4

State Estimation

4.1 Introduction

We now turn to the general problem of estimating the state of a noisy dynamic system given noisy measurements. We assume that the system generating the measurements is given by

$$\begin{aligned} x^+ &= f(x, w) \\ y &= h(x) + v \end{aligned} \tag{4.1}$$

in which the process disturbance, w, measurement disturbance, v, and system initial state, $x(0)$, are independent random variables with stationary probability densities. One of our main purposes is to provide a state estimate to the MPC regulator as part of a feedback control system, in which case the model changes to $x^+ = f(x, u, w)$ with both process disturbance w and control input u. But state estimation is a general technique that is often used in monitoring applications without any feedback control. So for simplicity of presentation, we start with state estimation as an independent subject and neglect the control input u as part of the system model as in (4.1).

Finally, in Section 4.6, we briefly treat the problem of combined MHE estimation and MPC regulation. In Chapter 5, we discuss the combined use of MHE and MPC in much more detail.

4.2 Full Information Estimation

Of the estimators considered in this chapter, full information estimation will prove to have the best theoretical properties in terms of stability and optimality. Unfortunately, it will also prove to be computationally intractable except for the simplest cases, such as a linear system model. Its value therefore lies in clearly defining what is *desirable* in a

	System variable	Decision variable	Optimal decision
state	x	χ	$\hat{x}$
process disturbance	w	ω	$\hat{w}$
measured output	y	η	$\hat{y}$
measurement disturbance	v	ν	$\hat{v}$

Table 4.1: System and state estimator variables.

state estimator. One method for practical estimator design therefore is to come as close as possible to the properties of full information estimation (FIE) while maintaining a tractable online computation. This design philosophy leads directly to moving horizon estimation (MHE).

First we define some notation necessary to distinguish the system variables from the estimator variables. We have already introduced the system variables (x, w, y, v). In the estimator optimization problem, these have corresponding decision variables, which we denote $(\chi, \omega, \eta, \nu)$. The *optimal* decision variables are denoted $(\hat{x}, \hat{w}, \hat{y}, \hat{v})$ and these optimal decisions are the estimates provided by the state estimator. This notation is summarized in Table 4.1. Next we summarize the relationships between these variables

$$\begin{aligned} x^+ &= f(x, w) & y &= h(x) + v \\ \chi^+ &= f(\chi, \omega) & y &= h(\chi) + \nu \\ \hat{x}^+ &= f(\hat{x}, \hat{w}) & y &= h(\hat{x}) + \hat{v} \end{aligned}$$

Notice that it is always the system measurement y that appears in the second column of equations. We can also define the decision variable output, $\eta = h(\chi)$, but notice that ν measures the fitting error, $\nu = y - h(\chi)$, and we must use the system measurement y and not η in this relationship. Therefore, we do not satisfy a relationship like $\eta = h(\chi) + \nu$, but rather

$$\begin{aligned} y &= h(\chi) + \nu & \eta &= h(\chi) \\ y &= h(\hat{x}) + \hat{v} & \hat{y} &= h(\hat{x}) \end{aligned}$$

We begin with a reasonably general definition of the full information estimator that produces an estimator that is *stable*, which we also shall define subsequently. The full information objective function is

$$V_T(\chi(0), \boldsymbol{\omega}) = \ell_x(\chi(0) - \overline{x}_0) + \sum_{i=0}^{T-1} \ell_i(\omega(i), \nu(i)) \tag{4.2}$$

subject to

$$\chi^+ = f(\chi, \omega) \qquad y = h(\chi) + \nu$$

in which T is the current time, $y(i)$ is the measurement at time i, and $\overline{x}_0$ is the prior information on the initial state.[1] Because $\nu = y - h(\chi)$ is the error in fitting the measurement y, $\ell_i(\omega, \nu)$ penalizes the model disturbance and the fitting error. These are the two error sources we reconcile in all state estimation problems.

The full information estimator is then defined as the solution to

$$\min_{\chi(0), \boldsymbol{\omega}} V_T(\chi(0), \boldsymbol{\omega}) \tag{4.3}$$

The solution to the optimization exists for all $T \in \mathbb{I}_{\geq 0}$ because $V_T(\cdot)$ is continuous, due to the continuity of $f(\cdot)$ and $h(\cdot)$, and because $V_T(\cdot)$ is an unbounded function of its arguments, as will be clear after stage costs $\ell_x(\cdot)$ and $\ell_i(\cdot)$ are defined. We denote the solution as $\hat{x}(0|T)$, $\hat{w}(i|T), 0 \leq i \leq T-1$, $T \geq 1$, and the optimal cost as V_T^0. We also use $\hat{x}(T) := \hat{x}(T|T)$ to simplify the notation. The optimal solution and cost also depend on the measurement sequence $\mathbf{y} = \big(y(0), y(1), \ldots, y(T-1)\big)$, and the prior $\overline{x}_0$, but this dependency is made explicit only when necessary. The choice of stage costs $\ell_x(\cdot)$ and $\ell_i(\cdot)$ is made after we define the class of disturbances affecting the system.

The next order of business is to decide what class of systems to consider if the goal is to obtain a stable state estimator. A standard choice in most nonlinear estimation literature is to assume system observability. The drawback with this choice is that it is overly restrictive, even for linear systems. As discussed in Chapter 1, for linear systems we require only detectability for stable estimation (Exercise 1.33). We therefore start instead with an assumption of detectability that is appropriate for nonlinear systems, called incremental input/output-to-state stability (i-IOSS) Sontag and Wang (1997).

[1]Notice we have dropped the final measurement $y(T)$ compared to the problem considered in Chapter 1 to formulate the prediction form rather than the filtering form of the state estimation problem. So what we denote here as $\hat{x}(T|T)$ would be $\hat{x}^-(T)$ in the notation of Chapter 1. This change is purely for notational convenience, and all results developed in this chapter can also be expressed in the filtering form of MHE.

Definition 4.1 (i-IOSS)**.** The system $x^+ = f(x, w), y = h(x)$ is *incrementally input/output-to-state stable* (i-IOSS) if there exist functions $\alpha(\cdot) \in \mathcal{KL}$ and $\gamma_w(\cdot)$, $\gamma_v(\cdot) \in \mathcal{K}$ such that for every two initial states z_1 and z_2, and any two disturbance sequences $\mathbf{w}_1$ and $\mathbf{w}_2$ generating state sequences $\mathbf{x}_1(z_1, \mathbf{w}_1)$ and $\mathbf{x}_2(z_2, \mathbf{w}_2)$, the following holds for all $k \in \mathbb{I}_{\geq 0}$

$$\begin{aligned} |x(k; z_1, \mathbf{w}_1) - x(k; z_2, \mathbf{w}_2)| \leq \alpha_1(|z_1 - z_2|, k) + \\ \gamma_w\left(\|\mathbf{w}_1 - \mathbf{w}_2\|_{0:k-1}\right) + \gamma_v\left(\|h(\mathbf{x}_1) - h(\mathbf{x}_2)\|_{0:k-1}\right) \end{aligned} \tag{4.4}$$

The notation $x(k; x_0, \mathbf{w})$ denotes the solution to $x^+ = f(x, w)$ with initial value $x(0) = x_0$ with disturbance sequence $\mathbf{w} = (w(0), w(1), \ldots)$. Also we use the notation $h(\mathbf{x}_1) := (h(x(0; z_1, \mathbf{w}_1)), h(x(1; z_1, \mathbf{w}_1)), \ldots)$.

One of the most important and useful implications of the i-IOSS property is the following proposition.

Proposition 4.2 (Convergence of state under i-IOSS)**.** *If system $x^+ = f(x, w)$, $y = h(x)$ is i-IOSS, $w_1(k) \to w_2(k)$ and $y_1(k) \to y_2(k)$ as $k \to \infty$, then*

$$x(k; z_1, \mathbf{w}_1) \to x(k; z_2, \mathbf{w}_2) \quad \textit{as } k \to \infty \quad \textit{for all } z_1, z_2$$

The proof of this proposition is discussed in Exercise 4.3.

The class of disturbances (w, v) affecting the system is defined next. Often we assume these are random variables with stationary probability densities, and often zero-mean normal densities. When we wish to establish estimator stability, however, we wish to show that if the disturbances affecting the measurement converge to zero, then the estimate error also converges to zero. So here we start off by restricting attention to *convergent* disturbances. We treat bounded disturbances in Section 4.4.

Definition 4.3 (β-convergent sequence)**.** Given a $\beta(\cdot) \in \mathcal{KL}$, a bounded sequence $(w(k))$, $k \in \mathbb{I}_{\geq 0}$ is termed β-convergent if

$$|w(k)| \leq \beta(\|\mathbf{w}\|, k) \qquad \text{for all } k \geq 0$$

The following result also is useful.

Proposition 4.4 (Bounded, convergent sequences are β-convergent)**.** *Every bounded convergent sequence is β-convergent for some $\beta \in \mathcal{KL}$.*

A proof is given in (Rawlings and Ji, 2012). The following proposition is useful for choosing an appropriate cost function for the estimator.

Proposition 4.5 (Bound for sum cost of convergent sequence.)**.** *Let* $\mathbf{w}$ *be an arbitrary* β*-convergent sequence. There exist* $\mathcal{K}$ *functions* $\overline{\gamma}_w(\cdot)$ *and* $\gamma_w(\cdot)$*, which depend on* $\beta(\cdot)$*, such that for any positive sequence of cost functions* $\ell_i : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *for* $i \in \mathbb{I}_{\geq 0}$*, satisfying the bound*

$$\ell_i(w) \leq \gamma_w(|w|) \qquad \text{for all } w \in \mathbb{R}^n, \quad i \in \mathbb{I}_{\geq 0}$$

Then the following holds

$$\sum_{i=0}^{\infty} \ell_i(w(i)) \leq \overline{\gamma}_w(\|\mathbf{w}\|)$$

A proof is given in (Rawlings and Ji, 2012).

Now we treat the case of convergent disturbances.

Assumption 4.6 (β-convergent disturbances)**.** The sequences $(\mathbf{w}, \mathbf{v})$ are β-convergent for some chosen $\beta \in \mathcal{KL}$.

Assumption 4.7 (Positive definite stage cost)**.** The initial state cost and stage costs are continuous functions and satisfy the following inequalities for all $x \in \mathbb{R}^n$, $w \in \mathbb{R}^g$, and $v \in \mathbb{R}^p$

$$\underline{\gamma}_x(|x|) \leq \ell_x(x) \quad \leq \gamma_x(|x|) \tag{4.5}$$

$$\underline{\gamma}_w(|(w,v)|) \leq \ell_i(w,v) \leq \gamma_w(|(w,v)|) \qquad i \geq 0 \tag{4.6}$$

in which $\underline{\gamma}_x, \underline{\gamma}_w, \gamma_x, \gamma_w, \in \mathcal{K}_\infty$. Also γ_w satisfies the requirements of Proposition 4.5, i.e., there exists $\overline{\gamma}_w \in \mathcal{K}$ such that the following holds for all β-convergent $(\mathbf{w}, \mathbf{v})$

$$\sum_{i=0}^{\infty} \ell_i(w(i), v(i)) \leq \overline{\gamma}_w(\|(\mathbf{w}, \mathbf{v})\|)$$

We require the following basic inequalities discussed in Appendix B to streamline the presentation. See also Exercise 4.7 for an alternative approach. For $\gamma(\cdot) \in \mathcal{K}$, the following holds for all $a_i \in \mathbb{R}_{\geq 0}$, $i \in \mathbb{I}_{1:n}$

$$\frac{1}{n}(\gamma(a_1) + \cdots + \gamma(a_n)) \leq \gamma(a_1 + \cdots + a_n) \leq \gamma(na_1) + \cdots + \gamma(na_n) \tag{4.7}$$

Similarly, for $\beta(\cdot) \in \mathcal{KL}$ the following holds for all $a_i \in \mathbb{R}_{\geq 0}$, $i \in \mathbb{I}_{1:n}$, and all $t \in \mathbb{R}_{\geq 0}$

$$\frac{1}{n}\left(\beta(a_1, t) + \cdots + \beta(a_n, t)\right) \leq \beta((a_1 + \cdots + a_n), t) \leq \\ \beta(na_1, t) + \beta(na_2, t) + \cdots + \beta(na_n, t) \quad (4.8)$$

Next we define *estimator stability*. Again, because the system is nonlinear, we must define stability of a solution. Consider the zero estimate error solution for all $k \geq 0$. This solution arises when the system's initial state is equal to the estimator's prior and there are zero disturbances, $x_0 = \overline{x}_0$, $(w(i), v(i)) = 0$ all $i \geq 0$. In this case, the optimal solution to the full information problem is $\hat{x}(0|T) = \overline{x}_0$ and $\hat{w}(i|T) = 0$ for all $0 \leq i \leq T$, $T \geq 1$, which also gives perfect agreement of estimate and measurement $h(\hat{x}(i|T)) = y(i)$ for $0 \leq i \leq T$, $T \geq 1$. The perturbation to this solution are: the system's initial state (distance from $\overline{x}_0$), and the process and measurement disturbances. We define a robustly globally asymptotically stable (RGAS) estimator in terms of the estimate error caused by $x_0 \neq \overline{x}_0$, and $(w(i), v(i)) \neq 0$.

Definition 4.8 (Robustly globally asymptotically stable estimation)**.** The estimate is based on the *noisy* measurement $\mathbf{y} = h(\mathbf{x}(x_0, \mathbf{w})) + \mathbf{v}$. The estimate is RGAS if for all x_0 and $\overline{x}_0$, and bounded $(\mathbf{w}, \mathbf{v})$, there exist functions $\alpha(\cdot) \in \mathcal{KL}$ and $\delta_w(\cdot) \in \mathcal{K}$ such that the following holds for all $k \in \mathbb{I}_{\geq 0}$

$$\left| x(k; x_0, \mathbf{w}) - x(k; \hat{x}(0|k), \hat{\mathbf{w}}_k) \right| \leq \\ \alpha(|x_0 - \overline{x}_0|, k) + \delta_w(\|(\mathbf{w}, \mathbf{v})\|_{0:k-1}) \quad (4.9)$$

Remark. The main import of the RGAS definition is that the dynamic system generating the estimate error is input-to-state stable (ISS) considering the disturbances (w, v) as the input (Sontag and Wang, 1995).

We require a preliminary proposition concerning the behavior of the estimate error to establish the main result of interest.

Proposition 4.9 (Boundedness and convergence of estimate error)**.** *Consider an i-IOSS (detectable) system and measurement sequence generated by* (4.1) *with disturbances satisfying Assumption 4.6, and stage cost satisfying Assumption 4.7. Then the full information estimate error satisfies the following.*

(a) For all $k \geq 0$, $j \leq k$

$$\left| x(j; x_0, \mathbf{w}) - x(j; \hat{x}(0|k), \widehat{\mathbf{w}}_k) \right| \leq \pi_x(|x_0 - \overline{x}_0|) + \pi_w(\|(\mathbf{w}, \mathbf{v})\|) \tag{4.10}$$

(b) As $k \to \infty$, $\left| x(k; x_0, \mathbf{w}) - x(k; \hat{x}(0|k), \widehat{\mathbf{w}}_k) \right| \to 0$

Proof.

(a) First we note that the optimal full information cost function is bounded above for all $T \geq 0$ by V_∞ defined by the following

$$V_T^0 = V_T(\hat{x}(0|T), \widehat{\mathbf{w}}_T) \leq$$
$$V_T(x(0), \mathbf{w}) = \ell_x(x_0 - \overline{x}_0) + \sum_{i=0}^{\infty} \ell_i(w(i), v(i)) =: V_\infty$$

From Assumption 4.7 we have the following upper bound for V_∞

$$V_\infty \leq \gamma_x(|x_0 - \overline{x}_0|) + \overline{\gamma}_w(\|(\mathbf{w}, \mathbf{v})\|) \tag{4.11}$$

Using the triangle inequality, we have for all $k \geq 0$

$$|x_0 - \hat{x}(0|k)| \leq |x_0 - \overline{x}_0| + |\hat{x}(0|k) - \overline{x}_0|$$

and since for all $k \geq 0$, $\ell_x(\hat{x}(0|k) - \overline{x}_0) \leq V_\infty$, the lower bound for $\ell_x(\cdot)$ in Assumption 4.7 gives $|\hat{x}(0|k) - \overline{x}_0| \leq \underline{\gamma}_x^{-1}(V_\infty)$. Substituting in (4.11), noting $\underline{\gamma}_x^{-1}(\cdot)$ is a $\mathcal{K}_\infty$ function, and using (4.7) for the sum, gives

$$|x_0 - \hat{x}(0|k)| \leq |x_0 - \overline{x}_0| + \underline{\gamma}_x^{-1}(2\gamma_x(|x_0 - \overline{x}_0|)) + \underline{\gamma}_x^{-1}(2\overline{\gamma}_w(\|(\mathbf{w}, \mathbf{v})\|))$$

Notice from the properties of $\mathcal{K}$ functions, that the right-hand side is the sum of $\mathcal{K}_\infty$ functions of the three disturbances, $x_0 - \overline{x}_0$, $\mathbf{w}$, and $\mathbf{v}$. So we have established the following bound valid for all $k \geq 0$

$$|x_0 - \hat{x}(0|k)| \leq \pi_x^x(|x_0 - \overline{x}_0|) + \pi_w^x(\|(\mathbf{w}, \mathbf{v})\|) \tag{4.12}$$

for $\pi_x^x, \pi_w^x \in \mathcal{K}_\infty$.

Next we develop a bound for the term $\left\|\mathbf{w} - \widehat{\mathbf{w}}_k\right\|_{0:k-1}$. From the triangle inequality and the definition of the sup norm we have

$$\left\|\mathbf{w} - \widehat{\mathbf{w}}_k\right\|_{0:k-1} \leq \|\mathbf{w}\|_{0:k-1} + \left\|\widehat{\mathbf{w}}_k\right\|_{0:k-1} \leq \|\mathbf{w}\| + \left\|\widehat{\mathbf{w}}_k\right\|_{0:k-1}$$

Now we require a bound for $\left\|\widehat{\mathbf{w}}_k\right\|_{0:k-1}$. We have from the definition of V_∞, $\underline{\gamma}_w(|\hat{w}(i|k)|) \leq V_\infty$ for $k \geq 0, i \leq k$. This implies $\left\|\widehat{\mathbf{w}}_k\right\|_{0:k-1} \leq$

$\underline{\gamma}_w^{-1}(V_\infty)$. Substituting (4.11) into this result and using the previous displayed equation and (4.7) gives

$$\left\|\mathbf{w} - \hat{\mathbf{w}}_k\right\|_{0:k-1} \leq \pi_x^w(|x_0 - \overline{x}_0|) + \pi_w^w(\|(\mathbf{w}, \mathbf{v})\|) \tag{4.13}$$

for $\pi_x^w, \pi_w^w \in \mathcal{K}_\infty$. Finally, notice that the same reasoning applies to $\left\|\mathbf{v} - \hat{\mathbf{v}}_k\right\|_{0:k-1}$ yielding

$$\left\|\mathbf{v} - \hat{\mathbf{v}}_k\right\|_{0:k-1} \leq \pi_x^v(|x_0 - \overline{x}_0|) + \pi_w^v(\|(\mathbf{w}, \mathbf{v})\|) \tag{4.14}$$

for $\pi_x^v, \pi_w^v \in \mathcal{K}_\infty$. The definition of i-IOSS gives for all $k \geq 0$, $j \leq k$

$$\begin{aligned} \left|x(j; x_0, \mathbf{w}) - x(j; \hat{x}(0|k), \hat{\mathbf{w}}_k)\right| &\leq \alpha_1(|x(0) - \hat{x}(0|k)|, j) + \\ &\quad \gamma_w(\left\|\mathbf{w} - \hat{\mathbf{w}}_k\right\|_{0:j-1}) + \gamma_v(\left\|\mathbf{v} - \hat{\mathbf{v}}_k\right\|_{0:j-1}) \end{aligned}$$

We substitute (4.12), (4.13), and (4.14) into this result, set $k = 0$ in the $\mathcal{KL}$ function $\alpha_1(\cdot, k)$ and use (4.7) to obtain

$$\left|x(j; x_0, \mathbf{w}) - x(j; \hat{x}(0|k), \hat{\mathbf{w}}_k)\right| \leq \pi_x(|x_0 - \overline{x}_0|) + \pi_w(\|(\mathbf{w}, \mathbf{v})\|)$$

for all $k \geq 0$, $j \leq k$, in which $\pi_x, \pi_w \in \mathcal{K}_\infty$, and we have established (4.10).

(b) First we establish that for any fixed $M \geq 1$, as $k \to \infty$

$$\hat{w}(j|k), \hat{v}(j|k) \to 0 \quad \text{for all } j \in \mathbb{I}_{k-M:k-1} \tag{4.15}$$

If we substitute the optimal $\hat{\mathbf{w}}_T, \hat{\mathbf{v}}_T$ at time T into the state estimation problem at time $T - M$ for $M \geq 1$, optimality at time $T - M$ implies

$$V_{T-M}^0 \leq V_T^0 - \sum_{i=T-M}^{T-1} \ell_i(\hat{w}(i|T), \hat{v}(i|T))$$

This shows that the sequence of optimal costs $\left(V_T^0\right)$ is nondecreasing with time T, and since V_T^0 is bounded above (by V_∞), the sequence converges as $T \to \infty$. Rearranging this inequality and using (4.6) gives

$$\sum_{i=T-M}^{T-1} \underline{\gamma}_w(|(\hat{w}(i|T), \hat{v}(i|T))|) \leq V_T^0 - V_{T-M}^0$$

Because the right-hand side goes to zero as $T \to \infty$ for all $M \geq 1$, using the properties of $\mathcal{K}$ functions then establishes (4.15). Next choose any $\varepsilon > 0$. From the i-IOSS assumption and the structure of the state model, (4.1), we have for all $k \geq 0$, $p \geq 1$

$$\begin{aligned}|x(k+p; x_0, \mathbf{w}) - x(k+p; \hat{x}(0|k+p), \widehat{\mathbf{w}}_{k+p})| \leq \\ \alpha_1(|x(k) - x(k; \hat{x}(0|k+p), \widehat{\mathbf{w}}_{k+p})|, p) + \\ \gamma_w(\|\mathbf{w} - \widehat{\mathbf{w}}_{k+p}\|_{k:k+p-1}) + \gamma_v(\|\mathbf{v} - \widehat{\mathbf{v}}_{k+p}\|_{k:k+p-1})\end{aligned}$$

From (4.10) we have that for all $k, p \geq 0$

$$\alpha_1(|x(k) - x(k; \hat{x}(0|k+p), \widehat{\mathbf{w}}_{k+p})|, p) \leq \alpha_1(a, p)$$

with positive scalar $a := \pi_x(|x_0 - \overline{x}_0|) + \pi_w(\|(\mathbf{w}, \mathbf{v})\|)$. Because $\alpha_1 \in \mathcal{KL}$, we can chose $P(a, \varepsilon)$ such that $\alpha_1(a, p) \leq \varepsilon/3$ and therefore

$$\alpha_1(|x(k) - x(k; \hat{x}(0|k+p), \widehat{\mathbf{w}}_{k+p})|, p) \leq \varepsilon/3$$

for all $p \geq P(a, \varepsilon)$. Because of Assumption 4.6 and (4.15), for any fixed $p \geq 1$, we can choose $K(p, \varepsilon) \geq 0$ such that

$$\gamma_w(\|\mathbf{w} - \widehat{\mathbf{w}}_{k+p}\|_{k:k+p-1}) \leq \varepsilon/3 \qquad \gamma_v(\|\mathbf{v} - \widehat{\mathbf{v}}_{k+p}\|_{k:k+p-1}) \leq \varepsilon/3$$

for all $k \geq K(p, \varepsilon)$. Combining these inequalities gives

$$\left|x(j) - \hat{x}(j)\right| \leq \varepsilon \qquad \text{for all } j \geq K(P(a, \varepsilon), \varepsilon) + P(a, \varepsilon)$$

and we have thus shown that $|x(k) - \hat{x}(k)| \to 0$ as $k \to \infty$, and the second part is established. ■

Finally, we present a theorem summarizing the theoretical properties of FIE for convergent disturbances.

Theorem 4.10 (FIE with β-convergent disturbances is RGAS)**.** *Consider an i-IOSS (detectable) system and measurement sequence generated by* (4.1) *with disturbances satisfying Assumption 4.6, and stage cost satisfying Assumption 4.7. Then the full information estimator is RGAS.*

Proof. Because Proposition 4.9 applies, we know that the estimate error is bounded and converges. Therefore from Proposition 4.4 there exists $\overline{\alpha}(\cdot) \in \mathcal{KL}$ such that for all $k \geq 0$

$$|x(k) - \hat{x}(k)| \leq \overline{\alpha}\left((\pi_x(|x_0 - \overline{x}_0|) + \pi_w(\|(\mathbf{w}, \mathbf{v})\|)), k\right)$$

From (4.8) we then have existence of $\alpha_x, \alpha_w \in \mathcal{KL}$ such that

$$|x(k) - \hat{x}(k)| \leq \alpha_x(|x_0 - \overline{x}_0|, k) + \alpha_w(\|(\mathbf{w}, \mathbf{v})\|, k) \tag{4.16}$$

Note that this result is stronger than we require for RGAS. To establish RGAS, we choose the $\mathcal{KL}$ function $\alpha(\cdot) = \alpha_x(\cdot)$, and the $\mathcal{K}$ functions $\delta_w(\cdot) = \alpha_w(\cdot, 0)$ to give

$$|x(k) - \hat{x}(k)| \leq \alpha(|x_0 - \overline{x}_0|, k) + \delta_w(\|(\mathbf{w}, \mathbf{v})\|)$$

Since $w(j), v(j)$ for $j \geq k$ affect neither $x(k)$ nor $\hat{x}(k)$, this result also implies that

$$|x(k) - \hat{x}(k)| \leq \alpha(|x_0 - \overline{x}_0|, k) + \delta_w(\|(\mathbf{w}, \mathbf{v})\|_{0:k-1})$$

and the estimate error satisfies (4.9) and RGAS has been established. ■

Notice that we can achieve $\mathcal{KL}$ functions of the sup norm of the disturbances in (4.16) because they are assumed *β-convergent* disturbances here. If we assume only *bounded* disturbances, however, then the $\mathcal{KL}$ functions have to become only $\mathcal{K}$ functions, but notice that is all that is required to establish RGAS as stated in (4.9).

4.2.1 State Estimation as Optimal Control of Estimate Error

Given the many structural similarities between estimation and regulation, the reader may wonder why the stability analysis of the full information estimator presented in the previous section looks rather different than the zero-state regulator stability analysis presented in Chapter 2. To provide some insight into essential *differences*, as well as similarities, between estimation and regulation, consider again the estimation problem in the simplest possible setting with a linear time invariant model and Gaussian noise

$$\begin{aligned} x^+ &= Ax + Gw \qquad & w &\sim N(0, Q) \\ y &= Cx + v \qquad & v &\sim N(0, R) \end{aligned} \tag{4.17}$$

and random initial state $x(0) \sim N(\overline{x}_0, P^-(0))$. In FIE, we define the objective function

$$V_T(\chi(0), \boldsymbol{\omega}) = \frac{1}{2}\left(|\chi(0) - \overline{x}_0|^2_{(P^-(0))^{-1}} + \sum_{i=0}^{T-1} |\omega(i)|^2_{Q^{-1}} + |\nu(i)|^2_{R^{-1}} \right)$$

subject to $\chi^+ = A\chi + G\omega$, $y = C\chi + \nu$. Denote the solution to this optimization as

$$(\hat{x}(0|T), \hat{\mathbf{w}}_T) = \arg \min_{\chi(0), \boldsymbol{\omega}} V_T(\chi(0), \boldsymbol{\omega})$$

and the trajectory of state estimates comes from the model $\hat{x}(i+1|T) = A\hat{x}(i|T) + G\hat{w}(i|T)$. We define estimate error as $\widetilde{x}(i|T) = x(i) - \hat{x}(i|T)$ for $0 \leq i \leq T-1$, $T \geq 1$.

Because the system is *linear*, the estimator is stable if and only if it is stable with zero process and measurement disturbances. So analyzing stability is equivalent to the following simpler question. If noise-free data are provided to the estimator, $(w(i), v(i)) = 0$ for all $i \geq 0$ in (4.17), is the estimate error asymptotically stable as $T \to \infty$ for all x_0? We next make this statement precise. First we note that the noise-free measurement satisfies $y(i) - C\hat{x}(i|T) = C\widetilde{x}(i|T), 0 \leq i \leq T$ and the initial condition term can be written in estimate error as $\hat{x}(0) - \overline{x}(0) = -(\widetilde{x}(0) - a)$ in which $a = x(0) - \overline{x}_0$. For the noise-free measurement we can therefore rewrite the cost function as

$$V_T(a, \widetilde{x}(0), \mathbf{w}) = \frac{1}{2}\left(|\widetilde{x}(0) - a|^2_{(P^-(0))^{-1}} + \sum_{i=0}^{T-1} |C\widetilde{x}(i)|^2_{R^{-1}} + |w(i)|^2_{Q^{-1}} \right) \tag{4.18}$$

in which we list explicitly the dependence of the cost function on parameter a. For estimation we solve

$$\min_{\widetilde{x}(0), \mathbf{w}} V_T(a, \widetilde{x}(0), \mathbf{w}) \tag{4.19}$$

subject to $\widetilde{x}^+ = A\widetilde{x} + Gw$. Now consider problem (4.19) as an optimal control problem (OCP) using w as the manipulated variable and minimizing an objective that measures size of estimate error $\widetilde{x}$ and control w. We denote the optimal solution as $\widetilde{x}^0(0; a)$ and $\mathbf{w}^0(a)$. Substituting these into the model equation gives optimal estimate error $\widetilde{x}^0(j|T; a), 0 \leq j \leq T, 0 \leq T$. Parameter a denotes how far $x(0)$, the system's initial state generating the measurement, is from $\overline{x}_0$, the prior. If we are lucky and $a = 0$, the optimal solution is $(\widetilde{x}^0, \mathbf{w}^0) = 0$, and we achieve zero cost in V_T^0 and zero estimate error $\widetilde{x}(j|T)$ at all time in the trajectory $0 \leq j \leq T$ for all time $T \geq 1$. The stability analysis in estimation is to show that the origin for $\widetilde{x}$ is asymptotically stable. In other words, we wish to show there exists a KL function β such that $\left|\widetilde{x}^0(T; a)\right| \leq \beta(|a|, T)$ for all $T \in \mathbb{I}_{\geq 0}$.

We note the following differences between standard regulation and the estimation problem (4.19). First we see that (4.19) is slightly nonstandard because it contains an extra decision variable, the initial state, and an extra term in the cost function, (4.18). Indeed, without this extra term, the regulator could choose $\widetilde{x}(0) = 0$ to zero the estimate error immediately, choose $\mathbf{w} = 0$, and achieve zero cost in $V_T^0(a)$ for all a. The nonstandard regulator allows $\widetilde{x}(0)$ to be manipulated as a decision variable, but penalizes its distance from a. Next we look at the stability question.

The stability analysis is to show there exists KL function β such that $\left|\widetilde{x}^0(T;a)\right| \leq \beta(|a|, T)$ for all $T \in \mathbb{I}_{\geq 0}$. Here convergence is a question about the terminal state in a sequence of *different* OCPs with increasing horizon length T. That is also not the standard regulator convergence question, which asks how the state trajectory evolves using the optimal control law. In standard regulation, we inject the optimal first input and ask whether we are successfully moving the system to the origin as time increases. In estimation, we do not inject anything into the system; we are provided more information as time increases and ask whether our explanation of the data is improving (terminal estimate error is decreasing) as time increases.

Because stability is framed around the behavior of the terminal state, we would not choose *backward* dynamic programming (DP) to solve (4.19), as in standard regulation. We do not seek the optimal first control move as a function of a known initial state. Rather we seek the optimal terminal state $\widetilde{x}^0(T;a)$ as a function of the parameter a appearing in the cost function. This problem is better handled by *forward* DP as discussed in Sections 1.3.2 and 1.4.3 of Chapter 1 when solving the full information state estimation problem. Exercise 4.16 discusses how to solve (4.19); we obtain the following recursion for the optimal terminal state

$$\widetilde{x}^0(k+1;a) = (A - \widetilde{L}(k)C)\,\widetilde{x}^0(k;a) \tag{4.20}$$

for $k \geq 0$. The initial condition for the recursion is $\widetilde{x}^0(0;a) = a$. The time-varying gains $\widetilde{L}(k)$ and associated cost matrices $P^-(k)$ required are

$$\begin{aligned} P^-(k+1) &= GQG' + AP^-(k)A' \\ &\quad - AP^-(k)C'(CP^-(k)C' + R)^{-1}CP^-(k)A' \end{aligned} \tag{4.21}$$

$$\widetilde{L}(k) = AP^-(k)C'(CP^-(k)C' + R)^{-1} \tag{4.22}$$

in which $P^-(0)$ is specified in the problem. As expected, these are the standard estimator recursions developed in Chapter 1. Asymptotic stability of the estimate error can be established by showing that $V(k,\widetilde{x}) := (1/2)\widetilde{x}'P(k)^{-1}\widetilde{x}$ is a Lyapunov function for (4.20) (Jazwinski, 1970, Theorem 7.4). Notice that this Lyapunov function is *not* the optimal cost of (4.19) as in a standard regulation problem. The optimal cost of (4.19), $V_T^0(a)$, is an *increasing* function of T rather than a decreasing function of T as required for a Lyapunov function. Also note that the argument used in Jazwinski (1970) to establish that $V(k, \widetilde{x})$ is a Lyapunov function for the *linear* system is *more complicated* than the argument used in Section 4.2 to prove stability of FIE for the *nonlinear* system.

Although one can find Lyapunov functions valid for estimation, they do not have the same simple connection to optimal cost functions as in standard regulation problems, even in the linear, unconstrained case. Stability arguments based instead on properties of $V_T^0(a)$ are simpler and more easily adapted to cover new situations arising in research problems.

4.2.2 Duality of Linear Estimation and Regulation

For linear systems, the estimate error $\widetilde{x}$ in FIE and state x in regulation to the origin display an interesting duality that we summarize briefly here. Consider the following steady-state estimation and infinite horizon regulation problems.

Estimator problem.

$$\begin{aligned} x(k+1) &= Ax(k) + Gw(k) \\ y(k) &= Cx(k) + v(k) \end{aligned}$$

$$R > 0 \qquad Q > 0 \qquad (A, C) \text{ detectable} \qquad (A, G) \text{ stabilizable}$$

$$\widetilde{x}(k+1) = \left(A - \widetilde{L}C\right)\widetilde{x}(k)$$

Regulator problem.

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned}$$

$$R > 0 \qquad Q > 0 \qquad (A, B) \text{ stabilizable} \qquad (A, C) \text{ detectable}$$

$$x(k+1) = (A + BK)\,x(k)$$

In Appendix A, we derive the dual dynamic system following the approach in Callier and Desoer (1991), and obtain the duality variables in regulation and estimation listed in Table 4.2.

We also have the following result connecting controllability of the original system and observability of the dual system.

Regulator	Estimator
A	A'
B	C'
C	G'
k	$l = N - k$
$\Pi(k)$	$P^-(l)$
$\Pi(k-1)$	$P^-(l+1)$
Π	P^-
Q	Q
R	R
P_f	$P^-(0)$
K	$-\widetilde{L}'$
$A + BK$	$(A - \widetilde{L}C)'$
x	$\widetilde{x}'$

Regulator	Estimator
$R > 0, \quad Q > 0$	$R > 0, \quad Q > 0$
(A, B) stabilizable	(A, C) detectable
(A, C) detectable	(A, G) stabilizable

Table 4.2: Duality variables and stability conditions for linear quadratic regulation and least squares estimation.

Lemma 4.11 (Duality of controllability and observability)**.** (A, B) *is controllable (stabilizable) if and only if* (A', B') *is observable (detectable).*

This result can be established directly using the Hautus lemma and is left as an exercise. This lemma and the duality variables allow us to translate stability conditions for infinite horizon regulation problems into stability conditions for FIE problems, and vice versa. For example, the following is a basic theorem covering convergence of Riccati equations in the form that is useful in establishing exponential stability of regulation as discussed in Chapter 1.

Theorem 4.12 (Riccati iteration and regulator stability)**.** *Given* (A, B) *stabilizable,* (A, C) *detectable,* $Q > 0$, $R > 0$, $P_f \geq 0$, *and the discrete Riccati equation*

$$\begin{aligned}
\Pi(k-1) &= C'QC + A'\Pi(k)A - \\
&\qquad A'\Pi(k)B(B'\Pi(k)B + R)^{-1}B'\Pi(k)A, \quad k = N, \ldots, 1 \\
\Pi(N) &= P_f
\end{aligned}$$

Then

(a) There exists $\Pi \geq 0$ *such that for every* $P_f \geq 0$

$$\lim_{k \to -\infty} \Pi(k) = \Pi$$

and Π *is the unique solution of the steady-state Riccati equation*

$$\Pi = C'QC + A'\Pi A - A'\Pi B(B'\Pi B + R)^{-1}B'\Pi A$$

among the class of positive semidefinite matrices.

(b) The matrix $A + BK$*, in which*

$$K = -(B'\Pi B + R)^{-1}B'\Pi A$$

is a stable matrix.

Bertsekas (1987, pp.59–64) provides a proof for a slightly different version of this theorem. Exercise 4.17 explores translating this theorem into the form that is useful for establishing exponential convergence of FIE.

4.3 Moving Horizon Estimation

As displayed in Figure 1.5 of Chapter 1, in MHE we consider only the N most recent measurements, $\mathbf{y}_N(T) = \big(y(T-N), y(T-N+1), \ldots, y(T-1)\big)$. For $T > N$, the MHE objective function is given by

$$\hat{V}_T(\chi(T-N), \boldsymbol{\omega}) = \Gamma_{T-N}(\chi(T-N)) + \sum_{i=T-N}^{T-1} \ell_i(\omega(i), \nu(i))$$

subject to $\chi^+ = f(\chi, \omega)$, $y = h(\chi) + \nu$. The MHE problem is defined to be

$$\min_{\chi(T-N), \boldsymbol{\omega}} \hat{V}_T(\chi(T-N), \boldsymbol{\omega}) \tag{4.23}$$

in which $\boldsymbol{\omega} = (\omega(T-N), \ldots, \omega(T-1))$. The designer chooses the prior weighting $\Gamma_k(\cdot)$ for $k > N$. Until the data horizon is full, i.e., for times $T \leq N$, we generally *define* the MHE problem to be the full information problem.

4.3.1 Zero Prior Weighting

Here we discount the early data completely and choose $\Gamma_i(\cdot) = 0$ for all $i \geq 0$. Because it discounts the past data completely, this form of MHE must be able to asymptotically reconstruct the state using only the most recent N measurements. The first issue is establishing existence of the solution. Unlike the full information problem, in which the positive definite initial penalty guarantees that the optimization takes place over a bounded (compact) set, here there is zero initial penalty. So we must restrict the system further than i-IOSS to ensure solution existence. We show next that observability is sufficient for this purpose.

Definition 4.13 (Observability). The system $x^+ = f(x, w), y = h(x)$ is *observable* if there exist finite $N_o \in \mathbb{I}_{\geq 1}$, $\gamma_w(\cdot)$, $\gamma_v(\cdot) \in \mathcal{K}$ such that for every two initial states z_1 and z_2, and any two disturbance sequences $\mathbf{w}_1, \mathbf{w}_2$, and all $k \geq N_o$

$$|z_1 - z_2| \leq \gamma_w\left(\|\mathbf{w}_1 - \mathbf{w}_2\|_{0:k-1} \right) + \gamma_v\left(\left\|\mathbf{y}_{z_1,\mathbf{w}_1} - \mathbf{y}_{z_2,\mathbf{w}_2}\right\|_{0:k-1} \right)$$

The MHE objective function $\hat{V}_T(\chi(T-N), \boldsymbol{\omega})$ is a continuous function of its arguments because $f(\cdot)$ and $h(\cdot)$ are continuous. We next show that $\hat{V}_T(\cdot)$ is an unbounded function of its arguments, which establishes existence of the solution of the MHE optimization problem. Let Assumptions 4.6–4.7 hold. Then we have that

$$\hat{V}_T(\chi(T-N), \boldsymbol{\omega}) = \sum_{i=T-N}^{T-1} \ell_i(\omega(i), \nu(i)) \geq \underline{\gamma}_w(\|(\boldsymbol{\omega}, \boldsymbol{\nu})\|_{T-N:T-1}) \quad (4.24)$$

From observability we have that for $N \geq N_o$

$$|x(T-N) - \chi(T-N)| \leq \gamma_w(\|\mathbf{w} - \boldsymbol{\omega}\|_{T-N:T-1}) + \gamma_v(\|\mathbf{v} - \boldsymbol{\nu}\|_{T-N:T-1}) \quad (4.25)$$

Consider arbitrary but fixed values of time T, horizon length $N \geq N_o$, and the system state and measurement sequence. Let the decision variables $|(\chi(T-N), \boldsymbol{\omega})| \to \infty$. Then we have that either $|\chi(T-N)| \to \infty$ or $|\boldsymbol{\omega}| \to \infty$. If $|\boldsymbol{\omega}| \to \infty$, we have directly from (4.24) that $\hat{V}_T \to \infty$. On the other hand, if $|\chi(T-N)| \to \infty$, then from (4.25), since $x(T-N)$, $\mathbf{w}$ and $\mathbf{v}$ are fixed, we have that either $\|\boldsymbol{\omega}\|_{T-N:T-1} \to \infty$ or $\|\boldsymbol{\nu}\|_{T-N:T-1} \to \infty$, which implies from (4.24) that $\hat{V}_T \to \infty$. We conclude that $\hat{V}_T(\chi(T-N), \boldsymbol{\omega}) \to \infty$ if $|(\chi(T-N), \boldsymbol{\omega})| \to \infty$. Therefore the objective function is a continuous and unbounded function of its arguments, and existence of the solution of the MHE problem can be established from the Weierstrass theorem (Proposition A.7). The solution does not have to be unique.

We show next that final-state observability is a less restrictive and more natural system requirement for MHE with zero prior weighting to provide stability and convergence.

Definition 4.14 (Final-state observability)**.** The system $x^+ = f(x,w)$, $y = h(x)$ is *final-state observable* (FSO) if there exist finite $N_o \in \mathbb{I}_{\geq 1}$, $\overline{\gamma}_w(\cdot), \overline{\gamma}_v(\cdot) \in \mathcal{K}$ such that for every two initial states z_1 and z_2, and any two disturbance sequences $\mathbf{w}_1, \mathbf{w}_2$, and all $k \geq N_o$

$$|x(k;z_1,\mathbf{w}_1) - x(k;z_2,\mathbf{w}_2)| \leq \overline{\gamma}_w\left(\|\mathbf{w}_1 - \mathbf{w}_2\|_{0:k-1} \right) + \overline{\gamma}_v\left(\left\|\mathbf{y}_{z_1,\mathbf{w}_1} - \mathbf{y}_{z_2,\mathbf{w}_2}\right\|_{0:k-1} \right)$$

Notice that FSO is not the same as observable. For sufficiently restricted $f(\cdot)$, FSO is weaker than observable and stronger than i-IOSS (detectable) as discussed in Exercises 4.13 and 4.14.

To ensure FSO, we restrict the system as follows.

Definition 4.15 (Globally $\mathcal{K}$-continuous)**.** A function $f : \mathcal{X} \to \mathcal{Y}$ is globally $\mathcal{K}$-continuous if there exists function $\sigma(\cdot) \in \mathcal{K}$ such that for all $x_1, x_2 \in \mathcal{X}$

$$\left| f(x_1) - f(x_2) \right| \leq \sigma(|x_1 - x_2|) \tag{4.26}$$

We then have the following result.

Proposition 4.16 (Observable and global $\mathcal{K}$-continuous imply FSO)**.** *An observable system $x^+ = f(x,w), y = h(x)$ with globally $\mathcal{K}$-continuous $f(\cdot)$ is final-state observable.*

The proof of this proposition is discussed in Exercise 4.14. Consider two equal disturbance sequences, $\mathbf{w}_1 = \mathbf{w}_2$, and two equal measurement sequences $\mathbf{y}_1 = \mathbf{y}_2$. FSO implies that for every pair z_1 and z_2, $x(N_o;z_1,\mathbf{w}_1) = x(N_o;z_2,\mathbf{w}_1)$; we know the *final* states at time $k = N_o$ are equal. FSO does not imply that the *initial* states are equal as required when the system is observable. We can of course add the nonnegative term $\beta(|z_1 - z_2|, k)$ to the right-hand side of the FSO inequality and obtain the i-IOSS inequality, so FSO implies i-IOSS. Exercise 4.11 treats observable, FSO, and detectable for the linear time-invariant system, which can be summarized compactly in terms of the eigenvalues of the partitioned state transition matrix corresponding to the unobservable modes.

Definition 4.17 (RGAS estimation (observable case))**.** The estimate is based on the *noisy* measurement $\mathbf{y} = h(\mathbf{x}(x_0,\mathbf{w})) + \mathbf{v}$. The estimator is RGAS (observable case) if for all x_0 and $\overline{x}_0$, and bounded $(\mathbf{w},\mathbf{v})$, there exist $N_o \in \mathbb{I}_{\geq 1}$ and function $\delta_w(\cdot) \in \mathcal{K}$ such that the following holds for all $k \in \mathbb{I}_{\geq N_o}$

$$\left| x(k;x_0,\mathbf{w}) - x(k;\hat{x}(0|k),\widehat{\mathbf{w}}_k) \right| \leq \delta_w(\|(\mathbf{w},\mathbf{v})\|_{0:k-1})$$

Remark. Notice that the definition of RGAS estimation in the observable case is silent about what happens to estimate error at early times, $k < N_o$, while the estimator is collecting enough measurements to obtain its first valid state estimate.

We have the following theorem for this estimator.

Theorem 4.18 (MHE is RGAS (observable case) with zero prior weighting)**.** *Consider an observable system with globally $\mathcal{K}$-continuous $f(\cdot)$, and measurement sequence generated by* (4.1) *with disturbances satisfying Assumption 4.6. The MHE estimate with zero prior weighting, $N \geq N_o$, and stage cost satisfying* (4.6)*, is RGAS (observability case).*

Proof. Consider the system to be at state $x(k-N)$ at time $k-N$ and subject to disturbances $\mathbf{w}_k = (w(k-N), \ldots w(k-1))$. Since the system is observable, the MHE problem has a solution for all $k \geq N \geq N_o$. Denote the estimator solution at time k as initial state $\hat{x}(k-N|k)$ and disturbance sequence $\hat{w}_k$. The system is FSO by Proposition 4.16 since the system is observable and $f(\cdot)$ is globally $\mathcal{K}$-continuous. The FSO property gives that for $k \geq N \geq N_o$

$$|x(k) - \hat{x}(k)| \leq \\ \overline{\gamma}_w\left(\left\|\mathbf{w}_k - \hat{\mathbf{w}}_k\right\|_{k-N:k-1}\right) + \overline{\gamma}_v\left(\left\|h(\mathbf{x}_k) - h(\hat{\mathbf{x}}_k)\right\|_{k-N:k-1}\right) \tag{4.27}$$

We also have the upper bound for the optimal MHE cost

$$\hat{V}_T^0 \leq \sum_{i=T-N}^{T-1} \ell_i(w(i), v(i)) \leq \overline{\gamma}_w(\|(\mathbf{w}, \mathbf{v})\|) \tag{4.28}$$

From the triangle inequality, we have that

$$\left\|\mathbf{w}_k - \hat{\mathbf{w}}_k\right\|_{k-N:k-1} \leq \|\mathbf{w}\| + \left\|\hat{\mathbf{w}}_k\right\|_{k-N:k-1}$$

Following the same steps as in the full information case, we can establish that

$$\left\|\hat{\mathbf{w}}_k\right\|_{k-N:k-1} \leq \underline{\gamma}_w^{-1}(\hat{V}_T^0)$$

and using the second inequality in (4.28) we have that

$$\left\|\mathbf{w}_k - \hat{\mathbf{w}}_k\right\|_{k-N:k-1} \leq \pi_w^w(\|(\mathbf{w}, \mathbf{v})\|)$$

with $\pi_w^w \in \mathcal{K}_\infty$. Similarly, we have that

$$\left\|\mathbf{v}_k - \hat{\mathbf{v}}_k\right\|_{k-N:k-1} \leq \pi_w^v(\|(\mathbf{w}, \mathbf{v})\|)$$

with $\pi_w^v \in \mathcal{K}_\infty$. Substituting these into (4.27) and collecting terms gives for all $k \geq N \geq N_0$

$$|x(k) - \hat{x}(k)| \leq \pi_w(\|(\mathbf{w}, \mathbf{v})\|)$$

with $\pi_w \in \mathcal{K}_\infty$. Again, because disturbances $(w(j), v(j))$ for $j \geq k-1$, do not affect $x(k)$ or $\hat{x}(k)$, we have that

$$|x(k) - \hat{x}(k)| \leq \pi_w(\|(\mathbf{w}, \mathbf{v})\|_{0:k-1})$$

and MHE with zero prior weighting is RGAS (observability case). ■

Notice that unlike in FIE, the estimate error bound does not require the initial error $x(0) - \overline{x}_0$ since we have zero prior weighting and as a result have assumed observability rather than detectability. Notice also that RGAS implies estimate error converges to zero for convergent disturbances.

4.3.2 Nonzero Prior Weighting

The two drawbacks of zero prior weighting are: the system had to be assumed *observable* rather than detectable to ensure existence of the solution to the MHE problem; and a large horizon N may be required to obtain performance comparable to full information estimation. We address these two disadvantages by using nonzero prior weighting. To get started, we use forward DP, as we did in Chapter 1 for the unconstrained linear case, to decompose the FIE problem exactly into the MHE problem (4.23) in which $\Gamma(\cdot)$ is chosen as arrival cost.

Definition 4.19 (Full information arrival cost)**.** The full information arrival cost is defined as

$$Z_T(p) = \min_{\chi(0),\boldsymbol{\omega}} V_T(\chi(0), \boldsymbol{\omega}) \tag{4.29}$$

subject to

$$\chi^+ = f(\chi, \omega) \qquad y = h(\chi) + \nu \qquad \chi(T; \chi(0), \boldsymbol{\omega}) = p$$

We have the following equivalence.

Lemma 4.20 (MHE and FIE equivalence)**.** *The MHE problem* (4.23) *is equivalent to the full information problem* (4.3) *for the choice* $\Gamma_k(\cdot) = Z_k(\cdot)$ *for all* $k > N$ *and* $N \geq 1$*.*

The proof is left as an exercise. This lemma is the essential insight provided by the DP recursion. But notice that evaluating arrival cost in (4.29) has the same computational complexity as solving a full information problem. So next we generate an MHE problem that has simpler computational requirements, but retains the excellent stability properties of full information estimation. We proceed as follows.

Definition 4.21 (MHE arrival cost)**.** The MHE arrival cost $\hat{Z}(\cdot)$ is defined for $T > N$ as

$$\begin{aligned}\hat{Z}_T(p) &= \min_{z,\boldsymbol{\omega}} \hat{V}_T(z, \boldsymbol{\omega}) \\ &= \min_{z,\boldsymbol{\omega}} \Gamma_{T-N}(z) + \sum_{i=T-N}^{T-1} \ell_i(\omega(i), \nu(i))\end{aligned} \tag{4.30}$$

subject to

$$\chi^+ = f(\chi, \omega) \qquad y = h(\chi) + \nu \qquad \chi(T; z, T-N, \boldsymbol{\omega}) = p$$

For $T \leq N$ we usually define the MHE problem to be the full information problem, so $\hat{Z}_T(\cdot) = Z_T(\cdot)$ and $\hat{V}_T^0 = V_T^0$. Notice from the second equality in the definition that the MHE arrival cost at T is defined in terms of the prior weighting at time $T - N$.

We next show that choosing a prior weighting that *underbounds* the MHE arrival cost is the key sufficient condition for stability and convergence of MHE.

Assumption 4.22 (Prior weighting)**.** We assume that $\Gamma_k(\cdot)$ is continuous and satisfies the following inequalities for all $k > N$

(a) Upper bound

$$\Gamma_k(p) \leq \hat{Z}_k(p) = \min_{z,\boldsymbol{\omega}} \Gamma_{k-N}(z) + \sum_{i=k-N}^{k-1} \ell_i(\omega(i), \nu(i)) \tag{4.31}$$

subject to $\chi^+ = f(\chi, \omega), y = h(\chi) + \nu, \chi(k; z, k-N, \boldsymbol{\omega}) = p$.

(b) Lower bound

$$\Gamma_k(p) \geq \hat{V}_k^0 + \underline{\gamma}_p(|p - \hat{x}(k)|) \tag{4.32}$$

in which $\underline{\gamma}_p \in \mathcal{K}_\infty$.

This assumption is depicted in Figure 4.1.

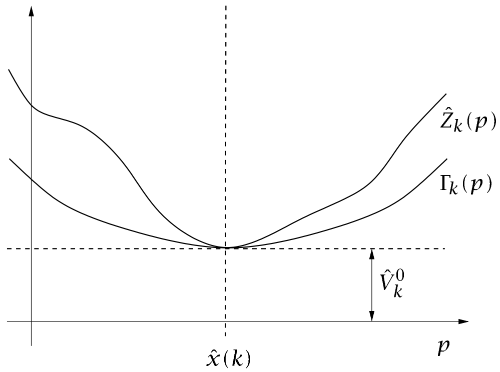

Figure 4.1: MHE arrival cost $\hat{Z}_k(p)$, underbounding prior weighting $\Gamma_k(p)$, and MHE optimal value $\hat{V}_k^0$; for all p and $k > N$, $\hat{Z}_k(p) \geq \Gamma_k(p) \geq \hat{V}_k^0$, and $\hat{Z}_k(\hat{x}(k)) = \Gamma_k(\hat{x}(k)) = \hat{V}_k^0$.

To establish convergence of the MHE estimates, it will prove useful to have an upper bound for the MHE optimal cost. Next we establish the stronger result that the MHE arrival cost is bounded above by the full information arrival cost as stated in the following proposition.

Proposition 4.23 (Arrival cost of full information greater than MHE)**.**

$$\hat{Z}_T(\cdot) \leq Z_T(\cdot) \qquad T \geq 1 \tag{4.33}$$

Proof. We know this result holds for $T \in \mathbb{I}_{1:N}$ because MHE is equivalent to full information for these T. Next we show that the inequality at T implies the inequality at $T + N$. Indeed, we have by the definition of the arrival costs

$$\hat{Z}_{T+N}(p) = \min_{z,\boldsymbol{\omega}} \Gamma_T(z) + \sum_{i=T}^{T+N-1} \ell_i(\omega(i), \nu(i))$$

$$Z_{T+N}(p) = \min_{z,\boldsymbol{\omega}} Z_T(z) + \sum_{i=T}^{T+N-1} \ell_i(\omega(i), \nu(i))$$

in which both optimizations are subject to the same constraints $\chi^+ = f(\chi, \omega)$, $y = h(\chi) + \nu$, $\chi(k; z, k-N, \boldsymbol{\omega}) = p$. From (4.31) $\Gamma_T(\cdot) \leq \hat{Z}_T(\cdot)$, and $\hat{Z}_T(\cdot) \leq Z_T(\cdot)$ by assumption. Together these imply the optimal values satisfy $\hat{Z}_{T+N}(p) \leq Z_{T+N}(p)$ for all p, and we have established $\hat{Z}_{T+N}(\cdot) \leq Z_{T+N}(\cdot)$. Therefore we have extended (4.33) from $T \in \mathbb{I}_{1:N}$ to $T \in \mathbb{I}_{1:2N}$. Continuing this recursion establishes (4.33) for $T \in \mathbb{I}_{\geq 1}$. ∎

Given (4.33) we also have the analogous inequality for the optimal costs of MHE and full information

$$\hat{V}_T^0 \leq V_T^0 \qquad T \geq 1 \tag{4.34}$$

Definition 4.24 (MHE-detectable system)**.** We say a system $x^+ = f(x, w)$, $y = h(x)$ is *MHE detectable* if the system augmented with an extra disturbance w_2

$$x^+ = f(x, w_1) + w_2 \qquad y = h(x)$$

is i-IOSS with respect to the augmented disturbance (w_1, w_2).

Note that MHE detectable is stronger than i-IOSS (detectable).

Theorem 4.25 (MHE is RGAS)**.** *Consider an MHE-detectable system and measurement sequence generated by* (4.1) *with disturbances satisfying Assumption 4.6. The MHE estimate defined by* (4.23) *using the prior weighting function* $\Gamma_k(\cdot)$ *satisfying Assumption 4.22 and stage cost satisfying Assumption 4.7 is RGAS.*

Proof. The MHE solution exists for $T \leq N$ by the existence of the full information solution, so we consider $T > N$. For disturbances satisfying Assumption 4.6, we established in the proof of Theorem 4.10 for the full information problem that $V_T^0 \leq V_\infty$ for all $T \geq 1$ including $T = \infty$. From Proposition 4.23 and (4.34), we have that the MHE optimal cost also has the upper bound $\hat{V}_T^0 \leq V_\infty$ for all $T \geq 1$ including $T = \infty$. Since we have assumed $f(\cdot)$ and $h(\cdot)$ are continuous, $\Gamma_i(\cdot)$ is continuous for $i > N$, and $\ell_i(\cdot)$ is continuous for all $i \geq 0$, the MHE cost function $\hat{V}_T(\cdot)$ is continuous for $T > N$. The lower bound on Γ_i for $i > N$ and ℓ_i for all $i \geq 0$ imply that for $T > N$, $\hat{V}_T(\chi(T-N), \boldsymbol{\omega})$ goes to infinity as either $\chi(T-N)$ or $\boldsymbol{\omega}$ goes to infinity. Therefore the MHE optimization takes place over a bounded, closed set for $T > N$, and the solution exists by the Weierstrass theorem. As in the proof of Proposition 4.9, we first establish (a) convergence and (b) boundedness of the MHE estimator.

(a) Consider the solution to the MHE problem at time T, $(\hat{x}(T-N|T), \hat{\mathbf{w}}_T)$. We have that

$$\hat{V}_T^0 = \Gamma_{T-N}(\hat{x}(T-N|T)) + \sum_{i=T-N}^{T-1} \ell_i(\hat{w}(i|T), \hat{v}(i|T))$$

From (4.32) we have

$$\Gamma_{T-N}(\hat{x}(T-N|T)) \geq \hat{V}_{T-N}^0 + \underline{\gamma}_p(|\hat{x}(T-N|T) - \hat{x}(T-N|T-N)|)$$

Using this inequality in the previous equation we have

$$\hat{V}_T^0 \geq \hat{V}_{T-N}^0 + \underline{\gamma}_p(|\hat{x}(T-N|T) - \hat{x}(T-N|T-N)|) + \sum_{i=T-N}^{T-1} \ell_i(\hat{w}(i|T), \hat{v}(i|T)) \tag{4.35}$$

and we have established that the sequence $\left(\hat{V}_{T+iN}^0\right)$ is a nondecreasing sequence in $i = 1, 2, \ldots$ for any fixed $T \geq 1$. Since $\hat{V}_k^0$ is bounded above for all $k \geq 1$, the sequence $\hat{V}_{T+iN}^0$ converges as $i \to \infty$ for any $T \geq 1$. This convergence gives as $T \to \infty$

$$\underline{\gamma}_p\left(|\hat{x}(T-N|T) - \hat{x}(T-N|T-N)|\right) \to 0 \tag{4.36}$$

$$\sum_{i=T-N}^{T-1} \ell_i(\hat{w}(i|T), \hat{v}(i|T)) \to 0 \tag{4.37}$$

Next we create a single estimate sequence by concatenating MHE sequences from times $N, 2N, 3N, \ldots$. This gives the state sequence and corresponding $\overline{\mathbf{w}}_1$ and $\overline{\mathbf{w}}_2$ sequences listed in the following table so that

$$\overline{x}^+ = f(\overline{x}, \overline{w}_1) + \overline{w}_2 \quad \text{for } k \geq 0 \qquad y = h(\overline{x}) + \overline{v}$$

$\overline{\mathbf{x}}$	$\overline{\mathbf{w}}_1$	$\overline{\mathbf{w}}_2$	$\overline{\mathbf{v}}$
$\hat{x}(0\|N)$	$\hat{w}(0\|N)$	0	$\hat{v}(0\|N)$
$\hat{x}(1\|N)$	$\hat{w}(1\|N)$	0	$\hat{v}(1\|N)$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\hat{x}(N-1\|N)$	$\hat{w}(N-1\|N)$	$\hat{x}(N\|2N) - \hat{x}(N\|N)$	$\hat{v}(N-1\|N)$
$\hat{x}(N\|2N)$	$\hat{w}(N\|2N)$	0	$\hat{v}(N\|2N)$
$\hat{x}(N+1\|2N)$	$\hat{w}(N+1\|2N)$	0	$\hat{v}(N+1\|2N)$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\hat{x}(2N-1\|2N)$	$\hat{w}(2N-1\|2N)$	$\hat{x}(2N\|3N) - \hat{x}(2N\|2N)$	$\hat{v}(2N-1\|2N)$
$\hat{x}(2N\|3N)$	$\hat{w}(2N\|3N)$	0	$\hat{v}(2N\|3N)$
$\hat{x}(2N+1\|3N)$	$\hat{w}(2N+1\|3N)$	0	$\hat{v}(2N+1\|3N)$
$\cdots$	$\cdots$	$\cdots$	$\cdots$

Notice that every N rows in the array, there is a nonzero entry in the $\mathbf{w}_2$ column. That disturbance is required to move from one MHE sequence to the next as shown in Figure 4.2. But (4.36) implies that $\overline{w}_2(k) \to 0$ as integer $k \to \infty$, and (4.37) implies that $\overline{w}_1(k) \to 0$ as $k \to \infty$. Therefore $|(w_1(k), 0) - (\overline{w}_1(k), \overline{w}_2(k))| \to 0$ as $k \to \infty$. We also have from (4.37) that $h(x(k)) - h(\overline{x}(k)) = \overline{v}(k) - v(k) \to 0$ as $k \to \infty$. Next we apply the MHE-detectability assumption to the $\mathbf{x}$ and $\overline{\mathbf{x}}$ sequences, to obtain

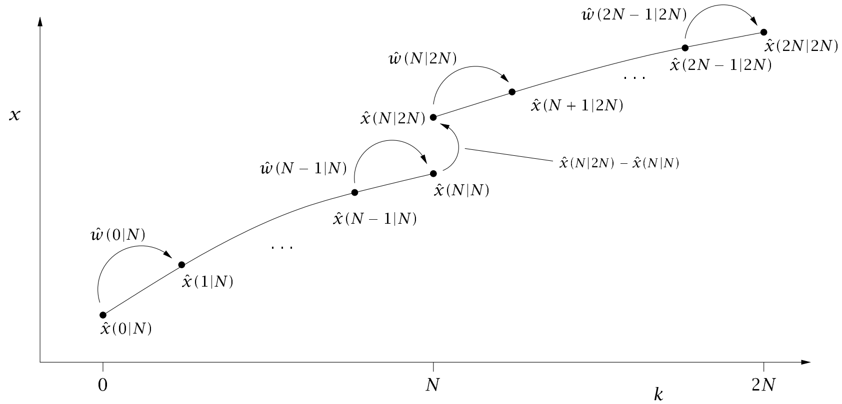

Figure 4.2: Concatenating two MHE sequences to create a single state estimate sequence from time 0 to $2N$.

the inequality

$$\begin{aligned}|x(k)-\overline{x}(k)| &\leq \beta(|x(0)-\hat{x}(0|N)|,k)+\\ &\gamma_w\left(\|(\mathbf{w}_1,0)-(\overline{\mathbf{w}}_1,\overline{\mathbf{w}}_2)\|_{0:k-1}\right)+\gamma_v\left(\|h(\mathbf{x})-h(\overline{\mathbf{x}})\|_{0:k-1}\right)\end{aligned} \tag{4.38}$$

From Proposition 4.2 we conclude that $\overline{x}(k)\to x(k)$ as $k\to\infty$. Therefore we have that $\hat{x}(iN+j|(i+1)N)\to x(iN+j)$ for all $j=0,1,\ldots,N-1$ as integer $i\to\infty$. Note that only $\hat{x}(iN|iN)$ is missing from this argument. But $x(iN)=f(x(iN-1),w(iN-1))$ and $\hat{x}(iN|iN)=f(\hat{x}(iN-1|iN),\hat{w}(iN-1|iN))$. Since $\hat{x}(iN-1|iN)\to x(iN-1)$, $\hat{w}(iN-1|iN)\to w(iN-1)$, and $f(\cdot)$ is continuous, we have that $\hat{x}(iN|iN)\to x(iN)$ as well. We can repeat this concatenation construction using the MHE sequences $N+j,2N+j,3N+j,\ldots$ for $j=1,\ldots,N-1$ to conclude that $\hat{x}(k)\to x(k)$ as $k\to\infty$, and convergence is established.

(b) Using (4.11), we have the following cost bounds for all $T\geq 1$

$$\hat{V}_T\leq V_T^0\leq V_\infty\leq\delta:=\gamma_x(|x_0-\overline{x}_0|)+\overline{\gamma}_w(\|(\mathbf{w},\mathbf{v})\|)$$

Boundedness for $k\leq N$ is already covered by the full information result, so we consider $k>N$. We have from the optimal MHE cost at $T=N$

$$\underline{\gamma}_x(|\hat{x}(0|N)-\overline{x}_0|)\leq\delta \qquad \gamma_x(|x_0-\overline{x}_0|)\leq\delta$$

which gives $|x(0) - \hat{x}(0|N)| \leq (\underline{\gamma}_x^{-1} + \gamma_x^{-1})(\delta)$. From (4.35) and the fact that $\hat{V}_T^0 \leq \delta$, we know that

$$\begin{aligned} \underline{\gamma}_p\big(|\hat{x}((i+1)N|iN) - \hat{x}(iN|iN)|\big) &\leq \delta \\ |\hat{x}((i+1)N|iN) - \hat{x}(iN|iN)| &\leq \underline{\gamma}_p^{-1}(\delta) \end{aligned}$$

which implies $|\overline{w}_2(k)| \leq \underline{\gamma}_p^{-1}(\delta)$ for all $k \geq 0$. Examining the terms in the $\overline{\mathbf{w}}_1$ column, we conclude as before that $|w(k) - \overline{w}_1(k)| \leq (\underline{\gamma}_p^{-1} + \underline{\gamma}_w^{-1})(\delta)$ for all $k \geq 0$. The $\overline{\mathbf{v}}$ column gives the bound

$$|v(k) - \overline{v}(k)| = |h(x(k)) - h(\overline{x}(k))| \leq (\underline{\gamma}_w^{-1} + \gamma_w^{-1})(\delta)$$

We also have the bounds

$$\begin{aligned} \|(\mathbf{w}_1, 0) - (\overline{\mathbf{w}}_1, \overline{\mathbf{w}}_2)\| &= \max_{k\geq 0} |(w_1(k), 0) - (\overline{w}_1(k), \overline{w}_2(k))| \\ &= \max_{k\geq 0} |(w_1(k) - \overline{w}_1(k), -\overline{w}_2(k))| \\ &\leq \max_{k\geq 0} |w_1(k) - \overline{w}_1(k)| + |\overline{w}_2(k)| \\ &\leq (2\underline{\gamma}_p^{-1} + \underline{\gamma}_w^{-1})(\delta) \end{aligned}$$

Substituting these into (4.38) gives

$$\begin{aligned} |x(k) - \overline{x}(k)| \leq \overline{\beta}\big((\underline{\gamma}_x^{-1} + \gamma_x^{-1})(\delta)\big) + \gamma_w\big((2\underline{\gamma}_p^{-1} + \underline{\gamma}_w^{-1})(\delta)\big) + \\ \gamma_v\big((\underline{\gamma}_w^{-1} + \gamma_w^{-1})(\delta)\big) \end{aligned}$$

with $\overline{\beta}(s) := \beta(s, 0)$, which is a $\mathcal{K}$ function. Therefore, from the definition of δ, the right-hand side defines a sum of $\mathcal{K}$ functions of $|x_0 - \overline{x}_0|$ and $\|(\mathbf{w}, \mathbf{v})\|$. This gives a bound for $\big|x(iN) - \hat{x}(iN + j|iN)\big|$ for all $i \geq 1$ and j satisfying $0 \leq j \leq N - 1$. Next we use the continuity of $f(\cdot)$ to obtain the bound for $|x(iN) - \hat{x}(iN|iN)|$ for all $i \geq 0$. Finally we repeat the concatenation construction using the MHE sequences $N + j, 2N + j, 3N + j, \ldots$ for $j = 1, \ldots, N - 1$ to obtain the bound for $|x(k) - \hat{x}(k)|$ for all $k > N$, and boundedness is established.

With convergence and boundedness established, we proceed as in the proof of Theorem 4.10 to establish that MHE is RGAS. ■

Satisfying the prior weighting *inequality* (4.31) is computationally less complex than satisfying the equality in the MHE arrival cost recursion (Definition 4.21), as we show subsequently in the constrained, linear case. But for the general nonlinear case, ensuring satisfaction of even (4.31) remains a key technical challenge for MHE research. We discuss a different approach to finding an appropriate MHE prior weighting that does *not* require underbounding the arrival cost in Section 4.4 on bounded disturbances.

4.3.3 Constrained Estimation

Constraints in estimation may be a useful way to add information to the estimation problem. We may wish to enforce physically known facts such as: concentrations of impurities, although small, must be nonnegative; fluxes of mass and energy must have the correct sign given temperature and concentration gradients; and so on. Unlike the regulator, the estimator has no way to enforce these constraints on the *system*. Therefore, it is important that any constraints imposed on the estimator are satisfied by the system generating the measurements. Otherwise we may prevent convergence of the estimated state to the system state. For this reason, care should be used in adding constraints to estimation problems.

Because we have posed state estimation as an optimization problem, it is straightforward to add constraints to the formulation. We assume that the system generating the data satisfy the following constraints.

Assumption 4.26 (Estimator constraint sets)**.**

(a) For all $k \in \mathbb{I}_{\geq 0}$, the sets $\mathbb{W}_k$, $\mathbb{X}_k$, and $\mathbb{V}_k$ are nonempty and closed, and $\mathbb{W}_k$ and $\mathbb{V}_k$ contain the origin.

(b) For all $k \in \mathbb{I}_{\geq 0}$, the disturbances and state satisfy

$$x(k) \in \mathbb{X}_k \qquad w(k) \in \mathbb{W}_k \qquad v(k) \in \mathbb{V}_k$$

(c) The prior satisfies $\overline{x}_0 \in \mathbb{X}_0$.

Constrained full information. The constrained full information estimation objective function is

$$V_T(\chi(0), \boldsymbol{\omega}) = \ell_x\big(\chi(0) - \overline{x}_0\big) + \sum_{i=0}^{T-1} \ell_i(\omega(i), \nu(i)) \tag{4.39}$$

subject to

$$\chi^+ = f(\chi, \omega) \qquad y = h(\chi) + \nu$$
$$\chi(i) \in \mathbb{X}_i \qquad \omega(i) \in \mathbb{W}_i \qquad \nu(i) \in \mathbb{V}_i \qquad i \in \mathbb{I}_{0:T-1}$$

The constrained full information problem is

$$\min_{\chi(0),\boldsymbol{\omega}} V_T(\chi(0), \boldsymbol{\omega}) \tag{4.40}$$

Theorem 4.27 (Constrained full information is RGAS)**.** *Consider an i-IOSS (detectable) system and measurement sequence generated by* (4.1) *with constrained, convergent disturbances satisfying Assumptions 4.6 and 4.26. The constrained full information estimator* (4.40) *with stage cost satisfying Assumption 4.7 is RGAS.*

Constrained MHE. The constrained moving horizon estimation objective function is

$$\hat{V}_T(\chi(T-N), \boldsymbol{\omega}) = \Gamma_{T-N}(\chi(T-N)) + \sum_{i=T-N}^{T-1} \ell_i(\omega(i), \nu(i)) \tag{4.41}$$

subject to

$$\chi^+ = f(\chi, \omega) \qquad y = h(\chi) + \nu$$
$$\chi(i) \in \mathbb{X}_i \qquad \omega(i) \in \mathbb{W}_i \qquad \nu(i) \in \mathbb{V}_i \qquad i \in \mathbb{I}_{T-N:T-1}$$

The constrained MHE is given by the solution to the following problem

$$\min_{\chi(T-N),\boldsymbol{\omega}} \hat{V}_T(\chi(T-N), \boldsymbol{\omega}) \tag{4.42}$$

Theorem 4.28 (Constrained MHE is RGAS)**.** *Consider an MHE-detectable system and measurement sequence generated by* (4.1) *with convergent, constrained disturbances satisfying Assumptions 4.6 and 4.26. The constrained MHE estimator* (4.42) *using the prior weighting function* $\Gamma_k(\cdot)$ *satisfying Assumption 4.22 and stage cost satisfying Assumption 4.7 is RGAS.*

Because the *system* satisfies the state and disturbance constraints due to Assumption 4.26, both full information and MHE optimization problems are feasible at all times. Therefore the proofs of Theorems 4.27 and 4.28 closely follow the proofs of their respective unconstrained versions, Theorems 4.10 and 4.25, and are omitted.

4.3.4 Smoothing and Filtering Update

We next focus on *constrained linear systems*

$$x^+ = Ax + Gw \qquad y = Cx + v \tag{4.43}$$

We proceed to strengthen several results of the previous sections for this special case. First, the i-IOSS assumption of full information estimation and the MHE detectability assumption both reduce to the assumption that (A, C) *is detectable* in this case. We usually choose a constant quadratic function for the estimator stage cost for all $i \in \mathbb{I}_{\geq 0}$

$$\ell_i(w, v) = (1/2)(|w|^2_{Q^{-1}} + |v|^2_{R^{-1}}) \qquad Q, R > 0 \tag{4.44}$$

In the unconstrained linear problem, we can of course find the full information arrival cost exactly; it is

$$Z_k(z) = V_k^0 + (1/2)\,|z - \hat{x}(k)|_{(P^-(k))^{-1}} \qquad k \geq 0$$

in which $P^-(k)$ satisfies the recursion (4.21) and $\hat{x}(k)$ is the full information estimate at time k. We use this quadratic function for the MHE prior weighting.

Assumption 4.29 (Prior weighting for linear system)**.**

$$\Gamma_k(z) = \hat{V}_k^0 + (1/2)\,|z - \hat{x}(k)|_{(P^-(k))^{-1}} \qquad k > N \tag{4.45}$$

in which $\hat{V}_k^0$ is the optimal MHE cost at time k.

Because the unconstrained arrival cost is available, we usually choose it to be the prior weighting in MHE, $\Gamma_k(\cdot) = Z_k(\cdot)$, $k \geq 0$. This choice implies the MHE estimator is RGAS also for the *constrained case* as we next demonstrate. To ensure the form of the estimation problem to be solved online is a quadratic program, we specialize the constraint sets to be polyhedral regions.

Assumption 4.30 (Polyhedral constraint sets)**.** For all $k \in \mathbb{I}_{\geq 0}$, the sets $\mathbb{W}_k$, $\mathbb{X}_k$, and $\mathbb{V}_k$ in Assumption 4.26 are nonempty, closed polyhedral regions containing the origin.

Corollary 4.31 (Constrained MHE is RGAS)**.** *Consider a detectable linear system and measurement sequence generated by* (4.43) *with convergent, constrained disturbances satisfying Assumptions 4.6 and 4.30. The constrained MHE estimator* (4.42) *using a prior weighting function satisfying* (4.45) *and stage cost satisfying* (4.44) *is RGAS.*

This corollary follows as a special case of Theorem 4.28.

The MHE approach discussed to this point uses at all time $T > N$ the MHE estimate $\hat{x}(T - N)$ and prior weighting function $\Gamma_{T-N}(\cdot)$ derived from the unconstrained arrival cost as shown in (4.45). We call this

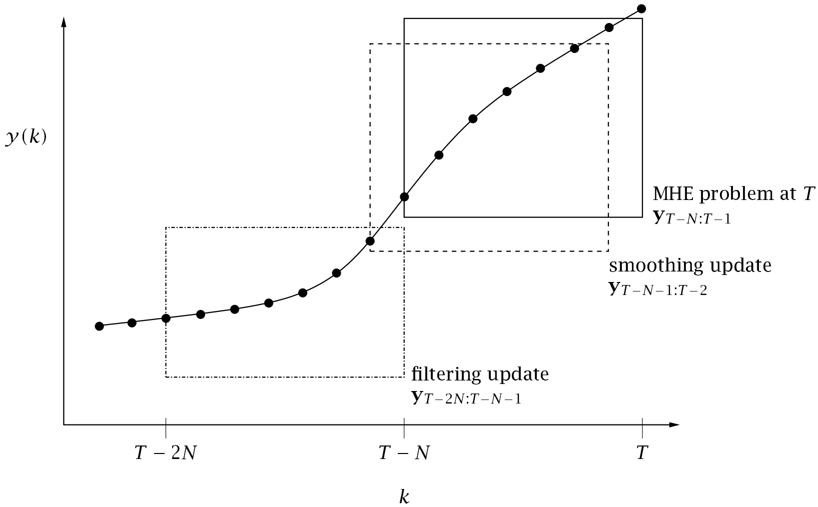

Figure 4.3: Smoothing update.

approach a "filtering update" because the prior weight at time T is derived from the solution of the MHE "filtering problem" at time $T-N$, i.e., the estimate of $\hat{x}(T-N) := \hat{x}(T-N|T-N)$ given measurements up to time $T-N-1$. For implementation, this choice requires storage of a window of N prior filtering estimates to be used in the prior weighting functions as time progresses.

Next we describe a "smoothing update" that can be used instead. As depicted in Figure 4.3, in the smoothing update we wish to use $\hat{x}(T-N|T-1)$ (instead of $\hat{x}(T-N|T-N)$) for the prior and wish to find an appropriate prior weighting based on this choice. For the linear *unconstrained* problem we can find an exact prior weighting that gives an equivalence to the full information problem. When constraints are added to the problem, however, the smoothing update provides a different MHE than the filtering update. Like the filtering prior, the smoothing prior weighting does give an underbound for the constrained full information problem, and therefore maintains the excellent stability properties of MHE with the filtering update. As mentioned previously the unconstrained full information arrival cost is given by

$$Z_{T-N}(z) = V^0_{T-N} + (1/2)\left|z - \hat{x}(T-N)\right|^2_{(P^-(T-N))^{-1}} \qquad T > N \quad (4.46)$$

in which $\hat{x}(T-N)$ is the optimal estimate for the unconstrained full information problem.

Next we consider using $\hat{x}(T-N|T-2)$ in place of $\hat{x}(T-N) := \hat{x}(T-N|T-1)$. We might guess that the proper weight for this prior estimate would be the smoothed covariance $P(T-N|T-2)$ instead of $P^-(T-N) := P(T-N|T-1)$, and that guess is correct, but not complete. Notice that the smoothed prior $\hat{x}(T-N|T-2)$ is influenced by the measurements $\mathbf{y}_{0:T-2}$. But the sum of stage costs in the MHE problem at time T depends on measurements $\mathbf{y}_{T-N:T-1}$, so we have to adjust the prior weighting so we do not double count the data $\mathbf{y}_{T-N:T-2}$. The correct prior weighting for the smoothing update has been derived by Rao, Rawlings, and Lee (2001), which we summarize next. The following notation is useful; for any square matrix R and integer $k \geq 1$, define $\mathrm{diag}_k(R)$ to be the following

$$\mathrm{diag}_k(R) := \underbrace{\begin{bmatrix} R & & & \\ & R & & \\ & & \ddots & \\ & & & R \end{bmatrix}}_{k \text{ times}} \qquad \mathcal{O}_k = \begin{bmatrix} 0 & & & \\ C & & & \\ CA & C & & \\ \vdots & \vdots & \ddots & \\ CA^{k-2} & CA^{k-3} & \cdots & C \end{bmatrix}$$

$$W_k = \mathrm{diag}_k(R) + \mathcal{O}_k(\mathrm{diag}_k(Q))\mathcal{O}_k'$$

We require the smoothed covariance $P(T-N|T-2)$, which we can obtain from the following recursion (Rauch, Tung, and Striebel, 1965; Bryson and Ho, 1975)

$$\begin{aligned} P(k|T) &= P(k) + \\ &P(k)A'(P^-(k+1))^{-1}\Big(P(k+1|T) - P^-(k+1)\Big)(P^-(k+1))^{-1}AP(k) \end{aligned}$$

We iterate this equation backward $N-1$ times starting from the known value $P(T-1|T-2) := P^-(T-1)$ to obtain $P(T-N|T-2)$. The smoothing arrival cost is then given by

$$\begin{aligned} \widetilde{Z}_{T-N}(z) = \hat{V}^0_{T-1} &+ (1/2)\left|z - \hat{x}(T-N|T-2)\right|^2_{(P(T-N|T-2))^{-1}} \\ &- (1/2)\left|\mathbf{y}_{T-N:T-2} - \mathcal{O}_{N-1}z\right|^2_{(W_{N-1})^{-1}} \qquad T > N \end{aligned} \tag{4.47}$$

See Rao et al. (2001) and Rao (2000, pp.80–93) for a derivation that shows $\widetilde{Z}_T(\cdot) = Z_T(\cdot)$ for $T > N$.[2]

[2]Note that Rao et al. (2001) and Rao (2000) contain some minor typos in the smoothed covariance recursion and the formula for W_k.

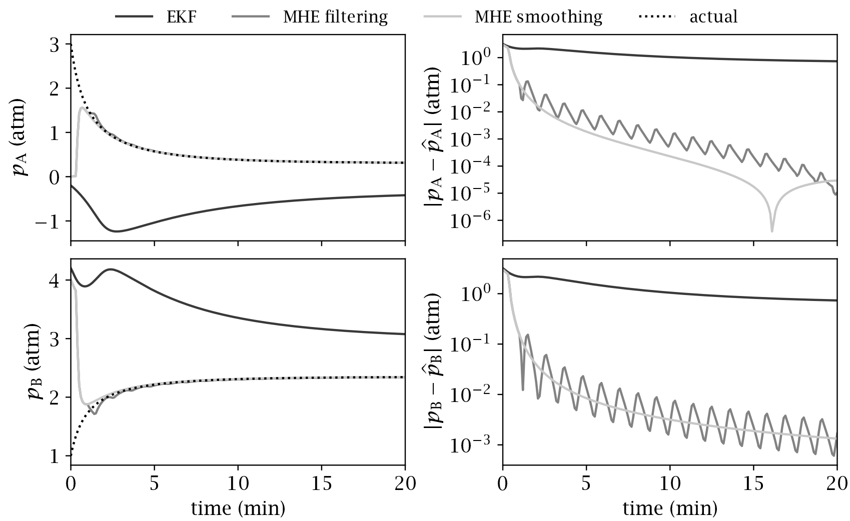

Figure 4.4: Comparison of filtering and smoothing updates for the batch reactor system. Second column shows absolute estimate error.

Examining this alternative expression for arrival cost, we see that the second term accounts for the use of the smoothed covariance and the smoothed estimate, and the third term subtracts the effect of the measurements that have been double counted in the MHE objective as well as the smoothed prior estimate. Setting the prior weighting $\Gamma_{T-N}(\cdot) = Z_{T-N}(\cdot)$ from (4.46), or $\Gamma_{T-N}(\cdot) = \widetilde{Z}_{T-N}(\cdot)$ from (4.47), gives the same result as the Kalman filter for the unconstrained linear problem. But the two arrival costs are approximations of the true arrival cost, and give different results once constraints are added to the problem or we use a nonlinear system model. Since the unconstrained arrival cost $\widetilde{Z}_k(\cdot)$ is also an underbound for the constrained arrival cost, MHE based on the smoothing update also provides an RGAS estimator for constrained linear systems satisfying the conditions of Theorem 4.31.

Example 4.32: Filtering and smoothing updates

Consider a constant-volume batch reactor in which the reaction $2A \rightleftharpoons B$ takes place (Tenny and Rawlings, 2002). The system state x consists of the partial pressures (p_A, p_b) that evolve according to

$$\frac{dp_A}{dt} = -2k_1 p_A^2 + 2k_2 p_B$$
$$\frac{dp_B}{dt} = k_1 p_A - k_2 p_B$$

with $k_1 = 0.16 \text{ min}^{-1} \text{ atm}^{-1}$ and $k_2 = 0.0064 \text{ min}^{-1}$. The only measurement is total pressure, $y = p_A + p_B$.

Starting from initial condition $x = (3, 1)$, the system is measured with sample time $\Delta = 0.1$ min. The model is exact and there are no disturbances. Using a poor initial estimate $\overline{x}_0 = (0.1, 4.5)$, parameters

$$Q = \begin{bmatrix} 10^{-4} & 0 \\ 0 & 0.01 \end{bmatrix} \qquad R = \begin{bmatrix} 0.01 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and horizon $N = 10$, MHE is performed on the system using the filtering and smoothing updates for the prior weighting. For comparison, the EKF is also used. The resulting estimates are plotted in Figure 4.4.

In this simulation, MHE performs well with either update formula. Due to the structure of the filtering update, every $N = 10$ time steps, a poor state estimate is used as the prior, which leads to undesirable periodic behavior in the estimated state. Due to the poor initial state estimate, the EKF produces negative pressure estimates, leading to large estimate errors throughout the simulation. □

4.4 Bounded Disturbances

It is of great practical interest to extend robust asymptotic stability of full information estimation (FIE) and MHE to the case of *bounded* disturbances. This goal is stated formally as Conjecture 1 of Rawlings and Ji (2012). Indeed, we hope that the RAS results presented previously for *convergent* disturbances turn out to be only a convenient special case generating enough insight to handle the more general class of bounded disturbances.

The problem has resisted analysis for the following simple reason. We cannot easily establish that the infinite horizon, full information problem has a solution. If we evaluate the cost of the actual plant disturbances in the full information optimization problem, the value of the cost function $V_N(x(0), \mathbf{w})$ increases without bound as $N \to \infty$. So we do not have an upper bound for the solution to the infinite horizon full information problem. That by itself does not mean there is not a solution, only that we do not have a convenient upper bound to use for analysis.

At the time of this writing there is not a complete solution for the case of bounded disturbances, so we summarize next what partial results are available in the literature. In this summary treatment of bounded disturbances, we mainly follow the recent treatment of MHE in Allan and Rawlings (2017), which was motivated by the recent results of Müller (2017); Hu (2017).

MHE setup. We define the MHE objective function for some horizon length N to be

$$V_N(\chi, \boldsymbol{\omega}, \boldsymbol{\nu}, \overline{x}) := \rho\Gamma(\chi, \overline{x}) + \sum_{k=0}^{N-1} \ell(\omega(k), \nu(k))$$

in which $\Gamma(\cdot)$ is a chosen prior weighting and $\rho > 0$ is some constant to be determined later to ensure estimator stability. We then define the optimal estimation problem using a filtering prior

$$\begin{aligned} \min_{\chi, \boldsymbol{\omega}, \boldsymbol{\nu}, \overline{x}} \; & V_N(\chi(t-N), \boldsymbol{\omega}, \boldsymbol{\nu}, \overline{x}) \\ \text{s.t. } & \overline{x} = \hat{x}(t-N) \\ & \chi^+ = f(\chi, \omega) \\ & y = h(\chi) + \nu \end{aligned}$$

Note that when there are $k \leq N$ measurements, we use a sum from zero to $k-1$ with a prior $\overline{x}_0$, i.e., we define the MHE problem to be the full information problem.

For this section, we modify the form of the detectability to compress the notation. We take the following max rather than sum form of i-IOSS. Note that i-IOSS and i-IOSS (max form) are equivalent definitions and one can work with whichever proves convenient. (See also Exercises 4.6 and 4.7.) In the following, we use the suggestive $\oplus$ notation to denote the maximum of two scalars, $a \oplus b := \max(a, b)$.

Definition 4.33 (i-IOSS (max form))**.** The system $x^+ = f(x, w), y = h(x)$ is *incrementally input/output-to-state stable (max form)* if there exist functions $\alpha(\cdot) \in \mathcal{KL}$ and $\gamma_w(\cdot)$, $\gamma_v(\cdot) \in \mathcal{K}$ such that for every two initial states z_1 and z_2, and any two disturbance sequences $\mathbf{w}_1$ and $\mathbf{w}_2$ generating state sequences $\mathbf{x}_1(z_1, \mathbf{w}_1)$ and $\mathbf{x}_2(z_2, \mathbf{w}_2)$, the following holds for all $k \in \mathbb{I}_{\geq 0}$

$$|x(k; z_1, \mathbf{w}_1) - x(k; z_2, \mathbf{w}_2)| \leq \alpha_1(|z_1 - z_2|, k) \oplus \gamma_w\left(\|\mathbf{w}_1 - \mathbf{w}_2\|_{0:k-1}\right) \oplus \gamma_v\left(\|h(\mathbf{x}_1) - h(\mathbf{x}_2)\|_{0:k-1}\right) \quad (4.48)$$

Next we restrict the forms of the cost functions so that we can establish that an MHE estimator is RAS. We make the following assumption on the stage cost that is standard for FIE and MHE, and guarantees that the optimization problem is well posed.

Assumption 4.34 (Positive definite cost function)**.** There exist $\underline{\gamma}_a(\cdot)$, $\overline{\gamma}_a(\cdot), \underline{\gamma}_s(\cdot), \overline{\gamma}_s(\cdot) \in \mathcal{K}_\infty$ such that

$$\underline{\gamma}_a(|\chi - \overline{x}|) \leq \Gamma(\chi, \overline{x}) \leq \overline{\gamma}_a(|\chi - \overline{x}|)$$
$$\underline{\gamma}_s(|(\omega, \nu)|) \leq \ell(\omega, \nu) \leq \overline{\gamma}_s(|(\omega, \nu)|)$$

Next we make a mild assumption on the stage-cost bounding functions.

Assumption 4.35 (Lipschitz continuity of stage-cost bound compositions)**.** The function $\underline{\gamma}_a^{-1}(2\overline{\gamma}_a(s))$ is Lipschitz continuous at the origin.

Note that Assumption 4.35 can be satisfied for any i-IOSS system by proper choice of stage and terminal cost.

Let $\gamma_d(\cdot) := \gamma_w(\cdot) + \gamma_v(\cdot)$. The next assumption restricts the class of systems so that we can establish that MHE is RAS.

Assumption 4.36 (Uniform i-IOSS contractivity)**.** For every $\overline{s} > 0$ and $\eta \in (0, 1)$, there exist $\rho, T > 0$ such that for all $s \in [0, \overline{s}]$

$$\gamma_d(2\underline{\gamma}_s^{-1}(2\rho\overline{\gamma}_a(s))) \leq \eta s \quad (4.49)$$

$$\beta(s, T) \leq \eta s \quad (4.50)$$

Furthermore, $\beta(s, t)$ is locally Lipschitz.

Remark. Assumption 4.36 essentially requires systems to be locally exponentially detectable.

We require a few technical propositions to facilitate the proof of the main result.

Proposition 4.37 (Locally Lipschitz upper bound)**.** *If a function $V : C \to \mathbb{R}^n$, in which $C \subseteq \mathbb{R}^m$ is closed, locally bounded, and Lipschitz continuous at a point x_0, then there exists a locally Lipschitz function $\sigma \in \mathcal{K}$ such that $|V(x) - V(x_0)| \leq \sigma(|x - x_0|)$ for all $x \in C$.*

The following result proves useful in analyzing the MHE algorithm.

Proposition 4.38 (The nameless lemma)**.** *For any MHE problem satisfying Assumption 4.34, we have for all $k \leq N$ that*

$$\begin{aligned}|e(k)| \leq &\beta(2\underline{\gamma}_a^{-1}(2\overline{\gamma}_a(|e_a|)), k) \oplus \gamma_d(2\underline{\gamma}_s^{-1}(2\rho\overline{\gamma}_a(|e_a|))) \\ &\oplus \beta(2\underline{\gamma}_a^{-1}((2N/\rho)\overline{\gamma}_s(\|\mathbf{d}\|)), k) \oplus \gamma_d(2\underline{\gamma}_s^{-1}(2N\overline{\gamma}_s(\|\mathbf{d}\|)))\end{aligned}$$

in which $d := (w, v)$, $e_a := x(0) - \overline{x}_0$, and $e := x - \hat{x}$.

Proof. By Assumption 4.34, we have that

$$\rho\underline{\gamma}_a(|\hat{e}_a|) \oplus \underline{\gamma}_s(\|\hat{\mathbf{d}}\|) \leq \rho\underline{\gamma}_a(|\hat{e}_a|) + \underline{\gamma}_s(\|\hat{\mathbf{d}}\|) \leq V_N(\hat{x}(0), \hat{\mathbf{d}}, \overline{x})$$

in which $\hat{e}_a := \hat{x} - \overline{x}$. Furthermore, by optimality, we have that

$$\begin{aligned}V_N(\hat{x}(0), \hat{\mathbf{d}}, \overline{x}) &\leq V_N(x(0), \mathbf{d}, \overline{x}) \\ &\leq \rho\overline{\gamma}_a(|e_a|) + N\overline{\gamma}_s(\|\mathbf{d}\|) \\ &\leq 2\rho\overline{\gamma}_a(|e_a|) \oplus 2N\overline{\gamma}_s(\|\mathbf{d}\|)\end{aligned}$$

Combining these bounds and rearranging, we obtain the following bounds

$$\begin{aligned}|\hat{e}_a| &\leq \underline{\gamma}_a^{-1}(2\overline{\gamma}_a(|e_a|) \oplus (2N/\rho)\overline{\gamma}_s(\|\mathbf{d}\|)) \qquad (4.51) \\ &= \underline{\gamma}_a^{-1}(2\overline{\gamma}_a(|e_a|)) \oplus \underline{\gamma}_a^{-1}((2N/\rho)\overline{\gamma}_s(\|\mathbf{d}\|)) \\ \|\hat{\mathbf{d}}\| &\leq \underline{\gamma}_s^{-1}(2\rho\overline{\gamma}_a(|e_a|) \oplus 2N\overline{\gamma}_s(\|\mathbf{d}\|)) \qquad (4.52) \\ &= \underline{\gamma}_s^{-1}(2\rho\overline{\gamma}_a(|e_a|)) \oplus \underline{\gamma}_s^{-1}(2N\overline{\gamma}_s(\|\mathbf{d}\|))\end{aligned}$$

Recall that $\gamma_d(s) := \gamma_w(s) + \gamma_v(s)$. From the system's i-IOSS bound (suppressing the time subscript on $\|(\cdot)\|_{0:k-1}$), we have that

$$\begin{aligned}|e(k)| &\leq \beta(|e(0)|, k) \oplus \gamma_w(\|\mathbf{w} - \hat{\mathbf{w}}\|) \oplus \gamma_v(\|\mathbf{v} - \hat{\mathbf{v}}\|) \\ &\leq \beta(|e(0)|, k) \oplus \gamma_d(\|\mathbf{w} - \hat{\mathbf{w}}\|) \oplus \gamma_d(\|\mathbf{v} - \hat{\mathbf{v}}\|) \\ &\leq \beta(|e(0)|, k) \oplus \gamma_d(\|\mathbf{d} - \hat{\mathbf{d}}\|) \\ &= \beta(|\hat{x}(0) - \overline{x} + \overline{x} - x(0)|, k) \oplus \gamma_d(\|\mathbf{d} - \hat{\mathbf{d}}\|) \\ &\leq \beta(|e_a| + |\hat{e}_a|, k) \oplus \gamma_d(\|\mathbf{d}\| + \|\hat{\mathbf{d}}\|) \\ &\leq \beta(2\,|e_a| \oplus 2\,|\hat{e}_a|, k) \oplus \gamma_d(2\,\|\mathbf{d}\| \oplus 2\|\hat{\mathbf{d}}\|)\end{aligned}$$

We next substitute (4.51) and (4.52) into this expression, and obtain

$$\begin{aligned}|e(k)| \leq &\beta\Big(2\,|e_a| \oplus 2\underline{\gamma}_a^{-1}(2\overline{\gamma}_a(|e_a|) \oplus (2N/\rho)\overline{\gamma}_s(\|\mathbf{d}\|)), k\Big) \\ &\quad \oplus \gamma_d(2\,\|\mathbf{d}\| \oplus 2\underline{\gamma}_s^{-1}(2\rho\overline{\gamma}_a(|e_a|)) \oplus \underline{\gamma}_s^{-1}(2N\overline{\gamma}_s(\|\mathbf{d}\|))) \\ = &\beta(2\,|e_a|, k) \oplus \beta(2\underline{\gamma}_a^{-1}(2\overline{\gamma}_a(|e_a|)), k) \oplus \gamma_d(2\underline{\gamma}_s^{-1}(2\rho\overline{\gamma}_a(|e_a|))) \\ &\quad \oplus \beta(2\underline{\gamma}_a^{-1}((2N/\rho)\overline{\gamma}_s(\|\mathbf{d}\|)), k) \oplus \gamma_d(2\,\|\mathbf{d}\|) \\ &\quad\; \oplus \gamma_d(2\underline{\gamma}_s^{-1}(2N\overline{\gamma}_s(\|\mathbf{d}\|)))\end{aligned}$$

Note that because $\underline{\gamma}_a(s) \leq \overline{\gamma}_a(s) \leq 2\overline{\gamma}_a(s)$, we have that $s \leq \underline{\gamma}_a^{-1}(2\overline{\gamma}_a(s))$. A similar argument follows for $\underline{\gamma}_s(\cdot)$ and $\overline{\gamma}_s(\cdot)$. Thus we have that the term $\beta(2\,|e_a|, k) \leq \beta(2\underline{\gamma}_a^{-1}(2\overline{\gamma}_a(|e_a|)), k)$ and the term $\gamma_d(2\,\|\mathbf{d}\|) \leq \gamma_d(2\underline{\gamma}_s^{-1}(2N\overline{\gamma}_s(\|\mathbf{d}\|)))$, so we can eliminate them from the maximization. Thus we have

$$\begin{aligned}|e(k)| \leq &\beta(2\underline{\gamma}_a^{-1}(2\overline{\gamma}_a(|e_a|)), k) \oplus \gamma_d(2\underline{\gamma}_s^{-1}(2\rho\overline{\gamma}_a(|e_a|))) \\ &\quad \oplus \beta(2\underline{\gamma}_a^{-1}((2N/\rho)\overline{\gamma}_s(\|\mathbf{d}\|)), k) \oplus \gamma_d(2\underline{\gamma}_s^{-1}(2N\overline{\gamma}_s(\|\mathbf{d}\|)))\end{aligned}$$

for all $k \in \mathbb{I}_{0:N}$, which is the desired result. ∎

We can now state and prove the main result.

Theorem 4.39 (MHE is RAS)**.** *For every $\overline{s} > 0$, there exists a T and ρ such that if $N \geq T$ and $|e_a| \leq \overline{s}$, then there exist $\beta_e(\cdot) \in \mathcal{KL}$, $\gamma_e(\cdot) \in \mathcal{K}$, and $\delta > 0$ such that if $\|\mathbf{d}\| \leq \delta$, then*

$$|e(k)| \leq \beta_e(|e_a|, k) \oplus \gamma_e(\|\mathbf{d}\|)$$

for all $k \geq 0$.

Proof. Let $\widetilde{s} := \beta(2\underline{\gamma}_a^{-1}(2\overline{\gamma}_a(\overline{s})), 0)$. By Assumption 4.35 and Proposition 4.37 there exists some $L_a(\widetilde{s}) > 0$ such that for all $s \in [0, \widetilde{s}]$, we have that $2\underline{\gamma}_a^{-1}(2\overline{\gamma}_a(s)) \leq L_a(\widetilde{s})s$. Let $\check{s} := L_a(\widetilde{s})\widetilde{s}$. Choose $\lambda \in (0,1)$ and $\eta := \lambda/L_a(\widetilde{s}) > 0$. By Assumption 4.36, for every $\eta > 0$ and $\check{s}$, there exists $T \geq 0$ such that $\beta(s, T) \leq \eta s$ for all $s \in [0, \check{s}]$. We thus have that there exists some $T \geq 0$ such that

$$\begin{aligned}\beta(2\underline{\gamma}_a^{-1}(2\overline{\gamma}_a(s)), T) &\leq \eta 2\underline{\gamma}_a^{-1}(2\overline{\gamma}_a(s)) && s \in [0, \check{s}] \\ &\leq \eta L_a s && s \in [0, \widetilde{s}] \\ &\leq \lambda s && \end{aligned} \tag{4.53}$$

Given the definitions above, and noting that $\beta(s,0) \geq s$ for all s, we have the following ordering of $\check{s}$, $\tilde{s}$, and $\overline{s}$.

$$\check{s} := L_a(\tilde{s})\tilde{s} \geq 2\underline{\gamma}_a^{-1}(2\overline{\gamma}_a(\tilde{s})) \geq 2\tilde{s} \geq 2 \cdot 2\underline{\gamma}_a^{-1}(2\overline{\gamma}_a(\overline{s})) \geq 4\overline{s}$$

With this ordering and (4.53), we have that $\beta(2\underline{\gamma}_a^{-1}(2\overline{\gamma}_a(s)), T) \leq \lambda s$ for $s \leq \tilde{s}$.

Fix ρ sufficiently small such that $\gamma_d(2\underline{\gamma}_s^{-1}(2\rho\overline{\gamma}_a(s))) \leq \lambda s$ for all $s \in [0, \tilde{s}]$. Because we use a filtering prior in MHE, by Proposition 4.38 we have that (recall $N \geq T$)

$$\begin{aligned}|e(k+N)| \leq &\beta(2\underline{\gamma}_a^{-1}(2\overline{\gamma}_a(|e(k)|)), N) \oplus \gamma_d(2\underline{\gamma}_s^{-1}(2\rho\overline{\gamma}_a(|e(k)|)))\\ &\oplus \beta(2\underline{\gamma}_a^{-1}((2N/\rho)\overline{\gamma}_s(\|\mathbf{d}\|)), N) \oplus \gamma_d(2\underline{\gamma}_s^{-1}(2N\overline{\gamma}_s(\|\mathbf{d}\|)))\\ \leq &\lambda\,|e(k)| \oplus \lambda\,|e(k)|\\ &\oplus \beta(2\underline{\gamma}_a^{-1}((2N/\rho)\overline{\gamma}_s(\|\mathbf{d}\|)), 0) \oplus \gamma_d(2\underline{\gamma}_s^{-1}(2N\overline{\gamma}_s(\|\mathbf{d}\|)))\end{aligned}$$

for all $k \geq 0$.

Next, we require bounds on $|e(k)|$ for $k \in \mathbb{I}_{0:N-1}$. Because there are not enough measurements to fill the horizon, all of these MHE problems use $\overline{x}_0$ as their prior. Thus, by applying Proposition 4.38, we have that

$$\begin{aligned}|e(k)| \leq &\beta(2\underline{\gamma}_a^{-1}(2\overline{\gamma}_a(|e_a|)), 0) \oplus \gamma_d(2\underline{\gamma}_s^{-1}(2\rho\overline{\gamma}_a(|e_a|)))\\ &\oplus \beta(2\underline{\gamma}_a^{-1}((2k/\rho)\overline{\gamma}_s(\|\mathbf{d}\|)), 0) \oplus \gamma_d(2\underline{\gamma}_s^{-1}(2k\overline{\gamma}_s(\|\mathbf{d}\|)))\\ \leq &\beta(2\underline{\gamma}_a^{-1}(2\overline{\gamma}_a(|e_a|)), 0) \oplus \gamma_d(2\underline{\gamma}_s^{-1}(2\rho\overline{\gamma}_a(|e_a|)))\\ &\oplus \beta(2\underline{\gamma}_a^{-1}((2N/\rho)\overline{\gamma}_s(\|\mathbf{d}\|)), 0) \oplus \gamma_d(2\underline{\gamma}_s^{-1}(2N\overline{\gamma}_s(\|\mathbf{d}\|)))\end{aligned}$$

Let $\sigma_x := \beta(2\underline{\gamma}_a^{-1}(2\overline{\gamma}_a(s)), 0)$, and note both that $\sigma_x(\cdot) \in \mathcal{K}_\infty$ and that $\sigma_x(s) \geq s \geq \lambda s \geq \gamma_d(2\underline{\gamma}_s^{-1}(2\rho\overline{\gamma}_a(s)))$ for all $s \in [0, \tilde{s}]$. Let $\gamma_e(s) := \beta(2\underline{\gamma}_a^{-1}((2N/\rho)\overline{\gamma}_s(s)), 0) \oplus \gamma_d(\underline{\gamma}_s^{-1}(2N\overline{\gamma}_s(s)))$ and note that $\gamma_e(\cdot) \in \mathcal{K}$. Thus, we have that

$$|e(k)| \leq \sigma_x(|e_a|) \oplus \gamma_e(\|\mathbf{d}\|)$$

for all $s \in [0, \overline{s}]$ and $k \in \mathbb{I}_{0:N-1}$. Because $\gamma_e(\cdot) \in \mathcal{K}$, there exists some $\delta > 0$ such that if $\|\mathbf{d}\| \leq \delta$, then $\gamma_e(\|\mathbf{d}\|) \leq \tilde{s}$.

Finally, we assemble the $\mathcal{KL}$ function to establish stability. For $k \in \mathbb{I}_{0:N-1}$, we have that

$$|e(k+N)| \leq \lambda\sigma_x(|e_a|) \oplus \lambda\gamma_e(\|\mathbf{d}\|) \oplus \gamma_e(\|\mathbf{d}\|) \leq \lambda\sigma_x(|e_a|) \oplus \gamma_e(\|\mathbf{d}\|)$$

because $\sigma_x(|e_a|) \leq \tilde{s}$ and $\gamma_e(\|\mathbf{d}\|) \leq \tilde{s}$. Thus we have for all $j \geq 0$ that

$$\left| e(k+jN) \right| \leq \lambda^j \sigma_x(|e_a|) \oplus \gamma_e(\|\mathbf{d}\|)$$

Let $\lambda^{\lfloor k/N \rfloor}\sigma_x(s) := \beta_e(s,k)$, and note that $\beta_e(\cdot) \in \mathcal{KL}$. Thus we have for all $k \geq 0$ that

$$|e(k)| \leq \beta_e(|e(0)|, k) \oplus \gamma_e(\|\mathbf{d}\|)$$

and thus MHE is RAS. ∎

Remark. A similar argument can be used to establish also that estimate error converges to zero for convergent disturbances.

We illustrate the previous analysis by applying it to a linear, time-invariant system.

Example 4.40: MHE of linear time-invariant system

Here we show that all of the assumptions made to treat the bounded disturbance case hold for a detectable, linear system. We consider the system

$$x^+ = Ax + Gw \qquad y = Cx$$

First we derive the i-IOSS inequality. For a linear system, i-IOSS and IOSS are equivalent (see Exercise 4.8), so we need to find $\beta(\cdot) \in \mathcal{KL}$ and $\gamma_w, \gamma_v(\cdot) \in \mathcal{K}$ such that for all $x(0)$ and $k \in \mathbb{I}_{\geq 0}$, the following holds

$$|x(k; x(0), \mathbf{w})| \leq \beta(|x(0)|, k) \oplus \gamma_w(\|\mathbf{w}\|_{0:k-1}) \oplus \gamma_v(\|\mathbf{y}\|_{0:k-1}) \quad (4.54)$$

A linear time-invariant system is IOSS if and only if (A, C) is detectable (see Exercise 4.5). Because (A, C) is detectable, there exists L such that $A - LC$ is stable. We then have that

$$\begin{aligned} x^+ &= (A - LC)x + Gw + Ly \\ x(k) &= (A - LC)^k x_0 + \sum_{j=0}^{k-1} (A - LC)^{k-j-1} [Gw(j) + Ly(j)] \end{aligned}$$

Since $A - LC$ is stable, we have the bound (Horn and Johnson, 1985, p.299) $|(A - LC)^i| \leq c\lambda^i$ in which $\text{eig}(A - LC) < \lambda < 1$. Taking norms and using this bound gives for all $k \geq 0$

$$|x(k)| \leq c\lambda^k |x_0| + \frac{c}{1-\lambda} \left[|G| \, \|\mathbf{w}\|_{0:k-1} + |L| \, \|\mathbf{y}\|_{0:k-1} \right]$$

We next define $\beta'(r,k) := cr\lambda^k$, which is a linear/exponential $\mathcal{KL}$ function, and $\gamma'_w(r) = \frac{c}{1-\lambda}\,|G|\,r$ and $\gamma'_v(r) = \frac{c}{1-\lambda}\,|L|\,r$, which are linear $\mathcal{K}$ functions, so that

$$|x(k)| \leq \beta'(|x(0)|\,,k) + \gamma'_w(\|\mathbf{w}\|_{0:k-1}) + \gamma'_v(\|\mathbf{y}\|_{0:k-1})$$

To obtain the max form, we define (see Exercise 4.6)

$$\beta(r,k) := 3cr\lambda^k \qquad \gamma_w(r) = \frac{3c}{1-\lambda}\,|G|\,r \qquad \gamma_v(r) = \frac{3c}{1-\lambda}\,|L|\,r$$

and (4.54) is satisfied.

Next we check Assumptions 4.34–4.36. For least squares estimation (Kalman filtering), we choose quadratic penalties (inverse covariance weights)

$$\Gamma(x) = (1/2)\,|x|^2_{P^{-1}} \qquad \ell(\omega,\nu) = (1/2)(|\omega|^2_{Q^{-1}} + |\nu|^2_{R^{-1}})$$

with $P, Q, R > 0$. Using the singular value bounds $\underline{\sigma}(P^{-1})\,|x|^2 \leq |x|^2_{P^{-1}} \leq \overline{\sigma}(P^{-1})\,|x|^2$ gives for all x

$$\underline{a}_\gamma\,|x|^2 \leq \Gamma(x) \leq \overline{a}_\gamma\,|x|^2$$

with $\underline{\sigma}(P^{-1})$ ($\overline{\sigma}(P^{-1})$) denoting the smallest (largest) singular value of matrix P^{-1}, and $\underline{a}_\gamma := (1/2)\underline{\sigma}(P^{-1})$, $\overline{a}_\gamma := (1/2)\overline{\sigma}(P^{-1})$. Analogous reasoning gives

$$\underline{a}_\ell\,|(w,v)|^2 \leq \ell(w,v) \leq \overline{a}_\ell\,|(w,v)|^2$$

with $\underline{a}_\ell := (1/2)\min(\underline{\sigma}(Q^{-1}), \underline{\sigma}(R^{-1}))$ and $\overline{a}_\ell = (1/2)\max(\overline{\sigma}(Q^{-1}), \overline{\sigma}(R^{-1}))$. Summarizing, we have the following functions in the LQ estimation problem

$$\gamma_d(r) = \gamma_w(r) + \gamma_v(r) = a_d r$$

$$\overline{\gamma}_a(r) = \overline{a}_\gamma r^2 \qquad \underline{\gamma}_a(r) = \underline{a}_\gamma r^2 \qquad \underline{\gamma}_a^{-1}(r) = \frac{1}{\sqrt{\underline{a}_\gamma}}\sqrt{r}$$

$$\underline{\gamma}_s(r) = \underline{a}_\ell r^2 \qquad \underline{\gamma}_s^{-1}(r) = \frac{1}{\sqrt{\underline{a}_\ell}}\sqrt{r}$$

with $a_d := (3c)/(1-\lambda)(|G| + |L|)$. So we satisfy Assumption 4.34 with the standard LQ estimation cost functions.

Next we check Assumption 4.35. We have that

$$\underline{\gamma}_a^{-1}(2\overline{\gamma}_a(s)) = \frac{1}{\sqrt{\underline{a}_\gamma}}\sqrt{2\overline{a}_\gamma s^2} = \sqrt{2\overline{a}_\gamma/\underline{a}_\gamma}\,s$$

and we can see that this linear function is globally Lipschitz, thus satisfying Assumption 4.35.

Finally, we check Assumption 4.36. Using the defined cost function relations to evaluate the right-hand side of (4.49) gives

$$\gamma_d(2\underline{\gamma}_s^{-1}(2\rho\overline{\gamma}_a(s))) = 2\sqrt{2}a_d\sqrt{\overline{a}_\gamma/\underline{a}_\ell}\sqrt{\rho}s$$

Substituting this in (4.49) and solving for ρ gives

$$\rho \leq \frac{1}{8}\left(\frac{\underline{a}_\ell}{\overline{a}_\gamma a_d^2}\right)\eta^2 \tag{4.55}$$

To establish (4.50) we substitute for the detectable linear system $\beta(s, T) = 3cs\lambda^T$, which is globally Lipschitz, and solve for T giving

$$T \geq \max(1, \log(\eta/3c)/\log(\lambda)) \tag{4.56}$$

Therefore Assumption 4.36 holds for all $s \geq 0$ for a detectable, linear system when prior weighting factor, ρ, and horizon length, T, satisfy (4.55) and (4.56), respectively. □

The results presented in this section are representative of what is currently known about MHE for bounded disturbances, but we expect that this analysis is far from finished. An immediate goal of research is to weaken or remove Assumption 4.36. If that proves difficult, then we would like to establish whether something like Assumption 4.36 is *necessary* for RAS estimation using MHE, and characterize better what class of systems satisfy this type of assumption.

4.5 Other Nonlinear State Estimators

State estimation for nonlinear systems has a long history, and moving horizon estimation is a rather new approach to the problem. As with model predictive control, the optimal estimation problem on which moving horizon is based has a long history, but only the rather recent advances in computing technology have enabled moving horizon estimation to be considered as a viable option in online applications. It is therefore worthwhile to compare moving horizon estimation to other less computationally demanding nonlinear state estimators.

4.5.1 Particle Filtering

An extensive discussion and complete derivation of particle filtering appeared in the first edition of the text (Rawlings and Mayne, 2009, pp.301–355). This material is available electronically on the text's website. As with many sample-based procedures, however, it seems that all of the available sampling strategies in particle filtering do run into the "curse of dimensionality." The low density of samples in a reasonably large-dimensional space (say $n \geq 5$) lead to inaccurate state estimates. For this reason we omit further discussion of particle filtering in this edition.

4.5.2 Extended Kalman Filtering

The extended Kalman filter (EKF) generates estimates for *nonlinear* systems by first linearizing the nonlinear system, and then applying the linear Kalman filter equations to the linearized system. The approach can be summarized in a recursion similar in structure to the Kalman filter (Stengel, 1994, pp.387–388)

$$\begin{aligned}
\hat{x}^-(k+1) &= f(\hat{x}(k), 0) \\
P^-(k+1) &= \overline{A}(k)P(k)\overline{A}(k)' + \overline{G}(k)Q\overline{G}(k)' \\
\hat{x}^-(0) &= \overline{x}_0 \qquad P^-(0) = Q_0
\end{aligned}$$

The mean and covariance after measurement are given by

$$\begin{aligned}
\hat{x}(k) &= \hat{x}^-(k) + L(k)\big(y(k) - h(\hat{x}^-(k))\big) \\
L(k) &= P^-(k)\overline{C}(k)'(R + \overline{C}(k)P^-(k)\overline{C}(k)')^{-1} \\
P(k) &= P^-(k) - L(k)\overline{C}(k)P^-(k)
\end{aligned}$$

with the following linearizations

$$\overline{A}(k) = \left.\frac{\partial f(x,w)}{\partial x}\right|_{(\hat{x}(k),0)} \quad \overline{G}(k) = \left.\frac{\partial f(x,w)}{\partial w}\right|_{(\hat{x}(k),0)} \quad \overline{C}(k) = \left.\frac{\partial h(x)}{\partial x}\right|_{\hat{x}^-(k)}$$

The densities of w, v, and x_0 are assumed to be normal. Many variations on this theme have been proposed, such as the iterated EKF and the second-order EKF (Gelb, 1974, 190–192). Of the nonlinear filtering methods, the EKF method has received the most attention due to its relative simplicity and demonstrated effectiveness in handling some nonlinear systems. Examples of implementations include estimation for the production of silicon/germanium alloy films (Middlebrooks and Rawlings, 2006), polymerization reactions (Prasad, Schley, Russo, and Bequette, 2002), and fermentation processes (Gudi, Shah, and Gray, 1994). The EKF is at best an *ad hoc* solution to a difficult problem, however, and hence there exist many pitfalls to the practical implementation of EKFs (see, for example, (Wilson, Agarwal, and Rippin, 1998)). These problems include the inability to accurately incorporate physical state constraints, and the naive use of linearization of the nonlinear model.

Until recently, few properties regarding the stability and convergence of the EKF have been established. Recent research shows bounded estimation error and exponential convergence for the continuous and discrete EKF forms given observability, small initial estimation error, small noise terms, and no model error (Reif, Günther, Yaz, and Unbehauen, 1999; Reif and Unbehauen, 1999; Reif, Günther, Yaz, and Unbehauen, 2000). Depending on the system, however, the bounds on initial estimation error and noise terms may be unrealistic. Also, initial estimation error may result in bounded estimate error but not exponential convergence, as illustrated by Chaves and Sontag (2002).

Julier and Uhlmann (2004a) summarize the status of the EKF as follows.

> The extended Kalman filter is probably the most widely used estimation algorithm for nonlinear systems. However, more than 35 years of experience in the estimation community has shown that it is difficult to implement, difficult to tune, and only reliable for systems that are almost linear on the time scale of the updates.

We seem to be making a transition from a previous era in which new approaches to nonlinear filtering were criticized as overly complex because "the EKF works," to a new era in which researchers are demonstrating ever simpler examples in which the EKF fails completely. The unscented Kalman filter is one of the methods developed specifically to overcome the problems caused by the naive linearization used in the EKF.

4.5.3 Unscented Kalman Filtering

The linearization of the nonlinear model at the current state estimate may not accurately represent the dynamics of the nonlinear system behavior even for one sample time. In the EKF prediction step, the mean propagates through the full nonlinear model, but the covariance propagates through the linearization. The resulting error is sufficient to throw off the correction step and the filter can diverge even with a perfect model. The unscented Kalman filter (UKF) avoids this linearization at a single point by sampling the nonlinear response at several points. The points are called sigma points, and their locations and weights are chosen to satisfy the given starting mean and covariance (Julier and Uhlmann, 2004a,b).[3] Given $\hat{x}$ and P, choose sample points, z^i, and weights, w^i, such that

$$\hat{x} = \sum_i w^i z^i \qquad P = \sum_i w^i (z^i - \hat{x})(z^i - \hat{x})'$$

Similarly, given $w \sim N(0, Q)$ and $v \sim N(0, R)$, choose sample points n^i for w and m^i for v. Each of the sigma points is propagated forward at each sample time using the nonlinear system model. The locations and weights of the transformed points then update the mean and covariance

$$\begin{aligned} z^i(k+1) &= f(z^i(k), n^i(k)) \\ \eta^i &= h(z^i) + m^i \quad \text{all } i \end{aligned}$$

From these we compute the forecast step

$$\begin{aligned} \hat{x}^- &= \sum_i w^i z^i \qquad \hat{y}^- = \sum_i w^i \eta^i \\ P^- &= \sum_i w^i (z^i - \hat{x}^-)(z^i - \hat{x}^-)' \end{aligned}$$

After measurement, the EKF correction step is applied after first expressing this step in terms of the covariances of the innovation and state prediction. The output error is given as $\tilde{y} := y - \hat{y}^-$. We next rewrite the Kalman filter update as

$$\begin{aligned} \hat{x} &= \hat{x}^- + L(y - \hat{y}^-) \\ L &= \underbrace{\mathcal{E}((x - \hat{x}^-)\tilde{y}')}_{P^-C'} \; \underbrace{\mathcal{E}(\tilde{y}\tilde{y}')^{-1}}_{(R+CP^-C')^{-1}} \\ P &= P^- - L \; \underbrace{\mathcal{E}((x - \hat{x}^-)\tilde{y}')'}_{CP^-} \end{aligned}$$

[3]Note that this idea is fundamentally different than the idea of particle filtering. The sigma points are chosen deterministically, for example, as points on a selected covariance contour ellipse or a simplex. The particle filtering points are chosen by random sampling.

in which we approximate the two expectations with the sigma-point samples

$$\mathcal{E}((x - \hat{x}^-)\tilde{y}') \approx \sum_i w^i(z^i - \hat{x}^-)(\eta^i - \hat{y}^-)'$$
$$\mathcal{E}(\tilde{y}\tilde{y}') \approx \sum_i w^i(\eta^i - \hat{y}^-)(\eta^i - \hat{y}^-)'$$

See Julier, Uhlmann, and Durrant-Whyte (2000); Julier and Uhlmann (2004a); van der Merwe, Doucet, de Freitas, and Wan (2000) for more details on the algorithm. An added benefit of the UKF approach is that the partial derivatives $\partial f(x,w)/\partial x, \partial h(x)/\partial x$ are not required. See also Nørgaard, Poulsen, and Ravn (2000) for other derivative-free nonlinear filters of comparable accuracy to the UKF. See Lefebvre, Bruyninckx, and De Schutter (2002); Julier and Uhlmann (2002) for an interpretation of the UKF as a use of statistical linear regression.

The UKF has been tested in a variety of simulation examples taken from different application fields including aircraft attitude estimation, tracking and ballistics, and communication systems. In the chemical process control field, Romanenko and Castro (2004); Romanenko, Santos, and Afonso (2004) have compared the EKF and UKF on a strongly nonlinear exothermic chemical reactor and a pH system. The reactor has nonlinear dynamics and a linear measurement model, i.e., a subset of states is measured. In this case, the UKF performs significantly better than the EKF when the process noise is large. The pH system has linear dynamics but a strongly nonlinear measurement, i.e., the pH measurement. In this case, the authors show a modest improvement in the UKF over the EKF.

4.5.4 EKF, UKF, and MHE Comparison

One nice feature enjoyed by the EKF and UKF formulations is the recursive update equations. One-step recursions are computationally efficient, which may be critical in online applications with short sample times. The MHE computational burden may be reduced by shortening the length of the moving horizon, N. But use of short horizons may produce inaccurate estimates, especially after an unmodeled disturbance. This unfortunate behavior is the result of the system's nonlinearity. As we saw in Sections 1.4.3–1.4.4, for *linear systems*, the full information problem and the MHE problem are identical to a one-step recursion using the appropriate state penalty coming from the filtering Riccati equation. Losing the equivalence of a one-step recursion to full information or a finite moving horizon problem brings into question whether the one-step recursion can provide equivalent estimator performance. We show in the following example that the EKF and the UKF do not provide estimator performance comparable to MHE.

Example 4.41: EKF, UKF, and MHE performance comparison

Consider the following set of reversible reactions taking place in a well-stirred, isothermal, gas-phase batch reactor

$$\mathrm{A} \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} \mathrm{B} + \mathrm{C} \qquad 2\mathrm{B} \underset{k_{-2}}{\overset{k_2}{\rightleftharpoons}} \mathrm{C}$$

The material balance for the reactor is

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} c_A \\ c_B \\ c_C \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 c_A - k_{-1} c_B c_C \\ k_2 c_B^2 - k_{-2} c_C \end{bmatrix} \\ \frac{dx}{dt} &= f_c(x) \end{aligned}$$

with states and measurement

$$x = \begin{bmatrix} c_A & c_B & c_C \end{bmatrix}' \qquad y = RT \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x$$

in which c_j denotes the concentration of species j in mol/L, R is the gas constant, and T is the reactor temperature in K. The measurement is the reactor pressure in atm, and we use the ideal gas law to model the pressure. The model is nonlinear because of the two second-order reactions. We model the system plus disturbances with the following discrete time model

$$\begin{aligned} x^+ &= f(x) + w \\ y &= Cx + v \end{aligned}$$

in which f is the solution of the ordinary differential equations (ODEs) over the sample time, Δ, i.e, if $s(t, x_0)$ is the solution of $dx/dt = f_c(x)$ with initial condition $x(0) = x_0$ at $t = 0$, then $f(x) = s(\Delta, x)$. The state and measurement disturbances, w and v, are assumed to be zero-mean independent normals with constant covariances Q and R. The following parameter values are used in the simulations

$$RT = 32.84\,\mathrm{mol}\cdot\mathrm{atm/L}$$

$$\Delta = 0.25 \quad k_1 = 0.5 \quad k_{-1} = 0.05 \quad k_2 = 0.2 \quad k_{-2} = 0.01$$

$$C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} RT \qquad P(0) = (0.5)^2 I \qquad Q = (0.001)^2 I \qquad R = (0.25)^2$$

$$\overline{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \qquad x(0) = \begin{bmatrix} 0.5 \\ 0.05 \\ 0 \end{bmatrix}$$

The prior density for the initial state, $N(\overline{x}_0, P(0))$, is deliberately chosen to poorly represent the actual initial state to model a large initial disturbance to the system. We wish to examine how the different estimators recover from this large unmodeled disturbance.

Solution

Figure 4.5 (top) shows a typical EKF performance for these conditions. Note that the EKF cannot reconstruct the state for this system and that the estimates converge to incorrect steady states displaying negative concentrations of A and B. For some realizations of the noise sequences, the EKF may converge to the correct steady state. Even for these cases, however, negative concentration estimates still occur during the transient, which correspond to physically impossible states. Figure 4.5 (bottom) presents typical results for the clipped EKF, in which negative values of the filtered estimates are set to zero. Note that although the estimates converge to the system states, this estimator gives pressure estimates that are two orders of magnitude larger than the measured pressure before convergence is achieved.

The standard UKF achieves results similar to the EKF as shown in Figure 4.6 (top). Vachhani, Narasimhan, and Rengaswamy (2006) have proposed a modification to the UKF to handle constrained systems. In this approach, the sigma points that violate the constraints are scaled back to the feasible region boundaries and the sigma-point weights are modified accordingly. If this constrained version of the UKF is applied to this case study, the estimates do not significantly improve as shown in Figure 4.6 (bottom). The UKF formulations used here are based on the algorithm presented by Vachhani et al. (2006, Sections 3 and 4) with the tuning parameter κ set to $\kappa = 1$. Adjusting this parameter using other suggestions from the literature (Julier and Uhlmann, 1997; Qu and Hahn, 2009; Kandepu, Imsland, and Foss, 2008) and trial and error, does not substantially improve the UKF estimator performance.

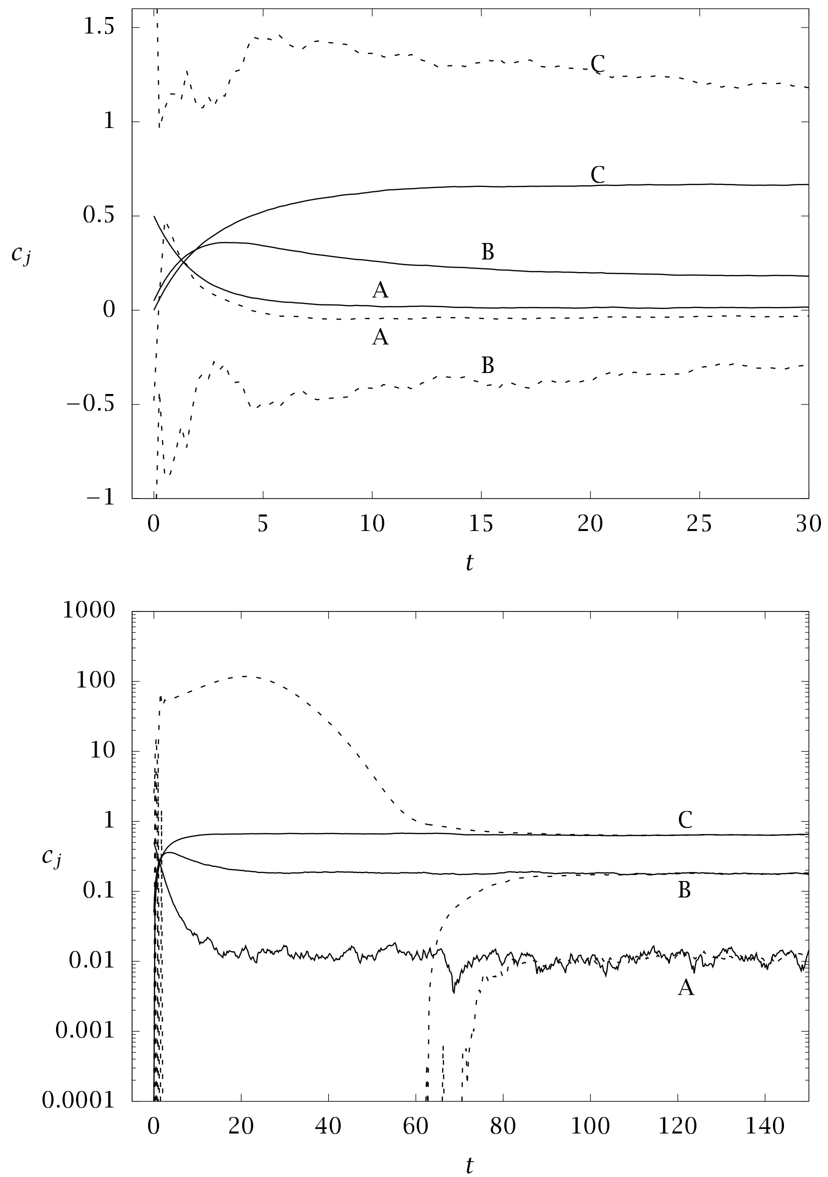

Figure 4.5: Evolution of the state (solid line) and EKF state estimate (dashed line). Top plot shows negative concentration estimates with the standard EKF. Bottom plot shows large estimate errors and slow convergence with the clipped EKF.

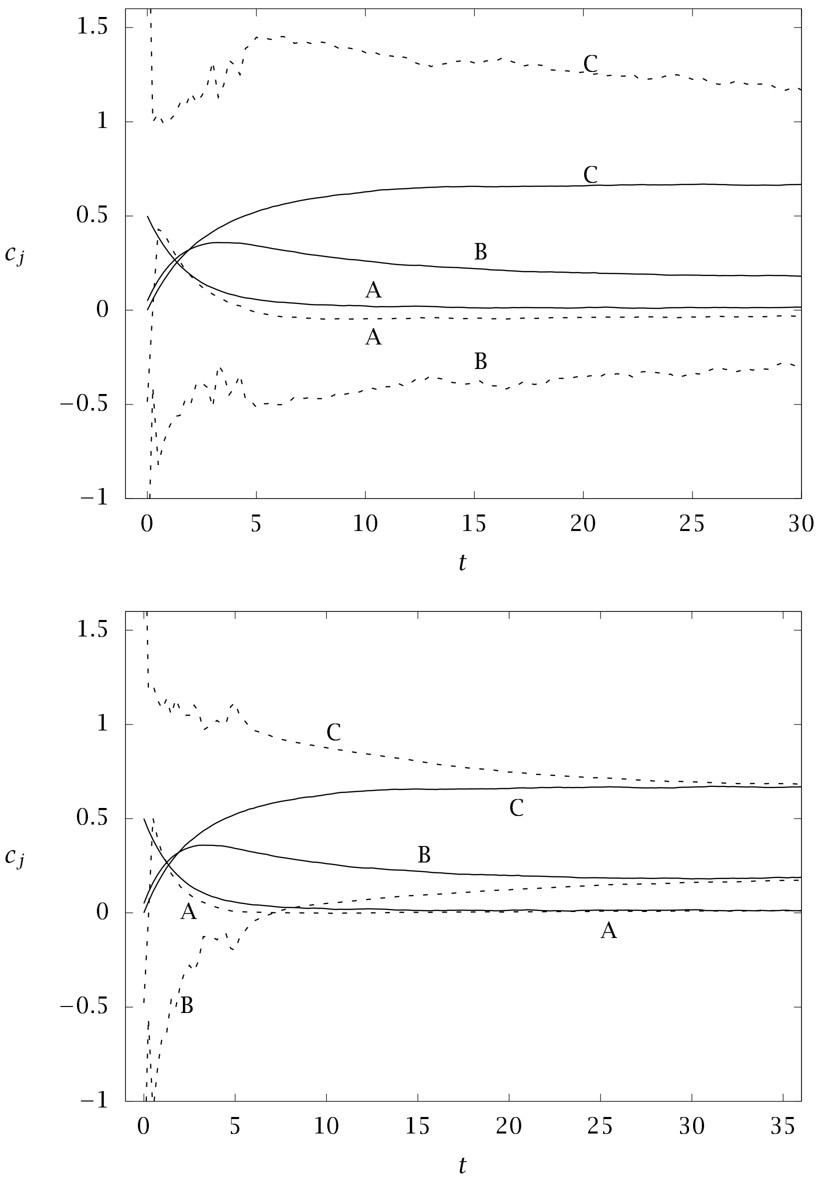

Figure 4.6: Evolution of the state (solid line) and UKF state estimate (dashed line). Top plot shows negative concentration estimates with the standard UKF. Bottom plot shows similar problems even if constraint scaling is applied.

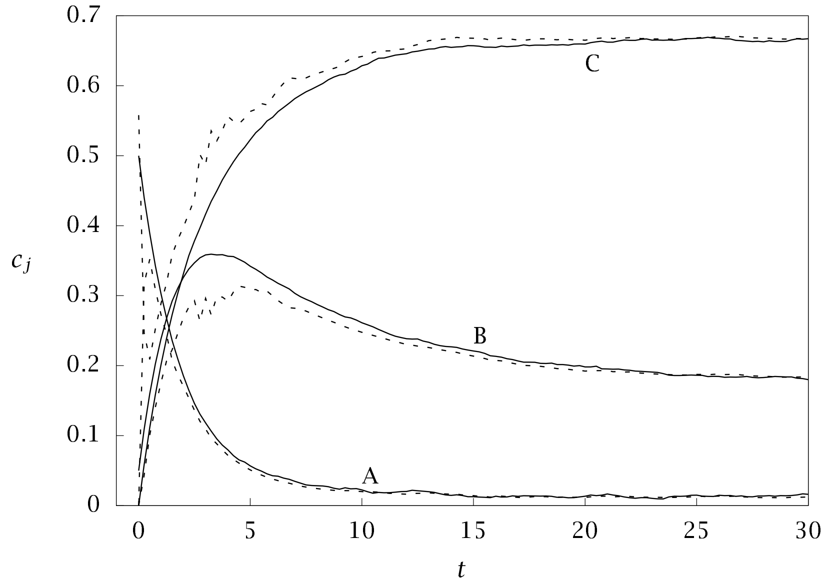

Figure 4.7: Evolution of the state (solid line) and MHE state estimate (dashed line).

Better performance is obtained in this example if the sigma points that violate the constraints are simply saturated rather than rescaled to the feasible region boundaries. But, this form of clipping still does not prevent the occurrence of negative concentrations in this example. Negative concentration estimates are not avoided by either scaling or clipping of the sigma points. As a solution to this problem, the use of constrained optimization for the sigma points is proposed (Vachhani et al., 2006; Teixeira, Tôrres, Aguirre, and Bernstein, 2008). If one is willing to perform online optimization, however, MHE with a short horizon is likely to provide more accurate estimates at similar computational cost compared to approaches based on optimizing the locations of the sigma points.

Finally, Figure 4.7 presents typical results of applying constrained MHE to this example. For this simulation we choose $N = 10$ and the smoothing update for the arrival cost approximation. Note that MHE recovers well from the poor initial prior. Comparable performance is obtained if the filtering update is used instead of the smoothing update to approximate the arrival cost. The MHE estimates are also insensitive to the choice of horizon length N for this example. □

The EKF, UKF, and all one-step recursive estimation methods, suffer from the "short horizon syndrome" by *design*. One can try to reduce the harmful effects of a short horizon through tuning various other parameters in the estimator, but the basic problem remains. Large initial state errors lead to inaccurate estimation and potential estimator divergence. The one-step recursions such as the EKF and UKF can be viewed as one extreme in the choice between speed and accuracy in that only a single measurement is considered at each sample. That is similar to an MHE problem in which the user chooses $N = 1$. Situations in which $N = 1$ lead to poor MHE performance often lead to unreliable EKF and UKF performance as well.

4.6 On combining MHE and MPC

Estimating the state of a system is an interesting problem in its own right, with many important applications having no connection to feedback control. But in some applications the goal of state estimation is indeed to provide a state feedback controller with a good estimate of the system state based on the available measurements. We close this chapter with a look at the properties of such control systems consisting of a moving horizon estimator that provides the state estimate to a model predictive controller.

What's desirable. Consider the evolution of the system $x^+ = f(x, u, w)$ and its measurement $y = h(x) + v$ when taking control using MPC based on the state estimate

$$x^+ = f(x, \kappa_N(\hat{x}), w) \qquad y = h(x) + v$$

with $f : \mathbb{Z} \times \mathbb{W} \to \mathbb{R}^n$, $h : \mathbb{R}^n \to \mathbb{R}^p$, in which $w \in \mathbb{W}$ is the process disturbance and $v \in \mathbb{V}$ is the measurement disturbance, $u = \kappa_N(\hat{x})$ is the control from the MPC regulator, and $\hat{x}$ is generated by the MHE estimator. We assume, as we have through the text, that $f(\cdot)$ and $h(\cdot)$ are continuous functions. Again we denote estimate error by $e := x - \hat{x}$, which gives for the state evolution

$$x^+ = f(x, \kappa_N(x - e), w) \qquad y = h(x) + v \tag{4.57}$$

The obvious difficulty with analyzing the effect of estimate error is the coupling of estimation and control. Unlike the problem studied earlier in the chapter, where $x^+ = f(x, w)$, we now have estimate error also influencing state evolution. This coupling precludes obtaining the simple bounds on $|e(k)|$ in terms of $(e(0), \mathbf{w}, \mathbf{v})$ as we did in the previous sections.

What's possible. Here we lower our sights from the analysis of the fully coupled problem and consider only the effect of *bounded estimate error* on the combined estimation/regulation problem. To make this precise, consider the following definition of an incrementally, uniformly input/output-to-state stable (i-UIOSS) system.

Definition 4.42 (i-UIOSS)**.** The system

$$x^+ = f(x, u, w) \qquad y = h(x)$$

is *incrementally uniformly input/output-to-state stable* (i-UIOSS) if there exist functions $\alpha(\cdot) \in \mathcal{KL}$ and $\gamma_w(\cdot)$, $\gamma_v(\cdot) \in \mathcal{K}$ such that for any two initial states z_1 and z_2, any input sequence $\mathbf{u}$, and any two disturbance sequences $\mathbf{w}_1$ and $\mathbf{w}_2$ generating state sequences $\mathbf{x}_1(z_1, \mathbf{u}, \mathbf{w}_1)$ and $\mathbf{x}_2(z_2, \mathbf{u}, \mathbf{w}_2)$, the following holds for all $k \in \mathbb{I}_{\geq 0}$

$$\begin{aligned} |x(k; z_1, \mathbf{u}, \mathbf{w}_1) - x(k; z_2, \mathbf{u}, \mathbf{w}_2)| &\leq \alpha(|z_1 - z_2|, k) \oplus \\ &\gamma_w(\|\mathbf{w}_1 - \mathbf{w}_2\|_{0:k-1}) \oplus \gamma_v(\|h(\mathbf{x}_1) - h(\mathbf{x}_2)\|_{0:k-1}) \end{aligned} \tag{4.58}$$

Notice that the bound is uniform in the sense that it is independent of the input sequence $\mathbf{u}$ generated by a controller. See Cai and Teel (2008, Definition 3.4) for similar definitions. Exercise 4.15 discusses how to establish that a detectable linear system $x^+ = Ax + Bu + Gw$, $y = Cx$ is i-UIOSS.

Given this strong form of detectability, we assume that we can derive an error bound of the form appearing in Theorem 4.39

Assumption 4.43 (Bounded estimate error)**.** There exists $\delta > 0$ such that for all $\|(\mathbf{w}, \mathbf{v})\| \leq \delta$ and for all $k \geq 0$ the following holds

$$|e(k)| \leq \beta(|e(0)|, k) + \sigma(\|(\mathbf{w}, \mathbf{v})\|)$$

Next we note that the evolution of the state in the form of (4.57) is not a compelling starting point for analysis because the estimate error perturbation appears inside a possibly discontinuous function, $\kappa_N(\cdot)$ (recall Example 2.8). Therefore, we instead express the equivalent evolution, but in terms of the state estimate as

$$\hat{x}^+ = f(\hat{x} + e, \kappa_N(\hat{x}), w) \qquad y = h(\hat{x} + e) + v$$

which is more convenient because the estimate error appears inside continuous functions $f(\cdot)$ and $h(\cdot)$.

We require that the system not leave an invariant set due to the disturbance.

Definition 4.44 (Robust positive invariance)**.** A set $\mathcal{X} \subseteq \mathbb{R}^n$ is robustly positive invariant with respect to a difference inclusion $x^+ \in f(x,d)$ if there exists some $\delta > 0$ such that $f(x,d) \subseteq \mathcal{X}$ for all $x \in \mathcal{X}$ and all disturbance sequences $\mathbf{d}$ satisfying $\|\mathbf{d}\| \leq \delta$.

So, we define robust asymptotic stability as input-to-state stability on a robust positive invariant set.

Definition 4.45 (Robust asymptotic stability)**.** The origin of a perturbed difference inclusion $x^+ \in f(x,d)$ is RAS in $\mathcal{X}$ if there exists some $\delta > 0$ such that for all disturbance sequences $\mathbf{d}$ satisfying $\|\mathbf{d}\| \leq \delta$ we have both that $\mathcal{X}$ is robustly positive invariant and that there exist $\beta(\cdot) \in \mathcal{KL}$ and $\gamma(\cdot) \in \mathcal{K}$ such that for each $x \in \mathcal{X}$, we have for all $k \in \mathbb{I}_{\geq 0}$ that all solutions $\phi(k;x,\mathbf{d})$ satisfy

$$\left|\phi(k;x,\mathbf{d})\right| \leq \beta(|x|\,,k) + \gamma(\|\mathbf{d}\|) \tag{4.59}$$

To establish input-to-state stability, we define an ISS Lyapunov function for a difference inclusion, similar to an ISS Lyapunov function defined in Jiang and Wang (2001); Lazar, Heemels, and Teel (2013). See also Definition B.45 in Appendix B.

Definition 4.46 (ISS Lyapunov function)**.** $V(\cdot)$ is an ISS Lyapunov function in the robust positive invariant set $\mathcal{X}$ for the difference inclusion $x^+ \in f(x,d)$ if there exists some $\delta > 0$, functions $\alpha_1(\cdot), \alpha_2(\cdot), \alpha_3(\cdot) \in \mathcal{K}_\infty$, and function $\sigma(\cdot) \in \mathcal{K}$ such that for all $x \in \mathcal{X}$ and $\|\mathbf{d}\| \leq \delta$

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{4.60}$$

$$\sup_{x^+ \in f(x,d)} V(x^+) \leq V(x) - \alpha_3(|x|) + \sigma(\|\mathbf{d}\|) \tag{4.61}$$

The value of an ISS Lyapunov function is analogous to having a Lyapunov function in standard stability analysis: it allows us to conclude input-to-state stability and therefore *robust* asymptotic stability. The following result is therefore highly useful in robustness analysis.

Proposition 4.47 (ISS Lyapunov stability theorem)**.** *If a difference inclusion $x^+ \in f(x,d)$ admits an ISS Lyapunov function in a robust positive*

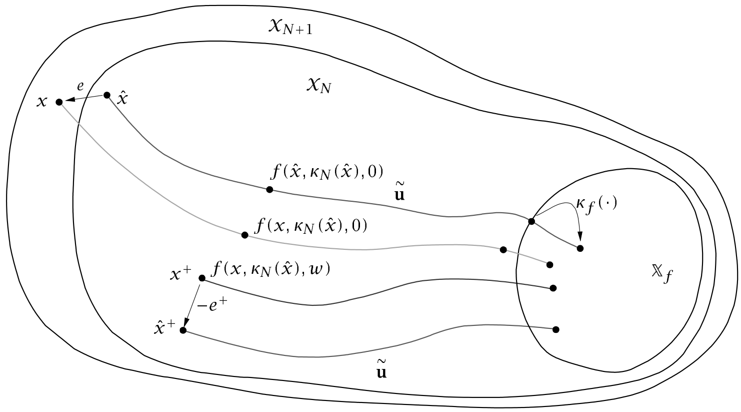

Figure 4.8: Although the nominal trajectory from $\hat{x}$ may terminate on the boundary of $\mathbb{X}_f$, the three perturbed trajectories, including the one from $\hat{x}^+$, terminate in $\mathbb{X}_f$. After Allan et al. (2017).

invariant set $\mathcal{X}$ for all $\|\mathbf{d}\| \leq \delta$ for some $\delta > 0$, then the origin is RAS in $\mathcal{X}$ for all $\|\mathbf{d}\| \leq \delta$.

The proof of this proposition follows Jiang and Wang (2001) as modified for a difference inclusion on a robust positive invariant set in Allan, Bates, Risbeck, and Rawlings (2017, Proposition 19).

Combined MHE/MPC is RAS. Our strategy now is to establish that $V_N^0(x)$ is an ISS Lyapunov function for the combined MHE/MPC system subject to process and measurement disturbances on a robust positive invariant set. We have already established the upper and lower bounding inequalities

$$\alpha_1(|x|) \leq V_N^0(x) \leq \alpha_2(|x|)$$

So we require only

$$\sup_{x^+ \in f(x,d)} V_N^0(x^+) \leq V_N^0(x) - \alpha_3(|x|) + \sigma(\|\mathbf{d}\|)$$

with disturbance d defined as $d := (e, w, e^+)$. That plus robust positive invariance establishes that the controlled system is RAS.

Figure 4.8 gives the picture of the argument we are going to make. We have that $\hat{x}^+ = f(\hat{x} - e, \kappa_N(\hat{x}), w) + e^+$ and $x^+ = f(x, \kappa_N(\hat{x}), w)$. We create the standard candidate input sequence by dropping the first input and applying the terminal control law to the terminal state, i.e., $\widetilde{\mathbf{u}} = \big(u^0(1;\hat{x}), \ldots, u^0(N-1;\hat{x}), \kappa_f(x^0(N;\hat{x}))\big)$. We then compute difference in cost of trajectories starting at x^+ and $\hat{x}^+$ using the same input sequence $\widetilde{\mathbf{u}}$. We choose the terminal region to be a sublevel set of the terminal cost, $\mathbb{X}_f = \mathrm{lev}_\tau V_f$, $\tau > 0$. Note that $\widetilde{\mathbf{u}}$ is feasible for both initial states, i.e., both trajectories terminate in $\mathbb{X}_f$, if $|(e, w, e^+)|$ is small enough.

As in Chapter 3, we make use of Proposition 3.4 to bound the size of the change to a continuous function (Allan et al., 2017, Proposition 20). Since $V_N(x, \mathbf{u})$ is continuous, Proposition 3.4 gives

$$\left| V_N(\hat{x}^+, \widetilde{\mathbf{u}}) - V_N\big(f(\hat{x}, \kappa_N(\hat{x})), \widetilde{\mathbf{u}}\big) \right| \leq \sigma_V(\left|\hat{x}^+ - f(\hat{x}, \kappa_N(\hat{x}))\right|)$$

with $\sigma_V(\cdot) \in \mathcal{K}$. Note that we are *not* using the possibly discontinuous $V_N^0(x)$ here). Since $f(x, u)$ is also continuous

$$\begin{aligned}
\left|\hat{x}^+ - f(\hat{x}, \kappa_N(\hat{x}))\right| &= \left| f(\hat{x} + e, \kappa_N(\hat{x}), w) + e^+ - f(\hat{x}, \kappa_N(\hat{x}))\right| \\
&\leq \left| f(\hat{x} + e, \kappa_N(\hat{x}), w) - f(\hat{x}, \kappa_N(\hat{x}), 0)\right| + \left|e^+\right| \\
&\leq \sigma_f(|(e, w)|) + \left|e^+\right| \\
&\leq \widetilde{\sigma}_f(|d|)
\end{aligned}$$

with $d := (e, w, e^+)$ and $\widetilde{\sigma}_f(\cdot) \in \mathcal{K}$. Therefore

$$\begin{aligned}
\left| V_N(\hat{x}^+, \widetilde{\mathbf{u}}) - V_N\big(f(\hat{x}, \kappa_N(\hat{x})), \widetilde{\mathbf{u}}\big) \right| &\leq \sigma_V \circ \widetilde{\sigma}_f(|d|) := \sigma(|d|) \\
V_N(\hat{x}^+, \widetilde{\mathbf{u}}) &\leq V_N\big(f(\hat{x}, \kappa_N(\hat{x})), \widetilde{\mathbf{u}}\big) + \sigma(|d|)
\end{aligned}$$

with $\sigma(\cdot) \in \mathcal{K}$. Note that for the candidate sequence, $V_N\big(f(\hat{x}, \kappa_N(\hat{x})), \widetilde{\mathbf{u}}\big) \leq V_N^0(\hat{x}) - \ell(\hat{x}, \kappa_N(\hat{x}))$, so we have that

$$V_N\big(f(\hat{x}, \kappa_N(\hat{x})), \widetilde{\mathbf{u}}\big) \leq V_N^0(\hat{x}) - \alpha_1(|\hat{x}|)$$

since $\alpha_1(|x|) \leq \ell(x, \kappa_N(x))$ for all x. Therefore, we finally have

$$\begin{aligned}
V_N(\hat{x}^+, \widetilde{\mathbf{u}}) &\leq V_N^0(\hat{x}) - \alpha_1(|\hat{x}|) + \sigma(|d|) \\
V_N^0(\hat{x}^+) &\leq V_N^0(\hat{x}) - \alpha_1(|\hat{x}|) + \sigma(\|\mathbf{d}\|)
\end{aligned} \tag{4.62}$$

and we have established that $V_N^0(\cdot)$ satisfies the inequality of an ISS-Lyapunov function. This analysis leads to the following main result.

Theorem 4.48 (Combined MHE/MPC is RAS)**.** *For the MPC regulator, let the standard Assumptions 2.2, 2.3, and 2.14 hold, and choose* $\mathbb{X}_f = \mathrm{lev}_\tau V_f$ *for some* $\tau > 0$. *For the moving horizon estimator, let Assumption 4.43 hold. Then for every* $\rho > 0$ *there exists* $\delta > 0$ *such that if* $\|\mathbf{d}\| \leq \delta$, *the origin is RAS for the system* $\hat{x}^+ = f(\hat{x} + e, \kappa_N(\hat{x}), w)$, $y = h(\hat{x} + e) + v$, *in the set* $\mathcal{X}_\rho = \mathrm{lev}_\rho \mathcal{X}_N$.

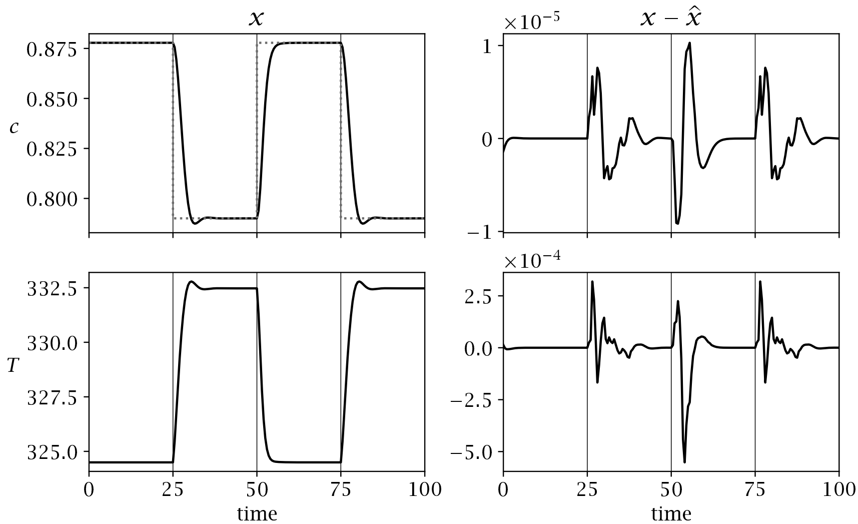

Figure 4.9: Closed-loop performance of combined nonlinear MHE/MPC with no disturbances. First column shows system states, and second column shows estimation error. Dashed line shows concentration setpoint. Vertical lines indicate times of setpoint changes.

A complete proof of this theorem, for the more general case of *suboptimal* MPC, is given in Allan et al. (2017, Theorem 21). The proof proceeds by first showing that $\mathcal{X}_\rho$ is robustly positive invariant for all $\rho > 0$. That argument is similar to the one presented in Chapter 3 before Proposition 3.5. The proof then establishes that inequality (4.62) holds for all $\hat{x} \in \mathcal{X}_\rho$. Proposition 4.47 is then invoked to establish that the origin is RAS.

Notice that neither $V_N^0(\cdot)$ nor $\kappa_N(\cdot)$ need be continuous for this combination of MHE and MPC to be inherently robust. Since $x = \hat{x} + e$, Theorem 4.48 also gives robust asymptotic stability of the evolution of x in addition to $\hat{x}$ for the closed-loop system with bounded disturbances.

Example 4.49: Combined MHE/MPC

Consider the nonlinear reactor system from Example 1.11 with sample time $\Delta = 0.5\,\text{min}$ and height h and inlet flow F fixed to their steady-state values. The resulting system has two states (concentration c and

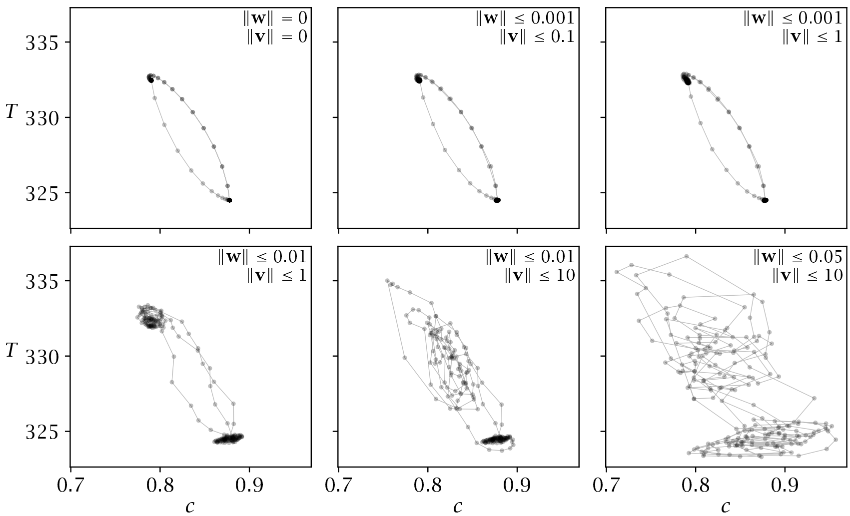

Figure 4.10: Closed-loop performance of combined nonlinear MHE/MPC for varying disturbance size. The system is controlled between two steady states.

temperature T) and one input (cooling temperature T_c). The only measured output is temperature, which means the reactor concentration must be estimated via MHE.

To illustrate the performance of combined MHE/MPC, closed-loop control to a changing setpoint is simulated. Figure 4.9 shows the changing states x and estimate errors $x - \hat{x}$. Note that each setpoint change leads to a temporary increase in estimate error, which eventually decays back to zero. Note that zero prior weighting is used in the MHE formulation.

To illustrate the response to disturbances, the simulation is repeated for varying disturbance sizes. The system itself is subject to a disturbance w, which is adds to the evolution of concentration, while the temperature measurement is subject to noise v. Figure 4.10 shows a phase plot of system evolution subject to the same setpoint changes as before. As the disturbances become larger, the system deviates further from its setpoint. Note that the same MHE objective function (with zero prior weight) is used for all cases. □

4.7 Notes

State estimation is a fundamental topic appearing in many branches of science and engineering, and has a large literature. A nice and brief annotated bibliography describing the early contributions to optimal state estimation of the *linear Gaussian* system is provided by Åström (1970, pp. 252-255). Kailath (1974) provides a comprehensive and historical review of *linear* filtering theory including the historical development of Wiener-Kolmogorov theory for filtering and prediction that preceded Kalman filtering (Wiener, 1949; Kolmogorov, 1941).

Jazwinski (1970) provides an early and comprehensive treatment of the optimal stochastic state estimation problem for linear and *nonlinear* systems. As mentioned in Section 4.2.1, Jazwinski (1970) proves stability of the optimal time-varying state estimator for the linear Gaussian case using $V(k,x) = x'P(k)^{-1}x$ as the Lyapunov function for the linear time-varying system governing estimate error. Note that this dynamic system is time varying even if the model is time invariant because the optimal estimator gains are time varying. This choice of Lyapunov function has been used to establish estimator stability in many subsequent textbooks (Stengel, 1994, pp.474-475). Kailath (1974, p.152) remarks that the known proofs that the optimal filter is stable "are somewhat difficult, and it is significant that only a small fraction of the vast literature on the Kalman filter deals with this problem." Perhaps the stability analysis developed in Section 4.2 can alleviate the difficulties associated with developing Lyapunov function arguments in optimal estimation.

For establishing stability of the *steady-state* optimal linear estimator, simpler arguments suffice because the estimate error equation is time invariant. Establishing duality with the optimal regulator is a favorite technique for establishing estimator stability in this case. See, for example, Kwakernaak and Sivan (1972, Theorem 4.11) for a general steady-state stability theorem for the linear Gaussian case. This result is proved by establishing properties of the steady-state solution to the Riccati equation for regulation and, by duality, estimation.

Many of the full information and MHE results in this chapter are motivated by results in Rao (2000) and Rao, Rawlings, and Mayne (2003). The full information analysis given here is more general because (i) we assume nonlinear detectability rather than nonlinear observability, and (ii) we establish asymptotic stability under process and measurement disturbances, which were neglected in previous analysis.

Muske, Rawlings, and Lee (1993) and Meadows, Muske, and Rawlings (1993) apparently were the first to use the increasing property of the optimal cost to establish asymptotic stability for full information estimation for linear models with constraints. Robertson and Lee (2002) present the interesting statistical interpretation of MHE for the constrained linear system. Michalska and Mayne (1995) establish stability of moving horizon estimation with zero prior weighting for the continuous time nonlinear system. Alessandri, Baglietto, and Battistelli (2008) also provide a stability analysis of MHE with an observability assumption and quadratic stage cost.

After the publication of the first edition of this text, Rawlings and Ji (2012) streamlined the presentation of the full information problem for the case of convergent disturbances, and pointed to MHE of bounded disturbances, and suboptimal MHE as two significant open research problems. Next Ji, Rawlings, Hu, Wynn, and Diehl (2016); Hu, Xie, and You (2015) provided the first analysis of full information estimation for bounded disturbances by introducing a max term in the estimation objective function, and assuming stronger forms of the i-IOSS detectability condition. This reformulation did provide RAS of full information estimation with bounded disturbances, but had the unfortunate side effect of removing convergent estimate error for convergent disturbances. In a significant step forward, Müller (2017) examined MHE with bounded disturbances for similarly restrictive i-IOSS conditions, and established bounds on arrival cost penalty and horizon length that provide both RAS for bounded disturbances and convergence of estimate error for convergent disturbances. The presentation in Section 4.4 is a slightly more general version of this analysis. Hu (2017) has generalized the detectability conditions in Ji et al. (2016) and treated both full information with the max term and MHE estimation. Moreover, numerous application papers using MHE have appeared in the last several years (see Hu (2017) for a brief summary) indicating a growing interest in this approach to state estimation.

For the case of output feedback, there are of course alternatives to simply combining independently designed MHE estimators and MPC regulators as briefly analyzed in Section 4.6. Recently Copp and Hespanha (2017) propose solving instead a single min-max optimization for simultaneous estimation and control. Because of the excellent resultant closed-loop properties, this class of approaches certainly warrants further attention and development.

4.8 Exercises

Exercise 4.1: Input-to-state stability and convergence

Assume the nonlinear system

$$x^+ = f(x, u)$$

is input-to-state stable (ISS) so that for all $x_0 \in \mathbb{R}^n$, input sequences $\mathbf{u}$, and $k \geq 0$

$$|x(k; x_0, \mathbf{u})| \leq \beta(|x_0|, k) + \gamma(\|\mathbf{u}\|)$$

in which $x(k; x_0, \mathbf{u})$ is the solution to the system equation at time k starting at state x_0 using input sequence $\mathbf{u}$, and $\gamma \in \mathcal{K}$ and $\beta \in \mathcal{KL}$.

(a) Show that the ISS property also implies

$$|x(k; x_0, \mathbf{u})| \leq \beta(|x_0|, k) + \gamma(\|\mathbf{u}\|_{0:k-1})$$

in which $\|\mathbf{u}\|_{a:b} = \max_{a \leq j \leq b} |u(j)|$.

(b) Show that the ISS property implies the "converging-input converging-state" property (Jiang and Wang, 2001), (Sontag, 1998, p. 330), i.e., show that if the system is ISS, then $u(k) \to 0$ implies $x(k) \to 0$.

Exercise 4.2: Output-to-state stability and convergence

Assume the nonlinear system

$$x^+ = f(x) \qquad y = h(x)$$

is output-to-state stable (OSS) so that for all $x_0 \in \mathbb{R}^n$ and $k \geq 0$

$$|x(k; x_0)| \leq \beta(|x_0|, k) + \gamma(\|\mathbf{y}\|_{0:k})$$

in which $x(k; x_0)$ is the solution to the system equation at time k starting at state x_0, and $\gamma \in \mathcal{K}$ and $\beta \in \mathcal{KL}$.

Show that the OSS property implies the "converging-output converging-state" property (Sontag and Wang, 1997, p. 281) i.e., show that if the system is OSS, then $y(k) \to 0$ implies $x(k) \to 0$.

Exercise 4.3: i-IOSS and convergence

Prove Proposition 4.2, which states that if system

$$x^+ = f(x, w) \qquad y = g(x)$$

is i-IOSS, and $w_1(k) \to w_2(k)$ and $y_1(k) \to y_2(k)$ as $k \to \infty$, then

$$x(k; z_1, \mathbf{w}_1) \to x(k; z_2, \mathbf{w}_2) \quad \text{as } k \to \infty \quad \text{for all } z_1, z_2$$

Exercise 4.4: Observability and detectability of linear time-invariant systems and OSS

Consider the linear time-invariant system

$$x^+ = Ax \qquad y = Cx$$

(a) Show that if the system is observable, then the system is OSS.

(b) Show that the system is detectable if and only if the system is OSS.

Exercise 4.5: Observability and detectability of linear time-invariant system and IOSS

Consider the linear time-invariant system with input

$$x^+ = Ax + Gw \qquad y = Cx$$

(a) Show that if the system is observable, then the system is IOSS.

(b) Show that the system is detectable if and only if the system is IOSS.

Exercise 4.6: Max or sum?

To facilitate complicated arguments involving $\mathcal{K}$ and $\mathcal{KL}$ functions, it is often convenient to interchange sum and max operations. First some suggestive notation: let the max operator over scalars be denoted with the $\oplus$ symbol so that

$$a \oplus b := \max(a, b)$$

(a) Show that the $\oplus$ operator is commutative and associative, i.e., $a \oplus b = b \oplus a$ and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ for all a, b, c, so that the following operation is well defined and the order of operation is inconsequential

$$a_1 \oplus a_2 \oplus a_3 \cdots \oplus a_n := \bigoplus_{i=1}^{n} a_i$$

(b) Find scalars d and e such that for all $a, b \geq 0$, the following holds

$$d(a + b) \leq a \oplus b \leq e(a + b)$$

(c) Find scalars $\overline{d}$ and $\overline{e}$ such that for all $a, b \geq 0$, the following holds

$$\overline{d}(a \oplus b) \leq a + b \leq \overline{e}(a \oplus b)$$

(d) Generalize the previous result to the n term sum; find $d_n, e_n, \overline{d}_n, \overline{e}_n$ such that the following holds for all $a_i \geq 0$, $i = 1, 2, \ldots, n$

$$d_n \sum_{i=1}^{n} a_i \leq \bigoplus_{i=1}^{n} a_i \leq e_n \sum_{i=1}^{n} a_i$$

$$\overline{d}_n \bigoplus_{i=1}^{n} a_i \leq \sum_{i=1}^{n} a_i \leq \overline{e}_n \bigoplus_{i=1}^{n} a_i$$

(e) Show that establishing convergence (divergence) in sum or max is equivalent, i.e., consider a time sequence $(s(k))_{k \geq 0}$

$$s(k) = \sum_{i=1}^{n} a_i(k) \qquad \overline{s}(k) = \bigoplus_{i=1}^{n} a_i(k)$$

Show that

$$\lim_{k \to \infty} s(k) = 0 \ (\infty) \text{ if and only if } \lim_{k \to \infty} \overline{s}(k) = 0 \ (\infty)$$

Exercise 4.7: Where did my constants go?

Once $\mathcal{K}$ and $\mathcal{KL}$ functions appear, we may save some algebra by switching from the sum to the max.

In the following, let $\gamma(\cdot)$ be any $\mathcal{K}$ function.

(a) If you choose to work with sum, derive the following bounding inequalities (Rawlings and Ji, 2012)

$$\gamma(a_1 + a_2 + \cdots + a_n) \leq \gamma(na_1) + \gamma(na_2) + \cdots + \gamma(na_n)$$
$$\gamma(a_1 + a_2 + \cdots + a_n) \geq \frac{1}{n}\Big(\gamma(a_1) + \gamma(a_2) + \cdots + \gamma(a_n)\Big)$$

(b) If you choose to work with max instead, derive instead the following simpler result

$$\gamma(a_1 \oplus a_2 \oplus \cdots \oplus a_n) = \gamma(a_1) \oplus \gamma(a_2) \oplus \cdots \oplus \gamma(a_n)$$

Notice that you have an equality rather than an inequality, which leads to tighter bounds.

Exercise 4.8: Linear systems and incremental stability

Show that for a linear time-invariant system, i-ISS (i-OSS, i-IOSS) is equivalent to ISS (OSS, IOSS).

Exercise 4.9: Nonlinear observability and Lipschitz continuity implies i-OSS

Consider the following definition of observability for nonlinear systems in which f and h are Lipschitz continuous. A system

$$x^+ = f(x) \qquad y = h(x)$$

is observable if there exists $N_o \in \mathbb{I}_{\geq 1}$ and $\mathcal{K}$ function γ such that

$$\sum_{k=0}^{N_o-1} \left| y(k; x_1) - y(k; x_2) \right| \geq \gamma(|x_1 - x_2|) \tag{4.63}$$

holds for all $x_1, x_2 \in \mathbb{R}^n$. This definition was used by Rao et al. (2003) in showing stability of nonlinear MHE to initial condition error under zero state and measurement disturbances.

(a) Show that this form of nonlinear observability implies i-OSS.

(b) Show that i-OSS does not imply this form of nonlinear observability and, therefore, i-OSS is a weaker assumption.

The i-OSS concept generalizes the linear system concept of detectability to nonlinear systems.

Exercise 4.10: Equivalance of detectability and IOSS for continuous time, linear, time-invariant system

Consider the continuous time, linear, time-invariant system with input

$$\dot{x} = Ax + Bu \qquad y = Cx$$

Show that the system is detectable if and only if the system is IOSS.

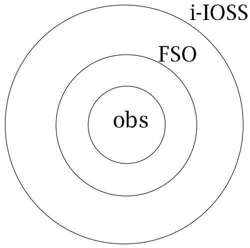

Figure 4.11: Relationships between i-IOSS, FSO, and observable for $\mathcal{K}$-continuous nonlinear systems.

Exercise 4.11: Observable, FSO, and detectable for linear systems

Consider the linear time-invariant system

$$x^+ = Ax \qquad y = Cx$$

and its observability canonical form. What conditions must the system satisfy to be

(a) observable?

(b) final-state observable (FSO)?

(c) detectable?

Exercise 4.12: Robustly GAS implies GAS in estimation

Show that an RGAS estimator is also GAS.

Exercise 4.13: Relationships between observable, FSO, and i-IOSS

Show that for the nonlinear system $x^+ = f(x, w)$, $y = h(x)$ with globally $\mathcal{K}$-continuous $f(\cdot)$ and $h(\cdot)$, i-IOSS, FSO, and observable are related as shown in the Venn diagram in Figure 4.11.

Exercise 4.14: Observabilty plus $\mathcal{K}$-continuity imply FSO

Prove Proposition 4.16. Hint: first try replacing global $\mathcal{K}$-continuity with the stronger assumption of global Lipschitz continuity to get a feel for the argument.

Exercise 4.15: Detectable linear time-invariant system and i-UIOSS

Show that the detectable linear time-invariant system $x^+ = Ax + Bu + Gw, y = Cx$ is i-UIOSS from Definition 4.42.

Exercise 4.16: Dynamic programming recursion for Kalman predictor

In the Kalman predictor, we use forward DP to solve at stage k

$$\min_{x,w} \ell(x, w) + V_k^-(x) \qquad \text{s.t. } z = Ax + w$$

in which x is the state at the current stage and z is the state at the next stage. The stage cost and arrival cost are given by

$$\ell(x,w) = (1/2)(\,|y(k) - Cx|^2_{R^{-1}} + w'Q^{-1}w) \qquad V^-_k(x) = (1/2)\,|x - \hat{x}^-(k)|^2_{(P^-(k))^{-1}}$$

and we wish to find the value function $V^0(z)$, which we denote $V^-_{k+1}(z)$ in the Kalman predictor estimation problem.

(a) Combine the two x terms to obtain

$$\min_{x,w} \frac{1}{2}\left(w'Q^{-1}w + (x - \hat{x}(k))'P(k)^{-1}(x - \hat{x}(k))\right) \qquad \text{s.t. } z = Ax + w$$

and, using the third part of Example 1.1, show

$$P(k) = P^-(k) - P^-(k)C'(CP^-(k)C' + R)^{-1}CP^-(k)$$
$$L(k) = P^-(k)C'(CP^-(k)C' + R)^{-1}C'R^{-1}$$
$$\hat{x}(k) = \hat{x}^-(k) + L(k)(y(k) - C\hat{x}^-(k))$$

(b) Add the w term and use the inverse form in Exercise 1.18 to show the optimal cost is given by

$$V^0(z) = (1/2)(z - A\hat{x}^-(k+1))'(P^-(k+1))^{-1}(z - A\hat{x}^-(k+1))$$
$$\hat{x}^-(k+1) = A\hat{x}(k)$$
$$P^-(k+1) = AP(k)A' + Q$$

Substitute the results for $\hat{x}(k)$ and $P(k)$ above and show

$$V^-_{k+1}(z) = (1/2)(z - \hat{x}^-(k+1))'(P^-(k+1))^{-1}(z - \hat{x}(k+1))$$
$$P^-(k+1) = Q + AP^-(k)A' - AP^-(k)C'(CP^-(k)C' + R)^{-1}CP^-(k)A$$
$$\hat{x}^-(k+1) = A\hat{x}^-(k) + \tilde{L}(k)(y(k) - C\hat{x}^-(k))$$
$$\tilde{L}(k) = AP^-(k)C'(CP^-(k)C' + R)^{-1}$$

(c) Compare and contrast this form of the estimation problem to the one given in Exercise 1.29 that describes the Kalman filter.

Exercise 4.17: Duality, cost to go, and covariance

Using the duality variables of Table 4.2, translate Theorem 4.12 into the version that is relevant to the state estimation problem.

Exercise 4.18: Estimator convergence for (A, G) not stabilizable

What happens to the stability of the optimal estimator if we violate the condition

$$(A, G) \text{ stabilizable}$$

(a) Is the steady-state Kalman filter a stable estimator? Is the full information estimator a stable estimator? Are these two answers contradictory? Work out the results for the case $A = 1, G = 0, C = 1, P^-(0) = 1, Q = 1, R = 1$.
Hint: you may want to consult de Souza, Gevers, and Goodwin (1986).

(b) Can this phenomenon happen in the LQ regulator? Provide the interpretation of the time-varying regulator that corresponds to the time-varying filter given above. Does this make sense as a regulation problem?

Exercise 4.19: Exponential stability of the Kalman predictor

Establish that the Kalman predictor defined in Section 4.2.1 is a globally exponentially stable estimator. What is the corresponding linear quadratic regulator?

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