# 3

# Robust and Stochastic Model Predictive Control

## 3.1 Introduction

### 3.1.1 Types of Uncertainty

Robust and stochastic control concern control of systems that are uncertain in some sense so that predicted behavior based on a *nominal* model is not identical to actual behavior. Uncertainty may arise in different ways. The system may have an additive disturbance that is unknown, the state of the system may not be perfectly known, or the model of the system that is used to determine control may be inaccurate.

A system with additive disturbance satisfies the following difference equation

$$x^+ = f(x, u, w)$$

If the disturbance $w$ in constrained optimal control problems is bounded it is often possible to design a model predictive controller that ensures the state and control constraints are satisfied for all possible disturbance sequences (robust MPC). If the disturbance $w$ is unbounded, it is impossible to ensure that the usual state and control constraints are satisfied for *all* disturbance sequences. The model predictive controller is then designed to ensure that the constraints are satisfied on average or, more usually, with a prespecified probability (stochastic MPC).

The situation in which the state is not perfectly measured may be treated in several ways. For example, inherent robustness is often studied using the model $x^+ = f(x + e, u, w)$ where $e$ denotes the error in measuring the state. In the stochastic optimal control literature, where the measured output is $y = Cx + v$ and the disturbance $w$ and measurement noise $v$ are usually assumed to be Gaussian white noise processes, the state or *hyperstate* of the optimal control problem is the conditional density of the state $x$ at time $k$ given prior measurements $(y(0), y(1), \ldots, y(k-1))$. Because this density usually is difficult to compute and use, except in the linear case when it is provided by the Kalman filter, a suboptimal procedure often is adopted. In this suboptimal approach, the state $x$ is replaced by its estimate $\hat{x}$ in a control law determined under the assumption that the state is accessible. This procedure is usually referred to as *certainty equivalence*, a term that was originally employed for the linear quadratic Gaussian (LQG) or similar cases when this procedure did not result in loss of optimality. When $f(\cdot)$ is linear, the evolution of the state estimate $\hat{x}$ may be expressed by a difference equation

$$\hat{x}^+ = g(\hat{x}, u) + \xi$$

in which $\xi$ is the *innovation process*. In controlling $\hat{x}$, we should ensure that the actual state $x$, which lies in a bounded, possibly time-varying set if the innovation process is bounded, satisfies the constraints of the optimal control problem certainly (robust MPC). If the innovation process is not bounded, the constraints should be satisfied with a prespecified probability (stochastic MPC).

A system that has parametric uncertainty may be modeled as

$$x^+ = f(x, u, \theta)$$

in which $\theta$ represents parameters of the system that are known only to the extent that they belong to a compact set $\Theta$. A much-studied example is

$$x^+ = Ax + Bu$$

in which $\theta := (A, B)$ may take any value in $\Theta := \text{co}\{(A_i, B_i) \mid i \in \mathcal{I}\}$ where $\mathcal{I} = \{1, 2, \ldots, I\}$, say, is an index set.

Finally the system description may not include all the dynamics. For example, fast dynamics may be ignored to simplify the system description, or a system described by a partial differential equation may be modeled by an ordinary differential equation (ODE).

It is possible, of course, for all these types of uncertainty to occur in a single application. In this chapter we focus on the effect of additive disturbance. Output MPC—in which the controller employs an estimate of the state, rather than the state itself—is discussed in Chapter 5.

### 3.1.2 Feedback Versus Open-Loop Control

It is well known that feedback is required only when uncertainty is present; in the absence of uncertainty, feedback control and open-loop control are equivalent. Indeed, when uncertainty is not present, as for the systems studied in Chapter 2, the optimal control for a given initial state may be computed using either dynamic programming (DP) that provides an optimal control policy or sequence of feedback control laws, or an open-loop optimal control that merely provides a sequence of control actions. A simple example illustrates this fact. Consider the deterministic linear dynamic system defined by

$$x^+ = x + u$$

The optimal control problem, with horizon $N = 3$, is

$$\mathbb{P}_3(x): \qquad V_3^0(x) = \min_{\mathbf{u}_3} V_3(x, \mathbf{u})$$

in which $\mathbf{u} = (u(0), u(1), u(2))$

$$V_3(x, \mathbf{u}) := (1/2) \sum_{i=0}^{2} [(x(i)^2 + u(i)^2)] + (1/2) x(3)^2$$

in which, for each $i$, $x(i) = \phi(i; x, \mathbf{u}) = x + u(0) + u(1) + \ldots + u(i-1)$, the solution of the difference equation $x^+ = x + u$ at time $i$ if the initial state is $x(0) = x$ and the control (input) sequence is $\mathbf{u} = (u(0), u(1), u(2))$; in matrix operations $\mathbf{u}$ is taken to be the column vector $[u(0), u(1), u(2)]'$. Thus

$$\begin{aligned} V_3(x, \mathbf{u}) &= (1/2)[x^2 + (x + u(0))^2 + (x + u(0) + u(1))^2 + \\ &\qquad (x + u(0) + u(1) + u(2))^2 + u(0)^2 + u(1)^2 + u(2)^2] \\ &= (3/2)x^2 + x \begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \mathbf{u} + (1/2)\mathbf{u}' P_3 \mathbf{u} \end{aligned}$$

in which

$$P_3 = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Therefore, the vector form of the optimal *open-loop* control sequence for an initial state of $x$ is

$$\mathbf{u}^0(x) = -P_3^{-1} \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}' x = -\begin{bmatrix} 0.615 & 0.231 & 0.077 \end{bmatrix}' x$$

and the optimal control and state sequences are

$$\mathbf{u}^0(x) = (-0.615x, -0.231x, -0.077x)$$
$$\mathbf{x}^0(x) = (x, 0.385x, 0.154x, 0.077x)$$

To compute the optimal *feedback* control, we use the DP recursions

$$V_i^0(x) = \min_{u \in \mathbb{R}} \{x^2/2 + u^2/2 + V_{i-1}^0(x+u)\}$$
$$\kappa_i^0(x) = \arg\min_{u \in \mathbb{R}} \{x^2/2 + u^2/2 + V_{i-1}^0(x+u)\}$$

with boundary condition

$$V_0^0(x) = (1/2)x^2$$

This procedure gives the value function $V_i^0(\cdot)$ and the optimal control law $\kappa_i^0(\cdot)$ at each $i$ where the subscript $i$ denotes time to go. Solving the DP recursion, for all $x \in \mathbb{R}$, all $i \in \{1, 2, 3\}$, yields

$$\begin{aligned} V_1^0(x) &= (3/4)x^2 & \kappa_1^0(x) &= -(1/2)x \\ V_2^0(x) &= (4/5)x^2 & \kappa_2^0(x) &= -(3/5)x \\ V_3^0(x) &= (21/26)x^2 & \kappa_3^0(x) &= -(8/13)x \end{aligned}$$

Starting at state $x$ at time 0, and applying the optimal control laws iteratively to the *deterministic* system $x^+ = x + u$ (recalling that at time $i$ the optimal control law is $\kappa_{3-i}^0(\cdot)$ since, at time $i$, $3 - i$ is the time to go) yields

$$\begin{aligned} x^0(0) &= x & u^0(0) &= -(8/13)x \\ x^0(1) &= (5/13)x & u^0(1) &= -(3/13)x \\ x^0(2) &= (2/13)x & u^0(2) &= -(1/13)x \\ x^0(3;x) &= (1/13)x \end{aligned}$$

so that the optimal control and state sequences are, respectively,

$$\mathbf{u}^0(x) = (-(8/13)x, -(3/13)x, -(1/13)x)$$
$$\mathbf{x}^0(x) = (x, (5/13)x, (2/13)x, (1/13)x)$$

which are identical with the optimal open-loop values computed above.

Consider next an uncertain version of the dynamic system in which uncertainty takes the simple form of an additive disturbance $w$; the system is defined by

$$x^+ = x + u + w$$

in which the only knowledge of $w$ is that it lies in the compact set $\mathbb{W} := [-1, 1]$. Let $\phi(i; x, \mathbf{u}, \mathbf{w})$ denote the solution of this system at time $i$ if the initial state is $x$ at time 0, and the input and disturbance sequences are, respectively, $\mathbf{u}$ and $\mathbf{w} := (w(0), w(1), w(2))$. The cost now depends on the disturbance sequence—but it also depends, in contrast to the deterministic problem discussed above, on whether the control is open-loop or feedback. To discuss the latter case, we define a feedback policy $\boldsymbol{\mu}$ to be a sequence of control laws

$$\boldsymbol{\mu} := (\mu_0(\cdot), \mu_1(\cdot), \mu_2(\cdot))$$

in which $\mu_i : \mathbb{R} \to \mathbb{R}$, $i = 0, 1, 2$; under policy $\boldsymbol{\mu}$, if the state at time $i$ is $x$, the control is $\mu_i(x)$. Let $\mathcal{M}$ denote the class of *admissible* policies, for example, those policies for which each control law $\mu_i(\cdot)$ is continuous. Then, $\phi(i; x, \boldsymbol{\mu}, \mathbf{w})$ denotes the solution at time $i \in \{0, 1, 2, 3\}$ of the following difference equation

$$x(i+1) = x(i) + \mu_i(x(i)) + w(i) \qquad x(0) = x$$

An open-loop control sequence $\mathbf{u} = (u(0), u(1), u(2))$ is then merely a degenerate policy $\boldsymbol{\mu} = (\mu_0(\cdot), \mu_1(\cdot), \mu_2(\cdot))$ where each control law $\mu_i(\cdot)$ satisfies

$$\mu_i(x) = u(i)$$

for all $x \in \mathbb{R}$ and all $i \in \{0, 1, 2\}$. The cost $V_3(\cdot)$ may now be defined

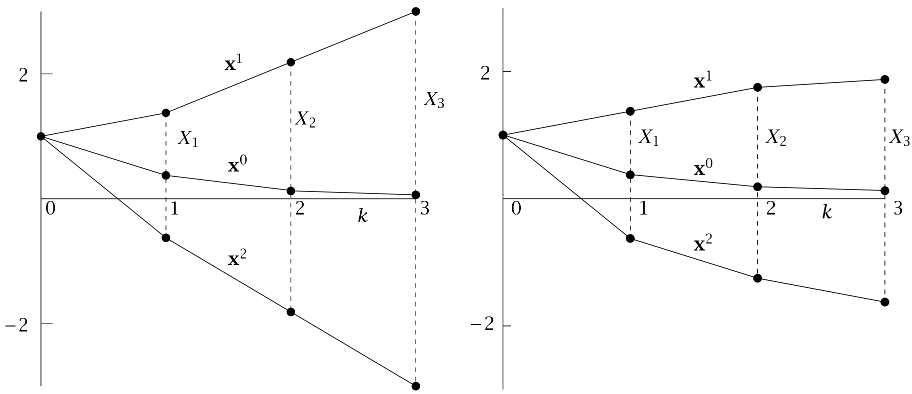
$$V_3(x, \boldsymbol{\mu}, \mathbf{w}) := (1/2) \sum_{i=0}^{2} [(x(i)^2 + u(i)^2)] + (1/2) x(3)^2$$

where, now, $x(i) = \phi(i; x, \boldsymbol{\mu}, \mathbf{w})$ and $u(i) = \mu_i(x(i))$. Since the disturbance is unpredictable, the value of $\mathbf{w}$ is not known at time 0, so the optimal control problem must "eliminate" it in some meaningful way so that the solution $\boldsymbol{\mu}^0(x)$ does not depend on $\mathbf{w}$. To eliminate $\mathbf{w}$, the optimal control problem $\mathbb{P}_3^*(x)$ is defined by

$$\mathbb{P}_3^*(x): \qquad V_3^0(x) := \inf_{\boldsymbol{\mu} \in \mathcal{M}} J_3(x, \boldsymbol{\mu})$$

in which the cost $J_3(\cdot)$ is defined in such a way that it does not depend on $\mathbf{w}$; inf is used rather than min in this definition since the minimum may not exist. The most popular choice for $J_3(\cdot)$ in the MPC literature is

$$J_3(x, \boldsymbol{\mu}) := \max_{\mathbf{w} \in \mathcal{W}} V_3(x, \boldsymbol{\mu}, \mathbf{w})$$




(a) Open-loop trajectories.

(b) Feedback trajectories.

**Figure 3.1:** Open-loop and feedback trajectories.

in which the disturbance $\mathbf{w}$ is assumed to lie in $\mathcal{W}$ a bounded class of admissible disturbance sequences. Alternatively, if the disturbance sequence is random, the cost $J_3(\cdot)$ may be chosen to be

$$J_3(x, \boldsymbol{\mu}) := \mathcal{E} V_3(x, \boldsymbol{\mu}, \mathbf{w})$$

in which $\mathcal{E}$ denotes "expectation" or average, over random disturbance sequences. For our purpose here, we adopt the simple cost

$$J_3(x, \boldsymbol{\mu}) := V_3(x, \boldsymbol{\mu}, \mathbf{0})$$

in which $\mathbf{0} := (0, 0, 0)$ is the zero disturbance sequence. In this case, $J_3(x, \boldsymbol{\mu})$ is the nominal cost, i.e., the cost associated with the nominal system $x^+ = x + u$ in which the disturbance is neglected. With this cost function, the solution to $\mathbb{P}_3^*(x)$ is the DP solution, obtained previously, to the deterministic *nominal* optimal control problem.

We now compare two solutions to $\mathbb{P}_3(x)$: the open-loop solution in which $\mathcal{M}$ is restricted to be the set of control sequences, and the feedback solution in which $\mathcal{M}$ is the class of admissible policies. The solution to the first problem is the solution to the deterministic problem discussed previously; the optimal control sequence is

$$\mathbf{u}^0(x) = (-(8/13)x, -(3/13)x, -(1/13)x)$$

in which $x$ is the initial state at time 0. The solution to the second problem is the sequence of control laws determined previously, also for the deterministic problem, using *dynamic programming*; the optimal policy is $\boldsymbol{\mu}^0 = \left(\mu_0^0(\cdot), \mu_1^0(\cdot), \mu_2(\cdot)\right)$ where the control laws (functions) $\mu_i(\cdot)$, $i = 0, 1, 2$, are defined by

$$\begin{aligned}
\mu_0^0(x) &:= \kappa_3^0(x) = -(8/13)x \quad \forall x \in \mathbb{R} \\
\mu_1^0(x) &:= \kappa_2^0(x) = -(3/5)x \quad \forall x \in \mathbb{R} \\
\mu_2^0(x) &:= \kappa_1^0(x) = -(1/2)x \quad \forall x \in \mathbb{R}
\end{aligned}$$

The two solutions, $\mathbf{u}^0(\cdot)$ and $\boldsymbol{\mu}^0$, when applied to the uncertain system $x^+ = x + u + w$, do *not* yield the same trajectories for all disturbance sequences. This is illustrated in Figure 3.1 for the three disturbance sequences, $\mathbf{w}^0 := (0, 0, 0)$, $\mathbf{w}^1 := (1, 1, 1)$, and $\mathbf{w}^2 := (-1, -1, -1)$; and initial state $x = 1$ for which the corresponding state trajectories, denoted $\mathbf{x}^0$, $\mathbf{x}^1$, and $\mathbf{x}^2$, are

**Open-loop solution.**

$$\begin{aligned}
\mathbf{x}^0 &= (1, (5/13), (2/13), (1/13)) \\
\mathbf{x}^1 &= (1, (18/13), (28/13), (40/13)) \\
\mathbf{x}^2 &= (1, -(8/13), -(24/13), -(38/13))
\end{aligned}$$

**Feedback solution.**

$$\begin{aligned}
\mathbf{x}^0 &= (1, (5/13), (2/13), (1/13)) \\
\mathbf{x}^1 &= (1, (18/13), (101/65), (231/130)) \\
\mathbf{x}^2 &= (1, -(8/13), -(81/65), -(211/130))
\end{aligned}$$

Even for the short horizon of 3, the superiority of the feedback solution can be seen although the feedback was designed for the deterministic (nominal) system and therefore did not take the disturbance into account. For the open-loop solution $\left|x^2(3) - x^1(3)\right| = 6$, whereas for the feedback case $\left|x^2(3) - x^1(3)\right| = 3.4$; the open-loop solution does not restrain the *spread* of the trajectories resulting from the disturbance $\mathbf{w}$. If the horizon length is $N$, for the open-loop solution, $\left|x^2(N) - x^1(N)\right| = 2N$, whereas for the feedback case $\left|x^2(N) - x^1(N)\right| \to 3.24$ as $N \to \infty$. The obvious and well-known conclusion is that feedback control is superior to open-loop control when uncertainty is present. Feedback control requires determination of a control *policy*, however, which is a difficult task if nonlinearity and/or constraints are features of the optimal control problem.

### 3.1.3 Robust and Stochastic MPC

An important feature of conventional, or deterministic, MPC discussed in Chapter 2 is that the solution of the open-loop optimal control problem solved online is identical to that obtained by DP for the given initial state. When uncertainty is present and the state is known or observations of the state are available, feedback control is superior to open-loop control. The optimal control problem solved online must, therefore, permit feedback in order for its solution to coincide with the DP solution. In robust and stochastic MPC, the decision variable is $\boldsymbol{\mu}$, a sequence of control *laws*, rather than $\mathbf{u}$, a sequence of control *actions*. MPC in which the decision variable is a policy has been termed *feedback* MPC to distinguish it from conventional MPC. Both forms of MPC naturally provide feedback control since the control that is implemented depends on the current state $x$ in both cases. But the control that is applied depends on whether the optimal control problem solved is open loop, in which case the decision variable is a control sequence, or feedback, in which case the decision variable is a feedback policy.

In feedback MPC the solution to the optimal control problem $\mathbb{P}_N^*(x)$ is the policy $\boldsymbol{\mu}^0(x) = \left(\mu_0^0(\cdot;x), \mu_1^0(\cdot;x), \ldots, \mu_{N-1}^0(\cdot;x)\right)$. The constituent control laws are restrictions of those determined by DP and therefore depend on the initial state $x$ as implied by the notation. Thus, only the value $u^0(x) = \mu_0(x;x)$ of the control law $\mu_0(\cdot;x)$ at the initial state $x$ need be determined, while successive laws need only be determined over a limited subset of the state space. In the example illustrated in Figure 3.1, $\mu_0(\cdot;x)$ need be determined only at the point $x = 1$, $\mu_1(\cdot;x)$ need only be determined in the subset $[-8/13, 18/13]$, and $\mu_2(\cdot;x)$ in the subset $[-81/65, 101/65]$, whereas in the DP solution these control laws are defined over the infinite interval $(-\infty, \infty)$.

While feedback MPC is superior in the presence of uncertainty, the associated optimal control problem is vastly more complex than the optimal control problem employed in deterministic MPC. The decision variable $\boldsymbol{\mu}$, being a sequence of control laws, is infinite dimensional; each law or function requires, in general, an infinite dimensional grid to specify it. The complexity is comparable to solving the DP equation, so that MPC, which in the deterministic case replaces DP with a solvable open-loop optimization problem, is not easily solved when uncertainty is present. Hence much research effort has been devoted to forms of feedback MPC that sacrifice optimality for simplicity. As in the early days of adaptive control, many different proposals have been made.

These proposals for robust MPC are all simpler to implement than the optimal solution provided by DP.

At the current stage of research it is perhaps premature to select a particular approach; we have, nevertheless, selected one approach, *tube-based* MPC that we describe here and in Chapter 5. There is a good reason for our choice. It is well known that standard mathematical optimization algorithms may be used to obtain an optimal open-loop control sequence for an optimal control problem. What is perhaps less well known is that there exist algorithms, the second variation algorithms, that provide not only an optimal control sequence but also a *local* time-varying feedback law of the form $u(k) = \bar{u}(k) + K(k)(x(k) - \bar{x}(k))$ in which $(\bar{u}(k))$ is the optimal open-loop control sequence and $(\bar{x}(k))$ the corresponding optimal open-loop state sequence. This policy provides feedback control for states $x(k)$ close to the nominal states $\bar{x}(k)$.

The second variation algorithms are perhaps too complex for routine use in MPC because they require computation of the second derivatives with respect to $(x, u)$ of $f(\cdot)$ and $\ell(\cdot)$. When the system is linear, the cost quadratic, and the disturbance additive, however, the optimal control law for the unconstrained infinite horizon case is $u = Kx$. This result may be expressed as a time-varying control law $u(k) = \bar{u}(k) + K(x(k) - \bar{x}(k))$ in which the state and control sequences $(\bar{x}(k))$ and $(\bar{u}(k))$ satisfy the nominal difference equations $\bar{x}^+ = A\bar{x} + B\bar{u}$, $\bar{u} = Kz$, i.e., the sequences $(\bar{x}(k))$ and $(\bar{u}(k))$ are optimal open-loop solutions for zero disturbance and some initial state. The time-varying control law $u(k) = \bar{u}(k) + K(x(k) - \bar{x}(k))$ is clearly optimal in the unconstrained case; it remains optimal for the constrained case in the neighborhood of the nominal trajectory $(\bar{x}(k))$ if $(\bar{x}(k))$ and $(\bar{u}(k))$ lie in the interior of their respective constraint sets.

These comments suggest that a time-varying policy of the form $u(x, k) = \bar{u}(k) + K(x - \bar{x}(k))$ might be adequate, at least when $f(\cdot)$ is linear. The nominal control and state sequences, $(\bar{u}(k))$ and $(z(k))$, respectively, can be determined by solving a standard open-loop optimal control problem of the form usually employed in MPC, and the feedback matrix $K$ can be determined offline. We show that this form of robust MPC has the same order of online complexity as that conventionally used for deterministic systems. It requires a modified form of the online optimal control problem in which the constraints are simply *tightened* to allow for disturbances, thereby constraining the trajectories of the uncertain system to lie in a tube centered on the nominal trajectories. Offline computations are required to determine the modified constraints and the feedback matrix $K$. We also present, in the last section of this chapter, a modification of this tube-based procedure for nonlinear systems for which a *nonlinear* local feedback policy is required.

A word of caution is necessary. Just as nominal model predictive controllers presented in Chapter 2 may fail in the presence of uncertainty, the controllers presented in this chapter may fail if the actual uncertainty does not satisfy our assumptions. In robust MPC this may occur when the disturbance that we assume to be bounded exceeds the assumed bounds; the controlled systems are robust only to the specified uncertainties. As always, online fault diagnosis and safe recovery procedures may be required to protect the system from unanticipated events.

### 3.1.4 Tubes

The approach that we adopt is motivated by the following observation. Both open-loop and feedback control generate, in the presence of uncertainty, a *bundle* or *tube* of trajectories, each trajectory in the bundle or tube corresponding to a particular realization of the uncertainty. In Figure 3.1(a), the tube corresponding to $\mathbf{u} = \mathbf{u}^0(x)$ and initial state $x = 1$, is $(X_0, X_1, X_2, X_3)$ where $X_0 = \{1\}$; for each $i$, $X_i = \{\phi(i; x, \mathbf{u}, \mathbf{w}) \mid \mathbf{w} \in \mathcal{W}\}$, the set of states at time $i$ generated by all possible realizations of the disturbance sequence. In robust MPC the state constraints must be satisfied by every trajectory in the tube. In stochastic MPC the tube has the property that state sequences lie within this tube with a prespecified probability.

Control of uncertain systems is best viewed as control of tubes rather than trajectories; the designer chooses, for each initial state, a tube in which all realizations of the state trajectory are controlled to lie (robust MPC), or in which the realizations lie with a given probability (stochastic MPC). By suitable choice of the tube, satisfaction of state and control constraints may be guaranteed for every realization of the disturbance sequence, or guaranteed with a given probability.

Determination of a suitable tube $(X_0, X_1, \ldots)$ corresponding to a given initial state $x$ and policy $\boldsymbol{\mu}$ is difficult even for linear systems, however, and even more difficult for nonlinear systems. Hence, in the sequel, we show for robust MPC how simple tubes that bound all realizations of the state trajectory may be constructed. For example, for linear systems with convex constraints, a tube $(X_0, X_1, \ldots, )$ may be designed to bound all realizations of the state trajectory; for each $i$,

$X_i = \{\bar{x}(i)\} \oplus S$, $\bar{x}(i)$ is the state at time $i$ of a deterministic system, $X_i$ is a polytope, and $S$ is a positive invariant set. This construction permits robust model predictive controllers to be designed with not much more computation online than that required for deterministic systems. The stochastic MPC controllers are designed to satisfy constraints with a given probability.

### 3.1.5 Difference Inclusion Description of Uncertain Systems

Here we introduce some notation that will be useful in the sequel. A deterministic discrete time system is usually described by a difference equation

$$x^+ = f(x, u) \tag{3.1}$$

We use $\phi(k; x, i, \mathbf{u})$ to denote the solution of (3.1) at time $k$ when the initial state at time $i$ is $x$ and the control sequence is $\mathbf{u} = (u(0), u(1), \ldots)$; if the initial time $i = 0$, we write $\phi(k; x, \mathbf{u})$ in place of $\phi(k; (x, 0), \mathbf{u})$. Similarly, an uncertain system may be described by the difference equation

$$x^+ = f(x, u, w) \tag{3.2}$$

in which the variable $w$ that represents the uncertainty takes values in a specified set $\mathbb{W}$. We use $\phi(k; x, i, \mathbf{u}, \mathbf{w})$ to denote the solution of (3.2) when the initial state at time $i$ is $x$ and the control and disturbance sequences are, respectively, $\mathbf{u} = (u(0), u(1), \ldots)$ and $\mathbf{w} = (w(0), w(1), \ldots)$. The uncertain system may alternatively be described by a *difference inclusion* of the form

$$x^+ \in F(x, u)$$

in which $F(\cdot)$ is a set-valued map. We use the notation $F : \mathbb{R}^n \times \mathbb{R}^m \rightsquigarrow \mathbb{R}^n$ or[1] $F : \mathbb{R}^n \times \mathbb{R}^m \to 2^{\mathbb{R}^n}$ to denote a function that maps points in $\mathbb{R}^n \times \mathbb{R}^m$ into subsets of $\mathbb{R}^n$. If the uncertain system is described by (3.2), then

$$F(x, u) = f(x, u, \mathbb{W}) := \{f(x, u, w) \mid w \in \mathbb{W}\}$$

If $x$ is the current state, and $u$ the current control, the successor state $x^+$ lies anywhere in the set $F(x, u)$. When the control policy $\boldsymbol{\mu} := (\mu_0(\cdot), \mu_1(\cdot), \ldots)$ is employed, the state evolves according to

$$x^+ \in F(x, \mu_k(x)), \quad k^+ = k + 1 \tag{3.3}$$

[1]For any set $X$, $2^X$ denotes the set of all subsets of $X$.

in which $x$ is the current state, $k$ the current time, and $x^+$ the successor state at time $k^+ = k + 1$. The system described by (3.3) does not have a single solution for a given initial state; it has a solution for each possible realization $\mathbf{w}$ of the disturbance sequence. We use $S(x, i)$ to denote the set of solutions of (3.3) if the initial state is $x$ at time $i$. If $\phi^*(\cdot) \in S(x, i)$ then

$$\phi^*(t) = \phi(t; x, i, \boldsymbol{\mu}, \mathbf{w})$$

for some admissible disturbance sequence $\mathbf{w}$ in which $\phi(t; x, i, \boldsymbol{\mu}, \mathbf{w})$ denotes the solution at time $t$ of

$$x^+ = f(x, \mu_k(x), w)$$

when the initial state is $x$ at time $i$ and the disturbance sequence is $\mathbf{w}$. The policy $\boldsymbol{\mu}$ is defined, as before, to be the sequence of control laws $(\mu_0(\cdot), \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot))$. The tube $\mathbf{X} = (X_0, X_1, \ldots)$, discussed in Section 3.5, generated when policy $\boldsymbol{\mu}$ is employed, satisfies

$$X_{k+1} = \mathbf{F}(X_k, \mu_k(\cdot)) := f(X_k, \mu_k(x), \mathbb{W})$$

## 3.2 Nominal (*Inherent*) Robustness

### 3.2.1 Introduction

Because feedback MPC is complex, it is natural to inquire if nominal MPC, i.e., MPC based on the nominal system ignoring uncertainty, is sufficiently robust to uncertainty. Before proceeding with a detailed analysis, a few comments may be helpful.

MPC uses, as a Lyapunov function, the value function of a parametric optimal control problem. Often the value function is continuous, but this is not necessarily the case, especially if state and/or terminal constraints are present. It is also possible for the value function to be continuous but the associated control law to be discontinuous; this can happen, for example, if the minimizing control is not unique.

It is important to realize that a control law may be stabilizing but not robustly stabilizing; arbitrary perturbations, no matter how small, can destabilize the system. This point is illustrated in Teel (2004) with the following discontinuous autonomous system ($n = 2$, $x = (x_1, x_2)$)

$$x^+ = f(x) \qquad f(x) = \begin{cases} (0, |x|) & x_1 \neq 0 \\ (0, 0) & \text{otherwise} \end{cases}$$

If the initial state is $x = (1, 1)$, then $\phi(1; x) = (0, \sqrt{2})$ and $\phi(2; x) = (0, 0)$, with similar behavior for other initial states. In fact, all solutions satisfy

$$\phi(k; x) \le \beta(|x|, k)$$

in which $\beta(\cdot)$, defined by

$$\beta(|x|, k) := 2(1/2)^k |x|$$

is a $\mathcal{KL}$ function, so that the origin is *globally asymptotically stable* (GAS). Consider now a perturbed system satisfying

$$x^+ = \begin{bmatrix} \delta \\ |x| + \delta \end{bmatrix}$$

in which $\delta > 0$ is a constant perturbation that causes $x_1$ to remain strictly positive. If the initial state is $x = \varepsilon(1, 1)$, then $x_1(k) = \delta$ for $k \ge 1$, and $x_2(k) > \varepsilon\sqrt{2} + k\delta \to \infty$ as $k \to \infty$, no matter how small $\delta$ and $\varepsilon$ are. Hence the origin is unstable in the presence of an arbitrarily small perturbation; global asymptotic stability is not a robust property of this system.

This example may appear contrived but, as Teel (2004) points out, a similar phenomenon can arise in receding horizon optimal control of a *continuous system*. Consider the following system

$$x^+ = \begin{bmatrix} x_1(1 - u) \\ |x|\, u \end{bmatrix}$$

in which the control $u$ is constrained to lie in the set $\mathbb{U} = [-1, 1]$. Suppose we choose a horizon length $N = 2$ and choose $\mathbb{X}_f$ to be the origin. If $x_1 \neq 0$, the only feasible control sequence steering $x$ to 0 in two steps is $\mathbf{u} = \{1, 0\}$; the resulting state sequence is $(x, (0, |x|), (0, 0))$. Since there is only one feasible control sequence, it is also optimal, and $\kappa_2(x) = 1$ for all $x$ such that $x_1 \neq 0$. If $x_1 = 0$, then the only optimal control sequence is $\mathbf{u} = (0, 0)$ and $\kappa_2(x) = 0$. The resultant closed-loop system satisfies

$$x^+ = f(x) := \begin{bmatrix} x_1(1 - \kappa_2(x)) \\ |x|\, \kappa_2(x) \end{bmatrix}$$

in which $\kappa_2(x) = 1$ if $x_1 \neq 0$, and $\kappa_2(x) = 0$ otherwise. Thus

$$f(x) = \begin{cases} (0, |x|) & x_1 \neq 0 \\ (0, 0) & \text{otherwise} \end{cases} \tag{3.4}$$

The system $x^+ = f(x)$ is the discontinuous system analyzed previously. Thus, receding horizon optimal control of a continuous system has resulted in a discontinuous system that is globally asymptotically stable (GAS) but has no robustness.

### 3.2.2 Difference Inclusion Description of Discontinuous Systems

Consider a system

$$x^+ = f(x)$$

in which $f(\cdot)$ is not continuous. An example of such a system occurred in the previous subsection where $f(\cdot)$ satisfies (3.4). Solutions of this system are very sensitive to the value of $x_1$. An infinitesimal change in $x_1$ at time 0, say, from 0 can cause a substantial change in the subsequent trajectory resulting, in this example, in a loss of robustness. To design a robust system, one must take into account, in the design process, the system's extreme sensitivity to variations in state. This can be done by *regularizing* the system (Teel, 2004). If $f(\cdot)$ is locally bounded,[2] the *regularization* $x^+ = f(x)$ is defined to be

$$x^+ \in F(x) := \bigcap_{\delta>0} \overline{f(\{x\} \oplus \delta\mathcal{B})}$$

in which $\mathcal{B}$ is the closed unit ball so that $\{x\} \oplus \delta\bar{\mathcal{B}} = \{z \mid |z - x| \leq \delta\}$ and $\overline{A}$ denotes the closure of set $A$. At points where $f(\cdot)$ is continuous, $F(x) = \{f(x)\}$, i.e., $F(x)$ is the single point $f(x)$. If $f(\cdot)$ is piecewise continuous, e.g., if $f(x) = x$ if $x < 1$ and $f(x) = 2x$ if $x \geq 1$, then $F(x) = \{\lim_{x_i \to x} f(x_i)\}$, the set of all limits of $f(x_i)$ as $x_i \to x$. For our example immediately above, $F(x) = \{x\}$ if $x < 1$ and $F(x) = \{2x\}$ if $x > 1$. When $x = 1$, the limit of $f(x_i)$ as $x_i \to 1$ from below is 1 and the limit of $f(x_i)$ as $x \to 1$ from above is 2, so that $F(1) = \{1, 2\}$. The regularization of $x^+ = f(x)$ where $f(\cdot)$ is defined in (3.4) is $x^+ \in F(x)$ where $F(\cdot)$ is defined by

$$F(x) = \left\{ \begin{bmatrix} 0 \\ |x| \end{bmatrix} \right\} \qquad x_1 \neq 0 \tag{3.5}$$

$$F(x) = \left\{ \begin{bmatrix} 0 \\ |x| \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \quad x_1 = 0 \tag{3.6}$$

---

[2]A function $f : \mathbb{R}^p \to \mathbb{R}^n$ is locally bounded if, for every $x \in \mathbb{R}^p$, there exists a neighborhood $\mathcal{N}$ of $x$ and a $c > 0$ such that $|f(z)| \leq c$ for all $z \in \mathcal{N}$.

If the initial state is $x = (1, 1)$, as before, then the difference inclusion generates the following tube

$$X_0 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}, \quad X_1 = \left\{ \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix} \right\}, \quad X_2 = \left\{ \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}, \quad \ldots$$

with $X_k = X_2$ for all $k \geq 2$. The set $X_k$ of possible states clearly does not converge to the origin even though the trajectory generated by the original system does.

### 3.2.3 When Is Nominal MPC Robust?

The discussion in Section 2.4.1 shows that nominal MPC is not necessarily robust. It is therefore natural to ask under what conditions nominal MPC is robust. To answer this, we have to define robustness precisely. In Appendix B, we define robust stability, and robust asymptotic stability, of a set. We employ this concept later in this chapter in the design of robust model predictive controllers that for a given initial state in the region of attraction, steer *every* realization of the state trajectory to this set. Here, however, we address a slightly different question: when is nominal MPC that steers every trajectory in the region of attraction to the origin robust? Obviously, the disturbance will preclude the controller from steering the state of the perturbed system to the origin; the best that can be hoped for is that the controller will steer the state to some small neighborhood of the origin. Let the nominal (controlled) system be described by $x^+ = f(x)$ in which $f(\cdot)$ is not necessarily continuous, and let the perturbed system be described by $x^+ = f(x + e) + w$. Also let $S_\delta(x)$ denote the set of solutions for the perturbed system with initial state $x$ and perturbation sequences $\mathbf{e} := (e(0), e(1), e(2), \ldots)$ and $\mathbf{w} := (w(0), w(1), w(2), \ldots)$ satisfying $\max\{\|\mathbf{e}\|, \|\mathbf{w}\|\} \leq \delta$ where, for any sequence $\boldsymbol{\nu}$, $\|\boldsymbol{\nu}\|$ denotes the sup norm, $\sup_{k \geq 0} |\nu(k)|$. The definition of robustness that we employ is (Teel, 2004)

**Definition 3.1** (Robust global asymptotic stability)**.** Let $\mathcal{A}$ be compact, and let $d(x, \mathcal{A}) := \min_a\{|a - x| \mid a \in \mathcal{A}\}$, and $|x|_{\mathcal{A}} := d(x, \mathcal{A})$. The set $\mathcal{A}$ is robustly globally asymptotically stable (RGAS) for $x^+ = f(x)$ if there exists a class $\mathcal{KL}$ function $\beta(\cdot)$ such that for each $\varepsilon > 0$ and each compact set $C$, there exists a $\delta > 0$ such that for each $x \in C$ and each $\phi \in S_\delta(x)$, there holds $|\phi(k; x)|_{\mathcal{A}} \leq \beta(|x|_{\mathcal{A}}, k) + \varepsilon$ for all $k \in \mathbb{I}_{\geq 0}$.

Taking the set $\mathcal{A}$ to be the origin ($\mathcal{A} = \{0\}$) so that $|x|_{\mathcal{A}} = |x|$, we see that if the origin is robustly asymptotically stable for $x^+ = f(x)$, then, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that every trajectory of the perturbed system $x^+ = f(x+e)+w$ with $\max\{\|\mathbf{e}\|, \|\mathbf{w}\|\} \leq \delta$ converges to $\varepsilon\mathcal{B}$ ($\mathcal{B}$ is the closed unit ball); this is the attractivity property. Also, if the initial state $x$ satisfies $|x| \leq \beta^{-1}(\varepsilon, 0)$, then $|\phi(k;x)| \leq \beta(\beta^{-1}(\varepsilon, 0), 0) + \varepsilon = 2\varepsilon$ for all $k \in \mathbb{I}_{\geq 0}$ and for all $\phi \in S_\delta$, which is the Lyapunov stability property. Here the function $\beta^{-1}(\cdot, 0)$ is the inverse of the function $\alpha \mapsto \beta(\alpha, 0)$.

We return to the question: under what conditions is asymptotic stability robust? We first define a slight extension to the definition of a Lyapunov function given in Chapter 2: A function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is defined to be a Lyapunov function for $x^+ = f(x)$ in $\mathbb{X}$ and set $\mathcal{A}$ if there exist functions $\alpha_i \in \mathcal{K}_\infty$, $i = 1, 2$ and a continuous, positive definite function $\alpha_3(\cdot)$ such that, for any $x \in \mathbb{X}$

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}})$$
$$V(f(x)) \leq V(x) - \alpha_3(|x|_{\mathcal{A}})$$

in which $|x|_{\mathcal{A}}$ is defined to be distance $d(x, \mathcal{A})$ of $x$ from the set $\mathcal{A}$. The following important result (Teel, 2004; Kellet and Teel, 2004) answers the important question, "When is asymptotic stability robust?"

**Theorem 3.2** (Lyapunov function and RGAS)**.** *Suppose $\mathcal{A}$ is compact and that $f(\cdot)$ is locally bounded.*[3] *The set $\mathcal{A}$ is RGAS for the system $x^+ = f(x)$ if and only if the system admits a* continuous *global Lyapunov function for $\mathcal{A}$.*

This result proves the existence of a $\delta > 0$ that specifies the permitted magnitude of the perturbations, but does not give a value for $\delta$. Robustness against perturbations of a specified magnitude may be required in practice; in the following section we show how to achieve this aim if it is possible.

In MPC, the value function of the finite horizon optimal control problem that is solved online is used as a Lyapunov function. In certain cases, such as linear systems with polyhedral constraints, the value function is known to be continuous; see Proposition 7.13. Theorem 3.2, suitably modified because the region of attraction is not global, then shows that asymptotic stability is robust, i.e., that asymptotic stability is not destroyed by *small* perturbations.

Theorem 3.2 characterizes robust stability of the set $\mathcal{A}$ for the system $x^+ = f(x)$ in the sense that it shows robust stability is equivalent

[3] A function $f : \mathcal{X} \to \mathcal{Y}$ is locally bounded if, for every $x \in \mathcal{X}$, there exists a neighborhood $\mathcal{N}$ of $x$ such that the set $f(\mathcal{N})$ in $\mathcal{Y}$ is bounded.

to the existence of a continuous global Lyapunov function for the system. It also is possible to characterize robustness of $x^+ = f(x)$ by global asymptotic stability of its regularization $x^+ \in F(x)$. It is shown in Appendix B that for the system $x^+ \in F(x)$, the set $\mathcal{A}$ is GAS if there exists a $\mathcal{KL}$ function $\beta(\cdot)$ such that for each $x \in \mathbb{R}^n$ and each solution $\phi(\cdot) \in S(x)$ of $x^+ \in F(x)$ with initial state $x$, $\phi(k) \leq \beta(|x|_{\mathcal{A}}, k)$ for all $k \in \mathbb{I}_{\geq 0}$. The following alternative characterization of robust stability of $\mathcal{A}$ for the system $x^+ = f(x)$ appears in (Teel, 2004).

**Theorem 3.3** (Robust global asymptotic stability and regularization)**.** *Suppose $\mathcal{A}$ is compact and that $f(\cdot)$ is locally bounded. The set $\mathcal{A}$ is RGAS for the system $x^+ = f(x)$ if and only if the set $\mathcal{A}$ is GAS for $x^+ \in F(x)$, the regularization of $x^+ = f(x)$.*

We saw previously that for $f(\cdot)$ and $F(\cdot)$ defined respectively in (3.4) and (3.6), the origin is not globally asymptotically stable for the regularization $x^+ \in F(x)$ of $x^+ = f(x)$ since not every solution of $x^+ \in F(x)$ converges to the origin. Hence the origin is not RGAS for this system.

### 3.2.4 Robustness of Nominal MPC

If the origin is asymptotically stable for the nominal version of an uncertain system, it is sometimes possible to establish that there exists a set $\mathcal{A}$ that is asymptotically stable for the uncertain system. We consider the uncertain system described by

$$x^+ = f(x, u, w) \tag{3.7}$$

in which $w$ is a bounded additive disturbance and $f(\cdot)$ is continuous. The system is subject to the state and control constraints

$$x(i) \in \mathbb{X} \qquad u(i) \in \mathbb{U} \quad \forall i \in \mathbb{I}_{\geq 0}$$

The set $\mathbb{X}$ is closed and the set $\mathbb{U}$ is compact. Each set contains the origin in its interior. The disturbance $w$ may take any value in the set $\mathbb{W}$. As before, $\mathbf{u}$ denotes the control sequence $(u(0), u(1), \ldots)$ and $\mathbf{w}$ the disturbance sequence $(w(0), w(1), \ldots)$; $\phi(i; x, \mathbf{u}, \mathbf{w})$ denotes the solution of (3.7) at time $i$ if the initial state is $x$, and the control and disturbance sequences are, respectively, $\mathbf{u}$ and $\mathbf{w}$. The *nominal* system is described by

$$x^+ = \bar{f}(x, u) := f(x, u, 0) \tag{3.8}$$

and $\bar{\phi}(i; x, \mathbf{u})$ denotes the solution of the nominal system (3.8) at time $i$ if the initial state is $x$ and the control sequence is $\mathbf{u}$. The *nominal* control problem, defined subsequently, includes, for reasons discussed in Chapter 2, a terminal constraint

$$x(N) \in \mathbb{X}_f$$

The *nominal* optimal control problem is

$$\mathbb{P}_N(x): \qquad V_N^0(x) = \min_{\mathbf{u}} \{V_N(x, \mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_N(x)\}$$
$$\mathbf{u}^0(x) = \arg\min_{\mathbf{u}} \{V_N(x, \mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_N(x)\}$$

in which $\mathbf{u}^0 = \left(u_0^0(x), u_1^0(x), \ldots, u_{N-1}^0(x)\right)$ and the nominal cost $V_N(\cdot)$ is defined by

$$V_N(x, \mathbf{u}) := \sum_{i=0}^{N-1} \ell(x(i), u(i)) + V_f(x(N)) \tag{3.9}$$

In (3.9) and (3.10), $x(i) := \bar{\phi}(i; x, \mathbf{u})$, the state of the nominal system at time $i$, for all $i \in \mathbb{I}_{0:N-1} = \{0, 1, 2, \ldots, N-1\}$. The set of *admissible* control sequences $\mathcal{U}_N(x)$ is defined by

$$\mathcal{U}_N(x) := \{\mathbf{u} \mid u(i) \in \mathbb{U}, \bar{\phi}(i : x, \mathbf{u}) \in \mathbb{X}\ \forall i \in \mathbb{I}_{0:N-1},\ x(N) \in \mathbb{X}_f \subset \mathbb{X}\} \tag{3.10}$$

which is the set of control sequences such that the nominal system satisfies the nominal control, state, and terminal constraints when the initial state at time 0 is $x$. Thus, $\mathcal{U}_N(x)$ is the set of feasible controls for the nominal optimal control problem $\mathbb{P}_N(x)$. The set $\mathcal{X}_N \subset \mathbb{R}^n$, defined by

$$\mathcal{X}_N := \{x \in \mathbb{R}^n \mid \mathcal{U}_N(x) \neq \emptyset\}$$

is the domain of the value function $V_N^0(\cdot)$, i.e., the set of $x \in \mathbb{X}$ for which $\mathbb{P}_N(x)$ has a solution; $\mathcal{X}_N$ is also the domain of the minimizer $\mathbf{u}^0(x)$. The value of the nominal control at state $x$ is $u^0(0; x)$, the first control in the sequence $\mathbf{u}^0(x)$. Hence the *implicit* nominal MPC control law is $\kappa_N : \mathcal{X}_N \to \mathbb{U}$ defined by

$$\kappa_N(x) = u^0(0; x)$$

We assume, as before, that $\ell(\cdot)$ and $V_f(\cdot)$ are defined by

$$\ell(x, u) := (1/2)(x'Qx + u'Ru) \qquad V_f(x) := (1/2)x'P_f x$$

in which $Q$, $R$, and $P_f$ are all positive definite. We also assume that $V_f(\cdot)$ and $\mathbb{X}_f := \{x \mid V_f(x) \leq c_f\}$ for some $c_f > 0$ satisfy the standard stability assumption that, for all $x \in \mathbb{X}_f$, there exists a $u = \kappa_f(x) \in \mathbb{U}$ such that $V_f(\bar{f}(x,u)) \leq V_f(x) - \ell(x,u)$ and $\bar{f}(x,u) \in \mathbb{X}_f$. Because $V_f(\cdot)$ is quadratic, there exist positive constants $c_1^f$ and $c_2^f$ such that $c_1^f \, |x|^2 \leq V_f(x) \leq c_2^f \, |x|^2$ and $V_f(f(x,\kappa_f(x))) \leq V_f(x) - c_1^f \, |x|^2$.

Under these assumptions, as shown in Chapter 2, there exist positive constants $c_1$ and $c_2$, $c_2 > c_1$, satisfying

$$c_1 \, |x|^2 \leq V_N^0(x) \leq c_2 \, |x|^2 \tag{3.11}$$

$$V_N^0(\bar{f}(x,\kappa_N(x))) \leq V_N^0(x) - c_1 \, |x|^2 \tag{3.12}$$

for all $x \in \mathcal{X}_N$. It then follows that

$$V_N^0(x^+) \leq \gamma V_N^0(x)$$

for all $x \in \bar{\mathcal{X}}_N$ with $x^+ := \bar{f}(x,\kappa_N(x))$ and $\gamma = (1 - c_1/c_2) \in (0, 1)$. Hence, $\bar{V}_N^0(x(i))$ decays exponentially to zero as $i \to \infty$; moreover, $V_N^0(x(i)) \leq \gamma^i V_N^0(x(0))$ for all $i \in \mathbb{I}_{\geq 0}$. From (3.11), the origin is exponentially stable, with a region of attraction $\bar{\mathcal{X}}_N$ for the nominal system under MPC.

We now examine the consequences of applying the nominal model predictive controller $\kappa_N(\cdot)$ to the *uncertain* system (3.7). The controlled uncertain system satisfies the difference equation

$$x^+ = f(x, \kappa_N(x), w) \tag{3.13}$$

in which $w$ can take any value in $\mathbb{W}$. It is obvious that the state $x(i)$ of the controlled system (3.13) cannot tend to the origin as $i \to \infty$; the best that can be hoped for is that $x(i)$ tends to and remains in some neighborhood $R_b$ of the origin. We shall establish this, if the disturbance $w$ is sufficiently small, using the value function $V_N^0(\cdot)$ of the nominal optimal control problem as a Lyapunov function for the controlled uncertain system (3.13).

To analyze the effect of the disturbance $w$ we employ the following useful technical result (Allan, Bates, Risbeck, and Rawlings, 2017, Proposition 20).

**Proposition 3.4** (Bound for continuous functions). *Let $C \subseteq D \subseteq \mathbb{R}^n$ with $C$ compact and $D$ closed. If $f(\cdot)$ is continuous, there exists an $\alpha(\cdot) \in \mathcal{K}_\infty$ such that, for all $x \in C$ and $y \in D$, we have that $|f(x) - f(y)| \leq \alpha(|x - y|)$.*

Since $\mathcal{X}_N$ is not necessarily robustly positive invariant (see Definition 3.6) for the uncertain system $x^+ = f(x, \kappa_N(x), w)$, we replace it by a subset, $R_c := \text{lev}_c V_N^0 = \{x \mid V_N^0(x) \le c\}$, the largest sublevel set of $V_N^0(\cdot)$ contained in $\mathcal{X}_N$. Let $R_b$ denote $\text{lev}_b V_N^0 = \{x \mid V_N^0(x) \le b\}$, the smallest sublevel set containing $\mathbb{X}_f$. Because $V_N^0(\cdot)$ is lower semicontinuous (see Appendix A.11) and $V_N^0(x) \ge c_1 |x|^2$, both $R_b$ and $R_c$ are compact. We show below, if $\mathbb{W}$ is sufficiently small, then $R_b$ and $R_c$ are robustly positive invariant for the uncertain system $x^+ = f(x, \kappa_N(x), w)$, $w \in \mathbb{W}$ and every trajectory of $x^+ = f(x, \kappa_N(x), w)$, commencing at a state $x \in R_c$, converges to $R_b$ and thereafter remains in this set.

**Satisfaction of the terminal constraint.** Our first task is to show that the terminal constraint $x(N) \in \mathbb{X}_f$ is satisfied by the uncertain system if $\mathbb{W}$ is sufficiently small. Let $\mathbf{u}^*(x) := \left(u_1^0(x), u_2^0(x), \ldots, u_{N-1}^0(x)\right)$ and let $\widetilde{\mathbf{u}}(x) := \left(\mathbf{u}^*(x), \kappa_f(x^0(N;x))\right)$. Since $V_f^*(\cdot)$ defined by

$$V_f^*(x, \mathbf{u}) := V_f(\bar{\phi}(N; x, \mathbf{u}))$$

is continuous, it follows from Proposition 3.4 that there exists a $\mathcal{K}_\infty$ function $\alpha_a(\cdot)$ such that

$$\left| V_f^*(x^+, \mathbf{u}) - V_f^*(\bar{x}^+, \mathbf{u}) \right| \le \alpha_a(|x^+ - \bar{x}^+|)$$

for all $(\bar{x}^+, \mathbf{u}) \in R_c \times \mathbb{U}^N$ and all $(x^+, \mathbf{u}) \in f(R_c, \mathbb{U}, \mathbb{W}) \times \mathbb{U}^N$. This result holds, in particular, for $\bar{x}^+ := f(x, \kappa_N(x), 0)$, $x^+ := f(x, \kappa_N(x), w)$ and $\mathbf{u} = \widetilde{\mathbf{u}}(x)$ with $x \in R_c$. As shown in Chapter 2, $x^0(N; \bar{x}^+) \in \mathbb{X}_f$; we wish to show $x^0(N; x^+) \in \mathbb{X}_f$.

Since $V_f^*(\bar{x}^+, \widetilde{\mathbf{u}}(x)) = V_f(f(x^0(N;x), \kappa_f(x^0(N;x)))) \le \gamma_f c_f$ and since $V_f^*(x^+, \widetilde{\mathbf{u}}(x)) \le V_f^*(\bar{x}^+, \widetilde{\mathbf{u}}(x)) + \alpha_a(|x^+ - \bar{x}^+|)$ it follows that $V_f(x^0(N;x)) \le V_f(x^0(N;\bar{x}^+)) + \alpha_a(|x^+ - \bar{x}^+|) \le \gamma_f c_f + \alpha_a(|x^+ - \bar{x}^+|)$. Hence, $x^0(N;x) \in \mathbb{X}_f$ implies $x^0(N;x^+) \in \mathbb{X}_f$ if $\alpha_a(|x^+ - \bar{x}^+|) \le (1 - \gamma_f)c_f$.

**Robust positive invariance of $R_c$ for the controlled uncertain system.** Suppose $x \in R_c$. Since $V_N(\cdot)$ is continuous, it follows from Proposition 3.4 that there exists a $\mathcal{K}_\infty$ function $\alpha_b(\cdot)$ such that

$$|V_N(x^+, \mathbf{u}) - V_N(\bar{x}^+, \mathbf{u})| \le \alpha_b(|x^+ - \bar{x}^+|)$$

for all $(x^+, \mathbf{u}) \in f(R_c, U, W) \times \mathbb{U}^N$, all $(\bar{x}^+, \mathbf{u}) \in R_c \times \mathbb{U}^N$. This result holds in particular for $x^+ = f(x, \kappa_N(x), w)$, $\bar{x}^+ = f(x, \kappa_N(x), 0)$ and $\mathbf{u} = \widetilde{\mathbf{u}}(x)$ with $x \in R_c$. Hence, if $x \in R_c$

$$V_N(x^+, \widetilde{\mathbf{u}}) \le V_N(\bar{x}^+, \widetilde{\mathbf{u}}(x)) + \alpha_b(|x^+ - \bar{x}^+|)$$

Since $V_N(x^+, \tilde{\mathbf{u}}) \leq V_N^0(x) - c_1 |x|^2$ and, since the control $\tilde{\mathbf{u}}(x)$, $x \in R_c$ satisfies both the control and terminal constraints if $\alpha_a(|x^+ - \bar{x}^+|) \leq (1 - \gamma_f)c_f$, it follows that

$$V_N^0(x^+) \leq V_N(x^+, \tilde{\mathbf{u}}) \leq V_N^0(x) - c_1 |x|^2 + \alpha_b(|x^+ - \bar{x}^+|)$$

so that

$$V_N^0(x^+) \leq \gamma V_N^0(x) + \alpha_b(|x^+ - \bar{x}^+|)$$

Hence $x \in R_c$ implies $x^+ = f(x, \kappa_N(x), w) \in R_c$ for all $w \in \mathbb{W}$ if $\alpha_a(|x^+ - \bar{x}^+|) \leq (1 - \gamma_f)c_f$ and $\alpha_b(|x^+ - \bar{x}^+|) \leq (1 - \gamma)c$.

**Robust positive invariant of $R_b$ for the controlled uncertain system.** Similarly, $x \in R_b$ implies $x^+ = f(x, \kappa_N(x), w) \in R_b$ for all $w \in \mathbb{W}$ if $\alpha_a(|x^+ - \bar{x}^+|) \leq (1 - \gamma_f)c_f$ and $\alpha_b(|x^+ - \bar{x}^+|) \leq (1 - \gamma)b$.

**Descent property of $V_N^0(\cdot)$ in $R_c \setminus R_b$.** Suppose that $x \in R_c \setminus R_b$ and that $\alpha_a(|x^+ - \bar{x}^+|) \leq (1 - \gamma_f)c_f$. Then because $\tilde{\mathbf{u}} \in \mathcal{U}_N(x^+)$, we have that $\mathbb{P}_N(x^+)$ is feasible and thus $V_N^0(x^+)$ is well defined. As above, we have that $V_N^0(x^+) \leq \gamma V_N^0(x) + \alpha_b(|x^+ - \bar{x}^+|)$. Let $\gamma^* \in (\gamma, 1)$. If $\alpha_b(|x^+ - \bar{x}^+|) \leq (\gamma^* - \gamma)b$, we have that

$$\begin{aligned} V_N^0(x^+) &\leq \gamma V_N^0(x) + (\gamma^* - \gamma)b \\ &< \gamma V_N^0(x) + (\gamma^* - \gamma)V_N^0(x) \\ &= \gamma^* V_N^0(x) \end{aligned}$$

because $V_N^0(x) > b$.

**Summary.** These conditions can be simplified if we assume that $f(\cdot)$ is uniformly Lipschitz continuous in $w$ with Lipschitz constant $L$ so that $|f(x^+ \kappa_N(x), w) - f(x, \kappa_N(x), 0)| \leq L|w|$ for all $(x, u) \in R_c \times \mathbb{U}$. The function $f(\cdot)$ has this property with $L = 1$ if $f(x, u, w) = f'(x, u) + w$. Under this assumption, the four conditions become

1. $\alpha_a(|Lw|) \leq (1 - \gamma_f)c_f$
2. $\alpha_a(|Lw|) \leq (1 - \gamma)c$
3. $\alpha_b(|Lw| \leq (1 - \gamma)c$
4. $\alpha_b(|Lw|) \leq (\gamma^* - \gamma)b$

Let $\delta^*$ denote the largest $\delta$ such that all four conditions are satisfied if $w \in \mathbb{W}$ with $|\mathbb{W}| \leq \delta$.[4] Condition 3 can be satisfied if $b \geq \delta^*/(1 - \gamma)$.

[4] $|\mathbb{W}| := \max_w \{|w| \mid w \in \mathbb{W}\}$

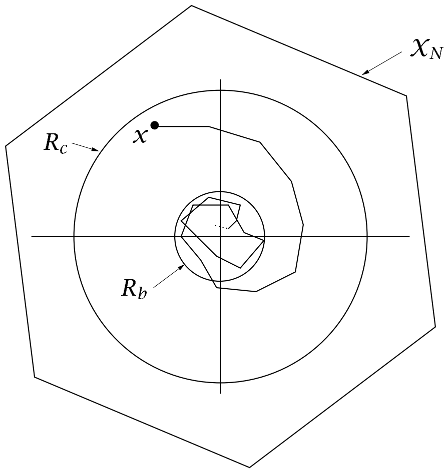



**Figure 3.2:** The sets $\mathcal{X}_N$, $R_b$, and $R_c$.

**Proposition 3.5** (Robustness of nominal MPC)**.** *Suppose all assumptions in Section 3.2.4 are satisfied and that* $|\mathbb{W}| \leq \delta^*$ *and* $c > b$. *Then, any initial state* $x \in R_c$ *of the controlled system* $x^+ = f(x, \kappa_N(x), w)$ *is steered to the set* $R_b$ *in finite time for all admissible disturbance sequences* $\mathbf{w}$ *satisfying* $w(i) \in \mathbb{W}$ *for all* $i \in \mathbb{I}_{\geq 0}$. *Thereafter, the state remains in* $R_b$ *for all admissible disturbance sequences.*

Figure 3.2 illustrates this result.

## 3.3 Min-Max Optimal Control: Dynamic Programming Solution

### 3.3.1 Introduction

In this section we show how robust control of an uncertain system may be achieved using dynamic programming (DP). Our purpose here is to use DP to gain insight. The results we obtain here are not of practical use for complex systems, but reveal the nature of the problem and show what the ideal optimal control problem solved online should be.

In Section 3.2 we examined the inherent robustness of an asymptotically stable system. If uncertainty is present, and it always is, it is preferable to design the controller to be *robust*, i.e., able to cope with some uncertainty. In this section we discuss the design of a robust controller for the system

$$x^+ = f(x, u, w) \tag{3.14}$$

in which a bounded disturbance input $w$ models the uncertainty. The disturbance is assumed to satisfy $w \in \mathbb{W}$ where $\mathbb{W}$ is compact convex, and contains the origin in its interior. The controlled system is required to satisfy the same state and control constraints as above, namely $(x, u) \in \mathbb{Z}$ as well as a terminal constraint $x(N) \in \mathbb{X}_f$. The constraint $(x, u) \in \mathbb{Z}$ may be expressed equivalently as $x \in \mathbb{X}$ and $u \in \mathbb{U}(x)$ in which $\mathbb{X} = \{x \mid \exists\, u \text{ such that } (x, u) \in \mathbb{Z}\}$ and $\mathbb{U}(x) = \{u \mid \exists\, x \text{ such that } (x, u) \in \mathbb{Z}\}$. Because of the disturbance, superior control may be achieved by employing feedback, in the form of a control *policy*, i.e., a sequence of control *laws* rather than employing open-loop control in the form of a sequence of control *actions*. Each control law is a function that maps states into control actions; if the control law at time $i$ is $\mu_i(\cdot)$, then the system at time $i$ satisfies $x(i+1) = f(x(i), \mu_i(x(i)))$. Because of uncertainty, feedback and open-loop control for a given initial state are not equivalent.

The solution at time $k$ of (3.14) with control and disturbance sequences $\mathbf{u} = (u(0), \ldots, u(N-1))$ and $\mathbf{w} = (w(0), \ldots, w(N-1))$ if the initial state is $x$ at time 0 is $\phi(k; x, \mathbf{u}, \mathbf{w})$. Similarly, the solution at time $k$ due to feedback policy $\boldsymbol{\mu} = (\mu_0(\cdot), \ldots, \mu_{N-1}(\cdot))$ and disturbance sequence $\mathbf{w}$ is denoted by $\phi(k; x, \boldsymbol{\mu}, \mathbf{w})$. As discussed previously, the cost may be taken to be that of the nominal trajectory, or the average, or maximum taken over all possible realizations of the disturbance sequence. Here we employ, as is common in the literature, the maximum over all realizations of the disturbance sequence $\mathbf{w}$, and define the cost due to policy $\mu$ with initial state $x$ to be

$$V_N(x, \boldsymbol{\mu}) := \max_{\mathbf{w}} \{J_N(x, \mu, \mathbf{w}) \mid \mathbf{w} \in \mathcal{W}\} \tag{3.15}$$

in which $\mathcal{W} = \mathbb{W}^N$ is the set of admissible disturbance sequences, and $J_N(x, \boldsymbol{\mu}, \mathbf{w})$ is the cost due to an individual realization $\mathbf{w}$ of the disturbance process and is defined by

$$J_N(x, \boldsymbol{\mu}, \mathbf{w}) := \sum_{i=0}^{N-1} \ell(x(i), u(i), w(i)) + V_f(x(N)) \tag{3.16}$$

in which $\boldsymbol{\mu} = (\mu_0(\cdot), \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot))$, $x(i) = \phi(i; x, \mu, \mathbf{w})$, and $u(i) = \mu_i(x(i))$. Let $\mathcal{M}(x)$ denote the set of feedback policies $\mu$ that for a given initial state $x$ satisfy: the state and control constraints, and the terminal constraint for every admissible disturbance sequence $\mathbf{w} \in \mathcal{W}$. The first control law $\mu_0(\cdot)$ in $\boldsymbol{\mu}$ may be replaced by a control action $u_0 = \mu_0(x)$ to simplify optimization, since the initial state $x$ is known whereas future states are uncertain. The set of admissible control policies $\mathcal{M}(x)$ is defined by

$$\mathcal{M}(x) := \{\mu \mid \mu_0(x) \in \mathbb{U}(x),\ \phi(i; x, \mu, \mathbf{w}) \in \mathbb{X},\ \mu_i(\phi(i; x, \mu, \mathbf{w})) \in \mathbb{U}(x) \\ \forall i \in \mathbb{I}_{0:N-1},\ \phi(N; x, \mu, \mathbf{w}) \in \mathbb{X}_f\ \forall \mathbf{w} \in \mathcal{W}\}$$

The robust optimal control problem is

$$\mathbb{P}_N(x): \quad \inf_{\boldsymbol{\mu}} \{V_N(x, \boldsymbol{\mu} \mid \boldsymbol{\mu} \in \mathcal{M}(x)\} \tag{3.17}$$

The solution to $\mathbb{P}_N(x)$, if it exists, is the policy $\mu^0(x)$

$$\boldsymbol{\mu}^0(x) = \left(\mu_0^0(\cdot; x), \mu_1^0(\cdot; x), \ldots, \mu_{N-1}^0(\cdot, x)\right)$$

and the value function is $V_N^0(x) = V_N(x, \boldsymbol{\mu}^0(x))$.

Dynamic programming solves problem $\mathbb{P}_N(x)$ with horizon $N$ for all $x$ such that the problem is feasible, yielding the optimal control policy $\boldsymbol{\mu}^0(\cdot) = \{\mu_0(\cdot), \ldots, \mu_{N-1}\}$ for the optimal control problem with horizon $N$. In doing so, it also solves, for each $i \in \mathbb{I}_{1:N}$, problem $\mathbb{P}_i(x)$ yielding the optimal control policy for the problem with horizon $i$.

### 3.3.2 Properties of the Dynamic Programming Solution

As for deterministic optimal control, the value function and implicit control law may, in principle, be obtained by DP. But DP is, in most cases, impossible to use because of its large computational demands. There are, of course, important exceptions such as $H_2$ and $H_\infty$ optimal control for unconstrained linear systems with quadratic cost functions. DP also can be used for low dimensional constrained optimal control problems when the system is linear, the constraints are affine, and the cost is affine or quadratic. Even when DP is computationally prohibitive, however, it remains a useful tool because of the insight it provides. Because of the cost definition, min-max DP is required. For each $i \in \{0, 1, \ldots, N\}$, let $V_i^0(\cdot)$ and $\kappa_i(\cdot)$ denote, respectively, the partial value function and the optimal solution to the optimal control problem $\mathbb{P}_i$ defined by (3.17) with $i$ replacing $N$. The DP recursion equations for computing these functions are

$$V_i^0(x) = \min_{u \in \mathbb{U}(x)} \max_{w \in \mathbb{W}} \{\ell(x,u,w) + V_{i-1}^0(f(x,u,w)) \mid f(x,u,\mathbb{W}) \subseteq \mathcal{X}_{i-1}\}$$
$$\kappa_i(x) = \arg \min_{u \in \mathbb{U}(x)} \max_{w \in \mathbb{W}} \{\ell(x,u,w) + V_{i-1}^0(f(x,u)) \mid f(x,u,\mathbb{W}) \subseteq \mathcal{X}_{i-1}\}$$
$$\mathcal{X}_i = \{x \in \mathbb{X} \mid \exists\ u \in \mathbb{U}(x) \text{ such that } f(x,u,\mathbb{W}) \subseteq \mathcal{X}_{i-1}\}$$

with boundary conditions

$$V_0^0(x) = V_f(x) \qquad \mathcal{X}_0 = \mathbb{X}_f$$

In these equations, the subscript $i$ denotes time to go so that $\kappa_i(\cdot) := \mu_{N-i}(\cdot)$ (equivalently $\mu_i(\cdot) := \kappa_{N-i}(\cdot)$). In particular, $\kappa_N(\cdot) = \mu_0(\cdot)$. For each $i$, $\mathcal{X}_i$ is the domain of $V_i^0(\cdot)$ (and $\kappa_i(\cdot)$) and is therefore the set of states $x$ for which a solution to problem $\mathbb{P}_i(x)$ exists. Thus $\mathcal{X}_i$ is the set of states that can be *robustly* steered by state feedback, i.e., by a policy $\boldsymbol{\mu} \in \mathcal{M}(x)$, to $\mathbb{X}_f$ in $i$ steps or less satisfying all constraints for all disturbance sequences. It follows from these definitions that

$$V_i^0(x) = \max_{w \in \mathbb{W}} \{\ell(x, \kappa_i(x), w) + V_{i-1}^0(f(x, \kappa_i(x), w))\} \tag{3.18}$$

as discussed in Exercise 3.1.

As in the deterministic case studied in Chapter 2, we are interested in obtaining conditions that ensure that the optimal finite horizon control law $\kappa_0^0(\cdot)$ is stabilizing. To do this we replace the stabilizing Assumption 2.14 in Section 2.4.2 of Chapter 2 by conditions appropriate to the robust control problem. The presence of a disturbance requires us to generalize some earlier definitions; we therefore define the terms *robustly control invariant* and *robustly positive invariant* that generalize our previous definitions of *control invariant* and *positive invariant* respectively.

**Definition 3.6** (Robust control invariance)**.** A set $X \subseteq \mathbb{R}^n$ is *robustly control invariant* for $x^+ = f(x,u,w)$, $w \in \mathbb{W}$ if, for every $x \in X$, there exists a $u \in \mathbb{U}(x)$ such that $f(x,u,\mathbb{W}) \subseteq X$.

**Definition 3.7** (Robust positive invariance)**.** A set $X$ is *robustly positive invariant* for $x^+ = f(x,w)$, $w \in \mathbb{W}$ if, for every $x \in X$, $f(x,\mathbb{W}) \subseteq X$.

As in Chapter 2, stabilizing conditions are imposed on the ingredients $\ell(\cdot), V_f(\cdot)$, and $\mathbb{X}_f$ of the optimal control problem to ensure that the resultant controlled system has desirable stability properties; the solution to a finite horizon optimal control problem does not necessarily ensure stability. Our new assumption is a robust generalization of the stabilizing Assumption 2.2 employed in Chapter 2.

**Assumption 3.8** (Basic stability assumption; robust case)**.**

(a) For all $x \in \mathbb{X}_f$ there exists a $u = \kappa_f(x) \in \mathbb{U}(x)$ such that

$$V_f(f(x,u,0)) \leq V_f(x) - \ell(x,u,0) \text{ and } f(x,u,w) \in \mathbb{X}_f \;\; \forall w \in \mathbb{W}$$

(b) $\mathbb{X}_f \subseteq \mathbb{X}$

(c) There exist $\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$ and $\alpha_f(\cdot)$ satisfying

$$\ell(x,u,w) \geq \alpha_1(|x|) \;\; \forall (x,w) \in \mathbb{R}^n \times \mathbb{W} \;\; \forall u \text{ such that } (x,u) \in \mathbb{Z}$$
$$V_f(x) \leq \alpha_f(|x|), \;\; \forall x \in \mathbb{X}_f$$

Assumption 3.8(a) replaces the unrealistic assumption in the first edition that, for each $x \in \mathbb{X}_f$, there exists a $u \in \mathbb{U}$ such that, for all $w \in \mathbb{W}$, $V_f(f(x,u,w)) \leq V_f(x) - \ell(x,u,w)$ and $f(x,u,w) \in \mathbb{X}_f$. Let $\delta \in \mathbb{R}_{\geq 0}$ be defined by

$$\delta := \max_{(x,w) \in \mathbb{X}_f \times \mathbb{W}} \{V_f(f(x,\kappa_f(x),w)) - V_f(x) + \ell(x,\kappa_f(x),w)\}$$

so that, if Assumption 3.8 holds

$$V_f(f(x,\kappa_f(x),w)) \leq V_f(x) - \ell(x,u,w) + \delta \;\; \forall (x,w) \in \mathbb{X}_f \times \mathbb{W} \tag{3.19}$$

If $\delta = 0$, the controller $\kappa_f(\cdot)$ can steer any $x \in \mathbb{X}_f$ to the origin despite the disturbance.

**Theorem 3.9** (Recursive feasibility of control policies)**.** *Suppose Assumption 3.8 holds. Then*

*(a)* $\mathcal{X}_N \supseteq \mathcal{X}_{N-1} \supseteq \ldots \supseteq \mathcal{X}_1 \supseteq \mathcal{X}_0 = \mathbb{X}_f$

*(b)* $\mathcal{X}_i$ *is robustly control invariant for* $x^+ = f(x,u,w) \;\; \forall i \in \mathbb{I}_{0:N}$

*(c)* $\mathcal{X}_i$ *is robustly positive invariant for* $x^+ = f(x,\kappa_i(x),w), \;\; \forall i \in \mathbb{I}_{0:N}$

*(d)* $[V_{i+1}^0 - V_i^0](x) \leq \max_{w \in \mathbb{W}}\{[V_i^0 - V_{i-1}^0](f(x,\kappa_i(x),w))\} \;\; \forall x \in \mathcal{X}_i$, $\forall i \in \mathbb{I}_{1:N-1}$*. Also* $V_i^0(x) - V_{i-1}^0(x) \leq \delta \;\; \forall x \in \mathcal{X}_{i-1}, \;\; \forall i \in \{1,\ldots,N\}$ *and* $V_i^0(x) \leq V_f(x) + i\delta \;\; \forall \; x \in \mathcal{X}_f, \forall i \in \mathbb{I}_{1:N}$

*(e) For any* $x \in \mathcal{X}_N$, $\Big(\kappa_N(x), \kappa_{N-1}(\cdot), \ldots, \kappa_1(\cdot), \kappa_f(\cdot)\Big)$ *is a feasible policy for* $\mathbb{P}_{N+1}(x)$*, and, for any* $x \in \mathcal{X}_{N-1}$, $(\kappa_{N-1}(x), \kappa_{N-2}(\cdot), \ldots, \kappa_1(\cdot), \kappa_f(\cdot))$ *is a feasible policy for* $\mathbb{P}_N(x)$*.*

*Proof.*

(a)–(c) Suppose, for some $i$, $\mathcal{X}_i$ is robust control invariant so that any point $x \in \mathcal{X}_i$ can be robustly steered into $\mathcal{X}_i$. By construction, $\mathcal{X}_{i+1}$ is the set of all points $x$ that can be robustly steered into $\mathcal{X}_i$. Also $\mathcal{X}_{i+1} \supseteq \mathcal{X}_i$ so that $\mathcal{X}_{i+1}$ is robust control invariant. But $\mathcal{X}_0 = \mathbb{X}_f$ is robust control invariant. Both (a) and (b) follow by induction. Part (c) follows from (b).

(d) From (3.18) we have

$$\begin{aligned}
[V_{i+1}^0 - V_i^0](x) &= \max_{w \in \mathbb{W}} \{\ell(x, \kappa_{i+1}(x), w) + V_i^0(f(x, \kappa_{i+1}(x), w))\} \\
&\quad - \max_{w \in \mathbb{W}} \{\ell(x, \kappa_i(x), w) + V_{i-1}^0(f(x, \kappa_i(x), w))\} \\
&\leq \max_{w \in \mathbb{W}} \{\ell(x, \kappa_i(x), w) + V_i^0(f(x, \kappa_i(x), w))\} \\
&\quad - \max_{w \in \mathbb{W}} \{\ell(x, \kappa_i(x), w) + V_{i-1}^0(f(x, \kappa_i(x), w))\}
\end{aligned}$$

for all $x \in \mathcal{X}_i$ since $\kappa_i(\cdot)$ may *not* be optimal for problem $\mathbb{P}_{i+1}(x)$. We now use the fact that $\max_w\{a(w)\} - \max_w\{b(w)\} \leq \max_w\{a(w) - b(w)\}$, which is discussed in Exercise 3.2, to obtain

$$[V_{i+1}^0 - V_i^0](x) \leq \max_{w \in \mathbb{W}} \{[V_i^0 - V_{i-1}^0](f(x, \kappa_i(x), w))\}$$

for all $x \in X_i$. Also, for all $x \in \mathcal{X}_0 = \mathbb{X}_f$

$$[V_1^0 - V_0^0](x) = \max_{w \in \mathbb{W}} \{\ell(x, \kappa_1(x), w) + V_f(f(x, \kappa_1(x), w)) - V_f(x)\} \leq \delta$$

in which the last inequality follows from Assumption 3.8. By induction, $V_i^0(x) - V_{i-1}^0(x) \leq \delta \ \forall x \in \mathcal{X}_{i-1}, \ \forall i \in \{1, \ldots, N\}$. It follows that $V_i^0(x) \leq V_f(x) + i\delta$ for all $x \in \mathbb{X}_f$, all $i \in \{1, \ldots, N\}$.

(e) Suppose $x \in \mathcal{X}_N$. Then $\boldsymbol{\kappa}^0(x) = (\kappa_N(x), \kappa_{N-1}(\cdot), \ldots, \kappa_1(\cdot))$ is a feasible and optimal policy for problem $\mathbb{P}_N(x)$, and steers every trajectory emanating from $x$ into $\mathcal{X}_0 = \mathbb{X}_f$ in $N$ time steps. Because $\mathbb{X}_f$ is robustly positive invariant for $x^+ = f(x, \kappa_f(x), w)$, $w \in \mathbb{W}$, the policy $\big(\kappa_N(x), \kappa_{N-1}(\cdot), \ldots, \kappa_1(\cdot), \kappa_f(\cdot)\big)$ is feasible for problem $\mathbb{P}_{N+1}(x)$. Similarly, the policy $(\kappa_{N-1}(x), \kappa_{N-2}(\cdot), \ldots, \kappa_1(\cdot))$ is feasible and optimal for problem $\mathbb{P}_{N-1}(x)$, and steers every trajectory emanating from $x \in \mathcal{X}_{N-1}$ into $\mathcal{X}_0 = \mathbb{X}_f$ in $N - 1$ time steps. Therefore the policy $\big(\kappa_{N-1}(x), \kappa_{N-2}(\cdot), \ldots, \kappa_1(\cdot), \kappa_f(\cdot)\big)$ is feasible for $\mathbb{P}_N(x)$ for any $x \in \mathcal{X}_{N-1}$. ∎

## 3.4 Robust Min-Max MPC

Because use of dynamic programming (DP) is usually prohibitive, obtaining an alternative, robust min-max model predictive control, is desirable. We present here an analysis that uses the improved stability condition Assumption 3.8. The system to be controlled is defined in (3.14) and the cost function $V_N(\cdot)$ in (3.15) and (3.16). The decision variable, which, in DP, is a sequence $\boldsymbol{\mu} = (\mu_0(\cdot), \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot))$ of control laws, each of which is an arbitrary function of the state $x$, is too complex for online optimization; so, we replace $\boldsymbol{\mu}$ by the simpler object $\boldsymbol{\mu}(\mathbf{v}) := (\mu(\cdot, v_0), \mu(\cdot, v_1), \ldots \mu(\cdot, v_{N-1}))$ in which $\mathbf{v} = (v_0, v_1, \ldots, v_{N-1})$ is a sequence of parameters with $\mu(\cdot)$ parameterized by $v_i$, $i \in \mathbb{I}_{0:N-1}$.

A simple parameterization is $\mu(\mathbf{v}) = \mathbf{v} = (v_0, v_1, \ldots, v_{N-1})$, a sequence of control *actions* rather than control *laws*. The decision variable $\mathbf{v}$ in this case is similar to the control sequence $\mathbf{u}$ used in deterministic MPC, and is simple enough for implementation; the disadvantage is that feedback is not allowed in the optimal control problem $\mathbb{P}_N(x)$. Hence the predicted trajectories may diverge considerably. An equally simple parameterization that has proved to be useful when the system being controlled is linear and time invariant is $\boldsymbol{\mu}(\mathbf{v}) = (\mu(\cdot, v_0) \ldots, \mu(\cdot, v_{N-1}))$ in which, for each $i$, $\mu(x, v_i) := v_i + Kx$; if $f(x, u, w) = Ax + Bu + w$, $K$ is chosen so that $A_K := A + BK$ is Hurwitz. More generally, $\mu(x, v_i) := \sum_{j \in J} v_i^j \theta_j(x) = \langle v_i, \theta(x) \rangle$, $\theta(x) := (\theta_1(x), \theta_2(x), \ldots, \theta_J(x))$. Hence the policy sequence $\boldsymbol{\mu}(\mathbf{v})$ is parameterized by the vector sequence $\mathbf{v} = (v_0, v_1, \ldots, v_{N-1})$. Choosing appropriate basis functions $\theta_j(\cdot)$, $j \in J$, is not simple. The decision variable is the vector sequence $\mathbf{v}$.

With this parameterization, the optimal control problem $\mathbb{P}_N(x)$ becomes

$$\mathbb{P}_N(x): \qquad V_N^0(x) = \min_{\mathbf{v}} \{V_N(x, \boldsymbol{\mu}(\mathbf{v})) \mid \mathbf{v} \in \mathcal{V}_N(x)\}$$

in which

$$V_N(x, \boldsymbol{\mu}(\mathbf{v})) := \max_{\mathbf{w}} \{J_N(x, \boldsymbol{\mu}(\mathbf{v}), \mathbf{w}) \mid \mathbf{w} \in \mathbb{W}^N\}$$

$$J_N(x, \boldsymbol{\mu}(\mathbf{v}), \mathbf{w}) := \sum_{i=0}^{N-1} \ell(x(i), u(i), w(i)) + V_f(x(N))$$

$$\mathcal{V}_N(x) := \{\mathbf{v} \mid (x(i), u(i)) \in \mathbb{Z},\ \forall i \in \mathbb{I}_{0:N-1},\ x(N) \in \mathbb{X}_f,\ \forall \mathbf{w} \in \mathbb{W}^N\}$$

with $x(i) := \phi(i; x, \boldsymbol{\mu}(\mathbf{v}), \mathbf{w})$ and $u(i) = \mu(x(i), v_i)$; $\phi(i; x, \boldsymbol{\mu}(\mathbf{v}), \mathbf{w})$ denotes the solution at time $i$ of $x(i+1) = f(x(i), u(i), w(i))$ with $x(0) = x$, $u(i) = \mu(x(i), v_i)$ for all $i \in \mathbb{I}_{0:N-1}$, and disturbance sequence $\mathbf{w}$. Let $\mathbf{v}^0(x)$ denote the minimizing value of the decision variable $\mathbf{v}$, $\boldsymbol{\mu}^0(x) := \boldsymbol{\mu}(\mathbf{v}^0(x))$ the corresponding optimal control policy, and let $V_N^0(x) := V_N(x, \boldsymbol{\mu}^0(x))$ denote the value function. We implicitly assume that a solution to $\mathbb{P}_N(x)$ exists for all $x \in \mathcal{X}_N(x) := \{x \mid \mathcal{V}_N(x) \neq \emptyset\}$ and that $\mathcal{X}_N$ is not empty. The MPC action at state $x$ is $\mu_0^0(x) = \mu(x, v_0^0(x))$, with $v_0^0(x)$ the first element of the optimal decision variable sequence $\mathbf{v}^0(x)$. The implicit MPC law is $\mu_0^0(\cdot)$. To complete the problem definition, we assume that $V_f(\cdot)$ and $\ell(\cdot)$ satisfy Assumption 3.8.

It follows from Assumption 3.8 that there exists a $\mathcal{K}_\infty$ function $\alpha_1(\cdot)$ such that $V_N^0(x) \geq \alpha_1(|x|)$ for all $x \in \mathcal{X}_N$, the domain of $V_N^0(\cdot)$. Determination of an upper bound for $V_N^0(\cdot)$ is difficult, so we *assume* that there exists a $\mathcal{K}_\infty$ function $\alpha_2(\cdot)$ such that $V_N^0(x) \leq \alpha_2(|x|)$ for all $x \in \mathcal{X}_N$. We now consider the descent condition, i.e., we determine an upper bound for $V_N^0(x^+) - V_N^0(x)$ as well as a *warm start* for obtaining, via optimization, the optimal decision sequence $\mathbf{v}^0(x^+)$ given $\mathbf{v}^0(x)$.

Suppose that, at state $x$, the value function $V_N^0(x)$ and the optimal decision sequence $\mathbf{v}^0(x)$ have been determined, as well as the control action $\mu_0^0(x)$. The subsequent state is $x^+ = f(x, \mu_0^0(x), w_0)$, with $w_0$ the value of the additive disturbance ($w(t)$ if the current time is $t$). Let

$$\boldsymbol{\mu}^*(x) := \boldsymbol{\mu}^0_{1:N-1}(x) = \Big(\mu(\,\cdot\,, v_1^0(x)), \mu(\,\cdot\,, v_2^0(x)), \ldots, \mu(\,\cdot\,, v_{N-1}^0(x))\Big)$$

denote $\boldsymbol{\mu}^0(x)$ with its first element $\mu(\,\cdot\,, v_0^0(x))$ removed; $\mu^*(x)$ is a sequence of $N-1$ control laws. In addition let $\tilde{\mathbf{u}}(x)$ be defined by

$$\tilde{\mathbf{u}}(x) := (\boldsymbol{\mu}^*(x), \kappa_f(\cdot))$$

$\tilde{\mathbf{u}}(x)$ is a sequence of $N$ control laws.

For any sequence $\mathbf{z}$ let $\mathbf{z}_{a:b}$ denote the subsequence $(z(a), z(a+1), \ldots, z(b))$; as above, $\mathbf{z} := \mathbf{z}_{0:N-1}$. Because $x \in \mathcal{X}_N$ is feasible for the optimal control problem $\mathbb{P}_N(x)$, every random trajectory with disturbance sequence $\mathbf{w} = \mathbf{w}_{0:N-1} \in \mathbb{W}^N$ emanating from $x \in \mathcal{X}_N$ under the control policy $\boldsymbol{\mu}^0(x)$ reaches the terminal state $x_N = \phi(N; x, \boldsymbol{\mu}^0(x), \mathbf{w}) \in \mathbb{X}_f$ in $N$ steps. Since $w(0)$ is the first element of $\mathbf{w}$, $\mathbf{w} = (w(0), \mathbf{w}_{1:N-1})$. Hence the random trajectory with control sequence $\boldsymbol{\mu}^0_{1:N-1}(x)$ and disturbance sequence $\mathbf{w}_{1:N-1}$ emanating from $x^+ = f(x, \mu_0^0(x), w(0))$ reaches $x_N \in \mathbb{X}_f$ in $N-1$ steps. Clearly

$$J_{N-1}(x^+, \boldsymbol{\mu}^0_{1:N-1}(x), \mathbf{w}_{1:N-1}) \leq J_N(x, \boldsymbol{\mu}^0(x), \mathbf{w}) - \ell(x, \mu_0^0(x), w_0)$$

By Assumption 3.8, $\ell(x, \mu_0^0(x), w(0)) = \ell(x, \kappa_N(x), w(0) \geq \alpha_1(|x|)$ and

$$J_{N-1}(x^+, \boldsymbol{\mu}^0_{1:N-1}(x), \mathbf{w}_{1:N-1}) \leq J_N(x, \boldsymbol{\mu}^0(x), \mathbf{w}) - \alpha_1(|x|)$$

The policy sequence $\tilde{\boldsymbol{\mu}}(x)$, which appends $\kappa_f(\cdot)$ to $\boldsymbol{\mu}^0_{1:N-1}(x)$, steers $x^+$ to $x_N$ in $N-1$ steps and then steers $x_N \in \mathbb{X}_f$ to $x(N+1) = f(x_N, \kappa_f(x_N), w_N)$ that lies in the interior of $\mathbb{X}_f$. Using Assumption 3.8, we obtain

$$J_N(x^+, \tilde{\boldsymbol{\mu}}(x), \mathbf{w}_{1:N}) \leq J_N(x, \boldsymbol{\mu}^0(x), \mathbf{w}) - \alpha_1(|x|) + \delta$$

Using this inequality with $\mathbf{w}_{0:N} = (w(0), \mathbf{w}^0(x^+))$[5] so that $\mathbf{w}_{1:N} = \mathbf{w}^0(x^+)$ and $\mathbf{w} = \mathbf{w}_{0:N-1} = \big(w(0), \mathbf{w}^0_{0,N-2}(x^+)\big)$ yields

$$\begin{aligned} V_N^0(x^+) &= J_N(x^+, \boldsymbol{\mu}^0(x^+), \mathbf{w}^0(x^+)) \leq J_N(x^+, \tilde{\boldsymbol{\mu}}(x), \mathbf{w}^0(x^+)) \\ &\leq J_N(x, \boldsymbol{\mu}^0(x), (w(0), \mathbf{w}^0_{0,N-2}(x^+)) - \alpha_1(|x|) + \delta \\ &\leq V_N^0(x) - \alpha_1(|x|) + \delta \end{aligned}$$

The last inequality follows from the fact that the disturbance sequence $\big(w(0), \mathbf{w}^0_{0,N-2}(x^+)\big)$ does not necessarily maximize $\mathbf{w} \mapsto J_N(x, \boldsymbol{\mu}^0(x), \mathbf{w})$.

Assume now that $\ell(\cdot)$ is quadratic and positive definite so that $\alpha_1(|x|) \geq c_1 |x|^2$. Assume also that $V_N^0(x) \leq c_2 |x|^2$ so that for all $x \in \mathcal{X}_N$

$$V_N^0(x^+) \leq \gamma V_N^0(x) + \delta$$

with $\gamma = 1 - c_1/c_2 \in (0, 1)$. Let $\varepsilon > 0$. It follows that for all $x \in \mathcal{X}_N$ such that $V_N^0(x) \geq c := (\delta + \varepsilon)/(1 - \gamma)$

$$V_N^0(x^+) \leq \gamma V_N^0(x) + \delta \leq V_N^0(x) - (1-\gamma)c + \delta \leq V_N^0(x) - \varepsilon$$

since $V_N^0(x) \geq c$ and, by definition, $(1-\gamma)c = \delta + \varepsilon$. Secondly, if $x$ lies in $\mathrm{lev}_c\, V_N^0$, then

$$V_N^0(x^+) \leq \gamma c + \delta \leq c - \varepsilon$$

since $V_N^0(x) \leq c$ and, by definition, $c = \gamma c + \delta + \varepsilon$. Hence $x \in \mathrm{lev}_c\, V_N^0$ implies $x^+ \in f(x, \mu_0^0(x), \mathbb{W}) \subset \mathrm{lev}_c\, V_N^0$.

[5] $\mathbf{w}^0(x^+) := \arg\max_{\mathbf{w} \in \mathbb{W}^N} J_N(x^+, \boldsymbol{\mu}^0(x^+), \mathbf{w})$.

**Summary.** If $\delta < (1-\gamma)c$ ($c > \delta/(1-\gamma)$) and $\text{lev}_c V_N^0 \subset \mathcal{X}_N$, every initial state $x \in \mathcal{X}_N$ of the closed-loop system $x^+ = f(x, \mu_0^0(x), w)$ is steered to the sublevel set $\text{lev}_c V_N^0$ in finite time for all disturbance sequences $\mathbf{w}$ satisfying $w(i) \in \mathbb{W}$, all $i \geq 0$, and thereafter remains in this set; the set $\text{lev}_c V_N^0$ is positive invariant for $x^+ = f(x, \mu_0^0(x), w)$, $w \in \mathbb{W}$. The policy sequence $\tilde{\mathbf{u}}(x)$, easily obtained from $\boldsymbol{\mu}^0(x)$, is feasible for $\mathbb{P}_N(x^+)$ and is a suitable warm start for computing $\boldsymbol{\mu}^0(x^+)$.

## 3.5 Tube-Based Robust MPC

### 3.5.1 Introduction

It was shown in Section 3.4 that it is possible to control an uncertain system robustly using a version of MPC that requires solving *online* an optimal control problem of minimizing a cost subject to satisfaction of state and control constraints for *all* possible disturbance sequences. For MPC with horizon $N$ and $q_x$ state constraints, the number of state constraints in the optimal control problem is $Nq_x$. Since the state constraints should be satisfied for *all* disturbance sequences, the number of state constraints for the uncertain case is $MNq_x$, with $M$ equal to the number of disturbance sequences. For linear MPC, $M$ can be as small as $V^N$ with $V$ equal to the number of vertices of $\mathbb{W}$ with $\mathbb{W}$ polytopic. For nonlinear MPC, Monte Carlo optimization must be employed, in which case $M$ can easily be several thousand to achieve constraint satisfaction with high probability. The number of constraints $MNq_x$ can thus exceed $10^6$ in process control applications.

It is therefore desirable to find approaches for which the online computational requirement is more modest. We describe, in this section, a *tube-based* approach. We show that all trajectories of the uncertain system lie in a bounded neighborhood of a nominal trajectory. This bounded neighborhood is called a tube. Determination of the tube enables satisfaction of the constraints by the uncertain system for *all* disturbance sequences to be obtained by ensuring that the nominal trajectory satisfies suitably tightened constraints. If the nominal trajectory satisfies the tightened constraints, every random trajectory in the associated tube satisfies the original constraints. Computation of the tightened constraints may be computationally expensive but can done *offline*; the *online* computational requirements are similar to those for nominal MPC.

To describe tube-based MPC, we use some concepts in set algebra. Given two subsets $A$ and $B$ of $\mathbb{R}^n$, we define set addition, set subtraction (sometimes called Minkowski or Pontryagin set subtraction), set multiplication, and Hausdorff distance between two sets as follows.

**Definition 3.10** (Set algebra and Hausdorff distance).

(a) Set addition: $A \oplus B := \{a + b \mid a \in A, b \in B\}$

(b) Set subtraction: $A \ominus B := \{x \in \mathbb{R}^n \mid \{x\} \oplus B \subseteq A\}$

(c) Set multiplication: Let $K \in \mathbb{R}^{m\times n}$; then $KA := \{Ka \mid a \in A\}$

(d) The Hausdorff distance $d_H(\cdot)$ between two subsets $A$ and $B$ of $\mathbb{R}^n$ is defined by

$$d_H(A, B) := \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\}$$

in which $d(x, S)$ denotes the distance of a point $x \in \mathbb{R}^n$ from a set $S \subset \mathbb{R}^n$ and is defined by

$$d(x, S) := \inf_{y}\{d(x, y) \mid y \in S\} \qquad d(x, y) := |x - y|$$

In these definitions, $\{x\}$ denotes the set consisting of a single point $x$, and $\{x\} \oplus B$ therefore denotes the set $\{x + b \mid b \in B\}$; the set $A \ominus B$ is the largest set $C$ such that $B \oplus C \subseteq A$. A sequence $(x(i))$ is said to converge to a set $S$ if $d(x(i), S) \to 0$ as $i \to \infty$. If $d_H(A, B) \leq \varepsilon$, then the distance of every point $a \in A$ from $B$ is less than or equal to $\varepsilon$, and that the distance of every point $b \in B$ from $A$ is less than or equal to $\varepsilon$. We say that the sequence of sets $(A(i))$ converges, in the Hausdorff metric, to the set $B$ if $d_H(A(i), B) \to 0$ as $i \to \infty$.

Our first task is to generate an outer-bounding tube. An excellent background for the following discussion is provided in Kolmanovsky and Gilbert (1998).

### 3.5.2 Outer-Bounding Tube for a Linear System with Additive Disturbance

Consider the following linear system

$$x^+ = Ax + Bu + w$$

in which $w \in \mathbb{W}$, a compact convex subset of $\mathbb{R}^n$ containing the origin. We assume that $\mathbb{W}$ contains the origin in its interior. Let $\phi(i; x, \mathbf{u}, \mathbf{w})$ denote the solution of $x^+ = Ax + Bu + w$ at time $i$ if the initial state at time 0 is $x$, and the control and disturbance sequences are, respectively, $\mathbf{u}$ and $\mathbf{w}$.

Let the nominal system be described by

$$\bar{x}^+ = A\bar{x} + B\bar{u}$$

and let $\bar{\phi}(i; \bar{x}, \mathbf{u})$ denote the solution of $\bar{x}^+ = A\bar{x} + B\bar{u}$ at time $i$ if the initial state at time 0 is $\bar{x}$. Then $e := x - \bar{x}$, the deviation of the actual state $x$ from the nominal state $\bar{x}$, satisfies the difference equation

$$e^+ = Ae + w$$

so that

$$e(i) = A^i e(0) + \sum_{j=0}^{i-1} A^j w(j)$$

in which $e(0) = x(0) - \bar{x}(0)$. If $e(0) = 0$, then $e(i) \in S(i)$ where the set $S(i)$ is defined by

$$S(i) := \sum_{j=0}^{i-1} A^j \mathbb{W} = \mathbb{W} \oplus A\mathbb{W} \oplus \ldots \oplus A^{i-1}\mathbb{W}$$

in which $\sum$ and $\oplus$ denote set addition. It follows from our assumptions on $\mathbb{W}$ that $S(i)$ contains the origin in its interior for all $i \geq n$.

We first consider the tube $\mathbf{X}(x, \mathbf{u})$ generated by the open-loop control sequence $\mathbf{u}$ when $x(0) = \bar{x}(0) = x$, and $e(0) = 0$. It is easily seen that $\mathbf{X}(x, \mathbf{u}) = (X(0; x), X(1; x, \mathbf{u}), \ldots, X(N; x, \mathbf{u}))$ with

$$X(i; x) := \{\bar{x}(i)\} \oplus S_i$$

and $\bar{x}(i) = \bar{\phi}(i; x, \mathbf{u})$, the state at time $i$ of the nominal system, is the center of the tube. So it is relatively easy to obtain the exact tube generated by an open-loop control if the system is linear and has a bounded additive disturbance, provided that one can compute the sets $S(i)$.

If $A$ is stable, then, as shown in Kolmanovsky and Gilbert (1998), $S(\infty) := \sum_{j=0}^{\infty} A^j \mathbb{W}$ exists and is positive invariant for $x^+ = Ax + w$, i.e., $x \in S(\infty)$ implies that $Ax + w \in S(\infty)$ for all $w \in \mathbb{W}$; also $S(i) \to S(\infty)$ in the Hausdorff metric as $i \to \infty$. The set $S(\infty)$ is known to be the minimal robust positive invariant set[6] for $x^+ = Ax + w$, $w \in \mathbb{W}$. Also

[6]Every other robust positive invariant set $X$ satisfies $X \supseteq S_\infty$.

$S(i) \subseteq S(i+1) \subseteq S(\infty)$ for all $i \in \mathbb{I}_{\geq 0}$ so that the tube $\hat{\mathbf{X}}(x, \mathbf{u})$ defined by

$$\hat{\mathbf{X}}(x, \mathbf{u}) := \Big(\hat{X}(0; x), \hat{X}(1; x, \mathbf{u}), \ldots, \hat{X}(N; x, \mathbf{u})\Big)$$

in which

$$\hat{X}(0; x) = \{x\} \oplus S(\infty) \qquad \hat{X}(i; x, \mathbf{u}) = \{\bar{x}(i)\} \oplus S(\infty)$$

is an outer-bounding tube with constant "cross section" $S(\infty)$ for the exact tube $\mathbf{X}(x, \mathbf{u})$ ($X(i; x, \mathbf{u}) \subseteq \hat{X}(i; x, \mathbf{u})$ for all $i \in \mathbb{I}_{\geq 0}$). It is sometimes more convenient to use the constant cross-section outer-bounding tube $\hat{\mathbf{X}}(x, \mathbf{u})$ in place of the exact tube $\mathbf{X}(x, \mathbf{u})$. If we restrict attention to the interval $[0, N]$ as we do in computing the MPC action, then replacing $S(\infty)$ by $S(N)$ yields a less conservative, constrained cross-section, outer-bounding tube for the interval $[0, N]$.

Use of the exact tube $\mathbf{X}(x, \mathbf{u})$ and the outer-bounding tube $\hat{\mathbf{X}}(x, \mathbf{u})$ may be limited for reasons discussed earlier—the sets $S(i)$ may be unnecessarily large simply because an open-loop control sequence rather than a feedback policy was employed to generate the tube. For example, if $\mathbb{W} = [-1, 1]$ and $x^+ = x + u + w$, then $S(i) = (i+1)\mathbb{W}$ increases without bound as time $i$ increases. We must introduce feedback to contain the size of $S(i)$, but wish to do so in a simple way because optimizing over arbitrary policies is prohibitive. The feedback policy we propose is

$$u = \bar{u} + K(x - \bar{x})$$

in which $x$ is the current state of the system $x^+ = Ax + Bu + w$, $\bar{x}$ is the current state of a nominal system defined below, and $\bar{u}$ is the current input to the nominal system. With this feedback policy, the state $x$ satisfies the difference equation

$$x^+ = Ax + B\bar{u} + BKe + w$$

in which $e := x - \bar{x}$ is the deviation of the actual state from the nominal state. The nominal system corresponding to the uncertain system $x^+ = Ax + B\bar{u} + BKe + w$ is

$$\bar{x}^+ = A\bar{x} + B\bar{u}$$

The deviation $e = x - \bar{x}$ now satisfies the difference equation

$$e^+ = A_K e + w \qquad A_K := A + BK$$

which is the same equation used previously except that $A$, which is possibly unstable, is replaced by $A_K$, which is stable by design. If $K$ is chosen so that $A_K$ is stable, then the corresponding uncertainty sets $S_K(i)$ defined by

$$S_K(i) := \sum_{j=0}^{i-1} A_K^j \mathbb{W}$$

can be expected to be smaller than the original uncertainty sets $S(i)$, $i \in \mathbb{I}_{\geq 0}$, considerably smaller if $A$ is unstable and $i$ is large. Our assumptions on $\mathbb{W}$ imply that $S_K(i)$, like $S(i)$, contains the origin in its interior for each $i$. Since $A_K$ is stable, the set $S_K(\infty) := \sum_{j=0}^{\infty} A_K^j \mathbb{W}$ exists and is positive invariant for $e^+ = A_K e + w$. Also, $S_K(i) \to S_K(\infty)$ in the Hausdorff metric as $i \to \infty$. Since $K$ is fixed, the feedback policy $u = K(x - \bar{x}) + \bar{u}$ is simply parameterized by the open-loop control sequence $\bar{\mathbf{u}}$. If $x(0) = \bar{x}(0) = x$, the tube generated by the feedback policy $u = \bar{u} + K(x - \bar{x})$ is $\mathbf{X}(x, \bar{\mathbf{u}}) = (X(0; x), X(1; x, \bar{\mathbf{u}}), \ldots, X(N; x, \bar{\mathbf{u}}))$ in which

$$X(0; x) = \{x\} \qquad X(i; x, \bar{\mathbf{u}}) := \{\bar{x}(i)\} \oplus S_K(i)$$

and $\bar{x}(i)$ is the solution of the nominal system $\bar{x}^+ = A\bar{x} + B\bar{u}$ at time $i$ if the initial state $\bar{x}(0) = x$, and the control sequence is $\bar{\mathbf{u}}$. For given initial state $x$ and control sequence $\bar{\mathbf{u}}$, the solution of $x^+ = Ax + B(\bar{u} + Ke) + w$ lies in the tube $\mathbf{X}(x, \bar{\mathbf{u}})$ for *every* admissible disturbance sequence $\mathbf{w}$. As before, $S_K(i)$ may be replaced by $S_K(\infty)$ to get an outer-bounding tube. If attention is confined to the interval $[0, N]$, $S_K(i)$ may be replaced by $S_K(N)$ to obtain a less conservative outer-bounding tube. If we consider again our previous example, $\mathbb{W} = [-1, 1]$ and $x^+ = x + u + w$, and choose $K = -(1/2)$, then $A_K = 1/2$, $S_K(i) = (1 + 0.5 + \ldots + 0.5^{i-1})\mathbb{W} \subset 2\mathbb{W}$, and $S_K(\infty) = 2\mathbb{W} = [-2, 2]$. In contrast, $S_K(i) \to [-\infty, \infty]$ as $i \to \infty$.

In the preceding discussion, we required $x(0) = \bar{x}(0)$ so that $e(0) = 0$ in order to ensure $e(i) \in S(i)$ or $e(i) \in S_K(i)$. When $A_K$ is stable, however, it is possible to relax this restriction. This follows from the previous statement that $S_K(\infty)$ exists and is robustly positive invariant for $e^+ = A_K e + w$, i.e., $e \in S_K(\infty)$ implies $e^+ \in S_K(\infty)$ for all $e^+ \in \{A_K e\} \oplus \mathbb{W}$. Hence, if $e(0) \in S_K(\infty)$, then $e(i) \in S_K(\infty)$ for all $i \in \mathbb{I}_{\geq 0}$, all $\mathbf{w} \in \mathbb{W}^i$.

In tube-based MPC, we ensure that $\bar{x}(i) \to 0$ as $i \to \infty$, so that $x(i)$, which lies in the sequence of sets $(\{\bar{x}(i)\} \oplus S_K(i))_{0:\infty}$, converges to the set $S_K(\infty)$ as $i \to \infty$. Figure 3.3 illustrates this result ($S := S_K(\infty)$). Even though $S_K(\infty)$ is difficult to compute, this is a useful theoretical property of the controlled system.

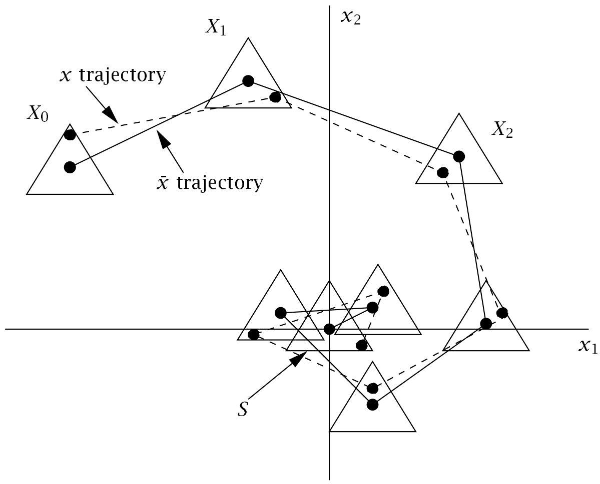



**Figure 3.3:** Outer-bounding tube $\mathbf{X}(\bar{x}, \bar{\mathbf{u}})$; $X_i = \{\bar{x}(i)\} \oplus S_K(\infty)$.

The controller is required to ensure that state-control constraint $(x, u) \in \mathbb{Z}$ is not transgressed. Let $\bar{\mathbb{Z}}$ be defined by

$$\bar{\mathbb{Z}} := \mathbb{Z} \ominus (S_K(\infty) \times KS_K(\infty))$$

since it follows from the definition of the set operation $\ominus$ that $\bar{\mathbb{Z}} \oplus (S_K(\infty) \times KS_K(\infty)) \subseteq \mathbb{Z}$. In the simple case when $\mathbb{Z} = \mathbb{X} \times \mathbb{U}$

$$\bar{\mathbb{Z}} = \bar{\mathbb{X}} \times \bar{\mathbb{U}} \qquad \bar{\mathbb{X}} = \mathbb{X} \ominus S_K(\infty) \qquad \bar{\mathbb{U}} = \mathbb{U} \ominus KS_K(\infty)$$

Computation of the set $S_K(\infty)$—which is known to be difficult—is not required, as we show later. It follows from the preceding discussion that if the nominal state and control trajectories $\bar{\mathbf{x}}$ and $\bar{\mathbf{u}}$ satisfy the *tightened* constraint $(\bar{x}(i), \bar{u}(i)) \in \hat{\mathbb{Z}} \subset \bar{\mathbb{Z}}$ for all $i \in \mathbb{I}_{0:N-1}$, the state and control trajectories $\mathbf{x}$ and $\mathbf{u}$ of the uncertain system then satisfy the original constraints $(x(i), u(i)) \in \mathbb{Z}$ for all $i \in \mathbb{I}_{0:N-1}$. This is the basis for tube-based robust MPC discussed next.

### 3.5.3 Tube-Based MPC of Linear Systems with Additive Disturbances

The tube-based controller has two components: (i) a nominal state-control trajectory $(\bar{x}(i), \bar{u}(i))_{i \in \mathbb{I}_{\geq 0}}$ that commences at the initial state $x$ and that satisfies the tightened constraint, and (ii) a feedback controller $u = \bar{u} + K(x - \bar{x})$ that attempts to steer the uncertain state-control trajectory to the nominal trajectory. The nominal state-control trajectory may be generated at the initial time or generated sequentially using standard MPC for deterministic systems. The latter gives more flexibility to cope with changing conditions, such as changing setpoint. Assume, then, that a controller $\bar{u} = \bar{\kappa}_N(\bar{x})$ for the nominal system $\bar{x}^+ = A\bar{x} + B\bar{u}$ has been determined using results in Chapter 2 by solving the standard optimal control problem of the form

$$\bar{\mathbb{P}}_N(\bar{x}): \quad \bar{V}_N^0(\bar{x}) = \min_{\bar{\mathbf{u}}}\{\bar{V}_N(\bar{x}, \bar{\mathbf{u}}) \mid \bar{\mathbf{u}} \in \bar{\mathcal{U}}_N(\bar{x})\}$$

$$\bar{V}_N(\bar{x}, \bar{\mathbf{u}}) = \sum_{i=0}^{N-1} \ell(\bar{x}(i), \bar{u}(i)) + V_f(\bar{x}(N))$$

$$\bar{\mathcal{U}}_N(\bar{x}) = \{\bar{\mathbf{u}} \mid (\bar{x}(i), \bar{u}(i)) \in \bar{\mathbb{Z}}, i \in \mathbb{I}_{0:N-1}, \bar{x}(N) \in \mathbb{X}_f\}$$

in which $\bar{x}(i) = \bar{\phi}(i; x, \bar{\mathbf{u}})$. Under usual conditions, the origin is asymptotically stable for the controlled nominal system described by

$$\bar{x}^+ = A\bar{x} + B\bar{\kappa}_N(\bar{x})$$

and the controlled system satisfies the constraint $(\bar{x}(i), \bar{u}(i)) \in \bar{\mathbb{Z}}$ for all $i \in \mathbb{I}_{\geq 0}$. Let $\bar{\mathcal{X}}_N$ denote the set $\{\bar{x} \mid \bar{\mathcal{U}}_N(\bar{x}) \neq \emptyset\}$. Of course, determination of the control $\bar{\kappa}_N(\bar{x})$ requires solving online the constrained optimal control problem $\mathbb{P}_N(\bar{x})$.

The feedback controller, given the state $x$ of the system being controlled, and the state $\bar{x}$ of the nominal system, generates the control $u = \bar{\kappa}_N(\bar{x}) + K(x - \bar{x})$. The composite system with state $(x, \bar{x})$ satisfies

$$x^+ = Ax + B\bar{\kappa}_N(\bar{x}) + K(x - \bar{x}) + w$$
$$\bar{x}^+ = A\bar{x} + B\bar{\kappa}_N(\bar{x})$$

The system with state $(e, \bar{x})$, $e := x - \bar{x}$, satisfies a simpler difference equation

$$e^+ = A_K e + w$$
$$\bar{x}^+ = A\bar{x} + B\bar{\kappa}_N(\bar{x})$$

The two states $(x, \bar{x})$ and $(e, \bar{x})$ are related by

$$\begin{bmatrix} e \\ \bar{x} \end{bmatrix} = T \begin{bmatrix} x \\ \bar{x} \end{bmatrix} \qquad T := \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix}$$

Since $T$ is invertible, the two systems with states $(x, \bar{x})$ and $(e, \bar{x})$ are equivalent. Hence, to establish robust stability it suffices to consider the simpler system with state $(e, \bar{x})$. First, we define robustly asymptotically stable (RAS).

**Definition 3.11** (Robust asymptotic stability of a set)**.** Suppose the sets $S_1$ and $S_2$, $S_2 \subset S_1$, are robustly positive invariant for the system $z^+ = f(z, w)$, $w \in \mathbb{W}$. The set $S_2$ is RAS for $z^+ = f(z, w)$ in $S_1$ if there exists a $\mathcal{KL}$ function $\beta(\cdot)$ such that every solution $\phi(\,\cdot\,; z, \mathbf{w})$ of $z^+ = f(z, w)$ with initial state $z \in S_1$ and any disturbance sequence $\mathbf{w} \in \mathbb{W}^\infty$ satisfies

$$|\phi(i; z, \mathbf{w})|_{S_2} \leq \beta(|z|_{S_2}, i) \;\; \forall i \in \mathbb{I}_{\geq 0}$$

In this definition, $|z|_S := d(z, S)$, the distance of $z$ from set $S$.

We now assume that $\bar{\kappa}_N(\cdot)$ and $\bar{\mathbb{Z}}$ have been determined to ensure the origin is asymptotically stable in a positive invariant set $\bar{\mathcal{X}}$ for the controlled nominal system $\bar{x}^+ = A\bar{x} + B\kappa_N(\bar{x})$. Under this assumption we have:

**Proposition 3.12** (Robust asymptotic stability of tube-based MPC for linear systems)**.** *The set $S_K(\infty) \times \{0\}$ is RAS for the composite system $(e^+ = A_K e + w, \bar{x}^+ = A\bar{x} + B\bar{\kappa}_N(\bar{x}))$ in the positive invariant set $S_K(\infty) \times \bar{\mathcal{X}}_N$.*

*Proof.* Because the origin is asymptotically stable for $\bar{x}^+ = A\bar{x} + B\bar{\kappa}_N(\bar{x})$, there exists a $\mathcal{KL}$ function $\beta(\cdot)$ such that every solution $\bar{\phi}(\,\cdot\,; \bar{x})$ of the controlled nominal system with initial state $\bar{x} \in \bar{\mathcal{X}}_N$ satisfies

$$\left|\bar{\phi}(i; \bar{x})\right| \leq \beta(|\bar{x}|, i) \;\; \forall i \in \mathbb{I}_{\geq 0}$$

Since $e(0) \in S_K(\infty)$ implies $e(i) \in S_K(\infty)$ for all $i \in \mathbb{I}_{\geq 0}$, it follows that

$$\left|(e(i), \bar{\phi}(i; \bar{x}))\right|_{S_K(\infty \times \{0\}} \leq |e(i)|_{S_K(\infty)} + \left|\bar{\phi}(i; \bar{x})\right| \leq \beta(|\bar{x}|, i)$$

Hence the set $S_K(\infty) \times \{0\}$ is RAS in $S_K(\infty) \times \bar{\mathcal{X}}_N$ for the composite system $(e^+ = A_K e + w, \bar{x}^+ = A\bar{x} + B\bar{\kappa}_N(\bar{x}))$. ■

It might be of interest to note that (see Exercise 3.4)

$$d_H(\{\bar{\phi}(i; \bar{x})\} \oplus S_K(\infty), S_K(\infty)) \leq \left|\bar{\phi}(i; \bar{x})\right| \leq \beta(|\bar{x}|, i)$$

for every solution $\bar{\phi}(\cdot)$ of the nominal system with initial state $\bar{x} \in \mathcal{X}_N$.

Finally we show how suitable tightened constraints may be determined. It was shown above that the nominal system should satisfy the tightened constraint $(\bar{x}, \bar{u}) \in \bar{\mathbb{Z}} = \mathbb{Z} \ominus (S_K(\infty), KS_K(\infty))$. Since $S_K(\infty)$ is difficult to compute and use, impossible for many process control applications, we present an alternative. Suppose $\mathbb{Z}$ is polytopic and is described by a set of scalar inequalities of the form $c'z \leq d$

($c_x'x + c_u'u \leq d$). We show next how each constraint of this form may be tightened so that satisfaction of the tightened constraint by the nominal system ensures satisfaction of original constraint by the uncertain system. For all $j \in \mathbb{I}_{\geq 0}$, let

$$\theta_j := \max_e \{c'(e, Ke) \mid e \in S_K(j)\} = \max_{\mathbf{w}} \{\sum_{i=0}^{j-1} c'(I, K)A_K^i w_i \mid \mathbf{w} \in \mathbb{W}_{0:j-1}\}$$

in which $c'(e, Ke) = c_x'e + c_u'Ke$ and $c'(I, K)A_K^i w_i = c_x'A_K^i w_i + c_u'KA_K^i w_i$. Satisfaction of the constraint $c'\bar{z} \leq d - \theta_\infty$ by the nominal system ensures satisfaction of $c'z \leq d$, $z = \bar{z} + (e, Ke)$, by the uncertain system; however, computation of $\theta_\infty$ is impractical so we adopt the approach in (Raković, Kerrigan, Kouramas, and Mayne, 2005a). Because $A_K$ is Hurwitz, for all $\alpha \in (0, 1)$ there exists a finite integer $N$ such that $A_K^N \mathbb{W} \subset \alpha\mathbb{W}$ and $KA_K^N \mathbb{W} \subset \alpha K\mathbb{W}$. It follows that

$$\theta_\infty \leq \theta_N + \alpha\theta_\infty$$

so that

$$\theta_\infty \leq (1 - \alpha)^{-1}\theta_N$$

Hence, satisfaction of the tightened constraint $c'\bar{z} \leq d - (1 - \alpha)^{-1}\theta_N$ by the nominal system ensures that the uncertain system satisfies the original constraint $c'z \leq d$. The tightened constraint set $\bar{\mathbb{Z}}$ is defined by these modified constraints.

**Example 3.13: Calculation of tightened constraints**

Consider the system

$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} + w$$

with $\mathbb{W} := \{w \mid |w|_\infty \leq 0.1\}$, $\mathbb{Z} := \{(x, u) \mid |x|_\infty \leq 1, |u| \leq 1\}$, and nominal control law $K := \begin{bmatrix} -0.4 & -1.2 \end{bmatrix}$. For increasing values of $N$, we calculate $\alpha$ such that $A_K^N \mathbb{W} \subset \alpha\mathbb{W}$ and $KA_K^N \mathbb{W} \subset \alpha K\mathbb{W}$.

Because $\mathbb{W}$ is a box, it is sufficient to check only its vertices, i.e., the four elements $w \in W := \{-0.1, 0.1\}^2$. Thus, we have

$$\alpha = \max \left( \frac{\max_{w \in W} \left| A_K^N w \right|_\infty}{\max_{w \in W} |w|_\infty}, \frac{\max_{w \in W} \left| KA_K^N w \right|_\infty}{\max_{w \in W} |Kw|_\infty} \right)$$

These values are shown in Figure 3.4. From here, we see that $N \geq 3$ is necessary for the approximation to hold. With the values of $\alpha$, the

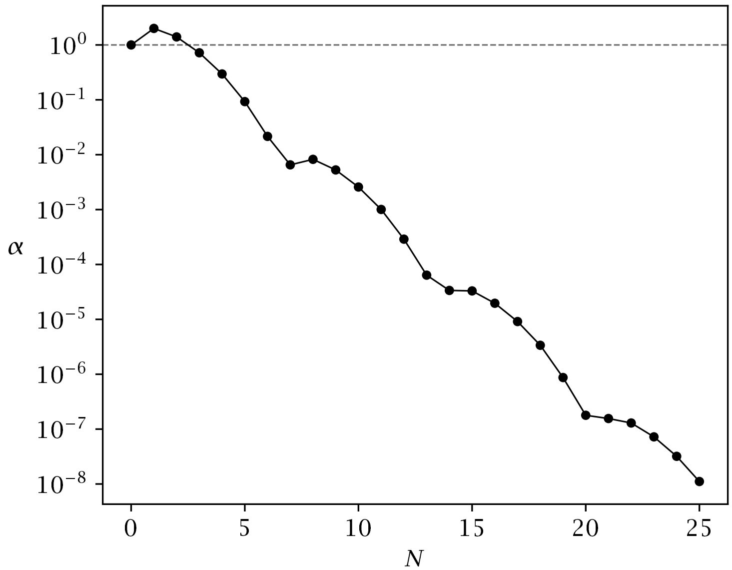



**Figure 3.4:** Minimum feasible $\alpha$ for varying $N$. Note that we require $\alpha \in [0, 1)$.

tightened constraint sets $\bar{\mathbb{Z}}$ can then be computed as above. Once again, because of the structure of $\mathbb{W}$, we need only check the vertices. Due to the symmetry of the system, each set is of the form

$$\bar{\mathbb{Z}} = \{(x, u) \mid |x_1| \leq \chi_1, |x_2| \leq \chi_2, |u| \leq \mu\}$$

The bounds $\chi_1$, $\chi_2$, and $\mu$ are shown in Figure 3.5. Note that while $N = 3$ gives a feasible value of $\alpha$, we require at least $N = 4$ for $\bar{\mathbb{Z}}$ to be nonempty. $\square$

**Time-varying constraint set $\bar{\mathbb{Z}}(i)$.** The tube-based model predictive controller is conservative in that the feasible set for $\bar{\mathbb{P}}_N(\bar{x})$ is unnecessarily small due to use of a constant constraint set $\bar{\mathbb{Z}} = \mathbb{Z} \ominus (S_K(\infty) \times KS_K(\infty))$. This reduces the region of attraction $\bar{\mathcal{X}}_N$, the set of states for which $\mathbb{P}_N(\bar{x})$ is feasible. Tube-based model predictive control can be made less conservative by using time-varying constraint set $\bar{\mathbb{Z}}(i) = \mathbb{Z} \ominus (S_K(i) \times KS_k(i))$, $i \in \mathbb{I}_{0:N-1}$ for the initial optimal control problem that generates the control sequence $\mathbf{u}^0(\bar{x})$. The control applied

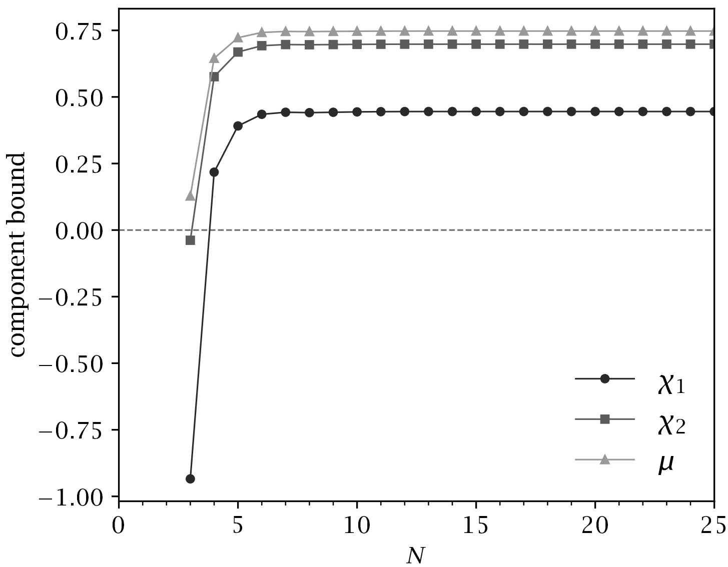



**Figure 3.5:** Bounds on tightened constraint set $\bar{\mathbb{Z}}$ for varying $N$. Bounds are $|x_1| \leq \chi_1$, $|x_2| \leq \chi_2$, and $|u| \leq \mu$.

to the uncertain system is $\bar{u}(i) + Ke(k)$; the infinite sequence $\bar{\mathbf{u}}$ is constructed as follows. The sequence $(\bar{u}(0), \bar{u}(1), \ldots, \bar{u}(N-1))$ is set equal to $\bar{\mathbf{u}}^0(\bar{x})$, the solution of the nominal optimal control problem at the initial state $\bar{x}$, with time-varying constraint sets $\bar{\mathbb{Z}}(i)$ and terminal constraint set $\bar{\mathbb{X}}(f)$. The associated state sequence is $(\bar{x}(0), \bar{x}(1), \ldots, \bar{x}(N))$ with $\bar{x}(N) \in \bar{\mathbb{X}}_f$. For $i \in \mathbb{I}_{\geq N}$, $\bar{u}(i)$ and $\bar{x}(i)$ are obtained as the solution at time $i$ of

$$\bar{x}^+ = A\bar{x} + B\kappa_f(\bar{x}), \quad u = \kappa_f(\bar{x})$$

with initial state $\bar{x}(N)$ at time $N$. We now assume that $\bar{\mathbb{X}}_f$ satisfies $\bar{\mathbb{X}}_f \oplus S_K(\infty) \subset \mathbb{X}$. Since $\bar{x}(N) \in \bar{\mathbb{X}}_f$ it follows that $\bar{x}(i) \in \bar{\mathbb{X}}_f$ and $x(i) \in \mathbb{X}$ for all $i \in \mathbb{I}_{\geq} N$. Also, for all $i \in \mathbb{I}_{0:N-1}$, $\bar{x}(i) \in \bar{\mathbb{X}}(i) = \mathbb{X} \ominus S_K(i)$ and $e(i) \in S_k(i)$ so that $x(i) = \bar{x}(i) + e(i) \in \mathbb{X}$. Hence $x(i) \in \mathbb{X}$ for all $i \in \mathbb{I}_{\geq 0}$. Since $\bar{x}(i) \to 0$, the state $x(i)$ of the uncertain system tends to $S_K(\infty)$ as $i \to \infty$. Since $\bar{\mathbb{Z}}(i) \supset \bar{\mathbb{Z}}$, the region of attraction is larger than that for tube-based MPC using a constant constraint set.

### 3.5.4 Improved Tube-Based MPC of Linear Systems with Additive Disturbances

In this section we describe a version of the tube-based model predictive controller that has pleasing theoretical properties. We omitted, in the previous section, to make use of an additional degree of freedom available to the controller, namely the ability to change the state $\bar{x}$ of the nominal system. In Chisci, Rossiter, and Zappa (2001), $\bar{x}$ is set equal to $x$, the current state of the uncertain system, but there is no guarantee that an initial state $x$ is superior to $\bar{x}$ in the sense of enhancing convergence to the origin of the nominal trajectory. To achieve more rapid convergence, we propose that an improved tube center $\bar{x}^*$ is chosen by minimizing the value function $\bar{V}_N^0(\cdot)$ of the nominal optimal control problem. It is necessary that the current state $x$ remains in the tube with new center $\bar{x}^*$. To achieve this, at state $(x, \bar{x})$, a new optimal control problem $\bar{\mathbb{P}}_N^*(x)$, is solved online, to determine an improved center $\bar{x}^*$ and, simultaneously the subsequent center $\bar{x}^+$. We assume, for simplicity, that $\mathbb{Z} = \mathbb{X} \times \mathbb{U}$ and $\bar{\mathbb{Z}} = \bar{\mathbb{X}} \times \bar{\mathbb{U}}$. The new optimal control problem $\mathbb{P}_N^*(x)$ that replaces $\bar{\mathbb{P}}_N(\bar{x})$ is defined by

$$\begin{aligned}
\mathbb{P}_N^*(x): \quad \bar{V}_N^*(x) &= \min_z \{\bar{V}_N^0(z) \mid x \in \{z\} \oplus S,\ z \in \bar{\mathbb{X}}\} \\
&= \min_{z, \bar{\mathbf{u}}} \{\bar{V}_N(z, \bar{\mathbf{u}}) \mid \bar{\mathbf{u}} \in \bar{\mathcal{U}}_N(z),\ x \in \{z\} \oplus S,\ z \in \bar{\mathbb{X}}\}
\end{aligned}$$

in which $S$ is an *inner* approximation of $S_K(\infty)$, e.g., $S = \mathbb{W}$, or $S = \sum_{i=0}^{j} \mathbb{W}^i$, with $j$ small enough to ensure the computation of $S$ is feasible. The solution to problem $\mathbb{P}_N^*(x)$ is $(\bar{x}^*(x), \bar{\mathbf{u}}^*(x))$. The constraint $x \in \{z\} \oplus S$ ensures that the current state $x$ lies in $\{\bar{x}^*(x)\} \oplus S_K(\infty)$, the first element of the "new tube." The argument of $\mathbb{P}_N^*(x)$ is $x$ because of the constraint $x \in \{z\} \oplus S$; the solution to the problem generates both the improved current nominal state $\bar{x}^*(x)$ as well as its successor $\bar{x}^+$. If $(x, \bar{x})$ satisfies $\bar{x} \in \bar{\mathcal{X}}_N$ and $x \in \{\bar{x}\} \oplus S_K(\infty)$, then $(\bar{x}, \widetilde{\mathbf{u}}(\bar{x}))$ is a warm start for $\mathbb{P}_N^*(x)$; here $\widetilde{\mathbf{u}}(\bar{x})$ is a warm start for $\bar{\mathbb{P}}_N(\bar{x})$. The successor nominal state is

$$\bar{x}^+ = (\bar{x}^*(x))^+ = A\bar{x}^*(x) + B\bar{\kappa}_N(\bar{x}^*(x))$$

in which, as usual, $\bar{\kappa}_N(\bar{x}^*(x))$ is the first element in the control sequence $\bar{\mathbf{u}}^*(x)$. It follows that

$$\bar{V}_N^*(x) = \bar{V}_N^0(\bar{x}^*(x)) \le \bar{V}_N^0(\bar{x}), \quad \bar{\mathbf{u}}^*(x) = \bar{\mathbf{u}}^0(\bar{x}^*(x))$$

The control applied to the uncertain system at state $x$ is

$$\bar{\kappa}_N^*(x) := \bar{\kappa}_N^*(x) + K(x - \bar{x}^*(x))$$

so the closed-loop uncertain system satisfies

$$x^+ = Ax + B\bar{\kappa}_N(\bar{x}^*(x)) + K(x - \bar{x}^*(x)) + w$$

and $e = x - \bar{x}^*(x)$ satisfies

$$e^+ = x^+ - (\bar{x}^*(x))^+ = Ae + BKe + w = A_K e + w$$

as before so that if $e \in S_K(\infty)$, then $e^+ \in S_K(\infty)$; hence $x \in \{\bar{x}^*(x)\} \oplus S_K(\infty)$ implies $x^+ \in \{(\bar{x}^*(x))^+\} \oplus S_K(\infty)$.

Suppose then that $\bar{x} \in \bar{\mathcal{X}}_N \subseteq \bar{\mathbb{X}}$ and $x \in \{\bar{x}\} \oplus S_K(\infty)$ so that $x \in \mathbb{X}$. If the usual assumptions for the nominal optimal control problem $\bar{\mathbb{P}}_N$ are satisfied and $\ell(\cdot)$ is quadratic and positive definite it follows that

$$\begin{aligned}
\bar{V}_N^*(x) &= \bar{V}_N^0(\bar{x}^*(x)) \geq c_1 \left|\bar{x}^*(x)\right|^2 \\
\bar{V}_N^*(x) &= \bar{V}_N^0(\bar{x}^*(x)) \leq c_2 \left|\bar{x}^*(x)\right|^2 \\
\bar{V}_N^*(x^+) &= \bar{V}_N^0(\bar{x}^*(x^+)) \leq \bar{V}_N^0((\bar{x}^*(x))^+) \leq \bar{V}_N^0(\bar{x}^*(x)) - c_1 \left|\bar{x}^*(x)\right|^2
\end{aligned}$$

The last inequality follows from the fact that $\bar{x}^+ = (\bar{x}^*(x))^+ = A\bar{x}^*(x) + B\bar{\kappa}_N^+(\bar{x}^*(x))$ and the descent property of the solution to $\bar{\mathbb{P}}_N^0(\bar{x}^*(x))$.

**Proposition 3.14** (Recursive feasibility of tube-based MPC)**.** *Suppose that at time* $0$, $(x, \bar{x}) \in (\{\bar{x}\} \oplus S_K(\infty)) \times \bar{\mathcal{X}}_N$. *Then, Problem* $\bar{\mathbb{P}}_N^*$ *is recursively feasible:* $(x, \bar{x}) \in (\{\bar{x}\} \oplus S_K(\infty)) \times \bar{\mathcal{X}}_N$ *implies* $(x, \bar{x})^+ = (x^+, \bar{x}^+) \in (\{\bar{x}^+\} \oplus S_K(\infty)) \times \bar{\mathcal{X}}_N$.

*Proof.* Suppose that $(x, \bar{x})$ satisfies $x \in \{\bar{x}\} \oplus S_K(\infty)$ and $\bar{x} \in \bar{\mathcal{X}}_N$. From the definition of $\bar{\mathbb{P}}_N^*$, any solution satisfies the tightened constraints so that $\bar{x}^*(x) \in \bar{\mathcal{X}}_N$ . The terminal conditions ensure, by the usual argument, that the successor state $\bar{x}^*(x)^+$ also lies in $\bar{\mathcal{X}}_N$. The condition $x \in \{z\} \oplus S$ in $\mathbb{P}_N^*(x)$ then implies that $x \in \{\bar{x}^*(x)\} \oplus S_K(\infty)$ so that $x^+ \in \{\bar{x}^+\} \oplus S_K(\infty)$ $(e^+ \in S_K(\infty))$. ∎

**Proposition 3.15** (Robust exponential stability of improved tube-based MPC)**.** *The set* $S_K(\infty)$ *is robustly exponentially stable in* $\bar{\mathcal{X}}_N \oplus S_K(\infty)$ *for the system* $x^+ = Ax + B\big(\bar{\kappa}_N(\bar{x}^*(x)) + K(x - \bar{x}^*(x))\big)$.

*Proof.* It follows from the upper and lower bounds on $\bar{V}_N^0(x^*(x))$, and the descent property listed above that

$$\bar{V}_N^0(x^*(x^+)) \leq \gamma \bar{V}_N^0(x^*(x))$$

with $\gamma = (1 - c_1/c_2) \in (0, 1)$. Hence, if $x(i)$ denotes the solution at time $i$ of $x^+ = Ax + B(\bar{\kappa}_N(\bar{x}^*(x)) + K(x - x^*(x))) + w$, $\bar{V}_N^0(\bar{x}^*(x(i)))$

decays exponentially fast to zero. It then follows from the upper bound on $\bar{V}_N^0(\bar{x}^*(x))$ that $x^*(x(i))$ also decays exponentially to zero. Because $x(i) \in \{\bar{x}^*(x(i))\}$ for all $i \in \mathbb{I}_{\geq 0}$, it follows, similarly to the proof of Proposition 3.12, that the set $S_K(\infty)$ is robustly exponentially stable in $\bar{\mathcal{X}}_N \oplus S_K(\infty)$ for the system $x^+ = Ax + B\bar{\kappa}_N(\bar{x}^*(x)) + K(x - \bar{x}^*(x)) + w$. ■

## 3.6 Tube-Based MPC of Nonlinear Systems

Satisfactory control in the presence of uncertainty requires feedback. As shown in Section 3.5, MPC of uncertain systems ideally requires optimization over control policies rather than control sequences, resulting in an optimal control problem that is often impossibly complex. Practicality demands simplification. Hence, in tube-based MPC of constrained *linear* systems we replace the general control policy $\boldsymbol{\mu} = (\mu_0(\cdot), \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot))$, in which each element $\mu_i(\cdot)$ is an arbitrary function, by the simpler policy $\boldsymbol{\mu}$ in which each element has the simple form $\mu_i(x) = \bar{u}(i) + K(x - \bar{x}(i))$; $\bar{u}(i)$ and $\bar{x}(i)$, the control and state of the nominal system at time $i$, are determined using conventional MPC.

The feedback gain $K$, which defines the local control law, is determined offline; it can be chosen so that all possible trajectories of the uncertain system lie in a tube centered on the nominal trajectory $(\bar{x}(0), \bar{x}(1), \ldots)$. The "cross section" of the tube is a constant set $S_K(\infty)$ so that every possible state of the uncertain system at time $i$ lies in the set $\{\bar{x}(i)\} \oplus S_K(\infty)$. This enables the nominal trajectory to be determined using MPC, to ensure that all possible trajectories of the uncertain system satisfy the state and control constraints, and that all trajectories converge to an invariant set centered on the origin.

It would be desirable to extend this methodology to the control of constrained nonlinear systems, but we face some formidable challenges. It is possible to define a nominal system and, as shown in Chapter 2, to determine, using MPC with "tightened" constraints, a nominal trajectory that can serve as the center of a tube. But it seems to be prohibitively difficult to determine a local control law that steers all trajectories of the uncertain system toward the nominal trajectory, and of a set centered on the nominal trajectory in which these trajectories can be guaranteed to lie.

We can overcome these difficulties by first generating a nominal trajectory—either by MPC as in the linear case or by a single solution of an optimal control problem—and then using MPC to steer the state of the uncertain system toward the nominal trajectory $\bar{x}(\cdot)$. The latter MPC controller replaces the linear controller $u = \bar{u} + K(x - \bar{x})$ employed in the linear case, and thereby avoids the difficulty of determining a local nonlinear version of this linear controller. The value function $(x, i) \mapsto V_N^0(x, i)$ of the optimal control problem that is used to determine the MPC controller is time varying and has the property that $V_N^0(\bar{x}(i), i) = 0$ for all $i$. The tube is now a sequence of sublevel sets $\left(\text{lev}_c\, V_N^0(\cdot, i)\right)_{i \in \mathbb{I}_{\geq 0}}$ and therefore, unlike the linear case, has a varying cross section. We show that if the initial state $x(0)$ lies in $\text{lev}_c\, V_N^0(\cdot, 0)$, then subsequent states $x(i)$ of the controlled system lie in $\text{lev}_c\, V_N^0(\cdot, i)$ for all $i \in \mathbb{I}_{\geq 0}$.

The system to be controlled is described by a nonlinear difference equation

$$x^+ = f(x, u, w) \tag{3.20}$$

in which the disturbance $w$ is assumed to lie in the compact set $\mathbb{W}$ that contains the origin. The state $x$ and the control $u$ are required to satisfy the constraints

$$x \in \mathbb{X} \qquad u \in \mathbb{U}$$

Both $\mathbb{X}$ and $\mathbb{U}$ are assumed to be compact and to contain the origin in their interiors. The solution of (3.20) at time $i$, if the initial state at time 0 is $x_0$ and the control is generated by policy $\boldsymbol{\mu}$, is $\phi(i; x_0, \boldsymbol{\mu}, \mathbf{w})$, in which $\mathbf{w}$ denotes, as usual, the disturbance sequence $(w(0), w(1), \ldots)$. Similarly, $\phi(i; x_0, \kappa, \mathbf{w})$ denotes the solution of (3.20) at time $i$, if the initial state at time 0 is $x$ and the control is generated by a time-invariant control law $\kappa(\cdot)$.

The nominal system is obtained by neglecting the disturbance $w$ and is therefore described by

$$\bar{x}^+ = \bar{f}(\bar{x}, \bar{u}) := f(\bar{x}, \bar{u}, 0)$$

Its solution at time $i$, if its initial state is $\bar{x}_0$, is denoted by $\bar{\phi}(i; \bar{x}_0, \bar{\mathbf{u}})$, in which $\bar{\mathbf{u}} := (\bar{u}(0), \bar{u}(1), \ldots)$ is the nominal control sequence. The deviation between the actual and nominal state is $e := x - \bar{x}$ and satisfies

$$e^+ = f(x, u, w) - f(\bar{x}, \bar{u}, 0) = f(x, u, w) - \bar{f}(\bar{x}, \bar{u})$$

Because $f(\cdot)$ is nonlinear, this difference equation cannot be simplified as in the linear case when $e^+$ is independent of $x$ and $\bar{x}$, and depends only on their difference $e$ and $w$.

### 3.6.1 The Nominal Trajectory

The nominal trajectory is a feasible trajectory for the nominal system that is sufficiently far from the boundaries of the original constraints to enable the model predictive controller for the uncertain system to satisfy these constraints. It is generated by the solution to a nominal optimal control problem $\bar{\mathbb{P}}_N(\bar{x})$ in which $\bar{x}$ is the state of the nominal system. The cost function $\bar{V}_N(\cdot)$ for the nominal optimal control problem is defined by

$$\bar{V}_N(\bar{x}, \bar{\mathbf{u}}) := \sum_{i=0}^{N-1} \ell(\bar{x}(i), \bar{u}(i)) \tag{3.21}$$

in which $\bar{x}(i) = \bar{\phi}(i; \bar{x}, \bar{\mathbf{u}})$ and $\bar{x}$ is the initial state. The function $\ell(\cdot)$ is defined by

$$\ell(\bar{x}, \bar{u}) := (1/2)\big(\,|\bar{x}|_Q^2 + |\bar{u}|_R^2\,\big)$$

in which $Q$ and $R$ are positive definite, $|\bar{x}|_Q^2 := \bar{x}^T Q \bar{x}$, and $|\bar{u}|_R^2 := \bar{u}^T Q \bar{u}$. We impose the following state and control constraints on the nominal system

$$\bar{x} \in \bar{\mathbb{X}} \qquad \bar{u} \in \bar{\mathbb{U}}$$

in which $\bar{\mathbb{X}} \subset \mathbb{X}$ and $\bar{\mathbb{U}} \subset \mathbb{U}$. The choice of $\bar{\mathbb{X}}$ and $\bar{\mathbb{U}}$ is more difficult than in the linear case because it is difficult to bound the deviation $e = x - \bar{x}$ of the state $x$ of the uncertain system from the state $\bar{x}$ of the nominal system; this is discussed below. The optimal nominal trajectories $\bar{\mathbf{u}}^0$ and $\bar{\mathbf{x}}^0$ are determined by minimizing $\bar{V}_N(\bar{x}_0, \bar{\mathbf{u}})$ subject to: $\bar{x}_0 = x_0$, the state and control constraints specified above, and the terminal constraint $\bar{x}(N) = 0$ (we omit the initial state $\bar{x}_0 = x_0$ in $\bar{\mathbf{u}}^0$ and $\bar{\mathbf{x}}^0$ to simplify notation). The state and control of the nominal system satisfy $\bar{x}(i) = 0$ and $\bar{u}(i) = 0$ for all $i \geq N$. This simplifies both analysis and implementation in that the control reverts to conventional MPC for all $i \geq N$.

### 3.6.2 Model Predictive Controller

The purpose of the model predictive controller is to maintain the state of the uncertain system $x^+ = f(x, u, w)$ close to the trajectory of the nominal system. This controller replaces the controller $u = v + K(x - z)$ employed in the linear case. Given the current state/time $(x, t)$ of the uncertain system, we determine a control sequence that minimizes with respect to the control sequence $\mathbf{u}$, the cost over a horizon $N$ of the deviation between the state and control of the *nominal* system, with initial state $x$ and control sequence $\mathbf{u}$, and the state and control of the *nominal* system, with initial state $\bar{x}^0(t)$ and control sequence $\bar{\mathbf{u}}_t^0 := (\bar{u}^0(t), \bar{u}^0(t+1), \ldots, \bar{u}^0(t+N-1))$. The cost $V_N(x, t, \mathbf{u})$ that measures the distance between these two trajectories is defined by

$$V_N(x, t, \mathbf{u}) := \sum_{i=0}^{N-1} \ell((x(i) - \bar{x}^0(t+i)), (u(i) - \bar{u}^0(t+i))) + V_f(x(N)) \tag{3.22}$$

in which $x(i) = \bar{\phi}(i; x, \mathbf{u})$. The optimal control problem $\mathbb{P}_N(x, t)$ solved online is defined by

$$V_N^0(x, t) := \min_{\mathbf{u}}\{V_N(x, t, \mathbf{u}) \mid \mathbf{u} \in \mathbb{U}^N\}$$

The *only* constraint in $\mathbb{P}_N(x, t)$ is the control constraint. The control applied to the uncertain system is $\kappa_N(x, t)$, the first element of $\mathbf{u}^0(x, t) = (u^0(0; x, t), u^0(1; x, t), \ldots, u^0(N-1; x, t))$, the optimizing control sequence. The associated optimal state sequence is $\mathbf{x}^0(x, t) = (x^0(0; x, t), \ldots, x^0(1; x, t), \ldots, x^0(N-1; x, t))$. The terminal penalty $V_f(\cdot)$, and the functions $\bar{f}(\cdot)$ and $\ell(\cdot)$ are assumed to satisfy the usual assumptions 2.2, 2.3, and 2.14 for the nominal system $\bar{x}^+ = \bar{f}(\bar{x}, \bar{u})$. In addition, $f : x \mapsto f(x, t, u)$ is assumed to be Lipschitz continuous for all $x \in \mathbb{R}^n$, uniformly in $(t, u) \in \mathbb{I}_{0:N} \times \mathbb{U}$, and $\ell(\cdot)$ is assumed to be quadratic and positive definite. Also, the linearization of $\bar{f}(\cdot)$ at $(0, 0)$ is assumed to be stabilizable.

We first address the problem that $\mathbb{P}_N(\cdot)$ has no terminal constraint. The function $V_f'(\cdot)$ and associated controller $\kappa_f(\cdot)$ is chosen, as in Section 2.5.5, to be a local Lyapunov function for the nominal system $\bar{x}^+ = \bar{f}(\bar{x}, \bar{u})$. The terminal cost $V_f(\cdot)$ is set equal to $\beta V_f'(\cdot)$ with $\beta$ chosen as shown in the following proposition. The associated terminal constraint $\mathbb{X}_f := \{x \mid V_f'(x) \leq \alpha\}$ for some $\alpha > 0$ is not employed in the optimal control problem, but is needed for analysis. For any state sequence $\mathbf{x}$ let $\mathbf{X}_c(\bar{\mathbf{x}})$ denote the tube (sequence of sets) $(X_0^c(\bar{\mathbf{x}}), X_1^c(\bar{\mathbf{x}}), \ldots)$ in which the $i$th element of the sequence is $X_i^c(\bar{\mathbf{x}}) := \{x \mid V_N^0(x, i) \leq c\}$. The tube $\mathbf{X}_d(\bar{\mathbf{x}})$ is similarly defined.

**Proposition 3.16** (Implicit satisfaction of terminal constraint)**.** *For all $c > 0$ there exists a $\beta_c := c/\alpha$ such that, for any $i \in \mathbb{I}_{\geq 0}$ and any $x \in X_i^c(\bar{\mathbf{x}}^0)$, the terminal state $x^0(N; x_0, i)$ lies in $\mathbb{X}_f$ if $\beta \geq \beta_c$.*

*Proof.* Since $x \in X_i^c(\bar{\mathbf{x}}^0)$ implies $V_N^0(x, i) \leq c$, we know $V_f(x^0(N; x, i)) = \beta V_f'(x^0(N; x, i)) \leq c$ so that $x^0(N; x_0, i) \in \mathbb{X}_f$ if $\beta \geq \beta_c$. ∎

Proposition 3.16 shows that the constraint that the terminal state lies in $\mathbb{X}_f$ is implicitly satisfied if $\beta \geq \beta_c$ and the initial state lies in $X_i^c(\bar{\mathbf{x}}^0)$ for any $i \in \mathbb{I}_{\geq 0}$. The next Proposition establishes important properties of the value function $V_N^0(\cdot)$.

**Proposition 3.17** (Properties of the value function)**.** *Suppose* $\beta \geq \beta_c$*. There exist constants* $c_1 > 0$ *and* $c_2 > 0$ *such that*

*(a)* $V_N^0(x,t) \geq c_1 \left| x - \bar{x}^0(t) \right|^2 \;\; \forall (x,t) \in \mathbb{R}^n \times \mathbb{I}_{\geq 0}$

*(b)* $V_N^0(x,t) \leq c_2 \left| x - \bar{x}^0(t) \right|^2 \;\; \forall (x,t) \in \mathbb{R}^n \times \mathbb{I}_{\geq 0}$

*(c)* $V_N^0((x,t)^+) \leq V_N^0(x,t) - c_1 \left| x - \bar{x}^0(t) \right|^2 \;\; \forall (x,t) \in X_i^c(\bar{x}^0) \times \mathbb{I}_{\geq 0}$

*in which* $(x,t)^+ = (x^+, t^+) = (\bar{f}(x, \kappa_N(x,t)), t+1)$*.*

It should be recalled that $\bar{x}^0(t) = 0$ and $\bar{u}^0(t) = 0$ for all $t \geq N$; the controller reverts to conventional MPC for $t \geq N$.

*Proof.*

(a) This follows from the fact that $V_N^0(x,t) \geq \ell(x - \bar{x}^0(t), u - \bar{u}^0(t))$ so that, by the assumptions on $\ell(\cdot)$, $V_N^0(x,t) \geq c_1 \left| x - \bar{x}^0(t) \right|^2$ for all $(x,t) \in \mathbb{R}^n \times \mathbb{I}_{\geq 0}$.

(b) We have that $V_N^0(x,t) = V_N(x, \mathbf{u}^0(x,t)) \leq V_N(x, \bar{\mathbf{u}}_t^0)$ with

$$V_N(x, \bar{\mathbf{u}}_t^0) = \sum_{i=0}^{N-1} \ell(x^0(i;x,t) - \bar{x}^0(t+i), 0) + V_f(x^0(N;x,t) - \bar{x}^0(t+N))$$

and $\bar{\mathbf{u}}_t^0 := (\bar{u}^0(t), \bar{u}^0(t+1), \bar{u}^0(t+2), \ldots)$. Lipschitz continuity of $f(\cdot)$ in $x$ gives $\left| \bar{\phi}(i;x,\bar{\mathbf{u}}_t^0) - \bar{x}^0(i+t) \right| \leq L^i \left| x - \bar{x}^0(t) \right|$. Since $\ell(\cdot)$ and $V_f(\cdot)$ are quadratic, it follows that $V_N^0(x,t) \leq c_2 \left| x - \bar{x}^0(t) \right|^2$ for all $(x,t) \in \mathbb{R}^n \times \mathbb{I}_{\geq 0}$, for some $c_2 > 0$.

(c) It follows from Proposition 3.16 that the terminal state $x^0(N;x,t) \in \mathbb{X}_f$ so that the usual stabilizing condition is satisfied and

$$V_N^0((x,t)^+) \leq V_N^0(x,t) - \ell(x, \kappa_N(x,t))$$

The desired result follows from the lower bound on $\ell(\cdot)$. ■

It follows that the origin is asymptotically stable in the tube $\mathbf{X}_c(\bar{\mathbf{x}}^0)$ for the time-varying nominal system $(x,i)^+ = \bar{f}(x, \kappa_N(x,i))$. However, our main interest is the behavior of the uncertain system with the controller $\kappa_N(\cdot)$. Before proceeding, we note that the tube $\mathbf{X}_c(\bar{\mathbf{x}}^0)$ is a "large" neighborhood of $\bar{\mathbf{x}}^0$ in the sense that any state/time $(x, i)$ in this set can be controlled to $\mathbb{X}_f$ in $N - i$ steps by a control subject only to the control constraint. We wish to determine, if possible, a "small" neighborhood $\mathbf{X}_d(\bar{\mathbf{x}})$ of $\bar{\mathbf{x}}^0$, $d < c$, in which the trajectories of the uncertain system are contained by the controller $\kappa_N(\cdot)$. The size of these neighborhoods, however, are dictated by the size of the disturbance set $\mathbb{W}$ as we show next.

**Proposition 3.18** (Neighborhoods of the uncertain system)**.** *Suppose $\beta \geq \beta_c$.*

*(a) $V_N^0((x,t)^+) \leq \gamma V_N^0(x,t)$ for all $(x,t) \in X_t^d(\bar{\mathbf{x}}^0) \times \mathbb{I}_{\geq 0}$, with $(x,t)^+ = (x^+, t^+) = (f(x, \kappa_N(x,t), 0), t+1)$ and $\gamma := 1 - c_1/c_2 \in (0,1)$.*

*(b) $x \mapsto V_N^0(\,\cdot\,; t)$ is Lipschitz continuous with Lipschitz constant $c_3 > 0$ in the compact set $X_t^c(\bar{\mathbf{x}}^0) = \{x \mid V_N^0(x,t) \leq c\}$ for all $t \in \mathbb{I}_{0:N}$.*

*(c) $V_N^0(f(x, \kappa_N(x,t), w), t+1) \leq \gamma V_N^0(x,t) + c_3 |w|$ for all $(x,t) \in (X_i^c(\bar{\mathbf{x}}^0) \oplus \mathbb{W}) \times \mathbb{I}_{\geq 0}$.*

*Proof.*

(a) This inequality follows directly from Proposition 3.17.

(b) This follows, as shown in Theorem C.29 in Appendix C, from the fact that $x \mapsto V_N^0(x,t)$ is Lipschitz continuous on bounded sets for each $t \in \mathbb{I}_{0:N}$, since $V_N(\cdot)$ is Lipschitz continuous on bounded sets and $\mathbf{u}$ lies in the compact set $\mathbb{U}^N$.

(c) The final inequality follows from (a), (b), and Proposition 3.17. ■

**Proposition 3.19** (Robust positive invariance of tube-based MPC for nonlinear systems)**.**

*(a) Suppose $\beta \geq \beta_c$ and $V_N^0(x,t) \leq d < c$ ($x \in X_t^d(\bar{\mathbf{x}}^0)$), then $V_N^0(x,t)^+ \leq d$ ($x \in X_{t+1}^d(\bar{\mathbf{x}}^0)$) with $(x,t)^+ = (x^+, t^+) = (f(x, \kappa_N(x,t), w), t+1)$ if $d \geq (c_3/(1-\gamma))\,|\mathbb{W}|$, $|\mathbb{W}| := \max_w\{|w| \mid w \in \mathbb{W}\}$.*

*(b) Suppose $\varepsilon > 0$. Then $V_N^0(x,t)^+ \leq V_N^0(x,t) - \varepsilon$ if $V_N^0(x) \geq d_\varepsilon := (c_3/(1-\gamma))\mathbb{W} + \varepsilon$.*

*Proof.*

(a) It follows from Proposition 3.17 that

$$V_N^0(f(x, \kappa_N(x,t), w), t+1) \leq \gamma d + c_3 |w|$$

If $d \geq (c_3/(1-\gamma))\,|\mathbb{W}|$, then

$$V_N^0(f(x, \kappa_N(x,t), w), t+1) \leq [(\gamma c_3)/(1-\gamma) + c_3]\,|\mathbb{W}| \leq [c_3/(1-\gamma)]\,|\mathbb{W}|$$

(b) $V_N^0(f(x, \kappa_N(x,t), w) \leq V_N^0(x) - \varepsilon$ if $\gamma V_N^0(x) + c_3 \mathbb{W} \leq V_N^0(x) - \varepsilon$, i.e., if $V_N^0(x) \geq [c_3/(1-\gamma)]\mathbb{W} + \varepsilon$. ■

These results show that—provided the inequalities $c \geq (c_3/(1-\gamma))\,|\mathbb{W}|$ and $d \geq (c_3/(1-\gamma))\,|\mathbb{W}|$ are satisfied—the tubes $\mathbf{X}_c(\bar{\mathbf{x}}^0)$ and $\mathbf{X}_d(\bar{\mathbf{x}}^0)$ are robustly positive invariant for $(x,t)^+ = (f(x, \kappa_N(x,t)), t+1)$, $w \in \mathbb{W}$ in the sense that if $x \in X_t^c(\bar{\mathbf{x}}^0)$ ($x \in X_t^d(\bar{\mathbf{x}}^0)$), then $x^+ \in X_{t+1}^c(\bar{\mathbf{x}}^0)$ ($x^+ \in X_{t+1}^d(\bar{\mathbf{x}}^0)$). The tubes $\mathbf{X}_c(\bar{\mathbf{x}}^0)$ and $\mathbf{X}_d(\bar{\mathbf{x}}^0)$ may be regarded as analogs of the sublevel sets $\mathrm{lev}_c\, V_N^0(\cdot)$ and $\mathrm{lev}_d\, V_N^0(\cdot)$ for time-invariant systems controlled by conventional MPC. If $d = d_\varepsilon$ and $c$ is large (which implies $\beta = c/\alpha$ is large), are such that tube $\mathbf{X}_d(\bar{\mathbf{x}}^0) \subset \mathbf{X}_c(\bar{\mathbf{x}}^0)$, then any trajectory commencing at $x \in X_t^c(\bar{\mathbf{x}}^0)$ converges to the tube $\mathbf{X}_d(\bar{\mathbf{x}}^0)$ in finite time and thereafter remains in the tube $\mathbf{X}_d(\bar{\mathbf{x}}^0)$. It follows that $d_H(X_i^d(\bar{\mathbf{x}}^0), X_N^c(\bar{\mathbf{x}}^0)$ becomes zero when $i$ exceeds some finite time not less than $N$.

### 3.6.3 Choosing the Nominal Constraint Sets $\bar{\mathbb{U}}$ and $\bar{\mathbb{X}}$

The first task is to choose $d$ as small as possible given the constraint $d \geq d_\varepsilon$, and to choose $c$ large. If the initial state $x_0$ lies in $X_0^c(\bar{\mathbf{x}}^0)$ (this can be ensured by setting $\bar{x}_0 = x_0$), then *all* state trajectories of the uncertain system lie in the tube $\mathbf{X}_c(\bar{\mathbf{x}}^0)$ and converge to the tube $\mathbf{X}_d(\bar{\mathbf{x}}^0)$. As $d \to 0$, the tube $\mathbf{X}_d(\bar{\mathbf{x}}^0)$ shrinks to the nominal trajectory $\bar{\mathbf{x}}^0$. If $d$ is sufficiently small, and if $\bar{\mathbb{X}}$ is a sufficiently small subset of $\mathbb{X}$, all state trajectories of the uncertain system lie in the state constraint set $\mathbb{X}$. This is, of course, a consequence of the fact that the nominal trajectory $\bar{\mathbf{x}}^0$ lies in the tightened constraint set $\bar{\mathbb{X}}$.

The set $\bar{\mathbb{U}}$ is chosen next. Since $\mathbb{U}$ is often a box constraint, a simple choice would be $\bar{\mathbb{U}} = \theta\mathbb{U}$ with $\theta \in (0,1)$. This choice determines how much control is devoted to controlling $\bar{x}_0$ to 0, and how much to reduce the effect of the disturbance $w$. It is possible to change this choice online.

The main task is to choose $\bar{\mathbb{X}}$. This can be done as follows. Assume that $\mathbb{X}$ is defined by a set of inequalities of the form $g_i(x) \leq h_i$, $i \in \mathbb{I}_{1:J}$. Then $\bar{\mathbb{X}}$ may be defined by the set of "tightened" inequalities $g_i(x) \leq \alpha_i h_i$, $i \in \mathbb{I}_{1:J}$, in which each $\alpha_i \in (0,1)$. Let $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_J)$. Then the "design parameter" $\alpha$ is chosen to satisfy the constraint that the state trajectory of the controlled uncertain system lies in $\mathbb{X}$ for all $x_0 \in$

$\mathcal{X}_0$ (the set of potential initial states), and all disturbance sequences $\mathbf{w} \in \mathbb{W}^N$. This is a complex semi-infinite optimization problem, but can be solved offline using recent results in Monte Carlo optimization that show the constraints can be satisfied with "practical certainty," i.e., with probability exceeding $1 - \beta$, $\beta << 1$, using a manageable number of random samples of $\mathbf{w}$.

**Example 3.20: Robust control of an exothermic reaction**

Consider the control of a continuous-stirred-tank reactor. We use a model derived in Hicks and Ray (1971) and modified by Kameswaran and Biegler (2006). The reactor is described by the second-order differential equation

$$\begin{aligned}
\dot{x}_1 &= (1/\theta)(1 - x_1) - kx_1 \exp(-M/x_2) \\
\dot{x}_2 &= (1/\theta)(x_f - x_2) + kx_1 \exp(-M/x_2) - \alpha u(x_2 - x_c) + w
\end{aligned}$$

in which $x_1$ is the product concentration, $x_2$ is the temperature, and $u$ is the coolant flowrate. The model parameters are $\theta = 20$, $k = 300$, $M = 5$, $x_f = 0.3947$, $x_c = 0.3816$, and $\alpha = 0.117$. The state, control, and disturbance constraint sets are

$$\begin{aligned}
\mathbb{X} &= \{x \in \mathbb{R}^2 \mid x_1 \in [0, 2],\ x_2 \in [0, 2]\} \\
\mathbb{U} &= \{u \in \mathbb{R} \mid u \in [0, 2]\} \\
\mathbb{W} &= \{w \in \mathbb{R} \mid w \in [-0.001, 0.001]\}
\end{aligned}$$

The controller is required to steer the system from a locally stable steady state $x(0) = (0.9831, 0.3918)$ at time 0, to a locally unstable steady state $z_e = (0.2632, 0.6519)$. Because the desired terminal state is $z_e$ rather than the origin, the stage cost $\ell(z, v)$ is replaced by $\ell(z - z_e, v - v_e)$ where $\ell(z, v) := (1/2)(|z|^2 + v^2)$ and $(z_e, v_e)$ is an equilibrium pair satisfying $z_e = f(z_e, v_e)$; the terminal constraint set $\mathbb{Z}_f$ is chosen to be $\{z_e\}$. The constraint sets for the nominal control problem are $\mathbb{Z} = \mathbb{X}$ and $\mathbb{V} = [0.02, 2]$. Since the state constraints are not activated, there is no need to tighten $\mathbb{X}$. The disturbance is chosen to be $w(t) = A \sin(\omega t)$ where $A$ and $\omega$ are independent uniformly distributed random variables, taking values in the sets $[0, 0.001]$ and $[0, 1]$, respectively. The horizon length is $N = 40$ and the sample time is $\Delta = 3$ giving a horizon time of 120. The model predictive controller uses $\ell_a(x, u) = (1/2)(|x|^2 + u^2)$, and the same horizon and sample time.

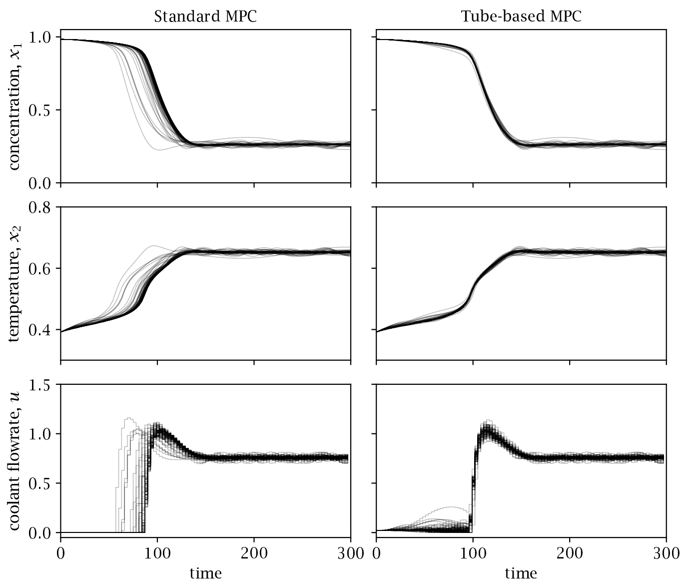



**Figure 3.6:** Comparison of 100 realizations of standard and tube-based MPC for the chemical reactor example.

For comparison, the performance of a standard MPC controller, using the same stage cost and the same terminal constraint set as that employed in the central-path controller, is simulated. Figure 3.6 (left) illustrates the performance of standard MPC, and Figure 3.6 (right) the performance of tube-based MPC for 100 realizations of the disturbance sequence. Tube-based MPC, as expected, has a smaller spread of state trajectories than is the case for standard MPC. Because each controller has the same stage cost and terminal constraint, the spread of trajectories in the steady-state phase is the same for the two controllers. Because the control constraint set for the central-path controller is tighter than that for the standard controller, the tube-based controller is somewhat slower than the standard controller.

The model predictive controller may be tuned to reduce more effectively the spread of trajectories due to the external disturbance. The main purpose of the central-path controller is to steer the system from one equilibrium state to another, while the purpose of the ancillary model predictive controller is to reduce the effect of the disturbance. These different objectives may require different stage costs.

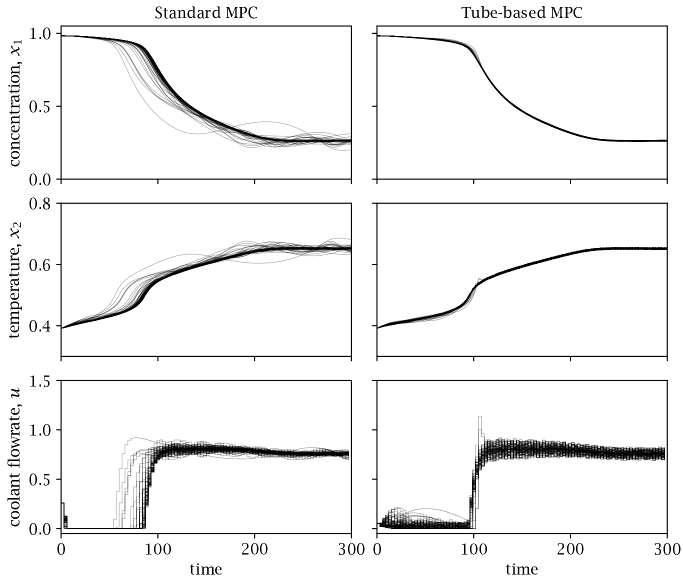



**Figure 3.7:** Comparison of standard and tube-based MPC with an aggressive model predictive controller.

The next simulation compares the performance of the standard and tube-based MPC when a more "aggressive" stage cost is employed for the model predictive controller. Figure 3.7 shows the performance of these two controllers when the central-path and standard MPC controller employ $\ell(z - z_e, v - v_e)$ with $\ell(z, v) := (1/2)\,|z|^2 + 5v^2$, and the ancillary model predictive controller employs $\ell_a(x, u) = 50\,|x|^2 + (1/20)u^2$. The tube-based MPC controller reduces the spread of the trajectories during both the transient *and* the steady-state phases.

It is also possible to tune the sample time of the ancillary model predictive controller. This feature may be useful when the disturbance frequency lies outside the pass band of the central-path controller. Figure 3.8 shows how concentration varies with time when the disturbance is $w(t) = 0.002\sin(0.4t)$, the sample time of the central-path controller is 12, whereas the sample time of the ancillary model predictive controller is 12 (left figure) and 8 (right figure). The central-path controller employs $\ell(z - z_e, v - v_e)$ where $\ell(z, v) := (1/2)(|z|^2 + v^2)$, and the model predictive controller employs the same stage cost $\ell_a(x,$

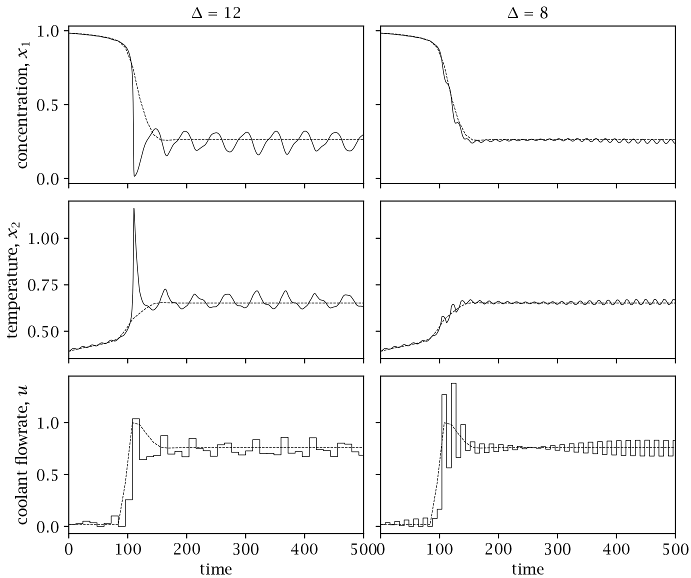



**Figure 3.8:** Concentration versus time for the ancillary model predictive controller with sample time $\Delta = 12$ (left) and $\Delta = 8$ (right).

$u) = \ell(x, u)$. The model predictive controller with the smaller sample time is more effective in rejecting the disturbance. □

## 3.7 Stochastic MPC

### 3.7.1 Introduction

In stochastic MPC, as in robust MPC, the system to be controlled is usually described by $x^+ = f(x, u, w)$, in which the disturbance $w$ is a random variable that is assumed to take values in $\mathbb{W}$. The constraint set $\mathbb{W}$ is not necessarily assumed to be bounded as it is in robust MPC. The decision variable $\boldsymbol{\mu}$ is usually assumed, as in robust MPC, to be a policy $\boldsymbol{\mu} = (\mu_0(\cdot), \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot))$, a sequence of control laws to contain the spread of trajectories that may result in a high cost and constraint violation. The functions $\mu_i(\cdot)$ are usually parameterized to simplify optimization. A parameterization that is widely used when the system being controlled is linear is $\mu_i(x) = v_i + Kx$, in which case the decision variable is simply the sequence $\mathbf{v} = (v_0, v_1, \ldots, v_{N-1})$. Let $\phi(i; x, \boldsymbol{\mu}, \mathbf{w})$ denote the solution of $x^+ = f(x, u, w)$ at time $i$ if the initial state at time 0 is $x$, the control at $(x, i)$ is $\mu_i(x)$, and the disturbance sequence is $\mathbf{w}$.

The cost that is minimized online is usually defined to be

$$V_N(x, \boldsymbol{\mu}) = E_{|x} J_N(x, \boldsymbol{\mu}, \mathbf{w})$$

$$J_N(x, \mathbf{u}, \mathbf{w}) = \sum_{i=0}^{N-1} \ell(\phi(i; x, \boldsymbol{\mu}, \mathbf{w}), \mu_i(\phi(i; x, \boldsymbol{\mu}, \mathbf{w}))) + V_f(\phi(N; x, \boldsymbol{\mu}, \mathbf{w}))$$

in which $E_{|x}(\cdot) = E(\cdot\,|x)$ and $E$ is expectation under the probability measure of the underlying probability space. For simplicity, the nominal cost $V_N(x, \boldsymbol{\mu}) = E_{|x} J_N(x, \boldsymbol{\mu}, \mathbf{0})$ is sometimes employed; here $\mathbf{0}$ is defined to be the sequence $(0, 0, \ldots, 0)$.

Because the random disturbance may be unbounded, it is not necessarily possible to satisfy the usual constraints $x \in \mathbb{X}$, $u \in \mathbb{U}$, and $x(N) \in \mathbb{X}_f$ almost surely; it is necessary to "soften" the constraints. Two methods for softening the constraints have been proposed. In the first method, hard constraints of the form $\phi(i; x, \boldsymbol{\mu}, \mathbf{w}) \in \mathbb{X}$ are replaced by average constraints of the form $E_{|x}\phi(i; x, \boldsymbol{\mu}, \mathbf{w}) \in \mathbb{X}$. In the second—more widely used—method a constraint of the form $\phi(i; x, \boldsymbol{\mu}, \mathbf{w}) \in \mathbb{X}$ is replaced by the probabilistic constraint

$$\mathcal{P}_{|x}\{\phi(i; x, \boldsymbol{\mu}, \mathbf{w}) \in \mathbb{X}\} \geq 1 - \varepsilon$$

for some suitably small $\varepsilon \in (0, 1)$; $\mathcal{P}$ denotes the probability measure of the underlying probability space. Hence the constraints in the optimal control problem $\mathbb{P}_N(x)$ solved online take the form

$$\begin{aligned} \mathcal{P}_{|x}\{\mu_i(\phi(i; x, \boldsymbol{\mu}, \mathbf{w})) \in \mathbb{U}\} &\geq 1 - \varepsilon \\ \mathcal{P}_{|x}\{\phi(i; x, \boldsymbol{\mu}, \mathbf{w}) \in \mathbb{X}\} &\geq 1 - \varepsilon \\ \mathcal{P}_{|x}\{\phi(N; x, \boldsymbol{\mu}, \mathbf{w}) \in \mathbb{X}_f\} &\geq 1 - \varepsilon \end{aligned}$$

for all $i \in \mathbb{I}_{0:N}$ *if* low probability of transgression of the control constraints is permissible. Let $\Pi_N(x)$ denote the set of parameterized policies that satisfy these constraints if the initial state is $x$.

The optimal control problem $\mathbb{P}_N(x)$ that is solved online can now be defined

$$\mathbb{P}_N(x): \qquad V_N^0(x) = \min_{\boldsymbol{\mu} \in \Pi_N(x)} V_N(x, \boldsymbol{\mu})$$

subject to the probabilistic constraints defined above. The solution to this problem, if it exists, is $\boldsymbol{\mu}^0(x) = \left(\mu_0^0(x), \mu_1^0(x), \ldots, \mu_{N-1}^0(x)\right)$. The control applied to the uncertain system at state $x$ is $\kappa_N(x) := \mu_0^0(x)$.

### 3.7.2 Stabilizing Conditions

Because the optimal control problem $\mathbb{P}_N(x)$ has a finite horizon $N$, the resultant control law $\kappa_N(\cdot)$ is neither optimal nor stabilizing. The stabilizing conditions, addition of a terminal penalty $V_f(\cdot)$ (a local control Lyapunov function) and a terminal constraint set $\mathbb{X}_f$ (a control invariant set), employed in deterministic MPC to stabilize the system, cannot be easily extended to the stochastic case. The existence of a local control law $\kappa_f(\cdot)$ that ensures $V_f(f(x, \kappa_f(x), w)) \leq V_f(x) - \ell(x, \kappa_f(x))$ for all $w \in \mathbb{W}$ would imply that any state in $\mathbb{X}_f$ could be steered to the origin despite the disturbance. Similarly, the condition that $x \in \mathbb{X}_f$ implies $x^+ = f(x, \kappa_f(x), w) \in \mathbb{X}_f$ for all $w \in \mathbb{W}$ cannot be satisfied if $\mathbb{W}$ is not bounded.

Nor is it possible, in attempting to solve $\mathbb{P}_N(x)$, to steer all trajectories emanating from $x$ to $\mathbb{X}_f$ if $\mathbb{W}$ is unbounded and $\mathbb{X}_f$ is bounded. Attempts in the literature to overcome these obstacles do not appear to be completely satisfactory. It is sometimes assumed that the disturbance, though random, is bounded—an assumption that permits the choice of a terminal constraint set that is robustly control positive invariant (there exists a local control law $\kappa_f(\cdot)$ such that $f(x, \kappa_f(x), w) \in \mathbb{X}_f$ for all $w \in \mathbb{W}$). To obtain a suitable terminal cost, it is also sometimes assumed that a *global* stochastic Lyapunov function $V_f(\cdot)$ and an associated global control law $\kappa_f(\cdot)$ can be found satisfying

$$E_{|x} V_f(f(x, \kappa_f(x), w)) \leq V_f(x) - \ell(x, \kappa_f(x)) \ \forall x \notin \mathbb{X}_f$$

Thus the stochastic descent property is assumed to exist outside $\mathbb{X}_f$. If $x \in \mathbb{X}_f$, the expected cost $E_{|x} V_f(x, \kappa_f(x), w)$ is allowed to increase. The requirement for a global Lyapunov function for a system with control constraints is strong. Indeed, one of the reasons for the success of deterministic MPC is that it avoided the need for a global Lyapunov function and an associated global control law $\kappa_f(\cdot)$. The stability condition Assumption 3.8 overcomes some of these difficulties in merely requiring a local Lyapunov function, but does require the disturbance to be bounded.

### 3.7.3 Stochastic Optimization

Problem $\mathbb{P}_N(x)$ defined above is a stochastic optimization problem, a field in which, fortunately, there has been considerable recent progress. Tempo, Calafiore, and Dabbene (2013) give an excellent exposition of this subject. Probabilistic design problems in this literature are usually formulated as $\min_\theta\{c'\theta \mid \mathcal{P}\{g(\theta,\omega) \leq \gamma\} \geq 1-\varepsilon\}$ in which $\theta$ is the decision variable and $\omega$ is the random variable; in our case $\omega = \mathbf{w}$, a random sequence with values in $\mathbb{W}^N$. The function $g(\cdot)$ is assumed to be convex in $\theta$ for any $\omega$. This problem is difficult to solve because it is hard to compute $\mathcal{P}\{g(\theta,\omega) \leq \gamma\}$. This difficulty motivates scenario optimization, which solves instead the problem $\min_\theta\{c'\theta \mid g(\theta, \omega^i) \leq \gamma,\ i \in \mathbb{I}_{1:M}\}$, in which $\omega^i$ is a random sample of $\omega$ and $M$ is the number of random samples. The scenario optimization problem is then much simpler to compute, especially if the problem is a standard convex program.

Moreover, given any $\beta \geq 0$ and any $\varepsilon \in (0,1)$, there exists a number $M^*(\varepsilon,\beta)$ such that if the number of samples $M$ used in the scenario optimization problem exceeds $M$ and the optimization is feasible, then the probabilistic constraint $\min_\theta\{c'\theta \mid \mathcal{P}\{g(\theta,\omega) \leq \gamma\} \geq 1-\varepsilon$ is satisfied with a probability greater than $1-\beta$. The significance of this result is that $M^*(\varepsilon,\beta)$ increases slowly as $\beta \to 0$ so that, for a reasonably small $M^*(\varepsilon,\beta)$, the probabilistic constraint $\min_\theta\{c'\theta \mid \mathcal{P}\{g(\theta, \omega) \leq \gamma\} \geq 1-\varepsilon\}$ holds with probability greater than $1-10^{-9}$, for example, and the constraint is then satisfied with *practical certainty*.

Tempo et al. (2013) give the value

$$M^*(\varepsilon,\beta) = \frac{2}{\varepsilon}\left(\log\left(\frac{1}{\beta}\right) + n_\theta\right) \tag{3.23}$$

in which $n_\theta$ is the dimension of $\theta$. These results are encouraging and useful, but do not automatically ensure that the problem $\mathbb{P}_N(x)$ can be solved *online*. For example, if $\varepsilon = 0.01$ (constraint should be satisfied with probability not less than 0.99) and $\beta = 10^{-9}$, then $M^*(\varepsilon,\beta) \approx 200(21+n_\theta) \geq 4400$. In a process control problem with, say, 50 state constraints and a horizon of 100, the number of constraints in $\mathbb{P}_N(x)$ exceeds $4400 \times 50 \times 100 = 2.2 \times 10^7$. It seems that these recent results should be used offline rather than online. This observation motivates the development below.

### 3.7.4 Tube-Based Stochastic MPC for Linear Constrained Systems

Although many papers propose that stochastic optimization be employed for solving online problems similar to $\mathbb{P}_N(x)$, even if the system being controlled is linear, it seems desirable for process control applications to develop a controller that is simpler to implement. We show below how, for linear constrained systems, it is possible to employ stochastic optimization *offline* to design a control strategy that is no more difficult to implement *online* than conventional MPC but is conservative; we show how conservatism may be reduced in the following section. Because the system being controlled is linear, the optimal control problem $\mathbb{P}_N(x)$ is convex and it is possible to use efficient scenario optimization for the stochastic optimization problem that is solved offline. The uncertain system to be controlled is described by

$$x^+ = Ax + Bu + w$$

and is subject to the constraints $x \in \mathbb{X}$ and $u \in \mathbb{U}$; $\mathbb{X}$ is closed, $\mathbb{U}$ is compact, and each set contains the origin in its interior. The disturbance $w$ is a stationary random process and is assumed to lie in the compact set $\mathbb{W}$ that contains the origin in its interior. The nominal system is described by

$$\bar{x}^+ = A\bar{x} + B\bar{u}$$

As in robust tube-based MPC, we employ the parameterized control

$$u = \bar{u} + Ke \qquad e := x - \bar{x}$$

in which case, as before

$$x^+ = Ax + B\bar{u} + BKe + w \qquad e^+ = A_K e + w \qquad A_K := A + BK$$

Under this widely chosen policy, it is necessary to assume that $e$ and, therefore, $w$ are bounded if the control is subject to hard constraints, as is generally the case in process control. The feedback matrix $K$ is chosen so that $A_K$ is Hurwitz. If $\mathbf{w}$ is an infinite sequence of independent, identically distributed, zero-mean random variables, then an optimal $K$ may be obtained from the solution to the unconstrained problem

$$\min \lim_{N\to\infty} E_{|x}(1/N) \sum_{i=0}^{N-1} \ell(x(i), u(i))$$

in which $\ell(x,u) = (1/2)x'Qx + u'Ru$ with both $Q$ and $R$ positive definite. Then $K = -(R + B'PB)^{-1}B'PA$, with $P$ the solution of the matrix Riccati equation $P = Q + A'P(R + B'PB)^{-1}PA$; $(1/2)x'Px$ is the minimum cost for the deterministic problem in which the disturbance $w$ is identically zero.

**Nominal control.** The strategy that we employ to control the uncertain systems is to generate nominal state and control trajectories, $\bar{\mathbf{x}}$ and $\bar{\mathbf{u}}$ respectively, that satisfy tightened constraints, and then to use the control $u = \bar{u} + K(x - \bar{x})$ to keep the state and control of the uncertain system close to the nominal trajectories. If the nominal trajectories are not too close to the boundaries of the state and control constraint sets, and if the trajectories of the uncertain system are close enough to the nominal trajectories (i.e., if $\mathbb{W}$ is sufficiently small), the probability of transgression of the state constraint is small. We ensure, as is usually required in practice, that the control constraint is always satisfied. To achieve these objectives we assume the following.

**Assumption 3.21** (Feasibility of robust control)**.** We assume $S_K(\infty) \subset \mathbb{X}$ and $KS_K(\infty) \subset \mathbb{U}$.

This assumption implies that robust control is possible. We do not employ the method usually employed in tube MPC for generating the nominal state and control trajectories. Instead, we require that the nominal state $\bar{x}(i)$ lies in $\mathbb{X}_f$ for all $i \geq N$. To achieve this we first solve the usual nominal optimal control problem at $\bar{x} = x$, the initial state of the system to be controlled, to obtain $\bar{u}$ over the interval $[0, N-1]$. Let $\bar{\mathbb{X}} = \mathbb{X} \ominus S_K(\infty)$ and $\bar{\mathbb{U}} = \mathbb{U} \ominus KS_K(\infty)$ denote the tightened constraint sets employed for robust MPC; they may be computed as shown in Section 3.5 with $\mathbb{Z} := \mathbb{X} \times \mathbb{U}$. The constraints $\bar{x} \in \bar{\mathbb{X}}$ and $\bar{u} \in \bar{\mathbb{U}}$, if satisfied by the nominal trajectories, ensure the original constraints $x \in \mathbb{X}$ and $u \in \mathbb{U}$ are satisfied by the uncertain system for *all* admissible disturbance sequences. Our purpose is to relax the state constraint to get a less conservative solution, i.e., we replace $\bar{\mathbb{X}}$ by $\bar{\mathbb{X}}_1 \subset \mathbb{X}$. Our nominal control problem $\bar{\mathbb{P}}_N(\bar{x})$ therefore has a control constraint $\bar{u} \in \bar{\mathbb{U}}$, a state constraint $\bar{x} \in \bar{\mathbb{X}}_1$, and a terminal constraint $\bar{x} \in \mathbb{X}_f$. The state constraint set $\bar{\mathbb{X}}_1$ satisfies $\bar{\mathbb{X}} \subset \bar{\mathbb{X}}_1 \subset \mathbb{X}$ and is less tight than the constraint set $\bar{\mathbb{X}}$ employed for robust MPC. The resultant nominal optimal control problem is

$$\bar{\mathbb{P}}_N(\bar{x}) : \quad \bar{V}_N^0(\bar{x}) = \min_{\bar{\mathbf{u}}} \{\bar{V}_N(\bar{x}, \bar{\mathbf{u}}) \mid \bar{\mathbf{u}} \in \bar{\mathcal{U}}_N(\bar{x})\}$$

in which

$$\bar{V}_N(\bar{x}, \bar{\mathbf{u}}) := \sum_{i=0}^{N-1} \ell(\bar{x}(i), \bar{u}(i)) + V_f(\bar{x}(N))$$

and

$$\bar{\mathcal{U}}_N(\bar{x}) := \{\bar{\mathbf{u}} \mid \bar{x}(i) \in \bar{\mathbb{X}}_1, \bar{u}(i) \in \bar{\mathbb{U}} \ \forall i \in \mathbb{I}_{0:N-1}, \ \bar{x}(N) \in \bar{\mathbb{X}}_f\}$$

with $\bar{x}(i) := \bar{\phi}(i; \bar{x}, \bar{\mathbf{u}})$ for all $i$. The constraint set $\bar{\mathbb{X}}_1 \subset \bar{\mathbb{X}}$ is determined, as shown below, to ensure that the constraint $x \in \mathbb{X}$ on the uncertain system is transgressed with low probability.

If $\bar{\mathbf{u}}^0(\bar{x})$ is the solution of $\mathbb{P}_N(\bar{x})$, we choose the nominal control $\bar{u}(i)$ to be equal to $\bar{u}^0(i;\bar{x})$ for all $i \in \mathbb{I}_{0:N-1}$ so that $\bar{x}(i) \in \bar{\mathbb{X}}_1 \subset \bar{\mathbb{X}}$ for all $i \in \mathbb{I}_{0:N-1}$ and $\bar{x}(N) \in \bar{\mathbb{X}}_f$. For $i \in \mathbb{I}_{\geq N}$, $\bar{u}(i)$ and $\bar{x}(i)$ are obtained as the solution at time $i$ of

$$\bar{x}^+ = A\bar{x} + B\kappa_f(\bar{x}) \qquad u = \kappa_f(\bar{x})$$

Hence $\bar{x}(i) \in \bar{\mathbb{X}}_f \subset \bar{\mathbb{X}}_1$ for all $i \in \mathbb{I}_{\geq N}$, $\bar{u}(i)$. It follows that $\bar{x}(i) \in \bar{\mathbb{X}}_f$ for all $i \in \mathbb{I}_{\geq 0}$. It follows that the corresponding state trajectory satisfies $\bar{x}(i) \in \bar{\mathbb{X}}_1$ for all $i \in \mathbb{I}_{0:N-1}$ and $\bar{x}(i) \in \mathbb{X}_f$ for all $i \in \mathbb{I}_{\geq N}$ since, by definition, the controller $\kappa_f(\cdot)$ keeps subsequent states in $\mathbb{X}_f$ if the initial state lies in $\mathbb{X}_f$.

We now make one further assumption.

**Assumption 3.22** (Robust terminal set condition)**.** The terminal set satisfies the following: $\mathbb{X}_f + S_K(\infty) \subset \mathbb{X}$.

Since $x(i) = \bar{x}(i) + e(i)$, since $\bar{x}(i) \in \mathbb{X}_f$ for all $i \in \mathbb{I}_{\geq N}$ and $e(i) \in S_K(\infty)$ for all $i \in \mathbb{I}_{\geq 0}$, it follows that $x(i) \in \mathbb{X}$ for all $i \in \mathbb{I}_{\geq N}$ for *all* possible disturbance sequences. Transgression of the state constraints takes place only if $i \in \mathbb{I}_{0:N-1}$. Next, we determine constraint sets $\bar{\mathbb{X}}_1$ and $\mathbb{U}$ that ensure the constraint $x(i) \in \mathbb{X}$ is transgressed with a sufficiently low probability for all $i \in \mathbb{I}_{0:N-1}$.

**Control of the uncertain system.** The controlled composite system, consisting of the uncertain and nominal systems, is described by

$$\begin{aligned} x^+ &= Ax + Bu + w \qquad u = \kappa_N(x,\bar{x}) := \bar{\kappa}_N(\bar{x}) + K(x - \bar{x}) \\ \bar{x}^+ &= A\bar{x} + B\bar{\kappa}_N(\bar{x}) \end{aligned}$$

with initial state $(x_0, x_0)$. Note this differs from some papers on tube-based control in which, at current state $x$, the control $\bar{u} = \bar{\kappa}_N(x)$ instead of $\bar{u} = \bar{\kappa}_N(\bar{x})$ is employed by solving $\bar{\mathbb{P}}_N(x)$ rather than $\bar{\mathbb{P}}_N(\bar{x})$. Solving $\bar{\mathbb{P}}_N(x)$ at time $t$ rather than $\bar{\mathbb{P}}_N(\bar{x})$ is equivalent to reinitializing $(\bar{x}, e)$ to $(x, 0)$ at time $t$. Unlike problem $\bar{\mathbb{P}}_N(\bar{x})$, however, problem $\bar{\mathbb{P}}_N(x)$ is not recursively feasible—necessitating ingenious, but complex, modifications of the controller to restore recursive feasibility. There is one advantage in using $\bar{\mathbb{P}}_N(x)$ instead of $\bar{\mathbb{P}}_N(\bar{x})$; reinitializing $e$ to zero means the predicted future deviations of $x$ from $\bar{x}$ are smaller. For analysis, it is convenient to consider the equivalent system with state $(e, \bar{x})$ described by

$$e^+ = A_K e + w \qquad \bar{x}^+ = A\bar{x} + B\bar{\kappa}_N(\bar{x})$$

Because $x(i) = \bar{x}(i) + e(i)$ and $e(i) = \sum_{j=0}^{i-1} A_K^{i-j-1} w(j)$ (since $e(0) = 0$), it is possible to determine $\bar{\mathbb{X}}$ and $\bar{\mathbb{U}}$ reasonably simply as we now show.

**Determination of $\bar{\mathbb{X}}_1$ and $\bar{\mathbb{U}}$.** We utilize $\bar{\mathbb{U}} := \mathbb{U} \ominus KS_K(\infty)$ as the tightened control constraint set for stochastic MPC. To reduce unnecessary conservatism it is usual in stochastic MPC to permit transgression of state constraints with a specified (low) probability; hence we choose a state constraint set $\bar{\mathbb{X}}_1$ satisfying $\bar{\mathbb{X}} \subset \bar{\mathbb{X}}_1 \subset \mathbb{X}$ as shown below. Because $A_K$ is stable and $\mathbb{W}$ is compact, $e(i)$ tends to a bounded stationary process. We have shown above that $x(i) \in \mathbb{X}$ for all $i \in \mathbb{I}_{\geq N}$ so that transgression of the state constraint can only take place when $i \in \mathbb{I}_{0:N-1}$.

To limit the probability of transgression of the state constraint in the interval $[0, N-1]$ and, hence, in the interval $[0, \infty)$, we would like to ensure that $\mathcal{P}\{x(i) \in \mathbb{X}\} \geq 1 - \varepsilon$ for all $i \in \mathbb{I}_{0:N-1}$; $\mathcal{P}\{A\}$ denotes probability of event $A$. Suppose there is a single state constraint of the form $c'x \leq d$ so that $\mathbb{X} = \{x \mid c'x \leq d\}$. We wish to determine a tighter constraint $c'\bar{x} \leq \bar{d} := d - f$, $f \in [0, d]$, such that $c'\bar{x}(i) \leq \bar{d}$ implies $c'x(i) = c'\bar{x}(i) + c'e(i) \leq d$ for all $i \in \mathbb{I}_{0:N-1}$ with probability not less than $1 - \varepsilon$. To achieve this objective, we would like to solve the stochastic problem $\mathbb{P}_N^s$ defined by

$$\min_{f \in [0,d]} \{f \mid \mathcal{P}\{c'e(i) \leq f^0 \ \forall i \in \mathbb{I}_{0:N-1}\} \geq 1 - \varepsilon\}$$

with $\varepsilon$ chosen to be suitably small; $c'e \leq f$ and $c'\bar{x} \leq d - f$ imply $c'x \leq d$. Problem $\mathbb{P}_N^s$ is complex. It can be replaced by a simpler problem. In Calafiore and Campi (2006); the probability constrained problem $\mathbb{P}_N^s$ is replaced by a scenario optimization problem $\mathbb{P}_N$; $\mathbb{P}_N$ is the convex optimization problem

$$\min_{f \in [0,d]} \{f \mid c'e(i; \mathbf{w}^j) \leq f, \ \forall i \in \mathbb{I}_{0:N-1}, \forall j \in \mathbb{I}_{1:M}\} \tag{3.24}$$

Here $\mathbf{w}^j$ denotes the $j^{th}$ sample of the finite sequence $\{w(0), w(1), w(2), \ldots, w(N-1\}$ and $e(i)$ is replaced by $e(i; \mathbf{w}^j)$ to denote its dependence on the random sequence $\mathbf{w}^j$.

It is shown in Calafiore and Campi (2006); Tempo et al. (2013) that given $(\varepsilon, \beta)$, there exists a relatively modest number of samples $M^*(\varepsilon, \beta)$ such that if $M \geq M^*$, one of the following two conditions hold. Either problem $\mathbb{P}_N$ is infeasible, in which case the robust control problem is infeasible; or its solution $f^0$ satisfies

$$\mathcal{P}\{c'e(i) \leq f^0 \ \forall i \in \mathbb{I}_{0:N-1}\} \geq 1 - \varepsilon$$

with probability $1 - \beta$ (i.e., with practical certainty if $\beta$ is chosen sufficiently small). The relaxed state constraint set is $\bar{\mathbb{X}}_1 = \{x \mid c'x \leq d - f^0\} \subset \bar{\mathbb{X}}$.

If $\mathbb{X} := \{x \mid Cx \leq d\}$ in which $d \in \mathbb{R}^p$, each row $c_k' x \leq d_k$ of the constraint can be relaxed as above by solving, if feasible, for all $k \in \mathbb{I}_{1:p}$, the convex problem

$$\min_{f_k \in [0,d_k]} \{\bar{d} \mid c_k' e(i; \mathbf{w}^j) \leq f_k \;\forall i \in \mathbb{I}_{0:N-1}, \forall j \in \mathbb{I}_{1:M}\}$$

yielding the relaxed state constraint $Cx \leq d - f^0$.

If the number of samples is chosen to be $M' > M^*(\varepsilon, \beta)$, then $M' = M^*(\varepsilon', \beta)$ with $\varepsilon' < \varepsilon$ so the probability of state constraint transgression is reduced from $\varepsilon$ to $\varepsilon'$, but the new nominal state constraint is now $c'\bar{x} \leq d - f'$, $f' \leq f^0$. The resultant control is more conservative. In the limit, as $M' \to \infty$, $\varepsilon' \to 0$ and $f'$ decreases to $f_r$, the value required for robust control.

The resultant controller ensures that $x(i)$ transgresses the state constraint only in the interval $[0, N-1]$, and then with a probability less than $\varepsilon$. Since $x(i)$ lies in $\bar{x}(i) + S_K(\infty)$ for all $i \in \mathbb{I}_{\geq 0}$, and since $\bar{x}(i) \to 0$ as $i \to \infty$, it follows that $x(i)$ tends to the set $S_K(\infty)$ as $i \to \infty$. The controller is considerably simpler to implement than many proposed in the literature, particularly those that require online stochastic optimization. Online implementation is no more complex than conventional (deterministic) MPC, and offline computation is complex but manageable. However, the state constraint $c'\bar{x} \leq d - f^0$ can be conservative; see the following example. Crucial assumptions underlying these results is that the disturbance $w$ is bounded, and the problem is such that robust MPC is possible.

A tentative proposal, which needs further research, for reducing conservatism follows the example.

**Example 3.23: Constraint tightening via sampling**

Consider the scalar system $x^+ = x + u + w$, with $\mathbb{X} = \mathbb{U} = [-1, 1]$ and $w$ uniformly distributed in $\mathbb{W} = [-1/2, 1/2]$. Using costs $Q = 1/2$, $R = 1$, the LQR gain is $K = 1/2$, which gives $A_K = 1/2$, and thus $S_K(\infty) = [-1, 1]$. Tightening the set $\mathbb{U}$, we have $\bar{\mathbb{U}} := \mathbb{U} \ominus KS_K(\infty) = [-1/2, 1/2]$.

To compute the tightened set $\mathbb{X}$, we apply the sampling procedure with horizon $N = 50$. For various values of $\varepsilon$, we compute the number of samples $M = M^*(\varepsilon, \beta)$ using (3.23) with $\beta = 0.01$. Then, we choose $M$ samples of $\mathbf{w}$ and solve (3.24). To evaluate the actual probability of constraint violation, we then test the constraint violation using $M_{\text{test}} = 25{,}000$ different samples $\mathbf{w}_{\text{test}}$. That is, we compute

$$\varepsilon_{\text{test}} := \mathcal{P}\{c'e(i; \mathbf{w}^j_{\text{test}}) > f \qquad i \in \mathbb{I}_{0:N-1}, j \in \mathbb{I}_{1:M_{\text{test}}}\}$$

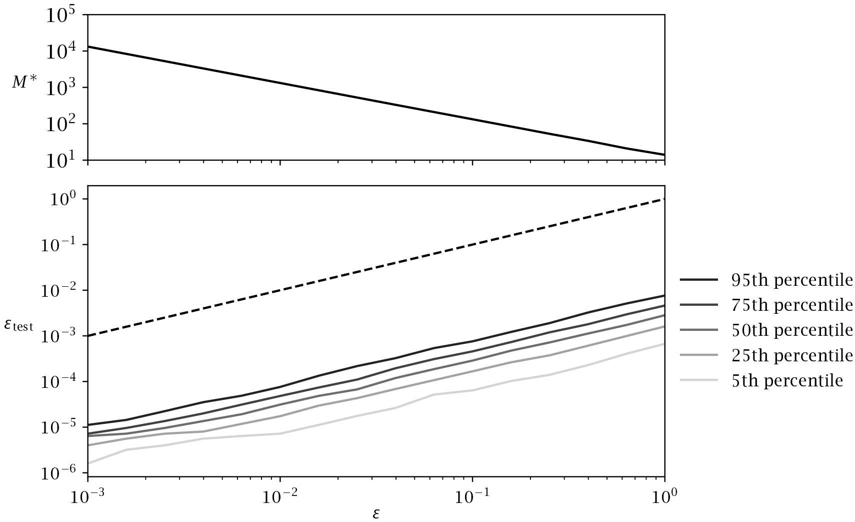



**Figure 3.9:** Observed probability $\varepsilon_{\text{test}}$ of constraint violation. Distribution is based on 500 trials for each value of $\varepsilon$. Dashed line shows the outcome predicted by formula (3.23), i.e., $\varepsilon_{\text{test}} = \varepsilon$.

Note that since $\varepsilon_{\text{test}}$ is now a random variable depending on the particular $M$ samples chosen, we repeat the process 500 times for each value of $M$. The distribution of $\varepsilon_{\text{test}}$ is shown in Figure 3.9. Notice that the formula (3.23) is conservative, i.e., that the observed probability $\varepsilon_{\text{test}}$ is smaller by two orders of magnitude than the chosen probability $\varepsilon$. This gap holds throughout the entire range of the test.

□

**Reducing conservatism.** The simple controller described above can be conservative for two reasons. The first is that the value $M^*(\varepsilon, \beta)$ is often too high, resulting in a smaller probability $\varepsilon$ of transgressing the state constraint than intended and, hence, a tighter state constraint than necessary. The second cause of conservatism is that $f$ is time invariant whereas it could be increased at those times when the state $\bar{x}(i)$ is far from the constraint boundary, allowing the constraint $c'e(i) \leq f$ to be more easily satisfied. This is automatically achieved in those computationally expensive methods that solve a stochastic optimal control problem at each state; see Calafiore and Fagiano (2013) for a comprehensive exposition of a method of this type. We achieve a similar but possibly smaller reduction of the second cause of conservatism by solving one stochastic optimal control problem at the initial state rather than at each subsequent state to obtain a time-varying state constraint set.

If $\mathbb{X} = \{x \mid c'x \leq d\}$, we first solve at time zero a nominal optimal control problem $\mathbb{P}_N(\bar{x})$ defined above under the heading "nominal control," but using the time-varying state constraint $\bar{x}(i) \in \bar{\mathbb{X}}^*(i) := \mathbb{X} \ominus S_K(i)$ for all $i \in \mathbb{I}_{0:N-1}$ rather than the constant state constraint $\bar{x}(i) \in \bar{\mathbb{X}}_1$ previously employed. Note that the time-varying constraint $\bar{x}(i) \in \bar{\mathbb{X}}^*(i)$, if satisfied for all $i \in \mathbb{I}_{0:N-1}$, ensures the original constraint $x \in \mathbb{X}$ is satisfied in the interval $[0, N-1]$ for *all* disturbance sequences. It is therefore tighter than the constraint we wish to determine.

The solution to our nominal control problem yields a preliminary nominal control sequence $\bar{\mathbf{u}}^*$ and state sequence $\bar{\mathbf{x}}^*$. Let $\phi(i; \bar{x}(N), \kappa_f)$ denote the solution at time $i \geq N$ of $\bar{x}^+ = A\bar{x} + B\kappa_f(\bar{x})$ if the initial state is $\bar{x}(N)$ at time $N$. The infinite control sequence $\bar{\mathbf{u}}$, defined by $\bar{u}(i) = \bar{u}^*(i)$ for all $i \in \mathbb{I}_{0:N-1}$ and by $\bar{u}(i) = \kappa_f \phi(i; \bar{x}(N), \kappa_f)$ for all $i \in \mathbb{I}_{\geq N}$ ensures that the resultant state $x(i) = \bar{x}(i) + e(i)$ satisfies $x(i) \in \mathbb{X}$ for all $i \in \mathbb{I}_{\geq 0}$ for *all* disturbance sequences. To obtain a stochastic solution, rather than a robust solution, to our problem, let $\alpha_i$ be defined by

$$\alpha_i := (d - c'\bar{x}^*(i)) \qquad i = 0, 1, \ldots N-1$$

The scenario problem associated with the stochastic control problem $\mathbb{P}_N^s$ defined above is now $\mathbb{P}_N^*(\bar{x})$ defined by

$$\min_{f \in [0,d]} \{f \mid c'e(i; \mathbf{w}^j) \leq f + \alpha_i \ \forall i \in \mathbb{I}_{0:N-1}, \forall j \in \mathbb{I}_{1:M}\}$$

By construction, the scenario problem is feasible; $f = 0$ corresponds to the robust solution (which is assumed to exist) since $c'e(i) \leq d - c'\bar{x}^*(i)$ implies $c'x(i) \leq c'\bar{x}^*(i) + e(i) \leq d$. Hence the solution $f^0$ of the scenario problem satisfies

$$\mathcal{P}\{c'e(i) \leq f^0 + \alpha_i \ \forall i \in \mathbb{I}_{0:N-1}\} \geq 1 - \varepsilon$$

with probability $1 - \beta$ (i.e., with practical certainty if $\beta$ is chosen sufficiently small). As $c'\bar{x}^*(i)$ varies from $d$ to zero, $\alpha_i$ varies from zero to $d$. The larger $\alpha_i$ is, the easier it is to satisfy the constraint $c'e(i; \mathbf{w}^j) \leq f^0 + \alpha_i$. Because $f + \alpha_i$ is larger than $f$, sometimes considerably larger, for all $i$, the constraint $c'e(i) \leq f + \alpha_i$ is more easily satisfied resulting in a lower value for $f^0$.

The nominal state and control sequences $\bar{\mathbf{x}}$ and $\bar{\mathbf{u}}$ are now computed as described above under the heading "nominal control" but using the time-varying state constraint set $\bar{\mathbb{X}}^*(i)$ instead of the constant set $\bar{\mathbb{X}}_1$ previously used. Then either the corresponding scenario problem $\mathbb{P}_N^*(\bar{x})$ is infeasible or its solution $f^0$ satisfies

$$\mathcal{P}\{c'e(i) \leq f^0 + \alpha_i \ \forall i \in \mathbb{I}_{0:N-1}\} \geq 1 - \varepsilon$$

with probability $1 - \beta$. Although this modification reduces conservatism, it incurs computational expense in the sense that a scenario problem has to be solved at time zero for the initial state $\bar{x} = x$. The computational expense is, however, less than that incurred in many proposals for stochastic MPC that require solution of a scenario problem for each time $i \in \mathbb{I}_{\geq 0}$.

## 3.8 Notes

**Robust MPC.** There is now a considerable volume of research on robust MPC; for a review of the literature up to 2000 see Mayne, Rawlings, Rao, and Scokaert (2000). Early literature examines robustness of nominal MPC under perturbations in Scokaert, Rawlings, and Meadows (1997); and robustness under model uncertainty in De Nicolao, Magni, and Scattolini (1996); and Magni and Sepulchre (1997). Sufficient conditions for robust stability of nominal MPC with modeling error are provided in Santos and Biegler (1999). Teel (2004) provides an excellent discussion of the interplay between nominal robustness and continuity of the Lyapunov function, and also presents some illuminating examples of nonrobust MPC. Robustness of the MPC controller described in Chen and Allgöwer (1998), when employed to control a system without state constraints, is established in Yu, Reble, Chen, and Allgöwer (2011). The theory of inherent robustness is usefully extended in Pannocchia, Rawlings, and Wright (2011); Allan et al. (2017); and applied to optimal and suboptimal MPC.

Many papers propose solving online an optimal control problem in which the decision variable is a sequence of control actions that takes into account future disturbances. Thus, it is shown in Limon, Álamo, and Camacho (2002) that it is possible to determine a sequence of constraints sets that become tighter with time, and that ensure the state constraint is not transgressed if the control sequence satisfies these tightened constraints. This procedure was extended in Grimm, Messina, Tuna, and Teel (2007), who do not require the value function to be continuous and do not require the terminal cost to be a control Lyapunov function.

Predicted trajectories when the decision variable is a control sequence can diverge considerably with time, making satisfaction of state and terminal constraints difficult or even impossible. This has motivated the introduction of "feedback" MPC, in which the decision variable is a *policy* (sequence of control laws) rather than a sequence of control actions (Mayne, 1995; Kothare, Balakrishnan, and Morari, 1996; Mayne, 1997; Lee and Yu, 1997; Scokaert and Mayne, 1998). If arbitrary control laws are admissible, the implicit MPC control law is identical to that obtained by dynamic programming; see Section 3.1.3 and papers such as Magni, De Nicolao, Scattolini, and Allgöwer (2003), where a $H_\infty$ MPC control law is obtained. But such results are *conceptual* because the decision variable is infinite dimensional. Hence practical controllers employ suboptimal policies that are finitely parameterized—an extreme example being nominal MPC. A widely used parameterization is $u = v + Kx$, particularly when the system being controlled is linear; this parameterization was first proposed in Rossiter, Kouvaritakis, and Rice (1998). The matrix $K$ is chosen to stabilize the unconstrained linear system, and the decision variable is the sequence $(v(i))_{0:N-1}$.

The robust suboptimal controllers discussed in this chapter employ the concept of tubes introduced in the pioneering papers by Bertsekas and Rhodes (1971a,b), and developed for continuous time systems by Aubin (1991) and Khurzhanski and Valyi (1997). In robust MPC, local feedback is employed to confine all trajectories resulting from the random disturbance to lie in a tube that surrounds a nominal trajectory chosen to ensure the whole tube satisfies the state and control constraints. Robustly positive invariant sets are employed to construct the tubes as shown in (Chisci et al., 2001) and (Mayne and Langson, 2001). Useful references are the surveys by Blanchini (1999), and Kolmanovsky and Gilbert (1995), as well as the recent book by Blanchini and Miani (2008). Kolmanovsky and Gilbert (1995) provide extensive coverage of the theory and computation of minimal and maximal robust (disturbance) invariant sets.

The computation of approximations to robust invariant sets that are themselves invariant is discussed in a series of papers by Raković and colleagues (Raković, Kerrigan, Kouramas, and Mayne, 2003; Raković et al., 2005a; Raković, Mayne, Kerrigan, and Kouramas, 2005b; Kouramas, Raković, Kerrigan, Allwright, and Mayne, 2005). The tube-based controllers described above are based on the papers (Langson, Chryssochoos, Raković, and Mayne, 2004; Mayne, Serón, and Raković, 2005). Construction of robust invariant sets is restricted to systems of relatively low dimension, and is avoided in Section 3.6.3 by employing optimization directly to determine tightened constraints. A tube-based controller for nonlinear systems is presented in Mayne, Kerrigan, van Wyk, and Falugi (2011).

Because robust MPC is still an active area of research, other methods for achieving robustness have been proposed. Diehl, Bock, and Kostina (2006) simplify the robust nonlinear MPC problem by using linearization, also employed in (Nagy and Braatz, 2004), and present some efficient numerical procedures to determine an approximately optimal control sequence. Goulart, Kerrigan, and Maciejowski (2006) propose a control that is an affine function of current and past states; the decision variables are the associated parameters. This method subsumes the tube-based controllers described in this chapter, and has the advantage that a separate nominal trajectory is not required. A disadvantage is the increased complexity of the decision variable, although an efficient computational procedure that reduces computational time per iteration from $O(N^6)$ to $O(N^3)$ has been developed in Goulart, Kerrigan, and Ralph (2008). Interesting extensions to tube-based MPC are presented in Raković (2012), and Raković, Kouvaritakis, Cannon, Panos, and Findeisen (2012).

Considerable attention has recently been given to input-to-state stability of uncertain systems. Thus Limon, Alamo, Raimondo, de la Peña, Bravo, and Camacho (2008) present the theory of input-to-state stability as a unifying framework for robust MPC, generalizes the tube-based MPC described in (Langson et al., 2004), and extends existing results on min-max MPC. Another example of research in this vein is the paper by Lazar, de la Peña, Heemels, and Alamo (2008) that utilizes input-to-state practical stability to establish robust stability of feedback min-max MPC. A different approach is described by Angeli, Casavola, Franže, and Mosca (2008) where it is shown how to construct, for each time $i$, an ellipsoidal inner approximation $\mathcal{E}_i$ to the set $\mathcal{T}_i$ of states that can be robustly steered in $i$ steps to a robust control invariant set $\mathcal{T}$. All that is required from the online controller is the determination of the minimum $i$ such that the current state $x$ lies in $\mathcal{E}_i$ and a control that steers $x \in \mathcal{E}_i$ into the set $\mathcal{E}_{i-1} \subset \mathcal{E}_i$.

**Stochastic MPC.** Interest in stochastic MPC has increased considerably. An excellent theoretical foundation is provided in Chatterjee and Lygeros (2015). Most papers address the stochastic constrained linear problem and propose that the online optimal control problem $\mathbb{P}_N(x)$ ($x$ is the current state) minimizes a suitable objective function subject to satisfaction of state constraints with a specified probability as discussed above. If time-invariant probabilistic state constraints are employed, a major difficulty with this approach, as pointed out in Kouvaritakis, Cannon, Raković, and Cheng (2010) in the context of stochastic MPC for constrained linear systems, is that recursive feasibility is lost unless further measures are taken. It is assumed in this paper, as well as in a later paper Lorenzen, Dabbene, Tempo, and Allgöwer (2015), that the disturbance is bounded, enabling a combination of stochastic and hard constraints to be employed.

In contrast to these papers, which employ the control policy parameterization $u = Kx + v$, Chatterjee, Hokayem, and Lygeros (2011) employ the parameterization, first proposed in Goulart et al. (2006), in which the control law is an affine function of finite number of past disturbances. This parameterization, although not parsimonious, results in a convex optimal control problem, which is advantageous. Recursive feasibility is easily achieved in the tube-based controller proposed above in Section 3.7.4, since it requires online solution of $\mathbb{P}_N(\bar{x})$ rather than $\mathbb{P}_N(x)$.

Another difficulty that arises in stochastic MPC, as pointed out above, is determination of suitable terminal conditions. It is impossible, for example, to obtain a terminal cost $V_f(\cdot)$ and local controller $\kappa_f(\cdot)$ such that $V_f(x^+) < V_f(x)$ for all $x \in X_f$, $x \neq 0$, and all $x^+ = f(x, \kappa_f(x), w)$. For this reason, Chatterjee and Lygeros (2015) propose that it should be possible to decrease $V_f(x)$ outside of the terminal constraint set $\mathbb{X}_f$, but that $V_f(x)$ should be permitted to increase by a bounded amount inside $\mathbb{X}_f$. The terminal ingredients, $V_f(\cdot)$ and $\mathbb{X}_f$, that we propose for robust MPC in Assumption 3.8 have this property a difference being that Chatterjee and Lygeros (2015) require $V_f(\cdot)$ to be a global (stochastic) Lyapunov function.

In most proposals, $\mathbb{P}_N(x)$ is a stochastic optimization problem, an area of study in which there have been recent significant advances discussed briefly above in Section 3.7.3. Despite this, the computational requirements for solving stochastic optimization problems online seems excessive for process control applications. It is therefore desirable that as much computation as possible is done offline as proposed in Kouvaritakis et al. (2010); Lorenzen et al. (2015); Mayne (2016); and in Section 3.7.4 above. In these papers, offline optimization is employed to choose tightened constraints that, if satisfied by the nominal system, ensure that the original constraints are satisfied by the uncertain system. It also is desirable, in process control applications, to avoid computation of polytopic sets, as in 3.6.3, since they cannot be reliably computed for complex systems.

Robustness against unstructured uncertainty has been considered in Løvaas, Serón, and Goodwin (2008); Falugi and Mayne (2011).

## 3.9 Exercises

### Exercise 3.1: Removing the outer min in a min-max problem

Show that $V_i^0 : \mathcal{X}_i \to \mathbb{R}$ and $\kappa_i : \mathcal{X}_i \to \mathbb{U}$ defined by

$$V_i^0(x) = \min_{u \in \mathbb{U}} \max_{w \in \mathbb{W}} \{\ell(x,u,w) + V_{i-1}^0(f(x,u,w)) \mid f(x,u,\mathbb{W}) \subset \mathcal{X}_{i-1}\}$$
$$\kappa_i(x) = \arg\min_{u \in \mathbb{U}} \max_{w \in \mathbb{W}} \{\ell(x,u,w) + V_{i-1}^0(f(x,u,w)) \mid f(x,u,\mathbb{W}) \subset \mathcal{X}_{i-1}\}$$
$$\mathcal{X}_i = \{x \in \mathbb{X} \mid \exists u \in \mathbb{U} \text{ such that } f(x,u,\mathbb{W}) \subset \mathcal{X}_{i-1}\}$$

satisfy

$$V_i^0(x) = \max_{w \in \mathbb{W}} \{\ell(x,\kappa_i(x),w) + V_{i-1}^0(f(x,\kappa_i(x),w))\}$$

### Exercise 3.2: Maximizing a difference

Prove the claim used in the proof of Theorem 3.9 that

$$\max_w \{a(w)\} - \max_w \{b(w)\} \leq \max_w \{a(w) - b(w)\}$$

Also show the following minimization version

$$\min_w \{a(w)\} - \min_w \{b(w)\} \geq \min_w \{a(w) - b(w)\}$$

### Exercise 3.3: Equivalent constraints

Assuming that $S$ is a polytope and, therefore, defined by linear inequalities, show that the constraint $x \in \{z\} \oplus S$ (on $z$ for given $x$) may be expressed as $Bz \leq b + Bx$, i.e., $z$ must lie in a polytope. If $S$ is symmetric ($x \in S$ implies $-x \in S$), show that $x \in \{z\} \oplus S$ is equivalent to $z \in \{x\} \oplus S$.

### Exercise 3.4: Hausdorff distance between translated sets

Prove that the Hausdorff distance between two sets $\{x\} \oplus S$ and $\{y\} \oplus S$, where $S$ is a compact subset of $\mathbb{R}^n$ and $x$ and $y$ are points in $\mathbb{R}^n$, is $|x - y|$.

### Exercise 3.5: Exponential convergence of $X(i)$

Complement the proof of Proposition 3.12 by proving the sequence of sets $(X(i))_{0:\infty}$, $X(i) := \{\bar{x}(i)\} \oplus S_K(\infty)$, converges exponentially fast to the set $S_K(\infty)$ as $i \to \infty$ if $\bar{x}(i)$ converges exponentially fast to 0 as $i \to \infty$.

### Exercise 3.6: Simulating a robust MPC controller

This exercise explores robust MPC for linear systems with an additive bounded disturbance

$$x^+ = Ax + Bu + w$$

The first task, using the tube-based controller described in Section 3.5.3 is to determine state and control constraint sets $\mathbb{Z}$ and $\mathbb{V}$ such that if the nominal system $z^+ = Az + Bv$ satisfies $z \in \mathbb{Z}$ and $v \in \mathbb{V}$, then the actual system $x^+ = Ax + Bu + w$ with $u = v + K(x - z)$ where $K$ is such that $A + BK$ is strictly stable, satisfies the constraints $x \in \mathbb{X}$ and $u \in \mathbb{U}$.

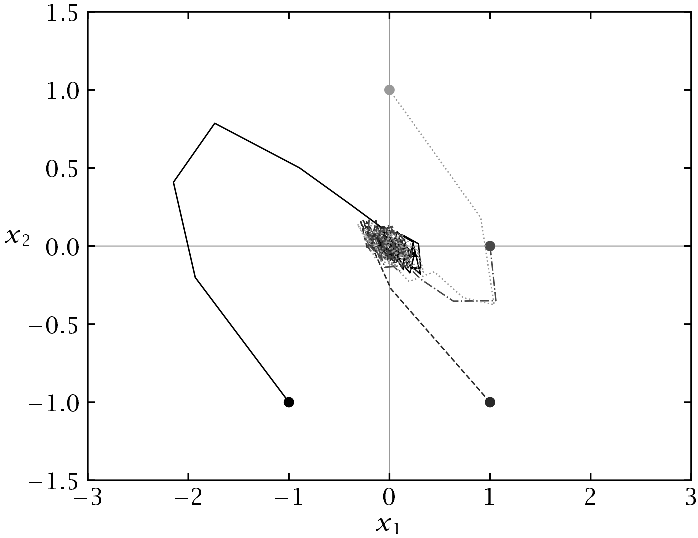



**Figure 3.10:** Closed-loop robust MPC state evolution with uniformly distributed $|w| \leq 0.1$ from four different $x_0$.

(a) To get started, consider the scalar system

$$x^+ = x + u + w$$

with constraint sets $\mathbb{X} = \{x \mid x \leq 2\}$, $\mathbb{U} = \{u \mid |u| \leq 1\}$, and $\mathbb{W} = \{w \mid |w| \leq 0.1\}$. Choose $K = -(1/2)$ so that $A_K = 1/2$. Determine $\mathbb{Z}$ and $\mathbb{V}$ so that if the nominal system $z^+ = z + v$ satisfies $z \in \mathbb{Z}$ and $v \in \mathbb{V}$, the uncertain system $x^+ = Ax + Bu + w$, $u = v + K(x - z)$ satisfies $x \in \mathbb{X}$, $u \in \mathbb{U}$.

(b) Repeat part (a) for the following uncertain system

$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + w$$

with the constraint sets $\mathbb{X} = \{x \in \mathbb{R}^2 \mid x_1 \leq 2\}$, $\mathbb{U} = \{u \in \mathbb{R} \mid |u| \leq 1\}$ and $\mathbb{W} = [-0.1, 0.1]$. Choose $K = \begin{bmatrix} -0.4 & -1.2 \end{bmatrix}$.

(c) Determine a model predictive controller for the nominal system and constraint sets $\mathbb{Z}$ and $\mathbb{V}$ used in (b).

(d) Implement robust MPC for the uncertain system and simulate the closed-loop system for a few initial states and a few disturbance sequences for each initial state. The phase plot for initial states $[-1, -1]$, $[1, 1]$, $[1, 0]$, and $[0, 1]$ should resemble Figure 3.10.

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