$$V_f(x) := (1/2)x'Px$$

Clearly $V_f(\cdot)$ is a global CLF for the linear system $x^+ = Ax + Bu$. Indeed, it follows from its definition that $V_f(\cdot)$ satisfies

$$V_f(A_K x) + (\mu/2)x'Q_K x - V_f(x) = 0 \quad \forall x \in \mathbb{R}^n \tag{2.22}$$

Consider now the nonlinear system $x^+ = f(x,u)$ with linear control $u = Kx$. The controlled system satisfies

$$x^+ = f(x,Kx)$$

We wish to show that $V_f(\cdot)$ is a local CLF for $x^+ = f(x,u)$ in some neighborhood of the origin; specifically, we wish to show there exists an $a \in (0,\infty)$ such that

$$V_f(f(x,Kx)) + (1/2)x'Q_K x - V_f(x) \le 0 \quad \forall x \in \operatorname{lev}_a V_f \tag{2.23}$$

in which, for all $a > 0$, $\operatorname{lev}_a V_f := \{x \mid V_f(x) \le a\}$ is a sublevel set of V_f. Since P is positive definite, $\operatorname{lev}_a V_f$ is an ellipsoid with the origin as its center. Comparing inequality (2.23) with (2.22), we see that (2.23) is satisfied if

$$V_f(f(x,Kx)) - V_f(A_K x) \le ((\mu-1)/2)x'Q_k x \quad \forall x \in \operatorname{lev}_a V_f \tag{2.24}$$

Let $e(\cdot)$ be defined as follows

$$e(x) := f(x,Kx) - A_K x$$

so that

$$V_f(f(x,Kx)) - V_f(A_K x) = (A_K x)'Pe(x) + (1/2)e(x)'Pe(x) \tag{2.25}$$

By definition, $e(0) = f(0,0) - A_K 0 = 0$ and $e_x(x) = f_x(x, Kx) + f_u(x, Kx)K - A_K$. It follows that $e_x(0) = 0$. Since $f(\cdot)$ is twice continuously differentiable, for any $\delta > 0$, there exists a $c_\delta > 0$ such that $|e_{xx}(x)| \leq c_\delta$ for all x in $\delta\mathcal{B}$. From Proposition A.11 in Appendix A

$$\begin{aligned} |e(x)| &= \left| e(0) + e_x(0)x + \int_0^1 (1-s)x' e_{xx}(sx)x ds \right| \\ &\leq \int_0^1 (1-s)c_\delta |x|^2 ds \leq (1/2)c_\delta |x|^2 \end{aligned}$$

for all x in $\delta\mathcal{B}$. From (2.25), we see that there exists an $\varepsilon \in (0, \delta]$ such that (2.24), and, hence, (2.23), is satisfied for all $x \in \varepsilon\mathcal{B}$. Because of our choice of $\ell(\cdot)$, there exists a $c_1 > 0$ such that $V_f(x) \geq \ell(x, Kx) \geq c_1 |x|^2$ for all $x \in \mathbb{R}^n$. It follows that $x \in \text{lev}_a V_f$ implies $|x| \leq \sqrt{a/c_1}$. We can choose a to satisfy $\sqrt{a/c_1} = \varepsilon$. With this choice, $x \in \text{lev}_a V_f$ implies $|x| \leq \varepsilon \leq \delta$, which, in turn, implies (2.23) is satisfied.

We conclude that there exists an $a > 0$ such that $V_f(\cdot)$ and $\mathbb{X}_f := \text{lev}_a V_f$ satisfy Assumptions 2.2 and 2.3. For each $x \in \mathbb{X}_f$ there exists a $u = \kappa_f(x) := Kx$ such that $V_f(x, u) \leq V_f(x) - \ell(x, u)$ since $\ell(x, Kx) = (1/2)x' Q_K x$ so that our assumption that $\ell(x, u) = (1/2)(x'Qx + u'Ru)$ where Q and R are positive definite, and our definition of $V_f(\cdot)$ ensure the existence of positive constants c_1, c_2 and c_3 such that $V_N^0(x) \geq c_1 |x|^2$ for all $\mathbb{R}^n$, $V_f(x) \leq c_2 |x|^2$ and $V_N^0(f(x, \kappa_f(x))) \leq V_N^0(x) - c_3 |x|^2$ for all $x \in \mathbb{X}_f$ thereby satisfying Assumption 2.14. Finally, by definition, the set $\mathbb{X}_f$ contains the origin in its interior. Summarizing, we have

> Assumptions 2.2, 2.3, and 2.14 are satisfied, and X_f contains the origin in its interior. In addition $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, and $\alpha_3(\cdot)$ satisfy the hypotheses of Theorem 2.21. Hence, by Theorems 2.19 and 2.21, the origin is exponentially stable for $x^+ = f(x, \kappa_N(x))$ in $\mathcal{X}_N$.

Asymptotic stability of the origin in $\mathcal{X}_N$ also may be established when $\mathbb{X}_f := \{0\}$ by assuming a $\mathcal{K}_\infty$ bound on $V_N^0(\cdot)$ as in Assumption 2.17.

2.5.6 Constrained Nonlinear Time-Varying Systems

Although Assumption 2.33 (the basic stability assumption) for the time-varying case suffices to ensure that $V_N^0(\cdot)$ has sufficient cost decrease, it can be asked if there exist a $V_f(\cdot)$ and $\mathbb{X}_f(\cdot)$ satisfying the hypotheses of this assumption, as well as Assumption 2.37. We give a few examples below.

Terminal equality constraint. Consider a linear time-varying system described by $x^+ = f(x(i), u(i), i) = A(i)x(i) + B(i)u(i)$ with $\ell(x, u, i) = (1/2)(x'Q(i)x(i) + u'R(i)u)$. Clearly $(\bar{x}, \bar{u}) = (0, 0)$ is an equilibrium pair since $f(0, 0, i) = 0$ for all $i \in \mathbb{I}_{\geq 0}$. The terminal constraint set is $\mathbb{X}_f(i) = \{0\}$ for all $i \in \mathbb{I}_{\geq 0}$, and the cost can be taken as $V_f(x, i) \equiv 0$. Assumption 2.33(a) is clearly satisfied. If, in addition, the matrices $A(i)$, $B(i)$, $Q(i)$, and $R(i)$ can be uniformly bounded from above, and the system is stabilizable (with $\mathbb{U}$ containing a neighborhood of the origin), then the weak controllability hypothesis implies that Assumption 2.37 is satisfied as well.

If $f(\cdot)$ is nonlinear, assumption 2.33(a) is satisfied if $f(0, 0, i) = 0$ for all $i \in \mathbb{I}_{\geq 0}$. Verifying Assumption 2.37 requires more work in the nonlinear case, but weak controllability is often the easiest way. In summary we have

> Given the terminal equality constraint and Assumption 2.37, Theorem 2.39 applies and the origin is asymptotically stable in $\mathcal{X}_N(i)$ at each time $i \geq 0$ for the time-varying system $x+ = f(x, \kappa_N(x, i), i)$.

Periodic target tracking. If the target is a periodic reference signal and the system is periodic with period T as in Limon, Alamo, de la Peña, Zeilinger, Jones, and Pereira (2012), Falugi and Mayne (2013b), and Rawlings and Risbeck (2017), it is possible, under certain conditions, to obtain terminal ingredients that satisfy Assumptions 2.33(a) and 2.37.

In the general case, terminal region synthesis is challenging. But given sufficient smoothness in the system model, we can proceed as follows. First we subtract the periodic state and input references and work in deviation variables so that the origin is again the target. Assuming $f(\cdot)$ is twice continuously differentiable in x and u at $(0, 0, i)$, we can linearize the system to determine

$$A(i) := \frac{\partial f}{\partial x}(0, 0, i) \qquad B(i) := \frac{\partial f}{\partial u}(0, 0, i)$$

Assuming the origin is in the interior of each $\mathbb{X}(i)$ (but not necessarily each $\mathbb{U}(i)$), we determine a subspace of unsaturated inputs $\widetilde{u}$ such that (i) $u(i) = F(i)\widetilde{u}(i)$, (ii) there exists $\epsilon > 0$ such that $F(i)\widetilde{u}(i) \in \mathbb{U}(i)$ for all $|\widetilde{u}| \leq \epsilon$, and (iii) the reduced linear system $(A(i), B(i)F(i))$ is stabilizable. These conditions ensure that the reduced linear system is locally unconstrained. Taking a positive definite stage cost

$$\ell(x, u, i) := \frac{1}{2}\left(x'Q(i)x + u'R(i)u\right)$$

we chose $\mu > 1$ and proceed as in the linear unconstrained case (Section 2.5.2) using the reduced model $(A(i), B(i)F(i))$ and adjusted cost matrices $\mu Q(i)$ and $\mu R(i)$. We thus have the relationship

$$V_f(A(i)x + B(i)u, i+1) \leq V_f(x, i) - \mu\ell(x, u, i)$$

with $u = \kappa_f(x, i) := K(i)x$ and $V_f(x, i) := (1/2)x'P(i)x$. Two issues remain: first, it is unlikely that $K(i)x \in \mathbb{U}(i)$ for all x and i; and second, the cost decrease holds only for the (approximate) linearized system.

To address the first issue, we start by defining the set

$$X(i) := \{x \in \mathbb{X}(i) \mid \kappa_f(x, i) \in \mathbb{U}(i) \text{ and } f(x, \kappa_f(u, i), i) \in \mathbb{X}(i+1)\}$$

on which $\kappa_f(\cdot)$ is valid. We require $\mathbb{X}_f(i) \subseteq X(i)$ for all $i \in \mathbb{I}_{\geq 0}$. By assumption, $X(i)$ contains a neighborhood of the origin, and so we can determine constants $a(i) > 0$ sufficiently small such that

$$\operatorname{lev}_{a(i)} V_f(\cdot, i) \subseteq X(i) \quad i \geq 0$$

For the second issue, we can appeal to Taylor's theorem as in Section 2.5.5 to find constants $b(i) \in (0, a(i)]$ such that

$$V_f(f(x, u, i), i+1) - V_f(A(i)x + B(i)u, i+1) \leq (\mu - 1)\ell(x, u, i)$$

for all $x \in \operatorname{lev}_{b(i)} V_f(\cdot, i)$ and $i \in \mathbb{I}_{\geq 0}$. That is, the approximation error of the linear system is sufficiently small. Thus, adding this inequality to the approximate cost decrease condition, we recover

$$V_f(f(x, u, i), i+1) - V_f(x, i) \leq -\ell(x, u, i)$$

on terminal regions $\mathbb{X}_f(i) = \operatorname{lev}_{b(i)} V_f(\cdot, i)$. That these terminal regions are positive invariant follows from the cost decrease condition. Note also that these sets $\mathbb{X}_f(i)$ contain the origin in their interiors, and thus Assumption 2.37 is satisfied. Summarizing we have

> Given sufficient smoothness in $f(x, u, i)$, terminal region synthesis can be accomplished for tracking a periodic reference. Then the assumptions of Theorem 2.39 are satisfied, and the origin (in deviation variables; hence, the periodic reference in the original variables) is asymptotically stable in $\mathcal{X}_N(i)$ at each time $i \geq 0$ for the time-varying system $x+ = f(x, \kappa_N(x, i), i)$.

2.6 Is a Terminal Constraint Set $\mathbb{X}_f$ Necessary?

While addition of a terminal cost $V_f(\cdot)$ does not materially affect the optimal control problem, addition of a terminal constraint $x(N) \in \mathbb{X}_f$, which is a state constraint, may have a significant effect. In particular, problems with only control constraints are usually easier to solve. So if state constraints are not present or if they are handled by penalty functions (soft constraints), it is highly desirable to avoid the addition of a terminal constraint. Moreover, it is possible to establish continuity of the value function for a range of optimal control problems *if* there are no state constraints; continuity of the value function ensures a degree of robustness (see Chapter 3). It is therefore natural to ask if the terminal constraint can be omitted without affecting stability.

A possible procedure is merely to omit the terminal constraint and to require that the initial state lies in a subset of $\mathcal{X}_N$ that is sufficiently small. We examine this alternative here and assume that $V_f(\cdot)$, $\mathbb{X}_f$ and $\ell(\cdot)$ satisfy Assumptions 2.2, 2.3, and 2.14, and that $\mathbb{X}_f := \{x \mid V_f(x) \leq a\}$ for some $a > 0$.

We assume, as in the examples of MPC discussed in Section 2.5, that the terminal cost function $V_f(\cdot)$, the constraint set $\mathbb{X}_f$, and the stage cost $\ell(\cdot)$ for the optimal control problem $\mathbb{P}_N(x)$ are chosen to satisfy Assumptions 2.2, 2.3, and 2.14 so that there exists a local control law $\kappa_f : \mathbb{X}_f \to \mathbb{U}$ such that $\mathbb{X}_f \subseteq \{x \in \mathbb{X} \mid \kappa_f(x) \in \mathbb{U}\}$ is positive invariant for $x^+ = f(x, \kappa_f(x))$ and $V_f(f(x, \kappa_f(x))) + \ell(x, \kappa_f(x)) \leq V_f(x)$ for all $x \in \mathbb{X}_f$. We assume that the function $V_f(\cdot)$ is defined on $\mathbb{X}$ even though it possesses the property $V_f(f(x, \kappa_f(x))) + \ell(x, \kappa_f(x)) \leq V_f(x)$ only in $\mathbb{X}_f$. In many cases, even if the system being controlled is nonlinear, $V_f(\cdot)$ is quadratic and positive definite, and $\kappa_f(\cdot)$ is linear. The set $\mathbb{X}_f$ may be chosen to be a sublevel set of $V_f(\cdot)$ so that $\mathbb{X}_f = W(a) := \{x \mid V_f(x) \leq a\}$ for some $a > 0$. We discuss in the sequel a modified form of the optimal control problem $\mathbb{P}_N(x)$ in which the terminal cost $V_f(\cdot)$ is replaced by $\beta V_f(\cdot)$ and the terminal constraint $\mathbb{X}_f$ is omitted, and show that if β is sufficiently large the solution of the modified optimal control problem is such that the optimal terminal state nevertheless lies in $\mathbb{X}_f$ so that terminal constraint is implicitly satisfied.

For all $\beta \geq 1$, let $\mathbb{P}_N^\beta(x)$ denote the modified optimal control problem defined by

$$\hat{V}_N^\beta(x) = \min_{\mathbf{u}} \{V_N^\beta(x, \mathbf{u}) \mid \mathbf{u} \in \hat{\mathcal{U}}_N(x)\}$$

in which the cost function to be minimized is now

$$V_N^\beta(x, \mathbf{u}) := \sum_{i=0}^{N-1} \ell(x(i), u(i)) + \beta V_f(x(N))$$

in which, for all i, $x(i) = \phi(i; x, \mathbf{u})$, the solution at time i of $x^+ = f(x, u)$ when the initial state is x and the control sequence is $\mathbf{u}$. The control constraint set $\hat{\mathcal{U}}_N(x)$ ensures satisfaction of the state and control constraints, but not the terminal constraint, and is defined by

$$\hat{\mathcal{U}}_N(x) := \{\mathbf{u} \mid (x(i), u(i)) \in \mathbb{Z}\ i \in \mathbb{I}_{0:N-1}, x(N) \in \mathbb{X}\}$$

The cost function $V_N^\beta(\cdot)$ with $\beta = 1$ is identical to the cost function $V_N(\cdot)$ employed in the standard problem $\mathbb{P}_N$ considered previously. Let $\hat{\mathcal{X}}_N := \{x \in \mathbb{X} \mid \hat{\mathcal{U}}_N(x) \neq \emptyset\}$ denote the domain of $\hat{V}_N^\beta(\cdot)$; let $\mathbf{u}^\beta(x)$ denote the solution of $\mathbb{P}_N^\beta(x)$; and let $\mathbf{x}^\beta(x)$ denote the associated optimal state trajectory. Thus

$$\mathbf{u}^\beta(x) = \left(u^\beta(0; x), u^\beta(1; x), \ldots, u^\beta(N-1; x)\right)$$
$$\mathbf{x}^\beta(x) = \left(x^\beta(0; x), x^\beta(1; x), \ldots, x^\beta(N; x)\right)$$

where $x^\beta(i; x) := \phi(i; x, \mathbf{u}^\beta(x))$ for all i. The implicit MPC control law is $\kappa_N^\beta(\cdot)$ where $\kappa_N^\beta(x) := u^\beta(0; x)$. Neither $\hat{\mathcal{U}}_N(x)$ nor $\hat{\mathcal{X}}_N$ depend on the parameter β. It can be shown (Exercise 2.11) that *the pair* $(\beta V_f(\cdot), \mathbb{X}_f)$ *satisfies Assumptions 2.2–2.14* if $\beta \geq 1$, since these assumptions are satisfied by the pair $(V_f(\cdot), \mathbb{X}_f)$. The absence of the terminal constraint $x(N) \in \mathbb{X}_f$ in problem $\mathbb{P}_N^\beta(x)$, which is otherwise the same as the normal optimal control problem $\mathbb{P}_N(x)$ when $\beta = 1$, ensures that $\hat{V}_N^1(x) \leq V_N^0(x)$ for all $x \in \mathcal{X}_N$ and that $\mathcal{X}_N \subseteq \hat{\mathcal{X}}_N$ where $V_N^0(\cdot)$ is the value function for $\mathbb{P}_N(x)$ and $\mathcal{X}_N$ is the domain of $V_N^0(\cdot)$.

Problem $\mathbb{P}_N^\beta(x)$ and the associated MPC control law $\kappa_N^\beta(\cdot)$ are defined below. Suppose $\mathbf{u}^\beta(x)$ is optimal for the terminally unconstrained problem $\mathbb{P}_N^\beta(x)$, $\beta \geq 1$, and that $\mathbf{x}^\beta(x)$ is the associated optimal state trajectory.

That the origin is asymptotically stable for $x^+ = f(x, \kappa_N^\beta(x))$ and each $\beta \geq 1$, with a region of attraction that depends on the parameter β is established by Limon, Alamo, Salas, and Camacho (2006) via the following results.

Lemma 2.40 (Entering the terminal region). *Suppose $\mathbf{u}^\beta(x)$ is optimal for the terminally unconstrained problem $\mathbb{P}_N^\beta(x)$, with $\beta \geq 1$, and that $\mathbf{x}^\beta(x)$ is the associated optimal state trajectory. If $x^\beta(N; x) \notin \mathbb{X}_f$, then $x^\beta(i; x) \notin \mathbb{X}_f$ for all $i \in \mathbb{I}_{0:N-1}$.*

Proof. Since, as shown in Exercise 2.11, $\beta V_f(x) \geq \beta V_f(f(x, \kappa_f(x))) + \ell(x, \kappa_f(x))$ and $f(x, \kappa_f(x)) \in \mathbb{X}_f$ for all $x \in \mathbb{X}_f$, all $\beta \geq 1$, it follows that for all $x \in \mathbb{X}_f$ and all $i \in \mathbb{I}_{0:N-1}$

$$\beta V_f(x) \geq \sum_{j=i}^{N-1} \ell(x^f(j; x, i), u^f(j; x, i)) + \beta V_f(x^f(N; x, i)) \geq \hat{V}_{N-i}^{\beta}(x)$$

in which $x^f(j; x, i)$ is the solution of $x^+ = f(x, \kappa_f(x))$ at time j if the initial state is x at time i, $u^f(j; x, i) = \kappa_f(x^f(j; x, i))$, and $\kappa_f(\cdot)$ is the local control law that satisfies the stability assumptions. The second inequality follows from the fact that the control sequence $\left(u^f(j; x, i)\right)$, $i \in \mathbb{I}_{i:N-1}$ is feasible for $\mathbb{P}_N^{\beta}(x)$ if $x \in \mathbb{X}_f$. Suppose contrary to what is to be proved, that there exists a $i \in \mathbb{I}_{0:N-1}$ such that $x^{\beta}(i; x) \in \mathbb{X}_f$. By the principle of optimality, the control sequence $(u^{\beta}(i; x), u^{\beta}(i+1; x), \ldots, u^{\beta}(N-1; x))$ is optimal for $\mathbb{P}_{N-i}^{\beta}(x^{\beta}(i; x))$. Hence

$$\beta V_f(x^{\beta}(i; x)) \geq \hat{V}_{N-i}^{\beta}(x^{\beta}(i; x)) \geq \beta V_f(x^{\beta}(N; x)) > \beta a$$

since $x^{\beta}(N; x) \notin \mathbb{X}_f$ contradicting the fact that $x^{\beta}(i; x) \in \mathbb{X}_f$. This proves the lemma. ∎

For all $\beta \geq 1$, let the set Γ_N^{β} be defined by

$$\Gamma_N^{\beta} := \{x \mid \hat{V}_N^{\beta}(x) \leq Nd + \beta a\}$$

We assume in the sequel that there exists a $d > 0$ such $\ell(x, u) \geq d$ for all $x \in \mathbb{X} \setminus \mathbb{X}_f$ and all $u \in \mathbb{U}$. The following result is due to Limon et al. (2006).

Theorem 2.41 (MPC stability; no terminal constraint)**.** *The origin is asymptotically or exponentially stable for the closed-loop system* $x^+ = f(x, \kappa_N^{\beta}(x))$ *with a region of attraction* Γ_N^{β}*. The set* Γ_N^{β} *is positive invariant for* $x^+ = f(x, \kappa_N^{\beta}(x))$*.*

Proof. From the Lemma, $x^{\beta}(N; x) \notin \mathbb{X}_f$ implies $x^{\beta}(i; x) \notin \mathbb{X}_f$ for all $i \in \mathbb{I}_{0:N}$. This, in turn, implies

$$\hat{V}_N^{\beta}(x) > Nd + \beta a$$

so that $x \notin \Gamma_N^{\beta}$. Hence $x \in \Gamma_N^{\beta}$ implies $x^{\beta}(N; x) \in \mathbb{X}_f$. It then follows, since $\beta V_f(\cdot)$ and $\mathbb{X}_f$ satisfy Assumptions 2.2 and 2.3, that the origin is asymptotically or exponentially stable for $x^+ = f(x, \kappa_N^\beta(x))$ with a region of attraction Γ_N^β. It also follows that $x \in \Gamma_N^\beta(x)$ implies

$$\hat{V}_N^\beta(x^\beta(1;x)) \leq \hat{V}_N^\beta(x) - \ell(x, \kappa_N^\beta(x)) \leq \hat{V}_N^\beta(x) \leq Nd + \beta a$$

so that $x^\beta(1;x) = f(x, \kappa_N^\beta(x)) \in \Gamma_N^\beta$. Hence Γ_N^β is positive invariant for $x^+ = f(x, \kappa_N^\beta(x))$. ∎

Limon et al. (2006) then proceed to show that Γ_N^β increases with β or, more precisely, that $\beta_1 \leq \beta_2$ implies that $\Gamma_N^{\beta_1} \subseteq \Gamma_N^{\beta_2}$. They also show that for any x steerable to the interior of $\mathbb{X}_f$ by a feasible control, there exists a β such that $x \in \Gamma_N^\beta$. We refer to requiring the initial state x to lie in Γ_N^β as an *implicit terminal constraint.*

If it is desired that the feasible sets for $\mathbb{P}_i(x)$ be nested ($\mathcal{X}_i \subset \mathcal{X}_{i+1}$, $i = 1, 2, \ldots N-1$) (thereby ensuring recursive feasibility), it is *necessary*, as shown in Mayne (2013), that $\mathbb{P}_N(x)$ includes a terminal constraint that is control invariant.

2.7 Suboptimal MPC

Overview. There is a significant practical problem that we have not yet addressed, namely that if the optimal control problem $\mathbb{P}_N(x)$ solved online is not convex, which is usually the case when the system is nonlinear, the global minimum of $V_N(x, \mathbf{u})$ in $\mathcal{U}_N(x)$ cannot usually be determined. Since we assume, in the stability theory given previously, that the global minimum *is* achieved, we have to consider the impact of this unpalatable fact. It is possible, as shown in Scokaert, Mayne, and Rawlings (1999); Pannocchia, Rawlings, and Wright (2011) to achieve stability *without* requiring globally optimal solutions of $\mathbb{P}_N(x)$. The basic idea behind the suboptimal model predictive controller is simple. Suppose the current state is x and that $\mathbf{u} = (u(0), u(1), \ldots, u(N-1)) \in \mathcal{U}_N(x)$ is a feasible control sequence for $\mathbb{P}_N(x)$. The first element $u(0)$ of $\mathbf{u}$ is applied to the system $x^+ = f(x, u)$; let $\kappa_N(x, \mathbf{u})$ denote this control. In the absence of uncertainty, the next state is equal to the predicted state $x^+ = f(x, u(0))$.

Consider the control sequence $\tilde{\mathbf{u}}$ defined by

$$\tilde{\mathbf{u}} = \Big(u(1), u(2), \ldots, u(N-1), \kappa_f(x(N))\Big) \tag{2.26}$$

in which $x(N) = \phi(N; x, \mathbf{u})$ and $\kappa_f(\cdot)$ is a local control law with the property that $u = \kappa_f(x)$ satisfies Assumption 2.2 for all $x \in \mathbb{X}_f$. The existence of such a $\kappa_f(\cdot)$, which is often of the form $\kappa_f(x) = Kx$, is implied by Assumption 2.2. Then, since $x(N) \in \mathbb{X}_f$ and since the stabilizing conditions 2.14 are satisfied, the control sequence $\tilde{\mathbf{u}} \in \mathcal{U}_N(x)$ satisfies

$$V_N(x^+, \tilde{\mathbf{u}}) \leq V_N(x, \mathbf{u}) - \ell(x, u(0)) \leq V_N(x, \mathbf{u}) - \alpha_1(|x|) \tag{2.27}$$

with $x^+ := f(x, u(0))$.

No optimization is required to get the cost reduction $\ell(x, u(0))$ given by (2.27); in practice the control sequence $\tilde{\mathbf{u}}$ can be improved by several iterations of an optimization algorithm. Inequality (2.27) is reminiscent of the inequality $V_N^0(x^+) \leq V_N^0(x) - \alpha_1(|x|)$ that provides the basis for establishing asymptotic stability of the origin for the controlled systems previously analyzed. This suggests that the simple algorithm described previously, which places very low demands on the online optimization algorithm, may also ensure asymptotic stability of the origin.

This is almost true. The obstacle to applying standard Lyapunov theory is that there is no obvious Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ because, at each state x^+, there exist many control sequences $\mathbf{u}^+$ satisfying $V_N(x^+, \mathbf{u}^+) \leq V_N(x, \mathbf{u}) - \alpha_1(|x|)$. The function $(x, \mathbf{u}) \mapsto V_N(x, \mathbf{u})$ is *not* a function of x only and may have many different values for each x; therefore it cannot play the role of the function $V_N^0(x)$ used previously. Moreover, the controller can generate, for a given initial state, many different trajectories, all of which have to be considered. We address these issues next following the recent development in Allan, Bates, Risbeck, and Rawlings (2017).

A key step is to consider suboptimal MPC as an evolution of an *extended state* consisting of the state and warm-start pair. Given a feasible warm start, optimization algorithms can produce an improved feasible sequence or, failing even that, simply return the warm start. The first input is injected and a new warm start can generated from the returned control sequence and terminal control law.

Warm start. An admissible warm start $\tilde{\mathbf{u}}$, must steer the current state x to the terminal region subject to the input constraints, i.e., $\tilde{\mathbf{u}} \in \mathcal{U}_N(x)$. It also must satisfy $V_N(x, \tilde{\mathbf{u}}) \leq V_f(x)$ if $x \in \mathbb{X}_f$, which ensures that $|x| \to 0$ implies $|\mathbf{u}| \to 0$. These two conditions define the set of admissible warm starts

$$\tilde{\mathcal{U}}_N(x) := \{\tilde{\mathbf{u}} \in \mathcal{U}_N(x) \mid V_N(x, \tilde{\mathbf{u}}) \leq V_f(x) \text{ if } x \in \mathbb{X}_f\} \tag{2.28}$$

When $x \in \mathbb{X}_f$ and $\tilde{\mathbf{u}} \in \mathcal{U}_N(x)$ but $V_N(x, \tilde{\mathbf{u}}) > V_f(x)$, an admissible warm start $\tilde{\mathbf{u}}_f(x)$ can be recovered using the terminal control law.

Proposition 2.42 (Admissible warm start in $\mathbb{X}_f$)**.** *For any $x \in \mathbb{X}_f$, the following warm start is feasible*

$$\tilde{\mathbf{u}}_f(x) := \big(\kappa_f(x), \kappa_f(f(x, \kappa_f(x))), \ldots\big) \in \tilde{\mathcal{U}}_N(x)$$

The proof of this proposition is discussed in Exercise 2.24.

We define the set of admissible control sequences $\check{\mathcal{U}}_N(x, \tilde{\mathbf{u}})$ as those feasible control sequences $\mathbf{u}$ that result in a lower cost than the warm start; the suboptimal control law is the set of first elements of admissible control sequences

$$\begin{aligned} \check{\mathcal{U}}_N(x, \tilde{\mathbf{u}}) &= \{\mathbf{u} \mid \mathbf{u} \in \tilde{\mathcal{U}}_N(x),\ V_N(x, \mathbf{u}) \le V_N(x, \tilde{\mathbf{u}})\} \\ \kappa_N(x, \tilde{\mathbf{u}}) &= \{u(0) \mid \mathbf{u} \in \check{\mathcal{U}}_N(x, \tilde{\mathbf{u}})\} \end{aligned}$$

From its definition, the suboptimal control law is a function of both the state x and the warm start $\tilde{\mathbf{u}} \in \tilde{\mathcal{U}}_N(x)$.

To complete the algorithm we require a successor warm start for the successor state $x^+ = f(x, u(0))$. First defining

$$\tilde{\mathbf{u}}_w(x, \mathbf{u}) := \Big(u(1), u(2), \ldots, u(N-1), \kappa_f(\phi(N; x, \mathbf{u}))\Big)$$

we choose the successor warm start $\tilde{\mathbf{u}}^+ \in \tilde{\mathcal{U}}_N(x^+)$ as follows

$$\tilde{\mathbf{u}}^+ := \begin{cases} \tilde{\mathbf{u}}_f(x^+) & \text{if } x^+ \in \mathbb{X}_f \text{ and} \\ & V_N(x^+, \tilde{\mathbf{u}}_f(x^+)) \le V_N(x^+, \tilde{\mathbf{u}}_w(x, \mathbf{u})) \\ \tilde{\mathbf{u}}_w(x, \mathbf{u}) & \text{else} \end{cases} \tag{2.29}$$

This mapping in (2.29) is denoted $\tilde{\mathbf{u}}^+ = \zeta(x, \mathbf{u})$, and Proposition 2.42 ensures that the warm start generated by $\zeta(x, \mathbf{u})$ is admissible for x^+. We have the following algorithm for suboptimal MPC.

Algorithm 2.43 (Suboptimal MPC)**.** First, choose $\mathbb{X}_f$ and $V_f(\cdot)$ satisfying Assumption 2.14 and obtain the initial state $x \in \mathcal{X}_N$ and any initial warm start $\tilde{\mathbf{u}} \in \tilde{\mathcal{U}}_N(x)$. Then repeat

1. Obtain current measurement of state x.
2. Compute any input $\mathbf{u} \in \check{\mathcal{U}}_N(x, \tilde{\mathbf{u}})$.
3. Inject the first element of the input sequence $\mathbf{u}$.
4. Compute the next warm start $\tilde{\mathbf{u}}^+ = \zeta(x, \mathbf{u})$.

Because the control law $\kappa_N(x, \tilde{\mathbf{u}})$ is a function of the warm start $\tilde{\mathbf{u}}$ as well as the state x, we extend the meaning of state to include the warm start.

2.7.1 Extended State

In Algorithm 2.43 we begin with a state and warm-start pair and proceed from this pair to the next at the start of each time step. We denote this extended state as $z := (x, \tilde{\mathbf{u}})$ for $x \in \mathcal{X}_N$ and $\tilde{\mathbf{u}} \in \tilde{\mathcal{U}}_N(x)$. The extended state evolves according to

$$\begin{aligned} z^+ \in H(z) := \{(x^+, \tilde{\mathbf{u}}^+) \mid x^+ &= f(x, u(0)), \\ \tilde{\mathbf{u}}^+ &= \zeta(x, \mathbf{u}),\ \mathbf{u} \in \check{\mathcal{U}}_N(z)\} \end{aligned} \tag{2.30}$$

in which $u(0)$ is the first element of $\mathbf{u}$. We denote by $\psi(k; z)$ any solution of (2.30) with initial extended state z and denote by $\phi(k; z)$ the accompanying x trajectory. We restrict $\mathcal{Z}_N$ to the set of z for which $\tilde{\mathbf{u}} \in \tilde{\mathcal{U}}_N(x)$.

$$\tilde{\mathcal{Z}}_N := \{(x, \tilde{\mathbf{u}}) \mid x \in \mathcal{X}_N \text{ and } \tilde{\mathbf{u}} \in \tilde{\mathcal{U}}_N(x)\}$$

To directly link the asymptotic behavior of z with that of x, the following proposition is necessary.

Proposition 2.44 (Linking warm start and state)**.** *There exists a function $\alpha_r(\cdot) \in \mathcal{K}_\infty$ such that $|\tilde{\mathbf{u}}| \leq \alpha_r(|x|)$ for any $(x, \tilde{\mathbf{u}}) \in \tilde{\mathcal{Z}}_N$.*

A proof is given in (Allan et al., 2017, Proposition 10).

2.7.2 Asymptotic Stability of Difference Inclusions

Because the extended state evolves as the difference inclusion (2.30), we present the following definitions of asymptotic stability and the associated Lyapunov functions. Consider the difference inclusion $z^+ \in H(z)$, such that $H(0) = \{0\}$.

Definition 2.45 (Asymptotic stability (difference inclusion))**.** We say the origin of the difference inclusion $z^+ \in H(z)$ is asymptotically stable in a positive invariant set $\mathcal{Z}$ if there exists a function $\beta(\cdot) \in \mathcal{KL}$ such that for any $z \in \mathcal{Z}$ and for all $k \in \mathbb{I}_{\geq 0}$, all solutions $\psi(k; z)$ satisfy

$$|\psi(k; z)| \leq \beta(|z|, k)$$

Definition 2.46 (Lyapunov function (difference inclusion))**.** $V(\cdot)$ is a Lyapunov function in the positive invariant set $\mathcal{Z}$ for the difference inclusion $z^+ \in H(z)$ if there exist functions $\alpha_1(\cdot), \alpha_2(\cdot), \alpha_3(\cdot) \in \mathcal{K}_\infty$ such that for all $z \in \mathcal{Z}$

$$\alpha_1(|z|) \leq V(z) \leq \alpha_2(|z|) \tag{2.31}$$

$$\sup_{z^+ \in H(z)} V(z^+) \leq V(z) - \alpha_3(|z|) \tag{2.32}$$

Although $V(\cdot)$ is not required to be continuous everywhere, (2.31) implies that it is continuous at the origin.

Proposition 2.47 (Asymptotic stability (difference inclusion))**.** *If the set $\mathcal{Z}$ contains the origin, is positive invariant for the difference inclusion $z^+ \in H(z)$, $H(0) = \{0\}$, and it admits a Lyapunov function $V(\cdot)$ in $\mathcal{Z}$, then the origin is asymptotically stable in $\mathcal{Z}$.*

A proof of this proposition is given in (Allan et al., 2017, Proposition 13); it is similar to the proof of Theorem B.15 in Appendix B.

Theorem 2.48 (Asymptotic stability of suboptimal MPC)**.** *Suppose Assumptions 2.2, 2.3, and 2.14 are satisfied, and that $\ell(x,u) \geq \alpha_\ell(|(x,u)|)$ for all $(x,u) \in \mathbb{Z}$, and $\mathbb{X}_f = \text{lev}_b V_f = \{x \in \mathbb{R}^n \mid V_f(x) \leq b\}$, for some $b > 0$. Then the function $V_N(z)$ is a Lyapunov function in the set $\tilde{\mathcal{Z}}_N$ for the closed-loop system* (2.30) *under Algorithm 2.43. Therefore the origin is asymptotically stable in $\tilde{\mathcal{Z}}_N$.*

Proof. First we show that $V_N(z)$ is a Lyapunov function for (2.30) on the positive invariant set $\tilde{\mathcal{Z}}_N$. Because $\mathbf{u} \in \check{\mathcal{U}}_N(z)$ and, by construction, $\tilde{\mathbf{u}}^+ \in \tilde{\mathcal{U}}_N(x^+)$, we have that $z^+ \in \tilde{\mathcal{Z}}_N$, so that $\tilde{\mathcal{Z}}_N$ is positive invariant. From the definition of the control law and the warm start, we have that for all $z \in \tilde{\mathcal{Z}}_N$

$$V_N(z) \geq V_N(x, \mathbf{u}) \geq \sum_{i=0}^{N-1} \ell(x(i), u(i)) \geq \sum_{i=0}^{N-1} \alpha_\ell(|(x(i), u(i))|)$$

Next we use (B.1) from Appendix B and the triangle inequality to obtain

$$\sum_{i=0}^{N-1} \alpha_\ell(|(x(i), u(i))|) \geq \alpha_\ell\left(\frac{1}{N} \sum_{i=0}^{N-1} |(x(i), u(i))|\right) \geq \alpha_\ell\big(|(\mathbf{x}, \mathbf{u})|/N\big)$$

Finally using the ℓ_p-norm property that for all vectors a, b, $|(a,b)| \geq |b|$, and noting that $x(0) = x$, so we have that

$$\alpha_\ell\big(|(\mathbf{x}, \mathbf{u})|/N\big) \geq \alpha_\ell\big(|(x, \mathbf{u})|/N\big) := \alpha_1\big(|(x, \mathbf{u})|\big) = \alpha_1(z)$$

with $\alpha_1 \in \mathcal{K}_\infty$. So we have established the lower bound $V_N(z) \geq \alpha_1(z)$ for all $z \in \tilde{\mathcal{Z}}_N$.

Because of Assumptions 2.2 and 2.3, the set $\mathcal{Z}_N$ is closed as shown in Proposition 2.10(c). The cost function $V_N(z)$ is continuous on $\mathcal{Z}_N$, which includes $z = 0$, so from Proposition B.25 we conclude that there exists $\alpha_2(\cdot) \in \mathcal{K}_\infty$ such that $V_N(z) \leq \alpha_2(|z|)$ for all $z \in \tilde{\mathcal{Z}}_N \subset \mathcal{Z}_N$, and the upper-bound condition of Definition 6.2 is satisfied.

As in standard MPC analysis, we have for all $z \in \tilde{\mathcal{Z}}_N$ that

$$V_N(z^+) \leq V_N(x, \mathbf{u}) - \ell(x, u(0)) \leq V_N(x, \mathbf{u}) - \alpha_\ell(|x, u(0)|)$$

Because $\tilde{\mathbf{u}} \in \tilde{\mathcal{U}}_N(x)$, from Proposition 2.44 we have that

$$\left|(x, \tilde{\mathbf{u}})\right| \leq |x| + \left|\tilde{\mathbf{u}}\right| \leq |x| + \alpha_r(|x|) := \alpha_{r'}(|x|) \leq \alpha_{r'}(|(x, u(0))|)$$

Therefore, $\alpha_\ell \circ \alpha_{r'}^{-1}(\left|(x, \tilde{\mathbf{u}})\right|) \leq \alpha_\ell(|(x, u(0))|)$. Defining $\alpha_3(\cdot) := \alpha_\ell \circ \alpha_{r'}^{-1}(\cdot)$ and because $V_N(x, \mathbf{u}) \leq V_N(x, \tilde{\mathbf{u}})$, we have that

$$V_N(z^+) \leq V_N(x, \tilde{\mathbf{u}}) - \alpha_3(|z|) = V_N(z) - \alpha_3(|z|)$$

for all $z \in \tilde{\mathcal{Z}}_N$ and $z^+ \in H(z)$. We conclude that $V_N(z)$ is a Lyapunov function for (2.30) in $\tilde{\mathcal{Z}}_N$. Asymptotic stability follows directly from Proposition 2.47. ■

From this result, a bound on just $x(k)$ rather than $z(k) = (x(k), \tilde{\mathbf{u}}(k))$ can also be derived. First we have that for all $k \geq 0$ and $z \in \tilde{\mathcal{Z}}_N$

$$|z(k; z)| \leq \beta(|z|, k) = \beta(\left|(x, \tilde{\mathbf{u}})\right|, k) \leq \beta(|x| + \left|\tilde{\mathbf{u}}\right|, k)$$

From Proposition 2.44 we then have that

$$\beta(|x| + \left|\tilde{\mathbf{u}}\right|, k) \leq \beta(|x| + \alpha_r(|x|), k) := \tilde{\beta}(|x|, k)$$

with $\tilde{\beta}(\cdot) \in \mathcal{KL}$. Combining these we have that

$$|z(k; z)| = \left|(x(k; z), \tilde{\mathbf{u}}(k; z))\right| \leq |x(k; z)| + \left|\tilde{\mathbf{u}}(k; z)\right| \leq \tilde{\beta}(|x|, k)$$

which implies $|x(k; z)| \leq \tilde{\beta}(|x|, k)$. So we have a bound on the evolution of $x(k)$ depending on *only* the x initial condition. Note that the evolution of $x(k)$ depends on the initial condition of $z = (x, \tilde{\mathbf{u}})$, so it depends on initial warm start $\tilde{\mathbf{u}}$ as well as initial x. We cannot ignore this dependence, which is why we had to analyze the extended state in the first place. For the same reason we also cannot define the invariant set in which the $x(k)$ evolution takes place without referring to $\tilde{\mathcal{Z}}_N$.

2.8 Economic Model Predictive Control

Many applications of control are naturally posed as tracking problems. Vehicle guidance, robotic motion guidance, and low-level objectives such as maintaining pressures, temperatures, levels, and flows in industrial processes are typical examples. MPC can certainly provide feedback control designs with excellent tracking performance for challenging multivariable, constrained, and nonlinear systems as we have explored thus far in the text. But feedback control derived from repeated online optimization of a process model enables other, higher-level goals to be addressed as well. In this section we explore using MPC for *optimizing economic performance* of a process rather than a simple tracking objective. As before, we assume the system dynamics are described by the model

$$x^+ = f(x,u)$$

But here the stage cost is some general function $\ell(x,u)$ that measures economic performance of the process. The stage cost is not positive definite with respect to some target equilibrium point of the model as in a tracking problem. We set up the usual MPC objective function as a sum of stage costs over some future prediction horizon

$$V_N(x,\mathbf{u}) = \sum_{k=0}^{N-1} \ell(x(k),u(k)) + V_f(x(N))$$

subject to the system model with $x(0) = x$, the initial condition. As before, we consider constraints on the states and inputs, $(x,u) \in \mathbb{Z}$. So the only significant change in the MPC problem has been the redefinition of the stage cost $\ell(x,u)$ to reflect the economics of the process. The terminal penalty $V_f(x)$ may be changed for the same reason. Typical stage-cost functions would be composed of a sum of prices of the raw materials and utilities, and the values of the products being manufactured.

We can also define the best steady-state solution of the system from the economic perspective. This optimal steady-state pair (x_s,u_s) is defined as the solution to the optimization problem $\mathbb{P}_s$

$$(x_s,u_s) := \arg\min_{(x,u)\in\mathbb{Z}} \{\ell(x,u) \mid x = f(x,u)\}$$

The standard industrial approach to addressing economic performance is to calculate this best economic steady state (often on a slower time scale than the process sample time), and then design an MPC controller with a different, tracking stage cost to reject disturbances and *track* this steady state. In this approach, a typical tracking stage cost would be the types considered thus far, e.g., $\ell_t(x,u) = (1/2)(|x - x_s|_Q^2 + |u - u_s|_R^2)$.

In economic MPC, we instead use the same economic stage cost directly in the dynamic MPC problem. Some relevant questions to be addressed with this change in design philosophy are: (i) how much economic performance improvement is possible, and (ii) how different is the closed-loop dynamic behavior. For example, we are not even guaranteed for a nonlinear system that operating at the steady state is the best possible dynamic behavior of the closed-loop system.

As an introduction to the topic, we next set up the simplest version of an economic MPC problem, in which we use a terminal constraint. In the Notes section, we comment on what generalizations are available in the literature. We now modify the basic assumptions given previously.

Assumption 2.49 (Continuity of system and cost)**.** The functions $f : \mathbb{Z} \to \mathbb{R}^n$ and $\ell : \mathbb{Z} \to \mathbb{R}_{\geq 0}$ are continuous. $V_f(\cdot) = 0$. There exists at least one point $(x_s, u_s) \in \mathbb{Z}$ satisfying $x_s = f(x_s, u_s)$.

Assumption 2.50 (Properties of constraint sets)**.** The set $\mathbb{Z}$ is closed. If there are control constraints, the set $\mathbb{U}(x)$ is compact and is uniformly bounded in $\mathbb{X}$.

Assumption 2.51 (Cost lower bound)**.**

(a) The terminal set is a single point, $\mathbb{X}_f = \{x_s\}$.

(b) The stage cost $\ell(x,u)$ is lower bounded for $(x,u) \in \mathbb{Z}$.

Note that since we are using a terminal equality constraint, we do not require the terminal penalty $V_f(\cdot)$, so it is set to zero. For clarity in this discussion, we do not assume that (x_s, u_s) has been shifted to the origin. The biggest change is that we do not assume here that the stage cost $\ell(x,u)$ is positive definite with respect to the optimal steady state, only that it is lower bounded.

Note that the set of steady states, $\mathbb{Z}_s := \{(x,u) \in \mathbb{Z} \mid x = f(x,u)\}$, is nonempty due to Assumption 2.49. It is closed because $\mathbb{Z}$ is closed (Assumption 2.50) and $f(\cdot)$ is continuous. But it may not be bounded so we are not guaranteed that the solution to $\mathbb{P}_s$ exists. So we consider (x_s, u_s) to be any element of $\mathbb{Z}_s$. We may want to choose (x_s, u_s) to be an element of the solution to $\mathbb{P}_s$, when it exists, but this is not necessary to the subsequent development.

The economic optimal control problem $\mathbb{P}_N(x)$, is the same as in (2.7)

$$\mathbb{P}_N(x): \qquad V_N^0(x) := \min_{\mathbf{u}}\{V_N(x,\mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_N(x)\}$$

Due to Assumptions 2.49 and 2.50, Proposition 2.4 holds, and the solution to the optimal control problem exists. The control law, $\kappa_N(\cdot)$ is therefore well defined; if it is not unique, we consider as before a fixed selection map, and the closed-loop system is again given by

$$x^+ = f(x, \kappa_N(x)) \tag{2.33}$$

2.8.1 Asymptotic Average Performance

We already have enough structure in this simple problem to establish that the average cost of economic MPC is better, i.e., not worse, than any steady-state performance $\ell(x_s, u_s)$.

Proposition 2.52 (Asymptotic average performance)**.** *Let Assumptions 2.49, 2.50, and 2.51 hold. Then for every $x \in \mathcal{X}_N$, the following holds*

$$\limsup_{t\to\infty} \sum_{k=0}^{t-1} \frac{\ell(x(k), u(k))}{t} \leq \ell(x_s, u_s)$$

in which $x(k)$ is the closed-loop solution to (2.33) *with initial condition x, and $u(k) = \kappa_N(x(k))$.*

Proof. Because of the terminal constraint, we have that

$$V_N^0(f(x, \kappa_N(x))) \leq V_N^0(x) - \ell(x, \kappa_N(x)) + \ell(x_s, u_s) \tag{2.34}$$

Performing a sum on this inequality gives

$$\sum_{k=0}^{t=1} \frac{\ell(x(k), u(k))}{t} \leq \ell(x_s, u_s) + (1/t)(V_N^0(x(0)) - V_N^0(x(t)))$$

The left-hand side may not have a limit, so we take lim sup of both sides. Note that from Assumption 2.51(b), $\ell(x,u)$ is lower bounded for $(x,u) \in \mathbb{Z}$, hence so is $V_N(x,u)$ for $(x,u) \in \mathbb{Z}$, and $V_N^0(x)$ for $x \in \mathcal{X}_N$. Denote this bound by M. Then $\lim_{t\to\infty} -(1/t)V_N^0(x(t)) \leq \lim_{t\to\infty} -M/t = 0$ and we have that

$$\limsup_{t\to\infty} \sum_{k=0}^{t=1} \frac{\ell(x(k), u(k))}{t} \leq \ell(x_s, u_s) \qquad \blacksquare$$

This result does not imply that the economic MPC controller stabilizes the steady state (x_s, u_s), only that the *average* closed-loop performance is better than the best steady-state performance. There are many examples of nonlinear systems for which the time-average of an oscillation is better than the steady state. For such systems, we would expect an optimizing controller to destabilize even a stable steady state to obtain the performance improvement offered by cycling the system.

Note also that the appearance in (2.34) of the term $-\ell(x, \kappa_N(x)) + \ell(x_s, u_s)$, which is sign indeterminate, destroys the cost decrease property of $V_N^0(\cdot)$ so it no longer can serve as a Lyapunov function in a closed-loop stability argument. We next examine the stability question.

2.8.2 Dissipativity and Asymptotic Stability

The idea of dissipativity proves insightful in understanding when economic MPC is stabilizing (Angeli, Amrit, and Rawlings, 2012). The basic idea is motivated by considering a thermodynamic system, mechanical energy, and work. Imagine we *supply* mechanical energy to a system by performing work on the system at some rate. We denote the mechanical energy as a *storage* function, i.e., as the way in which the work performed on the system is stored by the system. If the system has no *dissipation*, then the rate of change in storage function (mechanical energy) is equal to the supply rate (work). However, if the system also dissipates mechanical energy into heat, through friction for example, then the change in the storage function is strictly less than the work supplied. We make this physical idea precise in the following definition.

Definition 2.53 (Dissipativity)**.** The system $x^+ = f(x, u)$ is dissipative with respect to supply rate $s : \mathbb{Z} \to \mathbb{R}$ if there exists a storage function $\lambda : \mathbb{X} \to \mathbb{R}$ such that for all $(x, u) \in \mathbb{Z}$

$$\lambda(f(x, u)) - \lambda(x) \leq s(x, u) \tag{2.35}$$

The system is strictly dissipative with respect to supply rate s and steady-state x_s if there exists $\alpha(\cdot) \in \mathcal{K}_\infty$ such that for all $(x, u) \in \mathbb{Z}$

$$\lambda(f(x, u)) - \lambda(x) \leq s(x, u) - \alpha(|x - x_s|) \tag{2.36}$$

Note that we do *not* assume that $\lambda(\cdot)$ is continuous, and we define strict dissipativity with $\alpha(\cdot)$ a $\mathcal{K}_\infty$ function. In other literature, $\alpha(\cdot)$ is sometimes assumed to be a continuous, positive definite function.

We require one technical assumption; its usefulness will be apparent shortly.

Assumption 2.54 (Continuity at the steady state)**.** The function $V_N^0(\cdot) + \lambda(\cdot) : \mathcal{X}_N \to \mathbb{R}$ is continuous at x_s.

The following assumption is then sufficient to guarantee that economic MPC is stabilizing.

Assumption 2.55 (Strict dissipativity)**.** The system $x^+ = f(x,u)$ is strictly dissipative with supply rate

$$s(x,u) = \ell(x,u) - \ell(x_s,u_s)$$

Theorem 2.56 (Asymptotic stability of economic MPC)**.** *Let Assumptions 2.49, 2.50, 2.51, 2.54, and 2.55 hold. Then x_s is asymptotically stable in $\mathcal{X}_N$ for the closed-loop system $x^+ = f(x, \kappa_N(x))$.*

Proof. We know that $V_N^0(\cdot)$ is *not* a Lyapunov function for the given stage cost $\ell(\cdot)$, so our task is to construct one. We first introduce a *rotated* stage cost as follows (Diehl, Amrit, and Rawlings, 2011)

$$\tilde{\ell}(x,u) = \ell(x,u) - \ell(x_s,u_s) + \lambda(x) - \lambda(f(x,u))$$

Note from (2.36) and Assumption 2.55 that this stage cost then satisfies for all $(x,u) \in \mathbb{Z}$

$$\tilde{\ell}(x,u) \geq \alpha(|x - x_s|) \qquad \tilde{\ell}(x_s,u_s) = 0 \tag{2.37}$$

and we have the kind of stage cost required for a Lyapunov function. Next define an N-stage sum of this new stage cost as $\tilde{V}_N(x,\mathbf{u}) := \sum_{k=0}^{N-1} \tilde{\ell}(x(k),u(k))$ and perform the sum to obtain

$$\begin{aligned}
\tilde{V}_N(x,\mathbf{u}) &= \left(\sum_{k=0}^{N-1} \ell(x(k),u(k))\right) - N\ell(x_s,u_s) + \lambda(x) - \lambda(x_s) \\
&= V_N(x,\mathbf{u}) - N\ell(x_s,u_s) + \lambda(x) - \lambda(x_s)
\end{aligned} \tag{2.38}$$

Notice that $\tilde{V}_N(\cdot)$ and $V_N(\cdot)$ differ only by constant terms involving the steady state, (x_s,u_s), and the initial condition, x. Therefore because the optimization of $V_N(x,\mathbf{u})$ over $\mathbf{u}$ has a solution, so does the optimization of $\tilde{V}_N(x,\mathbf{u})$, and they are the *same* solution, giving the same control law $\kappa_N(x)$.

Because of the terminal constraint, we know that $\mathcal{X}_N$ is positive invariant for the closed-loop system. Next we verify that $\tilde{V}_N^0(x)$ is a

Lyapunov function for the closed-loop system. Since $\tilde{\ell}(x,u)$ is nonnegative, we have from (2.37) and the definition of $\tilde{V}_N$ as a sum of stage costs, that

$$\tilde{V}_N^0(x) \geq \alpha(|x - x_s|)$$

for all $x \in \mathcal{X}_N$, and we have established the required lower bound. The cost difference can be calculated to establish the required cost decrease

$$\tilde{V}_N^0(f(x, \kappa_N(x))) \leq \tilde{V}_N^0(x) - \tilde{\ell}(x, \kappa_N(x)) \leq \tilde{V}_N^0(x) - \alpha(|x - x_s|)$$

for all $x \in \mathcal{X}_N$. The remaining step is to verify the upper-bounding inequality. From Assumption 2.54 and (2.38), we know that $\tilde{V}_N^0(\cdot)$ is also continuous at x_s. Therefore, from Proposition 2.38, we have existence of $\alpha_2(\cdot) \in \mathcal{K}_\infty$ such that for all $x \in \mathcal{X}_N$

$$\tilde{V}_N^0(x) \leq \alpha_2(|x - x_s|)$$

We have established the three inequalities and $\tilde{V}_N^0(\cdot)$ is therefore a Lyapunov function in $\mathcal{X}_N$ for the system $x^+ = f(x, \kappa_N(x))$ and x_s. Theorem 2.13 then establishes that x_s is asymptotically stable in $\mathcal{X}_N$ for the closed-loop system. ■

These stability results can also be extended to time-varying and periodic systems.

Example 2.57: Economic MPC versus tracking MPC

Consider the linear system

$$f(x,u) = Ax + Bu \qquad A = \begin{bmatrix} 1/2 & 1 \\ 0 & 3/4 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

with economic cost function

$$\ell_{\text{econ}}(x,u) = q'x + r'u \qquad q = \begin{bmatrix} -2 \\ 2 \end{bmatrix} \qquad r = -10$$

and sets $\mathbb{X} = [-10, 10]^2$, $\mathbb{U} = [-1, 1]$. The economically optimal steady state is $x_s = (8, 4)$, $u_s = 1$. We compare economic MPC to tracking MPC with

$$\ell_{\text{track}}(x,u) = |x - x_s|_{10I}^2 + |u - u_s|_I^2$$

Figure 2.5 shows a phase plot of the closed-loop evolution starting from $x = (-8, 8)$. Both controllers use the terminal constraint $\mathbb{X}_f = \{x_s\}$.

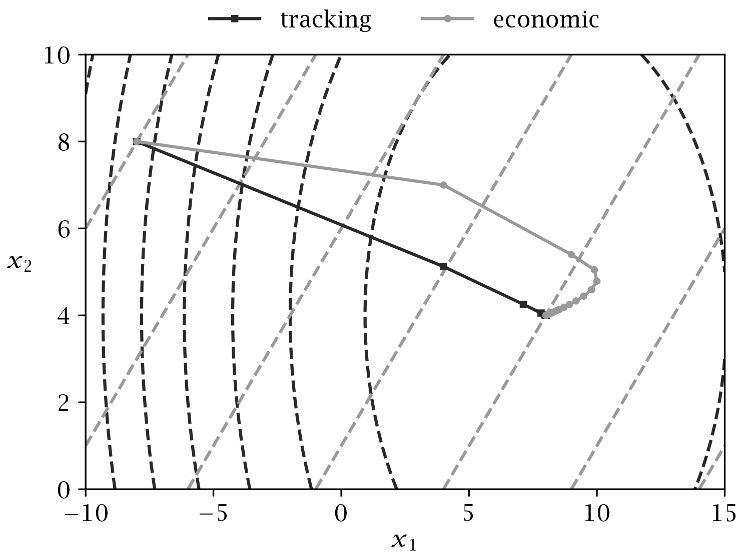


Figure 2.5: Closed-loop economic MPC versus tracking MPC starting at $x = (-8, 8)$ with optimal steady state $(8, 4)$. Both controllers asymptotically stabilize the steady state. Dashed contours show cost functions for each controller.

While tracking MPC travels directly to the setpoint, economic MPC takes a detour to achieve lower economic costs.

To prove that the economic MPC controller is stabilizing, we find a storage function. As a candidate storage function, we take

$$\lambda(x) = \mu'(x - x_s) + (x - x_s)'M(x - x_s)$$

which gives the rotated cost function

$$\widetilde{\ell}(x, u) = \ell_{\text{econ}}(x, u) + \lambda(x) - \lambda(f(x, u))$$

To start, we take $\mu = (4, 8)$ from the Lagrange multiplier of the steady-state problem. With $M = 0$, $\widetilde{\ell}(\cdot)$ is nonnegative but not positive definite, indicating that the system is dissipative but not strictly dissipative. To achieve strict dissipativity, we choose M such that $M - A'MA = 0.01I$. Although the resulting $\widetilde{\ell}(\cdot)$ function is nonconvex,

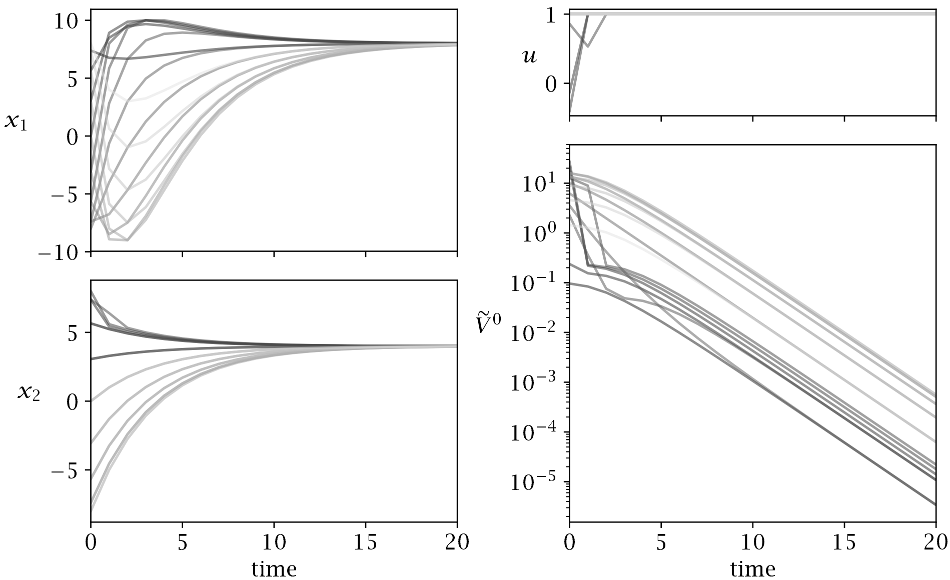


Figure 2.6: Closed-loop evolution under economic MPC. The rotated cost function $\widetilde{V}^0$ is a Lyapunov function for the system.

it is nevertheless positive definite on $\mathbb{Z}$, indicating strict dissipativity. To illustrate, we simulate a variety of initial conditions in Figure 2.6. Plotting the rotated cost function $\widetilde{V}^0(\cdot)$, we see that it is indeed a Lyapunov function for the system. □

2.9 Discrete Actuators

Discrete-valued actuators appear in nearly all large-scale industrial processes. These obviously include the on/off equipment switches. But, as discussed in Chapter 1, processes are often designed with multiple similar units such as furnaces, heaters, chillers, compressors, etc., operating in parallel. In these designs, an important aspect of the control problem is to choose how many and which of these several possible units to employ while the total feed flowrate to the process varies.

In industrial practice, these discrete decisions are usually removed from the MPC control layer and instead made at a different layer of the automation system using heuristics or other logical rules. If discrete inputs are chosen optimally, however, process performance can be greatly improved, and thus we would like to treat discrete decisions directly in MPC theory.

There are two basic issues brought about by including the discrete actuators in the control decision u. The first is theoretical: how much does the established MPC theory have to change to accommodate this class of decision variables? The second is computational: is it practical to solve the modified MPC optimal control problem in the available sample time? We address the theory question here, and find that the required changes to the existing theory are surprisingly minimal. The computational question is being addressed by the rapid development of mixed-integer solvers. It is difficult to predict what limits might emerge to slow this progress, but current mixed-integer solvers are already capable of addressing a not uninteresting class of industrial applications.

Figure 1.2 provides a representative picture of the main issue. From this perspective, if we embed the discrete decisions in the field of reals, we are merely changing the feasible region $\mathbb{U}$, from a simply connected set with an interior when describing only continuous actuators, to a disconnected set that may not have an interior when describing mixed continuous/discrete actuators. So one theoretical approach to the problem is to adjust the MPC theory to accommodate these types of $\mathbb{U}$ regions.

A careful reading of the assumptions made for the results presented thus far reveals that we have little work to do. We have not assumed that the equilibrium of interest lies in the interior of $\mathbb{U}$, or even that $\mathbb{U}$ has an interior. The main assumption about $\mathbb{U}$ are Assumption 2.3 for the time-invariant case, Assumption 2.26 for the time-varying case, and Assumption 2.50 for the economic MPC problem. The main restrictions are that $\mathbb{U}$ is closed, and sometimes compact, so that the optimization of $V_N(x, \mathbf{u})$ over $\mathbf{u}$ has a solution. All of these assumptions admit $\mathbb{U}$ regions corresponding to discrete variables. The first conclusion is that the results governing nominal closed-loop stability for various forms of MPC all pass through. These include Theorem 2.19 (time-invariant case), Theorem 2.39 (time-varying case), Theorem 2.24 ($\ell(y, u)$ stage cost), and Theorem 2.56 (economic MPC).

That does not mean that *nothing* has changed. The admissible region $\mathcal{X}_N$ in which the system is stabilized may change markedly, for example. Proposition 2.10 also passes through in the discrete-actuator case, so we know that the admissible sets are still nested, $\mathcal{X}_j \subseteq \mathcal{X}_{j+1}$

for all $j \geq 0$. But it is not unusual for systems with even linear dynamics to have *disconnected* admissible regions, which is not possible for linear systems with only continuous actuators and convex $\mathbb{U}$. When tracking a constant setpoint, the design of terminal regions and penalties must account for the fact that the discrete actuators usually remain at fixed values in a small neighborhood of the steady state of interest, and can be used only for rejecting larger disturbances and enhancing transient performance back to the steady state. Fine control about the steady state must be accomplished by the continuous actuators that are unconstrained in a neighborhood of the steady state. But this is the same issue that is faced when some subset of the continuous actuators are saturated at the steady state of interest (Rao and Rawlings, 1999), which is a routine situation in process control problems. We conclude the chapter with an example illustrating these issues.

Example 2.58: MPC with mixed continuous/discrete actuators

Consider a constant-volume tank that needs to be cooled. The system is diagrammed in Figure 2.7. The two cooling units operate such that they can be either on or off, and if on, the heat duty must be between $\dot{Q}_{\min}$ and $\dot{Q}_{\max}$. After nondimensionalizing, the system evolves according to

$$\frac{dT_1}{dt} = -\alpha(T_1 - T_0) - \rho_1(T_1 - T_2)$$
$$\frac{dT_2}{dt} = -\rho_2(T_2 - T_1) - \beta\dot{Q}$$

with $\alpha = 2$ and $\beta = \rho_1 = \rho_2 = 1$. The system states are (T_1, T_2), and the inputs are $(\dot{Q}, n_q)$ with

$$\mathbb{U} = \left\{(\dot{Q}, n_q) \in \mathbb{R} \times \{0, 1, 2\} \mid n_q\dot{Q}_{\min} \leq \dot{Q} \leq n_q\dot{Q}_{\max}\right\}$$

in which $\dot{Q}$ is the total cooling duty and n_q chooses the number of cooling units that are on at the given time. For $T_0 = 40$ and $\dot{Q}_{\max} = 10$, we wish to control the system to the steady state $x_s = (35, 25)$, $u_s = (10, 1)$, using costs $Q = I$ and $R = 10^{-3}I$. The system is discretized with $\Delta = 0.25$.

To start, we choose a terminal region and control law. Assuming $\dot{Q}_{\min} > 0$, both components of u are at constraints at the steady state, and thus we cannot use them in a linear terminal control law. The system is stable for $\kappa_f(x) = u_s$, however, and a valid terminal cost is $V_f(x) = (x - x_s)'P(x - x_s)$ with P satisfying $A'PA - P = Q$. As a terminal set we take $\mathbb{X}_f = \{x \mid V_f(x) \leq 1\}$, although any level set

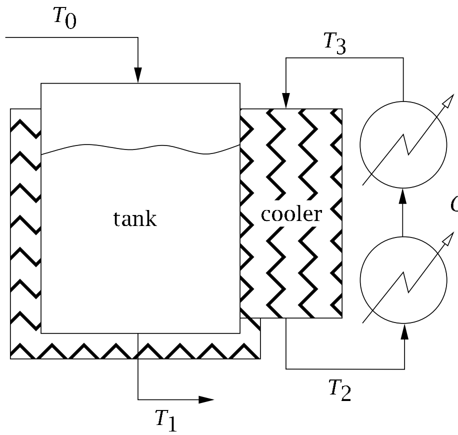


Figure 2.7: Diagram of tank/cooler system. Each cooling unit can be either on or off, and if on, it must be between its (possibly nonzero) minimum and maximum capacities.

would suffice. With this terminal region, Figure 2.8 shows the feasible sets for $\dot{Q}_{\min} = 0$ and $\dot{Q}_{\min} = 9$. Note that for $\dot{Q}_{\min} > 0$, the projection of $\mathbb{U}$ onto the total heat duty $\dot{Q}$ is a disconnected set of possible heat duties, leading to disconnected sets $\mathcal{X}_N$ for $N \leq 5$. (The sets $\mathcal{X}_N$ for $N \geq 6$ are connected.)

To control the system, we solve the standard MPC problem with horizon $N = 8$. Figure 2.9 shows a phase portrait of closed-loop evolution for various initial conditions with $\dot{Q}_{\min} = 9$. Each evaluation of the control law requires solving a mixed-integer, quadratically constrained QP (with the quadratic constraint due to the terminal region). In general, the controller chooses $u_2 = 1$ near the setpoint and $u_2 \in \{0, 2\}$ far from it, although this behavior is not global. Despite the disconnected nature of $\mathbb{U}$, all initial conditions are driven asymptotically to the setpoint. $\square$

2.10 Concluding Comments

MPC is an implementation, for practical reasons, of receding horizon control (RHC), in which offline determination of the RHC law $\kappa_N(\cdot)$ is replaced by online determination of its value $\kappa_N(x)$, the control action, at each state x encountered during its operation. Because the optimal

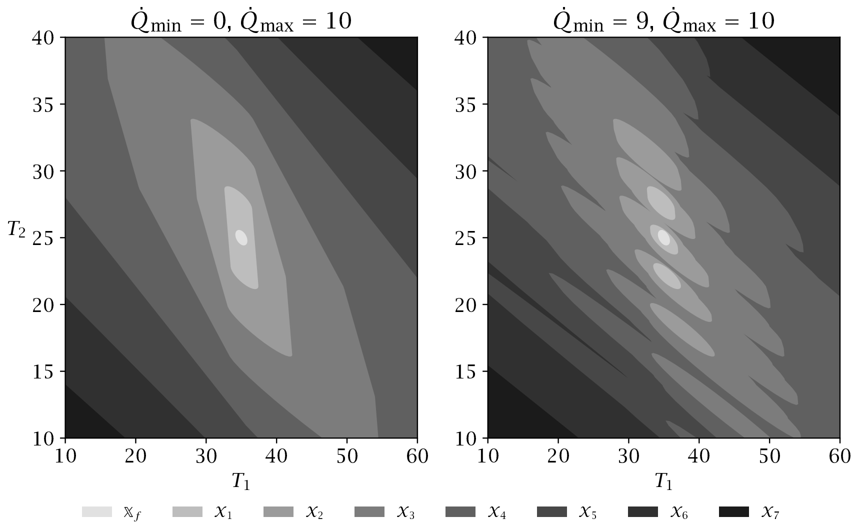


Figure 2.8: Feasible sets $\mathcal{X}_N$ for two values of $\dot{Q}_{\min}$. Note that for $\dot{Q}_{\min} = 9$ (right-hand side), $\mathcal{X}_N$ for $N \leq 4$ are disconnected sets.

control problem that defines the control is a finite horizon problem, neither stability nor optimality of the cost function is necessarily achieved by a receding horizon or model predictive controller.

This chapter shows how stability may be achieved by adding a terminal cost function and a terminal constraint to the optimal control problem. Adding a terminal cost function adds little or no complexity to the optimal control problem that has to be solved online, and usually improves performance. Indeed, the infinite horizon value function $V_\infty^0(\cdot)$ for the constrained problem would be an ideal choice for the terminal penalty because the value function $V_N^0(\cdot)$ for the online optimal control problem would then be equal to $V_\infty^0(\cdot)$, and the controller would inherit the performance advantages of the infinite horizon controller. In addition, the actual trajectories of the controlled system would be precisely equal, in the absence of uncertainty, to those predicted by the online optimizer. Of course, if we knew $V_\infty^0(\cdot)$, the optimal infinite horizon controller $\kappa_\infty(\cdot)$ could be determined and there would be no reason to employ MPC.

The infinite horizon cost $V_\infty^0(\cdot)$ is known globally only for special

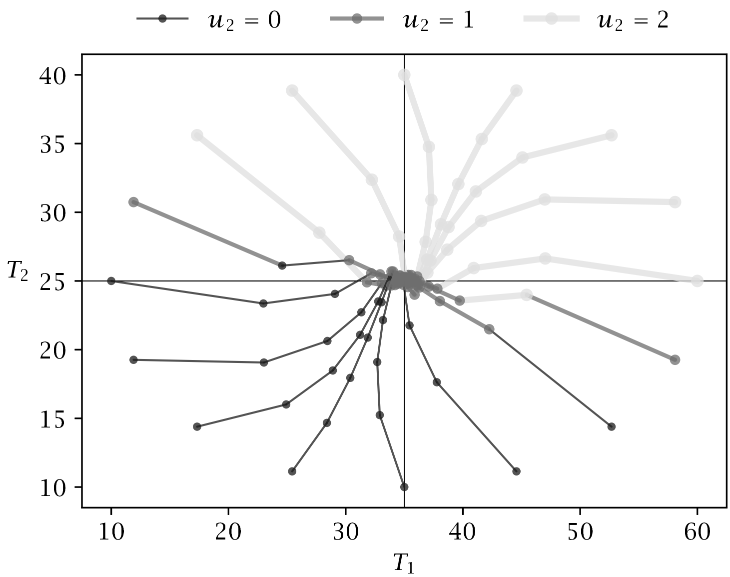


Figure 2.9: Phase portrait for closed-loop evolution of cooler system with $\dot{Q}_{\min} = 9$. Line colors show value of discrete actuator u_2.

cases, however, such as the linear quadratic (LQ) unconstrained problem. For more general problems in which constraints and/or nonlinearity are present, its value—or approximate value—in a neighborhood of the setpoint can usually be obtained and the use of this local control Lyapunov function (CLF) should, in general, enhance performance. Adding a terminal cost appears to be generally advantageous.

The reason for the terminal constraint is precisely the fact that the terminal penalty is usually merely a local CLF defined in the set $\mathbb{X}_f$; to benefit from the terminal cost, the terminal state must be constrained to lie in $\mathbb{X}_f$. Unlike the addition of a terminal penalty, however, addition of a terminal constraint may increase complexity of the optimal control problem considerably. Because efficient programs exist for solving quadratic programs (QPs), in which the cost function to be minimized is quadratic and the constraints polyhedral, there is an argument for using polyhedral constraints. Indeed, a potential terminal constraint set for the constrained LQ optimal control problem is the maximal constraint admissible set, which is polyhedral. This set is complex, however, i.e., defined by many linear inequalities, and would appear to be unsuitable for the complex control problems routinely encountered in industry.

A terminal constraint set that is considerably simpler is a suitable sublevel set of the terminal penalty, which is often a simple positive definite quadratic function resulting in a convex terminal constraint set. A disadvantage is that the terminal constraint set is now ellipsoidal rather than polytopic, and conventional QPs cannot be employed for the LQ constrained optimal control problem. This does not appear to be a serious disadvantage, however, because the optimal control problem remains convex, so interior point methods may be readily employed.

In the nonlinear case, adding an ellipsoidal terminal constraint set does not appreciably affect the complexity of the optimal control problem. A more serious problem, when the system is nonlinear, is that the optimal control problem is then usually nonconvex so that global solutions, on which many theoretical results are predicated, are usually too difficult to obtain. A method for dealing with this difficulty, which also has the advantage of reducing online complexity, is suboptimal MPC, described in this chapter and also in Chapter 6.

The current chapter also presents some results that contribute to an understanding of the subject but do not provide practical tools. For example, it is useful to know that the domain of attraction for many of the controllers described here is $\mathcal{X}_N$, the set of initial states controllable to the terminal constraint set, but this set cannot usually be computed. The set is, in principle, computable using the dynamic programming (DP) equations presented in this chapter, and may be computed if the system is linear and the constraints, including the terminal constraint, are polyhedral, provided that the state dimension and the horizon length are suitably small—considerably smaller than in problems routinely encountered in industry. In the nonlinear case, this set cannot usually be computed. Computation difficulties are not resolved if $\mathcal{X}_N$ is replaced by a suitable sublevel set of the value function $V_N^0(\cdot)$. Hence, in practice, both for linear and nonlinear MPC, this set has to be estimated by simulation.

2.11 Notes

MPC has an unusually rich history, making it impossible to summarize here the many contributions that have been made. Here we restrict attention to a subset of this literature that is closely related to the approach adopted in this book. A fuller picture is presented in the review paper (Mayne, Rawlings, Rao, and Scokaert, 2000).

The success of conventional MPC derives from the fact that for deterministic problems (no uncertainty), feedback is not required so the solution to the open-loop optimal control problem solved online for a particular initial state is the same as that obtained by solving the feedback problem using DP, for example. Lee and Markus (1967) pointed out the possibility of MPC in their book on optimal control.

> One technique for obtaining a feedback controller synthesis is to measure the current control process state and then compute very rapidly the open-loop control function. The first portion of this function is then used during a short time interval after which a new measurement of the process state is made and a new open-loop control function is computed for this new measurement. The procedure is then repeated.

Even earlier, Propoi (1963) proposed a form of MPC utilizing linear programming, for the control of linear systems with hard constraints on the control. A big surge in interest in MPC occurred when Richalet, Rault, Testud, and Papon (1978b) advocated its use for process control. A whole series of papers, such as (Richalet, Rault, Testud, and Papon, 1978a), (Cutler and Ramaker, 1980), (Prett and Gillette, 1980), (García and Morshedi, 1986), and (Marquis and Broustail, 1988) helped cement its popularity in the process control industries, and MPC soon became the most useful method in modern control technology for control problems with hard constraints—with thousands of applications to its credit.

The basic question of stability, an important issue since optimizing a finite horizon cost does not necessarily yield a stabilizing control, was not resolved in this early literature. Early academic research in MPC, reviewed in García, Prett, and Morari (1989), did not employ Lyapunov theory and therefore restricted attention to control of unconstrained linear systems, studying the effect of control and cost horizons on stability. Similar studies appeared in the literature on generalized predictive control (GPC) (Ydstie, 1984; Peterka, 1984; De Keyser and Van Cauwenberghe, 1985; Clarke, Mohtadi, and Tuffs, 1987) that arose to address deficiencies in minimum variance control. Interestingly enough, earlier research on RHC (Kleinman, 1970; Thomas, 1975; Kwon and Pearson, 1977) had shown indirectly that the imposition of a terminal equality constraint in the finite horizon optimal control problem ensured closed-loop stability for linear unconstrained systems. That a terminal equality constraint had an equally beneficial effect for constrained nonlinear discrete time systems was shown by Keerthi and Gilbert (1988) and for constrained nonlinear continuous time systems by Chen and Shaw (1982) and Mayne and Michalska (1990). In each of these papers, Lyapunov stability theory was employed in contrast to the then current literature on MPC and GPC.

The next advance showed that incorporation of a suitable terminal cost and terminal constraint in the finite horizon optimal control problem ensured closed-loop stability; the terminal constraint set is required to be control invariant, and the terminal cost function is required to be a local CLF. Perhaps the earliest proposal in this direction is the brief paper by Sznaier and Damborg (1987) for linear systems with polytopic constraints; in this prescient paper the terminal cost is chosen to be the value function for the *unconstrained* infinite horizon optimal control problem, and the terminal constraint set is the maximal constraint admissible set (Gilbert and Tan, 1991) for the optimal controlled system.[6] A suitable terminal cost and terminal constraint set for constrained nonlinear continuous time systems was proposed in Michalska and Mayne (1993) in the context of dual-mode MPC. In a paper that has had considerable impact, Chen and Allgöwer (1998) showed that similar ingredients may be employed to stabilize constrained nonlinear continuous time systems when conventional MPC is employed. Related results were obtained by Parisini and Zoppoli (1995), and De Nicolao, Magni, and Scattolini (1996).

Stability proofs for the form of MPC proposed, but not analyzed, in Sznaier and Damborg (1987) were finally provided by Chmielewski and Manousiouthakis (1996) and Scokaert and Rawlings (1998). These papers also showed that optimal control for the *infinite* horizon constrained optimal control problem for a specified initial state is achieved if the horizon is chosen sufficiently long. A terminal constraint is not required if a global, rather than a local, CLF is available for use as a terminal cost function. Thus, for the case when the system being controlled is linear and stable, and subject to a convex control constraint, Rawlings and Muske (1993) showed, in a paper that raised considerable interest, that closed-loop stability may be obtained if the terminal

[6] If the optimal infinite horizon controlled system is described by $x^+ = A_K x$ and if the constraints are $u \in \mathbb{U}$ and $x \in \mathbb{X}$, then the maximal constraint admissible set is $\{x \mid A_K^i x \in \mathbb{X},\ KA_K^i x \in \mathbb{U}\ \forall i \in \mathbb{I}_{\geq 0}\}$.

constraint is omitted and the infinite horizon cost using zero control is employed as the terminal cost. The resultant terminal cost is a global CLF.

The basic principles ensuring closed-loop stability in these and many other papers including (De Nicolao, Magni, and Scattolini, 1998), and (Mayne, 2000) were distilled and formulated as "stability axioms" in the review paper (Mayne et al., 2000); they appear as Assumptions 2.2, 2.3, and 2.14 in this chapter. These assumptions provide sufficient conditions for closed-loop stability for a given horizon. There is an alternative literature that shows that closed-loop stability may often be achieved if the horizon is chosen to be sufficiently long. Contributions in this direction include (Primbs and Nevistić, 2000), (Jadbabaie, Yu, and Hauser, 2001), as well as (Parisini and Zoppoli, 1995; Chmielewski and Manousiouthakis, 1996; Scokaert and Rawlings, 1998) already mentioned. An advantage of this approach is that it avoids addition of an explicit terminal constraint, although this may be avoided by alternative means as shown in Section 2.6. A significant development of this approach (Grüne and Pannek, 2011) gives a comprehensive investigation and extension of the conditions that ensure recursive feasibility and stability of MPC that does not have a terminal constraint. On the other hand, it has been shown (Mayne, 2013) that an explicit or implicit terminal constraint is necessary if positive invariance and the nested property $\mathcal{X}_{j+1} \supset \mathcal{X}_j$, $j \in \mathbb{I}_{\geq 0}$ of the feasible sets are required; the nested property ensures recursive feasibility.

Recently several researchers (Limon, Alvarado, Alamo, and Camacho, 2008, 2010; Fagiano and Teel, 2012; Falugi and Mayne, 2013a; Müller and Allgöwer, 2014; Mayne and Falugi, 2016) have shown how to extend the region of attraction $\mathcal{X}_N$, and how to solve the related problem of tracking a randomly varying reference—thereby alleviating the disadvantage caused by the reduction in the region of attraction due to the imposition of a terminal constraint. Attention has also been given to the problem of tracking a periodic reference using model predictive control (Limon et al., 2012; Falugi and Mayne, 2013b; Rawlings and Risbeck, 2017).

Regarding the analysis of nonpositive stage costs in Section 2.4.4, Grimm, Messina, Tuna, and Teel (2005) use a storage function like $\Lambda(\cdot)$ to compensate for a semidefinite stage cost. Cai and Teel (2008) give a discrete time converse theorem for IOSS for all $\mathbb{R}^n$. Allan and Rawlings (2017) give a converse theorem for IOSS on closed positive invariant sets and provide a lemma for changing the supply rate function.

Suboptimal MPC based on a warm start was proposed and analyzed by Scokaert et al. (1999). Pannocchia et al. (2011) establish that this form of suboptimal MPC is robustly stable for systems without state constraints if the terminal constraint is replaced with an enlarged terminal penalty. As noted by Yu, Reble, Chen, and Allgöwer (2014), however, the assumptions used for these results are strong enough to imply that the *optimal* value function is *continuous*. Allan et al. (2017) establish robustness for systems with discontinuous feedback and discontinuous optimal value function.

Lazar and Heemels (2009) analyze robustness of suboptimal MPC with respect to state disturbances under the condition that the suboptimal controller is able to find a solution within a specific degree of suboptimality from the global solution. Roset, Heemels, Lazar, and Nijmeijer (2008), show how to extend the analysis to treat measurement disturbances as well as state disturbances. Because this type of suboptimal MPC is defined in terms of the globally optimal cost, its implementation requires, in principle, global solvers.

Economic MPC was introduced in Rawlings and Amrit (2009), but designing process controllers other than MPC to optimize process economics has been a part of industrial practice for a long time. When using an economic (as opposed to tracking) stage cost, cost inequalities and conditions for asymptotic stability have been established for time-invariant systems with a steady state (Diehl et al., 2011; Amrit, Rawlings, and Angeli, 2011; Angeli et al., 2012; Ellis, Durand, and Christofides, 2014). Such results have been extended in Zanon, Gros, and Diehl (2013) to the time-varying periodic case under the assumptions of a linear storage function and Lipschitz continuity of the model and stage cost; Rawlings and Risbeck (2017) require only continuity of the model and stage cost, and allow a more general form for the storage function.

For the case of a time-invariant system with optimal periodic operation, convergence to the optimal periodic solution can be shown using similar notions of dissipativity (Müller and Grüne, 2015); but this case is different than the case treated by Rawlings and Risbeck (2017) because clock time does not appear. In Müller, Angeli, and Allgöwer (2015), the authors establish the interesting result that a certain dissipativity condition is also *necessary* for asymptotic stability. For periodic processes, stability has been investigated by converting to deviation variables (Huang, Harinath, and Biegler, 2011; Rawlings and Risbeck, 2017).

Various results on stability of MPC with discrete actuators have appeared in the literature. In Bemporad and Morari (1999), convergence to the origin is shown for mixed-logical-dynamical systems based on certain positive definite restrictions on the stage cost, although Lyapunov stability is not explicitly shown. For piecewise affine systems, Baotic, Christophersen, and Morari (2006) establish asymptotic stability for an infinite horizon control law via Lyapunov function arguments. In Di Cairano, Heemels, Lazar, and Bemporad (2014), a hybrid Lyapunov function is directly embedded within the optimal control problem, enforcing cost decrease as a hard constraint and ensuring closed-loop asymptotic stability. Alternatively, practical stability (i.e., boundedness) can often be shown by treating discretization of inputs as a disturbance and deriving error bounds with respect to the relaxed continuous-actuator system (Quevedo, Goodwin, and De Doná, 2004; Aguilera and Quevedo, 2013; Kobayshi, Shein, and Hiraishi, 2014). Finally, Picasso, Pancanti, Bemporad, and Bicchi (2003) shows asymptotic stability for open-loop stable linear systems with only practical stability for open-loop unstable systems. All of these results are concerned with stability of a steady state.

The approach presented in this chapter, which shows that current MPC asymptotic stability theorems based on Lyapunov functions also cover general nonlinear systems with mixed continuous/discrete actuators, was developed by Rawlings and Risbeck (2017).

2.12 Exercises

Exercise 2.1: Discontinuous MPC

In Example 2.8, compute $\mathcal{U}_3(x)$, $V_3^0(x)$, and $\kappa_3(x)$ at a few points on the unit circle.

Exercise 2.2: Boundedness of discrete time model

Complete the proof of Proposition 2.16 by showing that $f(\cdot)$ and $f_{\mathbb{Z}}^{-1}(\cdot)$ are bounded on bounded sets.

Exercise 2.3: Destabilization with state constraints

Consider a state feedback regulation problem with the origin as the setpoint (Muske and Rawlings, 1993). Let the system be

$$A = \begin{bmatrix} 4/3 & -2/3 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad C = [-2/3 \;\; 1]$$

and the controller objective function tuning matrices be

$$Q = I \qquad R = I \qquad N = 5$$

(a) Plot the unconstrained regulator performance starting from initial condition $x(0) = \begin{bmatrix} 3 & 3 \end{bmatrix}'$.

(b) Add the output constraint $y(k) \leq 0.5$. Plot the response of the constrained regulator (both input and output). Is this regulator stabilizing? Can you modify the tuning parameters Q, R to affect stability as in Section 1.3.4?

(c) Change the output constraint to $y(k) \leq 1 + \epsilon, \epsilon > 0$. Plot the closed-loop response for a variety of ϵ. Are any of these regulators destabilizing?

(d) Set the output constraint back to $y(k) \leq 0.5$ and add the terminal constraint $x(N) = 0$. What is the solution to the regulator problem in this case? Increase the horizon N. Does this problem eventually go away?

Exercise 2.4: Computing the projection of $\mathbb{Z}$ onto $\mathcal{X}_N$

Given a polytope

$$\mathbb{Z} := \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m \mid Gx + Hu \leq \psi\}$$

write an Octave or MATLAB program to determine $\mathcal{X}$, the projection of $\mathbb{Z}$ onto $\mathbb{R}^n$

$$\mathcal{X} = \{x \in \mathbb{R}^n \mid \exists u \in \mathbb{R}^m \text{ such that } (x, u) \in \mathbb{Z}\}$$

Use algorithms 3.1 and 3.2 in Keerthi and Gilbert (1987).

To check your program, consider a system

$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

subject to the constraints $\mathbb{X} = \{x \mid x_1 \leq 2\}$ and $\mathbb{U} = \{u \mid -1 \leq u \leq 1\}$. Consider the MPC problem with $N = 2$, $\mathbf{u} = (u(0), u(1))$, and the set $\mathbb{Z}$ given by

$$\mathbb{Z} = \{(x, \mathbf{u}) \mid x, \phi(1; x, \mathbf{u}), \phi(2; x, \mathbf{u}) \in \mathbb{X} \text{ and } u(0), u(1) \in \mathbb{U}\}$$

Verify that the set

$$\mathcal{X}_2 := \{x \in \mathbb{R}^2 \mid \exists \mathbf{u} \in \mathbb{R}^2 \text{ such that } (x, \mathbf{u}) \in \mathbb{Z}\}$$

is given by

$$\mathcal{X}_2 = \{x \in \mathbb{R}^2 \mid Px \leq p\} \qquad P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad p = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

Exercise 2.5: Computing the maximal output admissible set

Write an Octave or MATLAB program to determine the maximal constraint admissible set for the system $x^+ = Fx, y = Hx$ subject to the hard constraint $y \in Y$ in which $Y = \{y \mid Ey \leq e\}$. Use algorithm 3.2 in Gilbert and Tan (1991).

To check your program, verify for the system

$$F = \begin{bmatrix} 0.9 & 1 \\ 0 & 0.09 \end{bmatrix} \qquad H = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

subject to the constraint $Y = \{y \mid -1 \leq y \leq 1\}$, and that the maximal output admissible set is given by

$$O_\infty = \{x \in \mathbb{R}^2 \mid Ax \leq b\} \qquad A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 0.9 & 1.09 \\ -0.9 & -1.09 \\ 0.81 & 0.9981 \\ -0.81 & -0.9981 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Show that t^*, the smallest integer t such that $O_t = O_\infty$ satisfies $t^* = 2$.

What happens to t^* as F_{22} increases and approaches one? What do you conclude for the case $F_{22} \geq 1$?

Exercise 2.6: Terminal constraint and region of attraction

Consider the system

$$x^+ = Ax + Bu$$

subject to the constraints

$$x \in \mathbb{X} \qquad u \in \mathbb{U}$$

in which

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbb{X} = \{x \in \mathbb{R}^2 \mid x_1 \leq 5\} \qquad \mathbb{U} = \{u \in \mathbb{R}^2 \mid -\mathbf{1} \leq u \leq \mathbf{1}\}$$

and $\mathbf{1} \in \mathbb{R}^2$ is a vector of ones. The MPC cost function is

$$V_N(x, \mathbf{u}) = \sum_{i=0}^{N-1} \ell(x(i), u(i)) + V_f(x(N))$$

in which

$$\ell(x, u) = (1/2)\left(|x|_Q^2 + |u|^2\right) \qquad Q = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$$

and $V_f(\cdot)$ is the terminal penalty on the final state.

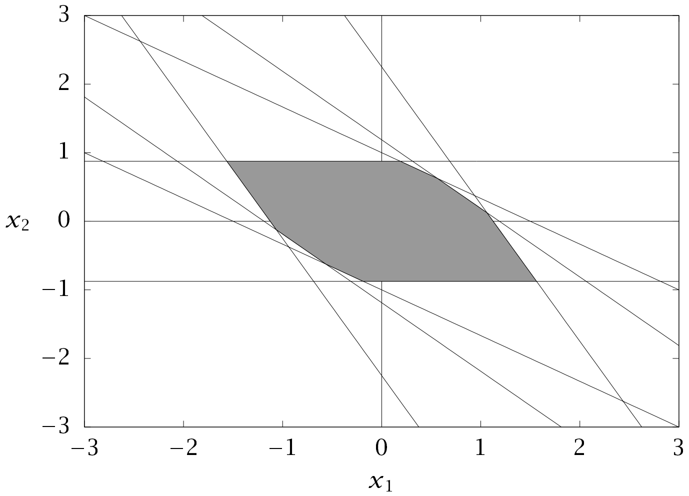


Figure 2.10: Region of attraction (shaded region) for constrained MPC controller of Exercise 2.6.

(a) Implement unconstrained MPC with no terminal cost ($V_f(\cdot) = 0$) for a few values of α. Choose a value of α for which the resultant closed loop is unstable. Try $N = 3$.

(b) Implement constrained MPC with no terminal cost or terminal constraint for the value of α obtained in the previous part. Is the resultant closed loop stable or unstable?

(c) Implement constrained MPC with terminal equality constraint $x(N) = 0$ for the same value of α. Find the region of attraction for the constrained MPC controller using the projection algorithm from Exercise 2.4. The result should resemble Figure 2.10.

Exercise 2.7: Infinite horizon cost to go as terminal penalty

Consider the system

$$x^+ = Ax + Bu$$

subject to the constraints

$$x \in \mathbb{X} \qquad u \in \mathbb{U}$$

in which

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\mathbb{X} = \{x \in \mathbb{R}^2 \mid -5 \le x_1 \le 5\} \qquad \mathbb{U} = \{u \in \mathbb{R}^2 \mid -\mathbf{1} \le u \le \mathbf{1}\}$$

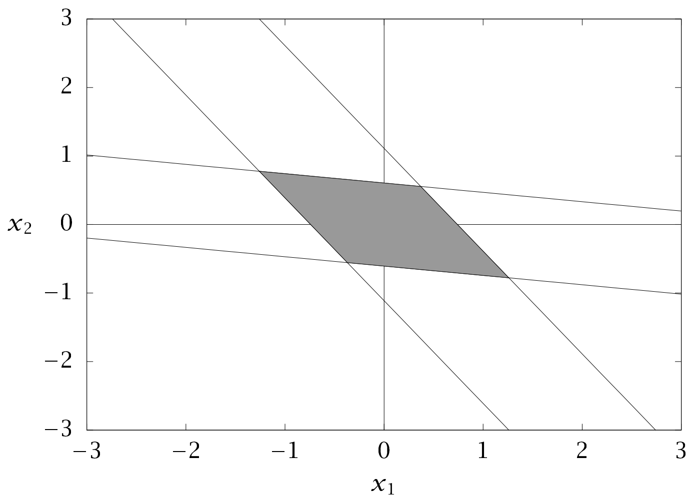


Figure 2.11: The region $\mathbb{X}_f$, in which the unconstrained LQR control law is feasible for Exercise 2.7.

The cost is

$$V_N(x, \mathbf{u}) := \sum_{i=0}^{N-1} \ell(x(i), u(i)) + V_f(x(N))$$

in which

$$\ell(x, u) = (1/2)(|x|_Q^2 + |u|^2) \qquad Q = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$$

and $V_f(\cdot)$ is the terminal penalty on the final state and $\mathbf{1} \in \mathbb{R}^2$ is a vector of all ones. Use $\alpha = 10^{-5}$ and $N = 3$ and terminal cost $V_f(x) = (1/2)x'\Pi x$ where Π is the solution to the steady-state Riccati equation.

(a) Compute the infinite horizon optimal cost and control law for the unconstrained system.

(b) Find the region $\mathbb{X}_f$, the maximal constraint admissible set using the algorithm in Exercise 2.5 for the system $x^+ = (A + BK)x$ with constraints $x \in \mathbb{X}$ and $Kx \in \mathbb{U}$. You should obtain the region shown in Figure 2.11.

(c) Add a terminal constraint $x(N) \in \mathbb{X}_f$ and implement constrained MPC. Find $\mathcal{X}_N$, the region of attraction for the MPC problem with $V_f(\cdot)$ as the terminal cost and $x(N) \in \mathbb{X}_f$ as the terminal constraint. Contrast it with the region of attraction for the MPC problem in Exercise 2.6 with a terminal constraint $x(N) = 0$.

(d) Estimate $\bar{\mathcal{X}}_N$, the set of initial states for which the MPC control sequence for horizon N is equal to the MPC control sequence for an infinite horizon.
Hint: $x \in \bar{\mathcal{X}}_N$ if $x^0(N; x) \in \operatorname{int}(\mathbb{X}_f)$. Why?

Exercise 2.8: Terminal penalty with and without terminal constraint

Consider the system

$$x^+ = Ax + Bu$$

subject to the constraints

$$x \in \mathbb{X} \qquad u \in \mathbb{U}$$

in which

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\mathbb{X} = \{x \in \mathbb{R}^2 \mid -15 \le x_1 \le 15\} \qquad \mathbb{U} = \{u \in \mathbb{R}^2 \mid -5 \cdot \mathbf{1} \le u \le 5 \cdot \mathbf{1}\}$$

The cost is

$$V_N(x, \mathbf{u}) = \sum_{i=0}^{N-1} \ell(x(i), u(i)) + V_f(x(N))$$

in which

$$\ell(x, u) = (1/2)(|x|_Q^2 + |u|)^2 \qquad Q = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$$

$V_f(\cdot)$ is the terminal penalty on the final state, and $\mathbf{1} \in \mathbb{R}^2$ is a vector of ones.

Use $\alpha = 10^{-5}$ and $N = 3$ and terminal cost $V_f(x) = (1/2)x'\Pi x$ where $V_f(\cdot)$ is the infinite horizon optimal cost for the unconstrained problem.

(a) Add a terminal constraint $x(N) \in \mathbb{X}_f$, in which $\mathbb{X}_f$ is the maximal constraint admissible set for the system $x^+ = (A + BK)x$ and K is the optimal controller gain for the unconstrained problem. Using the code developed in Exercise 2.7, estimate $\mathcal{X}_N$, the region of attraction for the MPC problem with this terminal constraint and terminal cost. Also estimate $\bar{\mathcal{X}}_N$, the region for which the MPC control sequence for horizon N is equal to the MPC control sequence for infinite horizon. Your results should resemble Figure 2.12.

(b) Remove the terminal constraint and *estimate* the domain of attraction $\hat{\mathcal{X}}_N$ (by simulation). Compare this $\hat{\mathcal{X}}_N$ with $\mathcal{X}_N$ and $\bar{\mathcal{X}}_N$ obtained previously.

(c) Change the terminal cost to $V_f(x) = (3/2)x'\Pi x$ and repeat the previous part.

Exercise 2.9: Decreasing property for the time-varying case

Consider the time-varying optimal control problem specified in 2.4.5. Suppose that $V_f(\cdot)$ and $\mathbb{X}_f$ satisfy the basic stability assumption, Assumption 2.33 Prove that the value function $V_N^0(\cdot)$ has the decreasing property

$$V_N^0((x,i)^+) \le V_N^0(x,i) - \ell(x, i, \kappa_N(x,i))$$

for all $(x, i) \in \mathbb{X}\mathbb{I}$.

Exercise 2.10: Terminal cost bound for the time-varying case

Refer to Section 2.4.5. Prove that the value function $V_N^0(\cdot)$ satisfies

$$V_N^0(x,i) \le V_f(x, i+N) \quad \forall (x,i) \in \mathbb{X}_f(i+N) \times \mathbb{I}_{\ge 0}$$

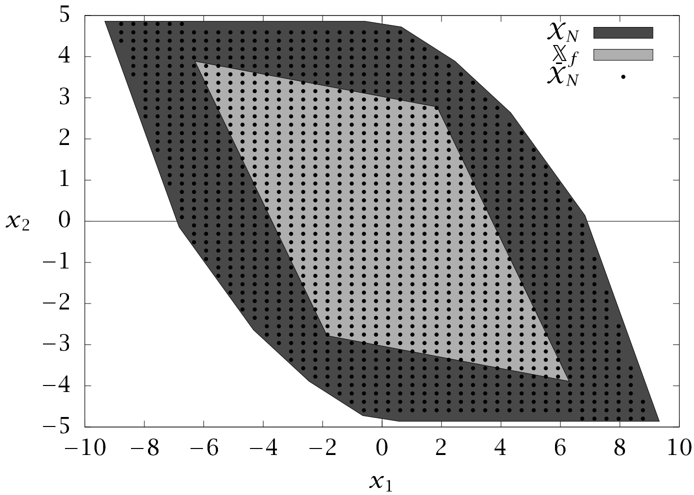


Figure 2.12: The region of attraction for terminal constraint $x(N) \in \mathbb{X}_f$ and terminal penalty $V_f(x) = (1/2)x'\Pi x$ and the estimate of $\bar{\mathcal{X}}_N$ for Exercise 2.8.

Exercise 2.11: Modification of terminal cost

Refer to Section 2.6. Show that the pair $(\beta V_f(\cdot), \mathbb{X}_f)$ satisfies Assumption 2.14 if $(V_f(\cdot), \mathbb{X}_f)$ satisfies this assumptions, $\beta \geq 1$, and $\ell(\cdot)$ satisfies Assumption 2.2.

Exercise 2.12: A Lyapunov theorem for asymptotic stability

Prove the asymptotic stability result for Lyapunov functions.

Theorem 2.59 (Lyapunov theorem for asymptotic stability)**.** *Given the dynamic system*

$$x^+ = f(x) \qquad 0 = f(0)$$

The origin is asymptotically stable if there exist $\mathcal{K}$ functions α, β, γ, and $r > 0$ such that Lyapunov function V satisfies for $x \in r\mathcal{B}$

$$\alpha(|x|) \leq V(x) \leq \beta(|x|)$$
$$V(f(x)) - V(x) \leq -\gamma(|x|)$$

Exercise 2.13: An MPC stability result

Given the following nonlinear model and objective function

$$x^+ = f(x, u), \qquad 0 = f(0, 0)$$
$$x(0) = x$$
$$V_N(x, \mathbf{u}) = \sum_{k=0}^{N-1} \ell(x(k), u(k))$$

Consider the terminal constraint MPC regulator

$$\min_{\mathbf{u}} V_N(x, \mathbf{u})$$

subject to

$$x^+ = f(x, u) \qquad x(0) = x \qquad x(N) = 0$$

and denote the first move in the optimal control sequence as $u^0(x)$. Given the closed-loop system

$$x^+ = f(x, u^0(x))$$

(a) Prove that the origin is asymptotically stable for the closed-loop system. State the cost function assumption and controllability assumption required so that the control problem is feasible for some set of defined initial conditions.

(b) What assumptions about the cost function $\ell(x, u)$ are required to strengthen the controller so that the origin is exponentially stable for the closed-loop system? How does the controllability assumption change for this case?

Exercise 2.14: Stability using observability instead of IOSS

Assume that the system $x^+ = f(x, u)$, $y = h(x)$ is ℓ-observable, i.e., there exists a $\alpha \in \mathcal{K}$ and an integer $N_o \geq 1$ such that

$$\sum_{j=0}^{N_o-1} \ell(y(i), u(i)) \geq \alpha(|x|)$$

for all x and all $\mathbf{u}$; here $x(i) := \phi(i; x, \mathbf{u})$ and $y(i) := h(x(i))$. Prove the result given in Section 2.4.4 that the origin is asymptotically stable for the closed-loop system $x^+ = f(x, \kappa_N(x))$ using the assumption that $x^+ = f(x, u)$, $y = h(x)$ is ℓ-observable rather than IOSS. Assume that $N \geq N_o$.

Exercise 2.15: Input/output-to-state stability (IOSS) and convergence

Proposition 2.60 (Convergence of state under IOSS)**.** *Assume that the system* $x^+ = f(x, u)$, $y = h(x)$ *is IOSS and that* $u(i) \to 0$ *and* $y(i) \to 0$ *as* $i \to \infty$. *Then* $x(i) = \phi(i; x, \mathbf{u}) \to 0$ *as* $i \to \infty$ *for any initial state* x.

Prove Proposition 2.60. Hint: consider the solution at time $k + l$ using the state at time k as the initial state.

Exercise 2.16: Equality for quadratic functions

Prove the following result which is useful for analyzing the unreachable setpoint problem.

Lemma 2.61 (An equality for quadratic functions)**.** *Let* $\mathbb{X}$ *be a nonempty compact subset of* $\mathbb{R}^n$, *and let* $\ell(\cdot)$ *be a strictly convex quadratic function on* $\mathbb{X}$ *defined by* $\ell(x) := (1/2)x'Qx + q'x + c, \quad Q > 0$. *Consider a sequence* $(x(i))_{i \in \mathbb{I}_{1:P}}$ *with mean* $\bar{x}_P := (1/P)\sum_{i=1}^{P} x(i)$. *Then the following holds*

$$\sum_{i=1}^{P} \ell(x(i)) = (1/2) \sum_{i=1}^{P} |x(i) - \bar{x}_P|_Q^2 + P\ell(\bar{x}_P)$$

It follows from this lemma that $\ell(\bar{x}_P) \leq (1/P)\sum_{i=1}^{P} \ell(x(i))$, which is Jensen's inequality for the special case of a quadratic function.

Exercise 2.17: Unreachable setpoint MPC and evolution in a compact set

Prove the following lemma, which is useful for analyzing the stability of MPC with an unreachable setpoint.

Lemma 2.62 (Evolution in a compact set)**.** *Suppose $x(0) = x$ lies in the set $\mathcal{X}_N$. Then the state trajectory $(x(i))$ where, for each i, $x(i) = \phi_f(i; x)$ of the controlled system $x^+ = f(x)$ evolves in a compact set.*

Exercise 2.18: MPC and multivariable, constrained systems

Consider a two-input, two-output process with the following transfer function

$$G(s) = \begin{bmatrix} \dfrac{2}{10s+1} & \dfrac{2}{s+1} \\ \dfrac{1}{s+1} & -\dfrac{4}{s+1} \end{bmatrix}$$

(a) Consider a unit setpoint change in the first output. Choose a reasonable sample time, Δ. Simulate the behavior of an offset-free discrete time MPC controller with $Q = I, S = I$ and large N.

(b) Add the constraint $-1 \leq u(k) \leq 1$ and simulate the response.

(c) Add the constraint $-0.1 \leq \Delta u/\Delta \leq 0.1$ and simulate the response.

(d) Add significant noise to both output measurements (make the standard deviation in each output about 0.1). Retune the MPC controller to obtain good performance. Describe which controller parameters you changed and why.

Exercise 2.19: LQR versus LAR

We are now all experts on the linear quadratic regulator (LQR), which employs a linear model and quadratic performance measure. Let's consider the case of a linear model but absolute value performance measure, which we call the linear absolute regulator (LAR)[7]

$$\min_{\mathbf{u}} \sum_{k=0}^{N-1} \left(q\,|x(k)| + r\,|u(k)|\right) + q(N)\,|x(N)|$$

For simplicity consider the following one-step controller, in which u and x are *scalars*

$$\min_{u(0)} V(x(0), u(0)) = |x(1)| + |u(0)|$$

subject to

$$x(1) = Ax(0) + Bu(0)$$

Draw a sketch of $x(1)$ versus $u(0)$ (recall $x(0)$ is a known parameter) and show the x-axis and y-axis intercepts on your plot. Now draw a sketch of $V(x(0), u(0))$ versus $u(0)$ in order to see what kind of optimization problem you are solving. You

[7]Laplace would love us for making this choice, but Gauss would not be happy.

may want to plot both terms in the objective function individually and then add them together to make your V plot. Label on your plot the places where the cost function V suffers discontinuities in slope. Where is the solution in your sketch? Does it exist for all $A, B, x(0)$? Is it unique for all $A, B, x(0)$?

The motivation for this problem is to change the quadratic program (QP) of the LQR to a linear program (LP) in the LAR, because the computational burden for LPs is often smaller than QPs. The absolute value terms can be converted into linear terms with the introduction of slack variables.

Exercise 2.20: Unreachable setpoints in constrained versus unconstrained linear systems

Consider the linear system with input constraint

$$x^+ = Ax + Bu \qquad u \in \mathbb{U}$$

We examine here both unconstrained systems in which $\mathbb{U} = \mathbb{R}^m$ and constrained systems in which $\mathbb{U} \subset \mathbb{R}^m$ is a convex polyhedron. Consider the stage cost defined in terms of setpoints for state and input x_{sp}, u_{sp}

$$\ell(x, u) = (1/2)\left(\left| x - x_{\text{sp}} \right|_Q^2 + \left| u - u_{\text{sp}} \right|_R^2 \right)$$

in which we assume for simplicity that $Q, R > 0$. For the setpoint to be unreachable in an unconstrained problem, the setpoint must be *inconsistent*, i.e., not a steady state of the system, or

$$x_{\text{sp}} \neq Ax_{\text{sp}} + Bu_{\text{sp}}$$

Consider also using the stage cost centered at the optimal steady state (x_s, u_s)

$$\ell_s(x, u) = (1/2)\left(|x - x_s|_Q^2 + |u - u_s|_R^2 \right)$$

The optimal steady state satisfies

$$(x_s, u_s) = \arg\min_{x,u} \ell(x, u)$$

subject to

$$\begin{bmatrix} I - A & -B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = 0 \qquad u \in \mathbb{U}$$

Figure 2.13 depicts an inconsistent setpoint, and the optimal steady state for unconstrained and constrained systems.

(a) For unconstrained systems, show that optimizing the cost function with terminal constraint

$$V(x, \mathbf{u}) := \sum_{k=0}^{N-1} \ell(x(k), u(k))$$

subject to

$$x^+ = Ax + Bu \qquad x(0) = x \qquad x(N) = x_s$$

gives the same solution as optimizing the cost function

$$V_s(x, \mathbf{u}) := \sum_{k=0}^{N-1} \ell_s(x(k), u(k))$$

subject to the same model constraint, initial condition, and terminal constraint.

Therefore, there is no reason to consider the unreachable setpoint problem further for an *unconstrained* linear system. Shifting the stage cost from $\ell(x, u)$ to $\ell_s(x, u)$ provides identical control behavior and is simpler to analyze.

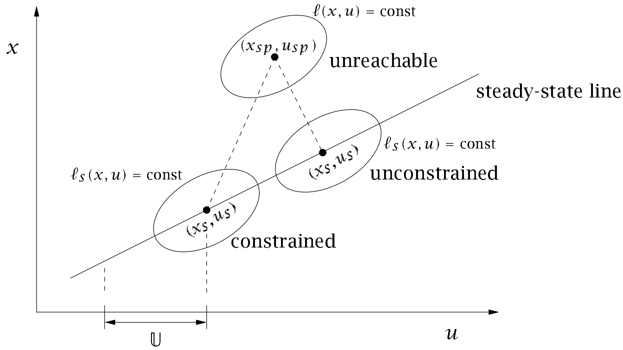


Figure 2.13: Inconsistent setpoint $(x_{\mathrm{sp}}, u_{\mathrm{sp}})$, unreachable stage cost $\ell(x,u)$, and optimal steady states (x_s, u_s), and stage costs $\ell_s(x,u)$ for constrained and unconstrained systems.

Hint. First define a third stage cost $l(x,u) = \ell(x,u) - \lambda'((I-A)x - Bu)$, and show, for any λ, optimizing with $l(x,u)$ as stage cost is the same as optimizing using $\ell(x,u)$ as stage cost. Then set $\lambda = \lambda_s$, the optimal Lagrange multiplier of the *steady-state* optimization problem.

(b) For *constrained* systems, provide a simple example that shows optimizing the cost function $V(x, \mathbf{u})$ subject to

$$x^+ = Ax + Bu \qquad x(0) = x \qquad x(N) = x_s \qquad u(k) \in \mathbb{U} \quad \text{for all } k \in \mathbb{I}_{0:N-1}$$

does *not* give the same solution as optimizing the cost function $V_s(x, \mathbf{u})$ subject to the same constraints. For *constrained* linear systems, these problems are different and optimizing the unreachable stage cost provides a new design opportunity.

Exercise 2.21: Filing for patent

An excited graduate student shows up at your office. He begins, "Look, I have discovered a great money-making scheme using MPC." You ask him to tell you about it. "Well," he says, "you told us in class that the optimal steady state is asymptotically stable even if you use the stage cost measuring distance from the unreachable setpoint, right?" You reply, "Yes, that's what I said." He continues, "OK, well look at this little sketch I drew," and he shows you a picture like Figure 2.14. "So imagine I use the infinite horizon cost function so the open-loop and closed-loop trajectories are identical. If the best steady state is asymptotically stable, then the stage cost asymptotically approaches $\ell(x_s, u_s)$, right?" You reply, "I guess that looks right." He then says, "OK, well if I look at the optimal cost using state x at time k and state x^+ at time $k+1$, by the principle of optimality I get the usual cost decrease"

$$V^0(x^+) \leq V^0(x) - \ell(x, u^0(x)) \tag{2.39}$$

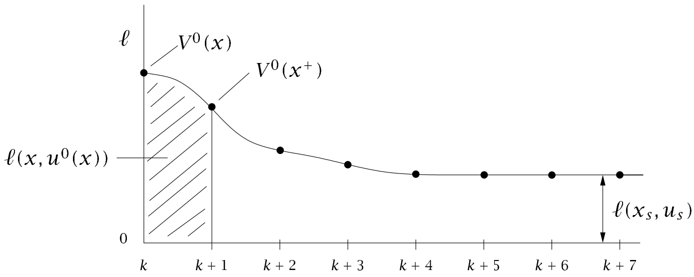


Figure 2.14: Stage cost versus time for the case of unreachable setpoint. The cost $V^0(x(k))$ is the area under the curve to the right of time k.

You interrupt, "Wait, these $V^0(\cdot)$ costs are not bounded in this case!" Unfazed, the student replies, "Yeah, I realize that, but this sketch is basically correct regardless. Say we just make the horizon *really long*; then the costs are all finite and this equation becomes closer and closer to being true as we make the horizon longer and longer." You start to feel a little queasy at this point. The student continues, "OK, so if this inequality basically holds, $V^0(x(k))$ is decreasing with k along the closed-loop trajectory, it is bounded below for all k, it converges, and, therefore, $\ell(x(k), u^0(x(k)))$ goes to zero as k goes to ∞." You definitely don't like where this is heading, and the student finishes with, "But $\ell(x,u) = 0$ implies $x = x_{\mathrm{sp}}$ and $u = u_{\mathrm{sp}}$, and the setpoint is *supposed* to be unreachable. But I have proven that infinite horizon MPC can reach an *unreachable* setpoint. We should patent this!"

How do you respond to this student? Here are some issues to consider.

1. Does the principle of optimality break down in the unreachable setpoint case?
2. Are the open-loop and closed-loop trajectories identical in the limit of an infinite horizon controller with an unreachable setpoint?
3. Does inequality (2.39) hold as $N \to \infty$? If so, how can you put it on solid footing? If not, why not, and with what do you replace it?
4. Do you file for patent?

Exercise 2.22: Stabilizable with small controls

Consider a time-varying system $x(i+1) = f(x,u,i)$ with stage cost $\ell(x,u,i)$ and terminal cost $V_f(x,i)$ satisfying Assumptions 2.25, 2.26, and 2.33. Suppose further that functions $f(\cdot)$ and $\ell(\cdot)$ are uniformly bounded by $\mathcal{K}_\infty$ functions α_f and α_ℓ, i.e.,

$$\begin{aligned} |f(x,u,i)| &\le \alpha_{fx}(|x|) + \alpha_{fu}(|u|) \\ \ell(x,u,i) &\le \alpha_{\ell x}(|x|) + \alpha_{\ell u}(|u|) \end{aligned}$$

for all $i \in \mathbb{I}_{\ge 0}$. Prove that if there exists a $\mathcal{K}_\infty$ function $\gamma(\cdot)$ such that for each $x \in \mathcal{X}_N(i)$, there exists $\mathbf{u} \in \mathcal{U}_N(x,i)$ with $|\mathbf{u}| \le \gamma(|x|)$, then there exists a $\mathcal{K}_\infty$ function $\alpha(\cdot)$ such that

$$V^0(x,i) \le \alpha(|x|)$$

for all $i \in \mathbb{I}_{\geq 0}$ and $x \in \mathcal{X}_N(i)$.

Hint: the following inequality may prove useful: for any $\mathcal{K}_\infty$ function α (see (B.1))

$$\alpha(s_1 + s_2 + \cdots + s_N) \leq \alpha(Ns_1) + \alpha(Ns_2) + \cdots + \alpha(Ns_N)$$

Exercise 2.23: Power lifting

Consider a stabilizable T-periodic linear system

$$x(i+1) = A(i)x(i) + B(i)u(i)$$

with positive definite stage cost

$$\ell(x, u, i) := \frac{1}{2}\left(x'Q(i)x + u'R(i)u\right)$$

Suppose there exist periodic control laws $K(i)$ and cost matrices $P(i)$ satisfying the periodic Riccati equation

$$\begin{aligned} P(i) &= Q(i) + A(i)'P(i+1)\left(A(i) + B(i)K(i)\right) \\ K(i) &= -\left(B(i)'P(i+1)B(i) + R(i)\right)^{-1} B(i)'P(i+1)A(i) \end{aligned}$$

Show that the control law $\mathbf{K} := \operatorname{diag}(K(0), \ldots, K(T-1))$ and cost $\mathbf{P} := \operatorname{diag}(P(0), \ldots, P(T-1))$ satisfy the Riccati equation for the time-invariant lifted system

$$\mathbf{A} := \begin{bmatrix} 0 & 0 & \cdots & 0 & A(T-1) \\ A(0) & 0 & \cdots & 0 & 0 \\ 0 & A(1) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A(T-2) & 0 \end{bmatrix}$$

$$\mathbf{B} := \begin{bmatrix} 0 & 0 & \cdots & 0 & B(T-1) \\ B(0) & 0 & \cdots & 0 & 0 \\ 0 & B(1) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B(T-2) & 0 \end{bmatrix}$$

$$\mathbf{Q} := \operatorname{diag}(Q(0), \ldots, Q(T-1))$$

$$\mathbf{R} := \operatorname{diag}(R(0), \ldots, R(T-1))$$

By uniqueness of solutions to the Riccati equation, this system can be used to synthesize control laws for periodic systems.

Exercise 2.24: Feasible warm start in $\mathbb{X}_f$

Establish Proposition 2.42, which states that for any $x \in \mathbb{X}_f$, the following warm start is feasible

$$\tilde{\mathbf{u}}_f(x) := \left(\kappa_f(x), \kappa_f(f(x, \kappa_f(x))), \ldots\right) \in \tilde{\mathcal{U}}_N(x)$$

Recall that a warm start $\tilde{\mathbf{u}}$ is a member of $\tilde{\mathcal{U}}_N(x)$ if all elements of the sequence of controls are members of $\mathbb{U}$, the state trajectory $\phi(k; x, \tilde{\mathbf{u}})$ terminates in $\mathbb{X}_f$, and $V_N(x, \tilde{\mathbf{u}})$ is less than $V_f(x)$.

Exercise 2.25: The geometry of cost rotation

Let's examine the rotated cost function in the simplest possible setting to understand what "rotation" means in this context. Consider the discrete time dynamic model and strictly convex quadratic cost function

$$x^+ = f(x,u) \qquad \ell(x,u) = (1/2)(\left|x - x_{\text{sp}}\right|_Q^2 + \left|u - u_{\text{sp}}\right|_R^2)$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $\ell: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$, $Q \in \mathbb{R}^{n\times n}, R \in \mathbb{R}^{m\times m}$ with $Q, R > 0$. We define the feasible control region as $u \in \mathbb{U}$ for some nonempty set $\mathbb{U}$. We wish to illustrate the ideas with the following simple linear system

$$f(x,u) = Ax + Bu \qquad A = 1/2 \qquad B = 1/2$$

subject to polyhedral constraint

$$\mathbb{U} = \{u \mid -1 \leq u \leq 1\}$$

We choose an *unreachable* setpoint that is not a steady state, and cost matrices as follows

$$(u_{\text{sp}}, x_{\text{sp}}) = (2,3) \qquad Q = R = 2$$

The optimal steady state (u_s, x_s) is given by the solution to the following optimization

$$(u_s, x_s) = \arg\min_{u,x}\{\ell(x,u) \mid u \in \mathbb{U},\ x = f(x,u)\} \tag{2.40}$$

(a) Solve this quadratic program and show that the solution is $(x_s, u_s) = (1,1)$. What is the Lagrange multiplier for the equality constraint?

(b) Next we define the rotated cost function following Diehl et al. (2011)

$$\tilde{\ell}(x,u) = \ell(x,u) - \lambda'(x - f(x,u)) - \ell(x_s,u_s)$$

Plot the contour of zero rotated cost $\tilde{\ell}(x,u) = 0$ for three λ values, $\lambda = 0, -8, -12$. Compare your contours to those shown in Figure 2.15.

Notice that as you decrease λ, you rotate (and enlarge) the zero cost contour of $\ell(x,u)$ about the point (x_s,u_s), hence the name *rotated* stage cost.

(c) Notice that the original cost function, which corresponds to $\lambda = 0$, has negative cost values (interior of the circle) that are in the feasible region. The zero contour for $\lambda = -8$ has become tangent to the feasible region, so the cost is nonnegative in the feasible region. But for $\lambda = -12$, the zero contour has been *over rotated* so that it again has negative values in the feasible region.

How does the value $\lambda = -8$ compare to the Lagrange multiplier of the optimal steady-state problem?

(d) Explain why MPC based on the rotated stage cost is a Lyapunov function for the closed-loop system.

Exercise 2.26: Strong duality implies dissipativity

Consider again the steady-state economic problem $\mathbb{P}_s$ for the optimal steady state (x_s, u_s)

$$\ell(x_s,u_s) := \min_{(x,u)\in\mathbb{Z}}\{\ell(x,u) \mid x = f(x,u)\}$$

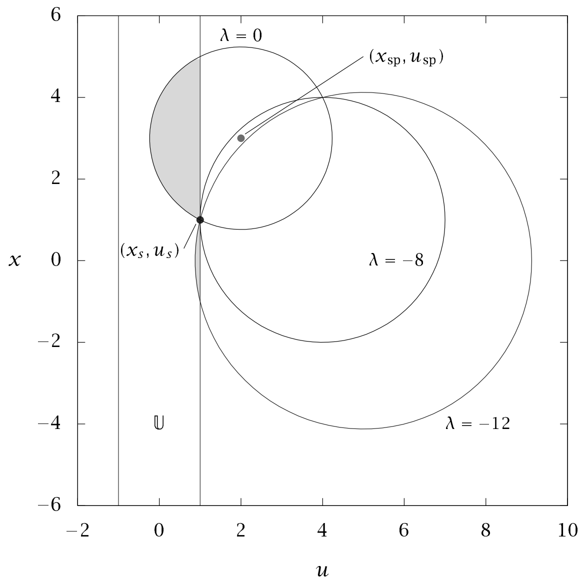


Figure 2.15: Rotated cost-function contour $\widetilde{\ell}(x,u) = 0$ (circles) for $\lambda = 0, -8, -12$. Shaded region shows feasible region where $\widetilde{\ell}(x,u) < 0$.

Form the Lagrangian and show that the solution is given also by

$$\ell(x_s, u_s) = \min_{(x,u)\in\mathbb{Z}} \max_{\lambda} \ell(x,u) - \lambda'(x - f(x,u))$$

Switching the order of min and max gives

$$\min_{(x,u)\in\mathbb{Z}} \max_{\lambda} \ell(x,u) - \lambda'(x - f(x,u)) \geq \max_{\lambda} \min_{(x,u)\in\mathbb{Z}} \ell(x,u) - \lambda'(x - f(x,u))$$

due to weak duality. The strong duality assumption states that equality is achieved in this inequality above, so that

$$\ell(x_s, u_s) = \max_{\lambda} \min_{(x,u)\in\mathbb{Z}} \ell(x,u) - \lambda'(x - f(x,u))$$

Let λ_s denote the optimal Lagrange multiplier in this problem. (For a brief review of these concepts, see also Exercises C.4, C.5, and C.6 in Appendix C.)

Show that the strong duality assumption implies that the system $x^+ = f(x,u)$ is dissipative with respect to the supply rate $s(x,u) = \ell(x,u) - \ell(x_s,u_s)$.

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