J. Nocedal and S. J. Wright. *Numerical Optimization.* Springer, New York, second edition, 2006.

A. Papoulis. *Probability, Random Variables, and Stochastic Processes.* McGraw-Hill, Inc., second edition, 1984.

E. Polak. *Optimization: Algorithms and Consistent Approximations.* Springer Verlag, New York, 1997. ISBN 0-387-94971-2.

R. T. Rockafellar. *Convex Analysis.* Princeton University Press, Princeton, N.J., 1970.

R. T. Rockafellar and R. J.-B. Wets. *Variational Analysis.* Springer-Verlag, 1998.

I. Schur. On the characteristic roots of a linear substitution with an application to the theory of integral equations (German). *Math Ann.*, 66:488–510, 1909.

G. Strang. *Linear Algebra and its Applications.* Academic Press, New York, second edition, 1980.

B

Stability Theory

B.1 Introduction

In this appendix we consider stability properties of discrete time systems. A good general reference for stability theory of continuous time systems is Khalil (2002). There are not many texts for stability theory of discrete time systems; a useful reference is LaSalle (1986). Recently stability theory for discrete time systems has received more attention in the literature. In the notes below we draw on Jiang and Wang (2001, 2002); Kellet and Teel (2004a,b).

We consider systems of the form

$$x^+ = f(x, u)$$

where the state x lies in $\mathbb{R}^n$ and the control (input) u lies in $\mathbb{R}^m$; in this formulation x and u denote, respectively, the current state and control, and x^+ the successor state. We assume in the sequel that the function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous. Let $\phi(k; x, \mathbf{u})$ denote the solution of $x^+ = f(x, u)$ at time k if the initial state is $x(0) = x$ and the control sequence is $\mathbf{u} = (u(0), u(1), u(2), \ldots)$; the solution exists and is unique. If a state-feedback control law $u = \kappa(x)$ has been chosen, the closed-loop system is described by $x^+ = f(x, \kappa(x))$, which has the same form $x^+ = f_c(x)$ where $f_c(\cdot)$ is defined by $f_c(x) := f(x, \kappa(x))$. Let $\phi(k; x, \kappa(\cdot))$ denote the solution of this difference equation at time k if the initial state at time 0 is $x(0) = x$; the solution exists and is unique (even if $\kappa(\cdot)$ is discontinuous). If $\kappa(\cdot)$ is not continuous, as may be the case when $\kappa(\cdot)$ is an implicit model predictive control (MPC) law, then $f_c(\cdot)$ may not be continuous. In this case we assume that $f_c(\cdot)$ is *locally bounded*.[1]

[1]A function $f : X \to X$ is locally bounded if, for any $x \in X$, there exists a neighborhood $\mathcal{N}$ of x such that $f(\mathcal{N})$ is a bounded set, i.e., if there exists a $M > 0$ such that $|f(x)| \leq M$ for all $x \in \mathcal{N}$.

We would like to be sure that the controlled system is "stable", i.e., that small perturbations of the initial state do not cause large variations in the subsequent behavior of the system, and that the state converges to a desired state or, if this is impossible due to disturbances, to a desired set of states. These objectives are made precise in Lyapunov stability theory; in this theory, the system $x^+ = f(x)$ is assumed given and conditions ensuring the stability, or asymptotic stability of a specified state or set are sought; the terms *stability* and *asymptotic stability* are defined below. If convergence to a specified state, x^* say, is sought, it is desirable for this state to be an *equilibrium* point:

Definition B.1 (Equilibrium point)**.** A point x^* is an equilibrium point of $x^+ = f(x)$ if $x(0) = x^*$ implies $x(k) = \phi(k; x^*) = x^*$ for all $k \geq 0$. Hence x^* is an equilibrium point if it satisfies

$$x^* = f(x^*)$$

An equilibrium point x^* is isolated if there are no other equilibrium points in a sufficiently small neighborhood of x^*. A linear system $x^+ = Ax + b$ has a single equilibrium point $x^* = (I - A)^{-1}b$ if $I - A$ is invertible; if not, the linear system has a continuum $\{x \mid (I - A)x = b\}$ of equilibrium points. A nonlinear system, unlike a linear system, may have several isolated equilibrium points.

In other situations, for example when studying the stability properties of an oscillator, convergence to a specified closed set $\mathcal{A} \subset \mathbb{R}^n$ is sought. In the case of a linear oscillator with state dimension 2, this set is an ellipse. If convergence to a set $\mathcal{A}$ is sought, it is desirable for the set $\mathcal{A}$ to be *positive invariant*:

Definition B.2 (Positive invariant set)**.** A closed set $\mathcal{A}$ is positive invariant for the system $x^+ = f(x)$ if $x \in \mathcal{A}$ implies $f(x) \in \mathcal{A}$.

Clearly, any solution of $x^+ = f(x)$ with initial state in $\mathcal{A}$, remains in $\mathcal{A}$. The closed set $\mathcal{A} = \{x^*\}$ consisting of a (single) equilibrium point is a special case; $x \in \mathcal{A}$ ($x = x^*$) implies $f(x) \in \mathcal{A}$ ($f(x) = x^*$). Define $|x|_{\mathcal{A}} := \inf_{z \in \mathcal{A}} |x - z|$ to be the distance of a point x from the set $\mathcal{A}$; if $\mathcal{A} = \{x^*\}$, then $|x|_{\mathcal{A}} = |x - x^*|$ which reduces to $|x|$ when $x^* = 0$.

Before introducing the concepts of stability and asymptotic stability and their characterization by Lyapunov functions, it is convenient to make a few definitions.

Definition B.3 ($\mathcal{K}$, $\mathcal{K}_\infty$, $\mathcal{KL}$, and $\mathcal{PD}$ functions)**.** A function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{K}$ if it is continuous, zero at zero, and strictly increasing; $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{K}_\infty$ if it is a class $\mathcal{K}$ and unbounded ($\sigma(s) \to \infty$ as $s \to \infty$). A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{I}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{KL}$ if it is continuous and if, for each $t \geq 0$, $\beta(\cdot, t)$ is a class $\mathcal{K}$ function and for each $s \geq 0$, $\beta(s, \cdot)$ is nonincreasing and satisfies $\lim_{t\to\infty} \beta(s,t) = 0$. A function $\gamma : \mathbb{R} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{PD}$ (is positive definite) if it is zero at zero and positive everywhere else.[2]

The following useful properties of these functions are established in Khalil (2002, Lemma 4.2): if $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are $\mathcal{K}$ functions ($\mathcal{K}_\infty$ functions), then $\alpha_1^{-1}(\cdot)$ and $(\alpha_1 \circ \alpha_2)(\cdot)$[3] are $\mathcal{K}$ functions[4] ($\mathcal{K}_\infty$ functions). Moreover, if $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are $\mathcal{K}$ functions and $\beta(\cdot)$ is a $\mathcal{KL}$ function, then $\sigma(r,s) = \alpha_1(\beta(\alpha_2(r), s))$ is a $\mathcal{KL}$ function.

The following properties prove useful when analyzing the robustness of perturbed systems.

1. For $\gamma(\cdot) \in \mathcal{K}$, the following holds for all $a_i \in \mathbb{R}_{\geq 0}$, $i \in \mathbb{I}_{1:n}$

$$\frac{1}{n}(\gamma(a_1) + \cdots + \gamma(a_n)) \leq \gamma(a_1 + \cdots + a_n) \leq \gamma(na_1) + \cdots + \gamma(na_n) \tag{B.1}$$

2. Similarly, for $\beta(\cdot) \in \mathcal{KL}$ the following holds for all $a_i \in \mathbb{R}_{\geq 0}$, $i \in \mathbb{I}_{1:n}$, and all $t \in \mathbb{R}_{\geq 0}$

$$\frac{1}{n}(\beta(a_1,t) + \cdots + \beta(a_n,t)) \leq \beta((a_1 + \cdots + a_n), t) \leq \beta(na_1,t) + \beta(na_2,t) + \cdots + \beta(na_n,t) \tag{B.2}$$

3. If $\alpha_i(\cdot) \in \mathcal{K}(\mathcal{K}_\infty)$ for $i \in \mathbb{I}_{1:n}$, then

$$\min_i\{\alpha_i(\cdot)\} := \underline{\alpha}(\cdot) \in \mathcal{K}(\mathcal{K}_\infty) \tag{B.3}$$

$$\max_i\{\alpha_i(\cdot)\} := \overline{\alpha}(\cdot) \in \mathcal{K}(\mathcal{K}_\infty) \tag{B.4}$$

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[2]Be aware that the existing stability literature sometimes includes continuity in the definition of a positive definite function. We used such a definition in the first edition of this text, for example. But in the second edition, we remove continuity and retain only the requirement of positivity in the definition of positive definite function.

[3]$(\alpha_1 \circ \alpha_2)(\cdot)$ is the composition of the two functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ and is defined by $(\alpha_1 \circ \alpha_2)(s) := \alpha_1(\alpha_2(s))$.

[4]Note, however, that the domain of $\alpha^{-1}(\cdot)$ may be restricted from $\mathbb{R}_{\geq 0}$ to $[0,a)$ for some $a > 0$.

4. Let $v_i \in \mathbb{R}^{n_i}$ for $i \in \mathbb{I}_{1:n}$, and $v := (v_1, \ldots, v_n) \in \mathbb{R}^{\sum n_i}$. If $\alpha_i(\cdot) \in \mathcal{K}(\mathcal{K}_\infty)$ for $i \in \mathbb{I}_{1:n}$, then there exist $\underline{\alpha}(\cdot), \overline{\alpha}(\cdot) \in \mathcal{K}(\mathcal{K}_\infty)$ such that
$$\underline{\alpha}(|v|) \leq \alpha_1(|v_1|) + \cdots + \alpha_n(|v_n|) \leq \overline{\alpha}(|v|) \tag{B.5}$$

See (Rawlings and Ji, 2012) for short proofs of (B.1) and (B.2), and (Allan, Bates, Risbeck, and Rawlings, 2017) for a short proof of (B.3). The result (B.4) follows similarly to (B.3). Result (B.5) follows from (B.1) and (B.3)-(B.4). See also Exercise B.10.

B.2 Stability and Asymptotic Stability

In this section we consider the stability properties of the autonomous system $x^+ = f(x)$; we assume that $f(\cdot)$ is locally bounded, and that the set $\mathcal{A}$ is closed and positive invariant for $x^+ = f(x)$ unless otherwise stated.

Definition B.4 (Local stability)**.** The (closed, positive invariant) set $\mathcal{A}$ is *locally stable* for $x^+ = f(x)$ if, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $|x|_{\mathcal{A}} < \delta$ implies $\left|\phi(i;x)\right|_{\mathcal{A}} < \varepsilon$ for all $i \in \mathbb{I}_{\geq 0}$.

See Figure B.1 for an illustration of this definition when $\mathcal{A} = \{0\}$; in this case we speak of stability of the origin.

Remark. Stability of the origin, as defined above, is equivalent to continuity of the map $x \mapsto \mathbf{x} := (x, \phi(1;x), \phi(2;x), \ldots)$, $\mathbb{R} \to \ell_\infty$ at the origin so that $\|\mathbf{x}\| \to 0$ as $x \to 0$ (a small perturbation in the initial state causes a small perturbation in the subsequent motion).

Definition B.5 (Global attraction)**.** The (closed, positive invariant) set $\mathcal{A}$ is *globally attractive* for the system $x^+ = f(x)$ if $\left|\phi(i;x)\right|_{\mathcal{A}} \to 0$ as $i \to \infty$ for all $x \in \mathbb{R}^n$.

Definition B.6 (Global asymptotic stability)**.** The (closed, positive invariant) set $\mathcal{A}$ is *globally asymptotically stable* (GAS) for $x^+ = f(x)$ if it is locally stable and globally attractive.

It is possible for the origin to be globally attractive but *not* locally stable. Consider a second order system
$$x^+ = Ax + \phi(x)$$

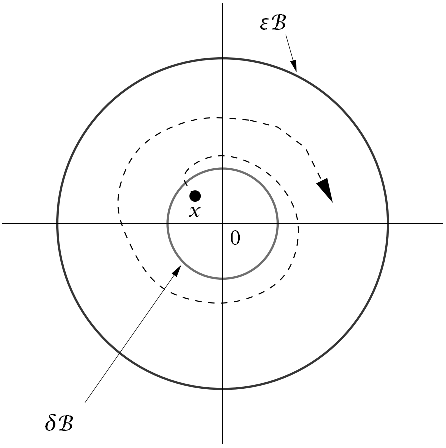

Figure B.1: Stability of the origin. $\mathcal{B}$ denotes the unit ball.

where A has eigenvalues $\lambda_1 = 0.5$ and $\lambda_2 = 2$ with associated eigenvectors w_1 and w_2, shown in Figure B.2; w_1 is the "stable" and w_2 the "unstable" eigenvector; the smooth function $\phi(\cdot)$ satisfies $\phi(0) = 0$ and $(\partial/\partial x)\phi(0) = 0$ so that $x^+ = Ax + \phi(x)$ behaves like $x^+ = Ax$ near the origin. If $\phi(x) \equiv 0$, the motion corresponding to an initial state αw_1, $\alpha \neq 0$, converges to the origin, whereas the motion corresponding to an initial state αw_2 diverges. If $\phi(\cdot)$ is such that it steers nonzero states toward the horizontal axis, we get trajectories of the form shown in Figure B.2. All trajectories converge to the origin but the motion corresponding to an initial state αw_2, *no matter how small*, is similar to that shown in Figure B.2 and cannot satisfy the ε, δ definition of local stability. The origin is globally attractive but not stable. A trajectory that joins an equilibrium point to itself, as in Figure B.2, is called a homoclinic orbit.

We collect below a set of useful definitions:

Definition B.7 (Various forms of stability)**.** The (closed, positive invariant) set $\mathcal{A}$ is

(a) locally stable if, for each $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $|x|_{\mathcal{A}} < \delta$ implies $|\phi(i;x)|_{\mathcal{A}} < \varepsilon$ for all $i \in \mathbb{I}_{\geq 0}$.

(b) unstable, if it is not locally stable.

(c) locally attractive if there exists $\eta > 0$ such that $|x|_{\mathcal{A}} < \eta$ implies $|\phi(i;x)|_{\mathcal{A}} \to 0$ as $i \to \infty$.

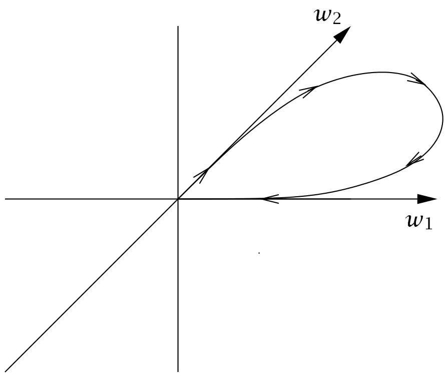

Figure B.2: An attractive but unstable origin.

(d) globally attractive if $\left|\phi(i;x)\right|_{\mathcal{A}} \to 0$ as $i \to \infty$ for all $x \in \mathbb{R}^n$.

(e) locally asymptotically stable if it is locally stable and locally attractive.

(f) globally asymptotically stable if it is locally stable and globally attractive.

(g) locally exponentially stable if there exist $\eta > 0$, $c > 0$, and $\gamma \in (0,1)$ such that $|x|_{\mathcal{A}} < \eta$ implies $\left|\phi(i;x)\right|_{\mathcal{A}} \leq c\,|x|_{\mathcal{A}}\,\gamma^i$ for all $i \in \mathbb{I}_{\geq 0}$.

(h) globally exponentially stable if there exists a $c > 0$ and a $\gamma \in (0,1)$ such that $\left|\phi(i;x)\right|_{\mathcal{A}} \leq c\,|x|_{\mathcal{A}}\,\gamma^i$ for all $x \in \mathbb{R}^n$, all $i \in \mathbb{I}_{\geq 0}$.

The following stronger definition of GAS has recently started to become popular.

Definition B.8 (Global asymptotic stability (KL version))**.** The (closed, positive invariant) set $\mathcal{A}$ is *globally asymptotically stable* (GAS) for $x^+ = f(x)$ if there exists a $\mathcal{KL}$ function $\beta(\cdot)$ such that, for each $x \in \mathbb{R}^n$

$$\left|\phi(i;x)\right|_{\mathcal{A}} \leq \beta(|x|_{\mathcal{A}}, i) \qquad \forall i \in \mathbb{I}_{\geq 0} \tag{B.6}$$

Proposition B.9 (Connection of classical and KL global asymptotic stability)**.** *Suppose $\mathcal{A}$ is compact (and positive invariant) and that $f(\cdot)$ is continuous. Then the classical and KL definitions of global asymptotic stability of $\mathcal{A}$ for $x^+ = f(x)$ are equivalent.*

The KL version of global asymptotic stability implies the classical version from (B.6) and the definition of a $\mathcal{KL}$ function. The converse is harder to prove but is established in Jiang and Wang (2002) where Proposition 2.2 establishes the equivalence of the existence of a $\mathcal{KL}$ function satisfying (2) with UGAS (uniform global asymptotic stability), and Corollary 3.3 which establishes the equivalence, *when* $\mathcal{A}$ is compact, of uniform global asymptotic stability and global asymptotic stability. Note that $f(\cdot)$ must be continuous for the two definitions to be equivalent. See Exercise B.9 for an example with discontinuous $f(\cdot)$ where the system is GAS in the classical sense but does not satisfy (B.6), i.e., is not GAS in the KL sense.

For a KL version of exponential stability, one simply restricts the form of the KL function $\beta(\cdot)$ of asymptotic stability to $\beta(|x|_{\mathcal{A}}, i) = c\,|x|_{\mathcal{A}}\,\lambda^i$ with $c > 0$ and $\lambda \in (0,1)$, but, as we see, that is exactly the classical definition so there is no distinction between the two forms for exponential stability.

In practice, global asymptotic stability of $\mathcal{A}$ often cannot be achieved because of state constraints. Hence we have to extend slightly the definitions given above. In the following, let $\mathcal{B}$ denote a unit ball in $\mathbb{R}^n$ with center at the origin.

Definition B.10 (Various forms of stability (constrained))**.** Suppose $X \subset \mathbb{R}^n$ is positive invariant for $x^+ = f(x)$, that $\mathcal{A} \subseteq X$ is closed and positive invariant for $x^+ = f(x)$. Then $\mathcal{A}$ is

(a) locally stable in X if, for each $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $x \in X \cap (\mathcal{A} \oplus \delta\mathcal{B})$, implies $|\phi(i;x)|_{\mathcal{A}} < \varepsilon$ for all $i \in \mathbb{I}_{\geq 0}$.

(b) locally attractive in X if there exists a $\eta > 0$ such that $x \in X \cap (\mathcal{A} \oplus \eta\mathcal{B})$ implies $|\phi(i;x)|_{\mathcal{A}} \to 0$ as $i \to \infty$.

(c) attractive in X if $|\phi(i;x)|_{\mathcal{A}} \to 0$ as $i \to \infty$ for all $x \in X$.

(d) locally asymptotically stable in X if it is locally stable in X and locally attractive in X.

(e) asymptotically stable in X if it is locally stable in X and attractive in X.

(f) locally exponentially stable in X if there exist $\eta > 0$, $c > 0$, and $\gamma \in (0,1)$ such that $x \in X \cap (\mathcal{A} \oplus \eta\mathcal{B})$ implies $|\phi(i;x)|_{\mathcal{A}} \leq c\,|x|_{\mathcal{A}}\,\gamma^i$ for all $i \in \mathbb{I}_{\geq 0}$.

(g) exponentially stable in X if there exists a $c > 0$ and a $\gamma \in (0,1)$ such that $|\phi(i;x)|_{\mathcal{A}} \leq c\,|x|_{\mathcal{A}}\,\gamma^i$ for all $x \in X$, all $i \in \mathbb{I}_{\geq 0}$.

The assumption that X is positive invariant for $x^+ = f(x)$ ensures that $\phi(i;x) \in X$ for all $x \in X$, all $i \in \mathbb{I}_{\geq 0}$. The KL version of asymptotic stability in X is the following.

Definition B.11 (Asymptotic stability (constrained, KL version))**.** Suppose that X is positive invariant and the set $\mathcal{A} \subseteq X$ is closed and positive invariant for $x^+ = f(x)$. The set $\mathcal{A}$ is *asymptotically stable in* X for $x^+ = f(x)$ if there exists a $\mathcal{KL}$ function $\beta(\cdot)$ such that, for each $x \in X$

$$\left|\phi(i;x)\right|_{\mathcal{A}} \leq \beta(|x|_{\mathcal{A}}, i) \qquad \forall i \in \mathbb{I}_{\geq 0} \tag{B.7}$$

Finally, we define the *domain of attraction* of an asymptotically stable set $\mathcal{A}$ for the system $x^+ = f(x)$ to be the set of all initial states x such that $\left|\phi(i;x)\right|_{\mathcal{A}} \to 0$ as $i \to \infty$. We use the term *region of attraction* to denote any set of initial states x such that $\left|\phi(i;x)\right|_{\mathcal{A}} \to 0$ as $i \to \infty$. From these definitions, if $\mathcal{A}$ is attractive *in* X, then X is a region of attraction of set $\mathcal{A}$ for the system $x^+ = f(x)$.

B.3 Lyapunov Stability Theory

Energy in a passive electrical or mechanical system provides a useful analogy to Lyapunov stability theory. In a lumped mechanical system, the total mechanical energy is the sum of the potential and kinetic energies. As time proceeds, this energy is dissipated by friction into heat and the total mechanical energy decays to zero at which point the system is in equilibrium. To establish stability or asymptotic stability, Lyapunov theory follows a similar path. If a real-valued function can be found that is positive and decreasing if the state does not lie in the set $\mathcal{A}$, then the state converges to this set as time tends to infinity. We now make this intuitive idea more precise.

B.3.1 Time-Invariant Systems

First we consider the time-invariant (autonomous) model $x^+ = f(x)$.

Definition B.12 (Lyapunov function (unconstrained and constrained))**.** Suppose that X is positive invariant and the set $\mathcal{A} \subseteq X$ is closed and positive invariant for $x^+ = f(x)$. A function $V : X \to \mathbb{R}_{\geq 0}$ is said to be a Lyapunov function in X for the system $x^+ = f(x)$ and set $\mathcal{A}$ if there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and *continuous* function $\alpha_3 \in \mathcal{PD}$ such that for any $x \in X$

$$V(x) \geq \alpha_1(|x|_{\mathcal{A}}) \tag{B.8}$$

$$V(x) \leq \alpha_2(|x|_{\mathcal{A}}) \tag{B.9}$$

$$V(f(x)) - V(x) \leq -\alpha_3(|x|_{\mathcal{A}}) \tag{B.10}$$

If $X = \mathbb{R}^n$, then we drop the restrictive phrase "in X."

Remark (Discontinuous f and V)**.** In MPC, the value function for the optimal control problem solved online is often employed as a Lyapunov function. The reader should be aware that many similar but different definitions of Lyapunov functions are in use in many different branches of the science and engineering literature. To be of the most use in MPC analysis, we do not assume here that $f(\cdot)$ or $V(\cdot)$ is continuous. We assume only that $f(\cdot)$ is locally bounded; $V(\cdot)$ is also locally bounded due to (B.9), and continuous on the set $\mathcal{A}$ (but not necessarily on a neighborhood of $\mathcal{A}$) due to (B.8)–(B.9).

Remark (Continuous (and positive definite) α_3)**.** One may wonder why $\alpha_3(\cdot)$ is assumed continuous in addition to positive definite in the definition of the Lyapunov function, when much of the classical literature leaves out continuity; see for example the autonomous case given in Kalman and Bertram (1960). Again, most of this classical literature assumes instead that $f(\cdot)$ is continuous, which we do not assume here. See Exercise B.8 for an example from Lazar, Heemels, and Teel (2009) with discontinuous $f(\cdot)$ for which removing continuity of $\alpha_3(\cdot)$ in Definition B.12 would give a Lyapunov function that fails to imply asymptotic stability.

For making connections to the wide body of existing stability literature, which mainly uses the classical definition of asymptotic stability, and because the proof is instructive, we first state and prove the classical version of the Lyapunov stability theorem.

Theorem B.13 (Lyapunov function and GAS (classical definition))**.** *Suppose that X is positive invariant and the set $\mathcal{A} \subseteq X$ is closed and positive invariant for $x^+ = f(x)$. Suppose $V(\cdot)$ is a Lyapunov function for $x^+ = f(x)$ and set $\mathcal{A}$. Then $\mathcal{A}$ is globally asymptotically stable (classical definition).*

Proof.

(a) Stability. Let $\varepsilon > 0$ be arbitrary and let $\delta := \alpha_2^{-1}(\alpha_1(\varepsilon))$. Suppose $|x|_{\mathcal{A}} < \delta$ so that, by (B.9), $V(x) \leq \alpha_2(\delta) = \alpha_1(\varepsilon)$. From (B.10),

$(V(x(i)))_{i\in\mathbb{I}_{\geq 0}}$, $x(i) := \phi(i;x)$, is a nonincreasing sequence so that, for all $i \in \mathbb{I}_{\geq 0}$, $V(x(i)) \leq V(x)$. From (B.8), $|x(i)|_{\mathcal{A}} \leq \alpha_1^{-1}(V(x)) \leq \alpha_1^{-1}(\alpha_1(\varepsilon)) = \varepsilon$ for all $i \in \mathbb{I}_{\geq 0}$.

(b) Attractivity. Let $x \in \mathbb{R}^n$ be arbitrary. From (B.9) $V(x)$ is finite, and from (B.8) and (B.10), the sequence $(V(x(i)))_{i\in\mathbb{I}_{\geq 0}}$ is nonincreasing and bounded below by zero and therefore converges to $\bar{V} \geq 0$ as $i \to \infty$. We next show that $\bar{V} = 0$. From (B.8) and (B.9) and the properties of K_∞ functions, we have that for all $i \geq 0$,

$$\alpha_2^{-1}(V(x(i))) \leq |x(i)|_{\mathcal{A}} \leq \alpha_1^{-1}(V(x(i))) \tag{B.11}$$

Assume for contradiction that $\bar{V} > 0$. Since $\alpha_3(\cdot)$ is continuous and positive definite and interval $\mathcal{I} := [\alpha_2^{-1}(\bar{V}), \alpha_1^{-1}(\bar{V})]$ is compact, the following optimization has a positive solution

$$\rho := \min_{|x|_{\mathcal{A}}\in\mathcal{I}} \alpha_3(|x|_{\mathcal{A}}) > 0$$

From repeated use of (B.10), we have that for all $i \geq 0$

$$V(x(i)) \leq V(x) - \sum_{j=0}^{i-1} \alpha_3(\left|x(j)\right|_{\mathcal{A}})$$

Since $|x(i)|_{\mathcal{A}}$ converges to interval $\mathcal{I}$ where $\alpha_3(|x(i)|_{\mathcal{A}})$ is underbounded by $\rho > 0$, $\alpha_3(\cdot)$ is continuous, and $V(x)$ is finite, the inequality above implies that $V(x(i)) \to -\infty$ as $i \to \infty$, which is a contradiction (see Exercise B.4 for further discussion). Therefore $V(x(i))$ converges to $\bar{V} = 0$ and (B.11) implies $x(i)$ converges to $\mathcal{A}$ as $i \to \infty$. ■

Next we establish the analogous Lyapunov stability theorem using the stronger KL definition of GAS, Definition B.8. Before establishing the Lyapunov stability theorem, it is helpful to present the following lemma established by Jiang and Wang (2002, Lemma 2.8) that enables us to assume when convenient that $\alpha_3(\cdot)$ in (B.10) is a $\mathcal{K}_\infty$ function rather than just a continuous $\mathcal{PD}$ function.

Lemma B.14 (From $\mathcal{PD}$ to $\mathcal{K}_\infty$ function (Jiang and Wang (2002))). *Assume $V(\cdot)$ is a Lyapunov function for system $x^+ = f(x)$ and set $\mathcal{A}$, and $f(\cdot)$ is locally bounded. Then there exists a smooth function*[5] *$\rho(\cdot) \in \mathcal{K}_\infty$ such that $W(\cdot) := \rho \circ V(\cdot)$ is also a Lyapunov function for system $x^+ = f(x)$ and set $\mathcal{A}$ that satisfies for all $x \in \mathbb{R}^n$*

$$W(f(x)) - W(x) \leq -\alpha(|x|_{\mathcal{A}})$$

[5] A smooth function has derivatives of all orders.

with $\alpha(\cdot) \in \mathcal{K}_\infty$*.*

Note that Jiang and Wang (2002) prove this lemma under the assumption that both $f(\cdot)$ and $V(\cdot)$ are continuous, but their proof remains valid if both $f(\cdot)$ and $V(\cdot)$ are only locally bounded.

We next establish the Lyapunov stability theorem in which we add the parenthetical (KL definition) purely for emphasis and to distinguish this result from the previous classical result, but we discontinue this emphasis after this theorem, and use exclusively the KL definition.

Theorem B.15 (Lyapunov function and global asymptotic stability (KL definition))**.** *Suppose that* X *is positive invariant and the set* $\mathcal{A} \subseteq X$ *is closed and positive invariant for* $x^+ = f(x)$*. Suppose* $V(\cdot)$ *is a Lyapunov function for* $x^+ = f(x)$ *and set* $\mathcal{A}$*. Then* $\mathcal{A}$ *is globally asymptotically stable (KL definition).*

Proof. Due to Lemma B.14 we assume without loss of generality that $\alpha_3 \in \mathcal{K}_\infty$. From (B.10) we have that

$$V(\phi(i+1;x)) \leq V(\phi(i;x)) - \alpha_3(\left|\phi(i;x)\right|_{\mathcal{A}}) \qquad \forall x \in \mathbb{R}^n \quad i \in \mathbb{I}_{\geq 0}$$

Using (B.9) we have that

$$\alpha_3(|x|_{\mathcal{A}}) \geq \alpha_3 \circ \alpha_2^{-1}(V(x)) \qquad \forall x \in \mathbb{R}^n$$

Combining these we have that

$$V(\phi(i+1;x)) \leq \sigma_1(V(\phi(i;x))) \qquad \forall x \in \mathbb{R}^n \quad i \in \mathbb{I}_{\geq 0}$$

in which

$$\sigma_1(s) := s - \alpha_3 \circ \alpha_2^{-1}(s)$$

We have that $\sigma_1(\cdot)$ is continuous on $\mathbb{R}_{\geq 0}$, $\sigma_1(0) = 0$, and $\sigma_1(s) < s$ for $s > 0$. But $\sigma_1(\cdot)$ may not be increasing. We modify σ_1 to achieve this property in two steps. First define

$$\sigma_2(s) := \max_{s' \in [0,s]} \sigma_1(s') \qquad s \in \mathbb{R}_{\geq 0}$$

in which the maximum exists for each $s \in \mathbb{R}_{\geq 0}$ because $\sigma_1(\cdot)$ is continuous. By its definition, $\sigma_2(\cdot)$ is nondecreasing, $\sigma_2(0) = 0$, and $0 \leq \sigma_2(s) < s$ for $s > 0$, and we next show that $\sigma_2(\cdot)$ is continuous on $\mathbb{R}_{\geq 0}$. Assume that $\sigma_2(\cdot)$ is discontinuous at a point $c \in \mathbb{R}_{\geq 0}$. Because it is a nondecreasing function, there is a positive jump in the function $\sigma_2(\cdot)$ at c (Bartle and Sherbert, 2000, p. 150). Define [6]

$$a_1 := \lim_{s \nearrow c} \sigma_2(s) \qquad a_2 := \lim_{s \searrow c} \sigma_2(s)$$

We have that $\sigma_1(c) \leq a_1 < a_2$ or we violate the limit of σ_2 from below. Since $\sigma_1(c) < a_2$, $\sigma_1(s)$ must achieve value a_2 for some $s < c$ or we violate the limit from above. But $\sigma_1(s) = a_2$ for $s < c$ also violates the limit from below, and we have a contradiction and $\sigma_2(\cdot)$ is continuous. Finally, define

$$\sigma(s) := (1/2)(s + \sigma_2(s)) \qquad s \in \mathbb{R}_{\geq 0}$$

and we have that $\sigma(\cdot)$ is a continuous, strictly increasing, and unbounded function satisfying $\sigma(0) = 0$. Therefore, $\sigma(\cdot) \in \mathcal{K}_\infty$, $\sigma_1(s) < \sigma(s) < s$ for $s > 0$ and therefore

$$V(\phi(i+1; x)) \leq \sigma(V(\phi(i; x))) \qquad \forall x \in \mathbb{R}^n \quad i \in \mathbb{I}_{\geq 0} \tag{B.12}$$

Repeated use of (B.12) and then (B.9) gives

$$V(\phi(i; x)) \leq \sigma^i \circ \alpha_2(|x|_{\mathcal{A}}) \qquad \forall x \in \mathbb{R}^n \quad i \in \mathbb{I}_{\geq 0}$$

in which σ^i represents the composition of σ with itself i times. Using (B.8) we have that

$$|\phi(i; x)|_{\mathcal{A}} \leq \beta(|x|_{\mathcal{A}}, i) \qquad \forall x \in \mathbb{R}^n \quad i \in \mathbb{I}_{\geq 0}$$

in which

$$\beta(s, i) := \alpha_1^{-1} \circ \sigma^i \circ \alpha_2(s) \qquad \forall s \in \mathbb{R}_{\geq 0} \quad i \in \mathbb{I}_{\geq 0}$$

For all $s \geq 0$, the sequence $w_i := \sigma^i(\alpha_2(s))$ is nonincreasing with i, bounded below (by zero), and therefore converges to a, say, as $i \to \infty$. Therefore, both $w_i \to a$ and $\sigma(w_i) \to a$ as $i \to \infty$. Since $\sigma(\cdot)$ is continuous we also have that $\sigma(w_i) \to \sigma(a)$ so $\sigma(a) = a$, which implies that $a = 0$, and we have shown that for all $s \geq 0$, $\alpha_1^{-1} \circ \sigma^i \circ \alpha_2(s) \to 0$ as $i \to \infty$. Since $\alpha_1^{-1}(\cdot)$ also is a $\mathcal{K}$ function, we also have that for all $s \geq 0$, $\alpha_1^{-1} \circ \sigma^i \circ \alpha_2(s)$ is nonincreasing with i. We have from the properties of $\mathcal{K}$ functions that for all $i \geq 0$, $\alpha_1^{-1} \circ \sigma^i \circ \alpha_2(s)$ is a $\mathcal{K}$ function, and can therefore conclude that $\beta(\cdot)$ is a $\mathcal{KL}$ function and the proof is complete. ∎

[6] The limits from above and below exist because $\sigma_2(\cdot)$ is nondecreasing (Bartle and Sherbert, 2000, p. 149). If the point $c = 0$, replace the limit from below by $\sigma_2(0)$.

Theorem B.15 provides merely a sufficient condition for global asymptotic stability that might be thought to be conservative. Next we establish a *converse* stability theorem that demonstrates necessity. In this endeavor we require a useful preliminary result on $\mathcal{KL}$ functions (Sontag, 1998b, Proposition 7)

Proposition B.16 (Improving convergence (Sontag (1998b)))**.** *Assume that $\beta(\cdot) \in \mathcal{KL}$. Then there exists $\theta_1(\cdot), \theta_2(\cdot) \in \mathcal{K}_\infty$ so that*

$$\beta(s,t) \leq \theta_1(\theta_2(s)e^{-t}) \qquad \forall s \geq 0, \quad \forall t \geq 0 \tag{B.13}$$

Theorem B.17 (Converse theorem for global asymptotic stability)**.** *Suppose that the (closed, positive invariant) set $\mathcal{A}$ is globally asymptotically stable for the system $x^+ = f(x)$. Then there exists a Lyapunov function for the system $x^+ = f(x)$ and set $\mathcal{A}$.*

Proof. Since the set $\mathcal{A}$ is GAS we have that for each $x \in \mathbb{R}^n$ and $i \in \mathbb{I}_{\geq 0}$

$$\left| \phi(i;x)_{\mathcal{A}} \right| \leq \beta(|x|_{\mathcal{A}}, i)$$

in which $\beta(\cdot) \in \mathcal{KL}$. Using (B.13) then gives for each $x \in \mathbb{R}^n$ and $i \in \mathbb{I}_{\geq 0}$

$$\theta_1^{-1}\left(\left| \phi(i;x) \right|_{\mathcal{A}} \right) \leq \theta_2(|x|_{\mathcal{A}})e^{-i}$$

in which $\theta_1^{-1}(\cdot) \in \mathcal{K}_\infty$. Propose as Lyapunov function

$$V(x) = \sum_{i=0}^{\infty} \theta_1^{-1}\left(\left| \phi(i;x) \right|_{\mathcal{A}} \right)$$

Since $\phi(0;x) = x$, we have that $V(x) \geq \theta_1^{-1}(|x|_{\mathcal{A}})$ and we choose $\alpha_1(\cdot) = \theta_1^{-1}(\cdot) \in \mathcal{K}_\infty$. Performing the sum gives

$$V(x) = \sum_{i=0}^{\infty} \theta_1^{-1}\left(\left| \phi(i;x)_{\mathcal{A}} \right| \right) \leq \theta_2(|x|_{\mathcal{A}}) \sum_{i=0}^{\infty} e^{-i} = \theta_2(|x|_{\mathcal{A}}) \frac{e}{e-1}$$

and we choose $\alpha_2(\cdot) = (e/(e-1))\theta_2(\cdot) \in \mathcal{K}_\infty$. Finally, noting that $f(\phi(i;x)) = \phi(i+1;x)$ for each $x \in \mathbb{R}^n$, $i \in \mathbb{I}_{\geq 0}$, we have that

$$\begin{aligned} V(f(x)) - V(x) &= \sum_{i=0}^{\infty} \theta_1^{-1}\left(\left| f(\phi(i;x)) \right|_{\mathcal{A}} \right) - \theta_1^{-1}\left(\left| \phi(i;x) \right|_{\mathcal{A}} \right) \\ &= -\theta_1^{-1}(\left| \phi(0;x) \right|_{\mathcal{A}}) \\ &= -\theta_1^{-1}(|x|_{\mathcal{A}}) \end{aligned}$$

and we choose $\alpha_3(\cdot) = \theta_1^{-1}(\cdot) \in \mathcal{K}_\infty$, and the result is established. ∎

The appropriate generalization of Theorem B.15 for the constrained case is:

Theorem B.18 (Lyapunov function for asymptotic stability (constrained))**.** *If there exists a Lyapunov function in X for the system $x^+ = f(x)$ and set $\mathcal{A}$, then $\mathcal{A}$ is asymptotically stable in X for $x^+ = f(x)$.*

The proof of this result is similar to that of Theorem B.15 and is left as an exercise.

Theorem B.19 (Lyapunov function for exponential stability)**.** *If there exists $V : X \to \mathbb{R}_{\geq 0}$ satisfying the following properties for all $x \in X$*

$$a_1 \left|x\right|_{\mathcal{A}}^{\sigma} \leq V(x) \leq a_2 \left|x\right|_{\mathcal{A}}^{\sigma}$$
$$V(f(x)) - V(x) \leq -a_3 \left|x\right|_{\mathcal{A}}^{\sigma}$$

in which $a_1, a_2, a_3, \sigma > 0$, then $\mathcal{A}$ is exponentially stable in X for $x^+ = f(x)$.

Linear time-invariant systems. We review some facts involving the discrete matrix Lyapunov equation and stability of the linear system

$$x^+ = Ax$$

in which $x \in \mathbb{R}^n$. The discrete time system is asymptotically stable if and only if the magnitudes of the eigenvalues of A are strictly less than unity. Such an A matrix is called stable, convergent, or discrete time Hurwitz.

In the following, $A, S, Q \in \mathbb{R}^{n \times n}$. The following matrix equation is known as a discrete matrix Lyapunov equation,

$$A'SA - S = -Q$$

The properties of solutions to this equation allow one to draw conclusions about the stability of A without computing its eigenvalues. Sontag (1998a, p. 231) provides the following lemma

Lemma B.20 (Lyapunov function for linear systems)**.** *The following statements are equivalent (Sontag, 1998a).*

(a) A is stable.

(b) For each $Q \in \mathbb{R}^{n \times n}$, there is a unique solution S of the discrete matrix Lyapunov equation

$$A'SA - S = -Q$$

and if $Q > 0$ then $S > 0$.

(c) There is some $S > 0$ such that $A'SA - S < 0$.

(d) There is some $S > 0$ such that $V(x) = x'Sx$ is a Lyapunov function for the system $x^+ = Ax$.

Exercise B.1 asks you to establish the equivalence of (a) and (b).

B.3.2 Time-Varying, Constrained Systems

Following the discussion in Rawlings and Risbeck (2017), we consider the nonempty sets $X(i) \subseteq \mathbb{R}^n$ indexed by $i \in \mathbb{I}_{\geq 0}$. We define the time-varying system

$$x^+ = f(x, i)$$

with $f(\cdot, i) : X(i) \to X(i+1)$. We assume that $f(\cdot, i)$ is locally bounded for all $i \in \mathbb{I}_{\geq 0}$. Note from the definition of f that the sets $X(i)$ satisfy positive invariance in the following sense: $x \in X(i)$ for any $i \geq 0$ implies $x(i+1) := f(x, i) \in X(i+1)$. We say that the set sequence $(X(i))_{i \geq 0}$ is *sequentially* positive invariant to denote this form of invariance.

Definition B.21 (Sequential positive invariance)**.** A sequence of sets $(X(i))_{i \geq 0}$ is sequentially positive invariant for the system $x^+ = f(x, i)$ if for any $i \geq 0$, $x \in X(i)$ implies $f(x, i) \in X(i+1)$.

We again assume that $\mathcal{A}$ is closed and positive invariant for the time-varying system, i.e, $x \in \mathcal{A}$ at any time $i \geq 0$ implies $f(x, i) \in \mathcal{A}$. We also assume that $\mathcal{A} \subseteq X(i)$ for all $i \geq 0$. We next define asymptotic stability of $\mathcal{A}$.

Definition B.22 (Asymptotic stability (time-varying, constrained))**.** Suppose that the sequence $(X(i))_{i \geq 0}$ is sequentially positive invariant and the set $\mathcal{A} \subseteq X(i)$ for all $i \geq 0$ is closed and positive invariant for $x^+ = f(x, i)$. The set $\mathcal{A}$ is *asymptotically stable* in $X(i)$ at each time $i \geq 0$ for $x^+ = f(x, i)$ if the following holds for all $i \geq i_0 \geq 0$, and $x \in X(i_0)$

$$\left|\phi(i; x, i_0)\right|_{\mathcal{A}} \leq \beta(|x|_{\mathcal{A}}, i - i_0) \tag{B.14}$$

in which $\beta \in \mathcal{KL}$ and $\phi(i; x, i_0)$ is the solution to $x^+ = f(x, i)$ at time $i \geq i_0$ with initial condition x at time $i_0 \geq 0$.

This stability definition is somewhat restrictive because $\phi(i; x, i_0)$ is bounded by a function depending on $i - i_0$ rather than on i. For example, to be more general we could define a time-dependent set of

$\mathcal{KL}$ functions, $\beta_j(\cdot)$, $j \geq 0$, and replace (B.14) with $|\phi(i; x, i_0)|_{\mathcal{A}} \leq \beta_{i_0}(|x|_{\mathcal{A}}, i)$ for all $i \geq i_0 \geq 0$.

We define a time-varying Lyapunov function for this system as follows.

Definition B.23 (Lyapunov function: time-varying, constrained case)**.** Let the sequence $(X(i))_{i\geq 0}$ be sequentially positive invariant, and the set $\mathcal{A} \subseteq X(i)$ for all $i \geq 0$ be closed and positive invariant. Let $V(\cdot, i) : X(i) \to \mathbb{R}_{\geq 0}$ satisfy for all $x \in X(i)$, $i \in \mathbb{I}_{\geq 0}$

$$\begin{aligned} \alpha_1(|x|_{\mathcal{A}}) \leq V(x, i) &\leq \alpha_2(|x|_{\mathcal{A}}) \\ V(f(x, i), i+1) - V(x, i) &\leq -\alpha_3(|x|_{\mathcal{A}}) \end{aligned}$$

with $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$. Then $V(\cdot, \cdot)$ is a time-varying Lyapunov function in the sequence $(X(i))_{i\geq 0}$ for $x^+ = f(x, i)$ and set $\mathcal{A}$.

Note that $f(x, i) \in X(i+1)$ since $x \in X(i)$ which verifies that $V(f(x, i), i+1)$ is well defined for all $x \in X(i)$, $i \geq 0$. We then have the following asymptotic stability result for the time-varying, constrained case.

Theorem B.24 (Lyapunov theorem for asymptotic stability (time-varying, constrained))**.** *Let the sequence $(X(i))_{i\geq 0}$ be sequentially positive invariant, and the set $\mathcal{A} \subseteq X(i)$ for all $i \geq 0$ be closed and positive invariant, and $V(\cdot, \cdot)$ be a time-varying Lyapunov function in the sequence $(X(i))_{i\geq 0}$ for $x^+ = f(x, i)$ and set $\mathcal{A}$. Then $\mathcal{A}$ is asymptotically stable in $X(i)$ at each time $i \geq 0$ for $x^+ = f(x, i)$.*

Proof. For $x \in X(i_0)$, we have that $(\phi(i; x, i_0), i) \in X(i)$ for all $i \geq i_0$. From the first and second inequalities we have that for all $i \geq i_0$ and $x \in X(i_0)$

$$\begin{aligned} V(\phi(i+1; x, i_0), i+1) &\leq V(\phi(i; x, i_0), i) - \alpha_3(|\phi(i; x, i_0)|_{\mathcal{A}}) \\ &\leq \sigma_1(V(\phi(i; x, i_0), i)) \end{aligned}$$

with $\sigma_1(s) := s - \alpha_3 \circ \alpha_2^{-1}(s)$. Note that $\sigma_1(\cdot)$ may not be $\mathcal{K}_\infty$ because it may not be increasing. But given this result we can find, as in the proof of Theorem B.15, $\sigma(\cdot) \in \mathcal{K}_\infty$ satisfying $\sigma_1(s) < \sigma(s) < s$ for all $s \in \mathbb{R}_{>0}$ such that $V(\phi(i+1; x, i_0), i+1) \leq \sigma(V(\phi(i; x, i_0), i))$. We then have that

$$|\phi(i; x, i_0)|_{\mathcal{A}} \leq \beta(|x|_{\mathcal{A}}, i - i_0) \qquad \forall x \in X(i_0), \quad i \geq i_0$$

in which $\beta(s, i) := \alpha_1^{-1} \circ \sigma^i \circ \alpha_2(s)$ for $s \in \mathbb{R}_{\geq 0}$, $i \geq 0$ is a $\mathcal{KL}$ function, and the result is established. ∎

B.3.3 Upper bounding $\mathcal{K}$ functions

In using Lyapunov functions for stability analysis, we often have to establish that the upper bound inequality holds on some closed set. The following result proves useful in such situations.

Proposition B.25 (Global K function overbound)**.** *Let $X \subseteq \mathbb{R}^n$ be closed and suppose that a function $V : X \to \mathbb{R}_{\geq 0}$ is continuous at $x_0 \in X$ and locally bounded on X, i.e., bounded on every compact subset of X. Then, there exists a K function α such that*

$$|V(x) - V(x_0)| \leq \alpha(|x - x_0|) \quad \textit{for all } x \in X$$

A proof is given in Rawlings and Risbeck (2015).

B.4 Robust Stability

We now turn to the task of obtaining stability conditions for discrete time systems subject to disturbances. There are two separate questions that should be addressed. The first is *nominal* robustness; is asymptotic stability of a set $\mathcal{A}$ for a (nominal) system $x^+ = f(x)$ maintained in the presence of arbitrarily small disturbances? The second question is the determination of conditions for asymptotic stability of a set $\mathcal{A}$ for a system perturbed by disturbances lying in a given compact set.

B.4.1 Nominal Robustness

Here we follow Teel (2004). The nominal system is $x^+ = f(x)$. Consider the perturbed system

$$x^+ = f(x + e) + w \tag{B.15}$$

where e is the state error and w the additive disturbance. Let $\mathbf{e} := (e(0), e(1), \ldots)$ and $\mathbf{w} := (w(0), w(1), \ldots)$ denote the disturbance sequences with norms $\|\mathbf{e}\| := \sup_{i \geq 0} |e(i)|$ and $\|\mathbf{w}\| := \sup_{i \geq 0} |w(i)|$. Let $M_\delta := \{(\mathbf{e}, \mathbf{w}) \mid \|\mathbf{e}\| \leq \delta, \|\mathbf{w}\| \leq \delta\}$ and, for each $x \in \mathbb{R}^n$, let S_δ denote the set of solutions $\phi(\cdot; x, \mathbf{e}, \mathbf{w})$ of (B.15) with initial state x (at time 0) and perturbation sequences $(\mathbf{e}, \mathbf{w}) \in M_\delta$. A closed, compact set $\mathcal{A}$ is *nominally* robustly asymptotically stable for the (nominal) system $x^+ = f(x)$ if a small neighborhood of $\mathcal{A}$ is locally stable and attractive for all sufficiently small perturbation sequences. We use the adjective *nominal* to indicate that we are examining how a system $x^+ = f(x)$ for which $\mathcal{A}$ is known to be asymptotically stable behaves when subjected to small disturbances. More precisely Teel (2004):

Definition B.26 (Nominal robust global asymptotic stability)**.** The closed, compact set $\mathcal{A}$ is said to be nominally robustly globally asymptotically stable (nominally RGAS) for the system $x^+ = f(x)$ if there exists a $\mathcal{KL}$ function $\beta(\cdot)$ and, for each $\varepsilon > 0$ and each compact set X, there exists a $\delta > 0$ such that, for each $x \in X$ and each solution $\phi(\cdot)$ of the perturbed system lying in S_δ, $\left|\phi(i)\right|_{\mathcal{A}} \leq \beta(|x|_{\mathcal{A}}, i) + \varepsilon$ for all $i \in \mathbb{I}_{\geq 0}$.

Thus, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that each solution $\phi(\cdot)$ of $x^+ = f(x+e)+w$ starting in a δ neighborhood of $\mathcal{A}$ remains in a $\beta(\delta, 0) + \varepsilon$ neighborhood of $\mathcal{A}$, and each solution starting anywhere in $\mathbb{R}^n$ converges to a ε neighborhood of $\mathcal{A}$. These properties are a necessary relaxation (because of the perturbations) of local stability and global attractivity.

Remark. What we call "nominally robustly globally asymptotically stable" in the above definition is called "robustly globally asymptotically stable" in Teel (2004); we use the term "nominal" to indicate that we are concerned with the effect of perturbations e and w on the stability properties of a "nominal" system $x^+ = f(x)$ for which asymptotic stability of a set $\mathcal{A}$ has been established (in the absence of perturbations). We use the expression "$\mathcal{A}$ is globally asymptotically stable for $x^+ = f(x + e) + w$" to refer to the case when asymptotic stability of a set $\mathcal{A}$ has been established for the perturbed system $x^+ = f(x+e)+w$.

The following result, where we add the adjective "nominal", is established in (Teel, 2004, Theorem 2):

Theorem B.27 (Nominal robust global asymptotic stability and Lyapunov function)**.** *Suppose set $\mathcal{A}$ is closed and compact and $f(\cdot)$ is locally bounded. Then the set $\mathcal{A}$ is nominally robustly globally asymptotically stable for the system $x^+ = f(x)$ if and only if there exists a continuous (in fact, smooth) Lyapunov function for $x^+ = f(x)$ and set $\mathcal{A}$.*

The significance of this result is that while a nonrobust system, for which $\mathcal{A}$ is globally asymptotically stable, has a Lyapunov function, that function is *not* continuous. For the globally asymptotically stable example $x^+ = f(x)$ discussed in Section 3.2 of Chapter 3, where $f(x) = (0, |x|)$ when $x_1 \neq 0$ and $f(x) = (0, 0)$ otherwise, one Lyapunov function $V(\cdot)$ is $V(x) = 2\,|x|$ if $x_1 \neq 0$ and $V(x) = |x|$ if $x_1 = 0$. That $V(\cdot)$ is a Lyapunov function follows from the fact that it satisfies $V(x) \geq |x|$, $V(x) \leq 2\,|x|$ and $V(f(x)) - V(x) = -\,|x|$ for all $x \in \mathbb{R}^2$.

It follows immediately from its definition that $V(\cdot)$ is not continuous; but we can also deduce from Theorem B.27 that every Lyapunov function for this system is not continuous since, as shown in Section 3.2 of Chapter 3, global asymptotic stability for this system is not robust. Theorem B.27 shows that existence of a continuous Lyapunov function guarantees nominal robustness. Also, it follows from Theorem B.17 that there exists a smooth Lyapunov function for $x^+ = f(x)$ if $f(\cdot)$ is continuous and $\mathcal{A}$ is GAS for $x^+ = f(x)$. Since $f(\cdot)$ is locally bounded if it is continuous, it then follows from Theorem B.27 that $\mathcal{A}$ is nominally robust GAS for $x^+ = f(x)$ if it is GAS and $f(\cdot)$ is continuous.

B.4.2 Robustness

We turn now to stability conditions for systems subject to bounded disturbances (not vanishingly small) and described by

$$x^+ = f(x, w) \tag{B.16}$$

where the disturbance w lies in the compact set $\mathbb{W}$. This system may equivalently be described by the difference inclusion

$$x^+ \in F(x) \tag{B.17}$$

where the set $F(x) := \{f(x, w) \mid w \in \mathbb{W}\}$. Let $S(x)$ denote the set of all solutions of (B.16) or (B.17) with initial state x. We require, in the sequel, that the closed set $\mathcal{A}$ is positive invariant for (B.16) (or for $x^+ \in F(x)$):

Definition B.28 (Positive invariance with disturbances)**.** The closed set $\mathcal{A}$ is positive invariant for $x^+ = f(x, w)$, $w \in \mathbb{W}$ if $x \in \mathcal{A}$ implies $f(x, w) \in \mathcal{A}$ for all $w \in \mathbb{W}$; it is positive invariant for $x^+ \in F(x)$ if $x \in \mathcal{A}$ implies $F(x) \subseteq \mathcal{A}$.

Clearly the two definitions are equivalent; $\mathcal{A}$ is positive invariant for $x^+ = f(x, w)$, $w \in \mathbb{W}$, if and only if it is positive invariant for $x^+ \in F(x)$.

Remark. In the MPC literature, but not necessarily elsewhere, the term robust positive invariant is often used in place of positive invariant to emphasize that positive invariance is maintained despite the presence of the disturbance w. However, since the uncertain system $x^+ = f(x, w)$, $w \in \mathbb{W}$ is specified ($x^+ = f(x, w)$, $w \in \mathbb{W}$ or $x^+ \in F(x)$) in the assertion that a closed set $\mathcal{A}$ is positive invariant, the word "robust" appears to be unnecessary. In addition, in the systems literature, the closed set $\mathcal{A}$ is said to be robust positive invariant for $x^+ \in F(x)$ if it satisfies conditions similar to those of Definition B.26 with $x^+ \in F(x)$ replacing $x^+ = f(x)$; see Teel (2004), Definition 3.

In Definitions B.29–B.31, we use "positive invariant" to denote "positive invariant for $x^+ = f(x, w)$, $w \in \mathbb{W}$" or for $x^+ \in F(x)$.

Definition B.29 (Local stability (disturbances))**.** The closed, positive invariant set $\mathcal{A}$ is *locally stable* for $x^+ = f(x, w)$, $w \in \mathbb{W}$ (or for $x^+ \in F(x)$) if, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that, for each x satisfying $|x|_{\mathcal{A}} < \delta$, each solution $\phi \in S(x)$ satisfies $|\phi(i)|_{\mathcal{A}} < \varepsilon$ for all $i \in \mathbb{I}_{\geq 0}$.

Definition B.30 (Global attraction (disturbances))**.** The closed, positive invariant set $\mathcal{A}$ is *globally attractive* for the system $x^+ = f(x, w)$, $w \in \mathbb{W}$ (or for $x^+ \in F(x)$) if, for each $x \in \mathbb{R}^n$, each solution $\phi(\cdot) \in S(x)$ satisfies $|\phi(i)|_{\mathcal{A}} \to 0$ as $i \to \infty$.

Definition B.31 (GAS (disturbances))**.** The closed, positive invariant set $\mathcal{A}$ is *globally asymptotically stable* for $x^+ = f(x, w)$, $w \in \mathbb{W}$ (or for $x^+ \in F(x)$) if it is locally stable and globally attractive.

An alternative definition of global asymptotic stability of closed set $\mathcal{A}$ for $x^+ = f(x, w)$, $w \in \mathbb{W}$, if $\mathcal{A}$ is compact, is the existence of a $\mathcal{KL}$ function $\beta(\cdot)$ such that for each $x \in \mathbb{R}^n$, each $\phi \in S(x)$ satisfies $|\phi(i)|_{\mathcal{A}} \leq \beta(|x|_{\mathcal{A}}, i)$ for all $i \in \mathbb{I}_{\geq 0}$. To cope with disturbances we require a modified definition of a Lyapunov function.

Definition B.32 (Lyapunov function (disturbances))**.** A function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be a Lyapunov function for the system $x^+ = f(x, w)$, $w \in \mathbb{W}$ (or for $x^+ \in F(x)$) and closed set $\mathcal{A}$ if there exist functions $\alpha_i \in \mathcal{K}_\infty$, $i = 1, 2, 3$ such that for any $x \in \mathbb{R}^n$,

$$V(x) \geq \alpha_1(|x|_{\mathcal{A}}) \tag{B.18}$$

$$V(x) \leq \alpha_2(|x|_{\mathcal{A}}) \tag{B.19}$$

$$\sup_{z \in F(x)} V(z) - V(x) \leq -\alpha_3(|x|_{\mathcal{A}}) \tag{B.20}$$

Remark. Without loss of generality, we can choose the function $\alpha_3(\cdot)$ in (B.20) to be a class $\mathcal{K}_\infty$ function if $f(\cdot)$ is continuous (see Jiang and Wang (2002), Lemma 2.8).

Inequality B.20 ensures $V(f(x,w)) - V(x) \leq -\alpha_3(|x|_{\mathcal{A}})$ for all $w \in \mathbb{W}$. The existence of a Lyapunov function for the system $x^+ \in F(x)$ and closed set $\mathcal{A}$ is a sufficient condition for $\mathcal{A}$ to be globally asymptotically stable for $x^+ \in F(x)$ as shown in the next result.

Theorem B.33 (Lyapunov function for global asymptotic stability (disturbances))**.** *Suppose $V(\cdot)$ is a Lyapunov function for $x^+ = f(x,w)$, $w \in \mathbb{W}$ (or for $x^+ \in F(x)$) and closed set $\mathcal{A}$ with $\alpha_3(\cdot)$ a $\mathcal{K}_\infty$ function. Then $\mathcal{A}$ is globally asymptotically stable for $x^+ = f(x,w)$, $w \in \mathbb{W}$ (or for $x^+ \in F(x)$).*

Proof. (i) Local stability: Let $\varepsilon > 0$ be arbitrary and let $\delta := \alpha_2^{-1}(\alpha_1(\varepsilon))$. Suppose $|x|_{\mathcal{A}} < \delta$ so that, by (B.19), $V(x) \leq \alpha_2(\delta) = \alpha_1(\varepsilon)$. Let $\phi(\cdot)$ be any solution in $S(x)$ so that $\phi(0) = x$. From (B.20), $(V(\phi(i)))_{i \in \mathbb{I}_{\geq 0}}$ is a nonincreasing sequence so that, for all $i \in \mathbb{I}_{\geq 0}$, $V(\phi(i)) \leq V(x)$. From (B.18), $|\phi(i)|_{\mathcal{A}} \leq \alpha_1^{-1}(V(x)) \leq \alpha_1^{-1}(\alpha_1(\varepsilon)) = \varepsilon$ for all $i \in \mathbb{I}_{\geq 0}$.
(ii) Global attractivity: Let $x \in \mathbb{R}^n$ be arbitrary. Let $\phi(\cdot)$ be any solution in $S(x)$ so that $\phi(0) = x$. From Equations B.18 and B.20, since $\phi(i+1) \in F(\phi(i))$, the sequence $(V(\phi(i)))_{i \in \mathbb{I}_{\geq 0}}$ is nonincreasing and bounded from below by zero. Hence both $V(\phi(i))$ and $V(\phi(i+1))$ converge to $\bar{V} \geq 0$ as $i \to \infty$. But $\phi(i+1) \in F(\phi(i))$ so that, from (B.20), $\alpha_3(|\phi(i)|_{\mathcal{A}}) \to 0$ as $i \to \infty$. Since $|\phi(i)|_{\mathcal{A}} = \alpha_3^{-1}(\alpha_3(|\phi(i)|_{\mathcal{A}}))$ where $\alpha_3^{-1}(\cdot)$ is a $\mathcal{K}_\infty$ function, $|\phi(i)|_{\mathcal{A}} \to 0$ as $i \to \infty$. ∎

B.5 Control Lyapunov Functions

A control Lyapunov function is a useful generalization, due to Sontag (1998a, pp.218–233), of a Lyapunov function; while a Lyapunov function is relevant for a system $x^+ = f(x)$ and provides conditions for the (asymptotic) stability of a set for this system, a control Lyapunov function is relevant for a control system $x^+ = f(x,u)$ and provides conditions for the existence of a controller $u = \kappa(x)$ that ensures (asymptotic) stability of a set for the controlled system $x^+ = f(x,\kappa(x))$. Consider the control system

$$x^+ = f(x,u)$$

where the control u is subject to the constraint

$$u \in \mathbb{U}$$

Our standing assumptions in this section are that $f(\cdot)$ is continuous and $\mathbb{U}$ is compact.

Definition B.34 (Global control Lyapunov function (CLF))**.** A function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a global control Lyapunov function for the system $x^+ = f(x, u)$ and closed set $\mathcal{A}$ if there exist $\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\alpha_3(\cdot)$ satisfying for all $x \in \mathbb{R}^n$:

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}})$$
$$\inf_{u \in \mathbb{U}} V(f(x,u)) - V(x) \leq -\alpha_3(|x|_{\mathcal{A}})$$

Definition B.35 (Global stabilizability)**.** Let set $\mathcal{A}$ be compact. The set $\mathcal{A}$ is globally stabilizable for the system $x^+ = f(x, u)$ if there exists a state-feedback function $\kappa : \mathbb{R}^n \to \mathbb{U}$ such that $\mathcal{A}$ is globally asymptotically stable for $x^+ = f(x, \kappa(x))$.

Remark. Given a global control Lyapunov function $V(\cdot)$, one can choose a control law $\kappa : \mathbb{R}^n \to \mathbb{U}$ satisfying

$$V(f(x, \kappa(x))) \leq V(x) - \alpha_3(|x|_{\mathcal{A}})/2$$

for all $x \in \mathbb{R}^n$ (seeTeel (2004)). Since $\mathbb{U}$ is compact, $\kappa(\cdot)$ is locally bounded and, hence, so is $x \mapsto f(x, \kappa(x))$. Thus we may use Theorem B.13 to deduce that $\mathcal{A}$ is globally asymptotically stable for $x^+ = f(x, \kappa(x))$. If $V(\cdot)$ is continuous, one can also establish nominal robustness properties.

In a similar fashion one can extend the concept of control Lyapunov functions to the case when the system is subject to disturbances. Consider the system

$$x^+ = f(x, u, w)$$

where the control u is constrained to lie in $\mathbb{U}$ and the disturbance takes values in the set $\mathbb{W}$. We assume that $f(\cdot)$ is continuous and that $\mathbb{U}$ and $\mathbb{W}$ are compact. The system may be equivalently defined by

$$x^+ \in F(x, u)$$

where the set-valued function $F(\cdot)$ is defined by

$$F(x, u) := \{f(x, u, w) \mid w \in \mathbb{W}\}$$

We can now make the obvious generalizations of the definitions in Section B.4.2.

Definition B.36 (Positive invariance (disturbance and control))**.** The closed set $\mathcal{A}$ is positive invariant for $x^+ = f(x, u, w)$, $w \in \mathbb{W}$ (or for $x^+ \in F(x, u)$) if for all $x \in \mathcal{A}$ there exists a $u \in \mathbb{U}$ such that $f(x, u, w) \in \mathcal{A}$ for all $w \in \mathbb{W}$ (or $F(x, u) \subseteq \mathcal{A}$).

Definition B.37 (CLF (disturbance and control))**.** A function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be a control Lyapunov function for the system $x^+ = f(x, u, w)$, $u \in \mathbb{U}$, $w \in \mathbb{W}$ (or $x^+ \in F(x, u)$, $u \in \mathbb{U}$) and set $\mathcal{A}$ if there exist functions $\alpha_i \in \mathcal{K}_\infty$, $i = 1, 2, 3$ such that for any $x \in \mathbb{R}^n$,

$$\begin{aligned} \alpha_1(|x|_{\mathcal{A}}) \leq V(x) &\leq \alpha_2(|x|_{\mathcal{A}}) \\ \inf_{u \in \mathbb{U}} \sup_{z \in F(x,u)} V(z) - V(x) &\leq -\alpha_3(|x|_{\mathcal{A}}) \end{aligned} \tag{B.21}$$

Remark (CLF implies control law)**.** Given a global control Lyapunov function $V(\cdot)$, one can choose a control law $\kappa : \mathbb{R}^n \to \mathbb{U}$ satisfying

$$\sup_{z \in F(x,\kappa(x))} V(z) \leq V(x) - \alpha_3(|x|_{\mathcal{A}})/2$$

for all $x \in \mathbb{R}^n$. Since $\mathbb{U}$ is compact, $\kappa(\cdot)$ is locally bounded and, hence, so is $x \mapsto f(x, \kappa(x))$. Thus we may use Theorem B.33 to deduce that $\mathcal{A}$ is globally asymptotically stable for $x^+ = f(x, \kappa(x), w)$, $w \in \mathbb{W}$ (for $x^+ \in F(x, \kappa(x))$).

These results can be further extended to deal with the constrained case. First, we generalize the definitions of positive invariance of a set.

Definition B.38 (Positive invariance (constrained))**.** The closed set $\mathcal{A}$ is control invariant for $x^+ = f(x, u)$, $u \in \mathbb{U}$ if, for all $x \in \mathcal{A}$, there exists a $u \in \mathbb{U}$ such that $f(x, u) \in \mathcal{A}$.

Suppose that the state x is required to lie in the closed set $\mathbb{X} \subset \mathbb{R}^n$. In order to show that it is possible to ensure a decrease of a Lyapunov function, as in (B.21), in the presence of the state constraint $x \in \mathbb{X}$, we assume that there exists a control invariant set $X \subseteq \mathbb{X}$ for $x^+ = f(x, u, w)$, $u \in \mathbb{U}$, $w \in \mathbb{W}$. This enables us to obtain a control law that keeps the state in X and, hence, in $\mathbb{X}$, and, under suitable conditions, to satisfy a variant of (B.21).

Definition B.39 (CLF (constrained))**.** Suppose the set X and closed set $\mathcal{A}$, $\mathcal{A} \subset X$, are control invariant for $x^+ = f(x, u)$, $u \in \mathbb{U}$. A function $V : X \to \mathbb{R}_{\geq 0}$ is said to be a control Lyapunov function in X for the system $x^+ = f(x,u)$, $u \in \mathbb{U}$, and closed set $\mathcal{A}$ in X if there exist functions $\alpha_i \in \mathcal{K}_\infty$, $i = 1,2,3$, defined on X, such that for any $x \in X$,

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}})$$
$$\inf_{u \in \mathbb{U}} \{V(f(x,u)) \mid f(x,u) \in X\} - V(x) \leq -\alpha_3(|x|_{\mathcal{A}})$$

Remark. Again, if $V(\cdot)$ is a control Lyapunov function in X for $x^+ = f(x,u)$, $u \in \mathbb{U}$ and closed set $\mathcal{A}$ in X, one can choose a control law $\kappa : \mathbb{R}^n \to \mathbb{U}$ satisfying

$$V(f(x,\kappa(x))) - V(x) \leq -\alpha_3(|x|_{\mathcal{A}})/2$$

for all $x \in X$. Since $\mathbb{U}$ is compact, $\kappa(\cdot)$ is locally bounded and, hence, so is $x \mapsto f(x,\kappa(x))$. Thus, when $\alpha_3(\cdot)$ is a $\mathcal{K}_\infty$ function, we may use Theorem B.18 to deduce that $\mathcal{A}$ is asymptotically stable for $x^+ = f(x, \kappa(x))$, $u \in \mathbb{U}$ in X; also $\phi(i;x) \in X \subset \mathbb{X}$ for all $x \in X$, all $i \in \mathbb{I}_{\geq 0}$.

Finally we consider the constrained case in the presence of disturbances. First we define control invariance in the presence of disturbances.

Definition B.40 (Control invariance (disturbances, constrained))**.** The closed set $\mathcal{A}$ is control invariant for $x^+ = f(x,u,w)$, $u \in \mathbb{U}$, $w \in \mathbb{W}$ if, for all $x \in \mathcal{A}$, there exists a $u \in \mathbb{U}$ such that $f(x,u,w) \in \mathcal{A}$ for all $w \in \mathbb{W}$ (or $F(x,u) \subseteq \mathcal{A}$ where $F(x,u) := \{f(x,u,w) \mid w \in \mathbb{W}\}$).

Next, we define what we mean by a control Lyapunov function in this context.

Definition B.41 (CLF (disturbances, constrained))**.** Suppose the set X and closed set $\mathcal{A}$, $\mathcal{A} \subset X$, are control invariant for $x^+ = f(x,u,w)$, $u \in \mathbb{U}$, $w \in \mathbb{W}$. A function $V : X \to \mathbb{R}_{\geq 0}$ is said to be a control Lyapunov function in X for the system $x^+ = f(x,u,w)$, $u \in \mathbb{U}$, $w \in \mathbb{W}$ and set $\mathcal{A}$ if there exist functions $\alpha_i \in \mathcal{K}_\infty$, $i = 1,2,3$, defined on X, such that for any $x \in X$,

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}})$$
$$\inf_{u \in \mathbb{U}} \sup_{z \in F(x,u) \cap X} V(z) - V(x) \leq -\alpha_3(|x|_{\mathcal{A}})$$

Suppose now that the state x is required to lie in the closed set $\mathbb{X} \subset \mathbb{R}^n$. Again, in order to show that there exists a condition similar to (B.21), we assume that there exists a control invariant set $X \subseteq \mathbb{X}$ for

$x^+ = f(x, u, w)$, $u \in \mathbb{U}$, $w \in \mathbb{W}$. This enables us to obtain a control law that keeps the state in X and, hence, in $\mathbb{X}$, and, under suitable conditions, to satisfy a variant of (B.21).

Remark. If $V(\cdot)$ is a control Lyapunov function in X for $x^+ = f(x, u)$, $u \in \mathbb{U}$, $w \in \mathbb{W}$ and set $\mathcal{A}$ in X, one can choose a control law $\kappa : X \to \mathbb{U}$ satisfying

$$\sup_{z \in F(x,\kappa(x))} V(z) - V(x) \leq -\alpha_3(|x|_{\mathcal{A}})/2$$

for all $x \in X$. Since $\mathbb{U}$ is compact, $\kappa(\cdot)$ is locally bounded and, hence, so is $x \mapsto f(x, \kappa(x))$. Thus, when $\alpha_3(\cdot)$ is a $\mathcal{K}_\infty$ function, we may use Theorem B.18 to deduce that $\mathcal{A}$ is asymptotically stable in X for $x^+ = f(x, \kappa(x), w)$, $w \in \mathbb{W}$ (or, equivalently, for $x^+ \in F(x, \kappa(x))$); also $\phi(i) \in X \subset \mathbb{X}$ for all $x \in X$, all $i \in \mathbb{I}_{\geq 0}$, all $\phi \in S(x)$.

B.6 Input-to-State Stability

We consider, as in the previous section, the system

$$x^+ = f(x, w)$$

where the disturbance w takes values in $\mathbb{R}^p$. In input-to-state stability (Sontag and Wang, 1995; Jiang and Wang, 2001) we seek a bound on the state in terms of a uniform bound on the disturbance sequence $\mathbf{w} := (w(0), w(1), \ldots)$. Let $\|\cdot\|$ denote the usual ℓ_∞ norm for sequences, i.e., $\|\mathbf{w}\| := \sup_{k \geq 0} |w(k)|$.

Definition B.42 (Input-to-state stable (ISS))**.** The system $x^+ = f(x, w)$ is (globally) input-to-state stable (ISS) if there exists a $\mathcal{KL}$ function $\beta(\cdot)$ and a $\mathcal{K}$ function $\sigma(\cdot)$ such that, for each $x \in \mathbb{R}^n$, and each disturbance sequence $\mathbf{w} = (w(0), w(1), \ldots)$ in ℓ_∞

$$\left|\phi(i; x, \mathbf{w}_i)\right| \leq \beta(|x|, i) + \sigma(\|\mathbf{w}_i\|)$$

for all $i \in \mathbb{I}_{\geq 0}$, where $\phi(i; x, \mathbf{w}_i)$ is the solution, at time i, if the initial state is x at time 0 and the input sequence is $\mathbf{w}_i := (w(0), w(1), \ldots, w(i-1))$.

We note that this definition implies the origin is globally asymptotically stable if the input sequence is identically zero. Also, the norm of the state is asymptotically bounded by $\sigma(\|\mathbf{w}\|)$ where $\mathbf{w} := (w(0), w(1), \ldots)$. As before, we seek a Lyapunov function that ensures input-to-state stability.

Definition B.43 (ISS-Lyapunov function)**.** A function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is an ISS-Lyapunov function for system $x^+ = f(x, w)$ if there exist $\mathcal{K}_\infty$ functions $\alpha_1(\cdot), \alpha_2(\cdot), \alpha_3(\cdot)$ and a $\mathcal{K}$ function $\sigma(\cdot)$ such that for all $x \in \mathbb{R}^n$, $w \in \mathbb{R}^p$

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$
$$V(f(x,w)) - V(x) \leq -\alpha_3(|x|) + \sigma(|w|)$$

The following result appears in Jiang and Wang (2001, Lemma 3.5)

Lemma B.44 (ISS-Lyapunov function implies ISS)**.** *Suppose $f(\cdot)$ is continuous and that there exists a continuous ISS-Lyapunov function for $x^+ = f(x, w)$. Then the system $x^+ = f(x, w)$ is ISS.*

The converse, i.e., input-to-state stability implies the existence of a smooth ISS-Lyapunov function for $x^+ = f(x, w)$ is also proved in Jiang and Wang (2002, Theorem 1). We now consider the case when the state satisfies the constraint $x \in \mathbb{X}$ where $\mathbb{X}$ is a closed subset of $\mathbb{R}^n$. Accordingly, we assume that the disturbance w satisfies $w \in \mathbb{W}$ where $\mathbb{W}$ is a compact set containing the origin and that $X \subset \mathbb{X}$ is a closed robust positive invariant set for $x^+ = f(x, w)$, $w \in \mathbb{W}$ or, equivalently, for $x^+ \in F(x, u)$.

Definition B.45 (ISS (constrained))**.** Suppose that $\mathbb{W}$ is a compact set containing the origin and that $X \subset \mathbb{X}$ is a closed robust positive invariant set for $x^+ = f(x, w)$, $w \in \mathbb{W}$. The system $x^+ = f(x, w)$, $w \in \mathbb{W}$ is ISS in X if there exists a class $\mathcal{KL}$ function $\beta(\cdot)$ and a class $\mathcal{K}$ function $\sigma(\cdot)$ such that, for all $x \in X$, all $\mathbf{w} \in \mathcal{W}$ where $\mathcal{W}$ is the set of infinite sequences $\mathbf{w}$ satisfying $w(i) \in \mathbb{W}$ for all $i \in \mathbb{I}_{\geq 0}$

$$\left|\phi(i; x, \mathbf{w}_i)\right| \leq \beta(|x|, i) + \sigma(\|\mathbf{w}_i\|)$$

Definition B.46 (ISS-Lyapunov function (constrained))**.** A function $V : X \to \mathbb{R}_{\geq 0}$ is an ISS-Lyapunov function in X for system $x^+ = f(x, w)$ if there exist $\mathcal{K}_\infty$ functions $\alpha_1(\cdot), \alpha_2(\cdot), \alpha_3(\cdot)$ and a $\mathcal{K}$ function $\sigma(\cdot)$ such that for all $x \in X$, all $w \in \mathbb{W}$

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$
$$V(f(x,w)) - V(x) \leq -\alpha_3(|x|) + \sigma(|w|)$$

The following result is a minor generalization of Lemma 3.5 in Jiang and Wang (2001).

Lemma B.47 (ISS-Lyapunov function implies ISS (constrained))**.** *Suppose that $\mathbb{W}$ is a compact set containing the origin and that $X \subset \mathbb{X}$ is a closed robust positive invariant set for $x^+ = f(x, w)$, $w \in \mathbb{W}$. If $f(\cdot)$ is continuous and there exists a continuous ISS-Lyapunov function in X for the system $x^+ = f(x, w)$, $w \in \mathbb{W}$, then the system $x^+ = f(x, w)$, $w \in \mathbb{W}$ is ISS in X.*

B.7 Output-to-State Stability and Detectability

We present some definitions and results that are discrete time versions of results due to Sontag and Wang (1997) and Krichman, Sontag, and Wang (2001). The output-to-state (OSS) property corresponds, informally, to the statement that "no matter what the initial state is, if the observed outputs are small, then the state must eventually be small". It is therefore a natural candidate for the concept of nonlinear (zero-state) detectability. We consider first the autonomous system

$$x^+ = f(x) \qquad y = h(x) \tag{B.22}$$

where $f(\cdot) : \mathbb{X} \to \mathbb{X}$ is locally Lipschitz continuous and $h(\cdot)$ is continuously differentiable where $\mathbb{X} = \mathbb{R}^n$ for some n. We assume $x = 0$ is an equilibrium state, i.e., $f(0) = 0$. We also assume $h(0) = 0$. We use $\phi(k; x_0)$ to denote the solution of (B.22) with initial state x_0, and $y(k; x_0)$ to denote $h(\phi(k; x_0))$. The function $y_{x_0}(\cdot)$ is defined by

$$y_{x_0}(k) := y(k; x_0)$$

We use $|\cdot|$ and $\|\cdot\|$ to denote, respectively the Euclidean norm of a vector and the sup norm of a sequence; $\|\cdot\|_{0:k}$ denotes the max norm of a sequence restricted to the interval $[0, k]$. For conciseness, $\mathbf{u}$, $\mathbf{y}$ denote, respectively, the sequences $(u(j))$, $(y(j))$.

Definition B.48 (Output-to-state stable (OSS))**.** The system (B.22) is output-to-state stable (OSS) if there exist functions $\beta(\cdot) \in \mathcal{KL}$ and $\gamma(\cdot) \in \mathcal{K}$ such that for all $x_0 \in \mathbb{R}^n$ and all $k \geq 0$

$$|x(k; x_0)| \leq \max\{\beta(|x_0|, k),\ \gamma(\|\mathbf{y}\|_{0:k})\}$$

Definition B.49 (OSS-Lyapunov function)**.** An OSS-Lyapunov function for system (B.22) is any function $V(\cdot)$ with the following properties

(a) There exist $\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$

for all x in $\mathbb{R}^n$.

(b) There exist $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\sigma(\cdot)$ such that for all $x \in \mathbb{R}^n$ either

$$V(x^+) \leq V(x) - \alpha(|x|) + \sigma(|y|)$$

or

$$V(x^+) \leq \rho V(x) + \sigma(|y|) \tag{B.23}$$

with $x^+ = f(x)$, $y = h(x)$, and $\rho \in (0,1)$.

Inequality (B.23) corresponds to an exponential-decay OSS-Lyapunov function.

Theorem B.50 (OSS and OSS-Lyapunov function)**.** *The following properties are equivalent for system* (B.22)*:*

(a) The system is OSS.

(b) The system admits an OSS-Lyapunov function.

(c) The system admits an exponential-decay OSS-Lyapunov function.

B.8 Input/Output-to-State Stability

Consider now a system with both inputs and outputs

$$x^+ = f(x, u) \qquad y = h(x) \tag{B.24}$$

Input/output-to-state stability corresponds roughly to the statement that, no matter what the initial state is, if the input and the output converge to zero, so does the state. We assume $f(\cdot)$ and $h(\cdot)$ are continuous. We also assume $f(0,0) = 0$ and $h(0) = 0$. Let $x(\cdot, x_0, \mathbf{u})$ denote the solution of (B.24) which results from initial state x_0 and control $\mathbf{u} = \big(u(j)\big)_{j \geq 0}$ and let $y_{x_0,\mathbf{u}}(k) := y(k; x_0, \mathbf{u})$ denote $h(x(k; x_0, \mathbf{u}))$.

Definition B.51 (Input/output-to-state stable (IOSS))**.** The system (B.24) is input/output-to-state stable (IOSS) if there exist functions $\beta(\cdot) \in \mathcal{KL}$ and $\gamma_1(\cdot)$, $\gamma_2(\cdot) \in \mathcal{K}$ such that

$$|x(k; x_0)| \leq \max\left\{\beta(|x_0|, k), \gamma_1\left(\|\mathbf{u}\|_{0:k-1}\right), \gamma_2\left(\|\mathbf{y}\|_{0:k}\right)\right\}$$

for every initial state $x_0 \in \mathbb{R}^n$, every control sequence $\mathbf{u} = \big(u(j)\big)$, and all $k \geq 0$.

Definition B.52 (IOSS-Lyapunov function)**.** An IOSS-Lyapunov function for system (B.24) is any function $V(\cdot)$ with the following properties:

(a) There exist $\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$

for all $x \in \mathbb{R}^n$.

(b) There exist $\mathcal{K}_\infty$ functions $\alpha(\cdot)$, $\sigma_1(\cdot)$, and $\sigma_2(\cdot)$ such that for every x and u either

$$V(x^+) \leq V(x) - \alpha(|x|) + \sigma_1(|u|) + \sigma_2(|y|)$$

or

$$V(x^+) \leq \rho V(x) + \sigma_1(|u|) + \sigma_2(|y|)$$

with $x^+ = f(x,u)$, $y = h(x)$, and $\rho \in (0,1)$.

The following result proves useful when establishing that MPC employing cost functions based on the inputs and outputs rather than inputs and states is stabilizing for IOSS systems. Consider the system $x^+ = f(x,u), y = h(x)$ with stage cost $\ell(y,u)$ and constraints $(x, u) \in \mathbb{Z}$. The stage cost satisfies $\ell(0,0) = 0$ and $\ell(y,u) \geq \alpha(|(y,u)|)$ for all $(y,u) \in \mathbb{R}^p \times \mathbb{R}^m$ with α_1 a $\mathcal{K}_\infty$ function. Let $\mathbb{X} := \{x \mid \exists u \text{ with } (x,u) \in \mathbb{Z}\}$.

Theorem B.53 (Modified IOSS-Lyapunov function)**.** *Assume that there exists an IOSS-Lyapunov function $V : \mathbb{X} \to \mathbb{R}_{\geq 0}$ for the constrained system $x^+ = f(x,u)$ such that the following holds for all $(x,u) \in \mathbb{Z}$ for which $f(x,u) \in \mathbb{X}$*

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$
$$V(f(x,u)) - V(x) \leq -\alpha_3(|x|) + \sigma(\ell(y,u))$$

with $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$.

For any $\overline{\alpha}_4 \in \mathcal{K}_\infty$, there exists another IOSS-Lyapunov function $\Lambda : \mathbb{X} \to \mathbb{R}_{\geq 0}$ for the constrained system $x^+ = f(x,u)$ such that the following holds for all $(x,u) \in \mathbb{Z}$ for which $f(x,u) \in \mathbb{X}$

$$\overline{\alpha}_1(|x|) \leq \Lambda(x) \leq \overline{\alpha}_2(|x|)$$
$$\Lambda(f(x,u)) - \Lambda(x) \leq -\rho(|x|) + \overline{\alpha}_4(\ell(y,u))$$

with $\overline{\alpha}_1, \overline{\alpha}_2 \in \mathcal{K}_\infty$ and continuous function $\rho \in \mathcal{PD}$. Note that $\Lambda = \gamma \circ V$ for some $\gamma \in \mathcal{K}$.

Conjecture B.54 (IOSS and IOSS-Lyapunov function)**.** *The following properties are equivalent for system* (B.24)*:*

(a) The system is IOSS.

(b) The system admits a smooth IOSS-Lyapunov function.

(c) The system admits an exponential-decay IOSS-Lyapunov function.

As discussed in the Notes section of Chapter 2, Grimm, Messina, Tuna, and Teel (2005) use a storage function like $\Lambda(\cdot)$ in Theorem B.53 to treat a semidefinite stage cost. Cai and Teel (2008) provide a discrete time converse theorem for IOSS that holds for all $\mathbb{R}^n$. Allan and Rawlings (2017) provide the converse theorem on closed positive invariant sets, and also provide a lemma for changing the supply rate function.

B.9 Incremental-Input/Output-to-State Stability

Definition B.55 (Incremental input/output-to-state stable)**.** The system (B.24) is incrementally input/output-to-state stable (i-IOSS) if there exists some $\beta(\cdot) \in \mathcal{KL}$ and $\gamma_1(\cdot)$, $\gamma_2(\cdot) \in \mathcal{K}$ such that, for every two initial states z_1 and z_2 and any two control sequences $\mathbf{u}_1 = (u_1(j))$ and $\mathbf{u}_2 = (u_2(j))$

$$\begin{aligned}&|x(k;z_1,\mathbf{u}_1) - x(k;z_2,\mathbf{u}_2)| \leq \\ &\max\left\{\beta(|z_1 - z_2|, k),\ \gamma_1(\|\mathbf{u}_1 - \mathbf{u}_2\|_{0:k-1}),\ \gamma_2(\left\|\mathbf{y}_{z_1,\mathbf{u}_1} - \mathbf{y}_{z_2,\mathbf{u}_2}\right\|_{0:k})\right\}\end{aligned}$$

B.10 Observability

Definition B.56 (Observability)**.** The system (B.24) is (uniformly) observable if there exists a positive integer N and an $\alpha(\cdot) \in \mathcal{K}$ such that

$$\sum_{j=0}^{k-1} \left|h(x(j;x,u)) - h(x(j;z,u))\right| \geq \alpha(|x - z|) \tag{B.25}$$

for all x, z, all $k \geq N$ and all control sequences u; here $x(j;z,u) = \phi(j;z,u)$, the solution of (B.24) when the initial state is z at time 0 and the control sequence is u.

When the system is linear, i.e., $f(x,u) = Ax + Bu$ and $h(x) = Cx$, this assumption is equivalent to assuming the observability Gramian $\sum_{j=0}^{n-1} CA^j(A^j)'C'$ is positive definite. Consider the system described by

$$z^+ = f(z,u) + w \qquad y + v = h(z) \tag{B.26}$$

with output $y_w = y + v$. Let $z(k; z, u, w)$ denote the solution, at time k of (B.26) if the state at time 0 is z, the control sequence is u and the disturbance sequence is w. We assume, in the sequel, that

Assumption B.57 (Lipschitz continuity of model)**.**

(a) The function $f(\cdot)$ is globally Lipschitz continuous in $\mathbb{R}^n \times \mathbf{U}$ with Lipschitz constant c.

(b) The function $h(\cdot)$ is globally Lipschitz continuous in $\mathbb{R}^n$ with Lipschitz constant c.

Lemma B.58 (Lipschitz continuity and state difference bound)**.** *Suppose Assumption B.57 is satisfied (with Lipschitz constant c). Then,*

$$|x(k; x, u) - z(k; z, u, w)| \leq c^k |x - z| + \sum_{i=0}^{k-1} c^{k-i-1} |w(i)|$$

Proof. Let $\delta(k) := |x(k; x, u) - z(k; z, u, w)|$. Then

$$\begin{aligned}\delta(k+1) &= \left| f(x(k; x, u), u(k)) - f(z(k; z, u, w), u(k)) - w(k) \right| \\ &\leq c\, |\delta(k)| + |w(k)|\end{aligned}$$

Iterating this equation yields the desired result. ∎

Theorem B.59 (Observability and convergence of state)**.** *Suppose* (B.24) *is (uniformly) observable and that Assumption B.57 is satisfied. Then, $w(k) \to 0$ and $v(k) \to 0$ as $k \to \infty$ imply $|x(k; x, u) - z(k; z, u, w)| \to 0$ as $k \to \infty$.*

Proof. Let $x(k)$ and $z(k)$ denote $x(k; x, u)$ and $z(k; z, u, w)$, respectively, in the sequel. Since (B.24) is observable, there exists an integer N satisfying (B.25). Consider the sum

$$\begin{aligned}S(k) = \sum_{j=k}^{k+N} v(k) &= \sum_{j=k}^{k+N} \left| h(x(j; x, u)) - h(z(j; z, u, w)) \right| \\ &\geq \sum_{j=k}^{k+N} \left| h(x(j; x(k), u)) - h(x(j; z(k), u)) \right| \\ &\quad - \sum_{j=k}^{k+N} \left| h(x(j; z(k), u)) - h(z(j; z(k), u, w)) \right| \end{aligned} \tag{B.27}$$

where we have used the fact that $|a+b| \geq |a| - |b|$. By the assumption of observability

$$\sum_{j=k}^{k+N} \left| h(x(j; x(k), u)) - h(x(j; z(k), u)) \right| \geq \alpha(|x(k) - z(k)|)$$

for all k. From Lemma B.58 and the Lipschitz assumption on $h(\cdot)$

$$\left| h(x(j; z(k), u)) - h(z(j; z(k), u, w)) \right| \leq \\ c \left| x(j; z(k), u) - z(j; z(k), u, w) \right| \leq c \sum_{i=k}^{j-1} c^{j-1-i} |w(i)|$$

for all j in $\{k+1, k+2, \ldots k+N\}$. Hence there exists a $d \in (0, \infty)$ such that the last term in (B.27) satisfies

$$\sum_{j=k}^{k+N} \left| h(x(j; x(k), u)) - h(x(j; z(k), u)) \right| \leq d \, \|w\|_{k-N:k}$$

Hence, (B.27) becomes

$$\alpha(|x(k) - z(k)|) \leq N \, \|v\|_{k-N:k} + d \, \|w\|_{k-N:k}$$

Since, by assumption, $w(k) \to 0$ and $v(k) \to 0$ as $k \to \infty$, and $\alpha(\cdot) \in \mathcal{K}$, it follows that $|x(k) - z(k)| \to 0$ as $k \to \infty$. ■

B.11 Exercises

Exercise B.1: Lyapunov equation and linear systems

Establish the equivalence of (a) and (b) in Lemma B.20.

Exercise B.2: Lyapunov function for exponential stability

Let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a Lyapunov function for the system $x^+ = f(x)$ with the following properties. For all $x \in \mathbb{R}^n$

$$a_1 |x|^\sigma \leq V(x) \leq a_2 |x|^\sigma$$
$$V(f(x)) - V(x) \leq -a_3 |x|^\sigma$$

in which $a_1, a_2, a_3, \sigma > 0$. Show that the origin of the system $x^+ = f(x)$ is globally exponentially stable.

Exercise B.3: A converse theorem for exponential stability

(a) Assume that the origin is globally exponentially stable (GES) for the system

$$x^+ = f(x)$$

in which $f(\cdot)$ is continuous. Show that there exists a continuous Lyapunov function $V(\cdot)$ for the system satisfying for all $x \in \mathbb{R}^n$

$$a_1 |x|^\sigma \leq V(x) \leq a_2 |x|^\sigma$$
$$V(f(x)) - V(x) \leq -a_3 |x|^\sigma$$

in which $a_1, a_2, a_3, \sigma > 0$.

Hint: Consider summing the solution $|\phi(i;x)|^\sigma$ on i as a candidate Lyapunov function $V(x)$.

(b) Establish that in the Lyapunov function defined above, any $\sigma > 0$ is valid, and also that the constant a_3 can be chosen as large as one wishes.

Exercise B.4: Proof of Theorem B.13

Revisit the argument establishing attractivity in Theorem B.13

Exercise B.5: Revisit Lemma 1.3 in Chapter 1

Establish Lemma 1.3 in Chapter 1 using the Lyapunov function tools established in this appendix. Strengthen the conclusion and establish that the closed-loop system is globally exponentially stable.

Exercise B.6: Continuity of Lyapunov function for asymptotic stability

Let X be a compact subset of $\mathbb{R}^n$ containing the origin in its interior that is positive invariant for the system $x^+ = f(x)$. If $f(\cdot)$ is continuous on X and the origin is asymptotically stable with a region of attraction X, show that the Lyapunov function suggested in Theorem B.17 is continuous on X.

Exercise B.7: A Lipschitz continuous converse theorem for exponential stability

Consider the system $x^+ = f(x)$, $f(0) = 0$, with function $f : D \to \mathbb{R}^n$ Lipschitz continuous on compact set $D \subset \mathbb{R}^n$ containing the origin in its interior. Choose $R > 0$ such that $B_R \subseteq D$. Assume that there exist scalars $c > 0$ and $\lambda \in (0,1)$ such that

$$|\phi(k;x)| \leq c\,|x|\,\lambda^k \qquad \text{for all } |x| \leq r, \quad k \geq 0$$

with $r := R/c$.

Show that there exists a *Lipschitz continuous* Lyapunov function $V(\cdot)$ satisfying for all $x \in B_r$

$$a_1 |x|^2 \leq V(x) \leq a_2 |x|^2$$
$$V(f(x)) - V(x) \leq -a_3 |x|^2$$

with $a_1, a_2, a_3 > 0$.

Hint: Use the proposed Lyapunov function of Exercise B.3 with $\sigma = 2$. See also (Khalil, 2002, Exercise 4.68).

Exercise B.8: Lyapunov function requirements: continuity of α_3

Consider the following scalar system $x^+ = f(x)$ with piecewise affine and discontinuous $f(\cdot)$ (Lazar et al., 2009)

$$f(x) = \begin{cases} 0, & x \in (-\infty, 1] \\ (1/2)(x+1), & x \in (1, \infty) \end{cases}$$

Note that the origin is a steady state

(a) Consider $V(x) = |x|$ as a candidate Lyapunov function. Show that this V satisfies (B.8)–(B.10) of Definition B.12, in which $\alpha_3(x)$ is positive definite but *not* continuous.

(b) Show by direction calculation that the origin is not globally asymptotically stable. Show that for initial conditions $x_0 \in (1, \infty)$, $x(k; x_0) \to 1$ as $k \to \infty$.

The conclusion here is that one cannot leave out continuity of α_3 in the definition of a Lyapunov function when allowing discontinuous system dynamics.

Exercise B.9: Difference between classical and KL stability definitions

Consider the *discontinuous* nonlinear scalar example (Teel and Zaccarian, 2006) $x^+ = f(x)$ with

$$f(x) = \begin{cases} \dfrac{1}{2}x & |x| \in [0, 1] \\ \dfrac{2x}{2 - |x|} & |x| \in (1, 2) \\ 0 & |x| \in [2, \infty) \end{cases}$$

Is this systems GAS under the classical definition? Is this system GAS under the KL definition? Discuss why or why not.

Exercise B.10: Combining $\mathcal{K}$ functions

Establish (B.5) starting from (B.3) and (B.4) and then using (B.1).

Bibliography

D. A. Allan and J. B. Rawlings. Input/output-to-state stability and nonlinear MPC with detectable stage costs. Technical Report 2017–01, TWCCC Technical Report, October 2017.

D. A. Allan, C. N. Bates, M. J. Risbeck, and J. B. Rawlings. On the inherent robustness of optimal and suboptimal nonlinear MPC. *Sys. Cont. Let.*, 106: 68–78, August 2017.

R. G. Bartle and D. R. Sherbert. *Introduction to Real Analysis.* John Wiley & Sons, Inc., New York, third edition, 2000.

C. Cai and A. R. Teel. Input–output-to-state stability for discrete-time systems. *Automatica*, 44(2):326 – 336, 2008.

G. Grimm, M. J. Messina, S. E. Tuna, and A. R. Teel. Model predictive control: For want of a local control Lyapunov function, all is not lost. *IEEE Trans. Auto. Cont.*, 50(5):546–558, 2005.

Z.-P. Jiang and Y. Wang. Input-to-state stability for discrete-time nonlinear systems. *Automatica*, 37:857–869, 2001.

Z.-P. Jiang and Y. Wang. A converse Lyapunov theorem for discrete-time systems with disturbances. *Sys. Cont. Let.*, 45:49–58, 2002.

R. E. Kalman and J. E. Bertram. Control system analysis and design via the "Second method" of Lyapunov, Part II: Discrete–time systems. *ASME J. Basic Engr.*, pages 394–400, June 1960.

C. M. Kellet and A. R. Teel. Discrete-time asymptotic controllability implies smooth control-Lyapunov function. *Sys. Cont. Let.*, 52:349–359, 2004a.

C. M. Kellet and A. R. Teel. Smooth Lyapunov functions and robustness of stability for difference inclusions. *Sys. Cont. Let.*, 52:395–405, 2004b.

H. K. Khalil. *Nonlinear Systems.* Prentice-Hall, Upper Saddle River, NJ, third edition, 2002.

M. Krichman, E. D. Sontag, and Y. Wang. Input-output-to-state stability. *SIAM J. Cont. Opt.*, 39(6):1874–1928, 2001.

J. P. LaSalle. *The stability and control of discrete processes*, volume 62 of *Applied Mathematical Sciences.* Springer-Verlag, 1986.