Chapter 3

Multistage Problems

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3.1 Problem Formulation

3.1.1 The General Setting

The two-stage stochastic programming models can be naturally extended to a multistage setting. We discussed examples of such decision processes in sections 1.2.3 and 1.4.2 for a multistage inventory model and a multistage portfolio selection problem, respectively. In the multistage setting, the uncertain data $\xi_1, \ldots, \xi_T$ is revealed gradually over time, in T periods, and our decisions should be adapted to this process. The decision process has the form

$$\text{decision } (x_1) \rightsquigarrow \text{observation } (\xi_2) \rightsquigarrow \text{decision } (x_2) \rightsquigarrow \cdots \rightsquigarrow \text{observation } (\xi_T) \rightsquigarrow \text{decision } (x_T).$$

We view the sequence $\xi_t \in \mathbb{R}^{d_t}$, $t = 1, \ldots, T$, of data vectors as a *stochastic process*, i.e., as a sequence of random variables with a specified probability distribution. We use notation $\xi_{[t]} := (\xi_1, \ldots, \xi_t)$ to denote the history of the process up to time t.

The values of the decision vector x_t, chosen at stage t, may depend on the information (data) $\xi_{[t]}$ available up to time t, but not on the results of future observations. This is the basic requirement of *nonanticipativity*. As x_t may depend on $\xi_{[t]}$, the sequence of decisions is a stochastic process as well.

We say that the process $\{\xi_t\}$ is *stagewise independent* if ξ_t is stochastically independent of $\xi_{[t-1]}$, $t = 2, \ldots, T$. It is said that the process is *Markovian* if for every $t = 2, \ldots, T$, the conditional distribution of ξ_t given $\xi_{[t-1]}$ is the same as the conditional distribution of ξ_t given ξ_{t-1}. Of course, if the process is stagewise independent, then it is Markovian. As before, we often use the same notation ξ_t to denote a random vector and its particular

realization. Which of these two meanings will be used in a particular situation will be clear from the context.

In a generic form a T-stage stochastic programming problem can be written in the nested formulation

$$\operatorname*{Min}_{x_1\in\mathcal{X}_1} f_1(x_1)+\mathbb{E}\left[\inf_{x_2\in\mathcal{X}_2(x_1,\xi_2)} f_2(x_2,\xi_2)+\mathbb{E}\Big[\cdots+\mathbb{E}\big[\inf_{x_T\in\mathcal{X}_T(x_{T-1},\xi_T)} f_T(x_T,\xi_T)\big]\Big]\right], \quad (3.1)$$

driven by the random data process $\xi_1, \xi_2, \ldots, \xi_T$. Here $x_t \in \mathbb{R}^{n_t}$, $t = 1, \ldots, T$, are decision variables, $f_t : \mathbb{R}^{n_t} \times \mathbb{R}^{d_t} \to \mathbb{R}$ are continuous functions and $\mathcal{X}_t : \mathbb{R}^{n_{t-1}} \times \mathbb{R}^{d_t} \rightrightarrows \mathbb{R}^{n_t}$, $t = 2, \ldots, T$, are measurable closed valued multifunctions. The first-stage data, i.e., the vector ξ_1, the function $f_1 : \mathbb{R}^{n_1} \to \mathbb{R}$, and the set $\mathcal{X}_1 \subset \mathbb{R}^{n_1}$ are deterministic. It is said that the multistage problem is *linear* if the objective functions and the constraint functions are linear. In a typical formulation,

$$f_t(x_t,\xi_t) := c_t^{\mathrm{T}} x_t, \quad \mathcal{X}_1 := \{x_1 : A_1 x_1 = b_1,\ x_1 \geq 0\},$$
$$\mathcal{X}_t(x_{t-1},\xi_t) := \{x_t : B_t x_{t-1} + A_t x_t = b_t,\ x_t \geq 0\},\ t = 2, \ldots, T.$$

Here, $\xi_1 := (c_1, A_1, b_1)$ is known at the first-stage (and hence is nonrandom), and $\xi_t := (c_t, B_t, A_t, b_t) \in \mathbb{R}^{d_t}$, $t = 2, \ldots, T$, are data vectors,[10] some (or all) elements of which can be random.

There are several equivalent ways to make this formulation precise. One approach is to consider decision variables $x_t = \boldsymbol{x}_t(\xi_{[t]})$, $t = 1, \ldots, T$, as functions of the data process $\xi_{[t]}$ up to time t. Such a sequence of (measurable) mappings $\boldsymbol{x}_t : \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_t} \to \mathbb{R}^{n_t}$, $t = 1, \ldots, T$, is called an *implementable policy* (or simply a *policy*) (recall that ξ_1 is deterministic). An implementable policy is said to be *feasible* if it satisfies the feasibility constraints, i.e.,

$$\boldsymbol{x}_t(\xi_{[t]}) \in \mathcal{X}_t(\boldsymbol{x}_{t-1}(\xi_{[t-1]}),\xi_t),\ t = 2, \ldots, T,\ \text{w.p. 1}. \quad (3.2)$$

We can formulate the multistage problem (3.1) in the form

$$\begin{array}{ll} \operatorname*{Min}\limits_{x_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_T} & \mathbb{E}\big[f_1(x_1) + f_2(\boldsymbol{x}_2(\xi_{[2]}),\xi_2) + \cdots + f_T\left(\boldsymbol{x}_T(\xi_{[T]}),\xi_T\right)\big] \\ \text{s.t.} & x_1 \in \mathcal{X}_1,\ \boldsymbol{x}_t(\xi_{[t]}) \in \mathcal{X}_t(\boldsymbol{x}_{t-1}(\xi_{[t-1]}),\xi_t),\ t = 2, \ldots, T. \end{array} \quad (3.3)$$

Note that optimization in (3.3) is performed over implementable and feasible policies and that policies $\boldsymbol{x}_2, \ldots, \boldsymbol{x}_T$ are *functions* of the data process, and hence are elements of appropriate functional spaces, while $x_1 \in \mathbb{R}^{n_1}$ is a deterministic vector. Therefore, unless the data process $\xi_1, \ldots, \xi_T$ has a finite number of realizations, formulation (3.3) leads to an infinite dimensional optimization problem. This is a natural extension of the formulation (2.66) of the two-stage problem.

Another possible way is to write the corresponding *dynamic programming* equations. That is, consider the last-stage problem

$$\operatorname*{Min}_{x_T\in\mathcal{X}_T(x_{T-1},\xi_T)} f_T(x_T,\xi_T).$$

[10] If data involves matrices, then their elements can be stacked columnwise to make it a vector.

The optimal value of this problem, denoted $Q_T(x_{T-1}, \xi_T)$, depends on the decision vector x_{T-1} and data ξ_T. At stage $t = 2, \ldots, T-1$, we formulate the problem

$$\begin{aligned} &\operatorname*{Min}_{x_t} \; f_t(x_t, \xi_t) + \mathbb{E}\left\{ Q_{t+1}\left(x_t, \xi_{[t+1]}\right) \middle| \xi_{[t]} \right\} \\ &\text{s.t. } x_t \in \mathcal{X}_t(x_{t-1}, \xi_t), \end{aligned}$$

where $\mathbb{E}\left[\cdot \,|\xi_{[t]}\right]$ denotes conditional expectation. Its optimal value depends on the decision x_{t-1} at the previous stage and realization of the data process $\xi_{[t]}$, and denoted $Q_t\left(x_{t-1}, \xi_{[t]}\right)$. The idea is to calculate the *cost-to-go* (or *value*) functions $Q_t\left(x_{t-1}, \xi_{[t])}\right)$, recursively, going backward in time. At the first stage we finally need to solve the problem:

$$\operatorname*{Min}_{x_1 \in \mathcal{X}_1} f_1(x_1) + \mathbb{E}\left[Q_2\left(x_1, \xi_2\right)\right].$$

The corresponding dynamic programming equations are

$$Q_t\left(x_{t-1}, \xi_{[t]}\right) = \inf_{x_t \in \mathcal{X}_t(x_{t-1}, \xi_t)} \left\{ f_t(x_t, \xi_t) + \mathcal{Q}_{t+1}\left(x_t, \xi_{[t]}\right) \right\}, \tag{3.4}$$

where

$$\mathcal{Q}_{t+1}\left(x_t, \xi_{[t]}\right) := \mathbb{E}\left\{ Q_{t+1}\left(x_t, \xi_{[t+1]}\right) \middle| \xi_{[t]} \right\}.$$

An implementable policy $\bar{\boldsymbol{x}}_t(\xi_{[t]})$ is *optimal* iff for $t = 1, \ldots, T$,

$$\bar{\boldsymbol{x}}_t(\xi_{[t]}) \in \arg \min_{x_t \in \mathcal{X}_t\left(\bar{\boldsymbol{x}}_{t-1}(\xi_{[t-1]}), \xi_t\right)} \left\{ f_t(x_t, \xi_t) + \mathcal{Q}_{t+1}\left(x_t, \xi_{[t]}\right) \right\}, \quad \text{w.p. 1}, \tag{3.5}$$

where for $t = T$ the term $\mathcal{Q}_{T+1}$ is omitted and for $t = 1$ the set X_1 depends only on ξ_1. In the dynamic programming formulation the problem is reduced to solving a family of finite dimensional problems, indexed by t and by $\xi_{[t]}$. It can be viewed as an extension of the formulation (2.61)–(2.62) of the two-stage problem.

If the process $\xi_1, \ldots, \xi_T$ is Markovian, then conditional distributions in the above equations, given $\xi_{[t]}$, are the same as the respective conditional distributions given ξ_t. In that case each cost-to-go function Q_t depends on ξ_t rather than the whole $\xi_{[t]}$ and we can write it as $Q_t\left(x_{t-1}, \xi_t\right)$. If, moreover, the stagewise independence condition holds, then each expectation function $\mathcal{Q}_t$ does not depend on realizations of the random process, and we can write it simply as $\mathcal{Q}_t\left(x_{t-1}\right)$.

3.1.2 The Linear Case

We discuss linear multistage problems in more detail. Let $x_1, \ldots, x_T$ be decision vectors corresponding to time periods (stages) $1, \ldots, T$. Consider the following linear programming problem:

$$\begin{array}{lccccccccccc} \text{Min} & c_1^{\mathrm{T}} x_1 & + & c_2^{\mathrm{T}} x_2 & + & c_3^{T} x_3 & + & \ldots & + & c_T^{\mathrm{T}} x_T & & \\ \text{s.t.} & A_1 x_1 & & & & & & & & & = & b_1, \\ & B_2 x_1 & + & A_2 x_2 & & & & & & & = & b_2, \\ & & & B_3 x_2 & + & A_3 x_3 & & & & & = & b_3, \\ & \multicolumn{11}{c}{\dotfill} \\ & & & & & & & B_T x_{T-1} & + & A_T x_T & = & b_T, \\ & x_1 \geq 0, & & x_2 \geq 0, & & x_3 \geq 0, & & \ldots & & x_T \geq 0. & & \end{array} \tag{3.6}$$

We can view this problem as a multiperiod stochastic programming problem where c_1, A_1 and b_1 are known, but some (or all) the entries of the cost vectors c_t, matrices B_t and A_t, and right-hand-side vectors b_t, $t = 2, \ldots, T$, are random. In the multistage setting, the values (realizations) of the random data become known in the respective time periods (stages), and we have the following sequence of actions:

$$\begin{gathered}\text{decision } (x_1)\\ \text{observation } \xi_2 := (c_2, B_2, A_2, b_2)\\ \text{decision } (x_2)\\ \vdots\\ \text{observation } \xi_T := (c_T, B_T, A_T, b_T)\\ \text{decision } (x_T).\end{gathered}$$

Our objective is to design the decision process in such a way that the expected value of the total cost is minimized while optimal decisions are allowed to be made at *every* time period $t = 1, \ldots, T$.

Let us denote by ξ_t the data vector, realization of which becomes known at time period t. In the setting of the multiperiod problem (3.6), ξ_t is assembled from the components of c_t, B_t, A_t, b_t, some (or all) of which can be random, while the data $\xi_1 = (c_1, A_1, b_1)$ at the first stage of problem (3.6) is assumed to be known. The important condition in the above *multistage decision process* is that every decision vector x_t may depend on the information available at time t (that is, $\xi_{[t]}$) but *not* on the results of observations to be made at later stages. This differs multistage stochastic programming problems from deterministic multiperiod problems, in which all the information is assumed to be available at the beginning of the decision process.

As it was outlined in section 3.1.1, there are several possible ways to formulate multistage stochastic programs in a precise mathematical form. In one such formulation $x_t = \boldsymbol{x}_t(\xi_{[t]})$, $t = 2, \ldots, T$, is viewed as a function of $\xi_{[t]}$, and the minimization in (3.6) is performed over appropriate functional spaces of such functions. If the number of scenarios is finite, this leads to a formulation of the linear multistage stochastic program as one large (deterministic) linear programming problem. We discuss that further in section 3.1.4. Another possible approach is to write dynamic programming equations, which we discuss next.

Let us look at our problem from the perspective of the last stage T. At that time the values of all problem data, $\xi_{[T]}$, are already known, and the values of the earlier decision vectors, $x_1, \ldots, x_{T-1}$, have been chosen. Our problem is, therefore, a simple linear programming problem

$$\begin{aligned}\operatorname*{Min}_{x_T}\ & c_T^{\mathrm{T}} x_T\\ \text{s.t. } & B_T x_{T-1} + A_T x_T = b_T,\\ & x_T \geq 0.\end{aligned}$$

The optimal value of this problem depends on the earlier decision vector $x_{T-1} \in \mathbb{R}^{n_{T-1}}$ and data $\xi_T = (c_T, B_T, A_T, b_T)$ and is denoted by $Q_T(x_{T-1}, \xi_T)$.

At stage $T-1$ we know x_{T-2} and $\xi_{[T-1]}$. We face, therefore, the following stochastic programming problem:

$$\begin{aligned} \operatorname*{Min}_{x_{T-1}} \; & c_{T-1}^{\mathrm{T}} x_{T-1} + \mathbb{E}\left[Q_T(x_{T-1}, \xi_T) \,\middle|\, \xi_{[T-1]} \right] \\ \text{s.t. } & B_{T-1} x_{T-2} + A_{T-1} x_{T-1} = b_{T-1}, \\ & x_{T-1} \geq 0. \end{aligned}$$

The optimal value of the above problem depends on $x_{T-2} \in \mathbb{R}^{n_{T-2}}$ and data $\xi_{[T-1]}$ and is denoted $Q_{T-1}(x_{T-2}, \xi_{[T-1]})$.

Generally, at stage $t = 2, \ldots, T-1$, we have the problem

$$\begin{aligned} \operatorname*{Min}_{x_t} \; & c_t^{\mathrm{T}} x_t + \mathbb{E}\left[Q_{t+1}(x_t, \xi_{[t+1]}) \,\middle|\, \xi_{[t]} \right] \\ \text{s.t. } & B_t x_{t-1} + A_t x_t = b_t, \\ & x_t \geq 0. \end{aligned} \tag{3.7}$$

Its optimal value, called *cost-to-go* function, is denoted $Q_t(x_{t-1}, \xi_{[t]})$.

On top of all these problems is the problem to find the first decisions, $x_1 \in \mathbb{R}^{n_1}$,

$$\begin{aligned} \operatorname*{Min}_{x_1} \; & c_1^{\mathrm{T}} x_1 + \mathbb{E}\left[Q_2(x_1, \xi_2) \right] \\ \text{s.t. } & A_1 x_1 = b_1, \\ & x_1 \geq 0. \end{aligned} \tag{3.8}$$

Note that all subsequent stages $t = 2, \ldots, T$ are absorbed in the above problem into the function $Q_2(x_1, \xi_2)$ through the corresponding expected values. Note also that since ξ_1 is not random, the optimal value $Q_2(x_1, \xi_2)$ does not depend on ξ_1. In particular, if $T = 2$, then (3.8) coincides with the formulation (2.1) of a two-stage linear problem.

The dynamic programming equations here take the form (compare with (3.4))

$$Q_t\left(x_{t-1}, \xi_{[t]}\right) = \inf_{x_t} \left\{ c_t^{\mathrm{T}} x_t + \mathcal{Q}_{t+1}\left(x_t, \xi_{[t]}\right) : B_t x_{t-1} + A_t x_t = b_t,\ x_t \geq 0 \right\},$$

where

$$\mathcal{Q}_{t+1}\left(x_t, \xi_{[t]}\right) := \mathbb{E}\left\{ Q_{t+1}\left(x_t, \xi_{[t+1]}\right) \middle| \xi_{[t]} \right\}.$$

Also an implementable policy $\bar{\boldsymbol{x}}_t(\xi_{[t]})$ is *optimal* if for $t = 1, \ldots, T$ the condition

$$\bar{\boldsymbol{x}}_t(\xi_{[t]}) \in \arg\min_{x_t} \left\{ c_t^{\mathrm{T}} x_t + \mathcal{Q}_{t+1}\left(x_t, \xi_{[t]}\right) : A_t x_t = b_t - B_t \bar{\boldsymbol{x}}_{t-1}(\xi_{[t-1]}),\ x_t \geq 0 \right\}$$

holds for almost every realization of the random process. (For $t = T$ the term $\mathcal{Q}_{T+1}$ is omitted and for $t = 1$ the term $B_t \bar{\boldsymbol{x}}_{t-1}$ is omitted.) If the process ξ_t is Markovian, then each cost-to-go function depends on ξ_t rather than $\xi_{[t]}$, and we can simply write $Q_t(x_{t-1}, \xi_t)$, $t = 2, \ldots, T$. If, moreover, the stagewise independence condition holds, then each expectation function $\mathcal{Q}_t$ does not depend on realizations of the random process, and we can write it as $\mathcal{Q}_t(x_{t-1})$, $t = 2, \ldots, T$.

The *nested formulation* of the linear multistage problem can be written as follows (compare with (3.1)):

$$\operatorname*{Min}_{\substack{A_1 x_1 = b_1 \\ x_1 \geq 0}} c_1^{\mathrm{T}} x_1 + \mathbb{E}\left[\min_{\substack{B_2 x_1 + A_2 x_2 = b_2 \\ x_2 \geq 0}} c_2^{\mathrm{T}} x_2 + \mathbb{E}\Big[\cdots + \mathbb{E}\big[\min_{\substack{B_T x_{T-1} + A_T x_T = b_T \\ x_T \geq 0}} c_T^{\mathrm{T}} x_T \big] \Big] \right]. \tag{3.9}$$

Suppose now that we deal with an underlying model with a full lower block triangular constraint matrix:

$$\begin{array}{lccccccccccc}
\text{Min} & c_1^{\mathrm{T}}x_1 & + & c_2^{\mathrm{T}}x_2 & + & c_3^{\mathrm{T}}x_3 & + & \cdots & + & c_T^{\mathrm{T}}x_T & & \\
\text{s.t.} & A_{11}x_1 & & & & & & & & & = & b_1, \\
 & A_{21}x_1 & + & A_{22}x_2 & & & & & & & = & b_2, \\
 & A_{31}x_1 & + & A_{32}x_2 & + & A_{33}x_3 & & & & & = & b_3, \\
 & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
 & A_{T1}x_1 & + & A_{T2}x_2 & + & \cdots & + & A_{T,T-1}x_{T-1} & + & A_{TT}x_T & = & b_T, \\
 & x_1 \geq 0, & & x_2 \geq 0, & & x_3 \geq 0, & & \cdots & & x_T \geq 0. & &
\end{array} \tag{3.10}$$

In the constraint matrix of (3.6), the respective blocks $A_{t1}, \ldots, A_{t,t-2}$ were assumed to be zeros. This allowed us to express there the optimal value Q_t of (3.7) as a function of the immediately preceding decision, x_{t-1}, rather than all earlier decisions $x_1, \ldots, x_{t-1}$. In the case of problem (3.10), each respective subproblem of the form (3.7) depends on the entire history of our decisions, $x_{[t-1]} := (x_1, \ldots, x_{t-1})$. It takes on the form

$$\begin{array}{l}
\underset{x_t}{\text{Min}}\; c_t^{\mathrm{T}}x_t + \mathbb{E}\left[Q_{t+1}(x_{[t]}, \xi_{[t+1]}) \,\middle|\, \xi_{[t]}\right] \\
\text{s.t. } A_{t1}x_1 + \cdots + A_{t,t-1}x_{t-1} + A_{t,t}x_t = b_t, \\
\qquad x_t \geq 0.
\end{array} \tag{3.11}$$

Its optimal value (i.e., the cost-to-go function) $Q_t(x_{[t-1]}, \xi_{[t]})$ is now a function of the whole history $x_{[t-1]}$ of the decision process rather than its last decision vector x_{t-1}.

Sometimes it is convenient to convert such a lower triangular formulation into the staircase formulation from which we started our presentation. This can be accomplished by introducing additional variables r_t which summarize the relevant history of our decisions. We shall call these variables the *model state* variables (to distinguish from information states discussed before). The relations that describe the next values of the state variables as a function of the current values of these variables, current decisions, and current random parameters are called *model state equations*.

For the general problem (3.10), the vectors $x_{[t]} = (x_1, \ldots, x_t)$ are sufficient model state variables. They are updated at each stage according to the state equation $x_{[t]} = (x_{[t-1]}, x_t)$ (which is linear), and the constraint in (3.11) can be formally written as

$$[\, A_{t1}\ A_{t2}\ \ldots\ A_{t,t-1}\,]x_{[t-1]} + A_{t,t}x_t = b_t.$$

Although it looks a little awkward in this general case, in many problems it is possible to define model state variables of reasonable size. As an example let us consider the structure

$$\begin{array}{lccccccccccc}
\text{Min} & c_1^{\mathrm{T}}x_1 & + & c_2^{\mathrm{T}}x_2 & + & c_3^{\mathrm{T}}x_3 & + & \cdots & + & c_T^{\mathrm{T}}x_T & & \\
\text{s.t.} & A_{11}x_1 & & & & & & & & & = & b_1, \\
 & B_1x_1 & + & A_{22}x_2 & & & & & & & = & b_2, \\
 & B_1x_1 & + & B_2x_2 & + & A_{33}x_3 & & & & & = & b_3, \\
 & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
 & B_1x_1 & + & B_2x_2 & + & \cdots & + & B_{T-1}x_{T-1} & + & A_{TT}x_T & = & b_T, \\
 & x_1 \geq 0, & & x_2 \geq 0, & & x_3 \geq 0, & & \cdots & & x_T \geq 0, & &
\end{array}$$

in which all blocks A_{it}, $i = t+1, \ldots, T$, are identical and observed at time t. Then we can define the state variables r_t, $t = 1, \ldots, T$, recursively by the state equation $r_t = r_{t-1} + B_t x_t$, $t = 1, \ldots, T-1$, where $r_0 = 0$. Subproblem (3.11) simplifies substantially:

$$\begin{aligned} \operatorname*{Min}_{x_t, r_t} \; & c_t^{\mathrm{T}} x_t + \mathbb{E}\left[Q_{t+1}(r_t, \xi_{[t+1]}) \,\middle|\, \xi_{[t]} \right] \\ \text{s.t. } & r_{t-1} + A_{t,t} x_t = b_t, \\ & r_t = r_{t-1} + B_t x_t, \\ & x_t \geq 0. \end{aligned}$$

Its optimal value depends on r_{t-1} and is denoted $Q_t(r_{t-1}, \xi_{[t]})$.

Let us finally remark that the simple sign constraints $x_t \geq 0$ can be replaced in our model by a general constraint $x_t \in X_t$, where X_t is a convex polyhedron defined by some linear equations and inequalities (local for stage t). The set X_t may be random, too, but has to become known at stage t.

3.1.3 Scenario Trees

In order to proceed with numerical calculations, one needs to make a discretization of the underlying random process. It is useful and instructive to discuss this in detail. That is, we consider in this section the case where the random process $\xi_1, \ldots, \xi_T$ has a finite number of realizations. It is useful to depict the possible sequences of data in a form of *scenario tree*. It has nodes organized in levels which correspond to stages $1, \ldots, T$. At level $t = 1$ we have only one *root node*, and we associate with it the value of ξ_1 (which is known at stage $t = 1$). At level $t = 2$ we have as many nodes as many different realizations of ξ_2 may occur. Each of them is connected with the root node by an arc. For each node ι of level $t = 2$ (which corresponds to a particular realization ξ_2^ι of ξ_2) we create at least as many nodes at level 3 as different values of ξ_3 may follow ξ_2^ι, and we connect them with the node ι, etc.

Generally, nodes at level t correspond to possible values of ξ_t that may occur. Each of them is connected to a unique node at level $t-1$, called the *ancestor node*, which corresponds to the identical first $t-1$ parts of the process $\xi_{[t]}$ and is also connected to nodes at level $t+1$, called *children nodes*, which correspond to possible continuations of history $\xi_{[t]}$. Note that in general realizations ξ_t^ι are vectors, and it may happen that some of the values ξ_t^ι, associated with nodes at a given level t, are equal to each other. Nevertheless, such equal values may be represented by different nodes, because they may correspond to different histories of the process. (See Figure 3.1 in Example 3.1 of the next section.)

We denote by Ω_t the set of all nodes at stage $t = 1, \ldots, T$. In particular, Ω_1 consists of a unique root node, Ω_2 has as many elements as many different realizations of ξ_2 may occur, etc. For a node $\iota \in \Omega_t$ we denote by $C_\iota \subset \Omega_{t+1}$, $t = 1, \ldots, T-1$, the set of all children nodes of ι, and by $a(\iota) \in \Omega_{t-1}$, $t = 2, \ldots, T$, the ancestor node of ι. We have that $\Omega_{t+1} = \cup_{\iota \in \Omega_t} C_\iota$ and the sets C_ι are disjoint, i.e., $C_\iota \cap C_{\iota'} = \emptyset$ if $\iota \neq \iota'$. Note again that with different nodes at stage $t \geq 3$ may be associated the same numerical values (realizations) of the corresponding data process ξ_t. *Scenario* is a path from the root note at stage $t = 1$ to a node at the last stage T. Each scenario represents a history of the process $\xi_1, \ldots, \xi_T$. By construction, there is one-to-one correspondence between scenarios and the set Ω_T, and hence the total number K of scenarios is equal to the cardinality[11] of the set Ω_T, i.e., $K = |\Omega_T|$.

[11] We denote by $|\Omega|$ the number of elements in a (finite) set Ω.

Next we should define a probability distribution on a scenario tree. In order to deal with the nested structure of the decision process we need to specify the conditional distribution of ξ_{t+1} given $\xi_{[t]}$, $t = 1, \ldots, T-1$. That is, if we are currently at a node $\iota \in \Omega_t$, we need to specify probability of moving from ι to a node $\eta \in C_\iota$. Let us denote this probability by $\rho_{\iota\eta}$. Note that $\rho_{\iota\eta} \geq 0$ and $\sum_{\eta \in C_\iota} \rho_{\iota\eta} = 1$, and that probabilities $\rho_{\iota\eta}$ are in one-to-one correspondence with arcs of the scenario tree. Probabilities $\rho_{\iota\eta}$, $\eta \in C_\iota$, represent conditional distribution of ξ_{t+1} given that the path of the process $\xi_1, \ldots, \xi_t$ ended at the node ι.

Every scenario can be defined by its nodes $\iota_1, \ldots \iota_T$, arranged in the chronological order, i.e., node ι_2 (at level $t = 2$) is connected to the root node, ι_3 is connected to the node ι_2, etc. The probability of that scenario is then given by the product $\rho_{\iota_1\iota_2}\rho_{\iota_2\iota_3} \cdots \rho_{\iota_{T-1}\iota_T}$. That is, a set of conditional probabilities defines a probability distribution on the set of scenarios. Conversely, it is possible to derive these conditional probabilities from scenario probabilities $p_k, k = 1, \ldots, K$, as follows. Let us denote by $\mathcal{S}^{(\iota)}$ the set of scenarios passing through node ι (at level t) of the scenario tree, and let $p^{(\iota)} := \mathsf{Pr}[\mathcal{S}^{(\iota)}]$, i.e., $p^{(\iota)}$ is the sum of probabilities of all scenarios passing through node ι. If $\iota_1, \iota_2, \ldots, \iota_t$, with ι_1 being the root node and $\iota_t = \iota$, is the history of the process up to node ι, then the probability $p^{(\iota)}$ is given by the product

$$p^{(\iota)} = \rho_{\iota_1\iota_2}\rho_{\iota_2\iota_3} \cdots \rho_{\iota_{t-1}\iota_t}$$

of the corresponding conditional probabilities. In another way, we can write this in the recursive form $p^{(\iota)} = \rho_{a\iota} p^{(a)}$, where $a = a(\iota)$ is the ancestor of the node ι. This equation defines the conditional probability $\rho_{a\iota}$ from the probabilities $p^{(\iota)}$ and $p^{(a)}$. Note that if $a = a(\iota)$ is the ancestor of the node ι, then $\mathcal{S}^{(\iota)} \subset \mathcal{S}^{(a)}$ and hence $p^{(\iota)} \leq p^{(a)}$. Consequently, if $p^{(a)} > 0$, then $\rho_{a\iota} = p^{(\iota)}/p^{(a)}$. Otherwise $\mathcal{S}^{(a)}$ is empty, i.e., no scenario is passing through the node a, and hence no scenario is passing through the node ι.

If the process $\xi_1, \ldots, \xi_T$ is stagewise independent, then the conditional distribution of ξ_{t+1} given $\xi_{[t]}$ is the same as the unconditional distribution of ξ_{t+1}, $t = 1, \ldots, T-1$. In that case at every stage $t = 1, \ldots, T-1$, with every node $\iota \in \Omega_t$ is associated an identical set of children, with the same set of respective conditional probabilities and with the same respective numerical values.

Recall that a stochastic process Z_t, $t = 1, 2, \ldots$, that can take a finite number $\{z_1, \ldots, z_m\}$ of different values is a *Markov chain* if

$$\mathsf{Pr}\left\{Z_{t+1} = z_j \,\middle|\, Z_t = z_i,\ Z_{t-1} = z_{i_{t-1}}, \ldots, Z_1 = z_{i_1}\right\} = \mathsf{Pr}\left\{Z_{t+1} = z_j \,\middle|\, Z_t = z_i\right\}$$

for all states $z_{i_{t-1}}, \ldots, z_{i_1}, z_i, z_j$ and all $t = 1, 2, \ldots$. Denote

$$p_{ij} := \mathsf{Pr}\left\{Z_{t+1} = z_j \,\middle|\, Z_t = z_i\right\}, \quad i, j = 1, \ldots, m.$$

In some situations, it is natural to model the data process as a Markov chain with the corresponding state space[12] $\{\zeta^1, \ldots, \zeta^m\}$ and probabilities p_{ij} of moving from state ζ^i to state ζ^j, $i, j = 1, \ldots, m$. We can model such a process by a scenario tree. At stage $t = 1$ there is one root node to which is assigned one of the values from the state space, say, ζ^i. At stage $t = 2$ there are m nodes to which are assigned values $\zeta^1, \ldots, \zeta^m$ with the

[12] In our model, values $\zeta^1, \ldots, \zeta^m$ can be numbers or vectors.

corresponding probabilities $p_{i1}, \ldots, p_{im}$. At stage $t = 3$ there are m^2 nodes, such that each node at stage $t = 2$, associated with a state ζ^a, $a = 1, \ldots, m$, is the ancestor of m nodes at stage $t = 3$ to which are assigned values $\zeta^1, \ldots, \zeta^m$ with the corresponding conditional probabilities $p_{a1}, \ldots, p_{am}$. At stage $t = 4$ there are m^3 nodes, etc. At each stage t of such T-stage Markov chain process there are m^{t-1} nodes, the corresponding random vector (variable) ξ_t can take values $\zeta^1, \ldots, \zeta^m$ with respective probabilities which can be calculated from the history of the process up to time t, and the total number of scenarios is m^{T-1}. We have here that the random vectors (variables) $\xi_1, \ldots, \xi_T$ are independently distributed iff $p_{ij} = p_{i'j}$ for any i, i', $j = 1, \ldots, m$, i.e., the conditional probability p_{ij} of moving from state ζ^i to state ζ^j does not depend on i.

In the above formulation of the Markov chain, the corresponding scenario tree represents the total history of the process with the number of scenarios growing exponentially with the number of stages. Now if we approach the problem by writing the cost-to-go functions $Q_t(x_{t-1}, \xi_t)$, going backward, then we do not need to keep track of the history of the process. That is, at every stage t the cost-to-go function $Q_t(\cdot, \xi_t)$ depends only on the current state (realization) $\xi_t = \zeta^i$, $i = 1, \ldots, m$, of the process. On the other hand, if we want to write the corresponding optimization problem (in the case of a finite number of scenarios) as one large linear programming problem, we still need the scenario tree formulation. This is the basic difference between the stochastic and dynamic programming approaches to the problem. That is, the stochastic programming approach does not necessarily rely on the Markovian structure of the process considered. This makes it more general at the price of considering a possibly very large number of scenarios.

An important concept associated with the data process is the corresponding filtration. We associate with the set Ω_T the sigma algebra $\mathcal{F}_T$ of *all* its subsets. The set Ω_T can be represented as the union of disjoint sets C_ι, $\iota \in \Omega_{T-1}$. Let $\mathcal{F}_{T-1}$ be the subalgebra of $\mathcal{F}_T$ generated by the sets C_ι, $\iota \in \Omega_{T-1}$. As they are disjoint, they are the elementary events of $\mathcal{F}_{T-1}$. By this construction, there is one-to-one correspondence between elementary events of $\mathcal{F}_{T-1}$ and the set Ω_{T-1} of nodes at stage $T - 1$. By continuing in this way we construct a sequence of sigma algebras $\mathcal{F}_1 \subset \cdots \subset \mathcal{F}_T$, called *filtration*. In this construction, elementary events of sigma algebra $\mathcal{F}_t$ are subsets of Ω_T which are in one-to-one correspondence with the nodes $\iota \in \Omega_t$. Of course, the cardinality $|\mathcal{F}_t| = 2^{|\Omega_t|}$. In particular, $\mathcal{F}_1$ corresponds to the unique root at stage $t = 1$ and hence $\mathcal{F}_1 = \{\emptyset, \Omega_T\}$.

3.1.4 Algebraic Formulation of Nonanticipativity Constraints

Suppose that in our basic problem (3.6) there are only finitely many, say, K, different scenarios the problem data can take. Recall that each scenario can be considered as a path of the respective scenario tree. With each scenario, numbered k, is associated probability p_k and the corresponding sequence of decisions[13] $\boldsymbol{x}^k = (x_1^k, x_2^k, \ldots, x_T^k)$. That is, with each possible scenario $k = 1, \ldots, K$ (i.e., a realization of the data process) we associate a sequence of decisions $\boldsymbol{x}^k$. Of course, it would not be appropriate to try to find the optimal

[13] To avoid ugly collisions of subscripts, we change our notation a little and put the index of the scenario, k, as a superscript.

values of these decisions by solving the relaxed version of (3.6):

$$\begin{array}{lllllllllll}
\text{Min} & \sum_{k=1}^{K} p_k \Big[c_1^\top x_1^k & + & (c_2^k)^\top x_2^k & + & (c_3^k)^\top x_3^k & + & \cdots & + & (c_T^k)^\top x_T^k \Big] & \\
\text{s.t.} & A_1 x_1^k & & & & & & & & = & b_1, \\
& B_2^k x_1^k & + & A_2^k x_2^k & & & & & & = & b_2^k, \\
& & & B_3^k x_2^k & + & A_3^k x_3^k & & & & = & b_3^k, \\
& \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
& & & & & & & B_T^k x_{T-1}^k & + & A_T^k x_T^k = & b_T^k, \\
& x_1^k \geq 0, & & x_2^k \geq 0, & & x_3^k \geq 0, & & \cdots & & x_T^k \geq 0, &
\end{array} \tag{3.12}$$

$$k = 1, \ldots, K.$$

The reason is the same as in the two-stage case. That is, in problem (3.12) all parts of the decision vector are allowed to depend on *all* parts of the random data, while each part x_t should be allowed to depend only on the data known up to stage t. In particular, problem (3.12) may suggest different values of x_1, one for each scenario k, while our first-stage decision should be independent of possible realizations of the data process.

In order to correct this problem we enforce the constraints

$$x_1^k = x_1^\ell, \quad \forall k, \ell \in \{1, \ldots, K\}, \tag{3.13}$$

similarly to the two-stage case (2.83). But this is not sufficient, in general. Consider the second part of the decision vector, x_2. It should be allowed to depend only on $\xi_{[2]} = (\xi_1, \xi_2)$, so it has to have the same value for all scenarios k for which $\xi_{[2]}^k$ are identical. We must, therefore, enforce the constraints

$$x_2^k = x_2^\ell, \quad \forall k, \ell \text{ for which } \xi_{[2]}^k = \xi_{[2]}^\ell.$$

Generally, at stage $t = 1, \ldots, T$, the scenarios that have the same history $\xi_{[t]}$ cannot be distinguished, so we need to enforce the *nonanticipativity constraints:*

$$x_t^k = x_t^\ell, \quad \forall k, \ell \text{ for which } \xi_{[t]}^k = \xi_{[t]}^\ell, \quad t = 1, \ldots, T. \tag{3.14}$$

Problem (3.12) together with the nonanticipativity constraints (3.14) becomes equivalent to our original formulation (3.6).

Remark 3. Let us observe that if in problem (3.12) only the constraints (3.13) are enforced, then from the mathematical point of view the problem obtained becomes a two-stage stochastic linear program with K scenarios. In this two-stage program the first-stage decision vector is x_1, the second-stage decision vector is $(x_2, \ldots, x_K)$, the technology matrix is B_2, and the recourse matrix is the block matrix

$$\begin{bmatrix} A_2 & 0 & \cdots & 0 & 0 \\ B_3 & A_3 & \cdots & 0 & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & B_T & A_T \end{bmatrix}.$$

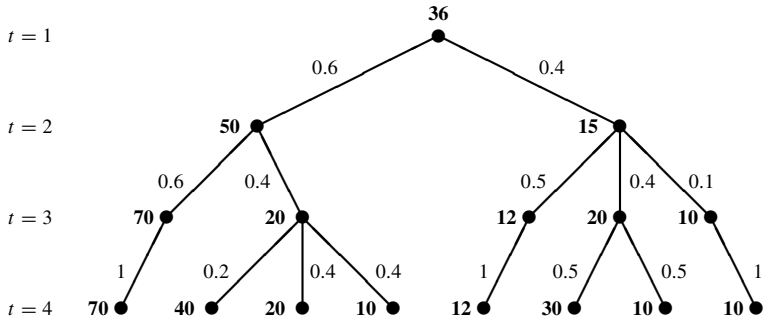

Figure 3.1. *Scenario tree. Nodes represent information states. Paths from the root to leaves represent scenarios. Numbers along the arcs represent conditional probabilities of moving to the next node. Bold numbers represent numerical values of the process.*

Since the two-stage problem obtained is a relaxation of the multistage problem (3.6), its optimal value gives a lower bound for the optimal value of problem (3.6) and in that sense it may be useful. Note, however, that this model does not make much sense in any application, because it assumes that at the end of the process, when all realizations of the random data become known, one can go back in time and make all decisions $x_2, \ldots, x_{K-1}$.

Example 3.1 (Scenario Tree). As discussed in section 3.1.3, it can be useful to depict the possible sequences of data $\xi_{[t]}$ in a form of a *scenario tree*. An example of such a scenario tree is given in Figure 3.1. Numbers along the arcs represent conditional probabilities of moving from one node to the next. The associated process $\xi_t = (c_t, B_t, A_t, b_t)$, $t = 1, \ldots, T$, with $T = 4$, is defined as follows. All involved variables are assumed to be one-dimensional, with c_t, B_t, A_t, $t = 2, 3, 4$, being fixed and only right-hand-side variables b_t being random. The values (realizations) of the random process $b_1, \ldots, b_T$ are indicated by the bold numbers at the nodes of the tree. (The numerical values of c_t, B_t, A_t are not written explicitly, although, of course, they also should be specified.) That is, at level $t = 1$, b_1 has the value 36. At level $t = 2$, b_2 has two values 15 and 50 with respective probabilities 0.4 and 0.6. At level $t = 3$ we have 5 nodes with which are associated the following numerical values (from right to left): 10, 20, 12, 20, 70. That is, b_3 can take 4 different values with respective probabilities $\Pr\{b_3 = 10\} = 0.4 \cdot 0.1$, $\Pr\{b_3 = 20\} = 0.4 \cdot 0.4 + 0.6 \cdot 0.4$, $\Pr\{b_3 = 12\} = 0.4 \cdot 0.5$, and $\Pr\{b_3 = 70\} = 0.6 \cdot 0.6$. At level $t = 4$, the numerical values associated with 8 nodes are defined, from right to left, as 10, 10, 30, 12, 10, 20, 40, 70. The respective probabilities can be calculated by using the corresponding conditional probabilities. For example,

$$\Pr\{b_4 = 10\} = 0.4 \cdot 0.1 \cdot 1.0 + 0.4 \cdot 0.4 \cdot 0.5 + 0.6 \cdot 0.4 \cdot 0.4.$$

Note that although some of the realizations of b_3, and hence of ξ_3, are equal to each other, they are represented by different nodes. This is necessary in order to identify different histories of the process corresponding to different scenarios. The same remark applies to b_4 and ξ_4. Altogether, there are eight scenarios in this tree. Figure 3.2 illustrates the way in which sequences of decisions are associated with scenarios from Figure 3.1.

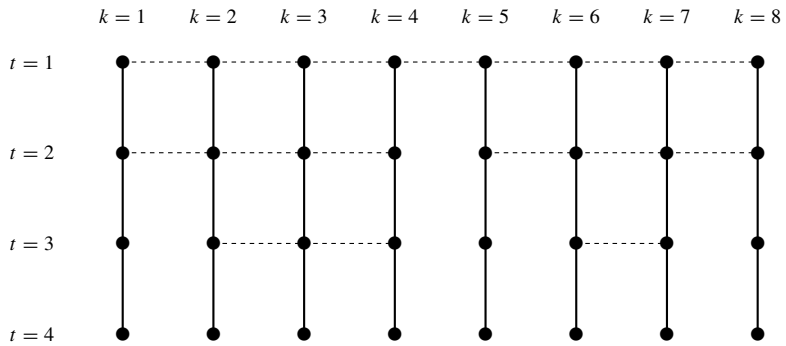

Figure 3.2. *Sequences of decisions for scenarios from Figure 3.1. Horizontal dotted lines represent the equations of nonanticipativity.*

The process b_t (and hence the process ξ_t) in this example is not Markovian. For instance,

$$\Pr\{b_4 = 10 \,|\, b_3 = 20, b_2 = 15, b_1 = 36\} = 0.5,$$

while

$$\begin{aligned}\Pr\{b_4 = 10 \,|\, b_3 = 20\} &= \frac{\Pr\{b_4 = 10, b_3 = 20\}}{\Pr\{b_3 = 20\}} \\ &= \frac{0.5 \cdot 0.4 \cdot 0.4 + 0.4 \cdot 0.4 \cdot 0.6}{0.4 \cdot 0.4 + 0.4 \cdot 0.6} = 0.44.\end{aligned}$$

That is, $\Pr\{b_4 = 10 \,|\, b_3 = 20\} \neq \Pr\{b_4 = 10 \,|\, b_3 = 20, b_2 = 15, b_1 = 36\}$. ∎

Relaxing the nonanticipativity constraints means that decisions $x_t = \boldsymbol{x}_t(\omega)$ are viewed as functions of all possible realizations (scenarios) of the data process. This was the case in formulation (3.12), where the problem was separated into K different problems, one for each scenario $\omega_k = (\xi_1^k, \ldots, \xi_T^k)$, $k = 1, \ldots, K$. The corresponding nonanticipativity constraints can be written in several way. One possible way is to write them, similarly to (2.84) for two-stage models, as

$$x_t = \mathbb{E}\left[x_t \middle| \xi_{[t]}\right], \quad t = 1, \ldots, T. \tag{3.15}$$

Another way is to use *filtration* associated with the data process. Let $\mathcal{F}_t$ be the sigma algebra generated by $\xi_{[t]}$, $t = 1, \ldots, T$. That is, $\mathcal{F}_t$ is the minimal subalgebra of the sigma algebra $\mathcal{F}$ such that $\xi_1(\omega), \ldots, \xi_t(\omega)$ are $\mathcal{F}_t$-measurable. Since ξ_1 is not random, $\mathcal{F}_1$ contains only two sets: $\emptyset$ and Ω. We have that $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_T \subset \mathcal{F}$. In the case of finitely many scenarios, we discussed construction of such a filtration at the end of section 3.1.3. We can write (3.15) in the following equivalent form

$$x_t = \mathbb{E}\left[x_t \middle| \mathcal{F}_t\right], \quad t = 1, \ldots, T. \tag{3.16}$$

(See section 7.2.2 for a definition of conditional expectation with respect to a sigma subalgebra.) Condition (3.16) holds iff $x_t(\omega)$ is measurable with respect to $\mathcal{F}_t$, $t = 1, \ldots, T$. One can use this measurability requirement as a definition of the nonanticipativity constraints.

Suppose, for the sake of simplicity, that there is a finite number K of scenarios. To each scenario corresponds a sequence $(x_1^k, \ldots, x_T^k)$ of decision vectors which can be considered as an element of a vector space of dimension $n_1 + \cdots + n_T$. The space of all such sequences $(x_1^k, \ldots, x_T^k)$, $k = 1, \ldots, K$, is a vector space, denoted $\mathfrak{X}$, of dimension $(n_1 + \cdots + n_T)K$.The nonanticipativity constraints (3.14) define a linear subspace of $\mathfrak{X}$, denoted $\mathfrak{L}$. Define the scalar product on the space $\mathfrak{X}$,

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle := \sum_{k=1}^{K} \sum_{t=1}^{T} p_k (x_t^k)^{\mathsf{T}} y_t^k, \tag{3.17}$$

and let $\boldsymbol{P}$ be the orthogonal projection of $\mathfrak{X}$ onto $\mathfrak{L}$ with respect to this scalar product. Then

$$\boldsymbol{x} = \boldsymbol{P}\boldsymbol{x}$$

is yet another way to write the nonanticipativity constraints.

A computationally convenient way of writing the nonanticipativity constraints (3.14) can be derived by using the following construction, which extends to the multistage case the system (2.87).

Let Ω_t be the set of nodes at level t. For a node $\iota \in \Omega_t$ we denote by $\mathcal{S}^{(\iota)}$ the set of scenarios that pass through node ι and are, therefore, indistinguishable on the basis of the information available up to time t. As explained before, the sets $\mathcal{S}^{(\iota)}$ for all $\iota \in \Omega_t$ are the atoms of the sigma-subalgebra $\mathcal{F}_t$ associated with the time stage t. We order them and denote them by $\mathcal{S}_t^1, \ldots, \mathcal{S}_t^{\gamma_t}$.

Let us assume that all scenarios $1, \ldots, K$ are ordered in such a way that each set $\mathcal{S}_t^\nu$ is a set of consecutive numbers $l_t^\nu, l_t^\nu + 1, \ldots, r_t^\nu$. Then nonanticipativity can be expressed by the system of equations

$$x_t^s - x_t^{s+1} = 0, \quad s = l_t^\nu, \ldots, r_t^\nu - 1, \quad t = 1, \ldots, T-1, \quad \nu = 1, \ldots, \gamma_t. \tag{3.18}$$

In other words, each decision is related to its neighbors from the left and from the right, if they correspond to the same node of the scenario tree.

The coefficients of constraints (3.18) define a giant matrix

$$M = [M^1 \ldots M^K],$$

whose rows have two nonzeros each: 1 and -1. Thus, we obtain an algebraic description of the nonanticipativity constraints:

$$M^1 x^1 + \cdots + M^K x^K = 0. \tag{3.19}$$

Owing to the sparsity of the matrix M, this formulation is very convenient for various numerical methods for solving linear multistage stochastic programming problems: the simplex method, interior point methods, and decomposition methods.

Example 3.2. Consider the scenario tree depicted in Figure 3.1. Let us assume that the scenarios are numbered from the left to the right. Our nonanticipativity constraints take on

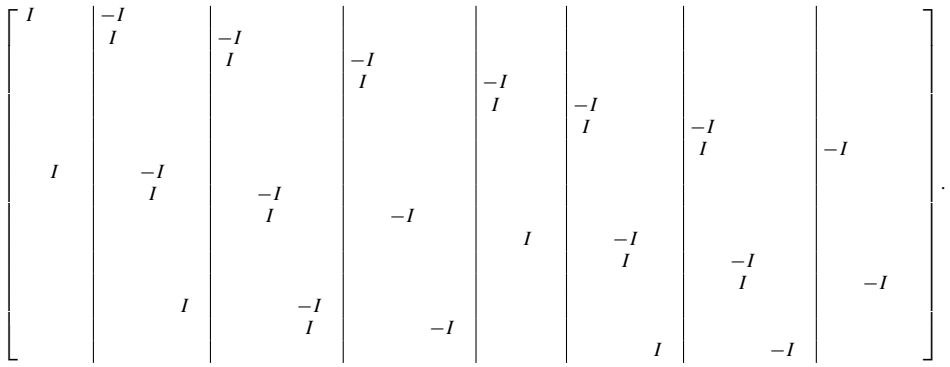

Figure 3.3. *The nonanticipativity constraint matrix M corresponding to the scenario tree from Figure 3.1. The subdivision corresponds to the scenario submatrices $M^1, \ldots, M^8$.*

the form

$$\begin{gathered} x_1^1 - x_1^2 = 0, \quad x_1^2 - x_1^3 = 0, \quad \ldots \quad , \; x_1^7 - x_1^8 = 0, \\ x_2^1 - x_2^2 = 0, \quad x_2^2 - x_2^3 = 0, \quad x_2^3 - x_2^4 = 0, \\ x_2^5 - x_2^6 = 0, \quad x_2^6 - x_2^7 = 0, \quad x_2^7 - x_2^8 = 0, \\ x_3^2 - x_3^3 = 0, \quad x_3^3 - x_3^4 = 0, \quad x_3^6 - x_3^7 = 0. \end{gathered}$$

Using I to denote the identity matrix of an appropriate dimension, we may write the constraint matrix M as shown in Figure 3.3. M is always a very sparse matrix: each row of it has only two nonzeros, each column at most two nonzeros. Moreover, all nonzeros are either 1 or -1, which is also convenient for numerical methods. ∎

3.2 Duality

3.2.1 Convex Multistage Problems

In this section we consider multistage problems of the form (3.1) with

$$\mathcal{X}_t(x_{t-1}, \xi_t) := \{x_t : B_t x_{t-1} + A_t x_t = b_t\}, \; t = 2, \ldots, T, \tag{3.20}$$

$\mathcal{X}_1 := \{x_1 : A_1 x_1 = b_1\}$ and $f_t(x_t, \xi_t)$, $t = 1, \ldots, T$, being random lower semicontinuous functions. We assume that functions $f_t(\cdot, \xi_t)$ are *convex* for a.e. ξ_t. In particular, if

$$f_t(x_t, \xi_t) := \begin{cases} c_t^{\mathrm{T}} x_t & \text{if } x_t \geq 0, \\ +\infty & \text{otherwise,} \end{cases} \tag{3.21}$$

then the problem becomes the linear multistage problem given in the nested formulation (3.9). All constraints involving only variables and quantities associated with stage

t are absorbed in the definition of the functions f_t. It is implicitly assumed that the data $(A_t, B_t, b_t) = (A_t(\xi_t), B_t(\xi_t), b_t(\xi_t))$, $t = 1, \ldots, T$, form a random process.

Dynamic programming equations take here the form

$$Q_t\left(x_{t-1}, \xi_{[t]}\right) = \inf_{x_t} \left\{ f_t(x_t, \xi_t) + \mathcal{Q}_{t+1}\left(x_t, \xi_{[t]}\right) : B_t x_{t-1} + A_t x_t = b_t \right\}, \tag{3.22}$$

where

$$\mathcal{Q}_{t+1}\left(x_t, \xi_{[t]}\right) := \mathbb{E}\left\{ Q_{t+1}\left(x_t, \xi_{[t+1]}\right) \middle| \xi_{[t]} \right\}.$$

For every $t = 1, \ldots, T$ and $\xi_{[t]}$, the function $Q_t(\cdot, \xi_{[t]})$ is convex. Indeed,

$$Q_T\left(x_{T-1}, \xi_T\right) = \inf_{x_T} \phi(x_T, x_{T-1}, \xi_T),$$

where

$$\phi(x_T, x_{T-1}, \xi_T) := \begin{cases} f_T(x_T, \xi_T) & \text{if } B_T x_{T-1} + A_T x_T = b_T, \\ +\infty & \text{otherwise.} \end{cases}$$

It follows from the convexity of $f_T(\cdot, \xi_T)$ that $\phi(\cdot, \cdot, \xi_T)$ is convex, and hence the optimal value function $Q_T(\cdot, \xi_T)$ is also convex. Convexity of functions $Q_t(\cdot, \xi_{[t]})$ can be shown in the same way by induction in $t = T, \ldots, 1$. Moreover, if the number of scenarios is finite and functions $f_t(x_t, \xi_t)$ are random *polyhedral*, then the cost-to-go functions $Q_t(x_{t-1}, \xi_{[t]})$ are also random polyhedral.

3.2.2 Optimality Conditions

Consider the cost-to-go functions $Q_t(x_{t-1}, \xi_{[t]})$ defined by the dynamic programming equations (3.22). With the optimization problem on the right-hand side of (3.22) is associated the following Lagrangian:

$$L_t(x_t, \pi_t) := f_t(x_t, \xi_t) + \mathcal{Q}_{t+1}\left(x_t, \xi_{[t]}\right) + \pi_t^{\mathsf{T}}\left(b_t - B_t x_{t-1} - A_t x_t\right).$$

This Lagrangian also depends on $\xi_{[t]}$ and x_{t-1}, which we omit for brevity of the notation. Denote

$$\psi_t(x_t, \xi_{[t]}) := f_t(x_t, \xi_t) + \mathcal{Q}_{t+1}\left(x_t, \xi_{[t]}\right).$$

The dual functional is

$$\begin{aligned} D_t(\pi_t) &:= \inf_{x_t} L_t(x_t, \pi_t) \\ &= -\sup_{x_t} \left\{ \pi_t^{\mathsf{T}} A_t x_t - \psi_t(x_t, \xi_{[t]}) \right\} + \pi_t^{\mathsf{T}}\left(b_t - B_t x_{t-1}\right) \\ &= -\psi_t^*(A_t^{\mathsf{T}}\pi_t, \xi_{[t]}) + \pi_t^{\mathsf{T}}\left(b_t - B_t x_{t-1}\right), \end{aligned}$$

where $\psi_t^*(\cdot, \xi_{[t]})$ is the conjugate function of $\psi_t(\cdot, \xi_{[t]})$. Therefore we can write the Lagrangian dual of the optimization problem on the right hand side of (3.22) as follows:

$$\operatorname*{Max}_{\pi_t} \left\{ -\psi_t^*(A_t^{\mathsf{T}}\pi_t, \xi_{[t]}) + \pi_t^{\mathsf{T}}\left(b_t - B_t x_{t-1}\right) \right\}. \tag{3.23}$$

Both optimization problems, (3.22) and its dual (3.23), are convex. Under various regularity conditions there is no duality gap between problems (3.22) and (3.23). In particular, we can formulate the following two conditions.

(D1) The functions $f_t(x_t, \xi_t)$, $t = 1, \ldots, T$, are random polyhedral, and the number of scenarios is finite.

(D2) For all sufficiently small perturbations of the vector b_t, the corresponding optimal value $Q_t(x_{t-1}, \xi_{[t]})$ is finite, i.e., there is a neighborhood of b_t such that for any b_t' in that neighborhood the optimal value of the right-hand side of (3.22) with b_t replaced by b_t' is finite.

We denote by $\mathfrak{D}_t\left(x_{t-1}, \xi_{[t]}\right)$ the set of optimal solutions of the dual problem (3.23). All subdifferentials in the subsequent formulas are taken with respect to x_t for an appropriate $t = 1, \ldots, T$.

Proposition 3.3. *Suppose that either condition* (D1) *holds and* $Q_t\left(x_{t-1}, \xi_{[t]}\right)$ *is finite or condition* (D2) *holds. Then,*

(i) *there is no duality gap between problems* (3.22) *and* (3.23)*, i.e.,*

$$Q_t\left(x_{t-1}, \xi_{[t]}\right) = \sup_{\pi_t}\left\{-\psi_t^*(A_t^{\mathsf{T}}\pi_t, \xi_{[t]}) + \pi_t^{\mathsf{T}}\left(b_t - B_t x_{t-1}\right)\right\}, \tag{3.24}$$

(ii) $\bar{x}_t$ *is an optimal solution of* (3.22) *iff there exists* $\bar{\pi}_t = \bar{\pi}_t(\xi_{[t]})$ *such that* $\bar{\pi}_t \in \mathfrak{D}(x_{t-1}, \xi_{[t]})$ *and*

$$0 \in \partial L_t\left(\bar{x}_t, \bar{\pi}_t\right), \tag{3.25}$$

(iii) *the function* $Q_t(\cdot, \xi_{[t]})$ *is subdifferentiable at* x_{t-1} *and*

$$\partial Q_t\left(x_{t-1}, \xi_{[t]}\right) = -B_t^{\mathsf{T}}\mathfrak{D}_t\left(x_{t-1}, \xi_{[t]}\right). \tag{3.26}$$

Proof. Consider the optimal value function

$$\vartheta(y) := \inf_{x_t}\left\{\psi_t(x_t, \xi_{[t]}) : A_t x_t = y\right\}.$$

Since $\psi_t(\cdot, \xi_{[t]})$ is convex, the function $\vartheta(\cdot)$ is also convex. Condition (D2) means that $\vartheta(y)$ is finite valued for all y in a neighborhood of $\bar{y} := b_t - B_t x_{t-1}$. It follows that $\vartheta(\cdot)$ is continuous and subdifferentiable at $\bar{y}$. By conjugate duality (see Theorem 7.8) this implies assertion (i). Moreover, the set of optimal solutions of the corresponding dual problem coincides with the subdifferential of $\vartheta(\cdot)$ at $\bar{y}$. Formula (3.26) then follows by the chain rule. Condition (3.25) means that $\bar{x}_t$ is a minimizer of $L(\cdot, \bar{\pi}_t)$, and hence the assertion (ii) follows by (i).

If condition (D1) holds, then the functions $Q_t\left(\cdot, \xi_{[t]}\right)$ are polyhedral, and hence $\vartheta(\cdot)$ is polyhedral. It follows that $\vartheta(\cdot)$ is lower semicontinuous and subdifferentiable at any point where it is finite valued. Again, the proof can be completed by applying the conjugate duality theory. □

Note that condition (D2) actually implies that the set $\mathfrak{D}_t\left(x_{t-1}, \xi_{[t]}\right)$ of optimal solutions of the dual problem is nonempty and *bounded*, while condition (D1) only implies that $\mathfrak{D}_t\left(x_{t-1}, \xi_{[t]}\right)$ is nonempty.

Now let us look at the optimality conditions (3.5), which in the present case can be written as follows:

$$\bar{\boldsymbol{x}}_t(\xi_{[t]}) \in \arg\min_{x_t}\left\{f_t(x_t, \xi_t) + \mathcal{Q}_{t+1}\left(x_t, \xi_{[t]}\right) : A_t x_t = b_t - B_t \bar{\boldsymbol{x}}_{t-1}(\xi_{[t-1]})\right\}. \tag{3.27}$$

Since the optimization problem on the right-hand side of (3.27) is convex, subject to linear constraints, we have that a feasible policy is optimal iff it satisfies the following optimality conditions: for $t = 1, \ldots, T$ and a.e. $\xi_{[t]}$ there exists $\bar{\boldsymbol{\pi}}_t(\xi_{[t]})$ such that the following condition holds:

$$0 \in \partial \left[f_t(\bar{\boldsymbol{x}}_t(\xi_{[t]}), \xi_t) + \mathcal{Q}_{t+1}\left(\bar{\boldsymbol{x}}_t(\xi_{[t]}), \xi_{[t]}\right)\right] - A_t^{\mathrm{T}} \bar{\boldsymbol{\pi}}_t(\xi_{[t]}). \tag{3.28}$$

Recall that all subdifferentials are taken with respect to x_t, and for $t = T$ the term $\mathcal{Q}_{T+1}$ is omitted.

We shall use the following regularity condition:

(D3) For $t = 2, \ldots, T$ and a.e. $\xi_{[t]}$ the function $\mathcal{Q}_t\left(\cdot, \xi_{[t-1]}\right)$ is finite valued.

The above condition implies, of course, that $Q_t\left(\cdot, \xi_{[t]}\right)$ is finite valued for a.e. $\xi_{[t]}$ conditional on $\xi_{[t-1]}$, which in turn implies relatively complete recourse. Note also that condition (D3) does not necessarily imply condition (D2), because in the latter the function $Q_t\left(\cdot, \xi_{[t]}\right)$ is required to be finite for all small perturbations of b_t.

Proposition 3.4. *Suppose that either conditions* (D2) *and* (D3) *or condition* (D1) *are satisfied. Then a* feasible *policy* $\bar{\boldsymbol{x}}_t(\xi_{[t]})$ *is optimal iff there exist mappings* $\bar{\boldsymbol{\pi}}_t(\xi_{[t]})$, $t = 1, \ldots, T$, *such that the condition*

$$0 \in \partial f_t(\bar{\boldsymbol{x}}_t(\xi_{[t]}), \xi_t) - A_t^{\mathrm{T}} \bar{\boldsymbol{\pi}}_t(\xi_{[t]}) + \mathbb{E}\left[\partial Q_{t+1}\left(\bar{\boldsymbol{x}}_t(\xi_{[t]}), \xi_{[t+1]}\right) \big| \xi_{[t]}\right] \tag{3.29}$$

holds true for a.e. $\xi_{[t]}$ *and* $t = 1, \ldots, T$. *Moreover, multipliers* $\bar{\boldsymbol{\pi}}_t(\xi_{[t]})$ *satisfy* (3.29) *iff for a.e.* $\xi_{[t]}$ *it holds that*

$$\bar{\boldsymbol{\pi}}_t(\xi_{[t]}) \in \mathfrak{D}(\bar{\boldsymbol{x}}_{t-1}(\xi_{[t-1]}), \xi_{[t]}). \tag{3.30}$$

Proof. Suppose that condition (D3) holds. Then by the Moreau–Rockafellar theorem (Theorem 7.4) we have that at $\bar{x}_t = \bar{\boldsymbol{x}}_t(\xi_{[t]})$,

$$\partial \left[f_t(\bar{x}_t, \xi_t) + \mathcal{Q}_{t+1}\left(\bar{x}_t, \xi_{[t]}\right)\right] = \partial f_t(\bar{x}_t, \xi_t) + \partial \mathcal{Q}_{t+1}\left(\bar{x}_t, \xi_{[t]}\right).$$

Also by Theorem 7.47 the subdifferential of $\mathcal{Q}_{t+1}\left(\bar{\boldsymbol{x}}_t, \xi_{[t]}\right)$ can be taken inside the expectation to obtain the last term in the right-hand side of (3.29). Note that conditional on $\xi_{[t]}$ the term $\bar{x}_t = \bar{\boldsymbol{x}}_t(\xi_{[t]})$ is fixed. Optimality conditions (3.29) then follow from (3.28). Suppose, further, that condition (D2) holds. Then there is no duality gap between problems (3.22) and (3.23), and the second assertion follows by (3.27) and Proposition 3.3(ii).

If condition (D1) holds, then functions $f_t(x_t, \xi_t)$ and $\mathcal{Q}_{t+1}\left(x_t, \xi_{[t]}\right)$ are random polyhedral, and hence the same arguments can be applied without additional regularity conditions. □

Formula (3.26) makes it possible to write optimality conditions (3.29) in the following form.

Theorem 3.5. *Suppose that either conditions* (D2) *and* (D3) *or condition* (D1) *are satisfied. Then a* feasible *policy* $\bar{\boldsymbol{x}}_t(\xi_{[t]})$ *is optimal iff there exist measurable* $\bar{\boldsymbol{\pi}}_t(\xi_{[t]})$, $t = 1, \ldots, T$, *such that*

$$0 \in \partial f_t(\bar{\boldsymbol{x}}_t(\xi_{[t]}), \xi_t) - A_t^{\mathrm{T}} \bar{\boldsymbol{\pi}}_t(\xi_{[t]}) - \mathbb{E}\left[B_{t+1}^{\mathrm{T}} \bar{\boldsymbol{\pi}}_{t+1}(\xi_{[t+1]}) \big| \xi_{[t]}\right] \tag{3.31}$$

for a.e. $\xi_{[t]}$ *and* $t = 1, \ldots, T$, *where for* $t = T$ *the corresponding term* $T + 1$ *is omitted.*

Proof. By Proposition 3.4 we have that a feasible policy $\bar{\boldsymbol{x}}_t(\xi_{[t]})$ is optimal iff conditions (3.29) and (3.30) hold true. For $t = 1$ this means the existence of $\bar{\pi}_1 \in \mathfrak{D}_1$ such that

$$0 \in \partial f_1(\bar{x}_1) - A_1^{\mathsf{T}}\bar{\pi}_1 + \mathbb{E}\left[\partial Q_2(\bar{x}_1, \xi_2)\right]. \tag{3.32}$$

Recall that ξ_1 is known, and hence the set $\mathfrak{D}_1$ is fixed. By (3.26) we have

$$\partial Q_2(\bar{x}_1, \xi_2) = -B_2^{\mathsf{T}}\mathfrak{D}_2(\bar{x}_1, \xi_2). \tag{3.33}$$

Formulas (3.32) and (3.33) mean that there exists a measurable selection

$$\bar{\boldsymbol{\pi}}_2(\xi_2) \in \mathfrak{D}_2(\bar{x}_1, \xi_2)$$

such that (3.31) holds for $t = 1$. By the second assertion of Proposition 3.4, the same selection $\bar{\boldsymbol{\pi}}_2(\xi_2)$ can be used in (3.29) for $t = 2$. Proceeding in that way we obtain existence of measurable selections

$$\bar{\boldsymbol{\pi}}_t(\xi_t) \in \mathfrak{D}_t\left(\bar{\boldsymbol{x}}_{t-1}(\xi_{[t-1]}), \xi_{[t]}\right)$$

satisfying (3.31). □

In particular, consider the multistage linear problem given in the nested formulation (3.9). That is, functions $f_t(x_t, \xi_t)$ are defined in the form (3.21), which can be written as

$$f_t(x_t, \xi_t) = c_t^{\mathsf{T}}x_t + \mathbb{I}_{\mathbb{R}^{n_t}_+}(x_t).$$

Then $\partial f_t(x_t, \xi_t) = \left\{c_t + \mathcal{N}_{\mathbb{R}^{n_t}_+}(x_t)\right\}$ at every point $x_t \geq 0$, and hence optimality conditions (3.31) take the form

$$0 \in \mathcal{N}_{\mathbb{R}^{n_t}_+}\left(\bar{\boldsymbol{x}}_t(\xi_{[t]})\right) + c_t - A_t^{\mathsf{T}}\bar{\boldsymbol{\pi}}_t(\xi_{[t]}) - \mathbb{E}\left[B_{t+1}^{\mathsf{T}}\bar{\boldsymbol{\pi}}_{t+1}(\xi_{[t+1]})\middle|\xi_{[t]}\right].$$

3.2.3 Dualization of Feasibility Constraints

Consider the linear multistage program given in the nested formulation (3.9). In this section we discuss dualization of that problem with respect to the feasibility constraints. As discussed before, we can formulate that problem as an optimization problem with respect to decision variables $x_t = \boldsymbol{x}_t(\xi_{[t]})$ viewed as functions of the history of the data process. Recall that the vector ξ_t of the data process of that problem is formed from some (or all) elements of (c_t, B_t, A_t, b_t), $t = 1, \ldots, T$. As before, we use the same symbols c_t, B_t, A_t, b_t to denote random variables and their particular realization. It will be clear from the context which of these meanings is used in a particular situation.

With problem (3.9) we associate the Lagrangian

$$\begin{aligned} L(\boldsymbol{x}, \boldsymbol{\pi}) &:= \mathbb{E}\left\{\sum_{t=1}^{T}\left[c_t^{\mathsf{T}}x_t + \pi_t^{\mathsf{T}}\left(b_t - B_t x_{t-1} - A_t x_t\right)\right]\right\} \\ &= \mathbb{E}\left\{\sum_{t=1}^{T}\left[c_t^{\mathsf{T}}x_t + \pi_t^{\mathsf{T}}b_t - \pi_t^{\mathsf{T}}A_t x_t - \pi_{t+1}^{\mathsf{T}}B_{t+1}x_t\right]\right\} \\ &= \mathbb{E}\left\{\sum_{t=1}^{T}\left[b_t^{\mathsf{T}}\pi_t + \left(c_t - A_t^{\mathsf{T}}\pi_t - B_{t+1}^{\mathsf{T}}\pi_{t+1}\right)^{\mathsf{T}}x_t\right]\right\} \end{aligned}$$

with the convention that $x_0 = 0$ and $B_{T+1} = 0$. Here the multipliers $\pi_t = \boldsymbol{\pi}_t(\xi_{[t]})$, as well as decisions $x_t = \boldsymbol{x}_t(\xi_{[t]})$, are functions of the data process up to time t.

The dual functional is defined as

$$D(\boldsymbol{\pi}) := \inf_{\boldsymbol{x} \geq 0} L(\boldsymbol{x}, \boldsymbol{\pi}),$$

where the minimization is performed over variables $x_t = \boldsymbol{x}_t(\xi_{[t]})$, $t = 1, \ldots, T$, in an appropriate functional space subject to the nonnegativity constraints. The Lagrangian dual of (3.9) is the problem

$$\operatorname*{Max}_{\boldsymbol{\pi}} D(\boldsymbol{\pi}), \tag{3.34}$$

where $\boldsymbol{\pi}$ lives in an appropriate functional space. Since, for a given $\boldsymbol{\pi}$, the Lagrangian $L(\cdot, \boldsymbol{\pi})$ is separable in $x_t = \boldsymbol{x}_t(\cdot)$, by the interchangeability principle (Theorem 7.80) we can move the operation of minimization with respect to x_t inside the conditional expectation $\mathbb{E}\left[\cdot \,|\xi_{[t]}\right]$. Therefore, we obtain

$$D(\boldsymbol{\pi}) = \mathbb{E}\left\{\sum_{t=1}^{T}\left[b_t^{\mathrm{T}}\pi_t + \inf_{x_t \in \mathbb{R}_+^{n_t}} \left(c_t - A_t^{\mathrm{T}}\pi_t - \mathbb{E}\left[B_{t+1}^{\mathrm{T}}\pi_{t+1}\big|\xi_{[t]}\right]\right)^{\mathrm{T}} x_t\right]\right\}.$$

Clearly we have that $\inf_{x_t \in \mathbb{R}_+^{n_t}} \left(c_t - A_t^{\mathrm{T}}\pi_t - \mathbb{E}\left[B_{t+1}^{\mathrm{T}}\pi_{t+1}\big|\xi_{[t]}\right]\right)^{\mathrm{T}} x_t$ is equal to zero if $A_t^{\mathrm{T}}\pi_t + \mathbb{E}\left[B_{t+1}^{\mathrm{T}}\pi_{t+1}|\xi_{[t]}\right] \leq c_t$, and to $-\infty$ otherwise. It follows that in the present case the dual problem (3.34) can be written as

$$\begin{aligned} &\operatorname*{Max}_{\boldsymbol{\pi}} \mathbb{E}\left[\sum_{t=1}^{T} b_t^{\mathrm{T}}\pi_t\right] \\ &\text{s.t. } A_t^{\mathrm{T}}\pi_t + \mathbb{E}\left[B_{t+1}^{\mathrm{T}}\pi_{t+1}|\xi_{[t]}\right] \leq c_t, \ t = 1, \ldots, T, \end{aligned} \tag{3.35}$$

where for the uniformity of notation we set all $T + 1$ terms equal to zero. Each multiplier vector $\pi_t = \boldsymbol{\pi}_t(\xi_{[t]})$, $t = 1, \ldots, T$, of problem (3.35) is a function of $\xi_{[t]}$. In that sense, these multipliers form a dual implementable policy. Optimization in (3.35) is performed over all implementable and feasible dual policies.

If the data process has a finite number of scenarios, then implementable policies $\boldsymbol{x}_t(\cdot)$ and $\boldsymbol{\pi}_t(\cdot)$, $t = 1, \ldots, T$, can be identified with finite dimensional vectors. In that case, the primal and dual problems form a pair of mutually dual linear programming problems. Therefore, the following duality result is a consequence of the general duality theory of linear programming.

Theorem 3.6. *Suppose that the data process has a finite number of possible realizations (scenarios). Then the optimal values of problems* (3.9) *and* (3.35) *are equal unless both problems are infeasible. If the (common) optimal value of these problems is finite, then both problems have optimal solutions.*

If the data process has a general distribution with an infinite number of possible realizations, then some regularity conditions are needed to ensure zero duality gap between problems (3.9) and (3.35).

3.2.4 Dualization of Nonanticipativity Constraints

In this section we deal with a problem which is slightly more general than linear problem (3.12). Let $f_t(x_t, \xi_t)$, $t = 1, \ldots, T$, be *random polyhedral* functions, and consider the problem

$$
\begin{array}{lllllllllll}
\text{Min} & \sum_{k=1}^{K} p_k \Big[f_1(x_1^k) & + & f_2^k(x_2^k) & + & f_3^k(x_3^k) & + & \cdots & + & f_T^k(x_T^k) \Big] & \\
\text{s.t.} & A_1 x_1^k & & & & & & & & = & b_1, \\
 & B_2^k x_1^k & + & A_2^k x_2^k & & & & & & = & b_2^k, \\
 & & & B_3^k x_2^k & + & A_3^k x_3^k & & & & = & b_3^k, \\
 & \multicolumn{9}{c}{\dotfill} \\
 & & & & & & & B_T^k x_{T-1}^k & + & A_T^k x_T^k = & b_T^k, \\
 & x_1^k \geq 0, & & x_2^k \geq 0, & & x_3^k \geq 0, & & \cdots & & x_T^k \geq 0, & \\
 & & & & & k = 1, \ldots, K. & & & & &
\end{array}
$$

Here $\xi_1^k, \ldots, \xi_T^k$, $k = 1, \ldots, K$, is a particular realization (scenario) of the corresponding data process, $f_t^k(x_t^k) := f_t(x_t^k, \xi_t^k)$ and $(B_t^k, A_t^k, b_t^k) := (B_t(\xi_t^k), A_t(\xi_t^k), b_t(\xi_t^k))$, $t = 2, \ldots, T$. This problem can be formulated as a multistage stochastic programming problem by enforcing the corresponding nonanticipativity constraints.

As discussed in section 3.1.4, there are many ways to write nonanticipativity constraints. For example, let $\mathfrak{X}$ be the linear space of all sequences $(x_1^k, \ldots, x_T^k)$, $k = 1, \ldots, K$, and $\mathfrak{L}$ be the linear subspace of $\mathfrak{X}$ defined by the nonanticipativity constraints. (These spaces were defined above (3.17).) We can write the corresponding multistage problem in the following lucid form:

$$
\operatorname*{Min}_{\boldsymbol{x} \in \mathfrak{X}} f(\boldsymbol{x}) := \sum_{k=1}^{K} \sum_{t=1}^{T} p_k f_t^k(x_t^k) \quad \text{s.t.} \quad \boldsymbol{x} \in \mathfrak{L}. \tag{3.36}
$$

Clearly, $f(\cdot)$ is a polyhedral function, so if problem (3.36) has a finite optimal value, then it has an optimal solution and the optimality conditions and duality relations hold true. Let us introduce the Lagrangian associated with (3.36),

$$
L(\boldsymbol{x}, \boldsymbol{\lambda}) := f(\boldsymbol{x}) + \langle \boldsymbol{\lambda}, \boldsymbol{x} \rangle,
$$

with the scalar product $\langle \cdot, \cdot \rangle$ defined in (3.17). By the definition of the subspace $\mathfrak{L}$, every point $\boldsymbol{x} \in \mathfrak{L}$ can be viewed as an implementable policy. By $\mathfrak{L}^{\perp} := \{\boldsymbol{y} \in \mathfrak{X} : \langle \boldsymbol{y}, \boldsymbol{x} \rangle = 0, \ \forall \boldsymbol{x} \in \mathfrak{L}\}$ we denote the orthogonal subspace to the subspace $\mathfrak{L}$.

Theorem 3.7. *A policy $\bar{\boldsymbol{x}} \in \mathfrak{L}$ is an optimal solution of* (3.36) *iff there exists a multiplier vector $\bar{\boldsymbol{\lambda}} \in \mathfrak{L}^{\perp}$ such that*

$$
\bar{\boldsymbol{x}} \in \arg\min_{\boldsymbol{x} \in \mathfrak{X}} L(\boldsymbol{x}, \bar{\boldsymbol{\lambda}}). \tag{3.37}
$$

Proof. Let $\bar{\boldsymbol{\lambda}} \in \mathfrak{L}^{\perp}$ and $\bar{\boldsymbol{x}} \in \mathfrak{L}$ be a minimizer of $L(\cdot, \bar{\boldsymbol{\lambda}})$ over $\mathfrak{X}$. Then by the first-order optimality conditions we have that $0 \in \partial_x L(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\lambda}})$. Note that there is no need here for a constraint qualification since the problem is polyhedral. Now $\partial_x L(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\lambda}}) = \partial f(\bar{\boldsymbol{x}}) + \bar{\boldsymbol{\lambda}}$.

Since $\mathcal{N}_{\mathfrak{L}}(\bar{\boldsymbol{x}}) = \mathfrak{L}^{\perp}$, it follows that $0 \in \partial f(\bar{\boldsymbol{x}}) + \mathcal{N}_{\mathfrak{L}}(\bar{\boldsymbol{x}})$, which is a sufficient condition for $\bar{\boldsymbol{x}}$ to be an optimal solution of (3.36). Conversely, if $\bar{\boldsymbol{x}}$ is an optimal solution of (3.36), then necessarily $0 \in \partial f(\bar{\boldsymbol{x}}) + \mathcal{N}_{\mathfrak{L}}(\bar{\boldsymbol{x}})$. This implies existence of $\bar{\boldsymbol{\lambda}} \in \mathfrak{L}^{\perp}$ such that $0 \in \partial_x L(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\lambda}})$. This, in turn, implies that $\bar{\boldsymbol{x}} \in \mathfrak{L}$ is a minimizer of $L(\cdot, \bar{\boldsymbol{\lambda}})$ over $\mathfrak{X}$. □

Also, we can define the dual function

$$D(\boldsymbol{\lambda}) := \inf_{\boldsymbol{x} \in \mathfrak{X}} L(\boldsymbol{x}, \boldsymbol{\lambda}),$$

and the dual problem

$$\operatorname*{Max}_{\boldsymbol{\lambda} \in \mathfrak{L}^{\perp}} D(\boldsymbol{\lambda}). \tag{3.38}$$

Since the problem considered is polyhedral, we have by the standard theory of linear programming the following results.

Theorem 3.8. *The optimal values of problems* (3.36) *and* (3.38) *are equal unless both problems are infeasible. If their (common) optimal value is finite, then both problems have optimal solutions.*

The crucial role in our approach is played by the requirement that $\boldsymbol{\lambda} \in \mathfrak{L}^{\perp}$. Let us decipher this condition. For $\boldsymbol{\lambda} = \left(\lambda_t^k\right)_{t=1,\ldots,T,\, k=1,\ldots,K}$, the condition $\boldsymbol{\lambda} \in \mathfrak{L}^{\perp}$ is equivalent to

$$\sum_{t=1}^{T} \sum_{k=1}^{K} p_k \langle \lambda_t^k, x_t^k \rangle = 0, \quad \forall \boldsymbol{x} \in \mathfrak{L}.$$

We can write this in a more abstract form as

$$\mathbb{E}\left[\sum_{t=1}^{T} \langle \lambda_t, x_t \rangle\right] = 0, \quad \forall \boldsymbol{x} \in \mathfrak{L}. \tag{3.39}$$

Since[14] $\mathbb{E}_{|t} x_t = x_t$ for all $\boldsymbol{x} \in \mathfrak{L}$, and $\langle \lambda_t, \mathbb{E}_{|t} x_t \rangle = \langle \mathbb{E}_{|t} \lambda_t, x_t \rangle$, we obtain from (3.39) that

$$\mathbb{E}\left[\sum_{t=1}^{T} \langle \mathbb{E}_{|t} \lambda_t, x_t \rangle\right] = 0, \quad \forall \boldsymbol{x} \in \mathfrak{L},$$

which is equivalent to

$$\mathbb{E}_{|t}[\lambda_t] = 0, \quad t = 1, \ldots, T. \tag{3.40}$$

We can now rewrite our necessary conditions of optimality and duality relations in a more explicit form. We can write the dual problem in the form

$$\operatorname*{Max}_{\boldsymbol{\lambda} \in \mathfrak{X}} D(\boldsymbol{\lambda}) \quad \text{s.t.} \quad \mathbb{E}_{|t}[\lambda_t] = 0, \quad t = 1, \ldots, T. \tag{3.41}$$

[14] In order to simplify notation, we denote in the remainder of this section by $\mathbb{E}_{|t}$ the conditional expectation, conditional on $\xi_{[t]}$.

Corollary 3.9. *A policy* $\bar{\boldsymbol{x}} \in \mathfrak{L}$ *is an optimal solution of* (3.36) *iff there exist multipliers vector* $\bar{\boldsymbol{\lambda}}$ *satisfying* (3.40) *such that*

$$\bar{\boldsymbol{x}} \in \arg\min_{\boldsymbol{x}\in\mathfrak{X}} L(\boldsymbol{x}, \bar{\boldsymbol{\lambda}}).$$

Moreover, problem (3.36) *has an optimal solution iff problem* (3.41) *has an optimal solution. The optimal values of these problems are equal unless both are infeasible.*

There are many different ways to express the nonanticipativity constraints, and thus there are many equivalent ways to formulate the Lagrangian and the dual problem. In particular, a dual formulation based on (3.18) is quite convenient for dual decomposition methods. We leave it to the reader to develop the particular form of the dual problem in this case.

Exercises

3.1. Consider the inventory model of section 1.2.3.

(a) Specify for this problem the variables, the data process, the functions, and the sets in the general formulation (3.1). Describe the sets $\mathcal{X}_t(x_{t-1}, \xi_t)$ as in formula (3.20).

(b) Transform the problem to an equivalent linear multistage stochastic programming problem.

3.2. Consider the cost-to-go function $Q_t(x_{t-1}, \xi_{[t]}), t = 2, \ldots, T$, of the linear multistage problem defined as the optimal value of problem (3.7). Show that $Q_t(x_{t-1}, \xi_{[t]})$ is convex in x_{t-1}.

3.3. Consider the assembly problem discussed in section 1.3.3 in the case when all demand has to be satisfied, by backlogging the orders. It costs b_i to delay delivery of a unit of product i by one period. Additional orders of the missing parts can be made after the last demand $D(T)$ is known. Write the dynamic programming equations of the problem. How they can be simplified, if the demand is stagewise independent?

3.4. A transportation company has n depots among which they move cargo. They are planning their operation in the next T days. The demand for transportation between depot i and depot $j \neq i$ on day t, where $t = 1, 2 \ldots, T$, is modeled as a random variable $D_{ij}(t)$. The total capacity of vehicles currently available at depot i is denoted $s_i, i = 1, \ldots, n$. Before each day t, the company considers repositioning their fleet to better prepare to the uncertain demand on the coming day. It costs c_{ij} to move a unit of capacity from location i to location j. After repositioning, the realization of the random variables $D_{ij}(t)$ is observed, and the demand is served, up to the limit determined by the transportation capacity available at each location. The profit from transporting a unit of cargo from location i to location j is equal q_{ij}. If the total demand at location i exceeds the capacity available at this location, the excessive demand is lost. It is up to the company to decide how much of each demand D_{ij} will be served, and which part will remain unsatisfied. For simplicity, we consider all capacity and transportation quantities as continuous variables.

After the demand is served, the transportation capacity of the vehicles at each location changes, as a result of the arrivals of vehicles with cargo from other locations.

Before the next day, the company may choose to reposition some of the vehicles to prepare for the next demand. On the last day, the vehicles are repositioned so that initial quantities s_i, $i = 1, \ldots, n$, are restored.

(a) Formulate the problem of maximizing the expected profit as a multistage stochastic programming problem.

(b) Write the dynamic programming equations for this problem. Assuming that the demand is stagewise independent, identify the state variables and simplify the dynamic programming equations.

(c) Develop a scenario-tree-based formulation of the problem.

3.5. Derive the dual problem to the linear multistage stochastic programming problem (3.12) with nonanticipativity constraints in the form (3.18).

3.6. You have initial capital C_0 which you may invest in a stock or keep in cash. You plan your investments for the next T periods. The return rate on cash is deterministic and equals r per each period. The price of the stock is random and equals S_t in period $t = 1, \ldots, T$. The current price S_0 is known to you and you have a model of the price process S_t in the form of a scenario tree. At the beginning, several American options on the stock prize are available. There are n put options with strike prices $p_1, \ldots, p_n$ and corresponding costs $c_1, \ldots, c_n$. For example, if you buy one put option i, at any time $t = 1, \ldots, T$ you have the right to exercise the option and cash $p_i - S_t$ (this makes sense only when $p_i > S_t$). Also, m call options are available, with strike prices $\pi_1, \ldots, \pi_m$ and corresponding costs $q_1, \ldots, q_m$. For example, if you buy one call option j, at any time $t = 1, \ldots, T$ you may exercise it and cash $S_t - \pi_j$ (this makes sense only when $\pi_j < S_t$). The options are available only at $t = 0$. At any time period t you may buy or sell the underlying stock. Borrowing cash and short selling, that is, selling shares which are not actually owned (with the hope of repurchasing them later with profit), are not allowed. At the end of period T all options expire. There are no transaction costs, and shares and options can be bought, sold (in the case of shares) or realized (in the case of options) in any quantities (not necessarily whole numbers). The amounts gained by exercising options are immediately available for purchasing shares.

Consider two objective functions:

(i) The expected value of your holdings at the end of period T.

(ii) The expected value of a piecewise linear utility function evaluated at the value of your final holdings. Its form is

$$u(C_T) = \begin{cases} C_T & \text{if } C_T \geq 0, \\ (1+R)C_T & \text{if } C_T < 0, \end{cases}$$

where $R > 0$ is some known constant.

For both objective functions,

(a) Develop a linear multistage stochastic programming model.

(b) Derive the dual problem by dualizing with respect to feasibility constraints.

Chapter 4

Optimization Models with Probabilistic Constraints

Darinka Dentcheva

4.1 Introduction

In this chapter, we discuss stochastic optimization problems with *probabilistic* (also called *chance*) constraints of the form

$$\begin{aligned} \text{Min} \;\; & c(x) \\ \text{s.t.} \;\; & \Pr\{g_j(x, Z) \le 0, \;\; j \in \mathcal{J}\} \ge p, \\ & x \in \mathcal{X}. \end{aligned} \tag{4.1}$$

Here $\mathcal{X} \subset \mathbb{R}^n$ is a nonempty set, $c : \mathbb{R}^n \to \mathbb{R}$, $g_j : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}$, $j \in \mathcal{J}$, where $\mathcal{J}$ is an index set, Z is an s-dimensional random vector, and p is a modeling parameter. We denote by P_Z the probability measure (probability distribution) induced by the random vector Z on $\mathbb{R}^s$. The event $A(x) = \{g_j(x, Z) \le 0, \;\; j \in \mathcal{J}\}$ in (4.1) depends on the decision vector x, and its probability $\Pr\{A(x)\}$ is calculated with respect to the probability distribution P_Z.

This model reflects the point of view that for a given decision x we do not reject the statistical hypothesis that the constraints $g_j(x, Z) \le 0$, $j \in \mathcal{J}$, are satisfied. We discussed examples and a motivation for such problems in Chapter 1 in the contexts of inventory, multiproduct, and portfolio selection models. We emphasize that imposing constraints on probability of events is particularly appropriate whenever high uncertainty is involved and reliability is a central issue. In such cases, constraints on the expected value may not be sufficient to reflect our attitude to undesirable outcomes.

We also note that the objective function $c(x)$ can represent an expected value function, i.e., $c(x) = \mathbb{E}[f(x, Z)]$; however, we focus on the analysis of the probabilistic constraints at the moment.

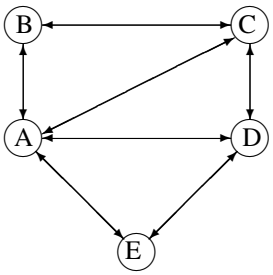

Figure 4.1. *Vehicle routing network*

We can write the probability $\Pr\{A(x)\}$ as the expected value of the characteristic function of the event $A(x)$, i.e., $\Pr\{A(x)\} = \mathbb{E}\left[\mathbf{1}_{A(x)}\right]$. The discontinuity of the characteristic function and the complexity of the event $A(x)$ make such problems qualitatively different from the expectation models. Let us consider two examples.

Example 4.1 (Vehicle Routing Problem). Consider a network with m arcs on which a random transportation demand arises. A set of n routes in the network is described by the incidence matrix T. More precisely, T is an $m \times n$ dimensional matrix such that

$$t_{ij} = \begin{cases} 1 & \text{if route } j \text{ contains arc } i, \\ 0 & \text{otherwise.} \end{cases}$$

We have to allocate vehicles to the routes to satisfy transportation demand. Figure 4.1 depicts a small network, and the table in Figure 4.2 provides the incidence information for 19 routes on this network. For example, route 5 consists of the arcs AB, BC, and CA.

Our aim is to satisfy the demand with high prescribed probability $p \in (0, 1)$. Let x_j be the number of vehicles assigned to route j, $j = 1, \ldots, n$. The demand for transportation on each arc is given by the random variables $Z_i, i = 1, \ldots, m$. We set $Z = (Z_1, \ldots, Z_m)^{\mathsf{T}}$. A cost c_j is associated with operating a vehicle on route j. Setting $c = (c_1, \ldots, c_n)^{\mathsf{T}}$, the model can be formulated as follows:[15]

$$\min_x \; c^{\mathsf{T}} x \tag{4.2}$$

$$\text{s.t. } \Pr\{Tx \geq Z\} \geq p, \tag{4.3}$$

$$x \in \mathbb{Z}_+^n. \tag{4.4}$$

In practical applications, we may have a heterogeneous fleet of vehicles with different capacities; we may consider imposing constraints on transportation time or other requirements. ∎

In the context of portfolio optimization, probabilistic constraints arise in a natural way, as discussed in Chapter 1.

[15]The notation $\mathbb{Z}_+$ is used to denote the set of nonnegative integer numbers.

Arc	*Route*																		
	1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*
AB	1				1						1				1		1		
AC		1				1	1						1					1	
AD			1					1	1			1							
AE				1						1				1		1			1
BA	1					1						1				1		1	
BC					1						1				1		1		
CA		1			1			1						1			1		
CB						1						1				1		1	
CD							1				1		1		1		1	1	
DA			1				1			1	1								
DC								1				1		1		1	1	1	
DE									1				1		1				1
EA				1					1				1		1				1
ED										1				1		1			1

Figure 4.2. *Vehicle routing incidence matrix*

Example 4.2 (Portfolio Optimization with Value-at-Risk Constraint). We consider n investment opportunities with random return rates $R_1, \ldots, R_n$ in the next year. We have certain initial capital and our aim is to invest it in such a way that the expected value of our investment after a year is maximized, under the condition that the chance of losing no more than a given fraction of this amount is at least p, where $p \in (0, 1)$. Such a requirement is called a *Value-at-Risk* (V@R) constraint (already discussed in Chapter 1).

Let $x_1, \ldots, x_n$ be the fractions of our capital invested in the n assets. After a year, our investment changes in value according to a rate that can be expressed as

$$g(x, R) = \sum_{i=1}^{n} R_i x_i.$$

We formulate the following stochastic optimization problem with a probabilistic constraint:

$$\begin{aligned} \text{Max} \ & \sum_{i=1}^{n} \mathbb{E}[R_i] x_i \\ \text{s.t.} \ & \Pr\Big\{ \sum_{i=1}^{n} R_i x_i \geq \eta \Big\} \geq p, \\ & \sum_{i=1}^{n} x_i = 1, \\ & x \geq 0. \end{aligned} \tag{4.5}$$

For example, $\eta = -0.1$ may be chosen if we aim at protecting against losses larger than 10%. ∎

The constraint

$$\Pr\{g_j(x, Z) \le 0,\ j \in \mathcal{J}\} \ge p$$

is called *a joint probabilistic constraint*, while the constraints

$$\Pr\{g_j(x, Z) \le 0\} \ge p_j,\ j \in \mathcal{J},\ \text{where } p_j \in [0, 1],$$

are called *individual probabilistic constraints*.

In the vehicle routing example, we have a joint probabilistic constraint. If we were to cover the demand on each arc separately with high probability, then the constraints would be formulated as follows:

$$\Pr\{T^i x \ge Z_i\} \ge p_i, \quad i = 1, \ldots, m,$$

where T^i denotes the ith row of the matrix T. However, the latter formulation would not ensure reliability of the network as a whole.

Infinitely many individual probabilistic constraints appear naturally in the context of stochastic orders. For an integrable random variable X, we consider its distribution function $F_X(\cdot)$.

Definition 4.3. *A random variable X* dominates in the first order *a random variable Y (denoted $X \succeq_{(1)} Y$) if*

$$F_X(\eta) \le F_Y(\eta), \quad \forall \eta \in \mathbb{R}.$$

The left-continuous inverse $F_X^{(-1)}$ of the cumulative distribution function of a random variable X is defined as follows:

$$F_X^{(-1)}(p) = \inf\ \{\eta : F_1(X; \eta) \ge p\}, \quad p \in (0, 1).$$

Given $p \in (0, 1)$, the number $q = q(X;\ p)$ is called a *p-quantile* of the random variable X if

$$\Pr\{X < q\} \le p \le \Pr\{X \le q\}.$$

For $p \in (0, 1)$ the set of p-quantiles is a closed interval and $F_X^{(-1)}(p)$ represents its left end. Directly from the definition of the first order dominance we see that

$$X \succeq_{(1)} Y \quad \Leftrightarrow \quad F_X^{(-1)}(p) \ge F_Y^{(-1)}(p), \quad \forall p \in (0, 1). \tag{4.6}$$

The first order dominance constraint can be interpreted as a continuum of probabilistic (chance) constraints.

Denoting $F_X^{(1)}(\eta) = F_X(\eta)$, we define higher order distribution functions of a random variable $X \in \mathcal{L}_{k-1}(\Omega, \mathcal{F}, P)$ as follows:

$$F_X^{(k)}(\eta) = \int_{-\infty}^{\eta} F_X^{(k-1)}(t)\, dt \quad \text{for } k = 2, 3, 4, \ldots.$$

We can express the integrated distribution function $F_X^{(2)}$ as the expected shortfall function. Integrating by parts, for each value η, we have the following formula:[16]

$$F_X^{(2)}(\eta) = \int_{-\infty}^{\eta} F_X(\alpha)\, d\alpha = \mathbb{E}\big[(\eta - X)_+\big]. \tag{4.7}$$

The function $F_X^{(2)}(\cdot)$ is well defined and finite for every integrable random variable. It is continuous, nonnegative, and nondecreasing. The function $F_X^{(2)}(\cdot)$ is also convex because its derivative is nondecreasing as it is a cumulative distribution function. By the same arguments, the higher order distribution functions are continuous, nonnegative, nondecreasing, and convex as well.

Due to (4.7), the second order dominance relation can be expressed in an equivalent way as follows:

$$X \succeq_{(2)} Y \text{ iff } \mathbb{E}\{[\eta - X]_+\} \leq \mathbb{E}\{[\eta - Y]_+\}, \quad \forall \eta \in \mathbb{R}. \tag{4.8}$$

The stochastic dominance relation generalizes to higher orders as follows.

Definition 4.4. *Given two random variables X and Y in $\mathcal{L}_{k-1}(\Omega, \mathcal{F}, P)$ we say that X* dominates *Y* in the *k*th order *if*

$$F_X^{(k)}(\eta) \leq F_Y^{(k)}(\eta), \quad \forall \eta \in \mathbb{R}.$$

We denote this relation by $X \succeq_{(k)} Y$.

We call the following semi-infinite (probabilistic) problem a *stochastic optimization problem with a stochastic ordering constraint*:

$$\begin{aligned} \operatorname*{Min}_x \ & c(x) \\ \text{s.t. } & \Pr\{g(x, Z) \leq \eta\} \leq F_Y(\eta), \quad \eta \in [a, b], \\ & x \in \mathcal{X}. \end{aligned} \tag{4.9}$$

Here the dominance relation is restricted to an interval $[a, b] \subset \mathbb{R}$. There are technical reasons for this restriction, which will become apparent later. In the case of discrete distributions with finitely many realizations, we can assume that the interval $[a, b]$ contains the entire support of the probability measures.

In general, we formulate the following semi-infinite probabilistic problem, which we refer to as a *stochastic optimization problem with a stochastic dominance constraint* of order $k \geq 2$:

$$\begin{aligned} \operatorname*{Min}_x \ & c(x) \\ \text{s.t. } & F_{g(x,Z)}^{(k)}(\eta) \leq F_Y^{(k)}(\eta), \quad \eta \in [a, b], \\ & x \in \mathcal{X}. \end{aligned} \tag{4.10}$$

[16] Recall that $[a]_+ = \max\{a, 0\}$.

Example 4.5 (Portfolio Selection Problem with Stochastic Ordering Constraints). Returning to Example 4.2, we can require that the net profit on our investment dominates certain benchmark outcome Y, which may be the return rate of our current portfolio or the return rate of some index. Then the Value-at-Risk constraint has to be satisfied at a continuum of points $\eta \in \mathbb{R}$. Setting $\Pr\{Y \le \eta\} = p_\eta$, we formulate the following model:

$$\begin{aligned} \text{Max} \ & \sum_{i=1}^{n} \mathbb{E}[R_i] x_i \\ \text{s.t.} \ & \Pr\left\{ \sum_{i=1}^{n} R_i x_i \le \eta \right\} \le p_\eta, \quad \forall \eta \in \mathbb{R}, \\ & \sum_{i=1}^{n} x_i = 1, \\ & x \ge 0. \end{aligned} \tag{4.11}$$

Using higher order stochastic dominance relations, we formulate a portfolio optimization model of form

$$\begin{aligned} \text{Max} \ & \sum_{i=1}^{n} \mathbb{E}[R_i] x_i \\ \text{s.t.} \ & \sum_{i=1}^{n} R_i x_i \succeq_{(k)} Y, \\ & \sum_{i=1}^{n} x_i = 1, \\ & x \ge 0. \end{aligned} \tag{4.12}$$

A second order dominance constraint on the portfolio return rate represents a constraint on the shortfall function:

$$\sum_{i=1}^{n} R_i x_i \succeq_{(2)} Y \quad \Longleftrightarrow \quad \mathbb{E}\left[\left(\eta - \sum_{i=1}^{n} R_i x_i \right)_+ \right] \le \mathbb{E}\big[(\eta - Y)_+\big], \quad \forall \eta \in \mathbb{R}.$$

The second order dominance constraint can also be viewed as a continuum of *Average Value-at-Risk*[17] (AV@R) constraints. For more information on this connection, see Dentcheva and Ruszczyński [56]. ∎

We stress that if $a = b$, then the semi-infinite model (4.9) reduces to a problem with a single probabilistic constraint, and problem (4.10) for $k = 2$ becomes a problem with a single shortfall constraint.

We shall pay special attention to problems with *separable* functions $g_i, i = 1, \ldots, m$, that is, functions of form $g_i(x, z) = \hat{g}_i(x) + h_i(z)$. The probabilistic constraint becomes

$$\Pr\{\hat{g}_i(x) \ge -h_i(Z), \ i = 1, \ldots, m\} \ge p.$$

[17] Average Value-at-Risk is also called Conditional Value-at-Risk.

We can view the inequalities under the probability as a deterministic vector function $\hat{g} : \mathbb{R}^n \to \mathbb{R}^m$, $\hat{g} = [\hat{g}_1, \ldots, \hat{g}_m]^{\mathrm{T}}$ constrained from below by a random vector Y with $Y_i = -h_i(Z)$, $i = 1, \ldots, m$. The problem can be formulated as

$$\begin{aligned} \operatorname*{Min}_{x} \ & c(x) \\ \text{s.t.} \ & \Pr\{\hat{g}(x) \geq Y\} \geq p, \\ & x \in \mathcal{X}, \end{aligned} \tag{4.13}$$

where the inequality $a \leq b$ for two vectors $a, b \in \mathbb{R}^n$ is understood componentwise.

We note again that the objective function can have a more specific form:

$$c(x) = \mathbb{E}[f(x, Z)].$$

By virtue of Theorem 7.43, we have that if the function $f(\cdot, Z)$ is continuous at x_0 w.p. 1 and there exists an integrable random variable $\hat{Z}$ such that $|f(x, Z(\omega))| \leq \hat{Z}(\omega)$ for P-almost every $\omega \in \Omega$ and for all x in a neighborhood of x_0, then for all x in a neighborhood of x_0 the expected value function $c(x)$ is well defined and continuous at x_0. Furthermore, convexity of $f(\cdot, Z)$ for a.e. Z implies convexity of the expectation function $c(x)$. Therefore, we can carry out the analysis of probabilistically constrained problems using a general objective function $c(x)$ with the understanding that in some cases it may be defined as an expectation function.

Problems with separable probabilistic constraints arise frequently in the context of serving certain demand, as in the vehicle routing Example 4.1. Another type of example is an inventory problem, as the following one.

Example 4.6 (Cash Matching with Probabilistic Liquidity Constraint). We have random liabilities L_t in periods $t = 1, \ldots, T$. We consider an investment in a bond portfolio from a basket of n bonds. The payment of bond i in period t is denoted by a_{it}. It is zero for the time periods t before purchasing of the bond is possible, as well as for t greater than the maturity time of the bond. At the time period of purchase, a_{it} is the negative of the price of the bond. At the following periods, a_{it} is equal to the coupon payment, and at the time of maturity it is equal to the face value plus the coupon payment. All prices of bonds and coupon payments are deterministic and no default is assumed. Our initial capital equals c_0.

The objective is to design a bond portfolio such that the probability of covering the liabilities over the entire period $1, \ldots, T$ is at least p. Subject to this condition, we want to maximize the final cash on hand, guaranteed with probability p.

Let us introduce the cumulative liabilities

$$Z_t = \sum_{\tau=1}^{t} L_\tau, \quad t = 1, \ldots, T.$$

Denoting by x_i the amount invested in bond i, we observe that the cumulative cash flows up to time t, denoted c_t, can be expressed as follows:

$$c_t = c_{t-1} + \sum_{i=1}^{n} a_{it} x_i, \quad t = 1, \ldots, T.$$

Using cumulative cash flows and cumulative liabilities permits the carryover of capital from one stage to the next, while keeping the random quantities at the right-hand side of the constraints. We represent the cumulative cash flow during the entire period by the vector $c = (c_1, \ldots, c_T)^{\mathrm{T}}$. Let us assume that we quantify our preferences by using concave utility function $U : \mathbb{R} \to \mathbb{R}$. We would like to maximize the final capital at hand in a risk-averse manner. The problem takes on the form

$$\begin{aligned} \max_{x,c} \ & \mathbb{E}\,[U(c_T - Z_T)] \\ \text{s.t. } & \Pr\{c_t \geq Z_t,\ t = 1, \ldots, T\} \geq p, \\ & c_t = c_{t-1} + \sum_{i=1}^{n} a_{it} x_i, \quad t = 1, \ldots, T, \\ & x \geq 0. \end{aligned}$$

This optimization problem has the structure of model (4.13). The first constraint can be called a *probabilistic liquidity constraint*. ∎

4.2 Convexity in Probabilistic Optimization

Fundamental questions for every optimization model concern convexity of the feasible set, as well as continuity and differentiability of the constraint functions. The analysis of models with probability functions is based on specific properties of the underlying probability distributions. In particular, the *generalized concavity* theory plays a central role in probabilistic optimization as it facilitates the application of powerful tools of convex analysis.

4.2.1 Generalized Concavity of Functions and Measures

We consider various nonlinear transformations of functions $f : \Omega \to \mathbb{R}_+$ defined on a convex set $\Omega \subset \mathbb{R}^n$.

Definition 4.7. *A nonnegative function $f(x)$ defined on a convex set $\Omega \subset \mathbb{R}^n$ is said to be α-concave, where $\alpha \in [-\infty, +\infty]$, if for all $x, y \in \Omega$ and all $\lambda \in [0, 1]$ the following inequality holds true:*

$$f(\lambda x + (1 - \lambda)y) \geq m_\alpha(f(x), f(y), \lambda),$$

where $m_\alpha : \mathbb{R}_+ \times \mathbb{R}_+ \times [0, 1] \to \mathbb{R}$ is defined as follows:

$$m_\alpha(a, b, \lambda) = 0 \quad \text{if} \quad ab = 0,$$

and if $a > 0$, $b > 0$, $0 \leq \lambda \leq 1$, then

$$m_\alpha(a, b, \lambda) = \begin{cases} a^\lambda b^{1-\lambda} & \text{if } \alpha = 0, \\ \max\{a, b\} & \text{if } \alpha = \infty, \\ \min\{a, b\} & \text{if } \alpha = -\infty, \\ (\lambda a^\alpha + (1 - \lambda) b^\alpha)^{1/\alpha} & \text{otherwise}. \end{cases}$$

In the case of $\alpha = 0$ the function f is called *logarithmically concave* or *log-concave* because $\ln f(\cdot)$ is a concave function. In the case of $\alpha = 1$, the function f is simply *concave*.

It is important to note that if f and g are two measurable functions, then the function $m_\alpha(f(\cdot), g(\cdot), \lambda)$ is a measurable function for all α and all $\lambda \in (0, 1)$. Furthermore, $m_\alpha(a, b, \lambda)$ has the following important property.

Lemma 4.8. *The mapping $\alpha \mapsto m_\alpha(a, b, \lambda)$ is nondecreasing and continuous.*

Proof. First we show the continuity of the mapping at $\alpha = 0$. We have the following chain of equations:

$$\begin{aligned}\ln m_\alpha(a, b, \lambda) &= \ln(\lambda a^\alpha + (1-\lambda)b^\alpha)^{1/\alpha} = \frac{1}{\alpha}\ln\left(\lambda e^{\alpha \ln a} + (1-\lambda)e^{\alpha \ln b}\right) \\ &= \frac{1}{\alpha}\ln\Big(1 + \alpha\big(\lambda \ln a + (1-\lambda)\ln b\big) + o(\alpha^2)\Big).\end{aligned}$$

Applying the l'Hôpital rule to the right-hand side in order to calculate its limit when $\alpha \to 0$, we obtain

$$\begin{aligned}\lim_{\alpha\to 0} \ln m_\alpha(a, b, \lambda) &= \lim_{\alpha\to 0} \frac{\lambda \ln a + (1-\lambda)\ln b + o(\alpha)}{1 + \alpha\big(\lambda \ln a + (1-\lambda)\ln b\big) + o(\alpha^2)} \\ &= \lim_{\alpha\to 0} \frac{\ln(a^\lambda b^{(1-\lambda)}) + o(\alpha)}{1 + \alpha \ln(a^\lambda b^{(1-\lambda)}) + o(\alpha^2)} = \ln(a^\lambda b^{(1-\lambda)}).\end{aligned}$$

Now we turn to the monotonicity of the mapping. First, let us consider the case of $0 < \alpha < \beta$. We set

$$h(\alpha) = m_\alpha(a, b, \lambda) = \exp\left(\frac{1}{\alpha}\ln\left[\lambda a^\alpha + (1-\lambda)b^\alpha\right]\right).$$

Calculating its derivative, we obtain

$$h'(\alpha) = h(\alpha)\Big(\frac{1}{\alpha} \cdot \frac{\lambda a^\alpha \ln a + (1-\lambda)b^\alpha \ln b}{\lambda a^\alpha + (1-\lambda)b^\alpha} - \frac{1}{\alpha^2}\ln\left[\lambda a^\alpha + (1-\lambda)b^\alpha\right]\Big).$$

We have to demonstrate that the expression on the right-hand side is nonnegative. Substituting $x = a^\alpha$ and $y = b^\alpha$, we obtain

$$h'(\alpha) = \frac{1}{\alpha^2}h(\alpha)\Big(\frac{\lambda x \ln x + (1-\lambda)y \ln y}{\lambda x + (1-\lambda)y} - \ln\left[\lambda x + (1-\lambda)y\right]\Big).$$

Using the fact that the function $z \mapsto z \ln z$ is convex for $z > 0$ and that both $x, y > 0$, we have that

$$\frac{\lambda x \ln x + (1-\lambda)y \ln y}{\lambda x + (1-\lambda)y} - \ln\left[\lambda x + (1-\lambda)y\right] \geq 0.$$

As $h(\alpha) > 0$, we conclude that $h(\cdot)$ is nondecreasing in this case. If $\alpha < \beta < 0$, we have the following chain of relations:

$$m_\alpha(a, b, \lambda) = \left[m_{-\alpha}\Big(\frac{1}{a}, \frac{1}{b}, \lambda\Big)\right]^{-1} \leq \left[m_{-\beta}\Big(\frac{1}{a}, \frac{1}{b}, \lambda\Big)\right]^{-1} = m_\beta(a, b, \lambda).$$

In the case of $0 = \alpha < \beta$, we can select a sequence $\{\alpha_k\}$ such that $\alpha_k > 0$ and $\lim_{k\to\infty} \alpha_k = 0$. We use the monotonicity of $h(\cdot)$ for positive arguments and the continuity at 0 to obtain the desired assertion. In the case $\alpha < \beta = 0$, we proceed in the same way, choosing appropriate sequence approaching 0.

If $\alpha < 0 < \beta$, then the inequality

$$m_\alpha(a, b, \lambda) \le m_0(a, b, \lambda) \le m_\beta(a, b, \lambda)$$

follows from the previous two cases. It remains to investigate how the mapping behaves when $\alpha \to \infty$ or $\alpha \to -\infty$. We observe that

$$\max\{\lambda^{1/\alpha} a, (1-\lambda)^{1/\alpha} b\} \le m_\alpha(a, b, \lambda) \le \max\{a, b\}.$$

Passing to the limit, we obtain that

$$\lim_{\alpha\to\infty} m_\alpha(a, b, \lambda) = \max\{a, b\}.$$

We also conclude that

$$\lim_{\alpha\to-\infty} m_\alpha(a, b, \lambda) = \lim_{\alpha\to-\infty} [m_{-\alpha}(1/a, 1/b, \lambda)]^{-1} = [\max\{1/a, 1/b\}]^{-1} = \min\{a, b\}.$$

This completes the proof. □

This statement has the very important implication that α-concavity entails β-concavity for all $\beta \le \alpha$. Therefore, all α-concave functions are $(-\infty)$-concave, that is, *quasi-concave*.

Example 4.9. Consider the density function of a nondegenerate multivariate normal distribution on $\mathbb{R}^s$:

$$\theta(x) = \frac{1}{\sqrt{(2\pi)^s \det(\Sigma)}} \exp\left\{-\tfrac{1}{2}(x-\mu)^{\mathrm{T}} \Sigma^{-1}(x-\mu)\right\},$$

where Σ is a positive definite symmetric matrix of dimension $s \times s$, $\det(\Sigma)$ denotes the determinant of the matrix Σ, and $\mu \in \mathbb{R}^s$. We observe that

$$\ln \theta(x) = -\tfrac{1}{2}(x-\mu)^{\mathrm{T}} \Sigma^{-1}(x-\mu) - \ln\left(\sqrt{(2\pi)^s \det(\Sigma)}\right)$$

is a concave function. Therefore, we conclude that θ is 0-concave, or log-concave. ■

Example 4.10. Consider a convex body (a convex compact set with nonempty interior) $\Omega \subset \mathbb{R}^s$. The uniform distribution on this set has density defined as follows:

$$\theta(x) = \begin{cases} \frac{1}{V_s(\Omega)}, & x \in \Omega, \\ 0, & x \notin \Omega, \end{cases}$$

where $V_s(\Omega)$ denotes the Lebesgue measure of Ω. The function $\theta(x)$ is quasi-concave on $\mathbb{R}^s$ and $+\infty$-concave on Ω. ■

We point out that for two Borel measurable sets A, B in $\mathbb{R}^s$, the Minkowski sum $A + B = \{x + y : x \in A,\ y \in B\}$ is Lebesgue measurable in $\mathbb{R}^s$.

Definition 4.11. *A probability measure P defined on the Lebesgue measurable subsets of a convex set $\Omega \subset \mathbb{R}^s$ is said to be* α-concave *if for any Borel measurable sets $A, B \subset \Omega$ and for all $\lambda \in [0, 1]$ we have the inequality*

$$P\left(\lambda A + (1-\lambda)B\right) \geq m_\alpha\big(P(A), P(B), \lambda\big),$$

where $\lambda A + (1-\lambda)B = \{\lambda x + (1-\lambda)y : x \in A,\ y \in B\}$.

We say that a random vector Z with values in $\mathbb{R}^n$ has an α-concave distribution if the probability measure P_Z induced by Z on $\mathbb{R}^n$ is α-concave.

Lemma 4.12. *If a random vector Z induces an α-concave probability measure on $\mathbb{R}^s$, then its cumulative distribution function F_Z is an α-concave function.*

Proof. Indeed, for given points $z^1, z^2 \in \mathbb{R}^s$ and $\lambda \in [0, 1]$, we define

$$A = \{z \in \mathbb{R}^s : z_i \leq z_i^1, i = 1, \ldots, s\} \quad \text{and} \quad B = \{z \in \mathbb{R}^s : z_i \leq z_i^2, i = 1, \ldots, s\}.$$

Then the inequality for F_Z follows from the inequality in Definition 4.11. □

Lemma 4.13. *If a random vector Z has independent components with log-concave marginal distributions, then Z has a log-concave distribution.*

Proof. For two Borel sets A, $B \subset \mathbb{R}^s$ and $\lambda \in (0, 1)$, we define the set $C = \lambda A + (1-\lambda)B$. Denote the projections of A, B and C on the coordinate axis by A_i, B_i and C_i, $i = 1, \ldots, s$, respectively. For any number $r \in C_i$ there is $c \in C$ such that $c_i = r$, which implies that we have $a \in A$ and $b \in B$ with $\lambda a + (1-\lambda)b = c$ and $r = \lambda a_i + (1-\lambda)b_i$. In other words, $r \in \lambda A_i + (1-\lambda)B_i$, and we conclude that $C_i \subset \lambda A_i + (1-\lambda)B_i$. On the other hand, if $r \in \lambda A_i + (1-\lambda)B_i$, then we have $a \in A$ and $b \in B$ such that $r = \lambda a_i + (1-\lambda)b_i$. Setting $c = \lambda a + (1-\lambda)b$, we conclude that $r \in C_i$. We obtain

$$\begin{aligned}
\ln[P_Z(C)] &= \sum_{i=1}^{s} \ln[P_{Z_i}(C_i)] = \sum_{i=1}^{s} \ln[P_{Z_i}(\lambda A_i + (1-\lambda)B_i)] \\
&\geq \sum_{i=1}^{s} \big(\lambda \ln[P_{Z_i}(A_i)] + (1-\lambda)\ln[P_{Z_i}(B_i)]\big) \\
&= \lambda \ln[P_Z(A)] + (1-\lambda)\ln[P_Z(B)]. \quad \square
\end{aligned}$$

As usually, concavity properties of a function imply a certain continuity of the function. We formulate without proof two theorems addressing this issue.

Theorem 4.14 (Borell [24]). *If P is a quasi-concave measure on $\mathbb{R}^s$ and the dimension of its support is s, then P has a density with respect to the Lebesgue measure.*

We can relate the α-concavity property of a measure to generalized concavity of its density. (See Brascamp and Lieb [26], Prékopa [159], Rinott [168], and the references therein.)

Theorem 4.15. *Let Ω be a convex subset of $\mathbb{R}^s$ and let $m > 0$ be the dimension of the smallest affine subspace L containing Ω. The probability measure P on Ω is γ-concave with $\gamma \in [-\infty, 1/m]$ iff its probability density function with respect to the Lebesgue measure on L is α-concave with*

$$\alpha = \begin{cases} \gamma/(1-m\gamma) & \text{if } \gamma \in (-\infty, 1/m), \\ -1/m & \text{if } \gamma = -\infty, \\ +\infty & \text{if } \gamma = 1/m. \end{cases}$$

Corollary 4.16. *Let an integrable function $\theta(x)$ be define and positive on a nondegenerate convex set $\Omega \subset \mathbb{R}^s$. Denote $c = \int_\Omega \theta(x)\,dx$. If $\theta(x)$ is α-concave with $\alpha \in [-1/s, \infty]$ and positive on the interior of Ω, then the measure P on Ω defined by setting that*

$$P(A) = \frac{1}{c}\int_A \theta(x)\,dx, \quad A \subset \Omega,$$

is γ-concave with

$$\gamma = \begin{cases} \alpha/(1+s\alpha) & \text{if } \alpha \in (-1/s, \infty), \\ 1/s & \text{if } \alpha = \infty, \\ -\infty & \text{if } \alpha = -1/s. \end{cases}$$

In particular, if a measure P on $\mathbb{R}^s$ has a density function $\theta(x)$ such that $\theta^{-1/s}$ is convex, then P is quasi-concave.

Example 4.17. We observed in Example 4.10 that the density of the unform distribution on a convex body Ω is a ∞-concave function. Hence, it generates a $1/s$-concave measure on Ω. On the other hand, the density of the normal distribution (Example 4.9) is log-concave, and, therefore, it generates a log-concave probability measure. ∎

Example 4.18. Consider positive numbers $\alpha_1, \ldots, \alpha_s$ and the simplex

$$S = \left\{ x \in \mathbb{R}^s : \sum_{i=1}^s x_i \le 1,\ x_i \ge 0,\ i = 1, \ldots, s \right\}.$$

The density function of the *Dirichlet distribution* with parameters $\alpha_1, \ldots, \alpha_s$ is defined as follows:

$$\theta(x) = \begin{cases} \dfrac{\Gamma(\alpha_1 + \cdots + \alpha_s)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_s)} x_1^{\alpha_1 - 1} x_2^{\alpha_2 - 1} \cdots x_s^{\alpha_s - 1} & \text{if } x \in \operatorname{int} S, \\ 0 & \text{otherwise.} \end{cases}$$

Here $\Gamma(\cdot)$ stands for the Gamma function $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$.

Assuming that $x \in \operatorname{int} S$, we consider

$$\ln \theta(x) = \sum_{i=1}^s (\alpha_i - 1) \ln x_i + \ln \Gamma(\alpha_1 + \cdots + \alpha_s) - \sum_{i=1}^s \ln \Gamma(\alpha_i).$$

If $\alpha_i \geq 1$ for all $i = 1, \ldots, s$, then $\ln \theta(\cdot)$ is a concave function on the interior of S and, therefore, $\theta(x)$ is log-concave on cl S. If all parameters satisfy $\alpha_i \leq 1$, then $\theta(x)$ is log-convex on cl (S). For other sets of parameters, this density function does not have any generalized concavity properties. ∎

The next results provide calculus rules for α-concave functions.

Theorem 4.19. *If the function $f : \mathbb{R}^n \to \mathbb{R}_+$ is α-concave and the function $g : \mathbb{R}^n \to \mathbb{R}_+$ is β-concave, where $\alpha, \beta \geq 1$, then the function $h : \mathbb{R}^n \to \mathbb{R}$, defined as $h(x) = f(x) + g(x)$ is γ-concave with $\gamma = \min\{\alpha, \beta\}$.*

Proof. Given points $x_1, x_2 \in \mathbb{R}^n$ and a scalar $\lambda \in (0, 1)$, we set $x_\lambda = \lambda x_1 + (1-\lambda)x_2$. Both functions f and g are γ-concave by virtue of Lemma 4.8. Using the Minkowski inequality, which holds true for $\gamma \geq 1$, we obtain

$$\begin{aligned} &f(x_\lambda) + g(x_\lambda) \\ &\quad \geq \big[\lambda\big(f(x_1)\big)^\gamma + (1-\lambda)\big(f(x_2)\big)^\gamma\big]^{\frac{1}{\gamma}} + \big[\lambda\big(g(x_1)\big)^\gamma + (1-\lambda)\big(g(x_2)\big)^\gamma\big]^{\frac{1}{\gamma}} \\ &\quad \geq \big[\lambda\big(f(x_1) + g(x_1)\big)^\gamma + (1-\lambda)\big(f(x_2) + g(x_2)\big)^\gamma\big]^{\frac{1}{\gamma}}. \end{aligned}$$

This completes the proof. □

Theorem 4.20. *Let f be a concave function defined on a convex set $C \subset \mathbb{R}^s$ and $g : \mathbb{R} \to \mathbb{R}$ be a nonnegative nondecreasing α-concave function, $\alpha \in [-\infty, \infty]$. Then the function $g \circ f$ is α-concave.*

Proof. Given $x, y \in \mathbb{R}^s$ and a scalar $\lambda \in (0, 1)$, we consider $z = \lambda x + (1-\lambda)y$. We have $f(z) \geq \lambda f(x) + (1-\lambda) f(y)$. By monotonicity and α-concavity of g, we obtain the following chain of inequalities:

$$[g \circ f](z) \geq g(\lambda f(x) + (1-\lambda) f(y)) \geq m_\alpha\big(g(f(x)), g(f(y)), \lambda\big).$$

This proves the assertion. □

Theorem 4.21. *Let the function $f : \mathbb{R}^m \times \mathbb{R}^s \to \mathbb{R}_+$ be such that for all $y \in Y \subset \mathbb{R}^s$ the function $f(\cdot, y)$ is α-concave ($\alpha \in [-\infty, \infty]$) on the convex set $X \subset \mathbb{R}^m$. Then the function $\varphi(x) = \inf_{y \in Y} f(x, y)$ is α-concave on X.*

Proof. Let $x_1, x_2 \in X$ and a scalar $\lambda \in (0, 1)$ be given. We set $z = \lambda x_1 + (1-\lambda)x_2$. We can find a sequence of points $y_k \in Y$ such that

$$\varphi(z) = \inf_{y \in Y} f(z, y) = \lim_{k \to \infty} f(z, y_k).$$

Using the α-concavity of the function $f(\cdot, y)$, we conclude that

$$f(z, y_k) \geq m_\alpha\big(f(x_1, y_k), f(x_2, y_k), \lambda\big).$$

The mapping $(a, b) \mapsto m_\alpha(a, b, \lambda)$ is monotone for nonnegative a and b and $\lambda \in (0, 1)$. Therefore, we have that

$$f(z, y_k) \geq m_\alpha\big(\varphi(x_1), \varphi(x_2), \lambda\big).$$

Passing to the limit, we obtain the assertion. □

Lemma 4.22. *If $\alpha_i > 0$, $i = 1, \ldots, m$, and $\sum_{i=1}^m \alpha_i = 1$, then the function $f : \mathbb{R}^m_+ \to \mathbb{R}$, defined as $f(x) = \prod_{i=1}^m x_i^{\alpha_i}$ is concave.*

Proof. We shall show the statement for the case of $m = 2$. For points $x, y \in \mathbb{R}^2_+$ and a scalar $\lambda \in (0, 1)$, we consider $\lambda x + (1 - \lambda)y$. Define the quantities

$$a_1 = (\lambda x_1)^{\alpha_1}, \quad a_2 = ((1 - \lambda)y_1)^{\alpha_1}, \quad b_1 = (\lambda x_2)^{\alpha_2}, \quad b_2 = ((1 - \lambda)y_2)^{\alpha_2}.$$

Using Hölder's inequality, we obtain the following:

$$\begin{aligned} f(\lambda x + (1 - \lambda)y) &= \left(a_1^{\frac{1}{\alpha_1}} + a_2^{\frac{1}{\alpha_1}}\right)^{\alpha_1} \left(b_1^{\frac{1}{\alpha_2}} + b_2^{\frac{1}{\alpha_2}}\right)^{\alpha_2} \\ &\geq a_1 b_1 + a_2 b_2 = \lambda x_1^{\alpha_1} x_2^{\alpha_2} + (1 - \lambda) y_1^{\alpha_1} y_2^{\alpha_2}. \end{aligned}$$

The assertion in the general case follows by induction. □

Theorem 4.23. *If the functions $f_i : \mathbb{R}^n \to \mathbb{R}_+$, $i = 1, \ldots, m$, are α_i-concave and α_i are such that $\sum_{i=1}^m {\alpha_i}^{-1} > 0$, then the function $g : \mathbb{R}^{nm} \to \mathbb{R}_+$, defined as $g(x) = \prod_{i=1}^m f_i(x_i)$ is γ-concave with $\gamma = \left(\sum_{i=1}^m {\alpha_i}^{-1}\right)^{-1}$.*

Proof. Fix points $x_1, x_2 \in \mathbb{R}^n_+$, a scalar $\lambda \in (0, 1)$ and set $x_\lambda = \lambda x_1 + (1 - \lambda)x_2$. By the generalized concavity of the functions f_i, $i = 1, \ldots, m$, we have the following inequality:

$$\prod_{i=1}^m f_i(x_\lambda) \geq \prod_{i=1}^m \Big(\lambda f_i(x_1)^{\alpha_i} + (1 - \lambda) f_i(x_2)^{\alpha_i}\Big)^{1/\alpha_i}.$$

We denote $y_{ij} = f_i(x_j)^{\alpha_i}$, $j = 1, 2$. Substituting into the last displayed inequality and raising both sides to power γ, we obtain

$$\left(\prod_{i=1}^m f_i(x_\lambda)\right)^\gamma \geq \prod_{i=1}^m \big(\lambda y_{i1} + (1 - \lambda) y_{i2}\big)^{\gamma/\alpha_i}.$$

We continue the chain of inequalities using Lemma 4.22:

$$\prod_{i=1}^m \big(\lambda y_{i1} + (1 - \lambda) y_{i2}\big)^{\gamma/\alpha_i} \geq \lambda \prod_{i=1}^m \big[y_{i1}\big]^{\gamma/\alpha_i} + (1 - \lambda) \prod_{i=1}^m \big[y_{i2}\big]^{\gamma/\alpha_i}.$$

Putting the inequalities together and using the substitutions at the right-hand side of the last inequality, we conclude that

$$\prod_{i=1}^m \big[f_1(x_\lambda)\big]^\gamma \geq \lambda \prod_{i=1}^m \big[f_i(x_1)\big]^\gamma + (1 - \lambda) \prod_{i=1}^m \big[f_i(x_2)\big]^\gamma,$$

as required. □

In the special case, when the functions $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, k$, are concave, we can apply Theorem 4.23 consecutively to conclude that $f_1 f_2$ is $\frac{1}{2}$-concave and $f_1 \cdots f_k$ is $\frac{1}{k}$-concave.

Lemma 4.24. *If A is a symmetric positive definite matrix of size $n \times n$, then the function $A \mapsto \det(A)$ is $\frac{1}{n}$ -concave.*

Proof. Consider two $n \times n$ symmetric positive definite matrices A, B and $\gamma \in (0, 1)$. We note that for every eigenvalue λ of A, $\gamma\lambda$ is an eigenvalue of γA, and, hence, $\det(\gamma A) = \gamma^n \det(A)$. We could apply the Minkowski inequality for matrices,

$$[\det(A + B)]^{\frac{1}{n}} \geq [\det(A)]^{\frac{1}{n}} + [\det(B)]^{\frac{1}{n}}, \tag{4.14}$$

which implies the $\frac{1}{n}$-concavity of the function. As inequality (4.14) is not well known, we provide a proof of it. First, we consider the case of diagonal matrices. In this case the determinants of A and B are products of their diagonal elements and inequality (4.14) follows from Lemma 4.22.

In the general case, let $A^{1/2}$ stand for the symmetric positive definite square root of A and let $A^{-1/2}$ be its inverse. We have

$$\begin{aligned} \det(A + B) &= \det\left(A^{1/2}A^{-1/2}(A + B)A^{-1/2}A^{1/2}\right) \\ &= \det\left(A^{-1/2}(A + B)A^{-1/2}\right)\det(A) \\ &= \det\left(I + A^{-1/2}BA^{-1/2}\right)\det(A). \end{aligned} \tag{4.15}$$

Notice that $A^{-1/2}BA^{-1/2}$ is symmetric positive definite and, therefore, we can choose an $n \times n$ orthogonal matrix R, which diagonalizes it. We obtain

$$\begin{aligned} \det\left(I + A^{-1/2}BA^{-1/2}\right) &= \det\left(R^{\mathrm{T}}\left(I + A^{-1/2}BA^{-1/2}\right)R\right) \\ &= \det\left(I + R^{\mathrm{T}}A^{-1/2}BA^{-1/2}R\right). \end{aligned}$$

At the right-hand side of the equation, we have a sum of two diagonal matrices and we can apply inequality (4.14) for this case. We conclude that

$$\begin{aligned} \left[\det\left(I + A^{-1/2}BA^{-1/2}\right)\right]^{\frac{1}{n}} &= \left[\det\left(I + R^{\mathrm{T}}A^{-1/2}BA^{-1/2}R\right)\right]^{\frac{1}{n}} \\ &\geq 1 + \left[\det\left(R^{\mathrm{T}}A^{-1/2}BA^{-1/2}R\right)\right]^{\frac{1}{n}} \\ &= 1 + [\det(B)]^{\frac{1}{n}}[\det(A)]^{-\frac{1}{n}}. \end{aligned}$$

Combining this inequality with (4.15), we obtain (4.14) in the general case. □

Example 4.25 (Dirichlet Distribution Continued). We return to Example 4.18. We see that the functions $x_i \mapsto x_i^{\beta_i}$ are $1/\beta_i$-concave, provided that $\beta_i > 0$. Therefore, the density function of the Dirichlet distribution is a product of $\frac{1}{\alpha_i - 1}$-concave functions, given that all parameters $\alpha_i > 1$. By virtue of Theorem 4.23, we obtain that this density is γ-concave with $\gamma = (\alpha_1 + \cdots \alpha_m - s)^{-1}$ provided that $\alpha_i > 1$, $i = 1, \ldots, m$. Due to Corollary 4.16, the Dirichlet distribution is a $(\alpha_1 + \cdots \alpha_m)^{-1}$-concave probability measure. ∎

Theorem 4.26. *If the s-dimensional random vector Z has an α-concave probability distribution, $\alpha \in [-\infty, +\infty]$, and T is a constant $m \times s$ matrix, then the m-dimensional random vector $Y = TZ$ has an α-concave probability distribution.*

Proof. Let $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^m$ be two Borel sets. We define

$$A_1 = \{z \in \mathbb{R}^s : Tz \in A\} \quad \text{and} \quad B_1 = \{z \in \mathbb{R}^s : Tz \in B\}.$$

The sets A_1 and A_2 are Borel sets as well due to the continuity of the linear mapping $z \mapsto Tz$. Furthermore, for $\lambda \in [0, 1]$ we have the relation

$$\lambda A_1 + (1-\lambda)B_1 \subset \{z \in \mathbb{R}^s : Tz \in \lambda A + (1-\lambda)B\}.$$

Denoting P_Z and P_Y the probability measure of Z and Y respectively, we obtain

$$\begin{aligned} P_Y\{\lambda A + (1-\lambda)B\} &\geq P_Z\{\lambda A_1 + (1-\lambda)B_1\} \\ &\geq m_\alpha\big(P_Z\{A_1\}, P_Z\{B_1\}, \lambda\big) \\ &= m_\alpha\big(P_Y\{A\}, P_Y\{B\}, \lambda\big). \end{aligned}$$

This completes the proof. □

Example 4.27. A univariate *gamma distribution* is given by the following probability density function:

$$f(z) = \begin{cases} \dfrac{\lambda^\vartheta z^{\vartheta-1} e^{-\lambda z}}{\Gamma(\vartheta)} & \text{for } z > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $\lambda > 0$ and $\vartheta > 0$ are constants. For $\lambda = 1$ the distribution is the standard gamma distribution. If a random variable Y has the gamma distribution, then ϑY has the standard gamma distribution. It is not difficult to check that this density function is log-concave, provided $\vartheta \geq 1$.

A *multivariate gamma distribution* can be defined by a certain linear transformation of m independent random variables $Z_1, \ldots, Z_m$ $(1 \leq m \leq 2^s - 1)$ that have the standard gamma distribution. Let an $s \times m$ matrix A with 0–1 elements be given. Setting $Z = (Z_1, \ldots, Z_{2^s-1})$, we define

$$Y = AZ.$$

The random vector Y has a multivariate standard gamma distribution.

We observe that the distribution of the vector Z is log-concave by virtue of Lemma 4.13. Hence, the *s-variate standard gamma distribution* is log-concave by virtue of Theorem 4.26. ∎

Example 4.28. The *Wishart distribution* arises in estimation of covariance matrices and can be considered as a multidimensional version of the χ^2- distribution. More precisely, let us assume that Z is an s-dimensional random vector having multivariate normal distribution with a nonsingular covariance matrix Σ and expectation μ. Given an iid sample $Z^1, \ldots, Z^N$ from this distribution, we consider the matrix

$$\sum_{i=1}^{N} (Z^i - \bar{Z})(Z^i - \bar{Z})^{\mathsf{T}},$$

where $\bar{Z}$ is the sample mean. This matrix has Wishart distribution with $N-1$ degrees of freedom. We denote the trace of a matrix A by $\mathrm{tr}(A)$.

If $N > s$, the Wishart distribution is a continuous distribution on the space of symmetric square matrices with probability density function defined by

$$f(A) = \begin{cases} \dfrac{\det(A)^{\frac{N-s-2}{2}} \exp\left(-\frac{1}{2}\mathrm{tr}(\Sigma^{-1}A)\right)}{2^{\frac{N-1}{2}s}\ \pi^{\frac{s(s-1)}{4}}\ \det(\Sigma)^{\frac{N-1}{2}} \prod_{i=1}^{s} \Gamma\left(\frac{N-i}{2}\right)} & \text{for } A \text{ positive definite,} \\ 0 & \text{otherwise.} \end{cases}$$

If $s = 1$ and $\Sigma = 1$, this density becomes the χ^2- distribution density with $N-1$ degrees of freedom.

If A_1 and A_2 are two positive definite matrices and $\lambda \in (0,1)$, then the matrix $\lambda A_1 + (1-\lambda)A_2$ is positive definite as well. Using Lemma 4.24 and Lemma 4.8 we conclude that function $A \mapsto \ln\det(A)$, defined on the set of positive definite Hermitian matrices, is concave. This implies that if $N \geq s+2$, then f is a log-concave function on the set of symmetric positive definite matrices. If $N = s+1$, then f is a log-convex on the convex set of symmetric positive definite matrices. ∎

Recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is called *regular in the sense of Clarke* or Clarke-regular, at a point x, if the directional derivative $f'(x;d)$ exists and

$$f'(x;d) = \lim_{y \to x, t \downarrow 0} \frac{f(y+td) - f(y)}{t}, \quad \forall d \in \mathbb{R}^n.$$

It is known that convex functions are regular in this sense. We call a concave function f regular with the understanding that the regularity requirement applies to $-f$. In this case, we have $\partial^\circ(-f)(x) = -\partial^\circ f(x)$, where $\partial^\circ f(x)$ refers to the Clarke generalized gradient of f at the point x. For convex functions $\partial^\circ f(x) = \partial f(x)$.

Theorem 4.29. *If $f : \mathbb{R}^n \to \mathbb{R}$ is α-concave ($\alpha \in \mathbb{R}$) on some open set $U \subset \mathbb{R}^n$ and $f(x) > 0$ for all $x \in U$, then $f(x)$ is locally Lipschitz continuous, directionally differentiable, and Clarke-regular. Its Clarke generalized gradients are given by the formula*

$$\partial^\circ f(x) = \begin{cases} \frac{1}{\alpha}\left[f(x)\right]^{1-\alpha} \partial\left[(f(x))^\alpha\right] & \text{if } \alpha \neq 0, \\ f(x)\partial\left(\ln f(x)\right) & \text{if } \alpha = 0. \end{cases}$$

Proof. If f is an α-concave function, then an appropriate transformation of f is a concave function on U. We define

$$\bar{f}(x) = \begin{cases} (f(x))^\alpha & \text{if } \alpha \neq 0, \\ \ln f(x) & \text{if } \alpha = 0. \end{cases}$$

If $\alpha < 0$, then $f^\alpha(\cdot)$ is convex. This transformation is well defined on the open subset U since $f(x) > 0$ for $x \in U$, and, thus, $\bar{f}(x)$ is subdifferentiable at any $x \in U$. Further, we represent f as follows:

$$f(x) = \begin{cases} \left(\bar{f}(x)\right)^{1/\alpha} & \text{if } \alpha \neq 0, \\ \exp(\bar{f}(x)) & \text{if } \alpha = 0. \end{cases}$$

In this representation, f is a composition of a continuously differentiable function and a concave function. By virtue of Clarke [38, Theorem 2.3.9(3)], the function f is locally Lipschitz continuous, directionally differentiable, and Clarke-regular. Its Clarke generalized gradient set is given by the formula

$$\partial^\circ f(x) = \begin{cases} \frac{1}{\alpha}\big(\bar{f}(x)\big)^{1/\alpha-1}\partial \bar{f}(x) & \text{if } \alpha \neq 0, \\ \exp\big(\bar{f}(x)\big)\partial \bar{f}(x) & \text{if } \alpha = 0. \end{cases}$$

Substituting the definition of $\bar{f}$ yields the result. □

For a function $f : \mathbb{R}^n \to \mathbb{R}$, we consider the set of points at which it takes positive values. It is denoted by $\operatorname{dompos} f$, i.e.,

$$\operatorname{dompos} f = \{x \in \mathbb{R}^n : f(x) > 0\}.$$

Recall that $\mathcal{N}_X(x)$ denotes the normal cone to the set X at $x \in X$.

Definition 4.30. *We call a point $\hat{x} \in \mathbb{R}^n$ a stationary point of an α-concave function f if there is a neighborhood U of $\hat{x}$ such that f is Lipschitz continuous on U, and $0 \in \partial^\circ f(\hat{x})$. Furthermore, for a convex set $X \subset \operatorname{dompos} f$, we call $\hat{x} \in X$ a stationary point of f on X if there is a neighborhood U of $\hat{x}$ such that f is Lipschitz continuous on U and $0 \in \partial^\circ f_X(\hat{x}) + \mathcal{N}_X(\hat{x})$.*

We observe that certain properties of the maxima of concave functions extend to generalized concave functions.

Theorem 4.31. *Let f be an α-concave function f and the set $X \subset \operatorname{dompos} f$ be convex. Then all the stationary points of f on X are global maxima and the set of global maxima of f on X is convex.*

Proof. First, assume that $\alpha = 0$. Let $\hat{x}$ be a stationary point of f on X. This implies that

$$0 \in f(\hat{x})\partial\big(\ln f(\hat{x})\big) + \mathcal{N}_X(\hat{x}). \tag{4.16}$$

Using that $f(\hat{x}) > 0$, we obtain

$$0 \in \partial\big(\ln f(\hat{x})\big) + \mathcal{N}_X(\hat{x}). \tag{4.17}$$

As the function $\bar{f}(x) = \ln f(x)$ is concave, this inclusion implies that $\hat{x}$ is a global maximal point of $\bar{f}$ on X. By the monotonicity of $\ln(\cdot)$, we conclude that $\hat{x}$ is a global maximal point of f on X. If a point $\tilde{x} \in X$ is a maximal point of $\bar{f}$ on X, then inclusion (4.17) is satisfied. It entails (4.16) as $X \subset \operatorname{dompos} f$, and, therefore, $\tilde{x}$ is a stationary point of f on X. Therefore, the set of maximal points of f on X is convex because this is the set of maximal points of the concave function $\bar{f}$.

In the case of $\alpha \neq 0$, the statement follows by the same line of argument using the function $\bar{f}(x) = \big[f(x)\big]^\alpha$. □

Another important property of α-concave measures is the existence of so-called floating body for all probability levels $p \in (\frac{1}{2}, 1)$.

Definition 4.32. *A measure P on $\mathbb{R}^s$ has a floating body at level $p > 0$ if there exists a convex body $C_p \subset \mathbb{R}^s$ such for all vectors $z \in \mathbb{R}^s$,*

$$P\{x \in \mathbb{R}^s : z^\mathsf{T} x \geq s_{C_p}(z)\} = 1 - p,$$

where $s_{C_p}(\cdot)$ is the support function of the set C_p. The set C_p is called the floating body of P at level p.

Symmetric log-concave measures have floating bodies. We formulate this result of Meyer and Reisner [128] without proof.

Theorem 4.33. *Any nondegenerate probability measure with symmetric log-concave density function has a floating body C_p at all levels $p \in \left(\frac{1}{2}, 1\right)$.*

We see that α-concavity as introduced so far implies continuity of the distribution function. As empirical distributions are very important in practical applications, we would like to find a suitable generalization of this notion applicable to discrete distributions. For this purpose, we introduce the following notion.

Definition 4.34. *A distribution function F is called α-concave on the set $\mathcal{A} \subset \mathbb{R}^s$ with $\alpha \in [-\infty, \infty]$ if*

$$F(z) \geq m_\alpha\big(F(x), F(y), \lambda\big)$$

for all $z, x, y \in \mathcal{A}$, and $\lambda \in (0, 1)$ such that $z \geq \lambda x + (1 - \lambda)y$.

Observe that if $\mathcal{A} = \mathbb{R}^s$, then this definition coincides with the usual definition of α-concavity of a distribution function.

To illustrate the relation between Definition 4.7 and Definition 4.34, let us consider the case of integer random vectors which are roundups of continuously distributed random vectors.

Remark 4. *If the distribution function of a random vector Z is α-concave on $\mathbb{R}^s$ then the distribution function of $Y = \lceil Z \rceil$ is α-concave on $\mathbb{Z}^s$.*

This property follows from the observation that at integer points both distribution functions coincide.

Example 4.35. Every distribution function of an s-dimensional *binary random vector* is α-concave on $\mathbb{Z}^s$ for all $\alpha \in [-\infty, \infty]$.

Indeed, let x and y be binary vectors, $\lambda \in (0, 1)$, and $z \geq \lambda x + (1 - \lambda)y$. As z is integer and x and y binary, then $z \geq x$ and $z \geq y$. Hence, $F(z) \geq \max\{F(x), F(y)\}$ by the monotonicity of the cumulative distribution function. Consequently, F is ∞-concave. Using Lemma 4.8 we conclude that F_Z is α-concave for all $\alpha \in [-\infty, \infty]$. ∎

For a random vector with independent components, we can relate concavity of the marginal distribution functions to the concavity of the joint distribution function. Note that the statement applies not only to discrete distributions, as we can always assume that the set $\mathcal{A}$ is the whole space or some convex subset of it.

Theorem 4.36. *Consider the s-dimensional random vector $Z = (Z^1, \ldots, Z^L)$, where the subvectors Z^l, $l = l, \ldots, L$, are s_l-dimensional and $\sum_{l=1}^{L} s_l = s$. Assume that Z^l, $l = l, \ldots, L$, are independent and that their marginal distribution functions $F_{Z^l} : \mathbb{R}^{s_l} \to [0, 1]$ are α_l-concave on the sets $\mathcal{A}_l \subset \mathbb{Z}^{s_l}$. Then the following statements hold true:*

1. *If $\sum_{l=1}^{L} \alpha_l^{-1} > 0$, $l = 1, \ldots, L$, then F_Z is α-concave on $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_L$ with $\alpha = (\sum_{l=1}^{L} \alpha_l^{-1})^{-1}$.*

2. *If $\alpha_l = 0$, $l = 1, \ldots, L$, then F_Z is log-concave on $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_L$.*

Proof. The proof of the first statement follows by virtue of Theorem 4.23 using the monotonicity of the cumulative distribution function.

For the second statement consider $\lambda \in (0, 1)$ and points $x = (x^1, \ldots, x^L) \in \mathcal{A}$, $y = (y^1, \ldots, y^L) \in \mathcal{A}$, and $z = (z^1, \ldots, z^L) \in \mathcal{A}$ such that $z \geq \lambda x + (1 - \lambda)y$. Using the monotonicity of the function $\ln(\cdot)$ and of $F_Z(\cdot)$, along with the log-concavity of the marginal distribution functions, we obtain the following chain of inequalities:

$$\begin{aligned}
\ln[F_Z(z)] &\geq \ln[F_Z(\lambda x + (1 - \lambda)y)] = \sum_{l=1}^{L} \ln\left[F_{Z^l}(\lambda x^l + (1 - \lambda)y^l)\right] \\
&\geq \sum_{l=1}^{L} \left[\lambda \ln[F_{Z^l}(x^l)] + (1 - \lambda) \ln[F_{Z^l}(y_l)]\right] \\
&\geq \lambda \sum_{l=1}^{L} \ln[F_{Z^l}(x^l)] + (1 - \lambda) \sum_{l=1}^{L} \ln[F_{Z^l}(y^l)] \\
&= \lambda[F_Z(x)] + (1 - \lambda)[F_Z(y)].
\end{aligned}$$

This concludes the proof. □

For integer random variables our definition of α-concavity is related to log-concavity of sequences.

Definition 4.37. *A sequence p_k, $k \in \mathbb{Z}$, is called log-concave if*

$$p_k^2 \geq p_{k-1} p_{k+1}, \quad \forall k \in \mathbb{Z}.$$

We have the following property. (See Prékopa [159, Theorem 4.7.2].)

Theorem 4.38. *Suppose that for an integer random variable Y the probabilities $p_k = \mathsf{Pr}\{Y = k\}$, $k \in \mathbb{Z}$, form a log-concave sequence. Then the distribution function of Y is α-concave on $\mathbb{Z}$ for every $\alpha \in [-\infty, 0]$.*

4.2.2 Convexity of Probabilistically Constrained Sets

One of the most general results in the convexity theory of probabilistic optimization is the following theorem.

Theorem 4.39. *Let the functions $g_j : \mathbb{R}^n \times \mathbb{R}^s$, $j \in \mathcal{J}$, be quasi-concave. If $Z \in \mathbb{R}^s$ is a random vector that has an α-concave probability distribution, then the function*

$$G(x) = P\{g_j(x, Z) \geq 0,\ j \in \mathcal{J}\} \tag{4.18}$$

is α-concave on the set

$$D = \{x \in \mathbb{R}^n : \exists z \in \mathbb{R}^s \text{ such that } g_j(x, z) \geq 0,\ j \in \mathcal{J}\}.$$

Proof. Given the points $x_1, x_2 \in D$ and $\lambda \in (0, 1)$, we define the sets

$$A_i = \{z \in \mathbb{R}^s : g_j(x_i, z) \geq 0,\ j \in \mathcal{J}\}, \quad i = 1, 2,$$

and $B = \lambda A_1 + (1 - \lambda)A_2$. We consider

$$G(\lambda x_1 + (1 - \lambda)x_2) = P\{g_j(\lambda x_1 + (1 - \lambda)x_2, Z) \geq 0,\ j \in \mathcal{J}\}.$$

If $z \in B$, then there exist points $z_i \in A_i$ such that $z = \lambda z_1 + (1 - \lambda)z_2$. By virtue of the quasi concavity of g_j we obtain that

$$g_j(\lambda x_1 + (1 - \lambda)x_2, \lambda z_1 + (1 - \lambda)z_2) \geq \min\{g_j(x_1, z_1), g_j(x_2, z_2)\} \geq 0, \quad \forall j \in \mathcal{J}.$$

This implies that $z \in \{z \in \mathbb{R}^s : g_j(\lambda x_1 + (1 - \lambda)x_2, z) \geq 0,\ j \in \mathcal{J}\}$, which entails that $\lambda x_1 + (1 - \lambda)x_2 \in D$ and that

$$G(\lambda x_1 + (1 - \lambda)x_2) \geq P\{B\}.$$

Using the α-concavity of the measure, we conclude that

$$G(\lambda x_1 + (1 - \lambda)x_2) \geq P\{B\} \geq m_\alpha\{P\{A_1\}, P\{A_2\}, \lambda\} = m_\alpha\{G(x_1), G(x_2), \lambda\},$$

as desired. □

Example 4.40 (The Log-Normal Distribution). The probability density function of the one-dimensional log-normal distribution with parameters μ and σ is given by

$$f(x) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma x} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

This density is neither log-concave nor log-convex. However, we can show that the cumulative distribution function is log-concave. We demonstrate it for the multidimensional case.

The m-dimensional random vector Z has the log-normal distribution if the vector $Y = (\ln Z_1, \ldots, \ln Z_m)^\mathsf{T}$ has a multivariate normal distribution. Recall that the normal distribution is log-concave. The distribution function of Z at a point $z \in \mathbb{R}^m$, $z > 0$, can be written as

$$F_Z(z) = \mathsf{Pr}\Big\{Z_1 \leq z_1, \ldots, Z_m \leq z_m\Big\} = \mathsf{Pr}\Big\{z_1 - e_1^Y \geq 0, \ldots, z_m - e_m^Y \geq 0\Big\}.$$

We observe that the assumptions of Theorem 4.39 are satisfied for the probability function on the right-hand side. Thus, F_Z is a log-concave function. ∎

As a consequence, under the assumptions of Theorem 4.39, we obtain convexity statements for sets described by probabilistic constraints.

Corollary 4.41. *Assume that the functions $g_j(\cdot,\cdot)$, $j \in \mathcal{J}$, are quasi-concave jointly in both arguments and that $Z \in \mathbb{R}^s$ is a random variable that has an α-concave probability distribution. Then the following set is convex and closed:*

$$X_0 = \big\{x \in \mathbb{R}^n : \Pr\{g_i(x, Z) \geq 0,\ i = 1, \ldots, m\} \geq p\big\}. \tag{4.19}$$

Proof. Let $G(x)$ be defined as in (4.18), and let $x_1, x_2 \in X_0$, $\lambda \in [0, 1]$. We have

$$G(\lambda x_1 + (1-\lambda)x_2) \geq m_\alpha\{G(x_1), G(x_2), \lambda\} \geq \min\{G(x_1), G(x_2)\} \geq p.$$

The closedness of the set follows from the continuity of α-concave functions. □

We consider the case of a separable mapping g when the random quantities appear only on the right-hand side of the inequalities.

Theorem 4.42. *Let the mapping $g : \mathbb{R}^n \to \mathbb{R}^m$ be such that each component g_i is a concave function. Furthermore, assume that the random vector Z has independent components and the one-dimensional marginal distribution functions F_{Z_i}, $i = 1, \ldots, m$, are α_i-concave. Furthermore, let $\sum_{i=1}^k \alpha_i^{-1} > 0$. Then the set*

$$X_0 = \Big\{x \in \mathbb{R}^n : \Pr\{g(x) \geq Z\} \geq p\Big\}$$

is convex.

Proof. Indeed, the probability function appearing in the definition of the set X_0 can be described as follows:

$$G(x) = P\{g_i(x) \geq Z_i,\ i = 1, \ldots, m\} = \prod_{i=1}^m F_{Z_i}(g_i(x_i)).$$

Due to Theorem 4.20, the functions $F_{Z_i} \circ g_i$ are α_i-concave. Using Theorem 4.23, we conclude that $G(\cdot)$ is γ-concave with $\gamma = (\sum_{i=1}^k \alpha_i^{-1})^{-1}$. The convexity of X_0 follows the same argument as in Corollary 4.41. □

Under the same assumptions, the set determined by the first order stochastic dominance constraint with respect to any random variable Y is convex and closed.

Theorem 4.43. *Assume that $g(\cdot,\cdot)$ is a quasi-concave function jointly in both arguments, and that Z has an α-concave distribution. Then the following sets are convex and closed:*

$$X_d = \big\{x \in \mathbb{R}^n : g(x, Z) \succeq_{(1)} Y\big\},$$
$$X_c = \big\{x \in \mathbb{R}^n : \Pr\big\{g(x, Z) \geq \eta\big\} \geq \Pr\big\{Y \geq \eta\big\},\quad \forall \eta \in [a, b]\big\}.$$

Proof. Let us fix $\eta \in \mathbb{R}$ and observe that the relation $g(x, Z) \succeq_{(1)} Y$ can be formulated in the following equivalent way:

$$\Pr\big\{g(x, Z) \geq \eta\big\} \geq \Pr\big\{Y \geq \eta\big\},\quad \forall \eta \in \mathbb{R}.$$

Therefore, the first set can be defined as follows:

$$X_d = \big\{x \in \mathbb{R}^n : \Pr\big\{g(x, Z) - \eta \geq 0\big\} \geq \Pr\big\{Y \geq \eta\big\}\ \forall \eta \in \mathbb{R}\big\}.$$

For any $\eta \in \mathbb{R}$, we define the set

$$X(\eta) = \big\{x \in \mathbb{R}^n : \Pr\big\{g(x, Z) - \eta \geq 0\big\} \geq \Pr\big\{Y \geq \eta\big\}\big\}.$$

This set is convex and closed by virtue of Corollary 4.41. The set X_d is the intersection of the sets $X(\eta)$ for all $\eta \in \mathbb{R}$, and, therefore, it is convex and closed as well. Analogously, the set X_c is convex and closed as $X_c = \bigcap_{\eta\in[a,b]} X(\eta)$. □

Let us observe that affine in each argument functions $g_i(x, z) = z^{\mathrm{T}}x + b_i$ are not necessarily quasi-concave in both arguments (x, z). We can apply Theorem 4.39 to conclude that the set

$$X_l = \big\{x \in \mathbb{R}^n : \Pr\{x^{\mathrm{T}}a_i \leq b_i(Z),\ i = 1, \ldots, m\} \geq p\big\} \tag{4.20}$$

is convex if a_i, $i = 1, \ldots, m$ are deterministic vectors. We have the following.

Corollary 4.44. *The set X_l is convex whenever $b_i(\cdot)$ are quasi-concave functions and Z has a quasi-concave probability distribution function.*

Example 4.45 (Vehicle Routing Continued). We return to Example 4.1. The probabilistic constraint (4.3) has the form

$$\Pr\big\{TX \geq Z\big\} \geq p_\eta.$$

If the vector Z of a random demand has an α-concave distribution, then this constraint defines a convex set. For example, this is the case if each component Z_i has a uniform distribution and the components (the demand on each arc) are independent of each other. ∎

If the functions g_i are not separable, we can invoke Theorem 4.33.

Theorem 4.46. *Let $p_i \in (\frac{1}{2}, 1)$ for all $i = 1, \ldots, n$. The set*

$$X_p = \big\{x \in \mathbb{R}^n :\ P_{Z_i}\{x^{\mathrm{T}}Z_i \leq b_i\} \geq p_i,\ i = 1, \ldots, m\big\} \tag{4.21}$$

is convex whenever Z_i has a nondegenerate log-concave probability distribution, which is symmetric around some point $\mu_i \in \mathbb{R}^n$.

Proof. If the random vector Z_i has a nondegenerate log-concave probability distribution, which is symmetric around some point $\mu_i \in \mathbb{R}^n$, then the vector $Y_i = Z_i - \mu_i$ has a symmetric and nondegenerate log-concave distribution.

Given points $x_1, x_2 \in X_p$ and a number $\lambda \in [0, 1]$, we define

$$K_i(x) = \{a \in \mathbb{R}^n : a^{\mathrm{T}}x \leq b_i\}, \quad i = 1, \ldots, n.$$

Let us fix an index i. The probability distribution of Y_i satisfies the assumptions of Theorem 4.33. Thus, there is a convex set C_{p_i} such that any supporting plane defines a half plane containing probability p_i:

$$P_{Y_i}\big\{y \in \mathbb{R}^n : y^{\mathrm{T}}x \leq s_{C_{p_i}}(x)\big\} = p_i \quad \forall x \in \mathbb{R}^n.$$

Thus,

$$P_{Z_i}\left\{z \in \mathbb{R}^n : z^{\mathrm{T}}x \le \mathrm{s}_{C_{p_i}}(x) + \mu_i^{\mathrm{T}}x\right\} = p_i \quad \forall x \in \mathbb{R}^n. \tag{4.22}$$

Since $P_{Z_i}\left\{K_i(x_1)\right\} \ge p_i$ and $P_{Z_i}\left\{K_i(x_2)\right\} \ge p_i$ by assumption, then

$$K_i(x_j) \subset \left\{z \in \mathbb{R}^n : z^{\mathrm{T}}x \le \mathrm{s}_{C_{p_i}}(x) + \mu_i^{\mathrm{T}}x\right\}, \quad j = 1, 2,$$
$$b_i \ge \mathrm{s}_{C_{p_i}}(x_1) + \mu_i^{\mathrm{T}}x_j, \quad j = 1, 2.$$

The properties of the support function entail that

$$\begin{aligned} b_i &\ge \lambda\left[\mathrm{s}_{C_{p_i}}(x_1) + \mu_i^{\mathrm{T}}x_1\right] + (1-\lambda)\left[\mathrm{s}_{C_{p_i}}(x_2) + \mu_i^{\mathrm{T}}x_2\right] \\ &= \mathrm{s}_{C_{p_i}}(\lambda x_1) + \mathrm{s}_{C_{p_i}}((1-\lambda)x_2) + \mu_i^{\mathrm{T}}\lambda x_1 + (1-\lambda)x_2 \\ &\ge \mathrm{s}_{C_{p_i}}(\lambda x_1 + (1-\lambda)x_2) + \mu_i^{\mathrm{T}}\lambda x_1 + (1-\lambda)x_2. \end{aligned}$$

Consequently, the set $K_i(x_\lambda)$ with $x_\lambda = \lambda x_1 + (1-\lambda)x_2$ contains the set

$$\left\{z \in \mathbb{R}^n : z^{\mathrm{T}}x_\lambda \le \mathrm{s}_{C_{p_i}}(x_\lambda) + \mu_i^{\mathrm{T}}x_\lambda\right\},$$

and, therefore, using (4.22) we obtain that

$$P_{Z_i}\left\{K_i(\lambda x_1 + (1-\lambda)x_2)\right\} \ge p_i.$$

Since i was arbitrary, we obtain that $\lambda x_1 + (1-\lambda)x_2 \in X_p$. □

Example 4.47 (Portfolio Optimization Continued). Let us consider the Portfolio Example 4.2. The probabilistic constraint has the form

$$\Pr\left\{\sum_{i=1}^{n} R_i x_i \le \eta\right\} \le p_\eta.$$

If the random vector $R = (R_1, \ldots, R_n)^{\mathrm{T}}$ has a multidimensional normal distribution or a uniform distribution, then the feasible set in this example is convex by virtue of the last corollary since both distributions are symmetric and log-concave. ∎

There is an important relation between the sets constrained by first and second order stochastic dominance relation to a benchmark random variable (see Dentcheva and Ruszczyński [53]). We denote the space of integrable random variables by $\mathcal{L}_1(\Omega, \mathcal{F}, P)$ and set

$$A_1(Y) = \{X \in \mathcal{L}_1(\Omega, \mathcal{F}, P) : X \succeq_{(1)} Y\},$$
$$A_2(Y) = \{X \in \mathcal{L}_1(\Omega, \mathcal{F}, P) : X \succeq_{(2)} Y\}.$$

Proposition 4.48. *For every $Y \in \mathcal{L}_1(\Omega, \mathcal{F}, P)$ the set $A_2(Y)$ is convex and closed.*

Proof. By changing the order of integration in the definition of the second order function $F^{(2)}$, we obtain

$$F_X^{(2)}(\eta) = \mathbb{E}[(\eta - X)_+]. \tag{4.23}$$

Therefore, an equivalent representation of the second order stochastic dominance relation is given by the relation

$$\mathbb{E}[(\eta - X)_+] \le \mathbb{E}[(\eta - Y)_+], \quad \forall \eta \in \mathbb{R}. \tag{4.24}$$

For every $\eta \in \mathbb{R}$ the functional $X \to \mathbb{E}[(\eta - X)_+]$ is convex and continuous in $\mathcal{L}_1(\Omega, \mathcal{F}, P)$, as a composition of a linear function, the "max" function, and the expectation operator. Consequently, the set $A_2(Y)$ is convex and closed. □

The set $A_1(Y)$ is closed, because convergence in $\mathcal{L}_1$ implies convergence in probability, but it is not convex in general.

Example 4.49. Suppose that $\Omega = \{\omega_1, \omega_2\}$, $P\{\omega_1\} = P\{\omega_2\} = 1/2$ and $Y(\omega_1) = -1$, $Y(\omega_2) = 1$. Then $X_1 = Y$ and $X_2 = -Y$ both dominate Y in the first order. However, $X = (X_1 + X_2)/2 = 0$ is not an element of $A_1(Y)$ and, thus, the set $A_1(Y)$ is not convex. We notice that X dominates Y in the second order. ∎

Directly from the definition we see that first order dominance relation implies the second order dominance. Hence, $A_1(Y) \subset A_2(Y)$. We have demonstrated that the set $A_2(Y)$ is convex; therefore, we also have

$$\operatorname{conv}(A_1(Y)) \subset A_2(Y). \tag{4.25}$$

We find sufficient conditions for the opposite inclusion.

Theorem 4.50. *Assume that $\Omega = \{\omega_1, \dots, \omega_N\}$, $\mathcal{F}$ contains all subsets of Ω, and $P\{\omega_k\} = 1/N$, $k = 1, \dots, N$. If $Y : (\Omega, \mathcal{F}, P) \to \mathbb{R}$ is a random variable, then*

$$\operatorname{conv}(A_1(Y)) = A_2(Y).$$

Proof. To prove the inverse inclusion to (4.25), suppose that $X \in A_2(Y)$. Under the assumptions of the theorem, we can identify X and Y with vectors $x = (x_1, \dots, x_N)$ and $y = (y_1, \dots, y_N)$ such that $x_i = X(i)$ and $y_i = Y(i)$, $i = 1, \dots, N$. As the probabilities of all elementary events are equal, the second order stochastic dominance relation coincides with the concept of *weak majorization*, which is characterized by the following system of inequalities:

$$\sum_{k=1}^{l} x_{[k]} \ge \sum_{k=1}^{l} y_{[k]}, \quad l = 1, \dots, N,$$

where $x_{[k]}$ denotes the kth smallest component of x.

As established by Hardy, Littlewood, and Polya [73], weak majorization is equivalent to the existence of a doubly stochastic matrix Π such that

$$x \ge \Pi y.$$

By Birkhoff's theorem [20], we can find permutation matrices $Q^1, \dots, Q^M$ and nonnegative reals $\alpha_1, \dots, \alpha_M$ totaling 1, such that

$$\Pi = \sum_{j=1}^{M} \alpha_j Q^j.$$

Setting $z^j = Q^j y$, we conclude that

$$x \geq \sum_{j=1}^{M} \alpha_j z^j.$$

Identifying random variables Z^j on $(\Omega, \mathcal{F}, P)$ with the vectors z^j, we also see that

$$X(\omega) \geq \sum_{j=1}^{M} \alpha_j Z^j(\omega)$$

for all $\omega \in \Omega$. Since each vector z^j is a permutation of y and the probabilities are equal, the distribution of Z^j is identical with the distribution of Y. Thus

$$Z^j \succeq_{(1)} Y, \quad j = 1, \ldots, M.$$

Let us define

$$\hat{Z}^j(\omega) = Z^j(\omega) + \left(X(\omega) - \sum_{k=1}^{M} \alpha_k Z^k(\omega) \right), \quad \omega \in \Omega, \quad j = 1, \ldots, M.$$

Then the last two inequalities render $\hat{Z}^j \in A_1(Y)$, $j = 1, \ldots, M$, and

$$X(\omega) = \sum_{j=1}^{M} \alpha_j \hat{Z}^j(\omega),$$

as required. □

This result does not extend to general probability spaces, as the following example illustrates.

Example 4.51. We consider the probability space $\Omega = \{\omega_1, \omega_2\}$, $P\{\omega_1\} = 1/3$, $P\{\omega_2\} = 2/3$. The benchmark variable Y is defined as $Y(\omega_1) = -1$, $Y(\omega_2) = 1$. It is easy to see that $X \succeq_{(1)} Y$ iff $X(\omega_1) \geq -1$ and $X(\omega_2) \geq 1$. Thus, $A_1(Y)$ is a convex set.

Now, consider the random variable $Z = \mathbb{E}[Y] = 1/3$. It dominates Y in the second order, but it does not belong to conv $A_1(Y) = A_1(Y)$. ■

It follows from this example that the probability space must be sufficiently rich to observe our phenomenon. If we could define a new probability space $\Omega' = \{\omega_1, \omega_{21}, \omega_{22}\}$, in which the event ω_2 is split in two equally likely events ω_{21}, ω_{22}, then we could use Theorem 4.50 to obtain the equality conv $A_1(Y) = A_2(Y)$. In the context of optimization however, the probability space has to be fixed at the outset and we are interested in sets of random variables as elements of $\mathcal{L}_p(\Omega, \mathcal{F}, P; \mathbb{R}^n)$, rather than in sets of their distributions.

Theorem 4.52. *Assume that the probability space* $(\Omega, \mathcal{F}, P)$ *is nonatomic. Then*

$$A_2(Y) = \text{cl}\{\text{conv}(A_1(Y))\}.$$

Proof. If the space $(\Omega, \mathcal{F}, P)$ is nonatomic, we can partition Ω into N disjoint subsets, each of the same P-measure $1/N$, and we verify the postulated equation for random variables

which are piecewise constant on such partitions. This reduces the problem to the case considered in Theorem 4.50. Passing to the limit with $N \to \infty$, we obtain the desired result. We refer the interested reader to Dentcheva and Ruszczyński [55] for technical details of the proof. □

4.2.3 Connectedness of Probabilistically Constrained Sets

Let $\mathcal{X} \subset \mathbb{R}^n$ be a closed convex set. In this section we focus on the following set:

$$X = \left\{x \in \mathcal{X} : \Pr\left[g_j(x, Z) \geq 0,\ j \in \mathcal{J}\right] \geq p\right\},$$

where $\mathcal{J}$ is an arbitrary index set. The functions $g_i : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}$ are continuous, Z is an s-dimensional random vector, and $p \in (0, 1)$ is a prescribed probability. It will be demonstrated later (Lemma 4.61) that the probabilistically constrained set X with separable functions g_j is a union of cones intersected by $\mathcal{X}$. Thus, X could be disconnected. The following result provides a sufficient condition for X to be topologically connected. A more general version of this result is proved in Henrion [84].

Theorem 4.53. *Assume that the functions $g_j(\cdot, Z)$, $j \in \mathcal{J}$ are quasi-concave and that they satisfy the following condition: for all $x^1, x^2 \in \mathbb{R}^n$ there exists a point $x^* \in \mathcal{X}$ such that*

$$g_j(x^*, z) \geq \min\{g_j(x^1, z), g_j(x^2, z)\}, \quad \forall z \in \mathbb{R}^s,\ \forall j \in \mathcal{J}.$$

Then the set X is connected.

Proof. Let $x^1,\ x^2 \in X$ be arbitrary points. We construct a path joining the two points, which is contained entirely in X. Let $x^* \in \mathcal{X}$ be the point that exists according to the assumption. We set

$$\pi(t) = \begin{cases} (1-2t)x^1 + 2tx^* & \text{for } 0 \leq t \leq 1/2, \\ 2(1-t)x^* + (2t-1)x^2 & \text{for } 1/2 < t \leq 1. \end{cases}$$

First, we observe that $\pi(t) \in \mathcal{X}$ for every $t \in [0, 1]$ since $x^1,\ x^2, x^* \in \mathcal{X}$ and the set $\mathcal{X}$ is convex. Furthermore, the quasi concavity of g_j, $j \in \mathcal{J}$, and the assumptions of the theorem imply for every j and for $0 \leq t \leq 1/2$ the following inequality:

$$g_j((1-2t)x^1 + 2tx^*, z) \geq \min\{g_j(x^1, z), g_j(x^*, z)\} = g_j(x^1, z).$$

Therefore,

$$\Pr\{g_j(\pi(t), Z) \geq 0,\ j \in \mathcal{J}\} \geq \Pr\{g(x^1) \geq 0,\ j \in \mathcal{J}\} \geq p \quad \text{for} \quad 0 \leq t \leq 1/2.$$

A similar argument applies for $1/2 < t \leq 1$. Consequently, $\pi(t) \in X$, and this proves the assertion. □

4.3 Separable Probabilistic Constraints

We focus our attention on problems with separable probabilistic constraints. The problem that we analyze in this section is

$$\begin{aligned} &\operatorname*{Min}_{x} \; c(x) \\ &\text{s.t. } \Pr\big\{g(x) \geq Z\big\} \geq p, \\ &\qquad x \in \mathcal{X}. \end{aligned} \tag{4.26}$$

We assume that $c : \mathbb{R}^n \to \mathbb{R}$ is a convex function and $g : \mathbb{R}^n \to \mathbb{R}^m$ is such that each component $g_i : \mathbb{R}^n \to \mathbb{R}$ is a concave function. We assume that the deterministic constraints are expressed by a closed convex set $\mathcal{X} \subset \mathbb{R}^n$. The vector Z is an m-dimensional random vector.

4.3.1 Continuity and Differentiability Properties of Distribution Functions

When the probabilistic constraint involves inequalities with random variables on the right-hand side only as in problem (4.26), we can express it as a constraint on a distribution function:

$$\Pr\big\{g(x) \geq Z\big\} \geq p \quad \Longleftrightarrow \quad F_Z\big(g(x)\big) \geq p.$$

Therefore, it is important to analyze the continuity and differentiability properties of distribution functions. These properties are relevant to the numerical solution of probabilistic optimization problems.

Suppose that Z has an α-concave distribution function with $\alpha \in \mathbb{R}$ and that the support of it, supp P_Z, has nonempty interior in $\mathbb{R}^s$. Then $F_Z(\cdot)$ is locally Lipschitz continuous on int supp P_Z by virtue of Theorem 4.29.

Example 4.54. We consider the following density function:

$$\theta(z) = \begin{cases} \frac{1}{2\sqrt{z}} & \text{for } z \in (0,1), \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding cumulative distribution function is

$$F(z) = \begin{cases} 0 & \text{for } z \leq 0, \\ \sqrt{z} & \text{for } z \in (0,1), \\ 1 & \text{for } z \geq 1. \end{cases}$$

The density θ is unbounded. We observe that F is continuous but it is not Lipschitz continuous at $z = 0$. The density θ is also not (-1)-concave and that means that the corresponding probability distribution is not quasi-concave. ∎

Theorem 4.55. *Suppose that all one-dimensional marginal distribution functions of an s-dimensional random vector Z are locally Lipschitz continuous. Then F_Z is locally Lipschitz continuous as well.*

Proof. The statement can be proved by straightforward estimation of the distribution function by its marginals for $s = 2$ and induction on the dimension of the space. □

It should be noted that even if the multivariate probability measure P_Z has a continuous and bounded density, then the distribution function F_Z is not necessarily Lipschitz continuous.

Theorem 4.56. *Assume that P_Z has a continuous density $\theta(\cdot)$ and that all one-dimensional marginal distribution functions are continuous as well. Then the distribution function F_Z is continuously differentiable.*

Proof. In order to simplify the notation, we demonstrate the statement for $s = 2$. It will be clear how to extend the proof for $s > 2$. We have that

$$F_Z(z_1, z_2) = \Pr(Z_1 \leq z_1, Z_2 \leq z_2) = \int_{-\infty}^{z_1} \int_{-\infty}^{z_2} \theta(t_1, t_2) dt_2 dt_1 = \int_{-\infty}^{z_1} \psi(t_1, z_2) dt_1,$$

where $\psi(t_1, z_2) = \int_{-\infty}^{z_2} \theta(t_1, t_2) dt_2$. Since $\psi(\cdot, z_2)$ is continuous, by the Newton–Leibnitz theorem we have that

$$\frac{\partial F_Z}{\partial z_1}(z_1, z_2) = \psi(z_1, z_2) = \int_{-\infty}^{z_2} \theta(z_1, t_2) dt_2.$$

In a similar way,

$$\frac{\partial F_Z}{\partial z_2}(z_1, z_2) = \int_{-\infty}^{z_1} \theta(t_1, z_2) dt_1.$$

Let us show continuity of $\frac{\partial F_Z}{\partial z_1}(z_1, z_2)$. Given the points $z \in \mathbb{R}^2$ and $y^k \in \mathbb{R}^2$, such that $\lim_{k\to\infty} y^k = z$, we have

$$\begin{aligned}\left| \frac{\partial F_Z}{\partial z_1}(z) - \frac{\partial F_Z}{\partial z_1}(y^k) \right| &= \left| \int_{-\infty}^{z_2} \theta(z_1, t) dt - \int_{-\infty}^{y_2^k} \theta(y_1^k, t) dt \right| \\ &\leq \left| \int_{z_2}^{y_2^k} \theta(y_1^k, t) dt \right| + \left| \int_{-\infty}^{z_2} [\theta(z_1, t) - \theta(y_1^k, t)] dt \right|.\end{aligned}$$

First, we observe that the mapping $(z_1, z_2) \mapsto \int_a^{z_2} \theta(z_1, t) dt$ is continuous for every $a \in \mathbb{R}$ by the uniform continuity of $\theta(\cdot)$ on compact sets in $\mathbb{R}^2$. Therefore, $|\int_{z_2}^{y_2^k} \theta(y_1^k, t) dt| \to 0$ whenever $k \to \infty$. Furthermore, $|\int_{-\infty}^{z_2} [\theta(z_1, t) - \theta(y_1^k, t)] dt| \to 0$ as well, due to the continuity of the one-dimensional marginal function F_{Z_1}. Moreover, by the same reason, the convergence is uniform about z_1. This proves that $\frac{\partial F_Z}{\partial z_1}(z)$ is continuous.

The continuity of the second partial derivative follows by the same line of argument. As both partial derivatives exist and are continuous, the function F_Z is continuously differentiable. □

4.3.2 *p*-Efficient Points

We concentrate on deriving an equivalent algebraic description for the feasible set of problem (4.26).

The p-level set of the distribution function $F_Z(z) = \Pr\{Z \leq z\}$ of Z is defined as follows:

$$\mathcal{Z}_p = \big\{z \in \mathbb{R}^m : F_Z(z) \geq p\big\}. \tag{4.27}$$

Clearly, problem (4.26) can be compactly rewritten as

$$\begin{aligned} &\operatorname*{Min}_x c(x) \\ &\text{s.t. } g(x) \in \mathcal{Z}_p, \\ &\quad x \in \mathcal{X}. \end{aligned} \tag{4.28}$$

Lemma 4.57. *For every $p \in (0, 1)$ the level set $\mathcal{Z}_p$ is nonempty and closed.*

Proof. The statement follows from the monotonicity and the right continuity of the distribution function. □

We introduce the key concept of a p-efficient point.

Definition 4.58. *Let $p \in (0, 1)$. A point $v \in \mathbb{R}^m$ is called a p-efficient point of the probability distribution function F if $F(v) \geq p$ and there is no $z \leq v$, $z \neq v$ such that $F(z) \geq p$.*

The p-efficient points are minimal points of the level set $\mathcal{Z}_p$ with respect to the partial order in $\mathbb{R}^m$ generated by the nonnegative cone $\mathbb{R}^m_+$.

Clearly, for a scalar random variable Z and for every $p \in (0, 1)$ there is exactly one p-efficient point, which is the smallest v such that $F_Z(v) \geq p$, i.e., $F_Z^{(-1)}(p)$.

Lemma 4.59. *Let $p \in (0, 1)$ and let*

$$l = \big(F_{Z_1}^{(-1)}(p), \ldots, F_{Z_m}^{(-1)}(p)\big). \tag{4.29}$$

Then every $v \in \mathbb{R}^m$ such that $F_Z(v) \geq p$ must satisfy the inequality $v \geq l$.

Proof. Let $v_i = F_{Z_i}^{(-1)}(p)$ be the p-efficient point of the ith marginal distribution function. We observe that $F_Z(v) \leq F_{Z_i}(v_i)$ for every $v \in \mathbb{R}^m$ and $i = 1, \ldots, m$, and, therefore, we obtain that the set of p-efficient points is bounded from below. □

Let $p \in (0, 1)$ and let v^j, $j \in \mathcal{E}$, be *all* p-efficient points of Z. Here $\mathcal{E}$ is an arbitrary index set. We define the cones

$$K_j = v^j + \mathbb{R}^m_+, \quad j \in \mathcal{E}.$$

The following result can be derived from Phelps theorem [150, Lemma 3.12] about the existence of conical support points, but we can easily prove it directly.

Theorem 4.60. *It holds that $\mathcal{Z}_p = \bigcup_{j \in \mathcal{E}} K_j$.*

Proof. If $y \in \mathcal{Z}_p$, then either y is p-efficient or there exists a vector w such that $w \leq y$, $w \neq y$, $w \in \mathcal{Z}_p$. By Lemma 4.59, one must have $l \leq w \leq y$. The set $Z_1 = \{z \in \mathcal{Z}_p : l \leq z \leq y\}$ is compact because the set $\mathcal{Z}_p$ is closed by virtue of Lemma 4.57. Thus, there exists $w^1 \in Z_1$ with the minimal first coordinate. If w^1 is a p-efficient point, then $y \in w^1 + \mathbb{R}^m_+$, what had to be shown. Otherwise, we define $Z_2 = \{z \in \mathcal{Z}_p : l \leq z \leq w^1\}$ and choose a point $w^2 \in Z_2$ with the minimal second coordinate. Proceeding in the same way, we shall find the minimal element w^m in the set $\mathcal{Z}_p$ with $w^m \leq w^{m-1} \leq \cdots \leq y$. Therefore, $y \in w^m + \mathbb{R}^m_+$, and this completes the proof. □

By virtue of Theorem 4.60 we obtain (for $0 < p < 1$) the following *disjunctive semi-infinite* formulation of problem (4.28):

$$\begin{aligned} &\operatorname*{Min}_x c(x) \\ &\text{s.t. } g(x) \in \bigcup_{j \in \mathcal{E}} K_j, \\ &x \in \mathcal{X}. \end{aligned} \tag{4.30}$$

This formulation provides insight into the structure of the feasible set and the nature of its nonconvexity. The main difficulty here is the implicit character of the disjunctive constraint.

Let S stand for the simplex in $\mathbb{R}^{m+1}$,

$$S = \left\{ \alpha \in \mathbb{R}^{m+1} : \sum_{i=1}^{m+1} \alpha_i = 1,\ \alpha_i \geq 0 \right\}.$$

Denote the convex hull of the p-efficient points by E, i.e., $E = \operatorname{conv}\{v^j,\ j \in \mathcal{E}\}$. We obtain a semi-infinite disjunctive representation of the convex hull of $\mathcal{Z}_p$.

Lemma 4.61. *It holds that*

$$\operatorname{conv}(\mathcal{Z}_p) = E + \mathbb{R}^m_+.$$

Proof. By Theorem 4.60, every point $y \in \operatorname{conv}\mathcal{Z}$ can be represented as a convex combination of points in the cones K_j. By the theorem of Caratheodory the number of these points is no more than $m + 1$. Thus, we can write $y = \sum_{i=1}^{m+1} \alpha_i (v^{j_i} + w^i)$, where $w^i \in \mathbb{R}^m_+$, $\alpha \in S$, and $j_i \in \mathcal{E}$. The vector $w = \sum_{i=1}^{m+1} \alpha_i w^i$ belongs to $\mathbb{R}^m_+$. Therefore, $y \in \sum_{i=1}^{m+1} \alpha_i v^{j_i} + \mathbb{R}^m_+$. □

We also have the representation $E = \left\{ \sum_{i=1}^{m+1} \alpha_i v^{j_i} : \alpha \in S,\ j_i \in \mathcal{E} \right\}$.

Theorem 4.62. *For every $p \in (0, 1)$ the set* $\operatorname{conv}\mathcal{Z}_p$ *is closed.*

Proof. Consider a sequence $\{z^k\}$ of points of $\operatorname{conv}\mathcal{Z}_p$ which is convergent to a point $\bar{z}$. Using Carathéodory's theorem again, we have

$$z^k = \sum_{i=1}^{m+1} \alpha_i^k y_i^k$$

with $y_i^k \in \mathcal{Z}_p$, $\alpha_i^k \geq 0$, and $\sum_{i=1}^{m+1} \alpha_i^k = 1$. By passing to a subsequence, if necessary, we can assume that the limits

$$\bar{\alpha}_i = \lim_{k\to\infty} \alpha_i^k$$

exist for all $i = 1, \ldots, m+1$. By Lemma 4.59, all points y_i^k are bounded below by some vector l. For simplicity of notation we may assume that $l = 0$.

Let $I = \{i : \bar{\alpha}_i > 0\}$. Clearly, $\sum_{i\in I} \bar{\alpha}_i = 1$. We obtain

$$z^k \geq \sum_{i\in I} \alpha_i^k y_i^k. \tag{4.31}$$

We observe that $0 \leq \alpha_i^k y_i^k \leq z^k$ for all $i \in I$ and all k. Since $\{z^k\}$ is convergent and $\alpha_i^k \to \bar{\alpha}_i > 0$, each sequence $\{y_i^k\}$, $i \in I$, is bounded. Therefore, we can assume that each of them is convergent to some limit $\bar{y}_i$, $i \in I$. By virtue of Lemma 4.57, $\bar{y}_i \in \mathcal{Z}_p$. Passing to the limit in inequality (4.31), we obtain

$$\bar{z} \geq \sum_{i\in I} \bar{\alpha}_i \bar{y}_i \in \text{conv}\mathcal{Z}.$$

Due to Lemma 4.61, we conclude that $\bar{z} \in \text{conv}\mathcal{Z}_p$. □

For a general random vector, the set of p-efficient points may be unbounded and not closed, as illustrated in Figure 4.3.

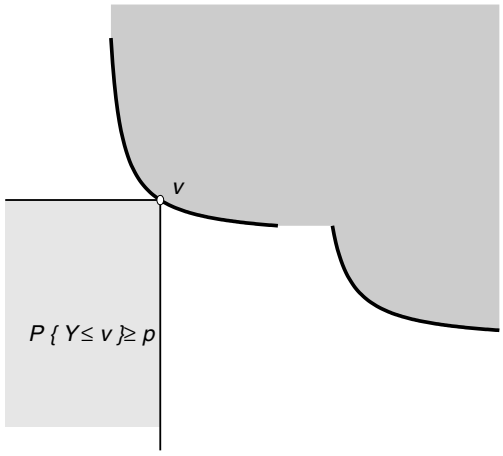

Figure 4.3. *Example of a set $\mathcal{Z}_p$ with p-efficient points v.*

We encounter also a relation between the p-efficient points and the extreme points of the convex hull of $\mathcal{Z}_p$.

Theorem 4.63. *For every $p \in (0, 1)$, the set of extreme points of* $\mathrm{conv}\mathcal{Z}_p$ *is nonempty and it is contained in the set of p-efficient points.*

Proof. Consider the lower bound l defined in (4.29). The set $\mathrm{conv}\mathcal{Z}_p$ is included in $l + \mathbb{R}^m_+$, by virtue of Lemmas 4.59 and 4.61. Therefore, it does not contain any line. Since $\mathrm{conv}\mathcal{Z}_p$ is closed by Theorem 4.62, it has at least one extreme point.

Let w be an extreme point of $\mathrm{conv}\mathcal{Z}_p$. Suppose that w is not a p-efficient point. Then Theorem 4.60 implies that there exists a p-efficient point $v \leq w$, $v \neq w$. Since $w + \mathbb{R}^m_+ \subset \mathrm{conv}\mathcal{Z}_p$, the point w is a convex combination of v and $w + (w - v)$. Consequently, w cannot be extreme. □

The representation becomes very handy when the vector Z has a discrete distribution on $\mathbb{Z}^m$, in particular, if the problem is of form (4.57). We shall discuss this special case in more detail. Let us emphasize that our investigations extend to the case when the random vector Z has a discrete distribution with values on a grid. Our further study can be adapted to the case of distributions on nonuniform grids for which a uniform lower bound on the distance of grid points in each coordinate exists. In this presentation, we assume that $Z \in \mathbb{Z}^m$. In this case, we can establish that the distribution function F_Z has finitely many p-efficient points.

Theorem 4.64. *For each $p \in (0, 1)$ the set of p-efficient points of an integer random vector is nonempty and finite.*

Proof. First we shall show that at least one p-efficient point exists. Since $p < 1$, there exists a point y such that $F_Z(y) \geq p$. By Lemma 4.59, the level set $\mathcal{Z}_p$ is bounded from below by the vector l of p-efficient points of one-dimensional marginals. Therefore, if y is not p-efficient, one of finitely many integer points v such that $l \leq v \leq y$ must be p-efficient.

Now we prove the finiteness of the set of p-efficient points. Suppose that there exists an infinite sequence of different p-efficient points v^j, $j = 1, 2, \ldots$. Since they are integer, and the first coordinate v^j_1 is bounded from below by l_1, with no loss of generality we may select a subsequence which is nondecreasing in the first coordinate. By a similar token, we can select further subsequences which are nondecreasing in the first k coordinates ($k = 1, \ldots, m$). Since the dimension m is finite, we obtain a subsequence of different p-efficient points which is nondecreasing in all coordinates. This contradicts the definition of a p-efficient point. □

Note the crucial role of Lemma 4.59 in this proof. In conclusion, we have obtained that the disjunctive formulation (4.30) of problem (4.28) has a *finite* index set $\mathcal{E}$.

Figure 4.4 illustrates the structure of the probabilistically constrained set for a discrete random variable.

The concept of α-concavity on a set can be used at this moment to find an equivalent representation of the set $\mathcal{Z}_p$ for a random vector with a discrete distribution.

Theorem 4.65. *Let $\mathcal{A}$ be the set of all possible values of an integer random vector Z. If the distribution function F_Z of Z is α-concave on $\mathcal{A} + \mathbb{Z}^m_+$ for some $\alpha \in [-\infty, \infty]$, then*

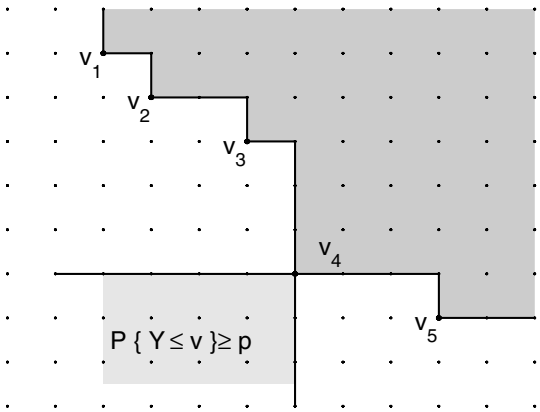

Figure 4.4. *Example of a discrete set $\mathcal{Z}_p$ with p-efficient points $v_1, \ldots, v_5$.*

for every $p \in (0, 1)$ one has

$$\mathcal{Z}_p = \left\{ y \in \mathbb{R}^m : \; y \geq z \geq \sum_{j \in \mathcal{E}} \lambda_j v^j, \; \sum_{j \in \mathcal{E}} \lambda_j = 1, \; \lambda_j \geq 0, \; z \in \mathbb{Z}^m \right\},$$

where v^j, $j \in \mathcal{E}$, are the p-efficient points of F.

Proof. The representation (4.30) implies that

$$\mathcal{Z}_p \subset \left\{ y \in \mathbb{R}^m : \; y \geq z \geq \sum_{j \in \mathcal{E}} \lambda_j v^j, \; \sum_{j \in \mathcal{E}} \lambda_j = 1, \; \lambda_j \geq 0, \; z \in \mathbb{Z}^m \right\}.$$

We have to show that every point y from the set at the right-hand side belongs to $\mathcal{Z}$. By the monotonicity of the distribution function F_Z, we have $F_Z(y) \geq F_Z(z)$ whenever $y \geq z$. Therefore, it is sufficient to show that $\Pr\{Z \leq z\} \geq p$ for all $z \in \mathbb{Z}^m$ such that $z \geq \sum_{j \in \mathcal{E}} \lambda_j v^j$ with $\lambda_j \geq 0$, $\sum_{j \in \mathcal{E}} \lambda_j = 1$. We consider five cases with respect to α.

Case 1: $\alpha = \infty$. It follows from the definition of α-concavity that

$$F_Z(z) \geq \max\{F_Z(v^j), \, j \in \mathcal{E} : \lambda_j \neq 0\} \geq p.$$

Case 2: $\alpha = -\infty$. Since $F_Z(v^j) \geq p$ for each index $j \in \mathcal{E}$ such that $\lambda_j \neq 0$, the assertion follows as in Case 1.

Case 3: $\alpha = 0$. By the definition of α-concavity, we have the following inequalities:

$$F_Z(z) \geq \prod_{j\in\mathcal{E}} [F_Z(v^j)]^{\lambda_j} \geq \prod_{j\in\mathcal{E}} p^{\lambda_j} = p.$$

Case 4: $\alpha \in (-\infty, 0)$. By the definition of α-concavity,

$$[F_Z(z)]^\alpha \leq \sum_{j\in\mathcal{E}} \lambda_j [F_Z(v^j)]^\alpha \leq \sum_{j\in\mathcal{E}} \lambda_j p^\alpha = p^\alpha.$$

Since $\alpha < 0$, we obtain $F_Z(z) \geq p$.

Case 5: $\alpha \in (0, \infty)$. By the definition of α-concavity,

$$[F_Z(z)]^\alpha \geq \sum_{j\in\mathcal{E}} \lambda_j [F_Z(v^j)]^\alpha \geq \sum_{j\in\mathcal{E}} \lambda_j p^\alpha = p^\alpha,$$

concluding that $z \in \mathcal{Z}$, as desired. □

The consequence of this theorem is that under the α-concavity assumption, all integer points contained in $\operatorname{conv}\mathcal{Z}_p = E + \mathbb{R}^m_+$ satisfy the probabilistic constraint. This demonstrates the importance of the notion of α-concavity for discrete distribution functions as introduced in Definition 4.34. For example, the set $\mathcal{Z}_p$ illustrated in Figure 4.4 does not correspond to any α-concave distribution function, because its convex hull contains integer points which do not belong to $\mathcal{Z}_p$. These are the points (3,6), (4,5), and (6,2).

Under the conditions of Theorem 4.65, problem (4.28) can be formulated in the following equivalent way:

$$\begin{aligned}
\operatorname*{Min}_{x,z,\lambda}\ & c(x) && (4.32)\\
\text{s.t. }\ & g(x) \geq z, && (4.33)\\
& z \geq \sum_{j\in\mathcal{E}} \lambda_j v^j, && (4.34)\\
& z \in \mathbb{Z}^m, && (4.35)\\
& \sum_{j\in\mathcal{E}} \lambda_j = 1, && (4.36)\\
& \lambda_j \geq 0,\ j \in \mathcal{E}, && (4.37)\\
& x \in \mathcal{X}. && (4.38)
\end{aligned}$$

In this way, we have replaced the probabilistic constraint by algebraic equations and inequalities, together with the integrality requirement (4.35). This condition cannot be dropped, in general. However, if other conditions of the problem imply that $g(x)$ is integer, then we may remove z entirely form the problem formulation. In this case, we replace constraints (4.33)–(4.35) with

$$g(x) \geq \sum_{j\in\mathcal{E}} \lambda_j v^j.$$

For example, if the definition of $\mathcal{X}$ contains the constraint $x \in \mathbb{Z}^n$, and, in addition, $g(x) = Tx$, where T is a matrix with integer elements, then we can dispose of the variable z.

If Z takes values on a nonuniform grid, condition (4.35) should be replaced by the requirement that z is a grid point.

Corollary 4.66. *If the distribution function F_Z of an integer random vector Z is α-concave on the set $\mathbb{Z}^m_+$ for some $\alpha \in [-\infty, \infty]$, then for every $p \in (0, 1)$ one has*

$$\mathcal{Z}_p \cap \mathbb{Z}^m_+ = \operatorname{conv}\mathcal{Z}_p \cap \mathbb{Z}^m_+.$$

4.3.3 Optimality Conditions and Duality Theory

In this section, we return to problem formulation (4.28). We assume that $c : \mathbb{R}^n \to \mathbb{R}$ is a convex function. The mapping $g : \mathbb{R}^n \to \mathbb{R}^m$ has concave components $g_i : \mathbb{R}^n \to \mathbb{R}$. The set $\mathcal{X} \subset \mathbb{R}^n$ is closed and convex; the random vector Z takes values in $\mathbb{R}^m$. The set $\mathcal{Z}_p$ is defined as in (4.27). We split variables and consider the following formulation of the problem:

$$\begin{aligned} &\operatorname*{Min}_{x,z} c(x) \\ &\text{s.t. } g(x) \geq z, \\ &\quad x \in \mathcal{X}, \\ &\quad z \in \mathcal{Z}_p. \end{aligned} \tag{4.39}$$

Associating a Lagrange multiplier $u \in \mathbb{R}^m_+$ with the constraint $g(x) \geq z$, we obtain the Lagrangian function:

$$L(x, z, u) = c(x) + u^{\mathrm{T}}(z - g(x)).$$

The dual functional has the form

$$\Psi(u) = \inf_{(x,z)\in\mathcal{X}\times\mathcal{Z}_p} L(x, z, u) = h(u) + d(u),$$

where

$$h(u) = \inf\{c(x) - u^{\mathrm{T}}g(x) \ : \ x \in \mathcal{X}\}, \tag{4.40}$$

$$d(u) = \inf\{u^{\mathrm{T}}z \ : \ z \in \mathcal{Z}_p\}. \tag{4.41}$$

For any $u \in \mathbb{R}^m_+$ the value of $\Psi(u)$ is a lower bound on the optimal value c^* of the original problem. The best Lagrangian lower bound will be given by the optimal value Ψ^* of the problem:

$$\sup_{u\geq 0} \Psi(u). \tag{4.42}$$

We call (4.42) the dual problem to problem (4.39). For $u \not\geq 0$ one has $d(u) = -\infty$, because the set $\mathcal{Z}_p$ contains a translation of $\mathbb{R}^m_+$. The function $d(\cdot)$ is concave. Note that $d(u) = -s_{\mathcal{Z}_p}(-u)$, where $s_{\mathcal{Z}_p}(\cdot)$ is the support function of the set $\mathcal{Z}_p$. By virtue of Theorem 4.62 and Hiriart-Urruty and Lemaréchal [89, Chapter V, Proposition 2.2.1], we have

$$d(u) = \inf\{u^{\mathrm{T}}z \ : \ z \in \operatorname{conv}\mathcal{Z}_p\}. \tag{4.43}$$

Let us consider the *convex hull problem*:

$$\begin{aligned} &\operatorname*{Min}_{x,z} c(x) \\ &\text{s.t. } g(x) \geq z, \\ &\quad x \in \mathcal{X}, \\ &\quad z \in \operatorname{conv} \mathcal{Z}_p. \end{aligned} \tag{4.44}$$

We impose the following constraint qualification condition:

$$\textit{There exist points } x^0 \in \mathcal{X} \textit{ and } z^0 \in \operatorname{conv} \mathcal{Z}_p \textit{ such that } g(x^0) > z^0. \tag{4.45}$$

If this constraint qualification condition is satisfied, then the duality theory in convex programming Rockafellar [174, Corollary 28.2.1] implies that there exists $\hat{u} \geq 0$ at which the minimum in (4.42) is attained, and $\Psi^* = \Psi(\hat{u})$ is the optimal value of the convex hull problem (4.44).

We now study in detail the structure of the dual functional Ψ. We shall characterize the solution sets of the two subproblems (4.40) and (4.41), which provide the values of the dual functional. Observe that the normal cone to the positive orthant at a point $u \geq 0$ is the following:

$$\mathcal{N}_{\mathbb{R}^m_+}(u) = \{d \in \mathbb{R}^m_- : d_i = 0 \text{ if } u_i > 0,\ i = 1, \ldots, m\}. \tag{4.46}$$

We define the set

$$V(u) = \{v \in \mathbb{R}^m : u^{\mathsf{T}} v = d(u) \text{ and } v \text{ is a } p\text{-efficient point}\}. \tag{4.47}$$

Lemma 4.67. *For every $u > 0$ the solution set of* (4.41) *is nonempty. For every $u \geq 0$ it has the following form: $\hat{Z}(u) = V(u) - \mathcal{N}_{\mathbb{R}^m_+}(u)$.*

Proof. First we consider the case $u > 0$. Then every recession direction q of $\mathcal{Z}_p$ satisfies $u^{\mathsf{T}} q > 0$. Since $\mathcal{Z}_p$ is closed, a solution to (4.41) must exist. Suppose that a solution z to (4.41) is not a p-efficient point. By virtue of Theorem 4.60, there is a p-efficient $v \in \mathcal{Z}_p$ such that $v \leq z$, and $v \neq z$. Thus, $u^{\mathsf{T}} v < u^{\mathsf{T}} z$, which is a contradiction. Therefore, we conclude that there is a p-efficient point v, which solves problem (4.41).

Consider the general case $u \geq 0$ and assume that the solution set of problem (4.41) is nonempty. In this case, the solution set always contains a p-efficient point. Indeed, if a solution z is not p-efficient, we must have a p-efficient point v dominated by z, and $u^{\mathsf{T}} v \leq u^{\mathsf{T}} z$ holds by the nonnegativity of u. Consequently, $u^{\mathsf{T}} v = u^{\mathsf{T}} z$ for all p-efficient $v \leq z$, which is equivalent to $z \in \{v\} - \mathcal{N}_{\mathbb{R}^m_+}(u)$, as required.

If the solution set of (4.41) is empty, then $V(u) = \emptyset$ by definition and the assertion is true as well. □

The last result allows us to calculate the subdifferential of the function d in a closed form.

Lemma 4.68. *For every $u \geq 0$ one has $\partial d(u) = \operatorname{conv}(V(u)) - \mathcal{N}_{\mathbb{R}^m_+}(u)$. If $u > 0$, then $\partial d(u)$ is nonempty.*

Proof. From (4.41) we obtain $d(u) = -\mathrm{s}_{\mathcal{Z}_p}(-u)$, where $\mathrm{s}_{\mathcal{Z}_p}(\cdot)$ is the support function of $\mathcal{Z}_p$ and, consequently, of $\operatorname{conv}\mathcal{Z}_p$. Consider the indicator function $\mathbb{I}_{\operatorname{conv}\mathcal{Z}_p}(\cdot)$ of the set $\operatorname{conv}\mathcal{Z}_p$. By virtue of Corollary 16.5.1 in Rockafellar [174], we have

$$\mathrm{s}_{\mathcal{Z}_p}(u) = \mathbb{I}^*_{\operatorname{conv}\mathcal{Z}_p}(u),$$

where the latter function is the conjugate of the indicator function $\mathbb{I}_{\operatorname{conv}\mathcal{Z}_p}(\cdot)$. Thus,

$$\partial d(u) = -\partial \mathbb{I}^*_{\operatorname{conv}\mathcal{Z}_p}(-u).$$

Recall that $\operatorname{conv}\mathcal{Z}_p$ is closed, by Theorem 4.62. Using Rockafellar [174, Theorem 23.5], we observe that $y \in \partial \mathbb{I}^*_{\operatorname{conv}\mathcal{Z}_p}(-u)$ iff $\mathbb{I}^*_{\operatorname{conv}\mathcal{Z}_p}(-u) + \mathbb{I}_{\operatorname{conv}\mathcal{Z}_p}(y) = -y^{\mathsf{T}}u$. It follows that $y \in \operatorname{conv}\mathcal{Z}_p$ and $\mathbb{I}^*_{\operatorname{conv}\mathcal{Z}_p}(-u) = -y^{\mathsf{T}}u$. Consequently,

$$y^{\mathsf{T}}u = d(u). \tag{4.48}$$

Since $y \in \operatorname{conv}\mathcal{Z}_p$ we can represent it as follows:

$$y = \sum_{j=1}^{m+1} \alpha_j e^j + w,$$

where e^j, $j = 1, \ldots, m+1$, are extreme points of $\operatorname{conv}\mathcal{Z}_p$ and $w \geq 0$. Using Theorem 4.63 we conclude that e^j are p-efficient points. Moreover, applying u, we obtain

$$y^{\mathsf{T}}u = \sum_{j=1}^{m+1} \alpha_j u^{\mathsf{T}} e^j + u^{\mathsf{T}} w \geq d(u), \tag{4.49}$$

because $u^{\mathsf{T}}e^j \geq d(u)$ and $u^{\mathsf{T}}w \geq 0$. Combining (4.48) and (4.49) we conclude that $u^{\mathsf{T}}e^j = d(u)$ for all j, and $u^{\mathsf{T}}w = 0$. Thus $y \in \operatorname{conv} V(u) - \mathcal{N}_{\mathbb{R}^m_+}(u)$.

Conversely, if $y \in \operatorname{conv} V(u) - \mathcal{N}_{\mathbb{R}^m_+}(u)$, then (4.48) holds true by the definitions of the set $V(u)$ and the normal cone. This implies that $y \in \partial d(u)$, as required.

Furthermore, the set $\partial d(u)$ is nonempty for $u > 0$ due to Lemma 4.67. □

Now, we analyze the function $h(\cdot)$. Define the set of minimizers in (4.40),

$$X(u) = \left\{x \in \mathcal{X} : c(x) - u^{\mathsf{T}} g(x) = h(u)\right\}.$$

Since the set $\mathcal{X}$ is convex and the objective function of problem (4.40) is convex for all $u \geq 0$, we conclude that the solution set $X(u)$ is convex for all $u \geq 0$.

Lemma 4.69. *Assume that the set $\mathcal{X}$ is compact. For every $u \in \mathbb{R}^m$, the subdifferential of the function h is described as follows:*

$$\partial h(u) = \operatorname{conv}\{-g(x) : x \in X(u)\}.$$

Proof. The function h is concave on $\mathbb{R}^m$. Since the set $\mathcal{X}$ is compact, c is convex, and g_i, $i = 1, \ldots, m$, are concave, the set $X(u)$ is compact. Therefore, the subdifferential of

$h(u)$ for every $u \in \mathbb{R}^m$ is the closure of conv $\{-g(x) : x \in X(u)\}$. (See Hiriart-Urruty and Lemaréchal [89, Chapter VI, Lemma 4.4.2].) By the compactness of $X(u)$ and concavity of g, the set $\{-g(x) : x \in X(u)\}$ is closed. Therefore, we can omit taking the closure in the description of the subdifferential of $h(u)$. □

This analysis provides the basis for the following necessary and sufficient optimality conditions for problem (4.42).

Theorem 4.70. *Assume that the constraint qualification condition* (4.45) *is satisfied and that the set $\mathcal{X}$ is compact. A vector $u \geq 0$ is an optimal solution of* (4.42) *iff there exists a point $x \in X(u)$, points $v^1, \ldots, v^{m+1} \in V(u)$ and scalars $\beta_1 \ldots, \beta_{m+1} \geq 0$ with $\sum_{j=1}^{m+1} \beta_j = 1$ such that*

$$\sum_{j=1}^{m+1} \beta_j v^j - g(x) \in \mathcal{N}_{\mathbb{R}^m_+}(u). \tag{4.50}$$

Proof. Using Rockafellar [174, Theorem 27.4], the necessary and sufficient optimality condition for (4.42) has the form

$$0 \in -\partial \Psi(u) + \mathcal{N}_{\mathbb{R}^m_+}(u). \tag{4.51}$$

Since int dom $d \neq \emptyset$ and dom $h = \mathbb{R}^m$, we have $\partial \Psi(u) = \partial h(u) + \partial d(u)$. Using Lemma 4.68 and Lemma 4.69, we conclude that there exist

$$\begin{aligned}
&p\text{-efficient points } v^j \in V(u), \quad j = 1, \ldots, m+1,\\
&\beta^j \geq 0, \quad j = 1, \ldots, m+1, \quad \sum_{j=1}^{m+1} \beta_j = 1,\\
&x^j \in X(u), \quad j = 1, \ldots, m+1,\\
&\alpha^j \geq 0, \quad j = 1, \ldots, m+1, \quad \sum_{j=1}^{m+1} \alpha_j = 1,
\end{aligned} \tag{4.52}$$

such that

$$\sum_{j=1}^{m+1} \alpha_j g(x^j) - \sum_{j=1}^{m+1} \beta_j v^j \in -\mathcal{N}_{\mathbb{R}^m_+}(u). \tag{4.53}$$

If the function c was strictly convex, or g was strictly concave, then the set $X(u)$ would be a singleton. In this case, all x^j would be identical and the above relation would immediately imply (4.50). Otherwise, let us define

$$x = \sum_{j=1}^{m+1} \alpha_j x^j.$$

By the convexity of $X(u)$ we have $x \in X(u)$. Consequently,

$$c(x) - \sum_{i=1}^{m} u_i g_i(x) = h(u) = c(x^j) - \sum_{i=1}^{m} u_i g_i(x^j), \quad j = 1, \ldots, m+1. \tag{4.54}$$

Multiplying the last equation by α_j and adding, we obtain

$$c(x) - \sum_{i=1}^{m} u_i g_i(x) = \sum_{j=1}^{m+1} \alpha_j \left[c(x^j) - \sum_{i=1}^{m} u_i g_i(x^j) \right] \geq c(x) - \sum_{i=1}^{m} u_i \sum_{j=1}^{m+1} \alpha_j g_i(x^j).$$

The last inequality follows from the convexity of c. We have the following inequality:

$$\sum_{i=1}^{m} u_i \left[g_i(x) - \sum_{j=1}^{m+1} \alpha_j g_i(x^j) \right] \leq 0.$$

Since the functions g_i are concave, we have $g_i(x) \geq \sum_{j=1}^{m+1} \alpha_j g_i(x^j)$. Therefore, we conclude that $u_i = 0$ whenever $g_i(x) > \sum_{j=1}^{m+1} \alpha_j g_i(x^j)$. This implies that

$$g(x) - \sum_{j=1}^{m+1} \alpha_j g(x^j) \in -\mathcal{N}_{\mathbb{R}^m_+}(u).$$

Since $\mathcal{N}_{\mathbb{R}^m_+}(u)$ is a convex cone, we can combine the last relation with (4.53) and obtain (4.50), as required.

Now, we prove the converse implication. Assume that we have $x \in X(u)$, points $v^1, \ldots, v^{m+1} \in V(u)$, and scalars $\beta_1 \ldots, \beta_{m+1} \geq 0$ with $\sum_{j=1}^{m+1} \beta_j = 1$ such that (4.50) holds true. By Lemma 4.68 and Lemma 4.69 we have

$$-g(x) + \sum_{j=1}^{m+1} \beta_j v^j \in \partial \Psi(u).$$

Thus (4.50) implies (4.51), which is a necessary and sufficient optimality condition for problem (4.42). □

Using these optimality conditions we obtain the following duality result.

Theorem 4.71. *Assume that the constraint qualification condition* (4.45) *for problem* (4.39) *is satisfied, the probability distribution of the vector Z is α-concave for some $\alpha \in [-\infty, \infty]$, and the set $\mathcal{X}$ is compact. If a point $(\hat{x}, \hat{z})$ is an optimal solution of* (4.39)*, then there exists a vector $\hat{u} \geq 0$, which is an optimal solution of* (4.42) *and the optimal values of both problems are equal. If $\hat{u}$ is an optimal solution of problem* (4.42)*, then there exist a point $\hat{x}$ such that $(\hat{x}, g(\hat{x}))$ is a solution of problem* (4.39)*, and the optimal values of both problems are equal.*

Proof. The α-concavity assumption implies that problems (4.39) and (4.44) are the same. If $\hat{u}$ is optimal solution of problem (4.42), we obtain the existence of points $\hat{x} \in X(\hat{u})$, $v^1, \ldots, v^{m+1} \in V(u)$ and scalars $\beta_1 \ldots, \beta_{m+1} \geq 0$ with $\sum_{j=1}^{m+1} \beta_j = 1$ such that the optimality conditions in Theorem 4.70 are satisfied. Setting $\hat{z} = g(\hat{x})$ we have to show that $(\hat{x}, \hat{z})$ is an optimal solution of problem (4.39) and that the optimal values of both problems are equal. First we observe that this point is feasible. We choose $y \in -\mathcal{N}_{\mathbb{R}^m_+}(\hat{u})$ such

that $y = g(\hat{x}) - \sum_{j=1}^{m+1} \beta_j v^j$. From the definitions of $X(\hat{u})$, $V(\hat{u})$, and the normal cone, we obtain

$$h(\hat{u}) = c(\hat{x}) - \hat{u}^{\mathrm{T}} g(\hat{x}) = c(\hat{x}) - \hat{u}^{\mathrm{T}} \left(\sum_{j=1}^{m+1} \beta_j v^j + y \right)$$

$$= c(\hat{x}) - \sum_{j=1}^{m+1} \beta_j d(\hat{u}) - \hat{u}^{\mathrm{T}} y = c(\hat{x}) - d(\hat{u}).$$

Thus,

$$c(\hat{x}) = h(\hat{u}) + d(\hat{u}) = \Psi^* \geq c^*,$$

which proves that $(\hat{x}, \hat{z})$ is an optimal solution of problem (4.39) and $\Psi^* = c^*$.

If $(\hat{x}, \hat{z})$ is a solution of (4.39), then by Rockafellar [174, Theorem 28.4] there is a vector $\hat{u} \geq 0$ such that $\hat{u}_i(\hat{z}_i - g_i(\hat{x})) = 0$ and

$$0 \in \partial c(\hat{x}) + \partial \hat{u}^{\mathrm{T}} g(\hat{x}) - \hat{z} + \mathcal{N}_{\mathcal{X} \times \mathcal{Z}}(\hat{x}, \hat{z}).$$

This means that

$$0 \in \partial c(\hat{x}) - \partial u^{\mathrm{T}} g(\hat{x}) + \mathcal{N}_{\mathcal{X}}(\hat{x}) \tag{4.55}$$

and

$$0 \in \hat{u} + \mathcal{N}_{\mathcal{Z}}(\hat{z}). \tag{4.56}$$

The first inclusion (4.55) is optimality condition for problem (4.40), and thus $x \in X(\hat{u})$. By virtue of Rockafellar [174, Theorem 23.5] the inclusion (4.56) is equivalent to $\hat{z} \in \partial \mathbb{I}^*_{\mathcal{Z}_p}(\hat{u})$. Using Lemma 4.68 we obtain that

$$\hat{z} \in \partial d(\hat{u}) = \operatorname{conv} V(\hat{u}) - \mathcal{N}_{\mathbb{R}^m_+}(\hat{u}).$$

Thus, there exists points $v^1, \ldots, v^{m+1} \in V(u)$ and scalars $\beta_1 \ldots, \beta_{m+1} \geq 0$ with $\sum_{j=1}^{m+1} \beta_j = 1$ such that

$$\hat{z} - \sum_{j=1}^{m+1} \beta_j v^j \in -\mathcal{N}_{\mathbb{R}^m_+}(\hat{u}).$$

Using the complementarity condition $\hat{u}_i(\hat{z}_i - g_i(\hat{x})) = 0$ we conclude that the optimality conditions of Theorem 4.70 are satisfied. Thus, $\hat{u}$ is an optimal solution of problem (4.42). □

For the special case of discrete distribution and linear constraints we can obtain more specific necessary and sufficient optimality conditions.

In the *linear* probabilistic optimization problem, we have $g(x) = Tx$, where T is an $m \times n$ matrix, and $c(x) = c^{\mathrm{T}} x$ with $c \in \mathbb{R}^n$. Furthermore, we assume that $\mathcal{X}$ is a closed

convex polyhedral set, defined by a system of linear inequalities. The problem reads as follows:

$$\begin{aligned} \operatorname*{Min}_{x} \ & c^{\mathrm{T}}x \\ \text{s.t.} \ & \Pr\{Tx \geq Z\} \geq p, \\ & Ax \geq b, \\ & x \geq 0. \end{aligned} \tag{4.57}$$

Here A is an $s \times n$ matrix and $b \in \mathbb{R}^s$.

Definition 4.72. *Problem* (4.57) *satisfies the dual feasibility condition if*

$$\Lambda = \{(u, w) \in \mathbb{R}^{m+s}_{+} \ : \ A^T w + T^T u \leq c\} \neq \emptyset.$$

Theorem 4.73. *Assume that the feasible set of* (4.57) *is nonempty and that Z has a discrete distribution on $\mathbb{Z}^m$. Then* (4.57) *has an optimal solution iff it satisfies the LQ condition, defined in* (4.72)*.*

Proof. If (4.57) has an optimal solution, then for some $j \in \mathcal{E}$ the linear optimization problem

$$\begin{aligned} \operatorname*{Min}_{x} \ & c^{\mathrm{T}}x \\ \text{s.t.} \ & Tx \geq v^j, \\ & Ax \geq b, \\ & x \geq 0, \end{aligned} \tag{4.58}$$

has an optimal solution. By duality in linear programming, its dual problem

$$\begin{aligned} \operatorname*{Max}_{u,w} \ & u^{\mathrm{T}}v^j + b^{\mathrm{T}}w \\ \text{s.t.} \ & T^{\mathrm{T}}u + A^{\mathrm{T}}w \leq c, \\ & u, w \geq 0, \end{aligned} \tag{4.59}$$

has an optimal solution and the optimal values of both programs are equal. Thus, the dual feasibility condition (4.72) must be satisfied. On the other hand, if the dual feasibility condition is satisfied, all dual programs (4.59) for $j \in \mathcal{E}$ have nonempty feasible sets, so the objective values of all primal problems (4.58) are bounded from below. Since at least one of them has a nonempty feasible set by assumption, an optimal solution must exist. □

Example 4.74 (Vehicle Routing Continued). We return to the vehicle routing Example 4.1, introduced at the beginning of the chapter. The convex hull problem reads

$$\operatorname*{Min}_{x,\lambda} \ c^{\mathrm{T}}x$$

$$\text{s.t.} \ \sum_{i=1}^{n} t_{il}x_i \geq \sum_{j\in\mathcal{E}} \lambda_j v^j, \tag{4.60}$$

$$\sum_{j\in\mathcal{E}} \lambda_j = 1, \tag{4.61}$$

$$x \geq 0, \ \lambda \geq 0.$$

We assign a Lagrange multiplier u to constraint (4.60) and a multiplier μ to constraint (4.61). The dual problem has the form

$$\begin{aligned} &\operatorname*{Max}_{u,\mu} \ \mu \\ &\text{s.t.} \ \sum_{l=1}^{m} t_{il} u_l \le c_i, \quad i = 1, 2, \ldots, n, \\ &\mu \le u^{\mathsf{T}} v^j, \quad j \in \mathcal{E}, \\ &u \ge 0. \end{aligned}$$

We see that u_l provides the increase of routing cost if the demand on arc l increases by one unit, μ is the minimum cost for covering the demand with probability p, and the p-efficient points v^j correspond to critical demand levels that have to be covered. The auxiliary problem $\operatorname{Min}_{z \in \mathcal{Z}} u^{\mathsf{T}} z$ identifies p-efficient points, which represent critical demand levels. The optimal value of this problem provides the minimum total cost of a critical demand. ∎

Our duality theory finds interesting interpretation in the context of the cash matching problem in Example 4.6.

Example 4.75 (Cash Matching Continued). Recall the problem formulation

$$\begin{aligned} &\operatorname*{Max}_{x,c} \ \mathbb{E}\big[U(c_T - Z_T)\big] \\ &\text{s.t.} \ \Pr\big\{c_t \ge Z_t, \ t = 1, \ldots, T\big\} \ge p, \\ &c_t = c_{t-1} + \sum_{i=1}^{n} a_{it} x_i, \quad t = 1, \ldots, T, \\ &x \ge 0. \end{aligned}$$

If the vector Z has a quasi-concave distribution (e.g., joint normal distribution), the resulting problem is convex.

The convex hull problem (4.44) can be written as follows:

$$\operatorname*{Max}_{x,\lambda,c} \ \mathbb{E}\big[U(c_T - Z_T)\big] \tag{4.62}$$

$$\text{s.t.} \ c_t = c_{t-1} + \sum_{i=1}^{n} a_{it} x_i, \quad t = 1, \ldots, T, \tag{4.63}$$

$$c_t \ge \sum_{j=1}^{T+1} \lambda_j v_t^j, \quad t = 1, \ldots, T, \tag{4.64}$$

$$\sum_{j=1}^{T+1} \lambda_j = 1, \tag{4.65}$$

$$\lambda \ge 0, \ x \ge 0. \tag{4.66}$$

In constraint (4.64) the vectors $v^j = (v_1^j, \ldots, v_T^j)$ for $j = 1, \ldots, T+1$ are *p-efficient trajectories* of the cumulative liabilities $Z = (Z_1, \ldots, Z_T)$. Constraints (4.64)–(4.66)

require that the cumulative cash flows are greater than or equal to some convex combination of p-efficient trajectories. Recall that by Lemma 4.61, no more than $T+1$ p-efficient trajectories are needed. Unfortunately, we do not know the optimal collection of these trajectories.

Let us assign nonnegative Lagrange multipliers $u = (u_1, \ldots, u_T)$ to the constraint (4.64), multipliers $w = (w_1, \ldots, w_T)$ to the constraints (4.63) and a multiplier $\rho \in \mathbb{R}$ to the constraint (4.65). To simplify notation, we define the function $\bar{U} : \mathbb{R} \to \mathbb{R}$ as follows:

$$\bar{U}(y) = \mathbb{E}[U(y - Z_T)].$$

It is a concave nondecreasing function of y due to the properties of $U(\cdot)$. We make the convention that its conjugate is defined as follows:

$$\bar{U}^*(u) = \inf_y \{uy - \bar{U}(y\}.$$

Consider the dual function of the convex hull problem:

$$\begin{aligned} D(w,u,\rho) = \min_{x\geq 0,\lambda\geq 0,c} \Bigg\{ & -\bar{U}(c_T) + \sum_{t=1}^{T} w_t \Bigg(c_t - c_{t-1} - \sum_{i=1}^{n} a_{it}x_i \Bigg) \\ & + \sum_{t=1}^{T} u_t \Bigg(\sum_{j=1}^{T+1} \lambda_j v_t^j - c_t \Bigg) + \rho \Bigg(1 - \sum_{j=1}^{T+1} \lambda_j \Bigg) \Bigg\} \\ = & -\max_{x\geq 0} \sum_{i=1}^{n}\sum_{t=1}^{T} a_{it} w_t x_i + \min_{\lambda\geq 0} \sum_{j=1}^{T+1} \left(\sum_{t=1}^{T} v_t^j u_t - \rho \right) \lambda_j + \rho \\ & + \min_{c} \Bigg\{ \sum_{t=1}^{T-1} c_t(w_t - u_t - w_{t+1}) - w_1 c_0 + c_T(w_T - u_T) - \bar{U}(c_T) \Bigg\} \\ = & \rho - w_1 c_0 + \bar{U}^*(w_T - u_T). \end{aligned}$$

The dual problem becomes

$$\min_{u,w,\rho} \; -\bar{U}^*(w_T - u_T) + w_1 c_0 - \rho \tag{4.67}$$

$$\text{s.t. } w_t = w_{t+1} + u_t, \quad t = T-1, \ldots, 1, \tag{4.68}$$

$$\sum_{t=1}^{\mathsf{T}} w_t a_{it} \leq 0, \quad i = 1, \ldots, n, \tag{4.69}$$

$$\rho \leq \sum_{t=1}^{\mathsf{T}} u_t v_t^j, \quad j = 1, \ldots, T+1. \tag{4.70}$$

$$u \geq 0. \tag{4.71}$$

We can observe that each dual variable u_t is the cost of borrowing a unit of cash for one time period, t. The amount u_t is to be paid at the end of the planning horizon. It follows from (4.68) that each multiplier w_t is the amount that has to be returned at the end of the planning horizon if a unit of cash is borrowed at t and held until time T.

The constraints (4.69) represent the *nonarbitrage condition*. For each bond i we can consider the following operation: borrow money to buy the bond and lend away its coupon payments, according to the rates implied by w_t. At the end of the planning horizon, we collect all loans and pay off the debt. The profit from this operation should be nonpositive for each bond in order to comply with the no-free-lunch condition, which is expressed via (4.69).

Let us observe that each product $u_t v_t^j$ is the amount that has to be paid at the end, for having a debt in the amount v_t^j in period t. Recall that v_t^j is the p-efficient cumulative liability up to time t. Denote the implied one-period liabilities by

$$\begin{aligned} L_t^j &= v_t^j - v_{t-1}^j, \quad t = 2, \ldots, T, \\ L_1^j &= v_1^j. \end{aligned}$$

Changing the order of summation, we obtain

$$\sum_{t=1}^{T} u_t v_t^j = \sum_{t=1}^{T} u_t \sum_{\tau=1}^{t} L_\tau^j = \sum_{\tau=1}^{T} L_\tau^j \sum_{t=\tau}^{T} u_t = \sum_{\tau=1}^{T} L_\tau^j (w_\tau + u_T - w_T).$$

It follows that the sum appearing on the right-hand side of (4.70) can be viewed as the extra cost of covering the jth p-efficient liability sequence by borrowed money, that is, the difference between the amount that has to be returned at the end of the planning horizon, and the total liability discounted by $w_T - u_T$.

If we consider the special case of a linear expected utility,

$$\hat{U}(c_T) = c_T - \mathbb{E}[Z_T],$$

then we can skip the constant $\mathbb{E}[Z_T]$ in the formulation of the optimization problem. The dual function of the convexified cash matching problem becomes

$$\begin{aligned} D(w, u, \rho) = & -\max_{x \geq 0} \sum_{i=1}^{n} \sum_{t=1}^{T} a_{it} w_t x_i + \min_{\lambda \geq 0} \sum_{j=1}^{T+1} \left(\sum_{t=1}^{T} v_t^j u_t - \rho \right) \lambda_j + \rho \\ & + \min_{c} \left\{ \sum_{t=1}^{T-1} c_t \left(w_t - u_t - w_{t+1} \right) - w_1 c_0 + c_T (w_T - u_T - 1) \right\} \\ = & \ \rho - w_1 c_0. \end{aligned}$$

The objective function of the dual problem takes on the form

$$\operatorname*{Min}_{u, w, \rho} w_1 c_0 - \rho,$$

and the constraints (4.68) extends to all time periods:

$$w_t = w_{t+1} + u_t, \quad t = T, T-1, \ldots, 1,$$

with the convention $w_{T+1} = 1$.

In this case, the sum on the right-hand side of (4.70) is the difference between the cost of covering the jth p-efficient liability sequence by borrowed money and the total liability.

The variable ρ represents the minimal cost of this form for all p-efficient trajectories. This allows us to interpret the dual objective function in this special case as the amount obtained at T for lending away our capital c_0 decreased by the extra cost of covering a p-efficient liability sequence by borrowed money. By duality this quantity is the same as c_T, which implies that both ways of covering the liabilities are equally profitable. In the case of a general utility function, the dual objective function contains an additional adjustment term. ∎

4.4 Optimization Problems with Nonseparable Probabilistic Constraints

In this section, we concentrate on the following problem:

$$\begin{aligned} \min_x \ & c(x) \\ \text{s.t.} \ & \Pr\{g(x, Z) \geq 0\} \geq p, \\ & x \in \mathcal{X}. \end{aligned} \tag{4.72}$$

The parameter $p \in (0, 1)$ denotes some probability level. We assume that the functions $c : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}$ and $g : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^m$ are continuous and the set $\mathcal{X} \subset \mathbb{R}^n$ is a closed convex set. We define the constraint function as follows:

$$G(x) = \Pr\{g(x, Z) \geq 0\}.$$

Recall that if $G(\cdot)$ is α-concave function, $\alpha \in \mathbb{R}$, then a transformation of it is a concave function. In this case, we define

$$\bar{G}(x) = \begin{cases} \ln p - \ln[G(x)] & \text{if} \quad \alpha = 0, \\ p^\alpha - [G(x)]^\alpha & \text{if} \quad \alpha > 0, \\ [G(x)]^\alpha - p^\alpha & \text{if} \quad \alpha < 0. \end{cases} \tag{4.73}$$

We obtain the following equivalent formulation of problem (4.72):

$$\begin{aligned} \min_x \ & c(x) \\ \text{s.t.} \ & \bar{G}(x) \leq 0, \\ & x \in \mathcal{X}. \end{aligned} \tag{4.74}$$

Assuming that $c(\cdot)$ is convex, we have a convex problem.

Recall that Slater's condition is satisfied for problem (4.72) if there is a point $x^s \in \operatorname{int}\mathcal{X}$ such that $\bar{G}(x^s) > 0$. Using optimality conditions for convex optimization problems, we can infer the following conditions for problem (4.72).

Theorem 4.76. *Assume that $c(\cdot)$ is a continuous convex function, the functions $g : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^m$ are quasi-concave, Z has an α-concave distribution, and the set $\mathcal{X} \subset \mathbb{R}^n$ is closed and convex. Furthermore, let Slater's condition be satisfied and* $\operatorname{int}\operatorname{dom} G \neq \emptyset$.

A point $\hat{x} \in \mathcal{X}$ is an optimal solution of problem (4.72) *iff there is a number $\lambda \in \mathbb{R}_+$ such that $\lambda[G(\hat{x}) - p] = 0$ and*

$$0 \in \partial c(\hat{x}) + \lambda \frac{1}{\alpha} G(\hat{x})^{1-\alpha} \partial G(\hat{x})^{\alpha} + \mathcal{N}_{\mathcal{X}}(\hat{x}) \qquad \text{if } \alpha \neq 0,$$

or

$$0 \in \partial c(\hat{x}) + \lambda G(\hat{x}) \partial \big(\ln G(\hat{x}) \big) + \mathcal{N}_{\mathcal{X}}(\hat{x}) \qquad \text{if } \alpha = 0.$$

Proof. Under the assumptions of the theorem, problem (4.72) can be reformulated in form (4.74), which is a convex optimization problem. The optimality conditions follow from the optimality conditions for convex optimization problems using Theorem 4.29. Due to Slater's condition, we have that $G(x) > 0$ on a set with nonempty interior, and therefore the assumptions of Theorem 4.29 are satisfied. □

4.4.1 Differentiability of Probability Functions and Optimality Conditions

We can avoid concavity assumptions and replace them by differentiability requirements. Under certain assumptions, we can differentiate the probability function and obtain optimality conditions in a differential form. For this purpose, we assume that Z has a probability density function $\theta(z)$ and that the support of P_Z is a closed set with a piecewise smooth boundary such that $\operatorname{supp} P_Z = \operatorname{cl}\{\operatorname{int}(\operatorname{supp} P_Z)\}$. For example, it can be the union of several disjoint sets but cannot contain isolated points, or surfaces of zero Lebesgue measure.

Consider the multifunction $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^s$, defined as follows:

$$H(x) = \big\{ z \in \mathbb{R}^s : g_i(x, z) \geq 0, i = 1, \ldots, m \big\}.$$

We denote the boundary of a set $H(x)$ by $\operatorname{bd} H(x)$. For an open set $U \subset \mathbb{R}^n$ containing the origin, we set

$$H_U = \operatorname{cl}\left(\bigcup_{x \in U} H(x) \right) \quad \text{and} \quad \triangle H_U = \operatorname{cl}\left(\bigcup_{x \in U} \operatorname{bd} H(x) \right),$$

$$V_U = \operatorname{cl} U \times H_U \quad \text{and} \quad \triangle V_U = \operatorname{cl} U \times \triangle H_U.$$

For any of these sets, we indicate with upper subscript r its restriction to the $\operatorname{supp} P_Z$, e.g., $H_U^r = H_U \cap \operatorname{supp} P_Z$. Let

$$S_i(x) = \big\{ z \in \operatorname{supp} P_Z : g_i(x, z) = 0,\ g_j(x, z) \geq 0,\ j \neq i \big\}, \quad i = 1, \ldots, m.$$

We use the notation

$$S(x) = \cup_{i=1}^{M} S_i(x), \quad \triangle H_i = \operatorname{int}\big(\cup_{x \in U} \left(\partial\{ g_i(x, z) \geq 0 \} \cap H^r(x) \right) \big).$$

The $(m-1)$-dimensional Lebesgue measure is denoted by P_{m-1}. We assume that the functions $g_i(x, z)$, $i = 1, \ldots, m$, are continuously differentiable and such that $\operatorname{bd} H(x) =$

$S(x)$ with $S(x)$ being the $(s-1)$-dimensional surface of the set $H(x) \subset \mathbb{R}^s$. The set H_U is the union of all sets $H(x)$ when $x \in U$, and, correspondingly, $\triangle H_U$ contains all surfaces $S(x)$ when $x \in U$.

First we formulate and prove a result about the differentiability of the probability function for a single constraint function $g(x, z)$, that is, $m = 1$. In this case we omit the index for the function g as well as for the set $S(x)$.

Theorem 4.77. *Assume that*

(i) *the vector functions $\nabla_x g(x, z)$ and $\nabla_z g(x, z)$ are continuous on $\triangle V_U^r$;*

(ii) *the vector functions $\nabla_z g(x, z) > 0$ (componentwise) on the set $\triangle V_U^r$;*

(iii) *the function $\|\nabla_x g(x, z)\| > 0$ on $\triangle V_U^r$.*

Then the probability function $G(x) = \Pr\{g(x, Z) \geq 0\}$ has partial derivatives for almost all $x \in U$ that can be represented as a surface integral,

$$\left(\frac{\partial G(x)}{\partial x_i}\right)_{i=1}^n = \int_{\mathrm{bd}\, H(x) \cap \mathrm{supp}\, P_Z} \frac{\theta(z)}{\|\nabla_z g(x, z)\|} \nabla_x g(x, z) dS.$$

Proof. Without loss of generality, we shall assume that $x \in U \subset \mathbb{R}$.

For two points $x, y \in U$, we consider the difference:

$$\begin{aligned} G(x) - G(y) &= \int_{H(x)} \theta(z) dz - \int_{H(y)} \theta(z) dz \\ &= \int_{H^r(x) \setminus H^r(y)} \theta(z) dz - \int_{H^r(y) \setminus H^r(x)} \theta(z) dz. \end{aligned} \tag{4.75}$$

By the implicit function theorem, the equation $g(x, z) = 0$ determines a differentiable function $x(z)$ such that

$$g(x(z), z) = 0 \quad \text{and} \quad \nabla_z x(z) = -\frac{\nabla_x g(x, z)}{\nabla_z g(x, z)}\bigg|_{x = x(z)}.$$

Moreover, the constraint $g(x, z) \geq 0$ is equivalent to $x \geq x(z)$ for all $(x, z) \in U \times \triangle H_U^r$, because the function $g(\cdot, z)$ strictly increases on this set due to the assumption (iii). Thus, for all points $x, y \in U$ such that $x < y$, we can write

$$\begin{aligned} H^r(x) \setminus H^r(y) &= \{z \in \mathbb{R}^s : g(x, z) \geq 0,\ g(y, z) < 0\} = \{z \in \mathbb{R}^s : x \geq x(z) > y\} = \emptyset, \\ H^r(y) \setminus H^r(x) &= \{z \in \mathbb{R}^s : g(y, z) \geq 0,\ g(x, z) < 0\} = \{z \in \mathbb{R}^s : y \geq x(z) > x\}. \end{aligned}$$

Hence, we can continue our representation of the difference (4.75) as follows:

$$G(x) - G(y) = -\int_{\{z \in \mathbb{R}^s : y \geq x(z) > x\}} \theta(z) dz.$$

Now, we apply Schwarz [194, Vol. 1, Theorem 108] and obtain

$$\begin{aligned} G(x) - G(y) &= -\int_x^y \int_{\{z \in \mathbb{R}^s : x(z) = t\}} \frac{\theta(z)}{\|\nabla_z x(z)\|} dS\, dt \\ &= \int_y^x \int_{\mathrm{bd}\, H(x)^r} \frac{|\nabla_x g(t, z)| \theta(z)}{\|\nabla_z g(x, z)\|} dS\, dt. \end{aligned}$$

By Fubini's theorem [194, Vol. 1, Theorem 77], the inner integral converges almost everywhere with respect to the Lebesgue measure. Therefore, we can apply Schwarz [194, Vol. 1, Theorem 90] to conclude that the difference $G(x) - G(y)$ is differentiable almost everywhere with respect to $x \in U$ and we have

$$\frac{\partial}{\partial x} G(x) = \int_{\mathrm{bd}\, H^r(x)} \frac{\nabla_x g(x, z)\theta(z)}{\|\nabla_z g(x, z)\|} dS.$$

We have used assumption (ii) to set $|\nabla_x g(x, z)| = \nabla_x g(x, z)$. □

Obviously, the statement remains valid if assumption (ii) is replaced by the opposite strict inequality, so that the function $g(x, z)$ would be strictly decreasing on $U \times \triangle H_U^r$.

We note that this result does not imply the differentiability of the function G at any fixed point $x_0 \in U$. However, this type of differentiability is sufficient for many applications, as is elaborated in Ermoliev [65] and Usyasev [216].

The conditions of this theorem can be slightly modified so that the result and the formula for the derivative are valid for piecewise smooth function.

Theorem 4.78 (Raik [166]). *Given a bounded open set $U \subset \mathbb{R}^n$, we assume that*

(i) *the density function $\theta(\cdot)$ is continuous and bounded on the set $\triangle H_i$ for each $i = 1, \ldots, m$;*

(ii) *the vector functions $\nabla_z g_i(x, z)$ and $\nabla_x g_i(x, z)$ are continuous and bounded on the set $U \times \triangle H_i$ for each $i = 1, \ldots, m$;*

(iii) *the function $\|\nabla_x g_i(x, z)\| \geq \delta > 0$ on the set $U \times \triangle H_i$ for each $i = 1, \ldots, m$;*

(iv)*the following conditions are satisfied for all $i = 1, \ldots, m$ and all $x \in U$:*

$$P_{m-1}\big\{S_i(x) \cap S_j(x)\big\} = 0, \quad i \neq j, \quad P_{m-1}\big\{\mathrm{bd}(\mathrm{supp} P_Z \cap S_i(x))\big\} = 0.$$

Then the probability function $G(x)$ is differentiable on U and

$$\nabla G(x) = \sum_{i=1}^{m} \int_{S_i(x)} \frac{\theta(z)}{\|\nabla_z g_i(x, z)\|} \nabla_x g_i(x, z) dS. \tag{4.76}$$

The precise proof of this theorem is omitted. We refer to Kibzun and Tretyakov [104] and Kibzun and Uryasev [105] for more information on this topic.

For example, if $g(x, Z) = x^{\mathrm{T}} Z$, $m = 1$, and Z has a nondegenerate multivariate normal distribution $\mathcal{N}(\bar{z}, \Sigma)$, then $g(x, Z) \sim \mathcal{N}\left(x^{\mathrm{T}}\bar{z}, x^{\mathrm{T}}\Sigma x\right)$, and hence the probability function $G(x) = \Pr\{g(x, Z) \geq 0\}$ can be written in the form

$$G(x) = \Phi\left(\frac{x^{\mathrm{T}}\bar{z}}{\sqrt{x^{\mathrm{T}}\Sigma x}}\right),$$

where $\Phi(\cdot)$ is the cdf of the standard normal distribution. In this case, $G(x)$ is continuously differentiable at every $x \neq 0$.

For problem (4.72), we impose the following constraint qualification at a point $\hat{x} \in \mathcal{X}$. There exists a point $x^r \in \mathcal{X}$ such that

$$\sum_{i=1}^{m} \int_{S_i(x)} \frac{\theta(z)}{\|\nabla_z g_i(\hat{x}, z)\|} (x^r - \hat{x})^{\mathrm{T}} \nabla_x g_i(\hat{x}, z) dS < 0. \tag{4.77}$$

This condition implies Robinson's condition. We obtain the following necessary optimality conditions.

Theorem 4.79. *Under the assumption of Theorem* 4.78*, let the constraint qualification* (4.77) *be satisfied, let the function $c(\cdot)$ be continuously differentiable, and let $\hat{x} \in \mathcal{X}$ be an optimal solution of problem* (4.72)*. Then there is a multiplier $\lambda \geq 0$ such that*

$$0 \in \nabla c(\hat{x}) - \lambda \sum_{i=1}^{m} \int_{S_i(x)} \frac{\theta(z)}{\|\nabla_z g_i(x,z)\|} \nabla_x g_i(x,z) dS + \mathcal{N}_{\mathcal{X}}(\hat{x}), \tag{4.78}$$

$$\lambda\big[G(\hat{x}) - p\big] = 0. \tag{4.79}$$

Proof. The statement follows from the necessary optimality conditions for smooth optimization problems and formula (4.76). □

4.4.2 Approximations of Nonseparable Probabilistic Constraints

Smoothing Approximation via Steklov Transformation

In order to apply the optimality conditions formulated in Theorem 4.76, we need to calculate the subdifferential of the probability function $\bar{G}$ defined by the formula (4.73). The calculation involves the subdifferential of the probability function and the characteristic function of the event

$$\{g_i(x,z) \geq 0,\ i = 1, \ldots, m\}.$$

The latter function may be discontinuous. To alleviate this difficulty, we shall approximate the function $G(x)$ by smooth functions.

Let $k : \mathbb{R} \to \mathbb{R}$ be a nonnegative integrable symmetric function such that

$$\int_{-\infty}^{+\infty} k(t)dt = 1.$$

It can be used as a density function of a random variable K, and, thus,

$$F_K(\tau) = \int_{-\infty}^{\tau} k(t)dt.$$

Taking the characteristic function of the interval $[0, \infty)$, we consider the Steklov–Sobolev average functions for $\varepsilon > 0$:

$$F_K^{\varepsilon}(\tau) = \int_{-\infty}^{+\infty} \mathbf{1}_{[0,\infty)}(\tau + \varepsilon t)k(t)dt = \frac{1}{\varepsilon}\int_{-\infty}^{+\infty} \mathbf{1}_{[0,\infty)}(t)k\left(\frac{t-\tau}{\varepsilon}\right)dt. \tag{4.80}$$

We see that by the definition of F_K^{ε} and $\mathbf{1}_{[0,\infty)}$, and by the symmetry of $k(\cdot)$ we have

$$\begin{aligned} F_K^{\varepsilon}(\tau) &= \int_{-\infty}^{+\infty} \mathbf{1}_{[0,\infty)}(\tau + \varepsilon t)k(t)dt = \int_{-\tau/\varepsilon}^{+\infty} k(t)dt \\ &= \int_{-\infty}^{\tau/\varepsilon} k(-t)dt = \int_{-\infty}^{\tau/\varepsilon} k(t)dt \\ &= F_K\left(\frac{\tau}{\varepsilon}\right). \end{aligned} \tag{4.81}$$

Setting

$$g_M(x,z) = \min_{1\le i\le m} g_i(x,z),$$

we note that g_M is quasi-concave, provided that all g_i are quasi-concave functions. If the functions $g_i(\cdot,z)$ are continuous, then $g_M(\cdot,z)$ is continuous as well.

Using (4.81), we can approximate the constraint function $G(\cdot)$ by the function

$$\begin{aligned} G_\varepsilon(x) &= \int_{\mathbb{R}^s} F_K^\varepsilon\big(g_M(x,z)-c\big)dP_z \\ &= \int_{\mathbb{R}^s} F_K\left(\frac{g_M(x,z)-c}{\varepsilon}\right)dP_z \\ &= \frac{1}{\varepsilon}\int_{\mathbb{R}^s}\int_{-\infty}^{-c} k\left(\frac{t+g_M(x,z)}{\varepsilon}\right)dt\,dP_z. \end{aligned} \tag{4.82}$$

Now, we show that the functions $G_\varepsilon(\cdot)$ uniformly converge to $G(\cdot)$ when ε converges to zero.

Theorem 4.80. *Assume that Z has a continuous distribution, the functions $g_i(\cdot,z)$ are continuous for almost all $z\in\mathbb{R}^s$ and that, for certain constant $c\in\mathbb{R}$, we have*

$$\Pr\{z\in\mathbb{R}^s : g_M(x,z)=c\}=0.$$

Then for any compact set $\mathbf{C}\subset\mathcal{X}$ the functions G_ε uniformly converge on $\mathbf{C}$ to G when $\varepsilon\to 0$, i.e.,

$$\lim_{\varepsilon\downarrow 0}\max_{x\in\mathbf{C}}\big|G_\varepsilon(x)-G(x)\big|=0.$$

Proof. Defining $\delta(\varepsilon)=\varepsilon^{1-\beta}$ with $\beta\in(0,1)$, we have

$$\lim_{\varepsilon\to 0}\delta(\varepsilon)=0 \quad\text{and}\quad \lim_{\varepsilon\to 0}F_K\left(\frac{\delta(\varepsilon)}{\varepsilon}\right)=1,\ \lim_{\varepsilon\to 0}F_K\left(\frac{-\delta(\varepsilon)}{\varepsilon}\right)=0. \tag{4.83}$$

Define for any $\delta>0$ the sets

$$\begin{aligned} A(x,\delta) &= \{z\in\mathbb{R}^s : g_M(x,z)-c\le -\delta\},\\ B(x,\delta) &= \{z\in\mathbb{R}^s : g_M(x,z)-c\ge \delta\},\\ C(x,\delta) &= \{z\in\mathbb{R}^s : |g_M(x,z)-c|\le \delta\}. \end{aligned}$$

On the set $A(x,\delta(\varepsilon))$ we have $\mathbf{1}_{[0,\infty)}\big(g_M(x,z)-c\big)=0$ and, using (4.81), we obtain

$$F_K^\varepsilon\big(g_M(x,z)-c\big)=F_K\left(\frac{g_M(x,z)-c}{\varepsilon}\right)\le F_K\left(\frac{-\delta(\varepsilon)}{\varepsilon}\right).$$

On the set $B(x,\delta(\varepsilon))$ we have $\mathbf{1}_{[0,\infty)}\big(g_M(x,z)-c\big)=1$ and

$$F_K^\varepsilon\big(g_M(x,z)-c\big)=F_K\left(\frac{g_M(x,z)-c}{\varepsilon}\right)\ge F_K\left(\frac{\delta(\varepsilon)}{\varepsilon}\right).$$

On the set $C(\delta(\varepsilon))$ we use the fact that $0 \le \mathbf{1}_{[0,\infty)}(t) \le 1$ and $0 \le F_K(t) \le 1$. We obtain the following estimate:

$$\begin{aligned}
&\big|G(x) - G_\varepsilon(x)\big| \\
&\quad \le \int_{\mathbb{R}^s} \big|\mathbf{1}_{[0,\infty)}\big(g_M(x,z) - c\big) - F_K^\varepsilon\big(g_M(x,z) - c\big)\big| dP_z \\
&\quad \le F_K\left(\frac{-\delta(\varepsilon)}{\varepsilon}\right) \int_{A(x,\delta(\varepsilon))} dP_Z + \left(1 - F_K\left(\frac{\delta(\varepsilon)}{\varepsilon}\right)\right) \int_{B(x,\delta(\varepsilon))} dP_Z + 2\int_{C(x,\delta(\varepsilon))} dP_Z \\
&\quad \le F_K\left(\frac{-\delta(\varepsilon)}{\varepsilon}\right) + \left(1 - F_K\left(\frac{\delta(\varepsilon)}{\varepsilon}\right)\right) + 2P_Z(C(x,\delta(\varepsilon))).
\end{aligned}$$

The first two terms on the right-hand side of the inequality converge to zero when $\varepsilon \to 0$ by the virtue of (4.83). It remains to show that $\lim_{\varepsilon\to 0} P_Z\{C(x,\delta(\varepsilon))\} = 0$ uniformly with respect to $x \in \mathbf{C}$. The function $(x, z, \delta) \mapsto |g_M(x,z) - c| - \delta$ is continuous in (x,δ) and measurable in z. Thus, it is uniformly continuous with respect to (x,δ) on any compact set $\mathbf{C} \times [-\delta_0, \delta_0]$ with $\delta_0 > 0$. The probability measure P_Z is continuous, and, therefore, the function

$$\Theta(x,\delta) = P\{|g_M(x,z) - c| - \delta \le 0\} = P\Big\{\cap_{\beta>\delta} C(x,\beta)\Big\}$$

is uniformly continuous with respect to (x,δ) on $\mathbf{C} \times [-\delta_0, \delta_0]$. By the assumptions of the theorem

$$\Theta(x,0) = P_Z\{z \in \mathbb{R}^s : |g_M(x,z) - c| = 0\} = 0,$$

and, thus,

$$\lim_{\varepsilon\to 0} P_Z\{z \in \mathbb{R}^s : |g_M(x,z) - c| \le \delta(\varepsilon)\} = \lim_{\delta\to 0} \Theta(x,\delta) = 0.$$

As $\Theta(\cdot,\delta)$ is continuous, the convergence is uniform on compact sets with respect to the first argument. □

Now, we derive a formula for the Clarke generalized gradients of the approximation G_ε. We define the index set

$$I(x,z) = \{i : g_i(x,z) = g_M(x,z),\ 1 \le i \le m\}.$$

Theorem 4.81. *Assume that the density function $k(\cdot)$ is nonnegative, bounded, and continuous. Furthermore, let the functions $g_i(\cdot,z)$ be concave for every $z \in \mathbb{R}^s$ and their subgradients be uniformly bounded as follows:*

$$\sup\{s \in \partial g_i(y,z),\ \|y - x\| \le \delta\} \le l_\delta(x,z),\ \delta > 0, \quad \forall i = 1,\dots,m,$$

where $l_\delta(x,z)$ is an integrable function of z for all $x \in \mathcal{X}$. Then $G_\varepsilon(\cdot)$ is Lipschitz continuous and Clarke-regular, and its Clarke generalized gradient set is given by

$$\partial^\circ G_\varepsilon(x) = \frac{1}{\varepsilon}\int_{\mathbb{R}^s} k\left(\frac{g_M(x,z) - c}{\varepsilon}\right) \operatorname{conv}\{\partial g_i(x,z) : i \in I(x,z)\}\, dP_Z.$$

Proof. Under the assumptions of the theorem, the function $F_K(\cdot)$ is monotone and continuously differentiable. The function $g_M(\cdot, z)$ is concave for every $z \in \mathbb{R}^s$ and its subdifferential are given by the formula

$$\partial g_M(y, z) = \operatorname{conv}\{s_i \in \partial g_i(y, z) : g_i(y, z) = g_M(y, z)\}.$$

Thus the subgradients of g_M are uniformly bounded:

$$\sup\{s \in \partial g_M(y, z), \|y - x\| \le \delta\} \le l_\delta(x, z),\ \delta > 0.$$

Therefore, the composite function $F_K\left(\frac{g_M(x,z)-c}{\varepsilon}\right)$ is subdifferentiable and its subdifferential can be calculated as

$$\partial^\circ F_K\left(\frac{g_M(x, z) - c}{\varepsilon}\right) = \frac{1}{\varepsilon} k\left(\frac{g_M(x, z) - c}{\varepsilon}\right) \cdot \partial g_M(x, z).$$

The mathematical expectation function

$$G_\varepsilon(x) = \int_{\mathbb{R}^s} F_K^\varepsilon(g_M(x, z) - c)dP_z = \int_{\mathbb{R}^s} F_K\left(\frac{g_M(x, z) - c}{\varepsilon}\right) dP_z$$

is regular by Clarke [38, Theorem 2.7.2], and its Clarke generalized gradient set has the form

$$\partial^\circ G_\varepsilon(x) = \int_{\mathbb{R}^s} \partial^\circ F_K\left(\frac{g_M(x, z) - c}{\varepsilon}\right) dP_Z = \frac{1}{\varepsilon}\int_{\mathbb{R}^s} k\left(\frac{g_M(x, z) - c}{\varepsilon}\right) \cdot \partial g_M(x, z)dP_Z.$$

Using the formula for the subdifferential of $g_M(x, z)$, we obtain the statement. □

Now we show that if we choose K to have an α-concave distribution, and all assumptions of Theorem 4.39 are satisfied, the generalized concavity property of the approximated probability function is preserved.

Theorem 4.82. *If the density function k is α-concave ($\alpha \ge 0$), Z has γ-concave distribution ($\gamma \ge 0$), the functions $g_i(\cdot, z)$, $i = 1, \ldots, m$, are quasi-concave, then the approximate probability function G_ε has a β-concave distribution, where*

$$\beta = \begin{cases} (\gamma^{-1} + (1 + s\alpha)/\alpha)^{-1} & \text{if } \alpha + \gamma > 0, \\ 0 & \text{if } \alpha + \gamma = 0. \end{cases}$$

Proof. If the density function k is α-concave ($\alpha \ge 0$), then K has a γ-concave distribution with $\gamma = \alpha/(1 + s\alpha)$. If Z has γ'-concave distribution ($\gamma \ge 0$), then the random vector $(Z, K)^\top$ has a β-concave distribution according to Theorem 4.36, where

$$\beta = \begin{cases} (\gamma^{-1} + \gamma'^{-1})^{-1} & \text{if } \gamma + \gamma' > 0, \\ 0 & \text{if } \gamma + \gamma' = 0. \end{cases}$$

Using the definition $G_\varepsilon(x)$ of (4.82), we can write

$$\begin{aligned} G_\varepsilon(x) &= \int_{\mathbb{R}^s} F_K^\varepsilon\left(\frac{g_M(x, z) - c}{\varepsilon}\right) dP_Z = \int_{\mathbb{R}^s}\int_{-\infty}^{(g_M(x,z)-c)/\varepsilon} k(t)dt\ dP_Z \\ &= \int_{\mathbb{R}^s}\int_{-\infty}^{\infty} \mathbf{1}_{\{(g_M(x,z)-c)/\varepsilon > t\}} dP_K\ dP_z = \int_{\mathbb{R}^s}\int_{H_\varepsilon(x)} dP_K\, dP_z, \end{aligned} \tag{4.84}$$

where

$$H_\varepsilon(x) = \{(z,t) \in \mathbb{R}^{s+1} : g_M(x,z) - \varepsilon t \geq c\}.$$

Since $g_M(\cdot, z)$ is quasi-concave, the set $H_\varepsilon(x)$ is convex. Representation (4.84) of G_ε and the β-concavity of (Z, K) imply the assumptions of Theorem 4.39, and, thus, the function G_ε is β-concave. □

This theorem shows that if the random vector Z has a generalized concave distribution, we can choose a suitable generalized concave density function $k(\cdot)$ for smoothing and obtain an approximate convex optimization problem.

Theorem 4.83. *In addition to the assumptions of Theorems* 4.80, 4.81*, and* 4.82*. Then on the set* $\{x \in \mathbb{R}^n : G(x) > 0\}$*, the function* G_ε *is Clarke-regular and the set of Clarke generalized gradients* $\partial^\circ G_\varepsilon(x^\varepsilon)$ *converge to the set of Clarke generalized gradients of* G*,* $\partial^\circ G(x)$ *in the following sense: if for any sequences* $\varepsilon \downarrow 0$*,* $x^\varepsilon \to x$ *and* $s^\varepsilon \in \partial^\circ G_\varepsilon(x^\varepsilon)$ *such that* $s^\varepsilon \to s$*, then* $s \in \partial^\circ G(x)$*.*

Proof. Consider a point x such that $G(x) > 0$ and points $x^\varepsilon \to x$ as $\varepsilon \downarrow 0$. All points x^ε can be included in some compact set containing x in its interior. The function G is generalized concave by virtue of Theorem 4.39. It is locally Lipschitz continuous, directionally differentiable, and Clarke-regular due to Theorem 4.29. It follows that $G(y) > 0$ for all point y in some neighborhood of x. By virtue of Theorem 4.80, this neighborhood can be chosen small enough, so that $G_\varepsilon(y) > 0$ for all ε small enough, as well. The functions G_ε are generalized concave by virtue of Theorem 4.82. It follows that G_ε are locally Lipschitz continuous, directionally differentiable, and Clarke-regular due to Theorem 4.29. Using the uniform convergence of G_ε on compact sets and the definition of Clarke generalized gradient, we can pass to the limit with $\varepsilon \downarrow 0$ in the inequality

$$\lim_{t \downarrow 0,\ y \to x^\varepsilon} \frac{1}{t}\big[G_\varepsilon(y + td) - G_\varepsilon(y)\big] \geq d^{\mathrm{T}} s^\varepsilon \quad \text{for any } d \in \mathbb{R}^n.$$

Consequently, $s \in \partial^\circ G(x)$. □

Using the approximate probability function we can solve the following approximation of problem (4.72):

$$\begin{aligned} \operatorname*{Min}_x \ & c(x) \\ \text{s.t.} \ & G_\varepsilon(x) \geq p, \\ & x \in \mathcal{X}. \end{aligned} \tag{4.85}$$

Under the conditions of Theorem 4.83 the function G_ε is β-concave for some $\beta \geq 0$. We can specify the necessary and sufficient optimality conditions for the approximate problem.

Theorem 4.84. *In addition to the assumptions of Theorem* 4.83*, assume that* $c(\cdot)$ *is a convex function, the Slater condition for problem* (4.85) *is satisfied, and* $\operatorname{int} G_\varepsilon \neq \emptyset$*. A point* $\hat{x} \in \mathcal{X}$

is an optimal solution of problem (4.85) *iff a nonpositive number* λ *exists such that*

$$0 \in \partial c(\hat{x}) + s\lambda \int_{\mathbb{R}^s} k\left(\frac{g_M(\hat{x}, z) - c}{\varepsilon}\right) \operatorname{conv}\left\{\partial g_i(\hat{x}, z) : i \in I(\hat{x}, z)\right\} dP_Z + \mathcal{N}_{\mathcal{X}}(\hat{x}),$$
$$\lambda[G_\varepsilon(\hat{x}) - p] = 0.$$

Here

$$s = \begin{cases} \alpha\varepsilon^{-1}\left[G_\varepsilon(\hat{x})\right]^{\alpha-1} & \text{if } \beta \neq 0, \\ \left[\varepsilon G_\varepsilon(\hat{x})\right]^{-1} & \text{if } \beta = 0. \end{cases}$$

Proof. We shall show the statement for $\beta = 0$. The proof for the other case is analogous. Setting $\bar{G}_\varepsilon(x) = \ln G_\varepsilon(x)$, we obtain a concave function $\bar{G}_\varepsilon$, and formulate the problem

$$\begin{aligned} &\operatorname*{Min}_x \ c(x) \\ &\text{s.t. } \ln p - \bar{G}_\varepsilon(x) \leq 0, \\ &\quad x \in \mathcal{X}. \end{aligned} \tag{4.86}$$

Clearly, $\hat{x}$ is a solution of the problem (4.86) iff it is a solution of problem (4.85). Problem (4.86) is a convex problem and Slater's condition is satisfied for it as well. Therefore, we can write the following optimality conditions for it. The point $\hat{x} \in \mathcal{X}$ is a solution iff a number $\lambda_0 > 0$ exists such that

$$0 \in \partial c(x) + \lambda_0 \partial\left[-\bar{G}_\varepsilon(\hat{x})\right] + \mathcal{N}_{\mathcal{X}}(\hat{x}), \tag{4.87}$$
$$\lambda_0[G_\varepsilon(\hat{x}) - p] = 0. \tag{4.88}$$

We use the formula for the Clarke generalized gradients of generalized concave functions to obtain

$$\partial^\circ \bar{G}_\varepsilon(\hat{x}) = \frac{1}{G_\varepsilon(\hat{x})} \partial^\circ G_\varepsilon(\hat{x}).$$

Moreover, we have a representation of the Clarke generalized gradient set of G_ε, which yields

$$\partial^\circ \bar{G}_\varepsilon(\hat{x}) = \frac{1}{\varepsilon G_\varepsilon(\hat{x})} \int_{\mathbb{R}^s} k\left(\frac{g_M(\hat{x}, z) - c}{\varepsilon}\right) \cdot \partial g_M(\hat{x}, z) dP_Z.$$

Substituting the last expression into (4.87), we obtain the result. □

Normal Approximation

In this section we analyze approximation for problems with individual probabilistic constraints, defined by linear inequalities. In this setting it is sufficient to consider a problem with a single probabilistic constraint of form

$$\begin{aligned} &\text{Max } \ c(x) \\ &\text{s.t. } \ \Pr\{x^{\mathrm{T}} Z \geq \eta\} \geq p, \\ &\quad x \in \mathcal{X}. \end{aligned} \tag{4.89}$$

Before developing the normal approximation for this problem, let us illustrate its potential on an example. We return to our Example 4.2, in which we formulated a portfolio optimization problem under a Value-at-Risk constraint.

$$\begin{aligned} \text{Max} \ & \sum_{i=1}^{n} \mathbb{E}[R_i] x_i \\ \text{s.t.} \ & \Pr\left\{ \sum_{i=1}^{n} R_i x_i \geq -\eta \right\} \geq p, \\ & \sum_{i=1}^{n} x_i \leq 1, \\ & x \geq 0. \end{aligned} \tag{4.90}$$

We denote the net increase of the value of our investment after a period of time by

$$G(x, R) = \sum_{i=1}^{n} \mathbb{E}[R_i] x_i.$$

Let us assume that the random return rates $R_1, \ldots, R_n$ have a joint normal probability distribution. Recall that the normal distribution is log-concave and the probabilistic constraint in problem (4.90) determines a convex feasible set, according to Theorem 4.39.

Another direct way to see that the last transformation of the probabilistic constraint results in a convex constraint is the following. We denote $\bar{r}_i = \mathbb{E}[R_i]$, $\bar{r} = (\bar{r}_1, \ldots, \bar{r}_n)^{\mathsf{T}}$, and assume that $\bar{r}$ is not the zero-vector. Further, let Σ be the covariance matrix of the joint distribution of the return rates. We observe that the total profit (or loss) $G(x, R)$ is a normally distributed random variable with expected value $\mathbb{E}\big[G(x, R)\big] = \bar{r}^{\mathsf{T}} x$ and variance $\mathbb{V}\text{ar}\big[G(x, R)\big] = x^{\mathsf{T}} \Sigma x$. Assuming that Σ is positive definite, the probabilistic constraint

$$\Pr\big\{G(x, R) \geq -\eta\big\} \geq p$$

can be written in the form (see the discussion on page 16)

$$z_p \sqrt{x^{\mathsf{T}} \Sigma x} - \bar{r}^{\mathsf{T}} x \leq \eta.$$

Hence problem (4.90) can be written in the following form:

$$\begin{aligned} \text{Max} \ & \bar{r}^{\mathsf{T}} x \\ \text{s.t.} \ & z_p \sqrt{x^{\mathsf{T}} \Sigma x} - \bar{r}^{\mathsf{T}} x \leq \eta, \\ & \sum_{i=1}^{n} x_i \leq 1, \\ & x \geq 0. \end{aligned} \tag{4.91}$$

Note that $\sqrt{x^{\mathsf{T}} \Sigma x}$ is a convex function, of x, and $z_p = \Phi^{-1}(p)$ is positive for $p > 1/2$, and hence (4.91) is a convex programming problem.

Now, we consider the general optimization problem (4.89). Assuming that the n-dimensional random vector Z has independent components and the dimension n is relatively large, we may invoke the central limit theorem. Under mild additional assumptions, we can conclude that the distribution of $x^{\mathsf{T}}Z$ is approximately normal and convert the probabilistic constraint into an algebraic constraint in a similar manner. Note that this approach is appropriate if Z has a substantial number of components and the vector x has appropriately large number of nonzero components, so that the central limit theorem would be applicable to $x^{\mathsf{T}}Z$. Furthermore, we assume that the probability parameter p is not too close to one, such as 0.9999.

We recall several versions of the central limit theorem (CLT). Let Z_i, $i = 1, 2, \ldots$, be a sequence of independent random variables defined on the same probability space. We assume that each Z_i has finite expected value $\mu_i = \mathbb{E}[Z_i]$ and finite variance $\sigma_i^2 = \mathbb{V}\mathrm{ar}[Z_i]$. Setting

$$s_n^2 = \sum_{i=1}^{n} \sigma_i^2 \quad \text{and} \quad r_n^3 = \sum_{i=1}^{n} \mathbb{E}\big(|Z_i - \mu_i|^3\big),$$

we assume that r_n^3 is finite for every n and that

$$\lim_{n\to\infty} \frac{r_n}{s_n} = 0. \tag{4.92}$$

Then the distribution of the random variable

$$\frac{\sum_{i=1}^{n}(Z_i - \mu_i)}{s_n} \tag{4.93}$$

converges toward the standard normal distribution as $n \to \infty$.

The condition (4.92) is called Lyapunov's condition. In the same setting, we can replace the Lyapunov's condition with the following weaker condition, proposed by Lindeberg. For every $\varepsilon > 0$ we define

$$Y_{in} = \begin{cases} (Z_i - \mu_i)^2/s_n^2 & \text{if } |Z_i - \mu_i| > \varepsilon s_n, \\ 0 & \text{otherwise.} \end{cases}$$

The Lindeberg's condition reads

$$\lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{E}(Y_{in}) = 0.$$

Let us denote $\bar{z} = (\mu_1, \ldots, \mu_n)^{\mathsf{T}}$. Under the conditions of the CLT, the distribution of our random variable $x^{\mathsf{T}}Z$ is close to the normal distribution with expected value $x^{\mathsf{T}}\bar{z}$ and variance $\sum_{i=1}^{n} \sigma_i^2 x_i^2$ for problems of large dimensions. Our probabilistic constraint takes on the form

$$\frac{\bar{z}^{\mathsf{T}}x - \eta}{\sqrt{\sum_{i=1}^{n} \sigma_i^2 x_i^2}} \geq z_p.$$

Define $\mathcal{X} = \left\{x \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i \leq 1\right\}$. Denoting the matrix with diagonal elements $\sigma_1, \ldots, \sigma_n$ by D, problem (4.89) can be replaced by the following approximate problem:

$$\begin{aligned} \operatorname*{Min}_x \ & c(x) \\ \text{s.t.} \ & z_p \|Dx\| \leq \bar{z}^{\mathsf{T}} x - \eta, \\ & x \in \mathcal{X}. \end{aligned}$$

The probabilistic constraint in this problem is approximated by an algebraic convex constraint. Due to the independence of the components of the random vector Z, the matrix D has a simple diagonal form. There are versions of the CLT which treat the case of sums of dependent variables, for instance, the n-dependent CLT, the martingale CLT, and the CLT for mixing processes. These statements will not be presented here. One can follow the same line of argument to formulate a normal approximation of the probabilistic constraint, which is very accurate for problems with large decision space.

4.5 Semi-infinite Probabilistic Problems

In this section, we concentrate on the semi-infinite probabilistic problem (4.9). We recall its formulation:

$$\begin{aligned} \operatorname*{Min}_x \ & c(x) \\ \text{s.t.} \ & \Pr\left\{g(x, Z) \geq \eta\right\} \geq \Pr\left\{Y \geq \eta\right\}, \quad \eta \in [a, b], \\ & x \in \mathcal{X}. \end{aligned}$$

Our goal is to derive necessary and sufficient optimality conditions for this problem. Denote the space of regular countably additive measures on $[a, b]$ having finite variation by $\mathcal{M}([a, b])$ and its subset of nonnegative measures by $\mathcal{M}_+([a, b])$.

We define the constraint function $G(x, \eta) = P\{z : g(x, z) \geq \eta\}$. As we shall develop optimality conditions in differential form, we impose additional assumptions on problem (4.9):

(i) The function c is continuously differentiable on $\mathcal{X}$.

(ii) The constraint function $G(\cdot, \cdot)$ is continuous with respect to the second argument and continuously differentiable with respect to the first argument.

(iii) The reference random variable Y has a continuous distribution.

The differentiability assumption on G may be enforced taking into account the results in section 4.4.1. For example, if the vector Z has a probability density $\theta(\cdot)$, the function $g(\cdot, \cdot)$ is continuously differentiable with nonzero gradient $\nabla_z g(x, z)$ and such that the quantity $\frac{\theta(z)}{\|\nabla_z g(x,z)\|} \nabla_x g(x, z)$ is uniformly bounded (in a neighborhood of x) by an integrable function, then the function G is differentiable. Moreover, we can express its gradient with respect to x a follows:

$$\nabla_x G(x, \eta) = \int_{\operatorname{bd} H(z,\eta)} \frac{\theta(z)}{\|\nabla_z g(x, z)\|} \nabla_x g(x, z) \, dP_{m-1},$$

where $\operatorname{bd} H(z, \eta)$ is the surface of the set $H(z, \eta) = \{z : g(x, z) \geq \eta\}$ and P_{m-1} refers to Lebesgue measure on the $(m-1)$-dimensional surface.

We define the set $\mathcal{U}([a,b])$ of functions $u(\cdot)$ satisfying the following conditions:

$$\begin{aligned} &u(\cdot) \text{ is nondecreasing and right continuous;}\\ &u(t) = 0, \quad \forall t \le a;\\ &u(t) = u(b), \quad \forall t \ge b. \end{aligned}$$

It is evident that $\mathcal{U}([a,b])$ is a convex cone.

First we derive a useful formula.

Lemma 4.85. *For any real random variable Z and any measure $\mu \in \mathcal{M}([a,b])$ we have*

$$\int_a^b \Pr\{Z \ge \eta\}\, d\mu(\eta) = \mathbb{E}\big[u(Z)\big], \tag{4.94}$$

where $u(z) = \mu([a,z])$.

Proof. We extend the measure μ to the entire real line by assigning measure 0 to sets not intersecting $[a,b]$. Using the probability measure P_Z induced by Z on $\mathbb{R}$ and applying Fubini's theorem, we obtain

$$\begin{aligned} \int_a^b \Pr\{Z \ge \eta\}\, d\mu(\eta) &= \int_a^\infty \Pr\{Z \ge \eta\}\, d\mu(\eta) = \int_a^\infty \int_\eta^\infty dP_Z(z)\, d\mu(\eta)\\ &= \int_a^\infty \int_a^z d\mu(\eta)\, dP_Z(z) = \int_a^\infty \mu([a,z])\, dP_Z(z) = \mathbb{E}\big[\mu([a,Z])\big]. \end{aligned}$$

We define $u(z) = \mu([a,z])$ and obtain the stated result. □

Let us observe that if the measure μ in the above lemma is nonnegative, then $u \in \mathcal{U}([a,b])$. Indeed, $u(\cdot)$ is nondecreasing since for $z_1 > z_2$ we have

$$u(z_1) = \mu([a,z_1]) = \mu([a,z_2]) + \mu((z_1,z_2]) \ge \mu([a,z_2]) = u(z_2).$$

Furthermore, $u(z) = \mu([a,z]) = \mu([a,b]) = u(b)$ for $z \ge b$.

We introduce the functional $L : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ associated with problem (4.9):

$$L(x,u) = c(x) + \mathbb{E}\Big[u(g(x,Z)) - u(Y)\Big].$$

We shall see that the functional L plays the role of a Lagrangian of the problem.

We also set $v(\eta) = \Pr\{Y \ge \eta\}$.

Definition 4.86. *Problem* (4.9) *satisfies the* differential uniform dominance condition *at the point $\hat{x} \in \mathcal{X}$ if there exists $x^0 \in \mathcal{X}$ such that*

$$\min_{a \le \eta \le b} \Big[G(\hat{x},\eta) + \nabla_x G(\hat{x},\eta)(x^0 - \hat{x}) - v(\eta)\Big] > 0.$$

Theorem 4.87. *Assume that $\hat{x}$ is an optimal solution of problem* (4.9) *and that the differential uniform dominance condition is satisfied at the point $\hat{x}$. Then there exists a function*

$\hat{u} \in \mathcal{U}$, *such that*

$$-\nabla_x L(\hat{x}, \hat{u}) \in \mathcal{N}_{\mathcal{X}}(\hat{x}), \tag{4.95}$$

$$\mathbb{E}\big[\hat{u}(g(\hat{x}, Z))\big] = \mathbb{E}\big[\hat{u}(Y)\big]. \tag{4.96}$$

Proof. We consider the mapping $\Gamma : \mathcal{X} \to \mathcal{C}([a, b])$ defined as follows:

$$\Gamma(x)(\eta) = \Pr\big\{g(x, Z) \geq \eta\big\} - v(\eta), \quad \eta \in [a, b]. \tag{4.97}$$

We define K as the cone of nonnegative functions in $\mathcal{C}([a, b])$. Problem (4.9) can be formulated as follows:

$$\begin{aligned} &\operatorname*{Min}_{x} \ c(x) \\ &\text{s.t.} \ \ \Gamma(x) \in K, \\ &\quad x \in \mathcal{X}. \end{aligned} \tag{4.98}$$

At first we observe that the functions $c(\cdot)$ and $\Gamma(\cdot)$ are continuously differentiable by the assumptions made at the beginning of this section. Second, the differential uniform dominance condition is equivalent to Robinson's constraint qualification condition:

$$0 \in \operatorname{int}\Big\{\Gamma(\hat{x}) + \nabla_x \Gamma(\hat{x})(\mathcal{X} - \hat{x}) - K\Big\}. \tag{4.99}$$

Indeed, it is easy to see that the uniform dominance condition implies Robinson's condition. On the other hand, if Robinson's condition holds true, then there exists $\varepsilon > 0$ such that the function identically equal to ε is an element of the set at the right-hand side of (4.99). Then we can find x^0 such that

$$\Gamma(\hat{x})(\eta) + \big[\nabla_x \Gamma(\hat{x})(\eta)\big](x^0 - \hat{x}) \geq \varepsilon, \quad \forall \eta \in [a, b].$$

Consequently, the uniform dominance condition is satisfied.

By the Riesz representation theorem, the space dual to $\mathcal{C}([a, b])$ is the space $\mathcal{M}([a, b])$ of regular countably additive measures on $[a, b]$ having finite variation. The Lagrangian $\Lambda : \mathbb{R}^n \times \mathcal{M}([a, b]) \to \mathbb{R}$ for problem (4.98) is defined as follows:

$$\Lambda(x, \mu) = c(x) + \int_a^b \Gamma(x)(\eta)\, d\mu(\eta). \tag{4.100}$$

The necessary optimality conditions for problem (4.98) have the form, *There exists a measure* $\hat{\mu} \in \mathcal{M}_+([a, b])$ *such that*

$$-\nabla_x \Lambda(\hat{x}, \hat{\mu}) \in \mathcal{N}_{\mathcal{X}}(\hat{x}), \tag{4.101}$$

$$\int_a^b \Gamma(\hat{x})(\eta)\, d\hat{\mu}(\eta) = 0. \tag{4.102}$$

Using Lemma 4.85, we obtain the equation for all x:

$$\begin{aligned} \int_a^b \Gamma(x)(\eta)\, d\hat{\mu}(\eta) &= \int_a^b \Big(\Pr\big\{g(x, z) \geq \eta\big\} - \Pr\big\{Y \geq \eta\big\}\Big)\, d\hat{\mu}(\eta) \\ &= \mathbb{E}\big[\hat{u}(g(x, Z))\big] - \mathbb{E}\big[\hat{u}(Y)\big], \end{aligned}$$

where $\hat{u}(\eta) = \hat{\mu}([a, \eta])$. Since $\hat{\mu}$ is nonnegative, the corresponding utility function $\hat{u}$ is an element of $\mathcal{U}([a, b])$. The correspondence between nonnegative measures $\mu \in \mathcal{M}([a, b])$ and utility functions $u \in \mathcal{U}$ and the last equation imply that (4.102) is equivalent to (4.96). Moreover,

$$\Lambda(x, \mu) = L(x, u),$$

and, therefore, (4.101) is equivalent to (4.95). □

We note that the functions $u \in \mathcal{U}([a, b])$ can be interpreted as von Neumann–Morgenstern utility functions of rational decision makers. The theorem demonstrates that one can view the maximization of expected utility as a dual model to the model with stochastic dominance constraints. Utility functions of decision makers are very difficult to elicit. This task becomes even more complicated when there is a group of decision makers who have to come to a consensus. Model (4.9) avoids these difficulties by requiring that a benchmark random outcome, considered reasonable, be specified. Our analysis, departing from the benchmark outcome, generates the utility function of the decision maker. It is implicitly defined by the benchmark used and by the problem under consideration.

We will demonstrate that it is sufficient to consider only the subset of $\mathcal{U}([a, b]$ containing piecewise constant utility functions.

Theorem 4.88. *Under the assumptions of Theorem* 4.87 *there exist piecewise constant utility function* $w(\cdot) \in \mathcal{U}$ *satisfying the necessary optimality conditions* (4.95)–(4.96). *Moreover, the function* $w(\cdot)$ *has at most* $n + 2$ *jump points: there exist numbers* $\eta_i \in [a, b], \quad i = 1, \ldots, k$, *such that the function* $w(\cdot)$ *is constant on the intervals* $(-\infty, \eta_1]$, $(\eta_1, \eta_2], \ldots, (\eta_k, \infty)$, *and* $0 \leq k \leq n + 2$.

Proof. Consider the mapping Γ defined by (4.97). As already noted in the proof of the previous theorem, it is continuously differentiable due to the assumptions about the probability function. Therefore, the derivative of the Lagrangian has the form

$$\nabla_x \Lambda(\hat{x}, \hat{\mu}) = \nabla_x c(\hat{x}) + \int_a^b \nabla_x \Gamma(\hat{x})(\eta)\, d\hat{\mu}(\eta).$$

The necessary condition of optimality (4.101) can be rewritten as follows:

$$-\nabla_x c(\hat{x}) - \int_a^b \nabla_x \Gamma(\hat{x})(\eta)\, d\hat{\mu}(\eta) \in \mathcal{N}_X(\hat{x}).$$

Considering the vector

$$g = \nabla_x c(\hat{x}) - \nabla_x \Lambda(\hat{x}, \hat{\mu}),$$

we observe that the optimal values of multipliers $\hat{\mu}$ have to satisfy the equation

$$\int_a^b \nabla_x \Gamma(\hat{x})(\eta)\, d\mu(\eta) = g. \tag{4.103}$$

At the optimal solution $\hat{x}$ we have $\Gamma(\hat{x})(\cdot) \leq 0$ and $\hat{\mu} \geq 0$. Therefore, the complementarity condition (4.102) can be equivalently expressed as the equation

$$\int_a^b \Gamma(\hat{x})(\eta)\, d\mu(\eta) = 0. \tag{4.104}$$

Every nonnegative solution μ of (4.103)–(4.104) can be used as the Lagrange multiplier satisfying conditions (4.101)–(4.102) at $\hat{x}$. Define

$$a = \int_a^b d\hat{\mu}(\eta).$$

We can add to (4.103)–(4.104) the condition

$$\int_a^b d\mu(\eta) = a. \tag{4.105}$$

The system of three equations (4.103)–(4.105) still has at least one nonnegative solution, namely, $\hat{\mu}$. If $\hat{\mu} \equiv 0$, then the dominance constraint is not active. In this case, we can set $w(\eta) \equiv 0$, and the statement of the theorem follows from the fact that conditions (4.103)–(4.104) are equivalent to (4.101)–(4.102).

Now, consider the case of $\hat{\mu} \not\equiv 0$. In this case, we have $a > 0$. Normalizing by a, we notice that (4.103)–(4.105) are equivalent to the following inclusion:

$$\begin{bmatrix} g/a \\ 0 \end{bmatrix} \in \operatorname{conv} \left\{ \begin{bmatrix} \nabla_x \Gamma(\hat{x})(\eta) \\ \Gamma(\hat{x})(\eta) \end{bmatrix} : \eta \in [a, b] \right\} \subset \mathbb{R}^{n+1}.$$

By Carathéodory's theorem, there exist numbers $\eta_i \in [a, b]$, and $\alpha_i \geq 0$, $i = 1, \ldots, k$, such that

$$\begin{bmatrix} g/a \\ 0 \end{bmatrix} = \sum_{i=1}^{k} \alpha_i \begin{bmatrix} \nabla_x \Gamma(\hat{x})(\eta_i) \\ \Gamma(\hat{x})(\eta_i) \end{bmatrix},$$

$$\sum_{i=1}^{k} \alpha_i = 1,$$

and

$$1 \leq k \leq n + 2.$$

We define atomic measure ν having atoms of mass $c\alpha_i$ at points η_i, $i = 1, \ldots, k$. It satisfies (4.103)–(4.104):

$$\int_a^b \nabla_x \Gamma(\hat{x})(\eta)\, d\nu(\eta) = \sum_{i=1}^{k} \nabla_x \Gamma(\hat{x})(\eta_i) c\alpha_i = g,$$

$$\int_a^b \Gamma(\hat{x})(\eta)\, d\nu(\eta) = \sum_{i=1}^{k} \Gamma(\hat{x})(\eta_i) c\alpha_i = 0.$$

Recall that (4.103)–(4.104) are equivalent to (4.101)–(4.102). Now, applying Lemma 4.85, we obtain the utility functions

$$w(\eta) = \nu[a, \eta], \quad \eta \in \mathbb{R}.$$

It is straightforward to check that $w \in \mathcal{U}([a, b])$ and the assertion of the theorem holds true. □

It follows from Theorem 4.88 that if the dominance constraint is active, then there exist at least one and at most $n+2$ *target values* η_i and *target probabilities* $v_i = \Pr\{Y \geq \eta_i\}$, $i = 1, \ldots, k$, which are critical for problem (4.9). They define a relaxation of (4.9) involving finitely many probabilistic constraints:

$$\begin{aligned} \min_x \ & c(x) \\ \text{s.t. } & \Pr\{g(x, Z) \geq \eta_i\} \geq v_i, \quad i = 1, \ldots, k, \\ & x \in \mathcal{X}. \end{aligned}$$

The necessary conditions of optimality for this relaxation yield a solution of the optimality conditions of the original problem (4.9). Unfortunately, the target values and the target probabilities are not known in advance.

A particular situation, in which the target values and the target probabilities can be specified in advance, occurs when Y has a discrete distribution with finite support. Denote the realizations of Y by

$$\eta_1 < \eta_2 < \cdots < \eta_k$$

and the corresponding probabilities by p_i, $i = 1, \ldots, k$. Then the dominance constraint is equivalent to

$$\Pr\{g(x, Z) \geq \eta_i\} \geq \sum_{j=i}^{k} p_j, \quad i = 1, \ldots, k.$$

Here, we use the fact that the probability distribution function of $g(x, Z)$ is continuous and nondecreasing.

Now, we shall derive sufficient conditions of optimality for problem (4.9). We assume additionally that the function g is jointly quasi-concave in both arguments and Z has an α-concave probability distribution.

Theorem 4.89. *Assume that a point $\hat{x}$ is feasible for problem* (4.9). *Suppose that there exists a function $\hat{u} \in \mathcal{U}$, $\hat{u} \neq 0$, such that conditions* (4.95)–(4.96) *are satisfied. If the function c is convex, the function g satisfies the concavity assumptions above and the variable Z has an α-concave probability distribution, then $\hat{x}$ is an optimal solution of problem* (4.9).

Proof. By virtue of Theorem 4.43, the feasible set of problem (4.98)) is convex and closed.

Let the operator Γ and the cone K be defined as in the proof of Theorem 4.87. Using Lemma 4.85, we observe that optimality conditions (4.101)–(4.102) for problem (4.98) are satisfied. Consider a feasible direction d at the point $\hat{x}$. As the feasible set is convex, we conclude that

$$\Gamma(\hat{x} + \tau d) \in K$$

for all sufficiently small $\tau > 0$. Since Γ is differentiable, we have

$$\frac{1}{\tau}\big[\Gamma(\hat{x} + \tau d) - \Gamma(\hat{x})\big] \to \nabla_x \Gamma(\hat{x})(d) \quad \text{whenever} \quad \tau \downarrow 0.$$

This implies that

$$\nabla_x \Gamma(\hat{x})(d) \in \mathcal{T}_K(\Gamma(\hat{x})),$$

where $\mathcal{T}_K(\gamma)$ denotes the tangent cone to K at γ. Since

$$\mathcal{T}_K(\gamma) = K + \{t\gamma : t \in \mathbb{R}\},$$

there exists $t \in \mathbb{R}$ such that

$$\nabla_x \Gamma(\hat{x})(d) + t\Gamma(\hat{x}) \in K. \tag{4.106}$$

Condition (4.101) implies that there exists $q \in \mathcal{N}_{\mathcal{X}}(\hat{x})$ such that

$$\nabla_x c(\hat{x}) + \int_a^b \nabla_x \Gamma(\hat{x})(\eta)\, d\mu(\eta) = -q.$$

Applying both sides of this equation to the direction d and using the fact that $q \in \mathcal{N}_{\mathcal{X}}(\hat{x})$ and $d \in \mathcal{T}_{\mathcal{X}}(\hat{x})$, we obtain

$$\nabla_x c(\hat{x})(d) + \int_a^b \big(\nabla_x \Gamma(\hat{x})(\eta)\big)(d)\, d\mu(\eta) \geq 0. \tag{4.107}$$

Condition (4.102), relation (4.106), and the nonnegativity of μ imply that

$$\int_a^b \big(\nabla_x \Gamma(\hat{x})(\eta)\big)(d)\, d\mu(\eta) = \int_a^b \Big[\big(\nabla_x \Gamma(\hat{x})(\eta)\big)(d) + t\big(\Gamma(\hat{x})\big)(\eta)\Big]\, d\mu(\eta) \leq 0.$$

Substituting into (4.107) we conclude that

$$d^{\mathsf{T}} \nabla_x c(\hat{x}) \geq 0$$

for every feasible direction d at $\hat{x}$. By the convexity of c, for every feasible point x we obtain the inequality

$$c(x) \geq c(\hat{x}) + d^{\mathsf{T}} \nabla_x c(\hat{x}) \geq c(\hat{x}),$$

as stated. □

Exercises

4.1. Are the following density functions α-concave and do they define a γ-concave probability measure? What are α and γ?

(a) If the m-dimensional random vector Z has the normal distribution with expected value $\mu = 0$ and covariance matrix Σ, the random variable Y is independent of Z and has the χ^2_k distribution, then the distribution of the vector X with components

$$X_i = \frac{Z_i}{\sqrt{Y/k}}, \quad i = 1, \ldots, m,$$

is called a *multivariate Student distribution*. Its density function is defined as follows:

$$\theta_m(x) = \frac{\Gamma(\frac{m+k}{2})}{\Gamma(\frac{k}{2})\sqrt{(2\pi)^m \det(\Sigma)}} \Big(1 + \frac{1}{k} x^{\mathsf{T}} \Sigma^{\frac{1}{2}} x\Big)^{-(m+k)/2}.$$

If $m = k = 1$, then this function reduces to the well-known univariate Cauchy density

$$\theta_1(x) = \frac{1}{\pi}\ \frac{1}{1+x^2}\ ,\ -\infty < x < \infty.$$

(b) The density function of the *m-dimensional F-distribution* with parameters $n_0, \ldots, n_m$, and $n = \sum_{i=1}^{m} n_i$, is defined as follows:

$$\theta(x) = c \prod_{i=1}^{m} x_i^{n_i/2-1} \Big(n_0 + \sum_{i=1}^{m} n_i x_i\Big)^{-n/2}, \quad x_i \geq 0,\ i = 1, \ldots, m,$$

where c is an appropriate normalizing constant.

(c) Consider another multivariate generalization of the *beta distribution*, which is obtained in the following way. Let S_1 and S_2 be two independent sampling covariance matrices corresponding to two independent samples of sizes $s_1 + 1$ and $s_2 + 1$, respectively, taken from the same q-variate normal distribution with covariance matrix Σ. The joint distribution of the elements on and above the main diagonal of the random matrix

$$(S_1 + S_2)^{\frac{1}{2}} S_2 (S_1 + S_2)^{-\frac{1}{2}}$$

is continuous if $s_1 \geq q$ and $s_2 \geq q$. The probability density function of this distribution is defined by

$$\theta(X) = \begin{cases} \dfrac{c(s_1, q)c(s_2, q)}{c(s_1 + s_2, q)} \det(X)^{\frac{1}{2}(s_2-q-1)} \det(I - X)^{\frac{1}{2}(s_1-q-1)} & \\ \qquad \text{for } X,\ I - X \text{ positive definite,} & \\ 0 \qquad \text{otherwise.} & \end{cases}$$

Here I stands for the identity matrix, and the function $c(\cdot, \cdot)$ is defined as follows:

$$\frac{1}{c(k, q)} = 2^{qk/2} \pi^{q(q-1)/2} \prod_{i=1}^{q} \Gamma\left(\frac{k - i + 1}{2}\right).$$

The number of independent variables in X is $s = \frac{1}{2}q(q + 1)$.

(d) The probability density function of *the Pareto distribution* is

$$\theta(x) = a(a + 1) \ldots (a + s - 1) \left(\prod_{j=1}^{s} \Theta_j\right)^{-1} \left(\sum_{j=1}^{s} \Theta_j^{-1} x_j - s + 1\right)^{-(a+s)}$$

for $x_i > \Theta_i,\ i = 1, \ldots, s$, and $\theta(x) = 0$ otherwise. Here $\Theta_i,\ i = 1, \ldots, s$ are positive constants.

4.2. Assume that P is an α-concave probability distribution and $A \subset \mathbb{R}^n$ is a convex set. Prove that the function $f(x) = P(A + x)$ is α-concave.

4.3. Prove that if $\theta : \mathbb{R} \to \mathbb{R}$ is a log-concave probability density function, then the functions

$$F(x) = \int_{t \le x} \theta(t)dt \quad \text{and} \quad \bar{F}(x) = 1 - F(x)$$

are log-concave as well.

4.4. Check that the binomial, the Poisson, the geometric, and the hypergeometric one-dimensional probability distributions satisfy the conditions of Theorem 4.38 and are, therefore, log-concave.

4.5. Let Z_1, Z_2, and Z_3 be independent exponentially distributed random variables with parameters λ_1, λ_2, and λ_3, respectively. We define $Y_1 = \min\{Z_1, Z_3\}$ and $Y_2 = \min\{Z_2, Z_3\}$. Describe $G(\eta_1, \eta_2) = P(Y_1 \ge \eta_1, Y_2 \ge \eta_2)$ for nonnegative scalars η_1 and η_2 and prove that $G(\eta_1, \eta_2)$ is log-concave on $\mathbb{R}^2$.

4.6. Let Z be a standard normal random variable, W be a χ^2-random variable with one degree of freedom, and A be an $n \times n$ positive definite matrix. Is the set

$$\Big\{x \in \mathbb{R}^n : \Pr\big(Z - \sqrt{(x^{\mathsf{T}}Ax)W} \ge 0\big) \ge 0.9\Big\}$$

convex?

4.7. If Y is an m-dimensional random vector with a log-normal distribution, and $g : \mathbb{R}^n \to \mathbb{R}^m$ is such that each component g_i is a concave function, show that the set

$$C = \Big\{x \in \mathbb{R}^n : \Pr\big(g(x) \ge Y\big) \ge 0.9\Big\}$$

is convex.

(a) Find the set of p-efficient points for $m = 1$, $p = 0.9$ and write an equivalent algebraic description of C.

(b) Assume that $m = 2$ and the components of Y are independent. Find a disjunctive algebraic formulation for the set C.

4.8. Consider the following optimization problem:

$$\begin{aligned} &\operatorname*{Min}_x \ c^{\mathsf{T}}x \\ &\text{s.t. } \Pr\big\{g_i(x) \ge Y_i,\ i = 1, 2\big\} \ge 0.9, \\ &\qquad x \ge 0. \end{aligned}$$

Here $c \in \mathbb{R}^n$, $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2$, is a concave function, and Y_1 and Y_2 are independent random variables that have the log-normal distribution with parameters $\mu = 0, \sigma = 2$.

Formulate necessary and sufficient optimality conditions for this problem.

4.9. Assuming that Y and Z are independent exponentially distributed random variables, show that the following set is convex:

$$\Big\{x \in \mathbb{R}^3 : \Pr\big(x_1^2 + x_2^2 + Yx_2 + x_2x_3 + Yx_3 \le Z\big) \ge 0.9\Big\}.$$

4.10. Assume that the random variable Z is uniformly distributed in the interval $[-1, 1]$ and $e = (1, \ldots, 1)^{\mathsf{T}}$. Prove that the following set is convex:

$$\Big\{x \in \mathbb{R}^n : \Pr\big(\exp(x^{\mathsf{T}}y) \ge (e^{\mathsf{T}}y)Z, \quad \forall y \in \mathbb{R}^n : \|y\| \le 1\big) \ge 0.95\Big\}.$$

4.11. Let Z be a two-dimensional random vector with Dirichlet distribution. Show that the following set is convex:

$$\Big\{x \in \mathbb{R}^2 : \mathrm{Pr}\big(\min(x_1 + 2x_2 + Z_1, x_1 Z_2 - x_1^2 - Z_2^2) \geq y\big) \geq e^{-y} \quad \forall y \in [\tfrac{1}{4}, 4]\Big\}.$$

4.12. Let Z be an n-dimensional random vector uniformly distributed on a set A. Check whether the set

$$\Big\{x \in \mathbb{R}^n : \mathrm{Pr}\big(x^{\mathsf{T}} Z \leq 1\big) \geq 0.95\Big\}$$

is convex for the following cases:

(a) $A = \{z \in \mathbb{R}^n : \|z\| \leq 1\}$.

(b) $A = \{z \in \mathbb{R}^n : 0 \leq z_i \leq i,\ i = 1, \ldots, m\}$.

(c) $A = \{z \in \mathbb{R}^n : Tz \leq 0,\ -1 \leq z_i \leq 1,\ i = 1, \ldots, m\}$, where T is an $(n-1) \times n$ matrix of form

$$T = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix}.$$

4.13. Assume that the two-dimensional random vector Z has independent components, which have the Poisson distribution with parameters $\lambda_1 = \lambda_2 = 2$. Find all p-efficient points of F_Z for $p = 0.8$.