7

Explicit Control Laws for Constrained Linear Systems

7.1 Introduction

In preceding chapters we show how model predictive control (MPC) can be derived for a variety of control problems with constraints. It is interesting to recall the major motivation for MPC; solution of a *feedback* optimal control problem for constrained and/or nonlinear systems to obtain a stabilizing control *law* is often prohibitively difficult. MPC sidesteps the problem of determining a control *law* $\kappa(\cdot)$ by determining, instead, at each state x encountered, a control *action* $u = \kappa(x)$ by solving a mathematical programming problem. This procedure, if repeated at *every* state x, yields an implicit control *law* $\kappa(\cdot)$ that solves the original feedback problem. In many cases, determining an explicit control law is impractical while solving a mathematical programming problem online for a given state is possible; this fact has led to the wide-scale adoption of MPC in the chemical process industry.

Some of the control problems for which MPC has been extensively used, however, have recently been shown to be amenable to analysis, at least for relatively simple systems. One such problem is control of linear discrete time systems with polytopic constraints, for which determination of a stabilizing control law was thought in the past to be prohibitively difficult. It has been shown that it is possible, in principle, to determine a stabilizing control law for some of these control problems. This result is often referred to as *explicit MPC* because it yields an explicit control law in contrast to MPC that yields a control action for each encountered state, thereby *implicitly* defining a control law. There are two objections to this terminology. First, determination of control laws for a wide variety of control problems has been the prime concern of control theory since its birth and certainly before the advent of MPC, an important tool in this endeavor being dynamic programming (DP). The new result shows that classical control-theoretic tools, such as DP, can be successfully applied to a wider range of problems than was previously thought possible. MPC is a useful method for implementing an implicit control law that can, in principle, be explicitly determined using control-theoretic tools.

Second, some authors using this terminology have, perhaps inadvertently, implied that these results can be employed in place of conventional MPC. This is far from the truth, since only relatively simple problems, far simpler than those routinely solved in MPC applications, can be solved. That said, the results may be useful in applications where models with low state dimension are sufficiently accurate and where it is important that the control be rapidly computed. A previously determined control law may yield the control action more rapidly than solving an optimal control problem. Potential applications include vehicle control.

In the next section we give a few simple examples of parametric programming. In subsequent sections we show how the solutions to parametric linear and quadratic programs may be obtained, and also show how these solutions may be used to solve optimal control problems when the system is linear, the cost quadratic or affine, and the constraints polyhedral.

7.2 Parametric Programming

A conventional optimization problem has the form $V^0 = \min_u\{V(u) \mid u \in \mathcal{U}\}$ where u is the "decision" variable, $V(u)$ is the cost to be minimized, and $\mathcal{U}$ is the constraint set. The solution to a conventional optimization is a *point* or *set* in $\mathcal{U}$; the value V^0 of the problem satisfies $V^0 = V(u^0)$ where u^0 is a minimizer. A simple example of such a problem is $V^0 = \min_u\{a + bu + (1/2)cu^2 \mid u \in [-1, 1]\}$ where the solution is required for only *one* value of the parameters a, b and c. The solution to this problem $u^0 = -b/c$ if $|b/c| \le 1$, $u^0 = -1$ if $b/c \ge 1$ and $u^0 = 1$ if $b/c \le -1$. This may be written more compactly as $u^0 = -\mathrm{sat}(b/c)$ where $\mathrm{sat}(\cdot)$ is the saturation function. The corresponding value is $V^0 = a - b^2/2c$ if $|b/c| \le 1$, $V^0 = a - b + c^2/2$ if $b/c \ge 1$ and $V^0 = a + b + c^2/2$ if $b/c \le -1$.

A parametric programming problem $\mathbb{P}(x)$ on the other hand, takes the form $V^0(x) = \min_u\{V(x, u) \mid u \in \mathcal{U}(x)\}$ where x is a *parameter* so that the optimization problem, and its solution, depend on the value of the parameter. Hence, the solution to a parametric programming problem $\mathbb{P}(x)$ is not a point or set but a *function* $x \mapsto u^0(x)$ that may be set valued; similarly the value of the problem is a function $x \mapsto V^0(x)$. At each x, the minimizer $u^0(x)$ may be a point or a set. Optimal control problems often take this form, with x being the state, and u, in open-loop discrete time optimal control, being a control sequence; $u^0(x)$, the optimal control sequence, is a function of the initial state. In state feedback optimal control, necessary when uncertainty is present, DP is employed yielding a sequence of parametric optimization problems in each of which x is the state and u a control action; see Chapter 2. The programming problem in the first paragraph of this section may be regarded as a parametric programming problem with the parameter $x := (a, b, c)$, $V(x, u) := (x_1 + x_2 u + (1/2)x_3 u^2/2)$ and $\mathcal{U}(x) := [-1, 1]$; $\mathcal{U}(x)$, in this example, does not depend on x. The solution to this problem yields the functions $u^0(\cdot)$ and $V^0(\cdot)$ defined by $u^0(x) = -\text{sat}(x_2/x_3)$ and $V^0(x) = V(x, u^0(x)) = x_1 + x_2 u^0(x) + (x_3/2)(u^0(x))^2$.

Because the minimizer and value of a parametric programming problem are *functions* rather than points or sets, we would not, in general, expect to be able to compute a solution. Surprisingly, parametric programs may be solved when the cost function $V(\cdot)$ is affine ($V(x, u) = a + b'x + c'u$) or quadratic ($V(x, u) = (1/2)x'Qx + x'Su + (1/2)u'Ru$) and $\mathcal{U}(x)$ is defined by a set of affine inequalities: $\mathcal{U}(x) = \{u \mid Mu \leq Nx + p\}$. The parametric constraint $u \in \mathcal{U}(x)$ may be conveniently expressed as $(x, u) \in \mathbb{Z}$ where $\mathbb{Z}$ is a subset of (x, u)-space which we will take to be $\mathbb{R}^n \times \mathbb{R}^m$; for each x

$$\mathcal{U}(x) = \{u \mid (x, u) \in \mathbb{Z}\}$$

We assume that $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Let $\mathcal{X} \subset \mathbb{R}^n$ be defined by

$$\mathcal{X} := \{x \mid \exists u \text{ such that } (x, u) \in \mathbb{Z}\} = \{x \mid \mathcal{U}(x) \neq \emptyset\}$$

The set $\mathcal{X}$ is the domain of $V^0(\cdot)$ and $u^0(\cdot)$ and is thus the set of points x for which a feasible solution of $\mathbb{P}(x)$ exists; it is the projection of $\mathbb{Z}$ (which is a set in (x, u)-space) onto x-space. See Figure 7.1, which illustrates $\mathbb{Z}$ and $\mathcal{U}(x)$ for the case when $\mathcal{U}(x) = \{u \mid Mu \leq Nx + p\}$; the set $\mathbb{Z}$ is thus defined by $\mathbb{Z} := \{(x, u) \mid Mu \leq Nx + p\}$. In this case, both $\mathbb{Z}$ and $\mathcal{U}(x)$ are polyhedral.

Before proceeding to consider parametric linear and quadratic programming, some simple examples may help the reader to appreciate the underlying ideas. Consider first a very simple parametric linear

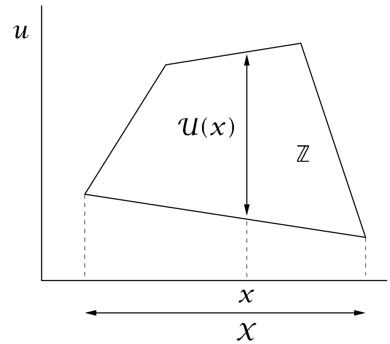

Figure 7.1: The sets $\mathbb{Z}$, $\mathcal{X}$, and $\mathcal{U}(x)$.

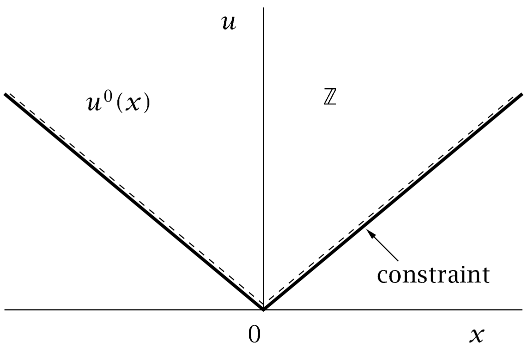

Figure 7.2: Parametric linear program.

program $\min_u\{V(x,u) \mid (x,u) \in \mathbb{Z}\}$ where $V(x,u) := x + u$ and $\mathbb{Z} := \{(x,u) \mid u + x \geq 0,\ u - x \geq 0\}$ so that $\mathcal{U}(x) = \{u \mid u \geq -x,\ u \geq x\}$. The problem is illustrated in Figure 7.2. The set $\mathbb{Z}$ is the region lying above the two solid lines $u = -x$ and $u = x$, and is convex. The gradient $\nabla_u V(x,u) = 1$ everywhere, so the solution, at each x, to the parametric program is the smallest u in $\mathcal{U}(x)$, i.e., the smallest u lying above the two lines $u = -x$ and $u = x$. Hence $u^0(x) = -x$ if $x \leq 0$ and $u^0(x) = x$ if $x \geq 0$, i.e., $u^0(x) = |x|$; the graph of $u^0(\cdot)$ is the dashed line in Figure 7.2. Both $u^0(\cdot)$ and $V^0(\cdot)$, in which $V^0(x) = x + u^0(x)$, are *piecewise affine*, being affine in each of the two regions $X_1 := \{x \mid x \leq 0\}$ and $X_2 := \{x \mid x \geq 0\}$.

Next consider an unconstrained parametric quadratic program (QP) $\min_u V(x,u)$ where $V(x,u) := (1/2)(x-u)^2 + u^2/2$. The problem is

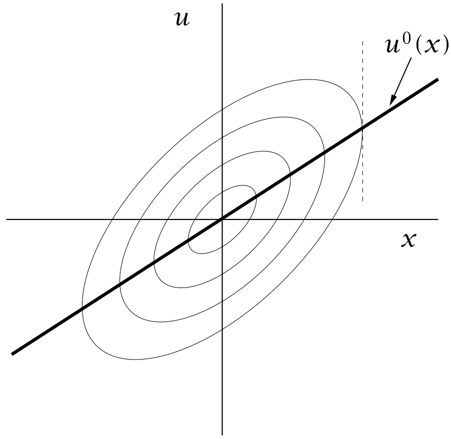

Figure 7.3: Unconstrained parametric quadratic program.

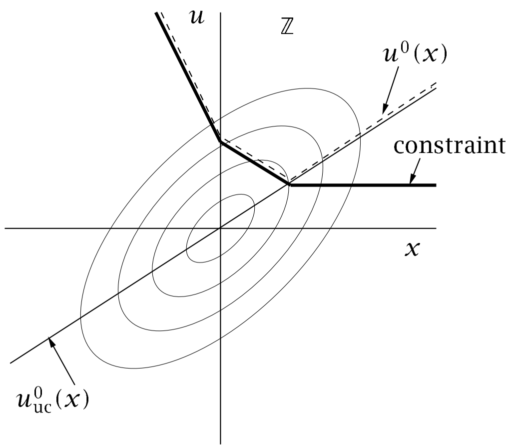

Figure 7.4: Parametric quadratic program.

illustrated in Figure 7.3. For each $x \in \mathbb{R}$, $\nabla_u V(x,u) = -x + 2u$ and $\nabla_{uu} V(x,u) = 2$ so that $u^0(x) = x/2$ and $V^0(x) = x^2/4$. Hence $u^0(\cdot)$ is affine and $V^0(\cdot)$ is quadratic in $\mathbb{R}$.

We now add the constraint set $\mathbb{Z} := \{(x,u) \mid u \geq 1,\ u + x/2 \geq 2,\ u + x \geq 2\}$; see Figure 7.4. The solution is defined on three regions, $X_1 := (-\infty, 0]$, $X_2 := [0,2]$, and $X_3 := [2, \infty)$. From the preceding example, the unconstrained minimum is achieved at $u^0_{\mathrm{uc}}(x) = x/2$ shown by the solid straight line in Figure 7.4. Since $\nabla_u V(x,u) = -x + 2u$, $\nabla_u V(x,$

$u) > 0$ for all $u > u^0_{\text{uc}}(x) = x/2$. Hence, in X_1, $u^0(x)$ lies on the boundary of $\mathbb{Z}$ and satisfies $u^0(x) = 2 - x$. Similarly, in X_2, $u^0(x)$ lies on the boundary of $\mathbb{Z}$ and satisfies $u^0(x) = 2 - x/2$. Finally, in X_3, $u^0(x) = u^0_{\text{uc}}(x) = x/2$, the unconstrained minimizer, and lies in the interior of $\mathbb{Z}$ for $x > 1$. The third constraint $u \geq 2 - x$ is active in X_1, the second constraint $u \geq 2 - x/2$ is active in X_2, while no constraints are active in X_3. Hence the minimizer $u^0(\cdot)$ is piecewise affine, being affine in each of the regions X_1, X_2 and X_3. Since $V^0(x) = (1/2)\left|x - u^0(x)\right|^2 + u^0(x)^2/2$, the value function $V^0(\cdot)$ is piecewise quadratic, being quadratic in each of the regions X_1, X_2 and X_3.

We require, in the sequel, the following definitions.

Definition 7.1 (Polytopic (polyhedral) partition)**.** A set $\mathcal{P} = \{\mathbb{Z}_i \mid i \in \mathcal{I}\}$, for some index set $\mathcal{I}$, is called a polytopic (polyhedral) partition of the polytopic (polyhedral) set $\mathbb{Z}$ if $\mathbb{Z} = \cup_{i\in\mathcal{I}}\mathbb{Z}_i$ and the sets $\mathbb{Z}_i$, $i \in \mathcal{I}$, are polytopes (polyhedrons) with nonempty interiors (relative to $\mathbb{Z}$)[1] that are nonintersecting: $\text{int}(\mathbb{Z}_i) \cap \text{int}(\mathbb{Z}_j) = \emptyset$ if $i \neq j$.

Definition 7.2 (Piecewise affine function)**.** A function $f : \mathbb{Z} \to \mathbb{R}^m$ is said to be piecewise affine on a polytopic (polyhedral) partition $\mathcal{P} = \{\mathbb{Z}_i \mid i \in \mathcal{I}\}$ if it satisfies, for some K_i, k_i, $i \in \mathcal{I}$, $f(x) = K_i x + k_i$ for all $x \in \mathbb{Z}_i$, all $i \in \mathcal{I}$. Similarly, a function $f : \mathbb{Z} \to \mathbb{R}$ is said to be piecewise quadratic on a polytopic (polyhedral) partition $\mathcal{P} = \{\mathbb{Z}_i \mid i \in \mathcal{I}\}$ if it satisfies, for some Q_i, r_i, and s_i, $i \in \mathcal{I}$, $f(x) = (1/2)x'Q_i x + r_i' x + s_i$ for all $x \in \mathbb{Z}_i$, all $i \in \mathcal{I}$.

Note the piecewise affine and piecewise quadratic functions defined this way are not necessarily continuous and may, therefore, be set valued at the intersection of the defining polyhedrons. An example is the piecewise affine function $f(\cdot)$ defined by

$$\begin{aligned} f(x) &:= -x - 1 \quad & x \in (-\infty, 0] \\ &:= \ x + 1 \quad & x \in [0, \infty) \end{aligned}$$

This function is set valued at $x = 0$ where it has the value $f(0) = \{-1, 1\}$. We shall mainly be concerned with continuous piecewise affine and piecewise quadratic functions.

We now generalize the points illustrated by our example above and consider, in turn, parametric quadratic programming and parametric

[1]The interior of a set $S \subseteq \mathbb{Z}$ relative to the set $\mathbb{Z}$ is the set $\{z \in S \mid \varepsilon(z)\mathcal{B} \cap \text{aff}(\mathbb{Z}) \subseteq \mathbb{Z}$ for some $\varepsilon > 0\}$ where $\text{aff}(\mathbb{Z})$ is the intersection of all affine sets containing $\mathbb{Z}$.

linear programming and their application to optimal control problems. We deal with parametric quadratic programming first because it is more widely used and because, with reasonable assumptions, the minimizer is unique making the underlying ideas somewhat simpler to follow.

7.3 Parametric Quadratic Programming

7.3.1 Preliminaries

The parametric QP $\mathbb{P}(x)$ is defined by

$$V^0(x) = \min_u \{V(x,u) \mid (x,u) \in \mathbb{Z}\}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. The cost function $V(\cdot)$ is defined by

$$V(x,u) := (1/2)x'Qx + u'Sx + (1/2)u'Ru + q'x + r'u + c$$

and the polyhedral constraint set $\mathbb{Z}$ is defined by

$$\mathbb{Z} := \{(x,u) \mid Mx \leq Nu + p\}$$

where $M \in \mathbb{R}^{r \times n}$, $N \in \mathbb{R}^{r \times m}$ and $p \in \mathbb{R}^r$; thus $\mathbb{Z}$ is defined by r affine inequalities. Let $u^0(x)$ denote the solution of $\mathbb{P}(x)$ if it exists, i.e., if $x \in \mathcal{X}$, the domain of $V^0(\cdot)$; thus

$$u^0(x) := \arg\min_u \{V(x,u) \mid (x,u) \in \mathbb{Z}\}$$

The solution $u^0(x)$ is unique if $V(\cdot)$ is strictly convex in u; this is the case if R is positive definite. Let the matrix $\mathcal{Q}$ be defined by

$$\mathcal{Q} := \begin{bmatrix} Q & S' \\ S & R \end{bmatrix}$$

For simplicity we assume the following in the sequel.

Assumption 7.3 (Strict convexity)**.** The matrix $\mathcal{Q}$ is positive definite.

Assumption 7.3 implies that both R and Q are positive definite. The cost function $V(\cdot)$ may be written in the form

$$V(x,u) = (1/2)(x,u)'\mathcal{Q}(x,u) + q'x + r'u + c$$

where, as usual, the vector (x,u) is regarded as a column vector $(x',$ $u')'$ in algebraic expressions. The parametric QP may also be expressed as

$$V^0(x) := \min_u \{V(x,u) \mid u \in \mathcal{U}(x)\}$$

where the parametric constraint set $\mathcal{U}(x)$ is defined by

$$\mathcal{U}(x) := \{u \mid (x, u) \in \mathbb{Z}\} = \{u \in \mathbb{R}^m \mid Mu \leq Nx + p\}$$

For each x the set $\mathcal{U}(x)$ is polyhedral. The domain $\mathcal{X}$ of $V^0(\cdot)$ and $u^0(\cdot)$ is defined by

$$\mathcal{X} := \{x \mid \exists u \in \mathbb{R}^m \text{ such that } (\mathrm{x},\mathrm{u}) \in \mathbb{Z}\} = \{x \mid \mathcal{U}(x) \neq \emptyset\}$$

For all $(x, u) \in \mathbb{Z}$, let the index set $I(x, u)$ specify the constraints that are *active* at (x, u), i.e.,

$$I(x, u) := \{i \in \mathbb{I}_{1:r} \mid M_i u = N_i x + p_i\}$$

where M_i, N_i, and p_i denote, respectively, the ith row of M, N, and p. Similarly, for any matrix or vector A and any index set I, A_I denotes the matrix or vector with rows A_i, $i \in I$. For any $x \in \mathcal{X}$, the indices set $I^0(x)$ specifies the constraints that are active at $(x, u^0(x))$, namely

$$I^0(x) := I(x, u^0(x)) = \{i \in \mathbb{I}_{1:r} \mid M_i u^0(x) = N_i x + p_i\}$$

Since $u^0(x)$ is unique, $I^0(x)$ is well defined. Thus $u^0(x)$ satisfies the equation

$$M_x^0 u = N_x^0 x + p_x^0$$

where

$$M_x^0 := M_{I^0(x)}, \; N_x^0 := N_{I^0(x)}, \; p_x^0 := p_{I^0(x)} \tag{7.1}$$

7.3.2 Preview

We show in the sequel that $V^0(\cdot)$ is piecewise quadratic and $u^0(\cdot)$ piecewise affine on a polyhedral partition of $\mathcal{X}$, the domain of both these functions. To do this, we take an arbitrary point x in $\mathcal{X}$, and show that $u^0(x)$ is the solution of an *equality* constrained QP $\mathbb{P}(x) : \min_u\{V(x, u) \mid M_x^0 u = N_x^0 x + p_x^0\}$ in which the equality constraint is $M_x^0 u = N_x^0 x + p_x^0$. We then show that there is a polyhedral region $R_x^0 \subset \mathcal{X}$ in which x lies and such that, for all $w \in R_x^0$, $u^0(w)$ is the solution of the equality constrained QP $\mathbb{P}(w) : \min_u\{V(w, u) \mid M_x^0 u = N_x^0 w + p_x^0\}$ in which the equality constraints are the same as those for $\mathbb{P}(x)$. It follows that $u^0(\cdot)$ is affine and $V^0(\cdot)$ is quadratic in R_x^0. We then show that there are only a finite number of such polyhedral regions so that $u^0(\cdot)$ is piecewise affine, and $V^0(\cdot)$ piecewise quadratic, on a polyhedral partition of $\mathcal{X}$. To carry out this program, we require a suitable characterization of optimality. We develop this in the next subsection. Some readers may prefer to jump to Proposition 7.8, which gives the optimality condition we employ in the sequel.

7.3.3 Optimality Condition for a Convex Program

Necessary and sufficient conditions for nonlinear optimization problems are developed in Section C.2 of Appendix C. Since we are concerned here with a relatively simple optimization problem where the cost is convex and the constraint set polyhedral, we give a self-contained exposition that uses the concept of a *polar cone*.

Definition 7.4 (Polar cone)**.** The *polar cone* of a cone $C \subseteq \mathbb{R}^n$ is the cone C^* defined by

$$C^* := \{g \in \mathbb{R}^n \mid \langle g, h \rangle \leq 0 \ \forall h \in C\}$$

We recall that a set $C \subseteq \mathbb{R}^n$ is a cone if $0 \in C$ and that $h \in C$ implies $\lambda h \in C$ for all $\lambda > 0$. A cone C is said to be *generated* by $\{a_i \mid i \in I\}$ where I is an index set if $C = \sum_{i \in I}\{\mu_i a_i \mid \mu_i \geq 0, i \in I\}$ in which case we write $C = \text{cone}\{a_i \mid i \in I\}$. We need the following result.

Proposition 7.5 (Farkas's lemma)**.** *Suppose C is a polyhedral cone defined by*

$$C := \{h \mid Ah \leq 0\} = \{h \mid \langle a_i, h \rangle \leq 0 \mid i \in \mathbb{I}_{1:m}\}$$

in which, for each i, a_i is the ith row of A. Then

$$C^* = \text{cone}\{a_i \mid i \in \mathbb{I}_{1:m}\}$$

A proof of this result is given in Section C.2 of Appendix C; that $g \in \text{cone}\{a_i \mid i \in \mathbb{I}_{1:m}\}$ implies $\langle g, h \rangle \leq 0$ for all $h \in C$ is easily shown. An illustration of Proposition 7.5 is given in Figure 7.5.

Next we make use of a standard necessary and sufficient condition of optimality for optimization problems in which the cost is convex and differentiable and the constraint set is convex.

Proposition 7.6 (Optimality conditions for convex set)**.** *Suppose, for each $x \in \mathcal{X}$, $u \mapsto V(x,u)$ is convex and differentiable and $\mathcal{U}(x)$ is convex. Then u is optimal for* $\min_u\{V(x,u) \mid u \in \mathcal{U}(x)\}$ *if and only if*

$$u \in \mathcal{U}(x) \text{ and } \langle \nabla_u V(x,u), v - u \rangle \geq 0 \quad \forall v \in \mathcal{U}(x)$$

Proof. This Proposition appears as Proposition C.9 in Appendix C where a proof is given. ■

In our case $\mathcal{U}(x)$, $x \in \mathcal{X}$, is polyhedral and is defined by

$$\mathcal{U}(x) := \{v \in \mathbb{R}^m \mid Mv \leq Nx + p\} \tag{7.2}$$

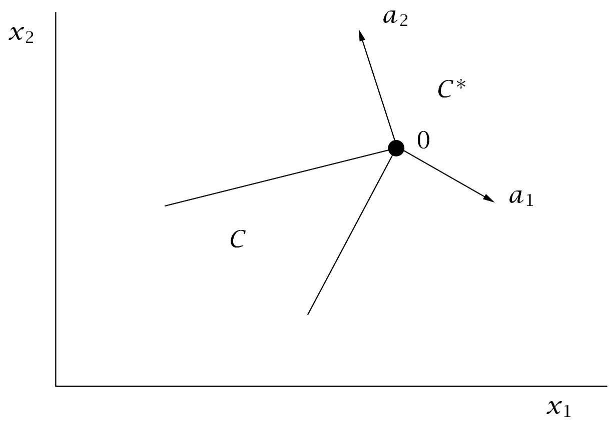

Figure 7.5: Polar cone.

so $v \in \mathcal{U}(x)$ if and only if, for all $u \in \mathcal{U}(x)$, $v - u \in \mathcal{U}(x) - \{u\} := \{v - u \mid v \in \mathcal{U}(x)\}$. With $h := v - u$

$$\mathcal{U}(x) - \{u\} = \left\{ h \in \mathbb{R}^m \;\middle|\; \begin{array}{ll} M_i h \leq 0, & i \in I(x,u) \\ M_j h < N_j x + p_j - M_j u, & j \in \mathbb{I}_{1:r} \setminus I(x,u) \end{array} \right\}$$

since $M_i u = N_i x + p_i$ for all $i \in I(x,u)$. For each $z = (x,u) \in \mathbb{Z}$, let $C(x,u)$ denote the cone of feasible directions[2] $h = v - u$ at u, i.e., $C(x,u)$ is defined by

$$C(x,u) := \{h \in \mathbb{R}^m \mid M_i h \leq 0,\ i \in I(x,u)\}$$

Clearly

$$\mathcal{U}(x) - \{u\} = C(x,u) \cap \{h \in \mathbb{R}^m \mid M_i h < N_i x + p_i - M_i u,\ i \in \mathbb{I}_{1:r} \setminus I(x,u)\}$$

so that $\mathcal{U}(x) - \{u\} \subseteq C(x,u)$; for any $(x,u) \in \mathbb{Z}$, any $h \in C(x,u)$, there exists an $\alpha > 0$ such that $u + \alpha h \in \mathcal{U}(x)$. Proposition 7.6 may be expressed as: u is optimal for $\min_u \{V(x,u) \mid u \in \mathcal{U}(x)\}$ if and only if

$$u \in \mathcal{U}(x) \text{ and } \langle \nabla_u V(x,u), h \rangle \geq 0 \quad \forall h \in \mathcal{U}(x) - \{u\}$$

We may now state a modified form of Proposition 7.6.

[2] A direction h at u is feasible if there exists an $\varepsilon > 0$ such that $u + \lambda h \in U(x)$ for all $\lambda \in [0, \varepsilon]$.

Proposition 7.7 (Optimality conditions in terms of polar cone)**.** *Suppose for each $x \in \mathcal{X}$, $u \mapsto V(x, \cdot)$ is convex and differentiable, and $\mathcal{U}(x)$ is defined by* (7.2)*. Then u is optimal for* $\min_u\{V(x,u) \mid u \in \mathcal{U}(x)\}$ *if and only if*

$$u \in \mathcal{U}(x) \text{ and } \langle \nabla_u V(x,u), h\rangle \geq 0 \quad \forall h \in C(x,u)$$

Proof. We show that the condition $\langle \nabla_u V(x,u), h\rangle \geq 0$ for all $h \in C(x, u)$ is equivalent to the condition $\langle \nabla_u V(x,u), h\rangle \geq 0$ for all $h \in \mathcal{U}(x) - \{u\}$ employed in Proposition 7.6. (i) Since $\mathcal{U}(x) - \{u\} \subseteq C(x,u)$, $\langle \nabla_u V(x,u), h\rangle \geq 0$ for all $h \in C(x,u)$ implies $\langle \nabla_u V(x,u), h\rangle \geq 0$ for all $h \in \mathcal{U}(x) - \{u\}$. (ii) $\langle \nabla_u V(x,u), h\rangle \geq 0$ for all $h \in \mathcal{U}(x) - \{u\}$ implies $\langle \nabla_u V(x,u), \alpha h\rangle \geq 0$ for all $h \in \mathcal{U}(x) - \{u\}$, all $\alpha > 0$. But, for any $h^* \in C(x,u)$, there exists an $\alpha \geq 1$ such that $h^* = \alpha h$ with $h := (1/\alpha)h^* \in \mathcal{U}(x) - \{u\}$. Hence $\langle \nabla_u V(x,u), h^*\rangle = \langle \nabla_u V(x,u), \alpha h\rangle \geq 0$ for all $h^* \in C(x,u)$. ■

We now make use of Proposition 7.7 to obtain the optimality condition in the form we use in the sequel. For all $(x,u) \in \mathbb{Z}$, let $C^*(x,u)$ denote the polar cone to $C(x,u)$.

Proposition 7.8 (Optimality conditions for linear inequalities)**.** *Suppose, for each $x \in \mathcal{X}$, $u \mapsto V(x,u)$ is convex and differentiable, and $\mathcal{U}(x)$ is defined by* (7.2)*. Then u is optimal for* $\min_u\{V(x,u) \mid u \in \mathcal{U}(x)\}$ *if and only if*

$$u \in \mathcal{U}(x) \text{ and } -\nabla_u V(x,u) \in C^*(x,u) = \operatorname{cone}\{M_i' \mid i \in I(x,u)\}$$

Proof. The desired result follows from a direct application of Proposition 7.5 to Proposition 7.7. ■

Note that $C(x,u)$ and $C^*(x,u)$ are both cones so that each set contains the origin. In particular, $C^*(x,u)$ is generated by the gradients of the constraints active at $z = (x,u)$, and may be defined by a set of affine inequalities: for each $z \in \mathbb{Z}$, there exists a matrix L_z such that

$$C^*(x,u) = C^*(z) = \{g \in \mathbb{R}^m \mid L_z g \leq 0\}$$

The importance of this result for us lies in the fact that the necessary and sufficient condition for optimality is satisfaction of two polyhedral constraints, $u \in \mathcal{U}(x)$ and $-\nabla_u V(x,u) \in C^*(x,u)$. Proposition 7.8 may also be obtained by direct application of Proposition C.12 of Appendix C; $C^*(x,u)$ may be recognized as $\mathcal{N}_{\mathcal{U}(x)}(u)$, the regular normal cone to the set $\mathcal{U}(x)$ at u.

7.3.4 Solution of the Parametric Quadratic Program

For the parametric programming problem $\mathbb{P}(x)$, the parametric cost function is

$$V(x,u) := (1/2)x'Qx + u'Sx + (1/2)u'Ru + q'x + r'u + c$$

and the parametric constraint set is

$$\mathcal{U}(x) := \{u \mid Mu \leq Nx + p\}$$

Hence, the cost gradient is

$$\nabla_u V(x,u) = Ru + Sx + r$$

in which, because of Assumption 7.3, R is positive definite. Hence, the necessary and sufficient condition for the optimality of u for the parametric QP $\mathbb{P}(x)$ is

$$\begin{aligned} &Mu \leq Nx + p \\ &-(Ru + Sx + r) \in C^*(x,u) \end{aligned}$$

in which $C^*(x,u) = \text{cone}\{M_i' \mid i \in I(x,u)\}$, the cone generated by the gradients of the active constraints, is polyhedral. We cannot use this characterization of optimality directly to solve the parametric programming problem since $I(x,u)$ and, hence, $C^*(x,u)$, varies with (x, u). Given any $x \in \mathcal{X}$, however, there exists the possibility of a region containing x such that $I^0(x) \subseteq I^0(w)$ for all w in this region. We make use of this observation as follows. It follows from the definition of $I^0(x)$ that the unique solution $u^0(x)$ of $\mathbb{P}(x)$ satisfies the equation

$$\begin{aligned} M_i u &= N_i x + p_i, \quad i \in I^0(x), \text{ i.e.,} \\ M_x^0 u &= N_x^0 x + p_x^0 \end{aligned}$$

where M_x^0, N_x^0, and p_x^0 are defined in (7.1). Hence $u^0(x)$ is the solution of the equality constrained problem

$$V^0(x) = \min_u \{V(x,u) \mid M_x^0 u = N_x^0 x + p_x^0\}$$

If the active constraint set remains constant near the point x or, more precisely, if $I^0(x) \subseteq I^0(w)$ for all w in some region in $\mathbb{R}^n$ containing x, then, for all w in this region, $u^0(w)$ satisfies the equality constraint

$M_x^0 u = N_x^0 w + p_x^0$. This motivates us to consider the simple equality constrained problem $\mathbb{P}_x(w)$ defined by

$$V_x^0(w) = \min_u \{V(w,u) \mid M_x^0 u = N_x^0 w + p_x^0\}$$
$$u_x^0(w) = \arg\min_u \{V(w,u) \mid M_x^0 u = N_x^0 w + p_x^0\}$$

The subscript x indicates that the equality constraints in $\mathbb{P}_x(w)$ depend on x. Problem $\mathbb{P}_x(w)$ is an optimization problem with a quadratic cost function and linear equality constraints and is, therefore, easily solved; see the exercises at the end of this chapter. Its solution is

$$V_x^0(w) = (1/2)w'Q_x w + r_x' w + s_x \tag{7.3}$$
$$u_x^0(w) = K_x w + k_x \tag{7.4}$$

for all w such that $I^0(w) = I^0(x)$ where $Q_x \in \mathbb{R}^{n\times n}$, $r_x \in \mathbb{R}^n$, $s_x \in \mathbb{R}$, $K_x \in \mathbb{R}^{m\times n}$ and $k_x \in \mathbb{R}^m$ are easily determined. Clearly, $u_x^0(x) = u^0(x)$; but, is $u_x^0(w)$, the optimal solution to $\mathbb{P}_x(w)$, the optimal solution $u^0(w)$ to $\mathbb{P}(w)$ in some region containing x and, if it is, what is the region? Our optimality condition answers this question. For all $x \in \mathcal{X}$, let the region R_x^0 be defined by

$$R_x^0 := \left\{ w \;\middle|\; \begin{array}{c} u_x^0(w) \in \mathcal{U}(w) \\ -\nabla_u V(w, u_x^0(w)) \in C^*(x, u^0(x)) \end{array} \right\} \tag{7.5}$$

Because of the equality constraint $M_x^0 u = N_x^0 w + p_x^0$ in problem $\mathbb{P}_x(w)$, it follows that $I(w, u_x^0(w)) \supseteq I(x, u^0(x))$, and that $C(w, u_x^0(w)) \subseteq C(x, u^0(x))$ and $C^*(w, u_x^0(w)) \supseteq C^*(x, u^0(x))$ for all $w \in R_x^0$. Hence $w \in R_x^0$ implies $u_x^0(w) \in \mathcal{U}(w)$ and $-\nabla_u V(w, u_x^0(w)) \in C^*(w, u_x^0(w))$ for all $w \in R_x^0$ which, by Proposition 7.8, is a necessary and sufficient condition for $u_x^0(w)$ to be optimal for $\mathbb{P}(w)$. In fact, $I(w, u_x^0(w)) = I(x, u^0(x))$ so that $C^*(w, u_x^0(w)) = C^*(x, u^0(x))$ for all w in the interior of R_x^0. The obvious conclusion of this discussion is the following.

Proposition 7.9 (Solution of $\mathbb{P}(w)$, $w \in R_x^0$). *For any $x \in \mathcal{X}$, $u_x^0(w)$ is optimal for $\mathbb{P}(w)$ for all $w \in R_x^0$.*

The constraint $u_x^0(w) \in \mathcal{U}(w)$ may be expressed as

$$M(K_x w + k_x) \leq Nw + p$$

which is an affine inequality in w. Similarly, since $\nabla_u V(w,u) = Ru + Sw + r$ and since $C^*(x, u^0(x)) = \{g \mid L_x^0 g \leq 0\}$ where $L_x^0 = L_{(x,u^0(x))}$, the constraint $-\nabla_u V(x, u_x^0(w)) \in C(x, u^0(x))$ may be expressed as

$$-L_x^0(R(K_x w + k_x) + Sw + r) \leq 0$$

which is also an affine inequality in the variable w. Thus, for each x, there exists a matrix F_x and vector f_x such that

$$R_x^0 = \{w \mid F_x w \leq f_x\}$$

so that R_x^0 is polyhedral. Since $u_x^0(x) = u^0(x)$, it follows that $u_x^0(x) \in \mathcal{U}(x)$ and $-\nabla_u V(x, u_x^0(x)) \in C^*(x, u^0(x))$ so that $x \in R_x^0$.

Our next task is to bound the number of distinct regions R_x^0 that exist as we permit x to range over $\mathcal{X}$. We note, from its definition, that R_x^0 is determined, through the constraint $M_x^0 u = N_x^0 w + p_x^0$ in $\mathbb{P}_x(w)$, through $u_x^0(\cdot)$ and through $C^*(x, u^0(x))$, by $I^0(x)$ so that $R_{x_1}^0 \neq R_{x_2}^0$ implies that $I^0(x_1) \neq I^0(x_2)$. Since the number of subsets of $\{1, 2, \ldots, p\}$ is finite, the number of distinct regions R_x^0 as x ranges over $\mathcal{X}$ is finite. Because each $x \in \mathcal{X}$ lies in the set R_x^0, there exists a discrete set of points $X \subset \mathcal{X}$ such that $\mathcal{X} = \cup\{R_x^0 \mid x \in X\}$. We have proved the following.

Proposition 7.10 (Piecewise quadratic (affine) cost (solution)).

(a) There exists a set X of a finite number of points in $\mathcal{X}$ such that $\mathcal{X} = \cup\{R_x^0 \mid x \in X\}$ and $\{R_x^0 \mid x \in X\}$ is a polyhedral partition of $\mathcal{X}$.

(b) The value function $V^0(\cdot)$ of the parametric piecewise QP $\mathbb{P}$ is piecewise quadratic in $\mathcal{X}$, being quadratic and equal to $V_x^0(\cdot)$, defined in (7.3) *in each polyhedron R_x, $x \in X$. Similarly, the minimizer $u^0(\cdot)$ is piecewise affine in $\mathcal{X}$, being affine and equal to $u_x^0(\cdot)$ defined in* (7.4) *in each polyhedron R_x^0, $x \in X$.*

Example 7.11: Parametric QP

Consider the example in Section 7.2. This may be expressed as

$$V^0(x) = \min_u V(x, u), \quad V(x, u) := \{(1/2)x^2 - ux + u^2 \mid Mu \leq Nx + p\}$$

where

$$M = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \quad N = \begin{bmatrix} 0 \\ 1/2 \\ 1 \end{bmatrix} \quad p = \begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix}$$

At $x = 1$, $u^0(x) = 3/2$ and $I^0(x) = \{2\}$. The equality constrained optimization problem $\mathbb{P}_x(w)$ is

$$V_x^0(w) = \min_u \{(1/2)w^2 - uw + u^2 \mid -u = (1/2)w - 2\}$$

so that $u^0(w) = 2 - w/2$. Hence

$$R_x^0 := \left\{ w \;\middle|\; \begin{matrix} Mu_x^0(w) \leq Nw + p(w) \\ -\nabla_u V(w, u_x^0(w)) \in C^*(x, u^0(x)) \end{matrix} \right\}$$

Since $M_2 = -1$, $C^*(x) = \text{cone}\{M_i' \mid i \in I^0(x)\} = \text{cone}\{M_2'\} = \{h \in \mathbb{R} \mid h \leq 0\}$; also

$$\nabla_u V(w, u_x^0(w)) = -w + 2u^0(w) = -w + 2(2 - w/2) = -2w + 4$$

so that R_x^0 is defined by the following inequalities

$$\begin{aligned} (1/2)w - 2 &\leq -1 && \text{or } w \leq 2 \\ (1/2)w - 2 &\leq (1/2)w - 2 && \text{or } w \in \mathbb{R} \\ (1/2)w - 2 &\leq w - 2 && \text{or } w \geq 0 \\ 2w - 4 &\leq 0 && \text{or } w \leq 2 \end{aligned}$$

which reduces to $w \in [0, 2]$ so $R_x^0 = [0, 2]$ when $x = 1$; $[0, 2]$ is the set X_2 determined in Section 7.2. □

Example 7.12: Explicit optimal control

We return to the MPC problem presented in Example 2.5 of Chapter 2

$$\begin{aligned} V^0(x, \mathbf{u}) &= \min_{\mathbf{u}}\{V(x, \mathbf{u}) \mid \mathbf{u} \in \mathcal{U}\} \\ V(x, \mathbf{u}) &:= (3/2)x^2 + [2x, x]\mathbf{u} + (1/2)\mathbf{u}'H\mathbf{u} \\ H &:= \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \\ \mathcal{U} &:= \{\mathbf{u} \mid M\mathbf{u} \leq p\} \end{aligned}$$

where

$$M := \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \qquad p := \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

It follows from the solution to Example 2.5 that

$$u^0(2) = \begin{bmatrix} -1 \\ -(1/2) \end{bmatrix}$$

and $I^0(x) = \{2\}$. The equality constrained optimization problem at $x = 2$ is

$$V_x^0(w) = \min_{\mathbf{u}}\{(3/2)w^2 + 2wu_1 + wu_2 + (1/2)\mathbf{u}'H\mathbf{u} \mid u_1 = -1\}$$

so that

$$u_x^0(w) = \begin{bmatrix} -1 \\ (1/2) - (1/2)w \end{bmatrix}$$

Hence $u_x^0(2) = [-1, -1/2]' = u^0(2)$ as expected. Since $M_x^0 = M_2 = [-1, 0]$, $C^*(x, u^0(x)) = \{g \in \mathbb{R}^2 \mid g_1 \leq 0\}$. Also

$$\nabla_{\mathbf{u}} V(w, \mathbf{u}) = \begin{bmatrix} 2w + 3u_1 + u_2 \\ w + u_1 + 2u_2 \end{bmatrix}$$

so that

$$\nabla_{\mathbf{u}} V(w, \mathbf{u}_x^0(w)) = \begin{bmatrix} (3/2)w - (5/2) \\ 0 \end{bmatrix}$$

Hence R_x^0, $x = 2$ is the set of w satisfying the following inequalities

$$\begin{aligned} (1/2) - (1/2)w &\leq 1 && \text{or } w \geq -1 \\ (1/2) - (1/2)w &\geq -1 && \text{or } w \leq 3 \\ -(3/2)w + (5/2) &\leq 0 && \text{or } w \geq (5/3) \end{aligned}$$

which reduces to $w \in [5/3, 3]$; hence $R_x^0 = [5/3, 3]$ when $x = 2$ as shown in Example 2.5. □

7.3.5 Continuity of $V^0(\cdot)$ and $u^0(\cdot)$

Continuity of $V^0(\cdot)$ and $u^0(\cdot)$ follows from Theorem C.34 in Appendix C. We present here a simpler proof based on the above analysis. We use the fact that the parametric quadratic problem is strictly convex, i.e., for each $x \in \mathcal{X}$, $u \mapsto V(x, u)$ is strictly convex and $\mathcal{U}(x)$ is convex, so that the minimizer $u^0(x)$ is unique as shown in Proposition C.8 of Appendix C.

Let $X = \{x_i \mid i \in \mathbb{I}_{1:I}\}$ denote the set defined in Proposition 7.10(a). For each $i \in \mathbb{I}_{i:I}$, let $R_i := R_{x_i}^0$, $V_i(\cdot) := V_{x_i}^0(\cdot)$ and $u_i(\cdot) := u_{x_i}^0(\cdot)$. From Proposition 7.10, $u^0(x) = u_i(x)$ for each $x \in R_i$, each $i \in \mathbb{I}_{1:I}$ so that $u^0(\cdot)$ is affine and hence continuous in the interior of each R_i, and also continuous at any point x on the boundary of $\mathcal{X}$ such that x lies in a single region R_i. Consider now a point x lying in the intersection of several regions, $x \in \cap_{i \in J} R_i$, where J is a subset of $\mathbb{I}_{1:I}$. Then, by Proposition 7.10, $u_i(x) = u^0(x)$ for all $x \in \cap_{i \in J} R_i$, all $i \in J$. Each $u_i(\cdot)$ is affine and, therefore, continuous, so that $u^0(\cdot)$ is continuous in $\cap_{i \in J} R_i$. Hence $u^0(\cdot)$ is continuous in $\mathcal{X}$. Because $V(\cdot)$ is continuous and $u^0(\cdot)$ is continuous in $\mathcal{X}$, the value function $V^0(\cdot)$ defined by $V^0(x) = V(x, u^0(x))$ is also continuous in $\mathcal{X}$. Let S denote any bounded subset of $\mathcal{X}$.

Then, since $V^0(x) = V_i(x) = (1/2)x'Q_i x + r_i' x + s_i$ for all $x \in R_i$, all $i \in \mathbb{I}_{1:I}$ where $Q_i := Q_{x_i}$, $r_i := r_{x_i}$ and $s_i := s_{x_i}$, it follows that $V^0(\cdot)$ is Lipschitz continuous in each set $R_i \cap S$ and, hence, Lipschitz continuous in $\mathcal{X} \cap S$. We have proved the following.

Proposition 7.13 (Continuity of cost and solution)**.** *The value function $V^0(\cdot)$ and the minimizer $u^0(\cdot)$ are continuous in $\mathcal{X}$. Moreover, the value function and the minimizer are Lipschitz continuous on bounded sets.*

7.4 Constrained Linear Quadratic Control

We now show how parametric quadratic programming may be used to solve the optimal receding horizon control problem when the system is linear, the constraints polyhedral, and the cost is quadratic. The system is described, as before, by

$$x^+ = Ax + Bu \tag{7.6}$$

and the constraints are, as before

$$x \in \mathbb{X} \qquad u \in \mathbb{U} \tag{7.7}$$

where $\mathbb{X}$ is a polyhedron containing the origin in its interior and $\mathbb{U}$ is a polytope also containing the origin in its interior. There may be a terminal constraint of the form

$$x(N) \in \mathbb{X}_f \tag{7.8}$$

where $\mathbb{X}_f$ is a polyhedron containing the origin in its interior. The cost is

$$V_N(x, \mathbf{u}) = \left[\sum_{i=0}^{N-1} \ell(x(i), u(i))\right] + V_f(x(N)) \tag{7.9}$$

in which, for all i, $x(i) = \phi(i; x, \mathbf{u})$, the solution of (7.6) at time i if the initial state at time 0 is x and the control sequence is $\mathbf{u} := (u(0), u(1), \ldots, u(N-1))$. The functions $\ell(\cdot)$ and $V_f(\cdot)$ are quadratic

$$\ell(x, u) := (1/2)x'Qx + (1/2)u'Ru, \quad V_f(x) := (1/2)x'Q_f x \tag{7.10}$$

The state and control constraints (7.7) induce, via the difference equation (7.6), an implicit constraint $(x, \mathbf{u}) \in \mathcal{Z}$ where

$$\mathbb{Z} := \{(x, \mathbf{u}) \mid x(i) \in \mathbb{X},\ u(i) \in \mathbb{U}, i \in \mathbb{I}_{0:N-1}, x(N) \in \mathbb{X}_f\} \tag{7.11}$$

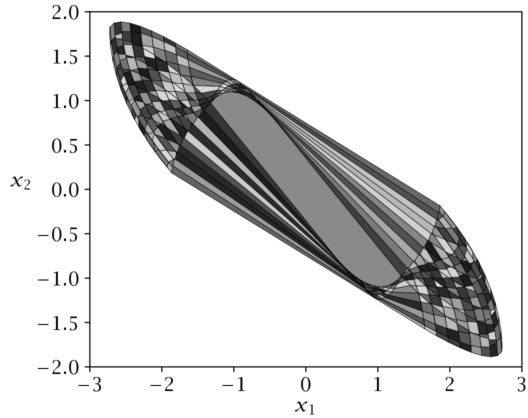

Figure 7.6: Regions R_x, $x \in X$ for a second-order example; after Mayne and Raković (2003).

where, for all i, $x(i) = \phi(i; x, \mathbf{u})$. It is easily seen that $\mathbb{Z}$ is polyhedral since, for each i, $x(i) = A^i x + M_i \mathbf{u}$ for some matrix M_i in $\mathbb{R}^{n \times Nm}$; here $\mathbf{u}$ is regarded as the column vector $\begin{bmatrix} u(0)' & u(1)' & \cdots & u(N-1)' \end{bmatrix}'$. Clearly $x(i) = \phi(i; x, \mathbf{u})$ is linear in $(x, \mathbf{u})$. The constrained linear optimal control problem may now be defined by

$$V_N^0(x) = \min_{\mathbf{u}} \{V_N(x, \mathbf{u}) \mid (x, \mathbf{u}) \in \mathbb{Z}\}$$

Using the fact that for each i, $x(i) = A^i x + M_i \mathbf{u}$, it is possible to determine matrices $\mathbf{Q} \in \mathbb{R}^{n \times n}$, $\mathbf{R} \in \mathbb{R}^{Nm \times Nm}$, and $\mathbf{S} \in \mathbb{R}^{Nm \times n}$ such that

$$V_N(x, \mathbf{u}) = (1/2)x'\mathbf{Q}x' + (1/2)\mathbf{u}'\mathbf{R}\mathbf{u} + \mathbf{u}'\mathbf{S}x \tag{7.12}$$

Similarly, as shown above, there exist matrices $\mathbf{M}$, $\mathbf{N}$ and a vector $\mathbf{p}$ such that

$$\mathbb{Z} = \{(x, \mathbf{u}) \mid \mathbf{M}\mathbf{u} \leq \mathbf{N}x + \mathbf{p}\} \tag{7.13}$$

This is precisely the parametric problem studied in Section 7.3, so that the solution $\mathbf{u}^0(x)$ to $\mathbb{P}(x)$ is piecewise affine on a polytopic partition $\mathcal{P} = \{R_x \mid x \in X\}$ of $\mathcal{X}$ the projection of $\mathbb{Z} \subset \mathbb{R}^n \times \mathbb{R}^{Nm}$ onto $\mathbb{R}^n$, being affine in each of the constituent polytopes of $\mathcal{P}$. The receding horizon control law is $x \mapsto u^0(0; x)$, the first element of $\mathbf{u}^0(x)$. An example is shown in Figure 7.6.

7.5 Parametric Piecewise Quadratic Programming

The dimension of the decision variable **u** in the constrained linear quadratic control problem discussed in Section 7.4 is Nm which is large. It may be better to employ dynamic programming by solving a sequence of problems $\mathbb{P}_1$, $\mathbb{P}_2$, ..., $\mathbb{P}_N$. Although $\mathbb{P}_1$ is a conventional parametric QP, each problem $\mathbb{P}_i$, $i = 2, 3, \ldots, N$, has the form

$$V_i^0(x) = \min_u \{V_{i-1}^0(Ax + Bu) + \ell(x, u) \mid u \in \mathbb{U}, Ax + Bu \in \mathcal{X}_{i-1}\}$$

in which $V_{i-1}^0(\cdot)$ is piecewise quadratic and $\mathcal{X}_{i-1}$ is polyhedral. The decision variable u in each problem $\mathbb{P}_i$ has dimension m. But each problem $\mathbb{P}_i(x)$, $x \in \mathcal{X}_i$, is a parametric piecewise QP rather than a conventional parametric QP. Hence a method for solving parametric piecewise quadratic programming problems is required if dynamic programming is employed to obtain a parametric solution to $\mathbb{P}_N$. Readers not concerned with this extension should proceed to Section 7.7.

The parametric QP $\mathbb{P}(x)$ is defined, as before, by

$$V^0(x) = \min_u \{V(x, u) \mid (x, u) \in \mathbb{Z}\} \tag{7.14}$$

where $x \in \mathcal{X} \subset \mathbb{R}^n$ and $u \in \mathbb{R}^m$, but now the cost function $V(\cdot)$ is assumed to be continuous, strictly convex, and piecewise quadratic on a polytopic partition $\mathcal{P} = \{\mathbb{Z}_i \mid i \in \mathcal{I}\}$ of the set $\mathbb{Z}$ so that

$$V(z) = V_i(z) = (1/2)z'\mathcal{Q}_i z + s_i' z + c_i$$

for all $z \in \mathbb{Z}_i$, all $i \in \mathcal{I}$ where $\mathcal{I}$ is an index set.[3] In (7.14), the matrix $\mathcal{Q}_i$ and the vector s_i have the structure

$$\mathcal{Q}_i = \begin{bmatrix} Q_i & S_i' \\ S_i & R_i \end{bmatrix} \qquad s_i = \begin{bmatrix} q_i \\ r_i \end{bmatrix}$$

so that for all $i \in \mathcal{I}$

$$V_i(x, u) = (1/2)x'Q_i x + u'S_i x + (1/2)u'R_i u + q_i' x + r_i' u + c$$

For each x, the function $u \mapsto V_i(x, u)$ is quadratic and depends on x. The constraint set $\mathbb{Z}$ is defined, as above, by

$$\mathbb{Z} := \{(x, u) \mid Mu \leq Nx + p\}$$

[3] Note that in this section the subscript i denotes partition i rather than "time to go."

Let $u^0(x)$ denote the solution of $\mathbb{P}(x)$, i.e.,

$$u^0(x) = \arg\min_u \{V(x,u) \mid (x,u) \in \mathbb{Z}\}$$

The solution $u^0(x)$ is unique if $V(\cdot)$ is strictly convex in u at each x; this is the case if each R_i is positive definite. The parametric piecewise QP may also be expressed, as before, as

$$V^0(x) = \min_u \{V(x,u) \mid u \in \mathcal{U}(x)\}$$
$$u^0(x) = \arg\min_u \{V(x,u) \mid u \in \mathcal{U}(x)\}$$

where the parametric constraint set $\mathcal{U}(x)$ is defined by

$$\mathcal{U}(x) := \{u \mid (x,u) \in \mathbb{Z}\} = \{u \mid Mu \leq Nx + p\}$$

Let $\mathcal{X} \subset \mathbb{R}^n$ be defined by

$$\mathcal{X} := \{x \mid \exists u \text{ such that } (x,u) \in \mathbb{Z}\} = \{x \mid \mathcal{U}(x) \neq \emptyset\}$$

The set $\mathcal{X}$ is the domain of $V^0(\cdot)$ and of $u^0(\cdot)$ and is thus the set of points x for which a feasible solution of $\mathbb{P}(x)$ exists; it is the projection of $\mathbb{Z}$, which is a set in (x,u)-space, onto x-space as shown in Figure 7.1. We make the following assumption in the sequel.

Assumption 7.14 (Continuous, piecewise quadratic function)**.** The function $V(\cdot)$ is continuous, strictly convex, and piecewise quadratic on the polytopic partition $\mathcal{P} = \{\mathbb{Z}_i \mid i \in \mathcal{I} := \mathbb{I}_{1:q}\}$ of the polytope $\mathbb{Z}$ in $\mathbb{R}^n \times \mathbb{R}^m$; $V(x,u) = V_i(x,u)$ where $V_i(\cdot)$ is a positive definite quadratic function of (x,u) for all $(x,u) \in \mathbb{Z}_i$, all $i \in \mathcal{I}$, and q is the number of constituent polytopes in $\mathcal{P}$.

The assumption of continuity places restrictions on the quadratic functions $V_i(\cdot)$, $i \in \mathcal{I}$. For example, we must have $V_i(z) = V_j(z)$ for all $z \in \mathbb{Z}_i \cap \mathbb{Z}_j$. Assumption 7.14 implies that the piecewise quadratic programming problem $\mathbb{P}(x)$ satisfies the hypotheses of Theorem C.34 so that the value function $V^0(\cdot)$ is continuous. It follows from Assumption 7.14 and Theorem C.34 that $V^0(\cdot)$ is strictly convex and continuous and that the minimizer $u^0(\cdot)$ is continuous. Assumption 7.14 implies that Q_i is positive definite for all $i \in \mathcal{I}$. For each x, let the set $\mathcal{U}(x)$ be defined by

$$\mathcal{U}(x) := \{u \mid (x,u) \in \mathbb{Z}\}$$

Thus $\mathcal{U}(x)$ is the set of admissible u at x, and $\mathbb{P}(x)$ may be expressed in the form $V^0(x) = \min_u \{V(x,u) \mid u \in \mathcal{U}(x)\}$.

For each $i \in \mathcal{I}$, we define an "artificial" problem $\mathbb{P}_i(x)$ as follows

$$V_i^0(x) := \min_u \{V_i(x,u) \mid (x,u) \in \mathbb{Z}_i\}$$
$$u_i^0(x) := \arg\min_u \{V_i(x,u) \mid (x,u) \in \mathbb{Z}_i\}$$

The cost $V_i(x,u)$ in the above equations may be replaced by $V(x,u)$ since $V(x,u) = V_i(x,u)$ in $\mathbb{Z}_i$. The problem is artificial because it includes constraints (the boundaries of $\mathbb{Z}_i$) that are not necessarily constraints of the original problem. We introduce this problem because it helps us to understand the solution of the original problem. For each $i \in \mathbb{I}_{1:p}$, let the set $\mathcal{U}_i(x)$ be defined as follows

$$\mathcal{U}_i(x) := \{u \mid (x,u) \in \mathbb{Z}_i\}$$

Thus the set $\mathcal{U}_i(x)$ is the set of admissible u at x, and problem $\mathbb{P}_i(x)$ may be expressed as $V_i^0(x) := \min_u\{V_i(x,u) \mid u \in \mathcal{U}_i(x)\}$; the set $\mathcal{U}_i(x)$ is polytopic. For each i, problem $\mathbb{P}_i(x)$ may be recognized as a standard parametric QP discussed in Section 7.4. Because of the piecewise nature of $V(\cdot)$, we require another definition.

Definition 7.15 (Active polytope (polyhedron))**.** A polytope (polyhedron) $\mathbb{Z}_i$ in a polytopic (polyhedral) partition $\mathcal{P} = \{\mathbb{Z}_i \mid i \in \mathcal{I}\}$ of a polytope (polyhedron) $\mathbb{Z}$ is said to be *active* at $z \in \mathbb{Z}$ if $z = (x,u) \in \mathbb{Z}_i$. The index set specifying the polytopes active at $z \in \mathbb{Z}$ is

$$S(z) := \{i \in \mathcal{I} \mid z \in \mathbb{Z}_i\}$$

A polytope $\mathbb{Z}_i$ in a polytopic partition $\mathcal{P} = \{\mathbb{Z}_i \mid i \in \mathcal{I}\}$ of a polytope $\mathbb{Z}$ is said to be *active* for problem $\mathbb{P}(x)$) if $(x, u^0(x)) \in \mathbb{Z}_i$. The index set specifying polytopes active at $(x, u^0(x))$ is $S^0(x)$ defined by

$$S^0(x) := S(x, u^0(x)) = \{i \in \mathcal{I} \mid (x, u^0(x)) \in \mathbb{Z}_i\}$$

Because we know how to solve the "artificial" problems $\mathbb{P}_i(x)$, $i \in \mathcal{I}$ that are parametric quadratic programs, it is natural to ask if we can recover the solution of the original problem $\mathbb{P}(x)$ from the solutions to these simpler problems. This question is answered by the following proposition.

Proposition 7.16 (Solving $\mathbb{P}$ using $\mathbb{P}_i$)**.** *For any $x \in \mathcal{X}$, u is optimal for $\mathbb{P}(x)$ if and only if u is optimal for $\mathbb{P}_i(x)$ for all $i \in S(x,u)$.*

Proof. (i) Suppose u is optimal for $\mathbb{P}(x)$ but, contrary to what we wish to prove, there exists an $i \in S(x,u) = S^0(x)$ such that u is not optimal for $\mathbb{P}_i(x)$. Hence there exists a $v \in \mathbb{R}^m$ such that $(x,v) \in \mathbb{Z}_i$ and $V(x,v) = V_i(x,v) < V_i(x,u) = V(x,u) = V^0(x)$, a contradiction of the optimality of u for $\mathbb{P}(x)$. (ii) Suppose u is optimal for $\mathbb{P}_i(x)$ for all $i \in S(x,u)$ but, contrary to what we wish to prove, u is not optimal for $\mathbb{P}(x)$. Hence $V^0(x) = V(x,u^0(x)) < V(x,u)$. If $u^0(x) \in \mathbb{Z}^{(x,u)} := \cup_{i \in S(x,u)} \mathbb{Z}_i$, we have a contradiction of the optimality of u in $\mathbb{Z}^{(x,u)}$. Assume then that $u^0(x) \in \mathbb{Z}_j$, $j \notin S(x,u)$; for simplicity, assume further that $\mathbb{Z}_j$ is adjacent to $\mathbb{Z}^{(x,u)}$. Then, there exists a $\lambda \in (0,1]$ such that $u^\lambda := u + \lambda(u^0(x) - u) \in \mathbb{Z}^{(x,u)}$; if not, $j \in S(x,u)$, a contradiction. Since $V(\cdot)$ is strictly convex, $V(x,u^\lambda) < V(x,u)$, which contradicts the optimality of u in $\mathbb{Z}^{(x,u)}$. The case when $\mathbb{Z}_j$ is not adjacent to $\mathbb{Z}^{(x,u)}$ may be treated similarly. ∎

To obtain a parametric solution, we proceed as before. We select a point $x \in \mathcal{X}$ and obtain the solution $u^0(x)$ to $\mathbb{P}(x)$ using a standard algorithm for convex programs. The solution $u^0(x)$ satisfies an equality constraint $E_x u = F_x x + g_x$, which we employ to define, for any $w \in \mathcal{X}$ near x an easily solved equality constrained optimization problem $\mathbb{P}_x(w)$ that is derived from the problems $\mathbb{P}_i(x)$, $i \in S^0(x)$. Finally, we show that the solution to this simple problem is also a solution to the original problem $\mathbb{P}(w)$ at all w in a polytope $R_x \subset \mathcal{X}$ in which x lies.

For each $i \in \mathcal{I}$, $\mathbb{Z}_i$ is defined by

$$\mathbb{Z}_i := \{(x,u) \mid M^i u \leq N^i x + p^i\}$$

Let M^i_j, N^i_j and q^i_j denote, respectively, the jth row of M^i, N^i and q^i, and let $I_i(x,u)$ and $I^0_i(x)$, defined by

$$I_i(x,u) := \{j \mid M^i_j u = N^i_j x + p^i_j\}, \quad I^0_i(x) := I_i(x, u^0_i(x))$$

denote, respectively, the active constraint set at $(x,u) \in \mathbb{Z}_i$ and the active constraint set for $\mathbb{P}_i(x)$. Because we now use subscript i to specify $\mathbb{Z}_i$, we change our notation slightly and now let $C_i(x,u)$ denote the cone of first-order feasible variations for $\mathbb{P}_i(x)$ at $u \in \mathcal{U}_i(x)$, i.e.,

$$C_i(x,u) := \{h \in \mathbb{R}^m \mid M^i_j h \leq 0 \ \forall j \in I_i(x,u)\}$$

Similarly, we define the polar cone $C^*_i(x,u)$ of the cone $C_i(x,u)$ at

$h = 0$ by

$$C_i^*(x,u) := \{v \in \mathbb{R}^m \mid v'h \leq 0 \ \forall h \in C_i(x,u)\}$$
$$= \left\{ \sum_{j \in I_i(x,u)} (M_j^i)'\lambda_j \;\middle|\; \lambda_j \geq 0, j \in I_i(x,u) \right\}$$

As shown in Proposition 7.7, a necessary and sufficient condition for the optimality of u for problem $\mathbb{P}_i(x)$ is

$$-\nabla_u V_i(x,u) \in C_i^*(x,u), \quad u \in \mathcal{U}_i(x) \tag{7.15}$$

If u lies in the interior of $\mathcal{U}_i(x)$ so that $I_i^0(x) = \emptyset$, condition (7.15) reduces to $\nabla_u V_i(x,u) = 0$. For any $x \in \mathcal{X}$, the solution $u^0(x)$ of the piecewise parametric program $\mathbb{P}(x)$ satisfies

$$M_j^i u = N_j^i x + p_j^i, \quad \forall j \in I_i^0(x),\ \forall i \in S^0(x) \tag{7.16}$$

To simplify our notation, we rewrite the equality constraint (7.16) as

$$E_x u = F_x x + g_x$$

where the subscript x denotes the fact that the constraints are precisely those constraints that are active for the problems $\mathbb{P}_i(x)$, $i \in S^0(x)$. The fact that $u^0(x)$ satisfies these constraints and is, therefore, the unique solution of the optimization problem

$$V^0(x) = \min_u \{V(x,u) \mid E_x u = F_x x + g_x\}$$

motivates us to define the equality constrained problem $\mathbb{P}_x(w)$ for $w \in \mathcal{X}$ near x by

$$V_x^0(w) = \min_u \{V_x(w,u) \mid E_x u = F_x w + g_x\}$$

where $V_x(w,u) := V_i(w,u)$ for all $i \in S^0(x)$ and is, therefore, a positive definite quadratic function of (x,u). The notation $V_x^0(w)$ denotes the fact that the parameter in the parametric problem $\mathbb{P}_x(w)$ is now w but the data for the problem, namely (E_x, F_x, g_x), is derived from the solution $u^0(x)$ of $\mathbb{P}(x)$ and is, therefore, x-dependent. Problem $\mathbb{P}_x(w)$ is a simple equality constrained problem in which the cost $V_x(\cdot)$ is quadratic and the constraints $E_x u = F_x w + g_x$ are linear. Let $V_x^0(w)$ denote the value of $\mathbb{P}_x(w)$ and $u_x^0(w)$ its solution. Then

$$\begin{aligned} V_x^0(w) &= (1/2)w'Q_x w + r_x' w + s_x \\ u_x^0(w) &= K_x w + k_x \end{aligned} \tag{7.17}$$

where Q_x, r_x, s_x, K_x and k_x are easily determined. It is easily seen that $u_x^0(x) = u^0(x)$ so that $u_x^0(x)$ is optimal for $\mathbb{P}(x)$. Our hope is that $u_x^0(w)$ is optimal for $\mathbb{P}(w)$ for all w in some neighborhood R_x of x. We now show this is the case.

Proposition 7.17 (Optimality of $u_x^0(w)$ in R_x)**.** *Let x be an arbitrary point in $\mathcal{X}$. Then*

(a) $u^0(w) = u_x^0(w)$ and $V^0(w) = V_x^0(w)$ for all w in the set R_x defined by

$$R_x := \left\{ w \in \mathbb{R}^n \,\middle|\, \begin{array}{c} u_x^0(w) \in \mathcal{U}_i(w)\ \forall i \in S^0(x) \\ -\nabla_u V_i(w, u_x^0(w)) \in C_i^*(x, u^0(x))\ \forall i \in S^0(x) \end{array} \right\}$$

(b) R_x is a polytope

(c) $x \in R_x$

Proof.

(a) Because of the equality constraint 7.16 it follows that $I_i(w, u_x(w)) \supseteq I_i(x, u^0(x))$ and that $S(w, u_x^0(w)) \supseteq S(x, u^0(x))$ for all $i \in S(x, u^0(x)) = S^0(x)$, all $w \in R_x$. Hence $C_i(w, u_x^0(w)) \subseteq C_i(x, u^0(x))$, which implies $C_i^*(w, u_x^0(w)) \supseteq C_i^*(x, u^0(x))$ for all $i \in S(x, u^0(x)) \subseteq S(w, u_x^0(w))$. It follows from the definition of R_x that $u_x^0(w) \in \mathcal{U}_i(w)$ and that $-\nabla_u V_i(w, u_x^0(w)) \in C_i^*(w, u_x^0(w))$ for all $i \in S(w, u_x^0(w))$. Hence $u = u_x^0(w)$ satisfies necessary and sufficient for optimality for $\mathbb{P}_i(w)$ for all $i \in S(w, u)$, all $w \in R_x$ and, by Proposition 7.16, necessary and sufficient conditions of optimality for $\mathbb{P}(w)$ for all $w \in R_x$. Hence $u_x^0(w) = u^0(w)$ and $V_x^0(w) = V^0(w)$ for all $w \in R_x$.

(b) That R_x is a polytope follows from the facts that the functions $w \mapsto u_x^0(w)$ and $w \mapsto \nabla_u V_i(w, u_x^0(w))$ are affine, the sets $\mathbb{Z}_i$ are polytopic and the sets $C_i^0(x, u^0(x))$ are polyhedral; hence $(w, u_x^0(w)) \in \mathbb{Z}_i$ is a polytopic constraint and $-\nabla_u V_i(w, u_x^0(w)) \in C_i^*(x, u^0(x))$ a polyhedral constraint on w.

(c) That $x \in R_x$ follows from Proposition 7.16 and the fact that $u_x^0(x) = u^0(x)$. ∎

Reasoning as in the proof of Proposition 7.10, we obtain the following.

Proposition 7.18 (Piecewise quadratic (affine) solution)**.** *There exists a finite set of points X in $\mathcal{X}$ such that $\{R_x \mid x \in X\}$ is a polytopic partition of $\mathcal{X}$. The value function $V^0(\cdot)$ for $\mathbb{P}(x)$ is strictly convex and* *piecewise quadratic and the minimizer $u^0(\cdot)$ is piecewise affine in $\mathcal{X}$ being equal, respectively, to the quadratic function $V_x^0(\cdot)$ and the affine function $u_x^0(\cdot)$ in each region R_x, $x \in \mathcal{X}$.*

7.6 DP Solution of the Constrained LQ Control Problem

A disadvantage in the procedure described in Section 7.4 for determining the piecewise affine receding horizon control law is the dimension Nm of the decision variable $\mathbf{u}$. It seems natural to inquire whether or not dynamic programming (DP), which replaces a multistage decision problem by a sequence of relatively simple single-stage problems, provides a simpler solution. We answer this question by showing how DP may be used to solve the constrained linear quadratic (LQ) problem discussed in Section 7.4. For all $j \in \mathbb{I}_{1:N}$, let $V_j^0(\cdot)$, the optimal value function at time-to-go j, be defined by

$$\begin{aligned}
V_j^0(x) &:= \min_u \{V_j(x,u) \mid (x,\mathbf{u}) \in \mathbb{Z}_j\} \\
V_j(x,\mathbf{u}) &:= \sum_{i=0}^{j-1} \ell(x(i),u(i)) + V_f(x(j)) \\
\mathbb{Z}_j &:= \{(x,\mathbf{u}) \mid x(i) \in \mathbb{X}, u(i) \in \mathbb{U}, i \in \mathbb{I}_{0:j-1},\ x(j) \in \mathbb{X}_f\}
\end{aligned}$$

with $x(i) := \phi(i; x, \mathbf{u})$; $V_j^0(\cdot)$ is the value function for $\mathbb{P}_j(x)$. As shown in Chapter 2, the constrained DP recursion is

$$V_{j+1}^0(x) = \min_u \{\ell(x,u) + V_j^0(f(x,u)) \mid u \in \mathbb{U}, f(x,u) \in \mathcal{X}_j\} \tag{7.18}$$

$$\mathcal{X}_{j+1} = \{x \in \mathbb{X} \mid \exists\, u \in \mathbb{U} \text{ such that } f(x,u) \in \mathcal{X}_j\} \tag{7.19}$$

where $f(x,u) := Ax + Bu$ with boundary condition

$$V_0^0(\cdot) = V_f(\cdot), \quad \mathcal{X}_0 = \mathbb{X}_f$$

The minimizer of (7.18) is $\kappa_{j+1}(x)$. In the equations, the subscript j denotes time to go, so that current time $i = N - j$. For each j, $\mathcal{X}_j$ is the domain of the value function $V_j^0(\cdot)$ and of the control law $\kappa_j(\cdot)$, and is the set of states that can be steered to the terminal set $\mathbb{X}_f$ in j steps or less by an admissible control that satisfies the state and control constraints. The time-invariant receding horizon control law for horizon j is $\kappa_j(\cdot)$ whereas the optimal policy for problem $\mathbb{P}_j(x)$ is $\{\kappa_j(\cdot), \kappa_{j-1}(\cdot), \ldots, \kappa_1(\cdot)\}$. The data of the problem are identical to the data in Section 7.4.

We know from Section 7.4 that $V_j^0(\cdot)$ is continuous, strictly convex and piecewise quadratic, and that $\kappa_j(\cdot)$ is continuous and piecewise affine on a polytopic partition $\mathcal{P}_{\mathcal{X}_j}$ of $\mathcal{X}_j$. Hence the function $(x,u) \mapsto V(x,u) := \ell(x,u) + V_j^0(Ax+Bu)$ is continuous, strictly convex and piecewise quadratic on a polytopic partition $\mathcal{P}_{\mathbb{Z}_{j+1}}$ of the polytope $\mathbb{Z}_{j+1}$ defined by

$$\mathbb{Z}_{j+1} := \{(x,u) \mid x \in \mathbb{X}, u \in \mathbb{U}, Ax+Bu \in \mathcal{X}_j\}$$

The polytopic partition $\mathcal{P}_{\mathbb{Z}_{j+1}}$ of $\mathbb{Z}_{j+1}$ may be computed as follows: if X is a constituent polytope of $\mathcal{X}_j$, then, from (7.19), the corresponding constituent polytope of $\mathcal{P}_{\mathbb{Z}_{j+1}}$ is the polytope Z defined by

$$Z := \{z = (x,u) \mid x \in \mathbb{X}, u \in \mathbb{U}, Ax+Bu \in X\}$$

Thus Z is defined by a set of linear inequalities; also $\ell(x,u) + V_j^0(f(x,u))$ is quadratic on Z. Thus the techniques of Section 7.5 can be employed for its solution, yielding the piecewise quadratic value function $V_{j+1}^0(\cdot)$, the piecewise affine control law $\kappa_{j+1}(\cdot)$, and the polytopic partition $\mathcal{P}_{\mathcal{X}_{j+1}}$ on which $V_{j+1}^0(\cdot)$ and $\kappa_{j+1}(\cdot)$ are defined. Each problem (7.18) is much simpler than the problem considered in Section 7.4 since m, the dimension of u, is much less than Nm, the dimension of $\mathbf{u}$. Thus, the DP solution is preferable to the direct method described in Section 7.4.

7.7 Parametric Linear Programming

7.7.1 Preliminaries

The parametric linear program $\mathbb{P}(x)$ is

$$V^0(x) = \min_u \{V(x,u) \mid (x,u) \in \mathbb{Z}\}$$

where $x \in \mathcal{X} \subset \mathbb{R}^n$ and $u \in \mathbb{R}^m$, the cost function $V(\cdot)$ is defined by

$$V(x,u) = q'x + r'u$$

and the constraint set $\mathbb{Z}$ is defined by

$$\mathbb{Z} := \{(x,u) \mid Mu \leq Nx + p\}$$

Let $u^0(x)$ denote the solution of $\mathbb{P}(x)$, i.e.,

$$u^0(x) = \arg\min_u \{V(x,u) \mid (x,u) \in \mathbb{Z}\}$$

The solution $u^0(x)$ may be set valued. The parametric linear program (LP) may also be expressed as

$$V^0(x) = \min_u \{V(x,u) \mid u \in \mathcal{U}(x)\}$$

where, as before, the parametric constraint set $\mathcal{U}(x)$ is defined by

$$\mathcal{U}(x) := \{u \mid (x,u) \in \mathbb{Z}\} = \{u \mid Mu \leq Nx + p\}$$

Also, as before, the domain of $V^0(\cdot)$ and $u^0(\cdot)$, i.e., the set of points x for which a feasible solution of $\mathbb{P}(x)$ exists, is the set $\mathcal{X}$ defined by

$$\mathcal{X} := \{x \mid \exists u \text{ such that } (x,u) \in \mathbb{Z}\} = \{x \mid \mathcal{U}(x) \neq \emptyset\}$$

The set $\mathcal{X}$ is the projection of $\mathbb{Z}$ (which is a set in (x,u)-space) onto x-space; see Figure 7.1. We assume in the sequel that the problem is well posed, i.e., for each $x \in \mathcal{X}$, $V^0(x) > -\infty$. This excludes problems like $V^0(x) = \inf_u\{x + u \mid -x \leq 1, x \leq 1\}$ for which $V^0(x) = -\infty$ for all $x \in \mathcal{X} = [-1, 1]$. Let $\mathbb{I}_{1:p}$ denote, as usual, the index set $\{1, 2, \ldots, p\}$. For all $(x,u) \in \mathbb{Z}$, let $I(x,u)$ denote the set of active constraints at (x,u), i.e.,

$$I(x,u) := \{i \in \mathbb{I}_{1:p} \mid M_i u = N_i x + p_i\}$$

where A_i denotes the ith row of any matrix (or vector) A. Similarly, for any matrix A and any index set I, A_I denotes the matrix with rows A_i, $i \in I$. If, for any $x \in \mathcal{X}$, $u^0(x)$ is unique, the set $I^0(x)$ of constraints active at $(x, u^0(x))$ is defined by

$$I^0(x) := I(x, u^0(x))$$

When $u^0(x)$ is unique, it is a vertex (a face of dimension zero) of the polyhedron $\mathcal{U}(x)$ and is the *unique* solution of

$$M_x^0 u = N_x^0 x + p_x^0$$

where

$$M_x^0 := M_{I^0(x)}, \; N_x^0 := N_{I^0(x)}, \; p_x^0 := p_{I^0(x)}$$

In this case, the matrix M_x^0 has rank m.

Any face F of $\mathcal{U}(x)$ with dimension $d \in \{1, 2, \ldots, m\}$ satisfies $M_i u = N_i x + p_i$ for all $i \in I_F$, all $u \in F$ for some index set $I_F \subseteq \mathbb{I}_{1:p}$. The matrix M_{I_F} with rows M_i, $i \in I_F$, has rank $m - d$, and the face F is defined by

$$F := \{u \mid M_i u = N_i x + p_i, i \in I_F\} \cap \mathcal{U}(x)$$

When $u^0(x)$ is not unique, it is a face of dimension $d \geq 1$ and the set $I^0(x)$ of active constraints is defined by

$$I^0(x) := \{i \mid M_i u = N_i x + p_i \ \forall u \in u^0(x)\} = \{i \mid i \in I(x,u) \ \forall u \in u^0(x)\}$$

The set $\{u \mid M_i u = N_i x + p_i, \ i \in I^0(x)\}$ is a hyperplane in which $u^0(x)$ lies. See Figure 7.7 where $u^0(x_1)$ is unique, a vertex of $\mathcal{U}(x_1)$, and $I^0(x_1) = \{2, 3\}$. If, in Figure 7.7, $r = -e_1$, then $u^0(x_1) = F_2(x_1)$, a face of dimension 1; $u^0(x_1)$ is, therefore, set valued. Since $u \in \mathbb{R}^m$ where $m = 2$, $u^0(x_1)$ is a facet, i.e., a face of dimension $m - 1 = 1$. Thus $u^0(x_1)$ is a set defined by $u^0(x_1) = \{u \mid M_1 u \leq N_1 x_1 + p_1, M_2 u = N_2 x_1 + p_2, \ M_3 u \leq N_3 x_1 + p_3\}$. At each $z = (x,u) \in \mathbb{Z}$, i.e., for each (x,u) such that $x \in \mathcal{X}$ and $u \in \mathcal{U}(x)$, the cone $C(z) = C(x,u)$ of first-order feasible variations is defined, as before, by

$$C(z) := \{h \in \mathbb{R}^m \mid M_i h \leq 0, \ i \in I(z)\} = \{h \in \mathbb{R}^m \mid M_{I(z)} h \leq 0\}$$

If $I(z) = I(x,u) = \emptyset$ (no constraints are active), $C(z) = \mathbb{R}^m$ (all variations are feasible).

Since $u \mapsto V(x,\cdot)$ is convex and differentiable, and $\mathcal{U}(x)$ is polyhedral for all x, the parametric LP $\mathbb{P}(x)$ satisfies the assumptions of Proposition 7.8. Hence, repeating Proposition 7.8 for convenience, we have

Proposition 7.19 (Optimality conditions for parametric LP)**.** *A necessary and sufficient condition for u to be a minimizer for the parametric LP $\mathbb{P}(x)$ is*

$$u \in \mathcal{U}(x) \text{ and } -\nabla_u V(x,u) \in C^*(x,u)$$

where $\nabla_u V(x,u) = r$ and $C^*(x,u)$ is the polar cone of $C(x,u)$.

An important difference between this result and that for the parametric QP is that $\nabla_u V(x,u) = r$ and, therefore, does not vary with x or u. We now use this result to show that both $V^0(\cdot)$ and $u^0(\cdot)$ are piecewise affine. We consider the simple case when $u^0(x)$ is unique for all $x \in \mathcal{X}$.

7.7.2 Minimizer $u^0(x)$ Is Unique for all $x \in \mathcal{X}$

Before proceeding to obtain the solution to a parametric LP when the minimizer $u^0(x)$ is unique for each $x \in \mathcal{X}$, we look first at the simple example illustrated in Figure 7.7, which shows the constraint set $\mathcal{U}(x)$ for various values of the parameter x in the interval $[x_1, x_3]$. The set

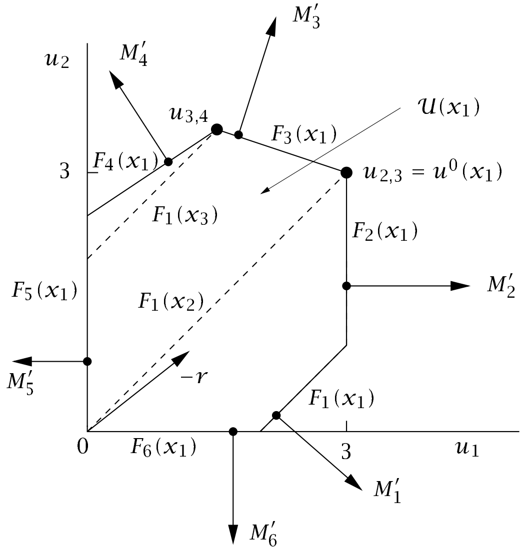

Figure 7.7: Solution to a parametric LP.

$\mathcal{U}(x_1)$ has six faces: $F_1(x_1)$, $F_2(x_1)$, $F_3(x_1)$, $F_4(x_1)$, $F_5(x_1)$, and $F_6(x_1)$. Face $F_1(x)$ lies in the hyperplane $\mathcal{H}_1(x)$ that varies linearly with x; each face $F_i(x)$, $i = 2, \ldots, 6$, lies in the hyperplane $\mathcal{H}_i$ that does *not* vary with x. All the faces vary with x as shown so that $\mathcal{U}(x_2)$ has four faces: $F_1(x_2)$, $F_3(x_2)$, $F_4(x_2)$, and $F_5(x_2)$; and $\mathcal{U}(x_3)$ has three faces: $F_1(x_3)$, $F_4(x_3)$, and $F_5(x_3)$. The face $F_1(x)$ is shown for three values of x: $x = x_1$ (the bold line), and $x = x_2$ and $x = x_3$ (dotted lines). It is apparent that for $x \in [x_1, x_2]$, $u^0(x) = u_{2,3}$ in which $u_{2,3}$ is the intersection of $\mathcal{H}_2$ and $\mathcal{H}_3$, and $u^0(x_3) = u_{3,4}$, in which $u_{3,4}$ is the intersection of $\mathcal{H}_3$ and $\mathcal{H}_4$. It can also be seen that $u^0(x)$ is unique for all $x \in \mathcal{X}$.

We now return to the general case. Suppose, for some $\in \mathcal{X}$, $u^0(x)$ is the unique solution of $\mathbb{P}(x)$; $u^0(x)$ is the unique solution of

$$M_x^0 u = N_x^0 x + p_x^0$$

It follows that $u^0(x)$ is the trivial solution of the simple equality constrained problem defined by

$$V^0(x) = \min_u \{V(x, u) \mid M_x^0 u = N_x^0 x + p_x^0\} \tag{7.20}$$

The solution $u^0(x)$ of this equality constrained problem is trivial because it is determined entirely by the equality constraints; the cost plays no part.

The optimization problem (7.20) motivates us, as in parametric quadratic programming, to consider, for any parameter w "close" to x, the simpler equality constrained problem $\mathbb{P}_x(w)$ defined by

$$\begin{aligned} V_x^0(w) &= \min_u \{V(w,u) \mid M_x^0 u = N_x^0 w + p_x^0\} \\ u_x^0(w) &= \arg\min_u \{V(w,u) \mid M_x^0 u = N_x^0 w + p_x^0\} \end{aligned}$$

Let $u_x^0(w)$ denote the solution of $\mathbb{P}_x(w)$. Because, for each $x \in \mathcal{X}$, the matrix M_x^0 has full rank m, there exists an index set I_x such that $M_{I_x} \in \mathbb{R}^{m\times m}$ is invertible. Hence, for each w, $u_x^0(w)$ is the unique solution of

$$M_{I_x} u = N_{I_x} w + p_{I_x}$$

so that for all $x \in \mathcal{X}$, all $w \in \mathbb{R}^m$

$$u_x^0(w) = K_x w + k_x \tag{7.21}$$

where $K_x := (M_{I_x})^{-1} N_{I_x}$ and $k_x := (M_{I_x})^{-1} p_{I_x}$. In particular, $u^0(x) = u_x^0(x) = K_x x + k_x$. Since $V_x^0(x) = V_x(x, u_x^0(w)) = q'x + r'u_x^0(w)$, it follows that

$$V_x^0(x) = (q' + r'K_x)x + r'k_x$$

for all $x \in \mathcal{X}$, all $w \in \mathbb{R}^m$. Both $V_x^0(\cdot)$ and $u_x^0(\cdot)$ are affine in x.

It follows from Proposition 7.19 that $-r \in C^*(x, u^0(x)) = \text{cone}\{M_i' \mid i \in I^0(x) = I(x, u^0(x))\} = \text{cone}\{M_i' \mid i \in I_x\}$. Since $\mathbb{P}_x(w)$ satisfies the conditions of Proposition 7.8, we may proceed as in Section 7.3.4 and define, for each $x \in \mathcal{X}$, the set R_x^0 as in (7.5)

$$R_x^0 := \left\{ w \in \mathbb{R}^n \;\middle|\; \begin{array}{c} u_x^0(w) \in \mathcal{U}(w) \\ -\nabla_u V(w, u_x^0(w)) \in C^*(x, u^0(x)) \end{array} \right\}$$

It then follows, as shown in Proposition 7.9, that for any $x \in \mathcal{X}$, $u_x^0(w)$ is optimal for $\mathbb{P}(w)$ for all $w \in R_x^0$. Because $\mathbb{P}(w)$ is a parametric LP, however, rather than a parametric QP, it is possible to simplify the definition of R_x^0. We note that $\nabla_u V(w, u_x^0(w)) = r$ for all $x \in \mathcal{X}$, all $w \in \mathbb{R}^m$. Also, it follows from Proposition 7.8, since $u^0(x)$ is optimal for $\mathbb{P}(x)$, that $-\nabla_u V(x, u^0(x)) = -r \in C^*(x)$ so that the second condition in the definition above for R_x^0 is automatically satisfied. Hence we may simplify our definition for R_x^0; for the parametric LP, R_x^0 may be defined by

$$R_x^0 := \{w \in \mathbb{R}^n \mid u_x^0(w) \in \mathcal{U}(w)\} \tag{7.22}$$

Because $u_x^0(\cdot)$ is affine, it follows from the definition of $\mathcal{U}(w)$ that R_x^0 is polyhedral. The next result follows from the discussion in Section 7.3.4.

Proposition 7.20 (Solution of $\mathbb{P}$)**.** *For any $x \in \mathcal{X}$, $u_x^0(w)$ is optimal for $\mathbb{P}(w)$ for all w in the set R_x^0 defined in* (7.22)*.*

Finally, the next result characterizes the solution of the parametric LP $\mathbb{P}(x)$ when the minimizer is unique.

Proposition 7.21 (Piecewise affine cost and solution)**.**

(a) There exists a finite set of points X in $\mathcal{X}$ such that $\{R_x^0 \mid x \in X\}$ is a polyhedral partition of $\mathcal{X}$.

(b) The value function $V^0(\cdot)$ for $\mathbb{P}(x)$ and the minimizer $u^0(\cdot)$ are piecewise affine in $\mathcal{X}$ being equal, respectively, to the affine functions $V_x^0(\cdot)$ and $u_x^0(\cdot)$ in each region R_x, $x \in \mathcal{X}$.

(c) The value function $V^0(\cdot)$ and the minimizer $u^0(\cdot)$ are continuous in $\mathcal{X}$.

Proof. The proof of parts (a) and (b) follows, apart from minor changes, the proof of Proposition 7.10. The proof of part (c) uses the fact that $u^0(x)$ is unique, by assumption, for all $x \in \mathcal{X}$ and is similar to the proof of Proposition 7.13. ∎

7.8 Constrained Linear Control

The previous results on parametric linear programming may be applied to obtain the optimal receding horizon control law when the system is linear, the constraints polyhedral, and the cost linear as is done in a similar fashion in Section 7.4 where the cost is quadratic. The optimal control problem is therefore defined as in Section 7.4, except that the stage cost $\ell(\cdot)$ and the terminal cost $V_f(\cdot)$ are now defined by

$$\ell(x,u) := q'x + r'u \qquad V_f(x) := q_f'x$$

As in Section 7.4, the optimal control problem $\mathbb{P}_N(x)$ may be expressed as

$$V_N^0(x) = \min_{\mathbf{u}}\{V_N(x,\mathbf{u}) \mid \mathbf{M}\mathbf{u} \leq \mathbf{N}x + \mathbf{p}\}$$

where, now

$$V_N(x,\mathbf{u}) = \mathbf{q}'x + \mathbf{r}'\mathbf{u}$$

Hence the problem has the same form as that discussed in Section 7.7 and may be solved as shown there.

It is possible, using a simple transcription, to use the solution of $\mathbb{P}_N(x)$ to solve the optimal control problem when the stage cost and terminal cost are defined by

$$\ell(x, u) := |Qx|_p + |Ru|_p\,, \quad V_f(x) := \left|Q_f x\right|_p$$

where $|\cdot|_p$ denotes the p-norm and p is either 1 or ∞.

7.9 Computation

Our main purpose above was to establish the structure of the solution of parametric linear or QPs and, hence, of the solutions of constrained linear optimal control problems when the cost is quadratic or linear. We have not presented algorithms for solving these problem although; there is now a considerable literature on this topic. One of the earliest algorithms (Serón, De Doná, and Goodwin, 2000) is enumeration based: checking every active set to determine if it defines a non-empty region in which the optimal control is affine. There has recently been a return to this approach because of its effectiveness in dealing with systems with relatively high state dimension but a low number of constraints (Feller, Johansen, and Olaru, 2013). The enumeration based procedures can be extended to solve mixed-integer problems. While the early algorithms for parametric linear and quadratic programming have exponential complexity, most later algorithms are based on a linear complementarity formulation and execute in polynomial time in the number of regions; they also use symbolic perturbation to select a unique and continuous solution when one exists (Columbano, Fukudu, and Jones, 2009). Some research has been devoted to obtaining approximate solutions with lower complexity but guaranteed properties such as stability (Borrelli, Bemporad, and Morari, 2017, Chapter 13).

Toolboxes for solving parametric linear and quadratic programming problems include the The Multi-Parametric Toolbox in MATLAB and MPT3 described in (Herceg, Kvasnica, Jones, and Morari, 2013).

A feature of parametric problems is that state dimension is not a reliable indicator of complexity. There exist problems with two states that require over 10^5 regions and problems with 80 states that require only hundreds of regions. While problems with state dimension less than, say, 4 can be expected to have reasonable complexity, higher dimension problems may or may not have manageable complexity.

7.10 Notes

Early work on parametric programming, e.g., (Dantzig, Folkman, and Shapiro, 1967) and (Bank, Guddat, Klatte, Kummer, and Tanner, 1983), was concerned with the sensitivity of optimal solutions to parameter variations. Solutions to the parametric linear programming problem were obtained relatively early (Gass and Saaty, 1955) and (Gal and Nedoma, 1972). Solutions to parametric QPs were obtained in (Serón et al., 2000) and (Bemporad, Morari, Dua, and Pistikopoulos, 2002) and applied to the determination of optimal control laws for linear systems with polyhedral constraints. Since then a large number of papers on this topic have appeared, many of which are reviewed in (Alessio and Bemporad, 2009). Most papers employ the Kuhn-Tucker conditions of optimality in deriving the regions R_x, $x \in X$. Use of the polar cone condition was advocated in (Mayne and Raković, 2002) in order to focus on the geometric properties of the parametric optimization problem and avoid degeneracy problems. Section 7.5, on parametric piecewise quadratic programming, is based on (Mayne, Raković, and Kerrigan, 2007). The example in Section 7.4 was first computed by Raković (Mayne and Raković, 2003). That results from parametric linear and quadratic programming can be employed, instead of maximum theorems, to establish continuity of $u^0(\cdot)$ and, hence, of $V^0(\cdot)$, was pointed out by Bemporad et al. (2002) and Borrelli (2003, p. 37).

Much research has been devoted to obtaining reliable algorithms; see the survey papers (Alessio and Bemporad, 2009) and (Jones, Barić, and Morari, 2007) and the references therein. Jones (2017, Chapter 13) provides a useful review of approximate explicit control laws of specified complexity that nevertheless guarantee stability and recursive feasibility.

7.11 Exercises

Exercise 7.1: QP with equality constraints

Obtain the solution u^0 and the value V^0 of the equality constrained optimization problem $V^0 = \min_u\{V(u) \mid h(u) = 0\}$ where $V(u) = (1/2)u'Ru + r'u + c$ and $h(u) := Mu - p$.

Exercise 7.2: Parametric QP with equality constraints

Show that the solution $u^0(x)$ and the value $V^0(x)$ of the parametric optimization problem $V^0(x) = \min_u\{V(x,u) \mid h(x,u) = 0\}$ where $V(x,u) := (1/2)x'Qx + u'Sx + (1/2)u'Ru + q'x + r'u + c$ and $h(x,u) := Mu - Nx - p$ have the form $u^0(x) = Kx + k$ and $V^0(x) = (1/2)x'\bar{Q}x + \bar{q}'x + s$. Determine $\bar{Q}$, $\bar{q}$, s, K, and k.

Exercise 7.3: State and input trajectories in constrained LQ problem

For the constrained linear quadratic problem defined in Section 7.4, show that $\mathbf{u} := (u(0), u(1), \ldots, u(N-1))$ and $\mathbf{x} := (x(0), x(1), \ldots, x(N))$, where $x(0) = x$ and $x(i) = \phi(i; x, \mathbf{u})$, $i = 0, 1, \ldots, N$, satisfy

$$\mathbf{x} = \mathbf{F}x + \mathbf{G}\mathbf{u}$$

and determine the matrices $\mathbf{F}$ and $\mathbf{G}$; in this equation $\mathbf{u}$ and $\mathbf{x}$ are column vectors. Hence show that $V_N(x, \mathbf{u})$ and $\mathbb{Z}$, defined respectively in (7.9) and (7.11), satisfy (7.12) and (7.13), and determine $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{M}$, $\mathbf{N}$, and $\mathbf{p}$.

Exercise 7.4: The parametric LP with unique minimizer

For the example of Figure 7.7, determine $u^0(x)$, $V^0(x)$, $I^0(x)$, and $C^*(x)$ for all x in the interval $[x_1, x_3]$. Show that $-r$ lies in $C^*(x)$ for all x in $[x_1, x_3]$.

Exercise 7.5: Cost function and constraints in constrained LQ control problem

For the constrained linear control problem considered in Section 7.8, determine the matrices $\mathbf{M}$, $\mathbf{N}$, and $\mathbf{p}$ that define the constraint set $\mathbb{Z}$, and the vectors $\mathbf{q}$ and $\mathbf{r}$ that define the cost $V_N(\cdot)$.

Exercise 7.6: Cost function in constrained linear control problem

Show that $|x|_p$, $p = 1$ and $p = \infty$, may be expressed as $\max_j\{s_j'x \mid j \in J\}$ and determine s_i, $i \in I$ for the two cases $p = 1$ and $p = \infty$. Hence show that the optimal control problem in Section 7.8 may be expressed as

$$V_N^0(x) = \min_{\mathbf{v}}\{V_N(x, \mathbf{v}) \mid \mathbf{M}\mathbf{v} \leq \mathbf{N}x + \mathbf{p}\}$$

where, now, $\mathbf{v}$ is a column vector whose components are $u(0), u(1), \ldots, u(N-1)$, $\ell_x(0)$, $\ell_x(1), \ldots, \ell_x(N)$, $\ell_u(0), \ell_u(1), \ldots, \ell_u(N-1)$ and f; the cost $V_N(x, \mathbf{v})$ is now defined by

$$V_N(x, \mathbf{v}) = \sum_{i=0}^{N-1} (\ell_x(i) + \ell_u(i)) + f$$

Finally, $\mathbf{Mv} \leq \mathbf{N}x + \mathbf{p}$ now specifies the constraints $u(i) \in \mathbb{U}$ and $x(i) \in \mathbb{X}$, $|Ru(i)|_p \leq \ell_u(i)$, $|Qx(i)|_p \leq \ell_x(i)$, $i = 0, 1, \ldots, N-1$, $x(N) \in \mathbb{X}_f$, and $\left|Q_f x(N)\right| \leq f$. As before, $\mathbf{x}^+ = \mathbf{Fx} + \mathbf{Gu}$.

Exercise 7.7: Is QP constraint qualification relevant to MPC?

Continuity properties of the MPC control law are often used to establish robustness properties of MPC such as robust asymptotic stability. In early work on continuity properties of linear model MPC, Scokaert, Rawlings, and Meadows (1997) used results on continuity of QPs with respect to parameters to establish MPC stability under perturbations. For example, Hager (1979) considered the following QP

$$\min_u (1/2)u'Hu + h'u + c$$

subject to

$$Du \leq d$$

and established that the QP solution u^0 and cost V^0 are Lipschitz continuous in the data of the QP, namely the parameters H, h, D, d. To establish this result Hager (1979) made the following assumptions.

- The solution is unique for all H, h, D, d in a chosen set of interest.
- The rows of D corresponding to the constraints active at the solution are linearly independent. The assumption of linear independence of active constraints is a form of *constraint qualification*.

(a) First we show that some form of constraint qualification is required to establish continuity of the QP solution with respect to matrix D. Consider the following QP example that does not satisfy Hager's constraint qualification assumption.

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \qquad d = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad h = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \qquad c = 1$$

Find the solution u^0 for this problem.

Next perturb the D matrix to

$$D = \begin{bmatrix} 1 & 1 \\ -1+\epsilon & -1 \end{bmatrix}$$

in which $\epsilon > 0$ is a small perturbation. Find the solution to the perturbed problem. Are V^0 and u^0 continuous in parameter D for this QP? Draw a sketch of the feasible region and cost contours for the original and perturbed problems. What happens to the feasible set when D is perturbed?

(b) Next consider MPC control of the following system with state inequality constraint and no input constraints

$$A = \begin{bmatrix} -1/4 & 1 \\ -1 & 1/2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \qquad x(k) \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad k \in \mathbb{I}_{0:N}$$

Using a horizon $N = 1$, eliminate the state $x(1)$ and write out the MPC QP for the input $u(0)$ in the form given above for $Q = R = I$ and zero terminal penalty. Find an initial condition x_0 such that the MPC constraint matrix D and vector d are identical to those given in the previous part. Is this $x_0 \in \mathcal{X}_N$?

Are the rows of the matrix of active constraints linearly independent in this MPC QP on the set $\mathcal{X}_N$? Are the MPC control law $\kappa_N(x)$ and optimal value function $V_N^0(x)$ Lipschitz continuous on the set $\mathcal{X}_N$ for this system? Explain the reason if these two answers differ.

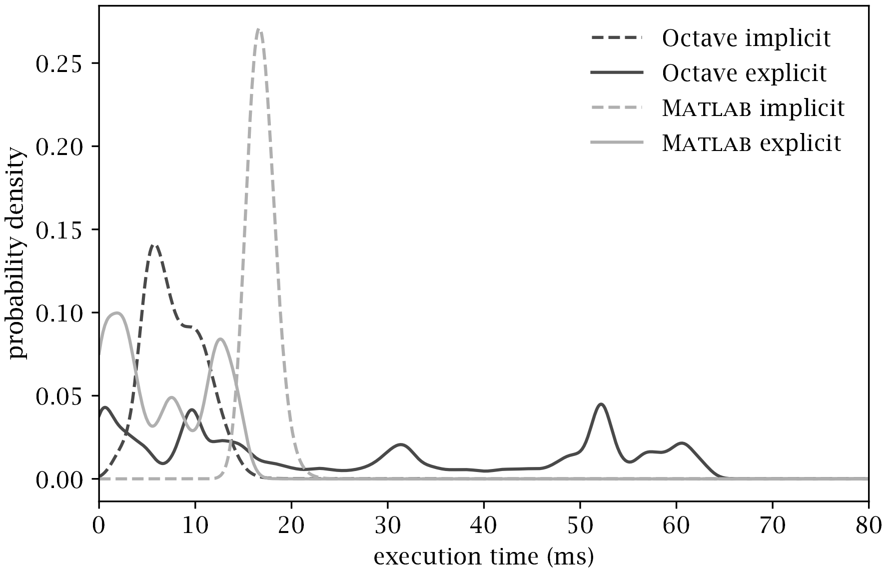

Figure 7.8: Solution times for explicit and implicit MPC for $N = 20$. Plot shows kernel density estimate for 10,000 samples using a Gaussian kernel ($\sigma = 1$ ms).

Exercise 7.8: Explicit versus implicit

Using the system from Figure 7.6, find the explicit control law for horizon $N = 20$ (you should find 1719 regions). Implement a simple lookup function for the explicit control law. Randomly sample a large number of points ($\geq$ 1000) from $\mathcal{X}_N$ and compare execution times for explicit MPC (via the lookup function) and implicit MPC (via solving a QP). Which method is better? Example results are shown in Figure 7.8, although your times may vary significantly. How could you improve your lookup function?

Exercise 7.9: Cascaded MPC and PID

Consider a Smart Tank$^{\text{TM}}$ of liquid whose height h evolves according to

$$\tau \frac{dh}{dt} + h = Kq, \quad \tau = 10, K = 1$$

with q the (net) inflow. The tank is Smart$^{\text{TM}}$ in that it has an integrated PI controller that computes

$$q = K_c \left(h_{\text{sp}} - h + \frac{1}{\tau_c} \epsilon \right)$$
$$\epsilon = \int h_{\text{sp}} - h \, dt$$

so that the height of the tank returns to h_{sp} automatically. Unfortunately, the controller parameters are not very Smart$^{\text{TM}}$, as they are fixed permanently at $K_c = 1/2$ and $\tau_c = 1$.

(a) Simulate the closed-loop behavior of the system starting from $h = -1$, $\epsilon = 0$ with $h_{\text{sp}} \equiv 0$.

(b) Design an MPC controller to choose h_{sp}. As a cost function take

$$\ell(h, \epsilon, q, h_{\text{sp}}) = 5(h^2 + \epsilon^2) + q^2 + 10h_{\text{sp}}^2$$

so that the controller drives the system to $h = \epsilon = 0$. Choose $\Delta = 1$. How does performance compare to the previous case? How much storage (i.e., how many floating-point numbers must be stored) to implement this controller?

(c) Add the constraint $q \in [-0.2, 0.2]$ to the MPC formulation, and design an explicit MPC controller valid for $h \in [-5, 5]$ and $\epsilon \in [-10, 10]$ (use `solvempqp.m` from Figure 7.6, and add constraints $Ep \leq e$ to only search the region of interest). How large does N have to be so that the full region is covered? How much storage is needed to implement this controller?

Exercise 7.10: Explicit economic MPC for electricity arbitrage

Electricity markets are often subject to real-time pricing, whereby the cost of purchasing electricity varies with time. Suppose that you have a large battery that allows you to buy electricity at one time and then sell it back to the grid at another. We can model this as a simple integrator system

$$x^+ = x + u$$

with x representing the amount of stored energy in the tank, and u giving the amount of electricity that is purchased for the battery $u > 0$ or discharged from the battery and sold back to the grid ($u < 0$). We wish to find an explicit control law based on the initial condition $x(0)$ a known forecast of electricity prices $c(0), c(1), \ldots, c(N-1)$.

(a) To start, suppose that u is constrained to the interval $[-1, 1]$ but x is unconstrained. A reasonable optimization problem is

$$\begin{aligned} \min_{\mathbf{u}} \quad & \sum_{k=0}^{N-1} c(k)u(k) + 0.1u(k)^2 \\ \text{s.t.} \quad & x(k+1) = x(k) + u(k) \\ & u(k) \in [-1, 1] \end{aligned}$$

where the main component of the objective function is the cost of electricity purchase/sale with a small penalty added to discourage larger transactions. By removing the state evolution equation, formulate an explicit quadratic programming problem with N variables (the $u(k)$) and $N + 1$ parameters ($x(0)$ and the price forecast $c(k)$). What is a theoretical upper bound on the number of regions in the explicit control law? Assuming that $x(0) \in [-10, 10]$ and each $c(k) \in [-1, -1]$, find the explicit control law for a few small values of N. (Consider using `solvempqp.m` from Figure 7.6; you will need to add constraints $Ep \leq e$ on the parameter vector to make sure the regions are bounded.) How many regions do you find?

(b) To make the problem more realistic, we add the constraint $x(k) \in [-10, 10]$ to the optimization, as well as an additional penalty on stored inventory. The optimization problem is then

$$\begin{aligned} \min_{\mathbf{u}} \quad & \sum_{k=0}^{N-1} c(k)u(k) + 0.1u(k)^2 + 0.01x(k)^2 \\ \text{s.t.} \quad & x(k+1) = x(k) + u(k) \\ & u(k) \in [-1, 1] \\ & x(k) \in [-10, 10] \end{aligned}$$

Repeat the previous part but using the new optimization problem.

(c) Suppose you wish to solve this problem with a 7-day horizon and a 1-hour time step. Can you use the explicit solution of either formulation? (Hint: for comparison, there are roughly 10^{80} atoms in the observable universe.)

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