5

Output Model Predictive Control

5.1 Introduction

In Chapter 2 we show how model predictive control (MPC) may be employed to control a *deterministic* system, that is, a system in which there are no uncertainties and the state is known. In Chapter 3 we show how to control an *uncertain* system in which uncertainties are present but the state is known. Here we address the problem of MPC of an uncertain system in which the state is *not* fully known. We assume that there are outputs available that may be used to estimate the state as shown in Chapter 4. These outputs are used by the model predictive controller to generate control actions; hence the name *output MPC*.

The state is not known, but a noisy measurement $y(t)$ of the state is available at each time t. Since the state x is not known, it is replaced by a hyperstate p that summarizes all prior information (previous inputs and outputs and the prior distribution of the initial state) and that has the "state" property: future values of p can be determined from the current value of p, and current and future inputs and outputs. Usually $p(t)$ is the conditional density of $x(t)$ given the prior density $p(0)$ of $x(0)$, and the current available "information" $I(t) := \big(y(0), y(1), \ldots, y(t-1), u(0), u(1), \ldots, u(t-1)\big)$.

For the purpose of control, future hyperstates have to be predicted since future noisy measurements of the state are not known. So the hyperstate satisfies an uncertain difference equation of the form

$$p^+ = \phi(p, u, \psi) \tag{5.1}$$

where $(\psi(t))_{t \in \mathbb{I}_{\geq 0}}$ is a sequence of random variables. The problem of controlling a system with unknown state x is transformed into the problem of controlling an uncertain system with known state p. For example, if the underlying system is described by

$$x^+ = Ax + Bu + w$$
$$y = Cx + \nu$$

where $(w(t))_{t \in \mathbb{I}_{\geq 0}}$ and $(\nu(t))_{t \in \mathbb{I}_{\geq 0}}$ are sequences of zero-mean, normal, independent random variables with variances Σ_w and Σ_ν, respectively, and if the prior density $p(0)$ of $x(0)$ is normal with density $n(\bar{x}_0, \Sigma_0)$, then, as is well known, $p(t)$ is the normal density $n(\hat{x}(t), \Sigma(t))$ so that the hyperstate $p(t)$ is finitely parameterized by $(\hat{x}(t), \Sigma(t))$. Hence the evolution equation for $p(t)$ may be replaced by the simpler evolution equation for $(\hat{x}, \Sigma)$, that is by

$$\hat{x}(t+1) = A\hat{x}(t) + Bu + L(t)\psi(t) \tag{5.2}$$
$$\Sigma(t+1) = \Phi(\Sigma(t)) \tag{5.3}$$

in which

$$\Phi(\Sigma) := A\Sigma A' - A\Sigma C'(C'\Sigma C + \Sigma_\nu)^{-1} C\Sigma A' + \Sigma_w$$
$$\psi(t) := y(t) - C\hat{x}(t) = C\tilde{x}(t) + \nu(t)$$
$$\tilde{x}(t) := x(t) - \hat{x}(t)$$

The initial conditions for (5.2) and (5.3) are

$$\hat{x}(0) = \bar{x}_0 \qquad \Sigma(0) = \Sigma_0$$

These are, of course, the celebrated Kalman filter equations derived in Chapter 1. The random variables $\tilde{x}$ and ψ have the following densities: $\tilde{x}(t) \sim n(0, \Sigma(t))$ and $\psi(t) \sim n(0, \Sigma_\nu + C'\Sigma(t)C)$. The finite dimensional equations (5.2) and (5.3) replace the difference equation (5.1) for the hyperstate p that is a conditional density and, therefore, infinite dimensional in general. The sequence $(\psi(t))_{t \in \mathbb{I}_{\geq 0}}$ is known as the *innovations* sequence; $\psi(t)$ is the “new” information contained in $y(t)$.

Output control, in general, requires control of the hyperstate p that may be computed with difficulty, since p satisfies a complex evolution equation $p^+ = \phi(p, u, \psi)$ where ψ is a random disturbance. Controlling p is a problem of the same type as that considered in Chapter 3, but considerably more complex since the function $p(\cdot)$ is infinite dimensional. Because of the complexity of the evolution equation for p, a simpler procedure is often adopted. Assuming that the state x is known, a stabilizing controller $u = \kappa(x)$ is designed. An observer or filter yielding an estimate $\hat{x}$ of the state is then separately designed and the control $u = \kappa(\hat{x})$ is applied to the plant. Indeed, this form of control is actually optimal for the linear quadratic Gaussian (LQG) optimal control problem considered in Chapter 1, but is not necessarily optimal and stabilizing when the system is nonlinear and constrained. We propose a variant of this procedure, modified to cope with state and control constraints.

The state estimate $\hat{x}$ satisfies an uncertain difference equation with an additive disturbance of the same type as that considered in Chapter 3. Hence we employ tube MPC, similar to that employed in Chapter 3, to obtain a nominal trajectory satisfying tightened constraints. We then construct a tube that has as its center the nominal trajectory, and which includes every possible realization of $\hat{\mathbf{x}} = (\hat{x}(t))_{t\in\mathbb{I}_{\geq 0}}$. We then construct a second tube that includes the first tube in its interior, and is such that every possible realization of the sequence $\mathbf{x} = (x(t))_{t\in\mathbb{I}_{\geq 0}}$ lies in its interior. The tightened constraints are chosen to ensure every possible realization of $\mathbf{x} = (x(t))_{t\in\mathbb{I}_{\geq 0}}$ does not transgress the original constraints. An advantage of the method presented here is that its online complexity is comparable to that of conventional MPC.

As in Chapter 3, a caveat is necessary. Because of the inherent complexity of output MPC, different compromises between simplicity and efficiency are possible. For this reason, output MPC remains an active research area and alternative methods, available or yet to be developed, may be preferred.

5.2 A Method for Output MPC

Suppose the system to be controlled is described by

$$x^+ = Ax + Bu + w$$
$$y = Cx + v$$

The state and control are required to satisfy the constraints $x(t) \in \mathbb{X}$ and $u(t) \in \mathbb{U}$ for all t, and the disturbance is assumed to lie in the compact set $\mathbb{W}$. It is assumed that the origin lies in the interior of the sets $\mathbb{X}$, $\mathbb{U}$, and $\mathbb{W}$. The state estimator $(\hat{x}, \Sigma)$ evolves, as shown in the sequel, according to

$$\hat{x}^+ = \phi(\hat{x}, u, \psi) \tag{5.4}$$

$$\Sigma^+ = \Phi(\Sigma) \tag{5.5}$$

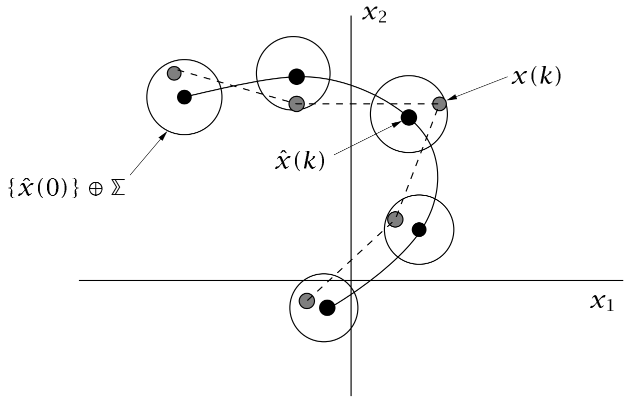

Figure 5.1: State estimator tube. The solid line $\hat{x}(t)$ is the center of the tube, and the dashed line is a sample trajectory of $x(t)$.

in which ψ is a random variable in the stochastic case, and a bounded disturbance taking values in Ψ when w and ν are bounded. In the latter case, $x \in \{\hat{x}\} \oplus \mathbb{\Sigma}$ implies $x^+ \in \{\hat{x}^+\} \oplus \mathbb{\Sigma}^+$ for all $\psi \in \Psi$.

As illustrated in Figure 5.1, the evolution equations generate a *tube*, which is the set sequence $(\{\hat{x}(t)\} \oplus \mathbb{\Sigma}(t))_{t\in\mathbb{I}_{\geq 0}}$; at time t the center of the tube is $\hat{x}(t)$ and the "cross section" is $\mathbb{\Sigma}(t)$. When the disturbances are bounded, which is the only case we consider in the sequel, all possible realizations of the state trajectory $(x(t))$ lie in the set $\{\hat{x}(t)\} \oplus \mathbb{\Sigma}(t)$ for all t; the dashed line is a sample trajectory of $x(t)$.

From (5.4), the estimator trajectory $(\hat{x}(t))_{t\in\mathbb{I}_{\geq 0}}$ is influenced both by the control that is applied and by the disturbance sequence $(\psi(t))_{t\in\mathbb{I}_{\geq 0}}$. If the trajectory were influenced only by the control, we could choose the control to satisfy the control constraints, and to cause the estimator tube to lie in a region such that the state constraints are satisfied by all possible realizations of the state trajectory. Hence the output MPC problem would reduce to a conventional MPC problem with modified constraints in which the state is $\hat{x}$, rather than x. The new state constraint is $\hat{x} \in \hat{\mathbb{X}}$ where $\hat{\mathbb{X}}$ is chosen to ensure that $\hat{x} \in \hat{\mathbb{X}}$ implies $x \in \mathbb{X}$ and, therefore, satisfies $\hat{\mathbb{X}} \subseteq \mathbb{X} \ominus \mathbb{\Sigma}$ if $\mathbb{\Sigma}$ does not vary with time t.

But the estimator state $\hat{x}(t)$ is influenced by the disturbance ψ (see (5.4)), so it cannot be precisely controlled. The problem of controlling the system described by (5.4) is the same type of problem studied in

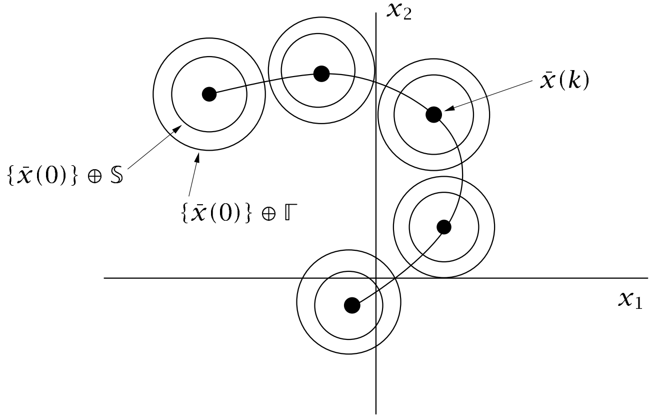

Figure 5.2: The system with disturbance. The state estimate lies in the inner tube, and the state lies in the outer tube.

Chapter 3, where the system was described by $x^+ = f(x, u, w)$ with the estimator state $\hat{x}$, which is accessible, replacing the state x. Hence we may use the techniques presented in Chapter 3 to choose a control that forces $\hat{x}$ to lie in another tube $(\{\bar{x}(t)\} \oplus \mathbb{S}(t))_{t\in\mathbb{I}_{\geq 0}}$ where the set sequence $(\mathbb{S}(t))_{t\in\mathbb{I}_{\geq 0}}$ that defines the cross section of the tube is precomputed. The sequence $(\bar{x}(t))_{t\in\mathbb{I}_{\geq 0}}$ that defines the center of the tube is the state trajectory of the nominal (deterministic) system defined by

$$\bar{x}^+ = \phi(\bar{x}, \bar{u}, 0) \tag{5.6}$$

the nominal version of (5.4). Thus we get two tubes, one embedded in the other. At time t the estimator state $\hat{x}(t)$ lies in the set $\{\bar{x}(t)\} \oplus \mathbb{S}(t)$, and $x(t)$ lies in the set $\{\hat{x}(t)\} \oplus \Sigma(t)$, so that for all t

$$x(t) \in \{\bar{x}(t)\} \oplus \mathbb{\Gamma}(t) \qquad \mathbb{\Gamma}(t) := \Sigma(t) \oplus \mathbb{S}(t)$$

Figure 5.2 shows the tube $(\{\bar{x}(t)\} \oplus \mathbb{S}(t))$, in which the trajectory $(\hat{x}(t))$ lies, and the tube $(\{\bar{x}(t)\} \oplus \mathbb{\Gamma}(t))$, in which the state trajectory $(x(t))$ lies.

5.3 Linear Constrained Systems: Time-Invariant Case

5.3.1 Introduction

We consider the following uncertain linear time-invariant system

$$\begin{aligned} x^+ &= Ax + Bu + w \\ y &= Cx + \nu \end{aligned} \tag{5.7}$$

in which $x \in \mathbb{R}^n$ is the current state, $u \in \mathbb{R}^m$ is the current control action, x^+ is the successor state, $w \in \mathbb{R}^n$ is an unknown state disturbance, $y \in \mathbb{R}^p$ is the current measured output, $\nu \in \mathbb{R}^p$ is an unknown output disturbance, the pair (A, B) is assumed to be controllable, and the pair (A, C) observable. The state and additive disturbances w and ν are known only to the extent that they lie, respectively, in the C-sets[1] $\mathbb{W} \subseteq \mathbb{R}^n$ and $\mathbb{N} \subseteq \mathbb{R}^p$. Let $\phi(i; x(0), \mathbf{u}, \mathbf{w})$ denote the solution of (5.7) at time i if the initial state at time 0 is $x(0)$, and the control and disturbance sequences are, respectively, $\mathbf{u} := (u(0), u(1), \ldots)$ and $\mathbf{w} := (w(0), w(1), \ldots)$. The system (5.7) is subject to the following set of hard state and control constraints

$$x \in \mathbb{X} \qquad u \in \mathbb{U} \tag{5.8}$$

in which $\mathbb{X} \subseteq \mathbb{R}^n$ and $\mathbb{U} \subseteq \mathbb{R}^m$ are polyhedral and polytopic sets respectively; both sets contain the origin as an interior point.

5.3.2 State Estimator

To estimate the state a Kalman filter or Luenberger observer is employed

$$\begin{aligned} \hat{x}^+ &= A\hat{x} + Bu + L(y - \hat{y}) \\ \hat{y} &= C\hat{x} \end{aligned} \tag{5.9}$$

in which $\hat{x} \in \mathbb{R}^n$ is the current observer state (state estimate), $u \in \mathbb{R}^m$ is the current control action, $\hat{x}^+$ is the successor state of the observer system, $\hat{y} \in \mathbb{R}^p$ is the current observer output, and $L \in \mathbb{R}^{n \times p}$. The output injection matrix L is chosen to satisfy $\rho(A_L) < 1$ where $A_L := A - LC$.

The estimated state $\hat{x}$ therefore satisfies the following uncertain difference equation

$$\hat{x}^+ = A\hat{x} + Bu + L(C\widetilde{x} + \nu)$$

[1] Recall, a C-set is a convex, compact set containing the origin.

The state estimation error $\tilde{x}$ is defined by $\tilde{x} := x - \hat{x}$ so that $x = \hat{x} + \tilde{x}$. Since $x^+ = Ax + Bu + w$, the state estimation error $\tilde{x}$ satisfies

$$\tilde{x}^+ = A_L\tilde{x} + \tilde{w} \qquad \tilde{w} := w - Lv \tag{5.10}$$

Because w and v are bounded, so is $\tilde{w}$; in fact, $\tilde{w}$ takes values in the C-set $\bar{\mathbb{W}}$ defined by

$$\bar{\mathbb{W}} := \mathbb{W} \oplus (-L\mathbb{N})$$

We recall the following standard definitions (Blanchini, 1999).

Definition 5.1 (Positive invariance; robust positive invariance)**.** A set $\Omega \subseteq \mathbb{R}^n$ is *positive invariant* for the system $x^+ = f(x)$ and the constraint set $\mathbb{X}$ if $\Omega \subseteq \mathbb{X}$ and $f(x) \in \Omega,\ \forall x \in \Omega$.
A set $\Omega \subseteq \mathbb{R}^n$ is *robust positive invariant* for the system $x^+ = f(x, w)$ and the constraint set $(\mathbb{X}, \mathbb{W})$ if $\Omega \subseteq \mathbb{X}$ and $f(x, w) \in \Omega,\ \forall w \in \mathbb{W}$, $\forall x \in \Omega$.

Since $\rho(A_L) < 1$ and $\tilde{\mathbb{W}}$ is compact, there exists, as shown in Kolmanovsky and Gilbert (1998), Theorem 4.1, a robust positive invariant set $\mathbb{\Sigma} \subseteq \mathbb{R}^n$, satisfying

$$A_L\mathbb{\Sigma} \oplus \bar{\mathbb{W}} = \mathbb{\Sigma} \tag{5.11}$$

Hence, for all $\tilde{x} \in \mathbb{\Sigma}$, $\tilde{x}^+ = A_L\tilde{x} + \tilde{w} \in \mathbb{\Sigma}$ for all $\tilde{w} \in \tilde{\mathbb{W}}$; the term *robust* in the description of $\mathbb{\Sigma}$ refers to this property. In fact, $\mathbb{\Sigma}$ is the *minimal* robust, positive invariant set for $\tilde{x}^+ = A_L\tilde{x} + \tilde{w}$, $\tilde{w} \in \tilde{\mathbb{W}}$, i.e., a set that is a subset of all robust positive invariant sets. There exist techniques (Raković, Kerrigan, Kouramas, and Mayne, 2005) for obtaining, for every $\epsilon > 0$, a polytopic, nonminimal, robust, positive invariant set $\mathbb{\Sigma}^0$ that satisfies $d_H(\mathbb{\Sigma}, \mathbb{\Sigma}^0) \leq \epsilon$ where $d_H(\cdot,\cdot)$ is the Hausdorff metric. However, it is not necessary to compute the set Σ or Σ^0 as shown in Chapter 3. An immediate consequence of (5.11) is the following.

Proposition 5.2 (Proximity of state and state estimate)**.** *If the initial system and observer states, $x(0)$ and $\hat{x}(0)$ respectively, satisfy $\{x(0)\} \in \{\hat{x}(0)\} \oplus \mathbb{\Sigma}$, then $x(i) \in \{\hat{x}(i)\} \oplus \mathbb{\Sigma}$ for all $i \in \mathbb{I}_{\geq 0}$, and all admissible disturbance sequences* $\mathbf{w}$ *and* $\boldsymbol{\nu}$*.*

The assumption that $\tilde{x}(i) \in \mathbb{\Sigma}$ for all i is a *steady-state* assumption; if $\tilde{x}(0) \in \mathbb{\Sigma}$, then $\tilde{x}(i) \in \mathbb{\Sigma}$ for all i. If, on the other hand, $\tilde{x}(0) \in \mathbb{\Sigma}(0)$ where $\mathbb{\Sigma}(0) \supseteq \mathbb{\Sigma}$, then it is possible to show that $\tilde{x}(i) \in \mathbb{\Sigma}(i)$ for all $i \in \mathbb{I}_{\geq 0}$ where $\mathbb{\Sigma}(i) \to \mathbb{\Sigma}$ in the Hausdorff metric as $i \to \infty$; the sequence $(\mathbb{\Sigma}(i))$ satisfies $\mathbb{\Sigma}(0) \supseteq \mathbb{\Sigma}(1) \supseteq \mathbb{\Sigma}(2) \supseteq \cdots \supseteq \mathbb{\Sigma}$. Hence, it is reasonable to assume that if the estimator has been running for a "long" time, it is in steady state.

Hence we have obtained a state estimator, with "state" $(\hat{x}, \mathbb{\Sigma})$ satisfying

$$\begin{aligned} \hat{x}^+ &= A\hat{x} + Bu + L(y - \hat{y}) \\ \mathbb{\Sigma}^+ &= \mathbb{\Sigma} \end{aligned} \tag{5.12}$$

and $x(i) \in \hat{x}(i) \oplus \mathbb{\Sigma}$ for all $i \in \mathbb{I}_{\geq 0}$, thus meeting the requirements specified in Section 5.2. Knowing this, our remaining task is to control $\hat{x}(i)$ so that the resultant closed-loop system is stable and satisfies all constraints.

5.3.3 Controlling $\hat{x}$

Since $\tilde{x}(i) \in \mathbb{\Sigma}$ for all i, we seek a method for controlling the observer state $\hat{x}(i)$ in such a way that $x(i) = \hat{x}(i) + \tilde{x}(i)$ satisfies the state constraint $x(i) \in \mathbb{X}$ for all i. The state constraint $x(i) \in \mathbb{X}$ will be satisfied if we control the estimator state to satisfy $\hat{x}(i) \in \mathbb{X} \ominus \mathbb{\Sigma}$ for all i. The estimator state satisfies (5.12) which can be written in the form

$$\hat{x}^+ = A\hat{x} + Bu + \delta \tag{5.13}$$

where the disturbance δ is defined by

$$\delta := L(y - \hat{y}) = L(C\tilde{x} + \nu)$$

and, therefore, always lies in the C-set $\mathbb{\Delta}$ defined by

$$\mathbb{\Delta} := L(C\mathbb{\Sigma} \oplus \mathbb{N})$$

The problem of controlling $\hat{x}$ is, therefore, the same as that of controlling an uncertain system with known state. This problem was extensively discussed in Chapter 3. We can therefore use the approach of Chapter 3 here with $\hat{x}$ replacing x, δ replacing w, $\mathbb{X} \ominus \mathbb{\Sigma}$ replacing $\mathbb{X}$ and $\mathbb{\Delta}$ replacing $\mathbb{W}$.

To control (5.13) we use, as in Chapter 3, a combination of open-loop and feedback control, i.e., we choose the control u as follows

$$u = \bar{u} + Ke \qquad e := \hat{x} - \bar{x} \tag{5.14}$$

where $\bar{x}$ is the state of a nominal (deterministic) system that we shall shortly specify; $\bar{u}$ is the feedforward component of the control u, and

Ke is the feedback component. The matrix K is chosen to satisfy $\rho(A_K) < 1$ where $A_K := A + BK$. The feedforward component v of the control u generates, as we show subsequently, a trajectory $(\bar{x}(i))$, which is the center of the tube in which the state estimator trajectory $(\hat{x}(i))$ lies. The feedback component Ke attempts to steer the trajectory $(\hat{x}(i))$ of the state estimate toward the center of the tube, and thereby controls the cross section of the tube. The controller is *dynamic* since it incorporates the nominal dynamic system.

With this control, $\hat{x}$ satisfies the following difference equation

$$\hat{x}^+ = A\hat{x} + B\bar{u} + BKe + \delta \qquad \delta \in \mathbb{\Delta} \tag{5.15}$$

The nominal (deterministic) system describing the evolution of $\bar{x}$ is obtained by neglecting the disturbances BKe and δ in (5.15) yielding

$$\bar{x}^+ = A\bar{x} + B\bar{u}$$

The deviation $e = \hat{x} - \bar{x}$ between the state $\hat{x}$ of the estimator and the state $\bar{x}$ of the nominal system satisfies

$$e^+ = A_K e + \delta \qquad A_K := A + BK \tag{5.16}$$

The feedforward component $\bar{u}$ of the control u generates the trajectory $(\bar{x}(i))$, which is the center of the tube in which the state estimator trajectory $(\hat{x}(i))$ lies. Because $\mathbb{\Delta}$ is a C-set and $\rho(A_K) < 1$, there exists a robust positive invariant C-set $\mathbb{S}$ satisfying

$$A_K\mathbb{S} \oplus \mathbb{\Delta} = \mathbb{S}$$

An immediate consequence is the following.

Proposition 5.3 (Proximity of state estimate and nominal state)**.** *If the initial states of the estimator and nominal system, $\hat{x}(0)$ and $\bar{x}(0)$ respectively, satisfy $\hat{x}(0) \in \{\bar{x}(0)\} \oplus \mathbb{S}$, then $\hat{x}(i) \in \{\bar{x}(i)\} \oplus \mathbb{S}$ and $u(i) \in \{\bar{u}(i)\} \oplus K\mathbb{S}$ for all $i \in \mathbb{I}_{\geq 0}$, and all admissible disturbance sequences* $\mathbf{w}$ *and* $\mathbf{v}$*.*

It follows from Proposition 5.3 that the state estimator trajectory $\hat{\mathbf{x}}$ remains in the tube $(\{\bar{x}(i)\} \oplus \mathbb{S})_{i\in\mathbb{I}_{\geq 0}}$ and the control trajectory $\bar{\mathbf{u}}$ remains in the tube $(\{\bar{u}(i)\} \oplus K\mathbb{S})_{i\in\mathbb{I}_{\geq 0}}$ provided that $e(0) \in \mathbb{S}$. Hence, from Propositions 5.2 and 5.3, the state trajectory $\mathbf{x}$ lies in the tube $(\{\bar{x}(i)\} \oplus \mathbb{\Gamma})_{i\in\mathbb{I}_{\geq 0}}$ where $\mathbb{\Gamma} := \mathbb{S} \oplus \mathbb{\Sigma}$ provided that $\widetilde{x}(0) = x(0) - \hat{x}(0) \in \mathbb{\Sigma}$ *and* $e(0) \in \mathbb{S}$. This information may be used to construct a robust output feedback model predictive controller using the procedures outlined in Chapter 3 for robust state feedback MPC of systems; the major difference is that we now control the estimator state $\hat{x}$ and use the fact that the actual state x lies in $\{\hat{x}\} \oplus \mathbb{\Sigma}$.

5.3.4 Output MPC

Model predictive controllers now can be constructed as described in Chapter 3, which dealt with robust control when the state was known. There is an obvious difference in that we now are concerned with controlling $\hat{x}$ whereas, in Chapter 3, our concern was control of x. We describe here the appropriate modification of the simple model predictive controller presented in Section 3.5.2. We adopt the same procedure of defining a nominal optimal control problem with tighter constraints than in the original problem. The solution to this problem defines the center of a tube in which solutions to the original system lie, and the tighter constraints in the nominal problem ensure that the original constraints are satisfied by the actual system.

The nominal system is described by

$$\bar{x}^+ = A\bar{x} + B\bar{u} \tag{5.17}$$

The nominal optimal control problem is the minimization of the cost function $\bar{V}_N(\bar{x}, \bar{\mathbf{u}})$ with

$$\bar{V}_N(\bar{x}, \bar{\mathbf{u}}) := \sum_{k=0}^{N-1} \ell(\bar{x}(k), \bar{u}(k)) + V_f(\bar{x}(N)) \tag{5.18}$$

subject to satisfaction by the state and control sequences of (5.17) and the *tighter* constraints

$$\bar{x}(i) \in \bar{\mathbb{X}} \subseteq \mathbb{X} \ominus \mathbb{\Gamma} \qquad \mathbb{\Gamma} := \mathbb{S} \oplus \mathbb{\Sigma} \tag{5.19}$$

$$\bar{u}(i) \in \bar{\mathbb{U}} \subseteq \mathbb{U} \ominus K\mathbb{S} \tag{5.20}$$

as well as a terminal constraint $\bar{x}(N) \in \bar{\mathbb{X}}_f \subseteq \bar{\mathbb{X}}$. Notice that $\mathbb{\Gamma}$ appears in (5.19) whereas $\mathbb{S}$, the set in which $e = \hat{x} - \bar{x}$ lies, appears in (5.20); this differs from the case studied in Chapter 3 where the same set appears in both equations. The sets $\mathbb{W}$ and $\mathbb{N}$ are assumed to be sufficiently small to ensure satisfaction of the following condition.

Assumption 5.4 (Constraint bounds)**.** $\mathbb{\Gamma} = \mathbb{S} \oplus \mathbb{\Sigma} \subseteq \mathbb{X}$ and $K\mathbb{S} \subseteq \mathbb{U}$.

If Assumption 5.4 holds, the sets on the right-hand side of (5.19) and (5.20) are not empty; it can be seen from their definitions that the sets $\mathbb{Z}$ and $\mathbb{S}$ tend to the set $\{0\}$ as $\mathbb{W}$ and $\mathbb{N}$ tend to the set $\{0\}$ in the sense that $d_H(\mathbb{W}, \{0\}) \to 0$ and $d_H(\mathbb{N}, \{0\}) \to 0$.

It follows from Propositions 5.2 and 5.3, if Assumption 5.4 holds, that satisfaction of the constraints (5.19) and (5.20) by the nominal system ensures satisfaction of the constraints (5.8) by the original system. The nominal optimal control problem is, therefore

$$\mathbb{P}_N(\bar{x}): \quad \bar{V}_N^0(\bar{x}) = \min_{\bar{\mathbf{u}}}\{\bar{V}_N(\bar{x}, \bar{\mathbf{u}}) \mid \bar{\mathbf{u}} \in \bar{\mathcal{U}}_N(\bar{x})\}$$

in which the constraint set $\bar{\mathcal{U}}_N(\bar{x})$ is defined by

$$\bar{\mathcal{U}}_N(\bar{x}) := \{\bar{\mathbf{u}} \mid \bar{u}(k) \in \bar{\mathbb{U}} \text{ and } \bar{\phi}(k; \bar{x}, \bar{\mathbf{u}}) \in \bar{\mathbb{X}}\ \forall k \in \{0, 1, \ldots, N-1\}, \\ \bar{\phi}(N; \bar{x}, \bar{\mathbf{u}}) \in \bar{\mathbb{X}}_f\} \tag{5.21}$$

In (5.21), $\bar{\mathbb{X}}_f \subseteq \bar{\mathbb{X}}$ is the terminal constraint set, and $\bar{\phi}(k; \bar{x}, \bar{\mathbf{u}})$ denotes the solution of $\bar{x}^+ = A\bar{x} + B\bar{u}$ at time k if the initial state at time 0 is $\bar{x}$ and the control sequence is $\bar{\mathbf{u}} = (\bar{u}(0), \bar{u}(1), \ldots, \bar{u}(N-1))$. The terminal constraint, which is not desirable in process control applications, may be omitted, as shown in Chapter 2, if the set of admissible initial states is suitably restricted. Let $\bar{\mathbf{u}}^0(\bar{x})$ denote the minimizing control sequence; the stage cost $\ell(\cdot)$ is chosen to ensure uniqueness of $\bar{\mathbf{u}}^0(\bar{x})$. The implicit model predictive control law for the nominal system is $\bar{\kappa}_N(\cdot)$ defined by

$$\bar{\kappa}_N(\bar{x}) := \bar{u}^0(0; \bar{x})$$

where $\bar{u}^0(0; \bar{x})$ is the first element in the sequence $\bar{\mathbf{u}}^0(\bar{x})$. The domain of $\bar{V}_N^0(\cdot)$ and $\bar{\mathbf{u}}^0(\cdot)$, and, hence, of $\bar{\kappa}_N(\cdot)$, is $\bar{\mathcal{X}}_N$ defined by

$$\bar{\mathcal{X}}_N := \{\bar{x} \in \bar{\mathbb{X}} \mid \bar{\mathcal{U}}_N(\bar{x}) \neq \emptyset\} \tag{5.22}$$

$\bar{\mathcal{X}}_N$ is the set of initial states $\bar{x}$ that can be steered to $\bar{\mathbb{X}}_f$ by an admissible control $\bar{\mathbf{u}}$ that satisfies the state and control constraints, (5.19) and (5.20), and the terminal constraint. From (5.14), the implicit control law for the state estimator $\hat{x}^+ = A\hat{x} + Bu + \delta$ is $\kappa_N(\cdot)$ defined by

$$\kappa_N(\hat{x}, \bar{x}) := \bar{\kappa}_N(\bar{x}) + K(\hat{x} - \bar{x})$$

The controlled composite system with state $(\hat{x}, \bar{x})$ satisfies

$$\hat{x}^+ = A\hat{x} + B\kappa_N(\hat{x}, \bar{x}) + \delta \tag{5.23}$$

$$\bar{x}^+ = A\bar{x} + B\bar{\kappa}_N(\bar{x}) \tag{5.24}$$

with initial state $(\hat{x}(0), \bar{x}(0))$ satisfying $\hat{x}(0) \in \{\bar{x}(0)\} \oplus \mathbb{S}$, $\bar{x}(0) \in \bar{\mathcal{X}}_N$. These constraints are satisfied if $\bar{x}(0) = \hat{x}(0) \in \bar{\mathcal{X}}_N$. The control algorithm may be formally stated as follows.

Algorithm 5.5 (Robust control algorithm (linear constrained systems))**.** First set $i = 0$, set $\hat{x} = \hat{x}(0)$, and set $\bar{x} = \hat{x}$. Then repeat

1. At time i, solve the nominal optimal control problem $\bar{\mathbb{P}}_N(\bar{x})$ to obtain the current nominal control action $\bar{u} = \bar{\kappa}_N(\bar{x})$ and the control $u = \bar{x} + K(\hat{x} - \bar{x})$.

2. Apply the control u to the system being controlled.

3. Compute the successor state estimate $\hat{x}^+$ and the successor state of the nominal system $\bar{x}^+$

$$\hat{x}^+ = A\hat{x} + Bu + L(y - C\hat{x}) \qquad \bar{x}^+ = A\bar{x} + B\bar{u}$$

4. Set $(\hat{x}, \bar{x}) = (\hat{x}^+, \bar{x}^+)$, set $i = i + 1$.

If the terminal cost $V_f(\cdot)$ and terminal constraint set $\bar{\mathbb{X}}_f$ satisfy the stability Assumption 2.14, and if Assumption 5.4 is satisfied, the value function $\bar{V}_N^0(\cdot)$ satisfies

$$\begin{aligned}
\bar{V}_N^0(\bar{x}) &\geq \ell(\bar{x}, \bar{\kappa}_N(\bar{x})) && \forall \bar{x} \in \bar{\mathcal{X}}_N \\
\bar{V}_N^0(\bar{x}) &\leq V_f(\bar{x}) && \forall \bar{x} \in \bar{\mathcal{X}}_N \\
\bar{V}_N^0(f(\bar{x}, \bar{\kappa}_N(\bar{x}))) &\leq \bar{V}_N^0(\bar{x}) - \ell(\bar{x}, \bar{\kappa}_N(\bar{x})) && \forall \bar{x} \in \bar{\mathcal{X}}_N
\end{aligned}$$

in which $\Delta\bar{V}_N^0(\bar{x}) := \bar{V}_N^0(f(\bar{x}, \bar{\kappa}_N(\bar{x}))) - \bar{V}_N^0(\bar{x})$.

As shown in Section 3.5.3, if, in addition to Assumption 5.4

1. the stability Assumption 2.14 is satisfied,

2. $\ell(\bar{x}, \bar{u}) = (1/2)(|\bar{x}|_Q^2 + |\bar{u}|_R^2)$ where Q and R are positive definite,

3. $V_f(\bar{x}) = (1/2)\,|\bar{x}|_{P_f}^2$ where P_f is positive definite, and

4. $\bar{\mathcal{X}}_N$ is a C-set,

then there exist positive constants c_1 and c_2 such that

$$\begin{aligned}
\bar{V}_N^0(\bar{x}) &\geq c_1\,|\bar{x}|^2 && \forall \bar{x} \in \bar{\mathcal{X}}_N \\
\bar{V}_N^0(\bar{x}) &\leq c_2\,|\bar{x}|^2 && \forall \bar{x} \in \bar{\mathcal{X}}_N \\
\bar{V}_N^0(f(\bar{x}, \bar{\kappa}_N(\bar{x}))) &\leq \bar{V}_N^0(\bar{x}) - c_1\,|\bar{x}|^2 && \forall \bar{x} \in \bar{\mathcal{X}}_N
\end{aligned}$$

It follows from Chapter 2 that the origin is exponentially stable for the nominal system $\bar{x}^+ = A\bar{x} + B\bar{\kappa}_N(\bar{x})$ with a region of attraction $\bar{\mathcal{X}}_N$ so that there exists a $c > 0$ and a $\gamma \in (0, 1)$ such that

$$|\bar{x}(i)| \leq c\,|\bar{x}(0)|\,\gamma^i$$

for all $\bar{x}(0) \in \bar{\mathcal{X}}_N$, all $i \in \mathbb{I}_{\geq 0}$. Also $\bar{x}(i) \in \bar{\mathcal{X}}_N$ for all $i \in \mathbb{I}_{\geq 0}$ if $\bar{x}(0) \in \bar{\mathcal{X}}_N$ so that problem $\mathbb{P}_N(\bar{x}(i))$ is always feasible. Because the state $\hat{x}(i)$ of the state estimator always lies in $\{\bar{x}(i)\} \oplus \mathbb{S}$, and the state $x(i)$ of the system being controlled always lies in $\{\bar{x}(i)\} \oplus \mathbb{F}$, it follows that $\hat{x}(i)$ converges robustly and exponentially fast to $\mathbb{S}$, and $x(i)$ converges robustly and exponentially fast to $\mathbb{F}$. We are now in a position to establish exponential stability of $\mathcal{A} := \mathbb{S} \times \{0\}$ with a region of attraction $(\bar{\mathcal{X}}_N \oplus \mathbb{S}) \times \bar{\mathcal{X}}_N$ for the composite system (5.23) and (5.24).

Proposition 5.6 (Exponential stability of output MPC)**.** *The set* $\mathcal{A} := \mathbb{S} \times \{0\}$ *is exponentially stable with a region of attraction* $(\bar{\mathcal{X}}_N \oplus \mathbb{S}) \times \bar{\mathcal{X}}_N$ *for the composite system* (5.23) *and* (5.24)*.*

Proof. Let $\phi := (\hat{x}, \bar{x})$ denote the state of the composite system. Then $|\phi|_{\mathcal{A}}$ is defined by

$$|\phi|_{\mathcal{A}} = |\hat{x}|_{\mathbb{S}} + |\bar{x}|$$

where $|\hat{x}|_{\mathbb{S}} := d(\hat{x}, \mathbb{S})$. But $\hat{x} \in \{\bar{x}\} \oplus \mathbb{S}$ implies $\hat{x} = \bar{x} + e$ for some $e \in S$ so that

$$|\hat{x}|_{\mathbb{S}} = d(\hat{x}, \mathbb{S}) = d(\bar{x} + e, \mathbb{S}) \leq d(\bar{x} + e, e) = |\bar{x}|$$

since $e \in \mathbb{S}$. Hence $|\phi|_{\mathcal{A}} \leq 2\,|\bar{x}|$ so that

$$|\phi(i)|_{\mathcal{A}} \leq 2\,|\bar{x}(i)| \leq 2c\,|\bar{x}(0)|\,\gamma^i \leq 2c\,|\phi(0)|\,\gamma^i$$

for all $\phi(0) \in (\bar{\mathcal{X}}_N \oplus \mathbb{S}) \times \bar{\mathcal{X}}_N$. Since for all $\bar{x}(0) \in \bar{\mathcal{X}}_N$, $\bar{x}(i) \in \bar{\mathbb{X}}$ and $\bar{u}(i) \in \bar{\mathbb{U}}$, it follows that $\hat{x}(i) \in \{\bar{x}(i)\} \oplus \mathbb{S}$, $x(i) \in \mathbb{X}$, and $u(i) \in \mathbb{U}$ for all $i \in \mathbb{I}_{\geq 0}$. Thus $\mathcal{A} := \mathbb{S} \times \{0\}$ is exponentially stable with a region of attraction $(\bar{\mathcal{X}}_N \oplus \mathbb{S}) \times \bar{\mathcal{X}}_N$ for the composite system (5.23) and (5.24). ∎

It follows from Proposition 5.6 that $x(i)$, which lies in the set $\{\bar{x}(i)\} \oplus \mathbb{F}$, $\mathbb{F} := \mathbb{S} \oplus \Sigma$, converges to the set $\mathbb{F}$. In fact $x(i)$ converges to a set that is, in general, smaller than $\mathbb{F}$ since $\mathbb{F}$ is a conservative bound on $\tilde{x}(i) + e(i)$. We determine this smaller set as follows. Let $\phi := (\tilde{x}, e)$ and let $\psi := (w, \nu)$; ϕ is the state of the two error systems and ψ is a bounded disturbance lying in a C-set $\Psi := \mathbb{W} \times \mathbb{N}$. Then, from (5.10) and (5.16), the state ϕ evolves according to

$$\phi^+ = \widetilde{A}\phi + \widetilde{B}\psi \tag{5.25}$$

where

$$\widetilde{A} := \begin{bmatrix} A_L & 0 \\ LC & A_K \end{bmatrix} \qquad \widetilde{B} := \begin{bmatrix} I & -L \\ 0 & L \end{bmatrix}$$

Because $\rho(A_L) < 1$ and $\rho(A_K) < 1$, it follows that $\rho(\widetilde{A}) < 1$. Since $\rho(\widetilde{A}) < 1$ and Ψ is compact, there exists a robust positive invariant set $\Phi \subseteq \mathbb{R}^n \times \mathbb{R}^n$ for (5.25) satisfying

$$\widetilde{A}\Phi \oplus \widetilde{B}\Psi = \Phi$$

Hence $\phi(i) \in \Phi$ for all $i \in \mathbb{I}_{\geq 0}$ if $\phi(0) \in \Phi$. Since $x(i) = \bar{x}(i) + e(i) + \widetilde{x}(i)$, it follows that $x(i) \in \{\bar{x}(i)\} \oplus H\Phi$, $H := \begin{bmatrix} I_n & I_n \end{bmatrix}$, for all $i \in \mathbb{I}_{\geq 0}$ provided that $x(0)$, $\hat{x}(0)$, and $\bar{x}(0)$ satisfy $(\widetilde{x}(0), e(0)) \in \Phi$ where $\widetilde{x}(0) = x(0) - \hat{x}(0)$ and $e(0) = \hat{x}(0) - \bar{x}(0)$. If these initial conditions are satisfied, $x(i)$ converges robustly and exponentially fast to the set $H\Phi$.

The remaining robust controllers presented in Section 3.5 may be similarly modified to obtain a robust output model predictive controller.

5.3.5 Computing the Tightened Constraints

The analysis above shows the tightened state and control constraint sets $\bar{\mathbb{X}}$ and $\bar{\mathbb{U}}$ for the nominal optimal control problem can, in principle, be computed using set algebra. Polyhedral set computations are not robust, however, and usually are limited to sets in $\mathbb{R}^n$ with $n \leq 15$. So we present here an alternative method for computing tightened constraints, similar to that described in 3.5.3.

We next show how to obtain a conservative approximation to $\bar{\mathbb{X}} \subseteq \mathbb{X} \ominus \Gamma$, $\Gamma = \mathbb{S} \oplus \Sigma$. Suppose $c'x \leq d$ is one of the constraints defining $\mathbb{X}$. Since $e = \hat{x} - \bar{x}$, which lies in $\mathbb{S}$, and $\widetilde{x} = x - \hat{x}$, which lies in Σ, satisfy $e^+ = A_K e + LC\widetilde{x} + Lv$ and $\widetilde{x}^+ = A_L\widetilde{x} + w - Lv$, the constraint $c'x \leq d$ (one of the constraints defining $\mathbb{X}$), the corresponding constraint in $\bar{\mathbb{X}}$ should be $c'x \leq d - \phi_\infty^{\bar{\mathbb{X}}}$ in which

$$\begin{aligned}
\phi_\infty^{\bar{\mathbb{X}}} &= \max\{c'e \mid e \in \mathbb{S}\} + \max\{c'\widetilde{x} \mid \widetilde{x} \in \Sigma\} \\
&= \max_{(w(i),v(i))} \sum_{j=0}^{\infty} A_K^j (LC\widetilde{x}(j) + Lv(j)) + \max_{(w(i),v(i))} \sum_{j=0}^{\infty} A_L^j (w(j) - Lv(j))
\end{aligned}$$

in which $\widetilde{x}(j) = \sum_{i=0}^{j-1} A_L^i (w(i) - Lv(i))$. The maximizations are subject to the constraints $w(i) \in \mathbb{W}$ and $v(i) \in \mathbb{N}$ for all $i \in \mathbb{I}_{\geq 0}$. Because maximization over an infinite horizon is impractical, we determine, as in 3.5.3, a horizon $N \in \mathbb{I}_{\geq 0}$ and an $\alpha \in (0,1)$ such that $A_K^N \mathbb{W} \subset \alpha \mathbb{W}$ and $A_L^N \mathbb{N} \subset \alpha \mathbb{N}$, and define the constraint in $\bar{\mathbb{X}}$ corresponding to the constraint $c'x \leq d$ in $\mathbb{X}$ to be $c'x \leq d - (1-\alpha^{-1})\phi_N^{\bar{\mathbb{X}}}$ with

$$\phi_N^{\bar{\mathbb{X}}} = \max_{(w(i),\nu(i))} \sum_{j=0}^{N-1} A_K^j (LC\tilde{x}(j) + L\nu(j)) + \max_{(w(i),\nu(i))} \sum_{j=0}^{N-1} A_L^j (w(j) - L\nu(j))$$

The tightened constraints yielding a conservative approximation to $\bar{\mathbb{U}} := \mathbb{U} \ominus KS$ may be similarly computed. The constraint $c'u \leq d$, one of the constraints defining $\mathbb{U}$, should be replaced by $c'u \leq d - (1-\alpha)^{-1}\phi_N^{\bar{\mathbb{U}}}$ with

$$\phi_N^{\bar{\mathbb{U}}} = \max\{c'e \mid e \in KS\} = \max_{(w(i),\nu(i))} \sum_{j=0}^{N-1} KA_K^j (LC\tilde{x}(j) + L\nu(j))$$

The maximizations for computing $\phi_N^{\bar{\mathbb{X}}}$ and $\phi_N^{\bar{\mathbb{U}}}$ are subject to the constraints $w(i) \in \mathbb{W}$ and $\nu(i) \in \mathbb{N}$ for all $i \in \mathbb{I}_{\geq 0}$.

5.4 Linear Constrained Systems: Time-Varying Case

The time-invariant case corresponds to the "steady-state" situation in which the sets $\mathbb{S}(t)$ and $\Sigma(t)$ have settled down to their steady-state values $\mathbb{S}$ and Σ, respectively. As a result the constraint sets $\bar{\mathbb{X}}$ and $\bar{\mathbb{U}}$ are also time invariant. When the state is accessible, the constraint $x \in \bar{\mathbb{X}}(i) := \mathbb{X} \ominus S(i)$ is less conservative than $x \in \bar{\mathbb{X}} = \mathbb{X} \ominus S$, in which $S = S(\infty)$. This relaxation of the constraint may be useful in some applications. The version of tube-based MPC employed here is such that $\mathbb{S}(t+1) \supset \mathbb{S}(t)$ for all t so that $\mathbb{S}(t)$ converges to $\mathbb{S}(\infty)$ as $t \to \infty$. In other versions of tube-based MPC, in which $\mathbb{P}_N(x)$ rather $\mathbb{P}_N(\bar{x})$ is solved online, $\mathbb{S}(t)$ is reset to the empty set so that advantage in using $\mathbb{S}(t)$ rather than $\mathbb{S}(\infty)$ is larger. On the other hand, the state estimation set $\Sigma(t)$ may increase or decrease with t depending on prior information. The time-varying version of tube-based MPC is fully discussed in Mayne, Raković, Findeisen, and Allgöwer (2009).

5.5 Offset-Free MPC

Offset-free MPC was introduced in Chapters 1 and 2 in a deterministic context; see also Pannocchia and Rawlings (2003). Suppose the system

Set	Definition	Membership
$\mathbb{X}$	state constraint set	$x \in \mathbb{X}$
$\mathbb{U}$	input constraint set	$u \in \mathbb{U}$
$\mathbb{W}_x$	state disturbance set	$w_x \in \mathbb{W}_x$
$\mathbb{W}_d$	integrating disturbance set	$w_d \in \mathbb{W}_d$
$\mathbb{W}$	total state disturbance set, $\mathbb{W}_x \times \mathbb{W}_d$	$w \in \mathbb{W}$
$\mathbb{N}$	measurement error set	$\nu \in \mathbb{N}$
$\widetilde{\mathbb{W}}$	estimate error disturbance set, $\mathbb{W} \oplus (-L\mathbb{N})$	$\widetilde{w} \in \widetilde{\mathbb{W}}$
Φ	total estimate error disturbance set, $\Phi = \widetilde{A}_L \Phi \oplus \widetilde{\mathbb{W}}$	$\phi \in \Phi$
Σ_x	state estimate error disturbance set, $\begin{bmatrix} I_n & 0 \end{bmatrix} \Phi$	$\widetilde{x} \in \Sigma_x$
Σ_d	integrating disturbance estimate error set, $\begin{bmatrix} 0 & I_p \end{bmatrix} \Phi$	$\widetilde{d} \in \Sigma_d$
$\mathbb{A}$	innovation set, $L(\widetilde{C}\Phi \oplus \mathbb{N})$	$L\widetilde{y} \in \mathbb{A}$
$\mathbb{A}_x$	set containing state component of innovation, $L_x(\widetilde{C}\Phi \oplus \mathbb{N})$	$L_x\widetilde{y} \in \mathbb{A}_x$
$\mathbb{A}_d$	set containing integrating disturbance component of innovation, $L_d(\widetilde{C}\Phi \oplus \mathbb{N})$	$L_d\widetilde{y} \in \mathbb{A}_d$
$\mathbb{S}$	nominal state tracking error invariance set, $A_K\mathbb{S} \oplus \mathbb{A}_x = \mathbb{S}$	$e \in \mathbb{S}$ $\hat{x} \in \{\bar{x}\} + \mathbb{S}$
$\mathbb{T}$	state tracking error invariance set, $\mathbb{S} + \Sigma_x$	$x \in \{\bar{x}\} + \mathbb{T}$
$\bar{\mathbb{U}}$	nominal input constraint set, $\bar{\mathbb{U}} = \mathbb{U} \ominus K\mathbb{S}$	$\bar{u} \in \bar{\mathbb{U}}$
$\bar{\mathbb{X}}$	nominal state constraint set, $\bar{\mathbb{X}} = \mathbb{X} \ominus \mathbb{T}$	: $\bar{x} \in \bar{\mathbb{X}}$

Table 5.1: Summary of the sets and variables used in output MPC.

to be controlled is described by

$$\begin{aligned} x^+ &= Ax + B_d d + Bu + w_x \\ y &= Cx + C_d d + \nu \\ r &= Hy \qquad \widetilde{r} = r - \bar{r} \end{aligned}$$

in which w_x and ν are unknown bounded disturbances taking values, respectively, in the compact sets $\mathbb{W}_x$ and $\mathbb{N}$ containing the origin in their interiors. In the following discussion, $y = Cx + C_d d$ is the output of the system being controlled, r is the controlled variable, and $\bar{r}$ is its setpoint. The variable $\widetilde{r}$ is the tracking error that we wish to minimize. We assume d is nearly constant but drifts slowly, and model its behavior by

$$d^+ = d + w_d$$

where w_d is a bounded disturbance taking values in the compact set $\mathbb{W}_d$; in practice d is bounded, although this is not implied by our model. We assume that $x \in \mathbb{R}^n$, $d \in \mathbb{R}^p$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^r$, and $e \in \mathbb{R}^q$, $q \leq r$, and that the system to be controlled is subject to the usual state and control constraints

$$x \in \mathbb{X} \qquad u \in \mathbb{U}$$

We assume $\mathbb{X}$ is polyhedral and $\mathbb{U}$ is polytopic.

Given the many sets that are required to specify the output feedback case we are about to develop, Table 5.1 may serve as a reference for the sets defined in the chapter and the variables that are members of these sets.

5.5.1 Estimation

Since both x and d are unknown, it is necessary to estimate them. For estimation purposes, it is convenient to work with the composite system whose state is $\phi := (x, d)$. This system may be described more compactly by

$$\begin{aligned} \phi^+ &= \widetilde{A}\phi + \widetilde{B}u + w \\ y &= \widetilde{C}\phi + \nu \end{aligned}$$

in which $w = (w_x, w_d)$ and

$$\widetilde{A} := \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} \qquad \widetilde{B} := \begin{bmatrix} B \\ 0 \end{bmatrix} \qquad \widetilde{C} := \begin{bmatrix} C & C_d \end{bmatrix}$$

and $w := (w_x, w_d)$ takes values in $\mathbb{W} = \mathbb{W}_x \times \mathbb{W}_d$. A necessary and sufficient condition for the detectability of $(\widetilde{A}, \widetilde{C})$ is given in Lemma 1.8. A sufficient condition is detectability of (A, C), coupled with invertibility of C_d. If $(\widetilde{A}, \widetilde{C})$ is detectable, the state may be estimated using the time-invariant observer or filter described by

$$\hat{\phi}^+ = \widetilde{A}\hat{\phi} + \widetilde{B}u + \delta \qquad \delta := L(y - \widetilde{C}\hat{\phi})$$

in which L is such that $\rho(\widetilde{A}_L) < 1$ where $\widetilde{A}_L := \widetilde{A} - L\widetilde{C}$. Clearly $\delta = L\widetilde{y}$ where $\widetilde{y} = \widetilde{C}\widetilde{\phi} + \nu$. The estimation error $\widetilde{\phi} := \phi - \hat{\phi}$ satisfies

$$\widetilde{\phi}^+ = \widetilde{A}\widetilde{\phi} + w - L(\widetilde{C}\widetilde{\phi} + \nu)$$

or, in simpler form

$$\widetilde{\phi}^+ = \widetilde{A}_L \widetilde{\phi} + \widetilde{w} \qquad \widetilde{w} := w - Lv$$

Clearly $\widetilde{w} = w - Lv$ takes values in the compact set $\widetilde{\mathbb{W}}$ defined by

$$\widetilde{\mathbb{W}} := \mathbb{W} \oplus (-L\mathbb{N})$$

If w and v are zero, $\widetilde{\phi}$ decays to zero exponentially fast so that $\hat{x} \to \bar{x}$ and $\hat{d} \to d$ exponentially fast. Since $\rho(\widetilde{A}_L) < 1$ and $\widetilde{\mathbb{W}}$ is compact, there exists a robust positive invariant set Φ for $\widetilde{\phi}^+ = \widetilde{A}_L\widetilde{\phi} + \widetilde{w}$, $\widetilde{w} \in \widetilde{\mathbb{W}}$ satisfying

$$\Phi = \widetilde{A}_L \Phi \oplus \widetilde{\mathbb{W}}$$

Hence $\widetilde{\phi}(i) \in \Phi$ for all $i \in \mathbb{I}_{\geq 0}$ if $\widetilde{\phi}(0) \in \Phi$. Since $\widetilde{\phi} = (\widetilde{x}, \widetilde{d}) \in \mathbb{R}^n \times \mathbb{R}^p$ where $\widetilde{x} := x - \hat{x}$ and $\widetilde{d} := d - \hat{d}$, we define the sets Σ_x and Σ_d as follows

$$\Sigma_x := \begin{bmatrix} I_n & 0 \end{bmatrix} \Phi \qquad \Sigma_d := \begin{bmatrix} 0 & I_p \end{bmatrix} \Phi$$

It follows that $\widetilde{x}(i) \in \Sigma_x$ and $\widetilde{d}(i) \in \Sigma_d$ so that $x(i) \in \{\hat{x}\} \oplus \Sigma_x$ and $d(i) \in \{\hat{d}(i)\} \oplus \Sigma_d$ for all $i \in \mathbb{I}_{\geq 0}$ if $\widetilde{\phi}(0) = (\widetilde{x}(0), \widetilde{d}(0)) \in \Phi$. That $\widetilde{\phi}(0) \in \Phi$ is a steady-state assumption.

5.5.2 Control

The estimation problem has a solution similar to previous solutions. The control problem is more difficult. As before, we control the estimator state, making allowance for state estimation error. The estimator state $\hat{\phi}$ satisfies the difference equation

$$\hat{\phi}^+ = \widetilde{A}\hat{\phi} + \widetilde{B}u + \delta$$

where the disturbance δ is defined by

$$\delta := L\widetilde{y} = L(\widetilde{C}\widetilde{\phi} + v)$$

The disturbance $\delta = (\delta_x, \delta_d)$ lies in the C−set $\mathbb{\Delta}$ defined by

$$\mathbb{\Delta} := L(\widetilde{C}\Phi \oplus \mathbb{N})$$

where the set Φ is defined in Section 5.5.1. The system $\hat{\phi}^+ = \widetilde{A}\hat{\phi} + \widetilde{B}u + \delta$ is not stabilizable, however, so we examine the subsystems with states $\hat{x}$ and $\hat{d}$

$$\begin{aligned} \hat{x}^+ &= A\hat{x} + B_d\hat{d} + Bu + \delta_x \\ \hat{d}^+ &= \hat{d} + \delta_d \end{aligned}$$

where the disturbances δ_x and δ_d are components of $\delta = (\delta_x, \delta_d)$ and are defined by

$$\delta_x := L_x \widetilde{y} = L_x(\widetilde{C}\widetilde{\phi} + \nu) \qquad \delta_d := L_d \widetilde{y} = L_d(\widetilde{C}\widetilde{\phi} + \nu)$$

The matrices L_x and L_d are the corresponding components of L. The disturbance δ_x and δ_d lie in the C−sets $\mathbb{\Delta}_x$ and $\mathbb{\Delta}_d$ defined by

$$\mathbb{\Delta}_x := \begin{bmatrix} I_n & 0 \end{bmatrix} \mathbb{\Delta} = L_x[\widetilde{C}\Phi \oplus \mathbb{N}] \qquad \mathbb{\Delta}_d := \begin{bmatrix} 0 & I_p \end{bmatrix} \mathbb{\Delta} = L_d[\widetilde{C}\Phi \oplus \mathbb{N}]$$

We assume that (A, B) is a stabilizable pair so the tube methodology may be employed to control $\hat{x}$. The system $\hat{d}^+ = \hat{d} + \delta_d$ is uncontrollable. The central trajectory is therefore chosen to be the nominal version of the difference equation for $(\hat{x}, \hat{d})$ and is described by

$$\begin{aligned} \bar{x}^+ &= A\bar{x} + B_d\hat{d} + B\bar{u} \\ \bar{d}^+ &= \bar{d} \end{aligned}$$

in which the initial state is $(\hat{x}, \hat{d})$. We obtain $\bar{u} = \bar{\kappa}_N(\bar{x}, \bar{d}, \bar{r})$ by solving a nominal optimal control problem defined later and set $u = \bar{u} + Ke$, $e := \hat{x} - \bar{x}$ where K is chosen so that $\rho(A_K) < 1$, $A_K := A + BK$; this is possible since (A, B) is assumed to be stabilizable. It follows that $e := \hat{x} - \bar{x}$ satisfies the difference equation

$$e^+ = A_K e + \delta_x \qquad \delta_x \in \mathbb{\Delta}_x$$

Because $\mathbb{\Delta}_x$ is compact and $\rho(A_K) < 1$, there exists a robust positive invariant set $\mathbb{S}$ for $e^+ = A_K e + \delta_x$, $\delta_x \in \mathbb{\Delta}_x$ satisfying

$$A_K \mathbb{S} \oplus \mathbb{\Delta}_x = \mathbb{S}$$

Hence $e(i) \in \mathbb{S}$ for all $i \in \mathbb{I}_{\geq 0}$ if $e(0) \in \mathbb{S}$. So, as in Proposition 5.3, the states and controls of the estimator and nominal system satisfy $\hat{x}(i) \in \{\bar{x}(i)\} \oplus \mathbb{S}$ and $u(i) \in \{\bar{u}(i)\} \oplus K\mathbb{S}$ for all $i \in \mathbb{I}_{\geq 0}$ if the initial states $\hat{x}(0)$ and $\bar{x}(0)$ satisfy $\hat{x}(0) \in \{\bar{x}(0)\} \oplus \mathbb{S}$. Using the fact established previously that $\widetilde{x}(i) \in \mathbb{\Sigma}_x$ for all i, we can also conclude that $x(i) = \bar{x}(i) + e(i) + \widetilde{x}(i) \in \{\bar{x}(i)\} \oplus \mathbb{\Gamma}$ and that $u(i) = \bar{u}(i) + Ke(i) \in \{\bar{u}(i)\} + K\mathbb{S}$ for all i where $\mathbb{\Gamma} := \mathbb{S} \oplus \mathbb{\Sigma}_x$ provided, of course, that $\phi(0) \in \{\hat{\phi}(0)\} \oplus \Phi$ and $x(0) \in \{\hat{x}(0)\} \oplus \mathbb{S}$. These conditions are equivalent to $\widetilde{\phi}(0) \in \Phi$ and $e(0) \in \mathbb{S}$ where, for all i, $e(i) := \hat{x}(i) - \bar{x}(i)$. Hence $x(i)$ lies in $\mathbb{X}$ and $u(i)$ lies in $\mathbb{U}$ if $\bar{x}(i) \in \bar{\mathbb{X}} := \mathbb{X} \ominus \mathbb{\Gamma}$ and $\bar{u}(i) \in \bar{\mathbb{U}} := \mathbb{U} \ominus K\mathbb{S}$.

Thus $\hat{x}(i)$ and $x(i)$ evolve in known neighborhoods of the central state $\bar{x}(i)$ that we can control. Although we know that the uncontrollable state $d(i)$ lies in the set $\{\hat{d}(i)\} \oplus i\mathbb{\Sigma}_d$ for all i, the evolution of $\hat{d}(i)$

is an uncontrollable random walk and is, therefore, unbounded. If the initial value of $\hat{d}$ at time 0 is $\hat{d}_0$, then $\hat{d}(i)$ lies in the set $\{\hat{d}_0\} \oplus i\Sigma_d$ that increases without bound as i increases. This behavior is a defect in our model for the disturbance d; the model is useful for estimation purposes, but is unrealistic in permitting unbounded values for d. Hence we assume in the sequel that d evolves in a compact C−set X_d. We can modify the observer to ensure that $\hat{d}$ lies in X_d, but find it simpler to observe that if d lies in X_d, $\hat{d}$ must lie in $X_d \oplus \Sigma_d$.

Target Calculation. Our first task is to determine the target state $\bar{x}_s$ and associated control $\bar{u}_s$; we require our estimate of the tracking error $\tilde{r} = r - \bar{r}$ to be zero in the absence of any disturbances. We follow the procedure outlined in Pannocchia and Rawlings (2003). Since our estimate of the measurement noise ν is 0 and since our best estimate of d when the target state is reached is $\hat{d}$, we require

$$\hat{r} - \bar{r} = H(C\bar{x}_s + C_d\hat{d}) - \bar{r} = 0$$

We also require the target state to be an equilibrium state satisfying, therefore, $\bar{x}_s = A\bar{x}_s + B_d\hat{d} + B\bar{u}_s$ for some control $\bar{u}_s$. Given $(\hat{d}, \bar{r})$, the target equilibrium pair $(\bar{x}_s, \bar{u}_s)(\hat{d}, \bar{r})$ is computed as follows

$$\begin{aligned}(\bar{x}_s, \bar{u}_s)(\hat{d}, \bar{r}) = \arg\min_{\bar{x},\bar{u}}\{L(\bar{x}, \bar{u}) \mid \bar{x} = A\bar{x} + B_d\hat{d} + B\bar{u},\\ H(C\bar{x} + C_d\hat{d}) = \bar{r}, \bar{x} \in \bar{\mathbb{X}},\ \bar{u} \in \bar{\mathbb{U}}\}\end{aligned}$$

where $L(\cdot)$ is an appropriate cost function; e.g., $L(\bar{x}, \bar{u}) = (1/2)\,|\bar{u}|^2_{\bar{R}}$. The equality constraints in this optimization problem can be satisfied if the matrix $\begin{bmatrix} I-A & -B \\ HC & 0 \end{bmatrix}$ has full rank. As the notation indicates, the target equilibrium pair $(\bar{x}_s, \bar{u}_s)(\hat{d}, \bar{r})$ is not constant, but varies with the estimate of the disturbance state d.

MPC algorithm. The control objective is to steer the central state $\bar{x}$ to a small neighborhood of the target state $\bar{x}_s(\hat{d}, \bar{r})$ while satisfying the state and control constraints $x \in \mathbb{X}$ and $u \in \mathbb{U}$. It is desirable that $\bar{x}(i)$ converges to $\bar{x}_s(\hat{d}, \bar{r})$ if $\hat{d}$ remains constant, in which case $x(i)$ converges to the set $\{\bar{x}_s(\hat{d}, \bar{r})\} \oplus \Gamma$. We are now in a position to specify the optimal control problem whose solution yields $\bar{u} = \bar{\kappa}_N(\bar{x}, \hat{d}, \bar{r})$ and, hence, $u = \bar{u} + K(\hat{x} - \bar{x})$.To achieve this objective, we define the deterministic optimal control problem

$$\bar{\mathbb{P}}_N(\bar{x}, \hat{d}, \bar{r}): \quad V_N^0(\bar{x}, \hat{d}, \bar{r}) := \min_{\bar{\mathbf{u}}}\{V_N(\bar{x}, \hat{d}, \bar{r}, \bar{\mathbf{u}}) \mid \bar{\mathbf{u}} \in \bar{\mathcal{U}}_N(\bar{x}, \hat{d}, \bar{r})\}$$

in which the cost $V_N(\cdot)$ and the constraint set $\bar{\mathcal{U}}_N(\bar{x}, \hat{d}, \bar{r})$ are defined by

$$V_N(\bar{x}, \hat{d}, \bar{r}, \bar{\mathbf{u}}) := \sum_{i=0}^{N-1} \ell(\bar{x}(i) - \bar{x}_s(\hat{d}, \bar{r}), \bar{u}(i) - \bar{u}_s(\hat{d}, \bar{r})) + V_f(\bar{x}(N), \bar{x}_s(\hat{d}, \bar{r}))$$

$$\bar{\mathcal{U}}_N(\bar{x}, \hat{d}, \bar{r}) := \{\bar{\mathbf{u}} \mid \bar{x}(i) \in \bar{\mathbb{X}}, \bar{u}(i) \in \bar{\mathbb{U}} \ \forall i \in \mathbb{I}_{0:N-1}, \bar{x}(N) \in \bar{\mathbb{X}}_f(\bar{x}_s(\hat{d}, \bar{r}))\}$$

and, for each i, $\bar{x}(i) = \bar{\phi}(i; \bar{x}, \hat{d}, \bar{\mathbf{u}})$, the solution of $\bar{x}^+ = A\bar{x} + B_d\hat{d} + B\bar{u}$ when the initial state is $\bar{x}$, the control sequence is $\bar{\mathbf{u}}$, and the disturbance $\hat{d}$ is constant, i.e., satisfies the nominal difference equation $\hat{d}^+ = \hat{d}$. The set of feasible $(\bar{x}, \hat{d}, \bar{r})$ and the set of feasible states $\bar{x}$ for $\bar{\mathbb{P}}_N(\bar{x}, \hat{d}, \bar{r})$ are defined by

$$\bar{\mathcal{F}}_N := \{(\bar{x}, \hat{d}, \bar{r}) \mid \mathcal{U}_N(\bar{x}, \hat{d}, \bar{r}) \neq \emptyset\} \qquad \bar{\mathcal{X}}_N(\hat{d}, \bar{r}) := \{\bar{x} \mid (\bar{x}, \hat{d}, \bar{r}) \in \bar{\mathcal{F}}_N\}$$

The terminal cost is zero when the terminal state is equal to the target state. The solution to $\bar{\mathbb{P}}_N(\bar{x}, \hat{d}, \bar{r})$ is

$$\bar{\mathbf{u}}^0(\bar{x}, \hat{d}, \bar{r}) = \{\bar{u}^0(0; \bar{x}, \hat{d}, \bar{r}), \bar{u}^0(1; \bar{x}, \hat{d}, \bar{r}), \ldots, \bar{u}^0(N-1; \bar{x}, \hat{d}, \bar{r})\}$$

and the implicit model control law $\bar{\kappa}_N(\cdot)$ is defined by

$$\bar{\kappa}_N(\bar{x}, \hat{d}, \bar{r}) := \bar{u}^0(0; \bar{x}, \hat{d}, \bar{r})$$

where $\bar{u}^0(0; \bar{x}, \hat{d}, \bar{r})$ is the first element in the sequence $\bar{\mathbf{u}}^0(\bar{x}, \hat{d}, \bar{r})$. The control u applied to the plant and the observer is $u = \kappa_N(\hat{x}, \bar{x}, \hat{d}, \bar{r})$ where $\kappa_N(\cdot)$ is defined by

$$\kappa_N(\hat{x}, \bar{x}, \hat{d}, \bar{r}) := \bar{\kappa}_N(\bar{x}, \hat{d}, \bar{r}) + K(\hat{x} - \bar{x})$$

Although the optimal control problem $\bar{\mathbb{P}}_N(\bar{x}, \hat{d}, \bar{r})$ is deterministic, $\hat{d}$ is random, so that the sequence $(\bar{x}(i))$, which satisfies $\bar{x}^+ = A\bar{x} + B_d\hat{d} + B\bar{\kappa}_N(\bar{x}, \hat{d}, \bar{r})$, is random, unlike the case discussed in Chapter 3. The control algorithm may now be formally stated.

Algorithm 5.7 (Robust control algorithm (offset-free MPC))**.**

1. At time 0, set $i = 0$, set $\hat{\phi} = \hat{\phi}(0)$ $(\hat{\phi} = (\hat{x}, \hat{d})$, and set $\bar{x} = \hat{x}$.

2. At time i, solve the "nominal" optimal control problem $\bar{\mathbb{P}}_N(\bar{x}, \hat{d}, \bar{r})$ to obtain the current "nominal" control action $\bar{u} = \bar{\kappa}_N(\bar{x}, \hat{d}, \bar{r})$ and the control action $u = \bar{u} + K(\hat{x} - \bar{x})$.

3. If $\bar{\mathbb{P}}_N(\bar{x}, \hat{d}, \bar{r})$ is infeasible, adopt safety/recovery procedure.

4. Apply the control u to the system being controlled.

5. Compute the successor state estimate $\hat{\phi}^+ = \widetilde{A}\hat{x} + \widetilde{B}u + L(y - \widetilde{C}\hat{\phi})$.

6. Compute the successor state $\bar{x}^+ = A\bar{x} + B_d\hat{d} + B\bar{u}$ of the nominal system.

7. Set $(\hat{\phi}, \bar{x}) = (\hat{\phi}^+, \bar{x}^+)$, set $i = i + 1$.

In normal operation, Step 2 is not activated; Propositions 5.2 and 5.3 ensure that the constraints $\hat{x} \in \{\bar{x}\} \oplus \mathbb{S}$ and $u \in \{\bar{u}\} \oplus K\mathbb{S}$ are satisfied. If an unanticipated event occurs and Step 2 is activated, the controller can be reinitialized by setting $\bar{u} = \bar{\kappa}_N(\hat{x}, \hat{d}, \bar{r})$, setting $u = \bar{u}$, and relaxing constraints if necessary.

5.5.3 Convergence Analysis

We give here an informal discussion of the stability properties of the controller. The controller described above is motivated by the following consideration: nominal MPC is able to handle "slow" uncertainties such as the drift of a target point."Fast" uncertainties, however, are better handled by the tube controller that generates, using MPC, a suitable central trajectory and a "fast" ancillary controller to steer trajectories of the uncertain system toward the central trajectory. As shown above, the controller ensures that $x(i) \in \{\bar{x}(i)\} \oplus \Gamma$ for all i; its success therefore depends on the ability of the controlled nominal system $\bar{x}^+ = A\bar{x} + B_d\hat{d} + B\bar{\kappa}_N(\bar{x}, \hat{d}, \bar{r})$, $\bar{u} = \bar{\kappa}_N(\bar{x}, \hat{d}, \bar{r})$, to track the target $\bar{x}_s(\hat{d}, \bar{r})$ that varies as $\hat{d}$ evolves.

Assuming that the standard stability assumptions are satisfied for the nominal optimal control problem $\bar{\mathbb{P}}_N(\bar{x}, \hat{d}, \bar{r})$ defined above, we have

$$V_N^0(\bar{x}, \hat{d}, \bar{r}) \geq c_1 \left| \bar{x} - \bar{x}_s(\hat{d}, \bar{r}) \right|^2$$
$$V_N^0(\bar{x}, \hat{d}, \bar{r}) \leq c_2 \left| \bar{x} - \bar{x}_s(\hat{d}, \bar{r}) \right|^2$$
$$V_N^0(\bar{x}^+, \hat{d}, \bar{r}) \leq V_N^0(\bar{x}, \hat{d}, \bar{r}) - c_1 \left| \bar{x} - \bar{x}_s(\hat{d}, \bar{r}) \right|^2$$

with $\bar{x}^+ = A\bar{x} + B_d\hat{d} + B\bar{\kappa}_N(\bar{x}, \hat{d}, \bar{r})$, for all $(\bar{x}, \hat{d}, \bar{r}) \in \bar{\mathcal{F}}_N$. The first and last inequalities follow from our assumptions; we assume the existence of the upper bound in the second inequality. The inequalities hold for all $(\bar{x}, \hat{d}, \bar{r}) \in \bar{\mathcal{F}}_N$. Note that the last inequality does NOT ensure $V_N^0(\bar{x}^+, \hat{d}^+, \bar{r}) \leq V_N^0(\bar{x}, \hat{d}, \bar{r}) - c_1 \left|\bar{x} - \bar{x}_s(\hat{d}, \bar{r})\right|^2$ with $\bar{x}^+ = A\bar{x} + B_d\hat{d} + B\bar{\kappa}_N(\bar{x}, \hat{d}, \bar{r})$ and $\hat{d}^+ := \hat{d} + \delta_d$. The perturbation due to δ_d has to be taken into account when analyzing stability.

Constant $\hat{d}$. If $\hat{d}$ remains constant, $\bar{x}_s(\hat{d}, \bar{r})$ is exponentially stable for $\bar{x}^+ = A\bar{x} + B_d\hat{d} + B\bar{\kappa}_N(\bar{x}, \hat{d}, \bar{r})$ with a region of attraction $\bar{\mathcal{X}}_N(\hat{d}, \bar{r})$. It can be shown, as in the proof of Proposition 5.6, that the set $\mathcal{A}(\hat{d}, \bar{r}) := (\{\bar{x}_s(\hat{d}, \bar{r})\} \oplus \mathbb{S}) \times \{\bar{x}_s(\hat{d}, \bar{r})\}$ is exponentially stable for the composite system $\hat{x}^+ = A\hat{x} + B_d\hat{d} + B\kappa_N(\hat{x}, \bar{x}, \hat{d}, \bar{r}) + \delta_x$, $\bar{x}^+ = A\bar{x} + B_d\hat{d} + B\bar{\kappa}_N(\bar{x}, \hat{d})$, $\delta_x \in \mathbb{A}_x$, with a region of attraction $(\bar{\mathcal{X}}_N(\hat{d}, \bar{r}) \oplus \mathbb{S}) \times \bar{\mathcal{X}}_N(\hat{d}, \bar{r})$. Hence $x(i) \in \{\bar{x}(i)\} \oplus \mathbb{\Gamma}$ tends to the set $\{\bar{x}_s(\hat{d}, \bar{r})\} \oplus \mathbb{\Gamma}$ as $i \to \infty$. If, in addition, $\mathbb{W} = \{0\}$ and $\mathbb{N} = \{0\}$, then $\mathbb{A} = \{0\}$ and $\mathbb{\Gamma} = \mathbb{\Sigma} = \mathbb{S} = \{0\}$ so that $x(i) \to \bar{x}_s(d, \bar{r})$ and $\tilde{r}(i) \to 0$ as $i \to \infty$.

Slowly varying $\hat{d}$. If $\hat{d}$ is varying, the descent property of $V_N^0(\cdot)$ is modified and it is necessary to obtain an upper bound for $V_N^0(A\bar{x} + B_d(\hat{d} + \delta_d) + B\bar{\kappa}_N(\bar{x}, \hat{d}, \bar{r}), \hat{d} + \delta_d, \bar{r})$. We make use of Proposition 3.4 in Chapter 3. If $\bar{\mathcal{X}}_N$ is compact and if $(\hat{d}, \bar{r}) \mapsto \bar{x}_s(\hat{d}, \bar{r})$ and $(\hat{d}, \bar{r}) \mapsto \bar{u}_s(\hat{d}, \bar{r})$ are both continuous in some compact domain, then, since $V_N(\cdot)$ is then continuous in a compact domain $\mathcal{A}$, it follows from the properties of $V_N^0(\cdot)$ and Proposition 3.4 that there exists a $\mathcal{K}_\infty$ function $\alpha(\cdot)$ such that

$$V_N^0(\bar{x}, \hat{d}, \bar{r}) \geq c_1 \left|\bar{x} - \bar{x}_s(\hat{d}, \bar{r})\right|^2$$

$$V_N^0(\bar{x}, \hat{d}, \bar{r}) \leq c_2 \left|\bar{x} - \bar{x}_s(\hat{d}, \bar{r})\right|^2$$

$$V_N^0(\bar{x}^+, \hat{d}^+, \bar{r}) \leq V_N^0(\bar{x}, \hat{d}, \bar{r}) - c_1 \left|\bar{x} - \bar{x}_s(\hat{d}, \bar{r})\right|^2 + \alpha(\delta_d)$$

for all $(\bar{x}, \hat{d}, \delta_d, \bar{r}) \in \mathcal{V}$; here $(\bar{x}, \hat{d})^+ := (\bar{x}^+, \hat{d}^+)$, $\bar{x}^+ = A\bar{x} + B_d(\hat{d} + \delta_d) + B\bar{\kappa}_N(\bar{x}, \hat{d}, \bar{r})$ and $\hat{d}^+ = \hat{d} + \delta_d$. A suitable choice for $\mathcal{A}$ is $\mathcal{V} \times \mathcal{D} \times \{\bar{r}\} \times \mathbb{U}^N$ with $\mathcal{V}$ the closure of $\operatorname{lev}_a V_N^0(\cdot)$ for some $a > 0$, and $\mathcal{D}$ a compact set containing d and $\hat{d}$. It follows that there exists a $\gamma \in (0, 1)$ such that

$$V_N^0((\bar{x}, \hat{d})^+, \bar{r}) \leq \gamma V_N^0(\bar{x}, \hat{d}, \bar{r}) + \alpha(\delta_d)$$

with $\gamma = 1 - c_1/c_2 \in (0, 1)$. Assuming that $\mathbb{P}_N(\bar{x}, \hat{d}, \bar{r})$ is recursively feasible

$$V_N^0(\bar{x}(i), \hat{d}(i), \bar{r}) \leq \gamma^i V_N^0(\bar{x}(0), \hat{d}(0), \bar{r}) + \alpha(\delta_d)(1 - \gamma^i)/(1 - \gamma)$$

in which $\bar{x}(0) = x(0)$ and $\hat{d}(0) = d(0)$. It then follows from the last inequality and the bounds on $V_N^0(\cdot)$ that

$$\left|\bar{x}(i) - \bar{x}_s(\hat{d}(i), \bar{r})\right| \leq \gamma^{i/2}(c_2/c_1)^{1/2}\left|\bar{x}(0) - \bar{x}_s(\hat{d}(0), \bar{r})\right| + c(i)$$

with $c(i) := [\alpha(\delta_d)(1-\gamma^i)/(1-\gamma)]^{1/2}$ so that $c(i) \to c := [\alpha(\delta_d)/(1-\gamma)]^{1/2}$ and $\left|\bar{x}(i) - \bar{x}_s(\hat{d}(i), \bar{r})\right| \to c$ as $i \to \infty$. Here we have made use of the fact that $(a+b)^{1/2} \leq a^{1/2} + b^{1/2}$.

Let $C \subset \mathbb{R}^n$ denote the set $\{x \mid |x| \leq c\}$. Then $\bar{x}(i) \to \{\bar{x}_s(\hat{d}(i), \bar{r})\} \oplus C$, $\hat{x}(i) \to \{\bar{x}_s(\hat{d}(i), \bar{r})\} \oplus C \oplus S$ and $x(i) \to \{\bar{x}_s(\hat{d}(i), \bar{r})\} \oplus C \oplus S \oplus \Sigma$ as $i \to \infty$. Since $c(i) = [\alpha(\delta_d)(1-\gamma^i)/(1-\gamma)]^{1/2} \to 0$ as $\delta_d \to 0$, it follows that $\bar{x}(i) \to \bar{x}_s(\hat{d}(i), \bar{r})$ as $i \to \infty$. The sizes of S and Σ are dictated by the process and measurement disturbances, w and ν respectively.

Recursive feasibility. The result that $x(i) \to \{\bar{x}_s(\hat{d}(i), \bar{r})\} \oplus C \oplus \Gamma$, $\Gamma := S \oplus \Sigma$, is useful because it gives an asymptotic bound on the tracking error. But it does depend on the recursive feasibility of the optimal control problem $\mathbb{P}_N(\cdot)$, which does not necessarily hold because of the variation of $\hat{d}$ with time. Tracking of a random reference signal has been considered in the literature, but not in the context of output MPC. We show next that $\mathbb{P}_N(\cdot)$ *is* recursively feasible and that the tracking error remains bounded if the estimate $\hat{d}$ of the disturbance d varies sufficiently slowly—that is if δ_d in the difference equation $\hat{d}^+ = \hat{d} + \delta_d$ is sufficiently small. This can be ensured by design of the state estimator.

To establish recursive feasibility, assume that the current "state" is $(\bar{x}, \hat{d}, \bar{r})$ and $\bar{x} \in \bar{\mathcal{X}}(\hat{d}, \bar{r})$. In other words, we assume $\bar{\mathbb{P}}_N(\bar{x}, \hat{d}, \bar{r})$ is feasible and $\bar{x}_N := \bar{\phi}(N; \bar{x}, \bar{\kappa}_N(\bar{x}, \hat{d}, \bar{r})) \in \mathbb{X}_f(\bar{x}_s(\hat{d}, \bar{r}))$. If the usual stability conditions are satisfied, problem $\mathbb{P}_N(\bar{x}^+, \hat{d}, \bar{r})$ is also feasible so that $\bar{x}^+ = A\bar{x} + B_d\hat{d} + B\bar{\kappa}_N(\bar{x}, \hat{d}, \bar{r}) \in \bar{\mathcal{X}}_N(\hat{d}, \bar{r})$. But $\hat{d}^+ = \hat{d} + \delta_d$ so that $\mathbb{P}_N(\bar{x}^+, \hat{d}^+, \bar{r})$ is *not* necessarily feasible since $\bar{x}_N$, which lies in $\mathbb{X}_f(\bar{x}_s(\hat{d}, \bar{r}))$, does not necessarily lie in $\mathbb{X}_f(\bar{x}_s(\hat{d}^+, \bar{r}))$. Let the terminal set $\mathbb{X}_f(\bar{x}_s(\hat{d}, \bar{r})) := \{x \mid V_f(x - \bar{x}_s(\hat{d}, \bar{r})) \leq c\}$. If the usual stability conditions are satisfied, for each $\bar{x}_N \in \mathbb{X}_f(\bar{x}_s(\hat{d}, \bar{r}))$, there exists a $u = \kappa_f(\bar{x}_N)$ that steers $\bar{x}_N$ to a state $\bar{x}_N^+$ in $\{x \mid V_f(x - \bar{x}_s(\hat{d}, \bar{r})) \leq e\}$, $e < c$. Consequently, there exists a feasible control sequence $\tilde{\mathbf{u}}(\bar{x}) \in \bar{\mathcal{U}}_N(\bar{x}, \hat{d}, \bar{r})$ that steers $\bar{x}^+$ to a state $\bar{x}_N^+ \in \{x \mid V_f(x - \bar{x}_s(\hat{d}, \bar{r})) \leq e\}$. If the map $\hat{d} \mapsto \bar{x}_s(\hat{d}, \bar{r})$ is uniformly continuous, there exists a constant $a > 0$ such that $|\delta_d| \leq a$ implies that $\bar{x}_N^+$ lies also in $\mathbb{X}_f(\bar{x}_s(\hat{d}^+, \bar{r})) = \{x \mid V_f(x - \bar{x}_s(\hat{d}^+, \bar{r})) \leq c\}$. Thus the control sequence $\tilde{\mathbf{u}}(\bar{x})$ also steers $\bar{x}^+$ to the set $\mathbb{X}_f(\bar{x}_s(\hat{d}^+, \bar{r}))$ and hence lies in $\bar{\mathcal{U}}_N(\bar{x}, \hat{d}^+, \bar{r})$. Hence problem $\bar{\mathbb{P}}_N(\bar{x}^+, \hat{d}^+, \bar{r})$ is feasible so that $\bar{\mathbb{P}}_N$ is recursively feasible if $\sup_{i \in \mathbb{I}_{0:\infty}} |\delta_d(i)| \leq e$.

Computing the tightened constraints. The first step in the control algorithm requires solution of the problem $\mathbb{P}_N(\bar{x}, \hat{d}, \bar{r})$, in which the state and control constraints are, respectively, $\bar{x} \in \bar{\mathbb{X}}$ and $\bar{u} \in \bar{\mathbb{U}}$. Since the sets $\bar{\mathbb{X}}$ and $\bar{\mathbb{U}}$ are difficult to compute, we replace them by tightened versions of the original constraints as described in Section 5.3.5.

> Summarizing, if the usual stability assumptions are satisfied, if $\hat{d}(i)$ remains in a compact set $X_{\hat{d}}$ for all i, if the map $\hat{d} \mapsto \bar{x}_s(\hat{d}, \bar{r})$ is continuous in $X_{\hat{d}}$, if $\ell(\cdot)$ and $V_f(\cdot)$ are quadratic and positive definite, and $|\delta_d(i)| \leq a$ for all i, then the asymptotic error $x(i) - \bar{x}_s(\hat{d}(i), \bar{r})$ lies in the compact set $C \oplus \Gamma$ ($\Gamma = S + \Sigma$) that converges to the set $\{0\}$ as the sets $\mathbb{W}$ and $\mathbb{N}$ that bound the disturbances converge to the zero set $\{0\}$. Similarly, the tracking error $r - \bar{r}$ is also bounded and converges to 0 as $\mathbb{W}$ and $\mathbb{N}$ converge to the zero set $\{0\}$.

5.6 Nonlinear Constrained Systems

When the system being controlled is nonlinear, the state can be estimated using moving horizon estimation (MHE), as described in Chapter 4. But establishing stability of nonlinear output MPC that employs MHE does not appear to have received much attention, with one important exception mentioned in Section 5.7.

5.7 Notes

The problem of output feedback control has been extensively discussed in the general control literature. For linear systems, it is well known that a stabilizing state feedback controller and an observer may be separately designed and combined to give a stabilizing output feedback controller (the separation principle). For nonlinear systems, Teel and Praly (1994) show that global stabilizability and complete uniform observability are sufficient to guarantee semiglobal stabilizability when a dynamic observer is used, and provide useful references to related work on this topic.

Although output MPC, in which nominal MPC is combined with a separately designed observer, is widely used in industry since the state is seldom available, it has received relatively little attention in the literature because of the inherent difficulty in establishing asymptotic stability. An extra complexity in MPC is the presence of hard constraints. A useful survey, more comprehensive than these notes, is provided in Findeisen, Imsland, Allgöwer, and Foss (2003). Earlier Michalska and Mayne (1995) show for deterministic systems that for any subset of the region of attraction of the full state feedback system, there exists a sampling time and convergence rate for the observer such that the subset also lies in the region of attraction of the output feedback system. A more sophisticated analysis in Imsland, Findeisen, Allgöwer, and Foss (2003) using continuous time MPC shows that the region of attraction and rate of convergence of the output feedback system can approach that of the state feedback system as observer gain increases.

We consider systems with input disturbances and noisy state measurement, and employ the "tube" methodology that has its roots in the work of Bertsekas and Rhodes (1971), and Glover and Schweppe (1971) on constrained discrete time systems subject to bounded disturbances. Reachability of a "target set" and a "target tube" are considered in these papers. These concepts were substantially developed in the context of continuous time systems in Khurzhanski and Valyi (1997); Aubin (1991); Kurzhanski and Filippova (1993).

The theory for discrete time systems is considerably simpler; a modern tube-based theory for optimal control of discrete time uncertain systems with imperfect state measurement appears in Moitié, Quincampoix, and Veliov (2002). As in this chapter, they regard a set X of states x that are consistent with past measurements as the "state" of the optimal control problem. The set X satisfies an uncertain "full information" difference equation of the form $X^+ = f^*(X, u, \mathbb{W}, v)$ so the output feedback optimal control problem reduces to robust control of an uncertain system with known state X.

The optimal control problem remains difficult because the state X, a subset of $\mathbb{R}^n$, is difficult to obtain numerically and determination of a control law as a function of (X, t) prohibitive. In Mayne, Raković, Findeisen, and Allgöwer (2006); Mayne et al. (2009) the output feedback problem is simplified considerably by replacing $X(t)$ by a simple outer approximation $\{\hat{x}(t)\} \oplus \mathbb{\Sigma}_x$ in the time-invariant case, and by $\{\hat{x}(t)\} \oplus \mathbb{\Sigma}_x(t)$ in the time-varying case. The set $\mathbb{\Sigma}_x$, or the sequence $(\mathbb{\Sigma}_x(t))$, may be precomputed so the difficult evolution equation for X is replaced by a simple evolution equation for $\hat{x}$; in the linear case, the Luenberger observer or Kalman filter describes the evolution of $\hat{x}$. The output feedback control problem reduces to control of an uncertain system with known state $\hat{x}$.

While the tube approach may be successfully employed for output MPC when the system being controlled is linear, there seems to be no literature on combining moving horizon estimation (MHE) with MPC when the system being controlled is nonlinear, except for the paper Copp and Hespanha (2014). The novel proposal in this paper is to replace separate solutions of the control and estimation problems by a single min-max problem in which the cost is, unusually, over the interval $(-\infty, \infty)$ or $[-T, T]$, and combines the cost of both estimation and control. The authors also propose an efficient interior point algorithm for solving the complex min-max problem.

The output MPC problem involves tracking of a possibly random reference, a problem that has extra difficulty when zero offset is required. There is a growing literature dealing with tracking random references not necessarily in the context of output MPC. Examples of papers dealing with this topic are Limon, Alvarado, Alamo, and Camacho (2008); Ferramosca, Limon, Alvarado, Alamo, and Camacho (2009); Falugi and Mayne (2013).

5.8 Exercises

Exercise 5.1: Hausdorff distance between a set and a subset

Show that $d_H(\mathbb{A}, \mathbb{B}) = \max_{a \in \mathbb{A}} d(a, \mathbb{B})$ if $\mathbb{A}$ and $\mathbb{B}$ are two compact subsets of $\mathbb{R}^n$ satisfying $\mathbb{B} \subseteq \mathbb{A}$.

Exercise 5.2: Hausdorff distance between sets $\mathbb{A} \oplus \mathbb{B}$ and $\mathbb{B}$

Show that $d_H(\mathbb{A} \oplus \mathbb{B}, \mathbb{A}) \leq |\mathbb{B}|$ if $\mathbb{A}$ and $\mathbb{B}$ are two compact subsets of $\mathbb{R}^n$ satisfying $0 \in \mathbb{B}$ in which $|\mathbb{B}| := \max_b \{|b| \mid b \in \mathbb{B}\}$.

Exercise 5.3: Hausdorff distance between sets $\{z\} \oplus \mathbb{B}$ and $\mathbb{A}$

Show that $d_H(\{z\} \oplus \mathbb{B}, \mathbb{A}) \leq |z| + d_H(\mathbb{A}, \mathbb{B})$ if $\mathbb{A}$ and $\mathbb{B}$ are two compact sets in $\mathbb{R}^n$.

Exercise 5.4: Hausdorff distance between sets $\{z\} \oplus \mathbb{A}$ and $\mathbb{A}$

Show that $d_H(\{z\} \oplus \mathbb{A}, \mathbb{A}) = |z|$ if z is a point and $\mathbb{A}$ is a compact set in $\mathbb{R}^n$.

Exercise 5.5: Hausdorff distance between sets $\mathbb{A} \oplus \mathbb{C}$ and $\mathbb{B} \oplus \mathbb{C}$

Show that $d_H(\mathbb{A} \oplus \mathbb{C}, \mathbb{B} \oplus \mathbb{C}) = d_H(\mathbb{A}, \mathbb{B})$ if $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$ are compact subsets of $\mathbb{R}^n$ satisfying $\mathbb{B} \subseteq \mathbb{A}$.

Exercise 5.6: Hausdorff distance between sets $F\mathbb{A}$ and $F\mathbb{B}$

Let $\mathbb{A}$ and $\mathbb{B}$ be two compact sets in $\mathbb{R}^n$ satisfying $\mathbb{A} \subseteq \mathbb{B}$, and let $F \in \mathbb{R}^{n \times n}$. Show that $d_H(F\mathbb{A}, F\mathbb{B}) \leq |F| \, d_H(\mathbb{A}, \mathbb{B})$ in which $|F|$ is the induced norm of F satisfying $|Fx| \leq |F| \; |x|$ and $|x| := d(x, 0)$.

Exercise 5.7: Linear combination of sets; $\lambda_1 \mathbb{W} \oplus \lambda_2 \mathbb{W} = (\lambda_1 + \lambda_2)\mathbb{W}$

If $\mathbb{W}$ is a convex set, show that $\lambda_1 \mathbb{W} \oplus \lambda_2 \mathbb{W} = (\lambda_1 + \lambda_2)\mathbb{W}$ for any $\lambda_1,\ \lambda_2 \in \mathbb{R}_{\geq 0}$. Hence show $\mathbb{W} \oplus \lambda \mathbb{W} \oplus \lambda^2 \mathbb{W} \oplus \ \cdots = (1 - \lambda)^{-1}\mathbb{W}$ if $\lambda \in [0, 1)$.

Exercise 5.8: Hausdorff distance between the sets $\Phi(i)$ and Φ

Show that there exist $c > 0$ and $\gamma \in (0, 1)$ such that

$$d_H(\Phi(i), \Phi) \leq c d_H(\Phi(0), \Phi)\gamma^i$$

in which

$$\Phi(i) = \widetilde{A}\Phi(i-1) + \widetilde{B}\Psi$$
$$\Phi = \widetilde{A}\Phi + \widetilde{B}\Psi$$

and $\widetilde{A}$ is a stable matrix ($\rho(\widetilde{A}) < 1$).

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