Chapter 1

Stochastic Programming Models

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1.1 Introduction

Readers familiar with the area of optimization can easily name several classes of optimization problems, for which advanced theoretical results exist and efficient numerical methods have been found. We can mention linear programming, quadratic programming, convex optimization, and nonlinear optimization. *Stochastic programming* sounds similar, but no specific formulation plays the role of the generic stochastic programming problem. The presence of random quantities in the model under consideration opens the door to a wealth of different problem settings, reflecting different aspects of the applied problem at hand. This chapter illustrates the main approaches that can be followed when developing a suitable stochastic optimization model. For the purpose of presentation, these are very simplified versions of problems encountered in practice, but we hope that they help us to convey our main message.

1.2 Inventory

1.2.1 The News Vendor Problem

Suppose that a company has to decide about order quantity x of a certain product to satisfy demand d. The cost of ordering is $c > 0$ per unit. If the demand d is larger than x, then the company makes an additional order for the unit price $b \geq 0$. The cost of this is equal to $b(d - x)$ if $d > x$ and is 0 otherwise. On the other hand, if $d < x$, then a holding cost of

$h(x-d) \geq 0$ is incurred. The total cost is then equal to[1]

$$F(x,d) = cx + b[d-x]_+ + h[x-d]_+. \tag{1.1}$$

We assume that $b > c$, i.e., the backorder penalty cost is *larger* than the ordering cost.

The objective is to minimize the total cost $F(x,d)$. Here x is the decision variable and the demand d is a parameter. Therefore, if the demand is known, the corresponding optimization problem can be formulated as

$$\operatorname*{Min}_{x \geq 0} F(x,d). \tag{1.2}$$

The objective function $F(x,d)$ can be rewritten as

$$F(x,d) = \max\big\{(c-b)x + bd,\, (c+h)x - hd\big\}, \tag{1.3}$$

which is a piecewise linear function with a minimum attained at $\bar{x} = d$. That is, if the demand d is known, then (as expected) the best decision is to order exactly the demand quantity d.

Consider now the case when the ordering decision should be made *before* a realization of the demand becomes known. One possible way to proceed in such a situation is to view the demand D as a *random variable*. By capital D, we denote the demand when viewed as a random variable in order to distinguish it from its particular realization d. We assume, further, that the probability distribution of D is *known*. This makes sense in situations where the ordering procedure repeats itself and the distribution of D can be estimated from historical data. Then it makes sense to talk about the expected value, denoted $\mathbb{E}[F(x,D)]$, of the total cost viewed as a function of the order quantity x. Consequently, we can write the corresponding optimization problem

$$\operatorname*{Min}_{x \geq 0} \big\{ f(x) := \mathbb{E}[F(x,D)] \big\}. \tag{1.4}$$

The above formulation approaches the problem by optimizing (minimizing) the total cost *on average*. What would be a possible justification of such approach? If the process repeats itself, then by the Law of Large Numbers, for a given (fixed) x, the average of the total cost, over many repetitions, will converge (with probability one) to the expectation $\mathbb{E}[F(x,D)]$, and, indeed, in that case the solution of problem (1.4) will be optimal on average.

The above problem gives a very simple example of a *two-stage problem* or a problem with a *recourse action*. At the first stage, before a realization of the demand D is known, one has to make a decision about the ordering quantity x. At the second stage, after a realization d of demand D becomes known, it may happen that $d > x$. In that case, the company takes the recourse action of ordering the required quantity $d - x$ at the higher cost of $b > c$.

The next question is how to solve the expected value problem (1.4). In the present case it can be solved in a closed form. Consider the cumulative distribution function (cdf) $H(x) := \Pr(D \leq x)$ of the random variable D. Note that $H(x) = 0$ for all $x < 0$, because the demand cannot be negative. The expectation $\mathbb{E}[F(x,D)]$ can be written in the following form:

$$\mathbb{E}[F(x,D)] = b\,\mathbb{E}[D] + (c-b)x + (b+h)\int_0^x H(z)dz. \tag{1.5}$$

[1] For a number $a \in \mathbb{R}$, $[a]_+$ denotes the maximum $\max\{a, 0\}$.

Indeed, the expectation function $f(x) = \mathbb{E}[F(x, D)]$ is a *convex* function. Moreover, since it is assumed that $f(x)$ is well defined and finite values, it is continuous. Consequently, for $x \geq 0$ we have

$$f(x) = f(0) + \int_0^x f'(z)dz,$$

where at nondifferentiable points the derivative $f'(z)$ is understood as the right-side derivative. Since $D \geq 0$, we have that $f(0) = b\mathbb{E}[D]$. Also, we have that

$$\begin{aligned} f'(z) &= c + \mathbb{E}\left[\frac{\partial}{\partial z}\left(b[D-z]_+ + h[z-D]_+\right)\right] \\ &= c - b\,\mathsf{Pr}(D \geq z) + h\,\mathsf{Pr}(D \leq z) \\ &= c - b(1 - H(z)) + hH(z) \\ &= c - b + (b+h)H(z). \end{aligned}$$

Formula (1.5) then follows.

We have that $\frac{d}{dx}\int_0^x H(z)dz = H(x)$, provided that $H(\cdot)$ is continuous at x. In this case, we can take the derivative of the right-hand side of (1.5) with respect to x and equate it to zero. We conclude that the optimal solutions of problem (1.4) are defined by the equation $(b+h)H(x) + c - b = 0$, and hence an optimal solution of problem (1.4) is equal to the quantile

$$\bar{x} = H^{-1}(\kappa) \quad \text{with} \quad \kappa = \frac{b-c}{b+h}. \tag{1.6}$$

Remark 1. Recall that for $\kappa \in (0, 1)$ the left-side κ-quantile of the cdf $H(\cdot)$ is defined as $H^{-1}(\kappa) := \inf\{t : H(t) \geq \kappa\}$. In a similar way, the right-side κ-quantile is defined as $\sup\{t : H(t) \leq \kappa\}$. If the left and right κ-quantiles are the same, then problem (1.4) has unique optimal solution $\bar{x} = H^{-1}(\kappa)$. Otherwise, the set of optimal solutions of problem (1.4) is given by the whole interval of κ-quantiles.

Suppose for the moment that the random variable D has a finitely supported distribution, i.e., it takes values $d_1, \dots, d_K$ (called *scenarios*) with respective probabilities $p_1, \dots, p_K$. In that case, its cdf $H(\cdot)$ is a step function with jumps of size p_k at each d_k, $k = 1, \dots, K$. Formula (1.6) for an optimal solution still holds with the corresponding left-side (right-side) κ-quantile, coinciding with one of the points d_k, $k = 1, \dots, K$. For example, the scenarios may represent historical data collected over a period of time. In such a case, the corresponding cdf is viewed as the *empirical* cdf, giving an approximation (estimation) of the true cdf, and the associated κ-quantile is viewed as the sample estimate of the κ-quantile associated with the true distribution.

It is instructive to compare the quantile solution $\bar{x}$ with a solution corresponding to one specific demand value $d := \bar{d}$, where $\bar{d}$ is, say, the mean (expected value) of D. As mentioned earlier, the optimal solution of such (deterministic) problem is $\bar{d}$. The mean $\bar{d}$ can be very different from the κ-quantile $\bar{x} = H^{-1}(\kappa)$. It is also worth mentioning that sample quantiles typically are much less sensitive than sample mean to random perturbations of the empirical data.

In applications, closed-form solutions for stochastic programming problems such as (1.4) are rarely available. In the case of finitely many scenarios, it is possible to model the stochastic program as a deterministic optimization problem by writing the expected value $\mathbb{E}[F(x, D)]$ as the weighted sum:

$$\mathbb{E}[F(x, D)] = \sum_{k=1}^{K} p_k F(x, d_k).$$

The deterministic formulation (1.2) corresponds to *one* scenario d taken with probability 1. By using the representation (1.3), we can write problem (1.2) as the linear programming problem

$$\begin{aligned} &\operatorname*{Min}_{x\geq 0,\, v}\ v \\ &\text{s.t. } v \geq (c-b)x + bd, \\ &\qquad v \geq (c+h)x - hd. \end{aligned} \tag{1.7}$$

Indeed, for fixed x, the optimal value of (1.7) is equal to $\max\{(c-b)x+bd, (c+h)x-hd\}$, which is equal to $F(x,d)$. Similarly, the expected value problem (1.4), with scenarios $d_1, \dots, d_K$, can be written as the linear programming problem:

$$\begin{aligned} &\operatorname*{Min}_{x\geq 0,\, v_1,\dots,v_K}\ \sum_{k=1}^{K} p_k v_k \\ &\text{s.t. } v_k \geq (c-b)x + bd_k,\ k = 1,\dots,K, \\ &\qquad v_k \geq (c+h)x - hd_k,\ k = 1,\dots,K. \end{aligned} \tag{1.8}$$

It is worth noting here the almost separable structure of problem (1.8). For a fixed x, problem (1.8) separates into the sum of optimal values of problems of the form (1.7) with $d = d_k$. As we shall see later, such a decomposable structure is typical for two-stage stochastic programming problems.

Worst-Case Approach

One can also consider the worst-case approach. That is, suppose that there are known lower and upper bounds for the demand, i.e., it is unknown that $d \in [l, u]$, where $l \leq u$ are given (nonnegative) numbers. Then the worst-case formulation is

$$\operatorname*{Min}_{x\geq 0} \max_{d\in[l,u]} F(x,d). \tag{1.9}$$

That is, while making decision x, one is prepared for the worst possible outcome of the maximal cost. By (1.3) we have that

$$\max_{d\in[l,u]} F(x,d) = \max\{F(x,l), F(x,u)\}.$$

Clearly we should look at the optimal solution in the interval $[l, u]$, and hence problem (1.9) can be written as

$$\operatorname*{Min}_{x\in[l,u]} \Big\{ \psi(x) := \max\big\{cx + h[x-l]_+, cx + b[u-x]_+\big\}\Big\}.$$

The function $\psi(x)$ is a piecewise linear convex function. Assuming that $b > c$, we have that the optimal solution of problem (1.9) is attained at the point where $h(x-l) =$

$b(u - x)$. That is, the optimal solution of problem (1.9) is

$$x^* = \frac{hl + bu}{h + b}.$$

The worst-case solution x^* can be quite different from the solution $\bar{x}$, which is optimal on average (given in (1.6)) and could be overall conservative. For instance, if $h = 0$, i.e., the holding cost is zero, then $x^* = u$. On the other hand, the optimal on average solution $\bar{x}$ depends on the distribution of the demand D which could be unavailable.

Suppose now that in addition to the lower and upper bounds of the demand, we know its mean (expected value) $\bar{d} = \mathbb{E}[D]$. Of course, we have that $\bar{d} \in [l, u]$. Then we can consider the following worst-case formulation:

$$\operatorname*{Min}_{x \geq 0} \sup_{H \in \mathfrak{M}} \mathbb{E}_H[F(x, D)], \tag{1.10}$$

where $\mathfrak{M}$ denotes the set of probability measures supported on the interval $[l, u]$ and having mean $\bar{d}$, and the notation $\mathbb{E}_H[F(x, D)]$ emphasizes that the expectation is taken with respect to the cumulative distribution function (probability measure) $H(\cdot)$ of D. We study minimax problems of the form (1.10) in section 6.6 (see also problem 6.8 on p. 330).

1.2.2 Chance Constraints

We have already observed that for a particular realization of the demand D, the cost $F(\bar{x}, D)$ can be quite different from the optimal-on-average cost $\mathbb{E}[F(\bar{x}, D)]$. Therefore, a natural question is whether we can control the risk of the cost $F(x, D)$ to be not "too high." For example, for a chosen value (threshold) $\tau > 0$, we may add to problem (1.4) the constraint $F(x, D) \leq \tau$ to be satisfied for *all* possible realizations of the demand D. That is, we want to make sure that the total cost will not be larger than τ in all possible circumstances. Assuming that the demand can vary in a specified uncertainty set $\mathfrak{D} \subset \mathbb{R}$, this means that the inequalities $(c - b)x + bd \leq \tau$ and $(c + h)x - hd \leq \tau$ should hold for all possible realizations $d \in \mathfrak{D}$ of the demand. That is, the ordering quantity x should satisfy the following inequalities:

$$\frac{bd - \tau}{b - c} \leq x \leq \frac{hd + \tau}{c + h} \quad \forall d \in \mathfrak{D}. \tag{1.11}$$

This could be quite restrictive if the uncertainty set $\mathfrak{D}$ is large. In particular, if there is at least one realization $d \in \mathfrak{D}$ greater than τ/c, then the system (1.11) is inconsistent, i.e., the corresponding problem has no feasible solution.

In such situations it makes sense to introduce the constraint that the probability of $F(x, D)$ being larger than τ is less than a specified value (significance level) $\alpha \in (0, 1)$. This leads to a *chance* (also called *probabilistic*) constraint which can be written in the form

$$\Pr\{F(x, D) > \tau\} \leq \alpha \tag{1.12}$$

or equivalently,

$$\Pr\{F(x, D) \leq \tau\} \geq 1 - \alpha. \tag{1.13}$$

By adding the chance constraint (1.13) to the optimization problem (1.4), we want to minimize the total cost on average while making sure that the risk of the cost to be excessive (i.e., the probability that the cost is larger than τ) is small (i.e., less than α). We have that

$$\Pr\{F(x,D) \le \tau\} = \Pr\left\{\tfrac{(c+h)x-\tau}{h} \le D \le \tfrac{(b-c)x+\tau}{b}\right\}. \tag{1.14}$$

For $x \le \tau/c$, the inequalities on the right-hand side of (1.14) are consistent, and hence for such x,

$$\Pr\{F(x,D) \le \tau\} = H\left(\tfrac{(b-c)x+\tau}{b}\right) - H\left(\tfrac{(c+h)x-\tau}{h}\right). \tag{1.15}$$

The chance constraint (1.13) becomes

$$H\left(\tfrac{(b-c)x+\tau}{b}\right) - H\left(\tfrac{(c+h)x-\tau}{h}\right) \ge 1-\alpha. \tag{1.16}$$

Even for small (but positive) values of α, it can be a significant relaxation of the corresponding worst-case constraints (1.11).

1.2.3 Multistage Models

Suppose now that the company has a planning horizon of T periods. We model the demand as a random process D_t indexed by the time $t = 1, \dots, T$. At the beginning, at $t = 1$, there is (known) inventory level y_1. At each period $t = 1, \dots, T$, the company first observes the current inventory level y_t and then places an order to replenish the inventory level to x_t. This results in order quantity $x_t - y_t$, which clearly should be nonnegative, i.e., $x_t \ge y_t$. After the inventory is replenished, demand d_t is realized,[2] and hence the next inventory level, at the beginning of period $t+1$, becomes $y_{t+1} = x_t - d_t$. We allow backlogging, and the inventory level y_t may become negative. The total cost incurred in period t is

$$c_t(x_t - y_t) + b_t[d_t - x_t]_+ + h_t[x_t - d_t]_+,$$

where c_t, b_t, h_t are the ordering, backorder penalty, and holding costs per unit, respectively, at time t. We assume that $b_t > c_t > 0$ and $h_t \ge 0$, $t = 1, \dots, T$. The objective is to minimize the expected value of the total cost over the planning horizon. This can be written as the following optimization problem:

$$\begin{aligned} &\operatorname*{Min}_{x_t \ge y_t} \sum_{t=1}^{T} \mathbb{E}\big\{c_t(x_t - y_t) + b_t[D_t - x_t]_+ + h_t[x_t - D_t]_+\big\} \\ &\text{s.t. } y_{t+1} = x_t - D_t,\ t = 1, \dots, T-1. \end{aligned} \tag{1.17}$$

For $T = 1$, problem (1.17) is essentially the same as the (static) problem (1.4). (The only difference is the assumption here of the initial inventory level y_1.) However, for $T > 1$, the situation is more subtle. It is not even clear what is the exact meaning of the formulation (1.17). There are several equivalent ways to give precise meaning to the above problem. One possible way is to write equations describing the dynamics of the corresponding optimization process. That is what we discuss next.

[2] As before, we denote by d_t a particular realization of the random variable D_t.

Consider the demand process D_t, $t = 1, \dots, T$. We denote by $D_{[t]} := (D_1, \dots, D_t)$ the history of the demand process up to time t, and by $d_{[t]} := (d_1, \dots, d_t)$ its particular realization. At each period (stage) t, our decision about the inventory level x_t should depend only on information available at the time of the decision, i.e., on an observed realization $d_{[t-1]}$ of the demand process, and not on future observations. This principle is called the *nonanticipativity* constraint. We assume, however, that the probability distribution of the demand process is known. That is, the conditional probability distribution of D_t, given $D_{[t-1]} = d_{[t-1]}$, is assumed to be known.

At the last stage $t = T$, for observed inventory level y_T, we need to solve the problem

$$\operatorname*{Min}_{x_T \geq y_T} c_T(x_T - y_T) + \mathbb{E}\left\{b_T[D_T - x_T]_+ + h_T[x_T - D_T]_+ \big| D_{[T-1]} = d_{[T-1]}\right\}. \tag{1.18}$$

The expectation in (1.18) is conditional on the realization $d_{[T-1]}$ of the demand process prior to the considered time T. The optimal value (and the set of optimal solutions) of problem (1.18) depends on y_T and $d_{[T-1]}$ and is denoted $Q_T(y_T, d_{[T-1]})$. At stage $t = T - 1$ we solve the problem

$$\begin{aligned} \operatorname*{Min}_{x_{T-1} \geq y_{T-1}} \; & c_{T-1}(x_{T-1} - y_{T-1}) \\ & + \mathbb{E}\Big\{b_{T-1}[D_{T-1} - x_{T-1}]_+ + h_{T-1}[x_{T-1} - D_{T-1}]_+ \\ & \quad + Q_T\left(x_{T-1} - D_{T-1}, D_{[T-1]}\right) \big| D_{[T-2]} = d_{[T-2]}\Big\}. \end{aligned} \tag{1.19}$$

Its optimal value is denoted $Q_{T-1}(y_{T-1}, d_{[T-2]})$. Proceeding in this way backward in time, we write the following *dynamic programming* equations:

$$\begin{aligned} Q_t(y_t, d_{[t-1]}) = \min_{x_t \geq y_t} c_t(x_t - y_t) + \mathbb{E}\Big\{ & b_t[D_t - x_t]_+ \\ & + h_t[x_t - D_t]_+ + Q_{t+1}\left(x_t - D_t, D_{[t]}\right) \big| D_{[t-1]} = d_{[t-1]}\Big\}, \end{aligned} \tag{1.20}$$

$t = T - 1, \dots, 2$. Finally, at the first stage we need to solve the problem

$$\operatorname*{Min}_{x_1 \geq y_1} c_1(x_1 - y_1) + \mathbb{E}\big\{b_1[D_1 - x_1]_+ + h_1[x_1 - D_1]_+ + Q_2(x_1 - D_1, D_1)\big\}. \tag{1.21}$$

Let us take a closer look at the above decision process. We need to understand how the dynamic programming equations (1.19)–(1.21) could be solved and what is the meaning of the solutions. Starting with the last stage, $t = T$, we need to calculate the value functions $Q_t(y_t, d_{[t-1]})$ going backward in time. In the present case, the value functions cannot be calculated in a closed form and should be approximated numerically. For a generally distributed demand process, this could be very difficult or even impossible. The situation simplifies dramatically if we assume that the random process D_t is *stagewise independent*, that is, if D_t is independent of $D_{[t-1]}$, $t = 2, \dots, T$. Then the conditional expectations in equations (1.18)–(1.19) become the corresponding unconditional expectations. Consequently, the value functions $Q_t(y_t)$ do not depend on demand realizations and become functions of the respective univariate variables y_t only. In that case, by discretization of y_t and the (one-dimensional) distribution of D_t, these value functions can be calculated in a recursive way.

Suppose now that somehow we can solve the dynamic programming equations (1.19)–(1.21). Let $\bar{x}_t$ be a corresponding optimal solution, i.e., $\bar{x}_T$ is an optimal solution of (1.18), $\bar{x}_t$ is an optimal solution of the right-hand side of (1.20) for $t = T-1, \dots, 2$, and $\bar{x}_1$ is an optimal solution of (1.21). We see that $\bar{x}_t$ is a function of y_t and $d_{[t-1]}$ for $t = 2, \dots, T$, while the first stage (optimal) decision $\bar{x}_1$ is independent of the data. Under the assumption of stagewise independence, $\bar{x}_t = \bar{x}_t(y_t)$ becomes a function of y_t alone. Note that y_t, in itself, is a function of $d_{[t-1]} = (d_1, \dots, d_{t-1})$ and decisions $(x_1, \dots, x_{t-1})$. Therefore, we may think about a sequence of possible decisions $x_t = x_t(d_{[t-1]})$, $t = 1, \dots, T$, as functions of realizations of the demand process available at the time of the decision (with the convention that x_1 is independent of the data). Such a sequence of decisions $x_t(d_{[t-1]})$ is called an *implementable policy*, or simply a *policy*. That is, an implementable policy is a rule which specifies our decisions, based on information available at the current stage, for any possible realization of the demand process. By definition, an implementable policy $x_t = x_t(d_{[t-1]})$ satisfies the nonanticipativity constraint. A policy is said to be *feasible* if it satisfies other constraints with probability one (w.p. 1). In the present case, a policy is feasible if $x_t \geq y_t$, $t = 1, \dots, T$, for almost every realization of the demand process.

We can now formulate the optimization problem (1.17) as the problem of minimization of the expectation in (1.17) with respect to all implementable feasible policies. An optimal solution of such problem will give us an optimal policy. We have that a policy $\bar{x}_t$ is optimal if it is given by optimal solutions of the respective dynamic programming equations. Note again that under the assumption of stagewise independence, an optimal policy $\bar{x}_t = \bar{x}_t(y_t)$ is a function of y_t alone. Moreover, in that case it is possible to give the following characterization of the optimal policy. Let x_t^* be an (unconstrained) minimizer of

$$c_t x_t + \mathbb{E}\big\{b_t[D_t - x_t]_+ + h_t[x_t - D_t]_+ + Q_{t+1}(x_t - D_t)\big\}, \quad t = T, \dots, 1, \tag{1.22}$$

with the convention that $Q_{T+1}(\cdot) = 0$. Since $Q_{t+1}(\cdot)$ is nonnegative valued and $c_t + h_t > 0$, we have that the function in (1.22) tends to $+\infty$ if $x_t \to +\infty$. Similarly, as $b_t > c_t$, it also tends to $+\infty$ if $x_t \to -\infty$. Moreover, this function is convex and continuous (as long as it is real valued) and hence attains its minimal value. Then by using convexity of the value functions, it is not difficult to show that $\bar{x}_t = \max\{y_t, x_t^*\}$ is an optimal policy. Such policy is called the *basestock* policy. A similar result holds without the assumption of stagewise independence, but then the critical values x_t^* depend on realizations of the demand process up to time $t-1$.

As mentioned above, if the stagewise independence condition is satisfied, then each value function $Q_t(y_t)$ is a function of the variable y_t. In that case, we can accurately represent $Q_t(\cdot)$ by discretization, i.e., by specifying its values at a finite number of points on the real line. Consequently, the corresponding dynamic programming equations can be accurately solved recursively going backward in time. The situation starts to change dramatically with an increase of the number of variables on which the value functions depend, like in the example discussed in the next section. The discretization approach may still work with several state variables, but it quickly becomes impractical when the dimension of the state vector increases. This is called the "curse of dimensionality." As we shall see it later, stochastic programming approaches the problem in a different way, by exploring convexity of the underlying problem and thus attempting to solve problems with a state vector of high dimension. This is achieved by means of discretization of the random process D_t in a form of a scenario tree, which may also become prohibitively large.

1.3 Multiproduct Assembly

1.3.1 Two-Stage Model

Consider a situation where a manufacturer produces n products. There are in total m different parts (or subassemblies) which have to be ordered from third-party suppliers. A unit of product i requires a_{ij} units of part j, where $i = 1, \dots, n$ and $j = 1, \dots, m$. Of course, a_{ij} may be zero for some combinations of i and j. The demand for the products is modeled as a random vector $D = (D_1, \dots, D_n)$. Before the demand is known, the manufacturer may preorder the parts from outside suppliers at a cost of c_j per unit of part j. After the demand D is observed, the manufacturer may decide which portion of the demand is to be satisfied, so that the available numbers of parts are not exceeded. It costs additionally l_i to satisfy a unit of demand for product i, and the unit selling price of this product is q_i. The parts not used are assessed salvage values $s_j < c_j$. The unsatisfied demand is lost.

Suppose the numbers of parts ordered are equal to x_j, $j = 1, \dots, m$. After the demand D becomes known, we need to determine how much of each product to make. Let us denote the numbers of units produced by z_i, $i = 1, \dots, n$, and the numbers of parts left in inventory by y_j, $j = 1, \dots, m$. For an observed value (a realization) $d = (d_1, \dots, d_n)$ of the random demand vector D, we can find the best production plan by solving the following linear programming problem:

$$
\begin{aligned}
\operatorname*{Min}_{z,y}\ & \sum_{i=1}^{n} (l_i - q_i) z_i - \sum_{j=1}^{n} s_j y_j \\
\text{s.t. } & y_j = x_j - \sum_{i=1}^{n} a_{ij} z_i, \quad j = 1, \dots, m, \\
& 0 \le z_i \le d_i, \quad i = 1, \dots, n, \quad y_j \ge 0, \quad j = 1, \dots, m.
\end{aligned}
$$

Introducing the matrix A with entries a_{ij}, where $i = 1, \dots, n$ and $j = 1, \dots, m$, we can write this problem compactly as follows:

$$
\begin{aligned}
\operatorname*{Min}_{z,y}\ & (l - q)^{\mathsf{T}} z - s^{\mathsf{T}} y \\
\text{s.t. } & y = x - A^{\mathsf{T}} z, \\
& 0 \le z \le d, \quad y \ge 0.
\end{aligned}
\tag{1.23}
$$

Observe that the solution of this problem, that is, the vectors z and y, depend on realization d of the demand vector D as well as on x. Let $Q(x, d)$ denote the optimal value of problem (1.23). The quantities x_j of parts to be ordered can be determined from the optimization problem

$$
\operatorname*{Min}_{x \ge 0}\ c^{\mathsf{T}} x + \mathbb{E}[Q(x, D)], \tag{1.24}
$$

where the expectation is taken with respect to the probability distribution of the random demand vector D. The first part of the objective function represents the ordering cost, while the second part represents the expected cost of the optimal production plan, given ordered quantities x. Clearly, for realistic data with $q_i > l_i$, the second part will be negative, so that some profit will be expected.

Problem (1.23)–(1.24) is an example of a *two-stage stochastic programming problem*, where (1.23) is called the *second-stage problem* and (1.24) is called the *first-stage problem*. As the second-stage problem contains random data (random demand D), its optimal value $Q(x, D)$ is a random variable. The distribution of this random variable depends on the first-stage decisions x, and therefore the first-stage problem cannot be solved without understanding of the properties of the second-stage problem.

In the special case of finitely many demand scenarios $d^1, \dots, d^K$ occurring with positive probabilities $p_1, \dots, p_K$, with $\sum_{k=1}^K p_k = 1$, the two-stage problem (1.23)–(1.24) can be written as one large-scale linear programming problem:

$$
\begin{aligned}
\text{Min } & c^{\mathsf{T}}x + \sum_{k=1}^{K} p_k \left[(l-q)^{\mathsf{T}} z^k - s^{\mathsf{T}} y^k\right] \\
\text{s.t. } & y^k = x - A^{\mathsf{T}} z^k, \quad k = 1, \dots, K, \\
& 0 \le z^k \le d^k, \quad y^k \ge 0, \quad k = 1, \dots, K, \\
& x \ge 0,
\end{aligned}
\tag{1.25}
$$

where the minimization is performed over vector variables x and z^k, y^k, $k = 1, \dots, K$. We have integrated the second-stage problem (1.23) into this formulation, but we had to allow for its solution (z^k, y^k) to depend on the scenario k, because the demand realization d^k is different in each scenario. Because of that, problem (1.25) has the numbers of variables and constraints roughly proportional to the number of scenarios K.

It is worth noticing the following. There are three types of decision variables here: the numbers of ordered parts (vector x), the numbers of produced units (vector z), and the numbers of parts left in the inventory (vector y). These decision variables are naturally classified as the *first*- and the *second*-stage decision variables. That is, the first-stage decisions x should be made *before* a realization of the random data becomes available and hence should be independent of the random data, while the second-stage decision variables z and y are made *after* observing the random data and are functions of the data. The first-stage decision variables are often referred to as *here-and-now* decisions (solution), and second-stage decisions are referred to as *wait-and-see* decisions (solution). It can also be noticed that the second-stage problem (1.23) is feasible for every possible realization of the random data; for example, take $z = 0$ and $y = x$. In such a situation we say that the problem has *relatively complete recourse*.

1.3.2 Chance Constrained Model

Suppose now that the manufacturer is concerned with the possibility of losing demand. The manufacturer would like the probability that all demand be satisfied to be larger than some fixed service level $1 - \alpha$, where $\alpha \in (0, 1)$ is small. In this case the problem changes in a significant way.

Observe that if we want to satisfy demand $D = (D_1, \dots, D_n)$, we need to have $x \ge A^{\mathsf{T}} D$. If we have the parts needed, there is no need for the production planning stage, as in problem (1.23). We simply produce $z_i = D_i$, $i = 1, \dots, n$, whenever it is feasible. Also, the production costs and salvage values do not affect our problem. Consequently, the requirement of satisfying the demand with probability at least $1 - \alpha$ leads to the following formulation of the corresponding problem:

$$\begin{aligned}&\operatorname*{Min}_{x\geq 0}\ c^{\mathsf T}x\\ &\text{ s.t. } \Pr\left\{A^{\mathsf T}D\leq x\right\}\geq 1-\alpha.\end{aligned} \tag{1.26}$$

The chance (also called probabilistic) constraint in the above model is more difficult than in the case of the news vendor model considered in section 1.2.2, because it involves a random vector $W = A^{\mathsf T}D$ rather than a univariate random variable.

Owing to the separable nature of the chance constraint in (1.26), we can rewrite this constraint as

$$H_W(x)\geq 1-\alpha, \tag{1.27}$$

where $H_W(x) := \Pr(W \leq x)$ is the cumulative distribution function of the n-dimensional random vector $W = A^{\mathsf T}D$. Observe that if $n = 1$ and $c > 0$, then an optimal solution $\bar{x}$ of (1.27) is given by the left-side $(1-\alpha)$-quantile of W, that is, $\bar{x} = H_W^{-1}(1-\alpha)$. On the other hand, in the case of multidimensional vector W, its distribution has many "smallest (left-side) $(1-\alpha)$-quantiles," and the choice of $\bar{x}$ will depend on the relative proportions of the cost coefficients c_j. It is also worth mentioning that even when the coordinates of the demand vector D are independent, the coordinates of the vector W can be dependent, and thus the chance constraint of (1.27) cannot be replaced by a simpler expression featuring one-dimensional marginal distributions.

The feasible set

$$\left\{x\in\mathbb{R}^m_+ : \Pr\left(A^{\mathsf T}D\leq x\right)\geq 1-\alpha\right\}$$

of problem (1.26) can be written in the following equivalent form:

$$\left\{x\in\mathbb{R}^m_+ : A^{\mathsf T}d\leq x,\ d\in\mathfrak{D},\ \Pr(\mathfrak{D})\geq 1-\alpha\right\}. \tag{1.28}$$

In the formulation (1.28), the set $\mathfrak{D}$ can be *any* measurable subset of $\mathbb{R}^n$ such that probability of $D \in \mathfrak{D}$ is at least $1-\alpha$. A considerable simplification can be achieved by choosing a *fixed* set $\mathfrak{D}_\alpha$ in such a way that $\Pr(\mathfrak{D}_\alpha) \geq 1-\alpha$. In that way we obtain a simplified version of problem (1.26):

$$\begin{aligned}&\operatorname*{Min}_{x\geq 0}\ c^{\mathsf T}x\\ &\text{ s.t. } A^{\mathsf T}d\leq x,\quad \forall\, d\in\mathfrak{D}_\alpha.\end{aligned} \tag{1.29}$$

The set $\mathfrak{D}_\alpha$ in this formulation is sometimes referred to as the *uncertainty set* and the whole formulation as the *robust optimization problem*. Observe that in our case we can solve this problem in the following way. For each part type j we determine x_j to be the minimum number of units necessary to satisfy *every* demand $d \in \mathfrak{D}_\alpha$, that is,

$$x_j = \max_{d\in\mathfrak{D}_\alpha}\sum_{i=1}^n a_{ij}d_i,\quad j=1,\dots,n.$$

In this case the solution is completely determined by the uncertainty set $\mathfrak{D}_\alpha$ and it does not depend on the cost coefficients c_j.

The choice of the uncertainty set, satisfying the corresponding chance constraint, is not unique and often is governed by computational convenience. In this book we shall be mainly concerned with stochastic models, and we shall not discuss models and methods of robust optimization.

1.3.3 Multistage Model

Consider now the situation when the manufacturer has a planning horizon of T periods. The demand is modeled as a stochastic process D_t, $t = 1, \dots, T$, where each $D_t = \big(D_{t1}, \dots, D_{tn}\big)$ is a random vector of demands for the products. The unused parts can be stored from one period to the next, and holding one unit of part j in inventory costs h_j. For simplicity, we assume that all costs and prices are the same in all periods.

It would not be reasonable to plan specific order quantities for the entire planning horizon T. Instead, one has to make orders and production decisions at successive stages, depending on the information available at the current stage. We use symbol $D_{[t]} := \big(D_1, \dots, D_t\big)$ to denote the history of the demand process in periods $1, \dots, t$. In every multistage decision problem it is very important to specify which of the decision variables may depend on which part of the past information.

Let us denote by $x_{t-1} = \big(x_{t-1,1}, \dots, x_{t-1,n}\big)$ the vector of quantities ordered at the beginning of stage t, *before* the demand vector D_t becomes known. The numbers of units produced in stage t will be denoted by z_t and the inventory level of parts at the end of stage t by y_t for $t = 1, \dots, T$. We use the subscript $t-1$ for the order quantity to stress that it may depend on the past demand realizations $D_{[t-1]}$ but not on D_t, while the production and storage variables at stage t may depend on $D_{[t]}$, which includes D_t. In the special case of $T = 1$, we have the two-stage problem discussed in section 1.3.1; the variable x_0 corresponds to the first stage decision vector x, while z_1 and y_1 correspond to the second-stage decision vectors z and y, respectively.

Suppose $T > 1$ and consider the last stage $t = T$, after the demand D_T has been observed. At this time, all inventory levels y_{T-1} of the parts, as well as the last order quantities x_{T-1}, are known. The problem at stage T is therefore identical to the second-stage problem (1.23) of the two-stage formulation:

$$\begin{aligned} \operatorname*{Min}_{z_T, y_T} \ & (l-q)^{\mathsf T} z_T - s^{\mathsf T} y_T \\ \text{s.t. } & y_T = y_{T-1} + x_{T-1} - A^{\mathsf T} z_T, \\ & 0 \le z_T \le d_T, \quad y_T \ge 0, \end{aligned} \tag{1.30}$$

where d_T is the observed realization of D_T. Denote by $Q_T(x_{T-1}, y_{T-1}, d_T)$ the optimal value of (1.30). This optimal value depends on the latest inventory levels, order quantities, and the present demand. At stage $T-1$ we know realization $d_{[T-1]}$ of $D_{[T-1]}$, and thus we are concerned with the conditional expectation of the last stage cost, that is, the function

$$\mathcal{Q}_T(x_{T-1}, y_{T-1}, d_{[T-1]}) := \mathbb{E}\big\{Q_T(x_{T-1}, y_{T-1}, D_T) \,\big|\, D_{[T-1]} = d_{[T-1]}\big\}.$$

At stage $T-1$ we solve the problem

$$\begin{aligned} \operatorname*{Min}_{z_{T-1}, y_{T-1}, x_{T-1}} \ & (l-q)^{\mathsf T} z_{T-1} + h^{\mathsf T} y_{T-1} + c^{\mathsf T} x_{T-1} + \mathcal{Q}_T(x_{T-1}, y_{T-1}, d_{[T-1]}) \\ \text{s.t. } & y_{T-1} = y_{T-2} + x_{T-2} - A^{\mathsf T} z_{T-1}, \\ & 0 \le z_{T-1} \le d_{T-1}, \quad y_{T-1} \ge 0. \end{aligned} \tag{1.31}$$

Its optimal value is denoted by $Q_{T-1}(x_{T-2}, y_{T-2}, d_{[T-1]})$. Generally, the problem at stage $t = T-1, \dots, 1$ has the form

$$\begin{array}{rl} \operatorname*{Min}\limits_{z_t, y_t, x_t} & (l-q)^{\mathsf T} z_t + h^{\mathsf T} y_t + c^{\mathsf T} x_t + \mathcal{Q}_{t+1}(x_t, y_t, d_{[t]}) \\ \text{s.t.} & y_t = y_{t-1} + x_{t-1} - A^{\mathsf T} z_t, \\ & 0 \le z_t \le d_t, \quad y_t \ge 0, \end{array} \tag{1.32}$$

with

$$\mathcal{Q}_{t+1}(x_t, y_t, d_{[t]}) := \mathbb{E}\big\{Q_{t+1}(x_t, y_t, D_{[t+1]}) \bigm| D_{[t]} = d_{[t]}\big\}.$$

The optimal value of problem (1.32) is denoted by $Q_t(x_{t-1}, y_{t-1}, d_{[t]})$, and the backward recursion continues. At stage $t = 1$, the symbol y_0 represents the initial inventory levels of the parts, and the optimal value function $Q_1(x_0, d_1)$ depends only on the initial order x_0 and realization d_1 of the first demand D_1.

The initial problem is to determine the first order quantities x_0. It can be written as

$$\operatorname*{Min}_{x_0 \ge 0} c^{\mathsf T} x_0 + \mathbb{E}[Q_1(x_0, D_1)]. \tag{1.33}$$

Although the first-stage problem (1.33) looks similar to the first-stage problem (1.24) of the two-stage formulation, it is essentially different since the function $Q_1(x_0, d_1)$ is not given in a computationally accessible form but in itself is a result of recursive optimization.

1.4 Portfolio Selection

1.4.1 Static Model

Suppose that we want to invest capital W_0 in n assets, by investing an amount x_i in asset i for $i = 1, \dots, n$. Suppose, further, that each asset has a respective return rate R_i (per one period of time), which is unknown (uncertain) at the time we need to make our decision. We address now a question of how to distribute our wealth W_0 in an optimal way. The total wealth resulting from our investment after one period of time equals

$$W_1 = \sum_{i=1}^n \xi_i x_i,$$

where $\xi_i := 1 + R_i$. We have here the balance constraint $\sum_{i=1}^n x_i \le W_0$. Suppose, further, that one possible investment is cash, so that we can write this balance condition as the equation $\sum_{i=1}^n x_i = W_0$. Viewing returns R_i as random variables, one can try to maximize the expected return on an investment. This leads to the following optimization problem:

$$\operatorname*{Max}_{x \ge 0} \mathbb{E}[W_1] \ \text{ s.t. } \sum_{i=1}^n x_i = W_0. \tag{1.34}$$

We have here that

$$\mathbb{E}[W_1] = \sum_{i=1}^n \mathbb{E}[\xi_i] x_i = \sum_{i=1}^n \mu_i x_i,$$

where $\mu_i := \mathbb{E}[\xi_i] = 1 + \mathbb{E}[R_i]$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Therefore, problem (1.34) has a simple optimal solution of investing everything into an asset with the largest expected return rate and has the optimal value of $\mu^* W_0$, where $\mu^* := \max_{1 \le i \le n} \mu_i$. Of course, from the practical point of view, such a solution is not very appealing. Putting everything into one asset can be very dangerous, because if its realized return rate is bad, one can lose much money.

An alternative approach is to maximize expected utility of the wealth represented by a concave nondecreasing function $U(W_1)$. This leads to the following optimization problem:

$$\operatorname*{Max}_{x \ge 0} \mathbb{E}[U(W_1)] \ \text{ s.t. } \sum_{i=1}^n x_i = W_0. \tag{1.35}$$

This approach requires specification of the utility function. For instance, let $U(W)$ be defined as

$$U(W) := \begin{cases} (1+q)(W-a) & \text{if } W \ge a, \\ (1+r)(W-a) & \text{if } W \le a \end{cases} \tag{1.36}$$

with $r > q > 0$ and $a > 0$. We can view the involved parameters as follows: a is the amount that we have to pay after return on the investment, q is the interest rate at which we can invest the additional wealth $W - a$, provided that $W > a$, and r is the interest rate at which we will have to borrow if W is less than a. For the above utility function, problem (1.35) can be formulated as the following two-stage stochastic linear program:

$$\operatorname*{Max}_{x \ge 0} \mathbb{E}[Q(x, \xi)] \ \text{ s.t. } \sum_{i=1}^n x_i = W_0, \tag{1.37}$$

where $Q(x, \xi)$ is the optimal value of the problem

$$\operatorname*{Max}_{y, z \in \mathbb{R}_+} (1+q)y - (1+r)z \ \text{ s.t. } \sum_{i=1}^n \xi_i x_i = a + y - z. \tag{1.38}$$

We can view the above problem (1.38) as the second-stage program. Given a realization $\xi = (\xi_1, \dots, \xi_n)$ of random data, we make an optimal decision by solving the corresponding optimization problem. Of course, in the present case the optimal value $Q(x, \xi)$ is a function of $W_1 = \sum_{i=1}^n \xi_i x_i$ and can be written explicitly as $U(W_1)$.

Yet another possible approach is to maximize the expected return while controlling the involved risk of the investment. There are several ways in which the concept of risk can be formalized. For instance, we can evaluate risk by variability of W measured by its *variance* $\mathbb{V}\mathrm{ar}[W] = \mathbb{E}[W^2] - (\mathbb{E}[W])^2$. Since W_1 is a linear function of the random variables ξ_i, we have that

$$\mathbb{V}\mathrm{ar}[W_1] = x^{\mathsf{T}} \Sigma x = \sum_{i,j=1}^n \sigma_{ij} x_i x_j,$$

where $\Sigma = [\sigma_{ij}]$ is the covariance matrix of the random vector ξ. (Note that the covariance matrices of the random vectors $\xi = (\xi_1, \dots, \xi_n)$ and $R = (R_1, \dots, R_n)$ are identical.) This leads to the optimization problem of maximizing the expected return subject to the additional constraint $\mathbb{V}\text{ar}[W_1] \le \nu$, where $\nu > 0$ is a specified constant. This problem can be written as

$$\operatorname*{Max}_{x \ge 0} \sum_{i=1}^{n} \mu_i x_i \ \text{ s.t. } \sum_{i=1}^{n} x_i = W_0, \quad x^{\mathsf{T}} \Sigma x \le \nu. \tag{1.39}$$

Since the covariance matrix Σ is positive semidefinite, the constraint $x^{\mathsf{T}} \Sigma x \le \nu$ is convex quadratic, and hence (1.39) is a convex problem. Note that problem (1.39) has at least one feasible solution of investing everything in cash, in which case $\mathbb{V}\text{ar}[W_1] = 0$, and since its feasible set is compact, the problem has an optimal solution. Moreover, since problem (1.39) is convex and satisfies the Slater condition, there is no duality gap between this problem and its dual:

$$\operatorname*{Min}_{\lambda \ge 0} \ \operatorname*{Max}_{\substack{\sum_{i=1}^{n} x_i = W_0 \\ x \ge 0}} \left\{ \sum_{i=1}^{n} \mu_i x_i - \lambda \left(x^{\mathsf{T}} \Sigma x - \nu \right) \right\}. \tag{1.40}$$

Consequently, there exists the Lagrange multiplier $\bar{\lambda} \ge 0$ such that problem (1.39) is equivalent to the problem

$$\operatorname*{Max}_{x \ge 0} \sum_{i=1}^{n} \mu_i x_i - \bar{\lambda} x^{\mathsf{T}} \Sigma x \ \text{ s.t. } \sum_{i=1}^{n} x_i = W_0. \tag{1.41}$$

The equivalence here means that the optimal value of problem (1.39) is equal to the optimal value of problem (1.41) plus the constant $\bar{\lambda}\nu$ and that any optimal solution of problem (1.39) is also an optimal solution of problem (1.41). In particular, if problem (1.41) has unique optimal solution $\bar{x}$, then $\bar{x}$ is also the optimal solution of problem (1.39). The corresponding Lagrange multiplier $\bar{\lambda}$ is given by an optimal solution of the dual problem (1.40). We can view the objective function of the above problem as a compromise between the expected return and its variability measured by its variance.

Another possible formulation is to minimize $\mathbb{V}\text{ar}[W_1]$, keeping the expected return $\mathbb{E}[W_1]$ above a specified value τ. That is,

$$\operatorname*{Min}_{x \ge 0} x^{\mathsf{T}} \Sigma x \ \text{ s.t. } \sum_{i=1}^{n} x_i = W_0, \quad \sum_{i=1}^{n} \mu_i x_i \ge \tau. \tag{1.42}$$

For appropriately chosen constants ν, $\bar{\lambda}$, and τ, problems (1.39)–(1.42) are equivalent to each other. Problems (1.41) and (1.42) are quadratic programming problems, while problem (1.39) can be formulated as a conic quadratic problem. These optimization problems can be efficiently solved. Note finally that these optimization problems are based on the first and second order moments of random data ξ and do not require complete knowledge of the probability distribution of ξ.

We can also approach risk control by imposing chance constraints. Consider the problem

$$\operatorname*{Max}_{x \ge 0} \sum_{i=1}^{n} \mu_i x_i \ \text{ s.t. } \sum_{i=1}^{n} x_i = W_0, \ \Pr\left\{ \sum_{i=1}^{n} \xi_i x_i \ge b \right\} \ge 1 - \alpha. \tag{1.43}$$

That is, we impose the constraint that with probability at least $1 - \alpha$ our wealth $W_1 = \sum_{i=1}^{n} \xi_i x_i$ should not fall below a chosen amount b. Suppose the random vector ξ has a multivariate normal distribution with mean vector μ and covariance matrix Σ, written $\xi \sim \mathcal{N}(\mu, \Sigma)$. Then W_1 has normal distribution with mean $\sum_{i=1}^n \mu_i x_i$ and variance $x^\mathsf{T} \Sigma x$, and

$$\Pr\{W_1 \ge b\} = \Pr\left\{Z \ge \frac{b - \sum_{i=1}^n \mu_i x_i}{\sqrt{x^\mathsf{T} \Sigma x}}\right\} = \Phi\left(\frac{\sum_{i=1}^n \mu_i x_i - b}{\sqrt{x^\mathsf{T} \Sigma x}}\right), \tag{1.44}$$

where $Z \sim \mathcal{N}(0, 1)$ has the standard normal distribution and $\Phi(z) = \Pr(Z \le z)$ is the cdf of Z.

Therefore, we can write the chance constraint of problem (1.43) in the form[3]

$$b - \sum_{i=1}^n \mu_i x_i + z_\alpha \sqrt{x^\mathsf{T} \Sigma x} \le 0, \tag{1.45}$$

where $z_\alpha := \Phi^{-1}(1 - \alpha)$ is the $(1 - \alpha)$-quantile of the standard normal distribution. Note that since matrix Σ is positive semidefinite, $\sqrt{x^\mathsf{T} \Sigma x}$ defines a seminorm on $\mathbb{R}^n$ and is a convex function. Consequently, if $0 < \alpha \le 1/2$, then $z_\alpha \ge 0$ and the constraint (1.45) is convex. Therefore, provided that problem (1.43) is feasible, there exists a Lagrange multiplier $\gamma \ge 0$ such that problem (1.43) is equivalent to the problem

$$\operatorname*{Max}_{x \ge 0} \sum_{i=1}^n \mu_i x_i - \eta \sqrt{x^\mathsf{T} \Sigma x} \ \text{ s.t. } \sum_{i=1}^n x_i = W_0, \tag{1.46}$$

where $\eta = \gamma z_\alpha / (1 + \gamma)$.

In financial engineering the (left-side) $(1 - \alpha)$-quantile of a random variable Y (representing losses) is called *Value-at-Risk*, i.e.,

$$\mathsf{V@R}_\alpha(Y) := H^{-1}(1 - \alpha), \tag{1.47}$$

where $H(\cdot)$ is the cdf of Y. The chance constraint of problem (1.43) can be written in the form of a Value-at-Risk constraint

$$\mathsf{V@R}_\alpha\left(b - \sum_{i=1}^n \xi_i x_i\right) \le 0. \tag{1.48}$$

It is possible to write a chance (Value-at-Risk) constraint here in a closed form because of the assumption of joint normal distribution. Note that in the present case the random variables ξ_i cannot be negative, which indicates that the assumption of normal distribution is not very realistic.

1.4.2 Multistage Portfolio Selection

Suppose we are allowed to rebalance our portfolio in time periods $t = 1, \dots, T - 1$ but without injecting additional cash into it. At each period t we need to make a decision about distribution of our current wealth W_t among n assets. Let $x_0 = (x_{10}, \dots, x_{n0})$ be initial

[3] Note that if $x^\mathsf{T} \Sigma x = 0$, i.e., $\mathbb{V}\mathrm{ar}(W_1) = 0$, then the chance constraint of problem (1.43) holds iff $\sum_{i=1}^n \mu_i x_i \ge b$. In that case equivalence to the constraint (1.45) obviously holds.

amounts invested in the assets. Recall that each x_{i0} is nonnegative and that the balance equation $\sum_{i=1}^n x_{i0} = W_0$ should hold.

We assume now that respective return rates $R_{1t}, \dots, R_{nt}$, at periods $t = 1, \dots, T$, form a random process with a known distribution. Actually, we will work with the (vector valued) random process $\xi_1, \dots, \xi_T$, where $\xi_t = (\xi_{1t}, \dots, \xi_{nt})$ and $\xi_{it} := 1 + R_{it}$, $i = 1, \dots, n$, $t = 1, \dots, T$. At time period $t = 1$ we can rebalance the portfolio by specifying the amounts $x_1 = (x_{11}, \dots, x_{n1})$ invested in the respective assets. At that time, we already know the actual returns in the first period, so it is reasonable to use this information in the rebalancing decisions. Thus, our second-stage decisions, at time $t = 1$, are actually functions of realizations of the random data vector ξ_1, i.e., $x_1 = x_1(\xi_1)$. Similarly, at time t our decision $x_t = (x_{1t}, \dots, x_{nt})$ is a function $x_t = x_t(\xi_{[t]})$ of the available information given by realization $\xi_{[t]} = (\xi_1, \dots, \xi_t)$ of the data process up to time t. A sequence of specific functions $x_t = x_t(\xi_{[t]})$, $t = 0, 1, \dots, T-1$, with x_0 being constant, defines an *implementable policy* of the decision process. It is said that such policy is *feasible* if it satisfies w.p. 1 the model constraints, i.e., the nonnegativity constraints $x_{it}(\xi_{[t]}) \geq 0$, $i = 1, \dots, n$, $t = 0, \dots, T-1$, and the balance of wealth constraints

$$\sum_{i=1}^n x_{it}(\xi_{[t]}) = W_t.$$

At period $t = 1, \dots, T$, our wealth W_t depends on the realization of the random data process and our decisions up to time t and is equal to

$$W_t = \sum_{i=1}^n \xi_{it} x_{i,t-1}(\xi_{[t-1]}).$$

Suppose our objective is to maximize the expected utility of this wealth at the last period, that is, we consider the problem

$$\text{Max } \mathbb{E}[U(W_T)]. \tag{1.49}$$

It is a multistage stochastic programming problem, where stages are numbered from $t = 0$ to $t = T - 1$. Optimization is performed over all implementable and feasible policies.

Of course, in order to complete the description of the problem, we need to define the probability distribution of the random process $R_1, \dots, R_T$. This can be done in many different ways. For example, one can construct a particular scenario tree defining time evolution of the process. If at every stage the random return of each asset is allowed to have just two continuations, independent of other assets, then the total number of scenarios is 2^{nT}. It also should be ensured that $1 + R_{it} \geq 0$, $i = 1, \dots, n$, $t = 1, \dots, T$, for all possible realizations of the random data.

In order to write dynamic programming equations, let us consider the above multistage problem backward in time. At the last stage $t = T-1$, a realization $\xi_{[T-1]} = (\xi_1, \dots, \xi_{T-1})$ of the random process is known and x_{T-2} has been chosen. Therefore, we have to solve the problem

$$\begin{aligned} &\operatorname*{Max}_{x_{T-1}\geq 0, W_T} \ \mathbb{E}\left\{U[W_T]\middle|\xi_{[T-1]}\right\} \\ &\text{s.t. } W_T = \sum_{i=1}^n \xi_{iT} x_{i,T-1}, \ \sum_{i=1}^n x_{i,T-1} = W_{T-1}, \end{aligned} \tag{1.50}$$

where $\mathbb{E}\{U[W_T]|\xi_{[T-1]}\}$ denotes the conditional expectation of $U[W_T]$ given $\xi_{[T-1]}$. The optimal value of the above problem (1.50) depends on W_{T-1} and $\xi_{[T-1]}$ and is denoted $Q_{T-1}(W_{T-1}, \xi_{[T-1]})$.

Continuing in this way, at stage $t = T-2, \dots, 1$, we consider the problem

$$\begin{aligned} \operatorname*{Max}_{x_t \geq 0, W_{t+1}} \; & \mathbb{E}\Big\{ Q_{t+1}(W_{t+1}, \xi_{[t+1]}) \Big| \xi_{[t]} \Big\} \\ \text{s.t. } & W_{t+1} = \sum_{i=1}^{n} \xi_{i,t+1} x_{i,t}, \; \sum_{i=1}^{n} x_{i,t} = W_t, \end{aligned} \tag{1.51}$$

whose optimal value is denoted $Q_t(W_t, \xi_{[t]})$. Finally, at stage $t = 0$ we solve the problem

$$\begin{aligned} \operatorname*{Max}_{x_0 \geq 0, W_1} \; & \mathbb{E}[Q_1(W_1, \xi_1)] \\ \text{s.t. } & W_1 = \sum_{i=1}^{n} \xi_{i1} x_{i0}, \; \sum_{i=1}^{n} x_{i0} = W_0. \end{aligned} \tag{1.52}$$

For a general distribution of the data process ξ_t, it may be hard to solve these dynamic programming equations. The situation simplifies dramatically if the process ξ_t is stagewise independent, i.e., ξ_t is (stochastically) independent of $\xi_1, \dots, \xi_{t-1}$ for $t = 2, \dots, T$. Of course, the assumption of stagewise independence is not very realistic in financial models, but it is instructive to see the dramatic simplifications it allows. In that case, the corresponding conditional expectations become unconditional expectations, and the cost-to-go (value) function $Q_t(W_t)$, $t = 1, \dots, T-1$, does not depend on $\xi_{[t]}$. That is, $Q_{T-1}(W_{T-1})$ is the optimal value of the problem

$$\begin{aligned} \operatorname*{Max}_{x_{T-1} \geq 0, W_T} \; & \mathbb{E}\{U[W_T]\} \\ \text{s.t. } & W_T = \sum_{i=1}^{n} \xi_{iT} x_{i,T-1}, \; \sum_{i=1}^{n} x_{i,T-1} = W_{T-1}, \end{aligned}$$

and $Q_t(W_t)$ is the optimal value of

$$\begin{aligned} \operatorname*{Max}_{x_t \geq 0, W_{t+1}} \; & \mathbb{E}\{Q_{t+1}(W_{t+1})\} \\ \text{s.t. } & W_{t+1} = \sum_{i=1}^{n} \xi_{i,t+1} x_{i,t}, \; \sum_{i=1}^{n} x_{i,t} = W_t \end{aligned}$$

for $t = T-2, \dots, 1$.

The other relevant question is what utility function to use. Let us consider the *logarithmic* utility function $U(W) := \ln W$. Note that this utility function is defined for $W > 0$. For positive numbers a and w and for $W_{T-1} = w$ and $W_{T-1} = aw$, there is a one-to-one correspondence $x_{T-1} \leftrightarrow ax_{T-1}$ between the feasible sets of the corresponding problem (1.50). For the logarithmic utility function, this implies the following relation between the optimal values of these problems:

$$Q_{T-1}(aw, \xi_{[T-1]}) = Q_{T-1}(w, \xi_{[T-1]}) + \ln a. \tag{1.53}$$

That is, at stage $t = T - 1$ we solve the problem

$$\operatorname*{Max}_{x_{T-1}\geq 0} \mathbb{E}\left\{ \ln\left(\sum_{i=1}^{n} \xi_{i,T} x_{i,T-1}\right) \Big| \xi_{[T-1]} \right\} \quad \text{s.t.} \quad \sum_{i=1}^{n} x_{i,T-1} = W_{T-1}. \tag{1.54}$$

By (1.53) its optimal value is

$$Q_{T-1}\left(W_{T-1}, \xi_{[T-1]}\right) = \nu_{T-1}\left(\xi_{[T-1]}\right) + \ln W_{T-1},$$

where $\nu_{T-1}\left(\xi_{[T-1]}\right)$ denotes the optimal value of (1.54) for $W_{T-1} = 1$. At stage $t = T - 2$ we solve the problem

$$\begin{aligned} \operatorname*{Max}_{x_{T-2}\geq 0} & \mathbb{E}\left\{ \nu_{T-1}\left(\xi_{[T-1]}\right) + \ln\left(\sum_{i=1}^{n} \xi_{i,T-1} x_{i,T-2}\right) \Big| \xi_{[T-2]} \right\} \\ \text{s.t.} & \sum_{i=1}^{n} x_{i,T-2} = W_{T-2}. \end{aligned} \tag{1.55}$$

Of course, we have that

$$\begin{aligned} &\mathbb{E}\left\{ \nu_{T-1}\left(\xi_{[T-1]}\right) + \ln\left(\sum_{i=1}^{n} \xi_{i,T-1} x_{i,T-2}\right) \Big| \xi_{[T-2]} \right\} \\ &\qquad = \mathbb{E}\left\{ \nu_{T-1}\left(\xi_{[T-1]}\right) \Big| \xi_{[T-2]} \right\} + \mathbb{E}\left\{ \ln\left(\sum_{i=1}^{n} \xi_{i,T-1} x_{i,T-2}\right) \Big| \xi_{[T-2]} \right\}, \end{aligned}$$

and hence by arguments similar to (1.53), the optimal value of (1.55) can be written as

$$Q_{T-2}\left(W_{T-2}, \xi_{[T-2]}\right) = \mathbb{E}\left\{ \nu_{T-1}\left(\xi_{[T-1]}\right) \Big| \xi_{[T-2]} \right\} + \nu_{T-2}\left(\xi_{[T-2]}\right) + \ln W_{T-2},$$

where $\nu_{T-2}\left(\xi_{[T-2]}\right)$ is the optimal value of the problem

$$\operatorname*{Max}_{x_{T-2}\geq 0} \mathbb{E}\left\{ \ln\left(\sum_{i=1}^{n} \xi_{i,T-1} x_{i,T-2}\right) \Big| \xi_{[T-2]} \right\} \quad \text{s.t.} \quad \sum_{i=1}^{n} x_{i,T-2} = 1.$$

An identical argument applies at earlier stages. Therefore, it suffices to solve at each stage $t = T - 1, \ldots, 1, 0$, the corresponding optimization problem

$$\operatorname*{Max}_{x_t\geq 0} \mathbb{E}\left\{ \ln\left(\sum_{i=1}^{n} \xi_{i,t+1} x_{i,t}\right) \Big| \xi_{[t]} \right\} \quad \text{s.t.} \quad \sum_{i=1}^{n} x_{i,t} = W_t \tag{1.56}$$

in a completely myopic fashion.

By definition, we set ξ_0 to be constant, so that for the first-stage problem, at $t = 0$, the corresponding expectation is unconditional. An optimal solution $\bar{x}_t = \bar{x}_t(W_t, \xi_{[t]})$ of problem (1.56) gives an optimal policy. In particular, the first-stage optimal solution $\bar{x}_0$ is given by an optimal solution of the problem

$$\operatorname*{Max}_{x_0\geq 0} \mathbb{E}\left\{ \ln\left(\sum_{i=1}^{n} \xi_{i1} x_{i0}\right) \right\} \quad \text{s.t.} \quad \sum_{i=1}^{n} x_{i0} = W_0. \tag{1.57}$$

We also have here that the optimal value, denoted ϑ^*, of the optimization problem (1.49) can be written as

$$\vartheta^* = \ln W_0 + \nu_0 + \sum_{t=1}^{T-1} \mathbb{E}\left[\nu_t(\xi_{[t]})\right], \tag{1.58}$$

where $\nu_t(\xi_{[t]})$ is the optimal value of problem (1.56) for $W_t = 1$. Note that $\nu_0 + \ln W_0$ is the optimal value of problem (1.57) with ν_0 being the (deterministic) optimal value of (1.57) for $W_0 = 1$.

If the random process ξ_t is stagewise independent, then conditional expectations in (1.56) are the same as the corresponding unconditional expectations, and hence optimal values $\nu_t(\xi_{[t]}) = \nu_t$ do not depend on $\xi_{[t]}$ and are given by the optimal value of the problem

$$\operatorname*{Max}_{x_t \geq 0} \ \mathbb{E}\left\{\ln\left(\sum_{i=1}^{n} \xi_{i,t+1} x_{i,t}\right)\right\} \ \text{ s.t. } \sum_{i=1}^{n} x_{i,t} = 1. \tag{1.59}$$

Also in the stagewise independent case, the optimal policy can be described as follows. Let $x_t^* = (x_{1t}^*, \dots, x_{nt}^*)$ be the optimal solution of (1.59), $t = 0, \dots, T-1$. Such optimal solution is unique by strict concavity of the logarithm function. Then

$$\bar{x}_t(W_t) := W_t x_t^*, \quad t = 0, \dots, T-1,$$

defines the optimal policy.

Consider now the *power* utility function $U(W) := W^\gamma$ with $1 \geq \gamma > 0$, defined for $W \geq 0$. Suppose again that the random process ξ_t is *stagewise independent*. Recall that this condition implies that the cost-to-go function $Q_t(W_t)$, $t = 1, \dots, T-1$, depends only on W_t. By using arguments similar to the analysis for the logarithmic utility function, it is not difficult to show that $Q_{T-1}(W_{T-1}) = W_{T-1}^\gamma Q_{T-1}(1)$, and so on. The optimal policy $\bar{x}_t = \bar{x}_t(W_t)$ is obtained in a myopic way as an optimal solution of the problem

$$\operatorname*{Max}_{x_t \geq 0} \mathbb{E}\left\{\left(\sum_{i=1}^{n} \xi_{i,t+1} x_{it}\right)^\gamma\right\} \ \text{ s.t. } \sum_{i=1}^{n} x_{it} = W_t. \tag{1.60}$$

That is, $\bar{x}_t(W_t) = W_t x_t^*$, where x_t^* is an optimal solution of problem (1.60) for $W_t = 1$, $t = 0, \dots, T-1$. In particular, the first-stage optimal solution $\bar{x}_0$ is obtained in a myopic way by solving the problem

$$\operatorname*{Max}_{x_0 \geq 0} \mathbb{E}\left\{\left(\sum_{i=1}^{n} \xi_{i1} x_{i0}\right)^\gamma\right\} \ \text{ s.t. } \sum_{i=1}^{n} x_{i0} = W_0.$$

The optimal value ϑ^* of the corresponding multistage problem (1.49) is

$$\vartheta^* = W_0^\gamma \prod_{t=0}^{T-1} \eta_t, \tag{1.61}$$

where η_t is the optimal value of problem (1.60) for $W_t = 1$.

The above myopic behavior of multistage stochastic programs is rather exceptional. A more realistic situation occurs in the presence of transaction costs. These are costs associated with the changes in the numbers of units (stocks, bonds) held. Introduction of transaction costs will destroy such myopic behavior of optimal policies.

1.4.3 Decision Rules

Consider the following policy. Let $x_t^* = (x_{1t}^*, \dots, x_{nt}^*)$, $t = 0, \dots, T-1$, be vectors such that $x_t^* \geq 0$ and $\sum_{i=1}^n x_{it}^* = 1$. Define the *fixed mix* policy

$$x_t(W_t) := W_t x_t^*, \quad t = 0, \dots, T-1. \tag{1.62}$$

As discussed above, under the assumption of stagewise independence, such policies are optimal for the logarithmic and power utility functions provided that x_t^* are optimal solutions of the respective problems (problem (1.59) for the logarithmic utility function and problem (1.60) with $W_t = 1$ for the power utility function). In other problems, a policy of form (1.62) may be nonoptimal. However, it is readily implementable, once the current wealth W_t is observed. As mentioned, rules for calculating decisions as functions of the observations gathered up to time t, similar to (1.62), are called policies or alternatively *decision rules*.

We analyze now properties of the decision rule (1.62) under the simplifying assumption of stagewise independence. We have

$$W_{t+1} = \sum_{i=1}^n \xi_{i,t+1} x_{it}(W_t) = W_t \sum_{i=1}^n \xi_{i,t+1} x_{it}^*. \tag{1.63}$$

Since the random process $\xi_1, \dots, \xi_T$ is stagewise independent, by independence of ξ_{t+1} and W_t we have

$$\mathbb{E}[W_{t+1}] = \mathbb{E}[W_t]\mathbb{E}\left(\sum_{i=1}^n \xi_{i,t+1} x_{it}^*\right) = \mathbb{E}[W_t] \underbrace{\sum_{i=1}^n \mu_{i,t+1} x_{it}^*}_{x_t^{*\mathsf{T}} \mu_{t+1}}, \tag{1.64}$$

where $\mu_t := \mathbb{E}[\xi_t]$. Consequently, by induction,

$$\mathbb{E}[W_t] = \prod_{\tau=1}^t \left(\sum_{i=1}^n \mu_{i\tau} x_{i,\tau-1}^*\right) = \prod_{\tau=1}^t \left(x_{\tau-1}^{*\mathsf{T}} \mu_\tau\right).$$

In order to calculate the variance of W_t we use the formula

$$\mathbb{V}\mathrm{ar}(Y) = \mathbb{E}(\underbrace{\mathbb{E}[(Y - \mathbb{E}(Y|X))^2|X]}_{\mathbb{V}\mathrm{ar}(Y|X)}) + \underbrace{\mathbb{E}([\mathbb{E}(Y|X) - \mathbb{E}Y]^2)}_{\mathbb{V}\mathrm{ar}[\mathbb{E}(Y|X)]}, \tag{1.65}$$

where X and Y are random variables. Applying (1.65) to (1.63) with $Y := W_{t+1}$ and $X := W_t$ we obtain

$$\mathbb{V}\mathrm{ar}[W_{t+1}] = \mathbb{E}[W_t^2]\mathbb{V}\mathrm{ar}\left(\sum_{i=1}^n \xi_{i,t+1} x_{it}^*\right) + \mathbb{V}\mathrm{ar}[W_t]\left(\sum_{i=1}^n \mu_{i,t+1} x_{it}^*\right)^2. \tag{1.66}$$

Recall that $\mathbb{E}[W_t^2] = \mathbb{V}\mathrm{ar}[W_t] + (\mathbb{E}[W_t])^2$ and $\mathbb{V}\mathrm{ar}\left(\sum_{i=1}^n \xi_{i,t+1} x_{it}^*\right) = x_t^{*\mathsf{T}} \Sigma_{t+1} x_t^*$, where Σ_{t+1} is the covariance matrix of ξ_{t+1}.

It follows from (1.64) and (1.66) that

$$\frac{\mathbb{V}\mathrm{ar}[W_{t+1}]}{(\mathbb{E}[W_{t+1}])^2} = \frac{x_t^{*\mathsf{T}} \Sigma_{t+1} x_t^*}{(x_t^{*\mathsf{T}} \mu_{t+1})^2} + \frac{\mathbb{V}\mathrm{ar}[W_t]}{(\mathbb{E}[W_t])^2} \tag{1.67}$$

and hence

$$\frac{\mathbb{V}\mathrm{ar}[W_t]}{(\mathbb{E}[W_t])^2} = \sum_{\tau=1}^{t} \frac{\mathbb{V}\mathrm{ar}\left(\sum_{i=1}^n \xi_{i,\tau} x^*_{i,\tau-1}\right)}{\left(\sum_{i=1}^n \mu_{i\tau} x^*_{i,\tau-1}\right)^2} = \sum_{\tau=1}^{t} \frac{x^{*\mathsf{T}}_{\tau-1} \Sigma_\tau x^*_{\tau-1}}{(x^{*\mathsf{T}}_{\tau-1}\mu_\tau)^2}, \quad t = 1, \dots, T. \tag{1.68}$$

This shows that if the terms $x^{*\mathsf{T}}_{\tau-1} \Sigma_\tau x^*_{\tau-1} / (x^{*\mathsf{T}}_{\tau-1}\mu_\tau)^2$ are of the same order for $\tau = 1, \dots, T$, then the ratio of the standard deviation $\sqrt{\mathbb{V}\mathrm{ar}[W_T]}$ to the expected wealth $\mathbb{E}[W_T]$ is of order $O(\sqrt{T})$ with an increase in the number of stages T.

1.5 Supply Chain Network Design

In this section we discuss a stochastic programming approach to modeling a supply chain network design. A supply chain is a network of suppliers, manufacturing plants, warehouses, and distribution channels organized to acquire raw materials, convert these raw materials to finished products, and distribute these products to customers. We first describe a deterministic mathematical formulation for the supply chain design problem.

Denote by $\mathcal{S}$, $\mathcal{P}$, and $\mathcal{C}$ the respective (finite) sets of suppliers, processing facilities, and customers. The union $\mathcal{N} := \mathcal{S} \cup \mathcal{P} \cup \mathcal{C}$ of these sets is viewed as the set of nodes of a directed graph $(\mathcal{N}, \mathcal{A})$, where $\mathcal{A}$ is a set of arcs (directed links) connecting these nodes in a way representing flow of the products. The processing facilities include manufacturing centers $\mathcal{M}$, finishing facilities $\mathcal{F}$, and warehouses $\mathcal{W}$, i.e., $\mathcal{P} = \mathcal{M} \cup \mathcal{F} \cup \mathcal{W}$. Further, a manufacturing center $i \in \mathcal{M}$ or a finishing facility $i \in \mathcal{F}$ consists of a set of manufacturing or finishing machines $\mathcal{H}_i$. Thus the set $\mathcal{P}$ includes the processing centers as well as the machines in these centers. Let $\mathcal{K}$ be the set of products flowing through the supply chain.

The supply chain configuration decisions consist of deciding which of the processing centers to build (major configuration decisions) and which processing and finishing machines to procure (minor configuration decisions). We assign a binary variable $x_i = 1$ if a processing facility i is built or machine i is procured, and $x_i = 0$ otherwise. The operational decisions consist of routing the flow of product $k \in \mathcal{K}$ from the supplier to the customers. By y^k_{ij} we denote the flow of product k from a node i to a node j of the network, where $(i, j) \in \mathcal{A}$. A deterministic mathematical model for the supply chain design problem can be written as follows:

$$\operatorname*{Min}_{x,y} \sum_{i\in\mathcal{P}} c_i x_i + \sum_{k\in\mathcal{K}} \sum_{(i,j)\in\mathcal{A}} q^k_{ij} y^k_{ij} \tag{1.69}$$

$$\text{s.t.} \sum_{i\in\mathcal{N}} y^k_{ij} - \sum_{\ell\in\mathcal{N}} y^k_{j\ell} = 0, \quad j \in \mathcal{P},\ k \in \mathcal{K}, \tag{1.70}$$

$$\sum_{i\in\mathcal{N}} y^k_{ij} \ge d^k_j, \quad j \in \mathcal{C},\ k \in \mathcal{K}, \tag{1.71}$$

$$\sum_{i \in \mathcal{N}} y_{ij}^k \le s_j^k, \quad j \in \mathcal{S},\ k \in \mathcal{K}, \tag{1.72}$$

$$\sum_{k \in \mathcal{K}} r_j^k \left(\sum_{i \in \mathcal{N}} y_{ij}^k \right) \le m_j x_j, \ \ j \in \mathcal{P}, \tag{1.73}$$

$$x \in \mathcal{X},\ \ y \ge 0. \tag{1.74}$$

Here c_i denotes the investment cost for building facility i or procuring machine i, q_{ij}^k denotes the per-unit cost of processing product k at facility i and/or transporting product k on arc $(i, j) \in \mathcal{A}$, d_j^k denotes the demand of product k at node j, s_j^k denotes the supply of product k at node j, r_j^k denotes per-unit processing requirement for product k at node j, m_j denotes capacity of facility j, $\mathcal{X} \subset \{0, 1\}^{|\mathcal{P}|}$ is a set of binary variables, and $y \in \mathbb{R}^{|\mathcal{A}| \times |\mathcal{K}|}$ is a vector with components y_{ij}^k. All cost components are annualized.

The objective function (1.69) is aimed at minimizing total investment and operational costs. Of course, a similar model can be constructed for maximizing profits. The set $\mathcal{X}$ represents logical dependencies and restrictions, such as $x_i \le x_j$ for all $i \in \mathcal{H}_j$ and $j \in \mathcal{P}$ or $j \in \mathcal{F}$, i.e., machine $i \in \mathcal{H}_j$ should be procured only if facility j is built (since x_i are binary, the constraint $x_i \le x_j$ means that $x_i = 0$ if $x_j = 0$). Constraints (1.70) enforce the flow conservation of product k across each processing node j. Constraints (1.71) require that the total flow of product k to a customer node j should exceed the demand d_j^k at that node. Constraints (1.72) require that the total flow of product k from a supplier node j should be less than the supply s_j^k at that node. Constraints (1.73) enforce capacity constraints of the processing nodes. The capacity constraints then require that the total processing requirement of all products flowing into a processing node j should be smaller than the capacity m_j of facility j if it is built ($x_j = 1$). If facility j is not built ($x_j = 0$), the constraint will force all flow variables $y_{ij}^k = 0$ for all $i \in \mathcal{N}$. Finally, constraint (1.74) enforces feasibility constraint $x \in \mathcal{X}$ and the nonnegativity of the flow variables corresponding to an arc $(ij) \in \mathcal{A}$ and product $k \in \mathcal{K}$.

It will be convenient to write problem (1.69)–(1.74) in the following compact form:

$$\underset{x \in \mathcal{X},\ y \ge 0}{\text{Min}} \ c^{\mathsf{T}} x + q^{\mathsf{T}} y \tag{1.75}$$

$$\text{s.t.} \quad Ny = 0, \tag{1.76}$$

$$Cy \ge d, \tag{1.77}$$

$$Sy \le s, \tag{1.78}$$

$$Ry \le Mx, \tag{1.79}$$

where vectors c, q, d, and s correspond to investment costs, processing/transportation costs, demands, and supplies, respectively; matrices N, C, and S are appropriate matrices corresponding to the summations on the left-hand side of the respective expressions. The notation R corresponds to a matrix of r_j^k, and the notation M corresponds to a matrix with m_j along the diagonal.

It is realistic to assume that at the time at which a decision about vector $x \in \mathcal{X}$ should be made, i.e., which facilities to built and machines to procure, there is an uncertainty about parameters involved in operational decisions represented by vector $y \in \mathbb{R}^{|\mathcal{A}| \times |\mathcal{K}|}$. This naturally classifies decision variables x as the first-stage decision variables and y as the second-stage decision variables. Note that problem (1.75)–(1.79) can be written in the following equivalent form as a two-stage program:

$$\operatorname*{Min}_{x\in\mathcal{X}} c^{\mathsf{T}}x + Q(x,\xi), \tag{1.80}$$

where $Q(x,\xi)$ is the optimal value of the second-stage problem

$$\begin{aligned} &\operatorname*{Min}_{y\geq 0} q^{\mathsf{T}}y && (1.81)\\ &\text{s.t. } Ny = 0, && (1.82)\\ &\quad\ Cy \geq d, && (1.83)\\ &\quad\ Sy \leq s, && (1.84)\\ &\quad\ Ry \leq Mx && (1.85)\end{aligned}$$

with $\xi = (q,d,s,R,M)$ being the vector of the involved parameters. Of course, the above optimization problem depends on the data vector ξ. If some of the data parameters are uncertain, then the deterministic problem (1.80) does not make much sense since it depends on unknown parameters.

Suppose now that we can model uncertain components of the data vector ξ as random variables with a specified joint probability distribution. Then we can formulate the stochastic programming problem

$$\operatorname*{Min}_{x\in\mathcal{X}} c^{\mathsf{T}}x + \mathbb{E}[Q(x,\xi)], \tag{1.86}$$

where the expectation is taken with respect to the probability distribution of the random vector ξ. That is, the cost of the second-stage problem enters the objective of the first-stage problem *on average*. A distinctive feature of the stochastic programming problem (1.86) is that the first-stage problem here is a combinatorial problem with binary decision variables and finite feasible set $\mathcal{X}$. On the other hand, the second-stage problem (1.81)–(1.85) is a linear programming problem and its optimal value $Q(x,\xi)$ is convex in x (if x is viewed as a vector in $\mathbb{R}^{|\mathcal{P}|}$).

It could happen that for some $x \in \mathcal{X}$ and some realizations of the data ξ, the corresponding second-stage problem (1.81)–(1.85) is infeasible, i.e., the constraints (1.82)–(1.85) define an empty set. In that case, by definition, $Q(x,\xi) = +\infty$, i.e., we apply an infinite penalization for infeasibility of the second-stage problem. For example, it could happen that demand d is not satisfied, i.e., $Cy \leq d$ with some inequalities strict, for any $y \geq 0$ satisfying constraints (1.82), (1.84), and (1.85). Sometimes this can be resolved by a recourse action. That is, if demand is not satisfied, then there is a possibility of supplying the deficit $d - Cy$ at a penalty cost. This can be modeled by writing the second-stage problem in the form

$$\begin{aligned} &\operatorname*{Min}_{y\geq 0, z\geq 0} q^{\mathsf{T}}y + h^{\mathsf{T}}z && (1.87)\\ &\text{s.t. } Ny = 0, && (1.88)\\ &\quad\ Cy + z \geq d, && (1.89)\\ &\quad\ Sy \leq s, && (1.90)\\ &\quad\ Ry \leq Mx, && (1.91)\end{aligned}$$

where h represents the vector of (positive) recourse costs. Note that the above problem (1.87)–(1.91) is always feasible, for example, $y = 0$ and $z \geq d$ clearly satisfy the constraints of this problem.

Exercises

1.1. Consider the expected value function $f(x) := \mathbb{E}[F(x, D)]$, where function $F(x, d)$ is defined in (1.1). (i) Show that function $F(x, d)$ is convex in x and hence that $f(x)$ is also convex. (ii) Show that $f(\cdot)$ is differentiable at a point $x > 0$ iff the cdf $H(\cdot)$ of D is continuous at x.

1.2. Let $H(z)$ be the cdf of a random variable Z and $\kappa \in (0, 1)$. Show that the minimum in the definition $H^{-1}(\kappa) = \inf\{t : H(t) \geq \kappa\}$ of the left-side quantile is always attained.

1.3. Consider the chance constrained problem discussed in section 1.2.2. (i) Show that system (1.11) has no feasible solution if there is a realization of d greater than τ/c. (ii) Verify equation (1.15). (iii) Assume that the probability distribution of the demand D is supported on an interval $[l, u]$ with $0 \leq l \leq u < +\infty$. Show that if the significance level $\alpha = 0$, then the constraint (1.16) becomes

$$\frac{bu - \tau}{b - c} \leq x \leq \frac{hl + \tau}{c + h}$$

and hence is equivalent to (1.11) for $\mathfrak{D} = [l, u]$.

1.4. Show that the optimal value functions $Q_t(y_t, d_{[t-1]})$, defined in (1.20), are convex in y_t.

1.5. Assuming the stagewise independence condition, show that the basestock policy $\bar{x}_t = \max\{y_t, x_t^*\}$, for the inventory model, is optimal (recall that x_t^* denotes a minimizer of (1.22)).

1.6. Consider the assembly problem discussed in section 1.3.1 in the case when all demand has to be satisfied, by making additional orders of the missing parts. In this case, the cost of each additionally ordered part j is $r_j > c_j$. Formulate the problem as a linear two-stage stochastic programming problem.

1.7. Consider the assembly problem discussed in section 1.3.3 in the case when all demand has to be satisfied, by backlogging the excessive demand, if necessary. In this case, it costs b_i to delay delivery of a unit of product i by one period. Additional orders of the missing parts can be made after the last demand D_T becomes known. Formulate the problem as a linear multistage stochastic programming problem.

1.8. Show that for utility function $U(W)$, of the form (1.36), problems (1.35) and (1.37)–(1.38) are equivalent.

1.9. Show that variance of the random return $W_1 = \xi^{\mathsf{T}}x$ is given by formula $\mathbb{V}\mathrm{ar}[W_1] = x^{\mathsf{T}}\Sigma x$, where $\Sigma = \mathbb{E}\left[(\xi - \mu)(\xi - \mu)^{\mathsf{T}}\right]$ is the covariance matrix of the random vector ξ and $\mu = \mathbb{E}[\xi]$.

1.10. Show that the optimal value function $Q_t(W_t, \xi_{[t]})$, defined in (1.51), is convex in W_t.

1.11. Let D be a random variable with cdf $H(t) = \Pr(D \leq t)$ and $D^1, \dots, D^N$ be an iid random sample of D with the corresponding empirical cdf $\widehat{H}_N(\cdot)$. Let $a = H^{-1}(\kappa)$ and $b = \sup\{t : H(t) \leq \kappa\}$ be respective left- and right-side κ-quantiles of $H(\cdot)$. Show that $\min\{|\widehat{H}_N^{-1}(\kappa) - a|, |\widehat{H}_N^{-1}(\kappa) - b|\}$ tends w.p. 1 to 0 as $N \to \infty$.