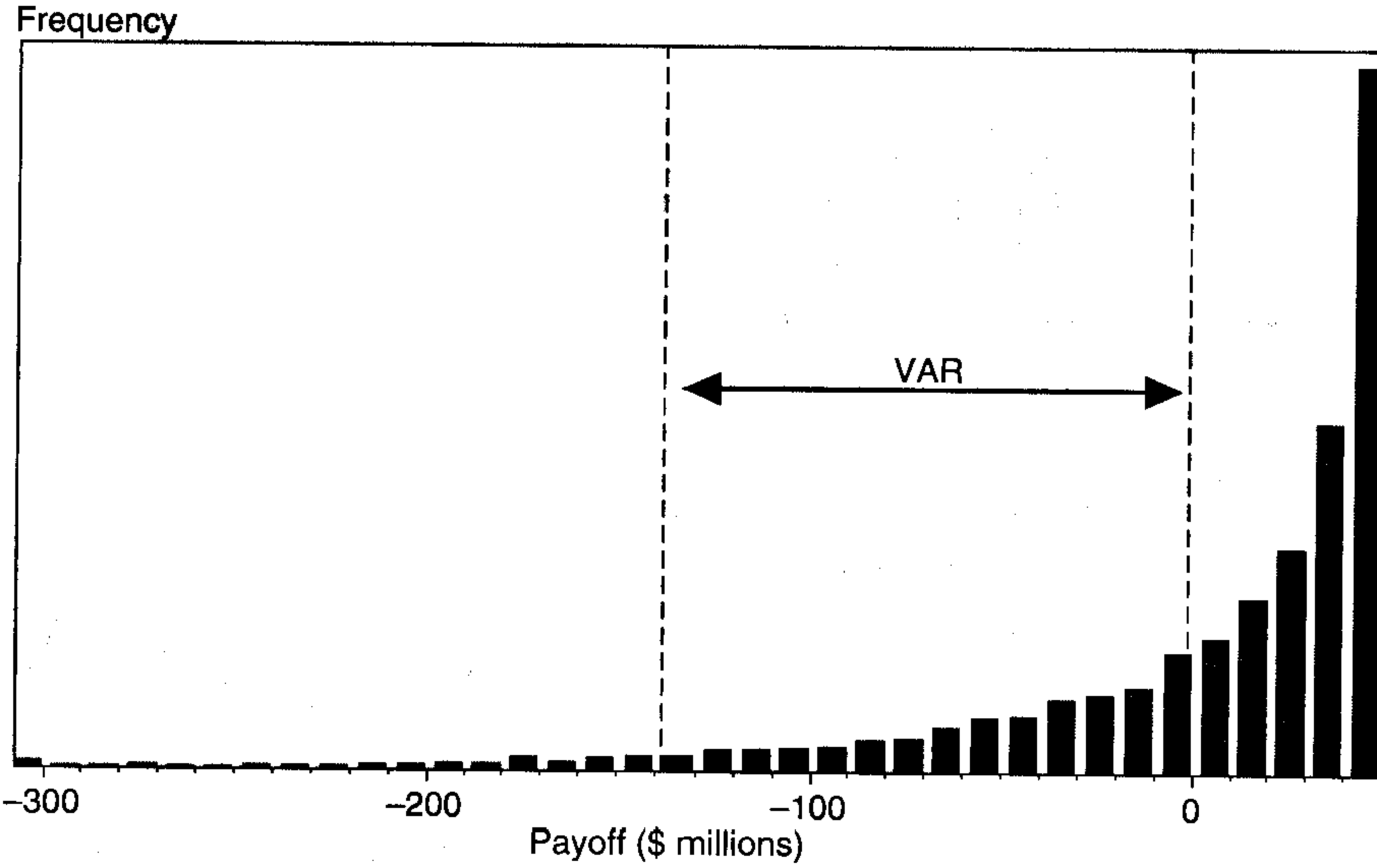
FIGURE 10-7

Distribution of 1-month payoff for straddle.



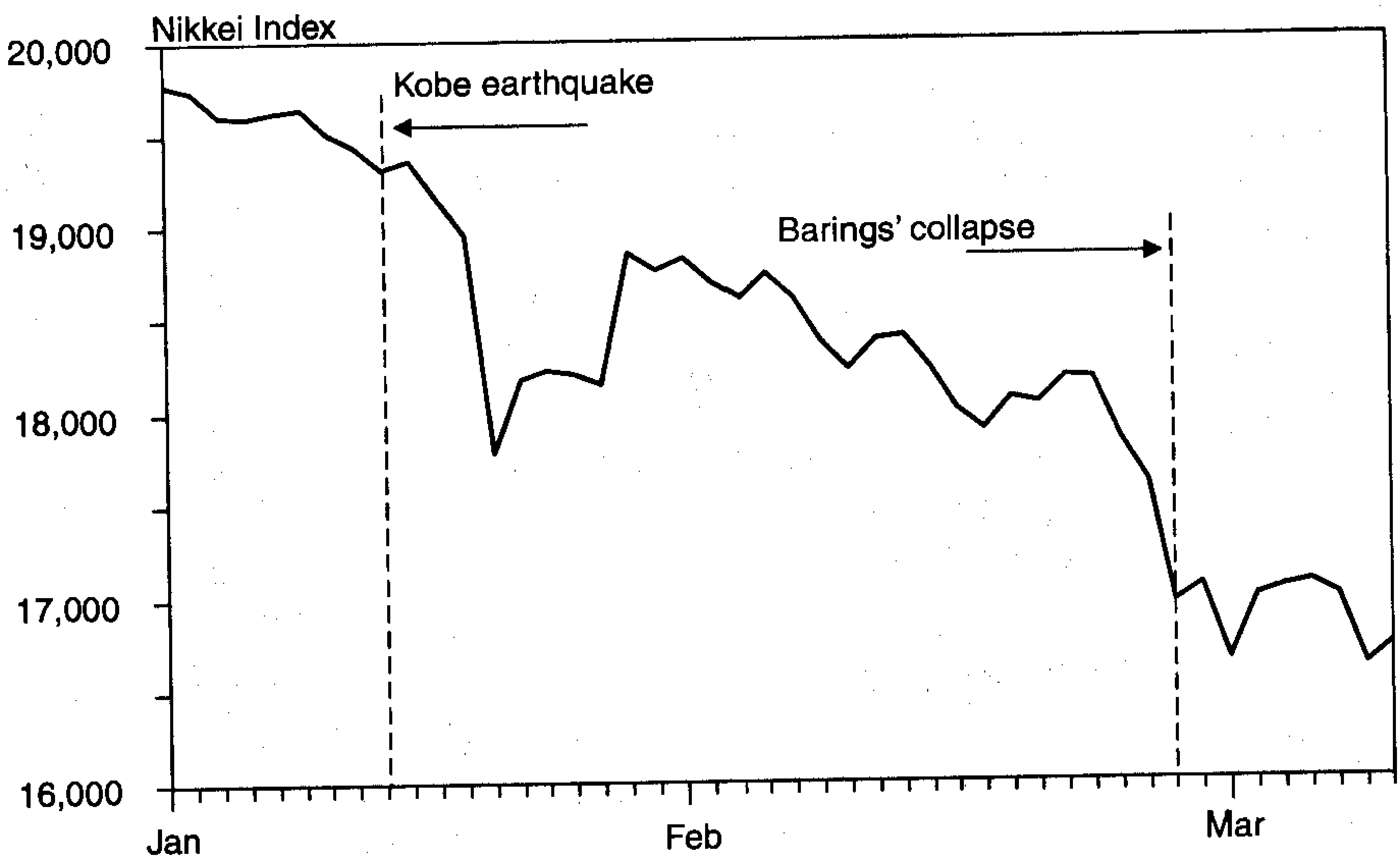
The risks involved are described in Figure 10-7, which plots the frequency distribution of payoffs on the straddle using a *full Monte Carlo simulation* with 10,000 replications. This distribution is obtained from a revaluation of the portfolio after a month over a range of values for the Nikkei. Each replication uses full valuation with a remaining maturity of 2 months (the 3-month original maturity minus the 1-month VAR horizon). The distribution looks highly skewed to the left. Its mean is −$1 million, and the 95th percentile is −$139 million. Hence the 1-month 95 percent VAR is $138 million.

Next, we can use the delta-gamma-Monte-Carlo approach, which consists of using the simulations of S but valuing the portfolio on the target date using only the partial derivatives. This yields a VAR of $128 million, not too far from the true value.

And indeed, the option position contributed to Barings' fall. As January 1995 began, the historical volatility on the Japanese market was very low, around 10 percent. At the time, the Nikkei was hovering around 19,000. The option position would have been profitable if the market had been stable. Unfortunately, this was not so. The Kobe earthquake struck Japan on January 17 and led to a drop in the Nikkei to 18,000, shown in Figure 10-8. To make

FIGURE 10-8

The Nikkei's fall.



things worse, options became more expensive as market volatility increased. Both the long futures and the straddle positions lost money. As losses ballooned, Leeson increased his exposure in a desperate attempt to recoup the losses, but to no avail. On February 27, the Nikkei dropped further to 17,000. Unable to meet the mounting margin calls, Barings went bust.

10.3 DELTA-NORMAL METHOD

10.3.1 Implementation

When the risk factors are jointly normally distributed and the positions can be represented by their delta exposures, the measurement of VAR is considerably simplified. We have N risk factors. Define $x_{i,t}$ as the exposures aggregated across all instruments for each risk factor i and measured in currency units. Equivalently, we could divide these by the current portfolio value W to obtain the portfolio weights $w_{i,t}$.

The portfolio *rate of return* is

$$R_{p,t+1} = \sum_{i=1}^{N} w_{i,t} R_{i,t+1} \qquad (10.12)$$

where the weights $w_{i,t}$ are indexed by time to indicate that this is the current portfolio. This method allows easy aggregation of risks for large portfolios because of the invariance property of normal variables: Portfolios of jointly normal variables are themselves normally distributed. The portfolio normality assumption is also justified by the *central limit theorem*, which states that the average of independent random variables converges to a normal distribution. For portfolios diversified across a number of risk factors that have modest correlations, these conditions could be approximately met.

Using matrix notations, as in Chapter 7, the portfolio variance is given by

$$\sigma^2(R_{p,t+1}) = w_t' \Sigma_{t+1} w_t \tag{10.13}$$

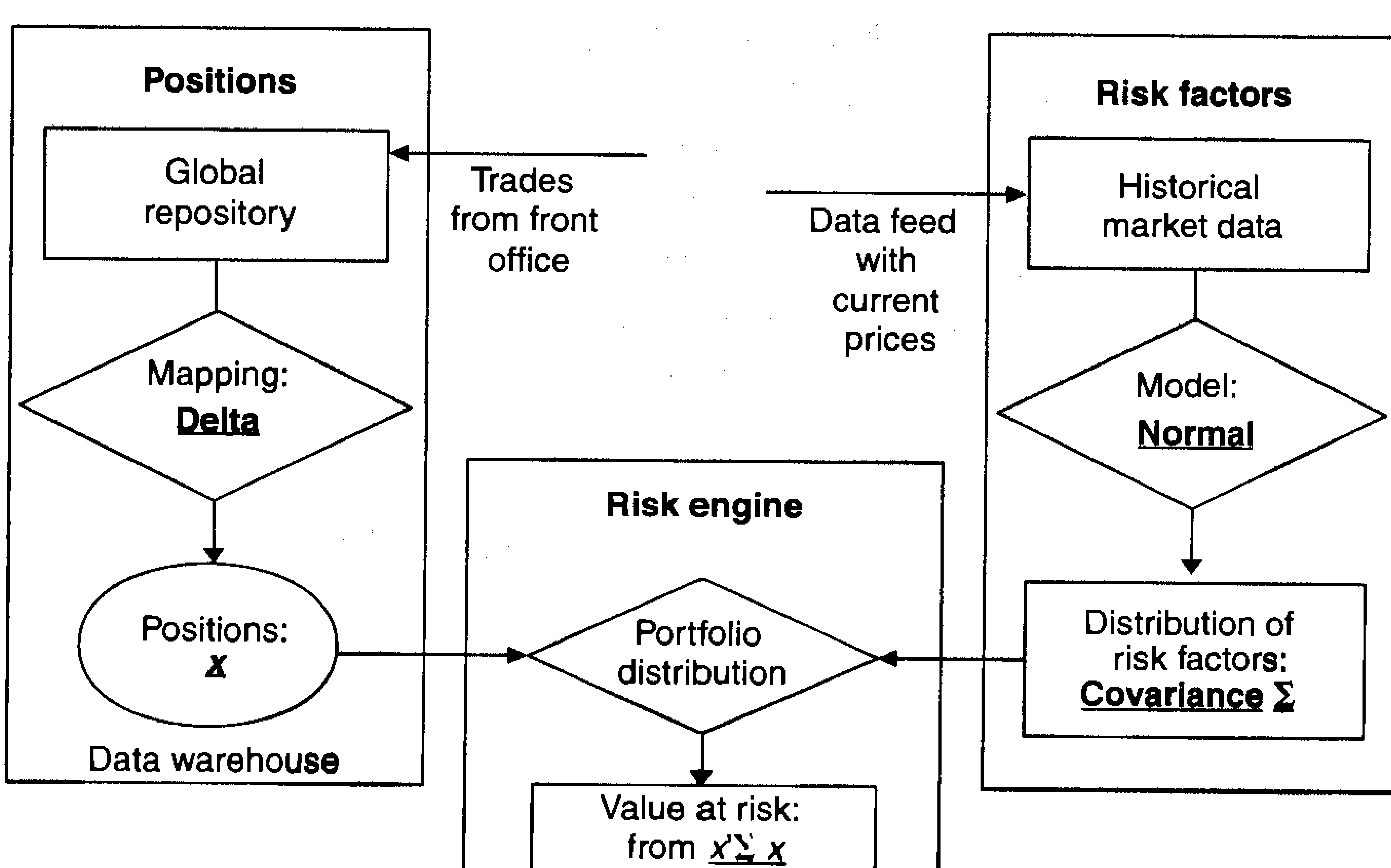
where Σ_{t+1} is the forecast of the covariance matrix over the VAR horizon, perhaps using models developed in Chapter 9. The portfolio VAR then is

$$\mathrm{VAR} = \alpha\sqrt{x_t' \Sigma_{t+1} x_t} = \alpha W \sqrt{w_t' \Sigma_{t+1} w_t} \tag{10.14}$$

where α is the deviate corresponding to the confidence level for the normal distribution or for another parametric distribution. Figure 10-9 details the steps involved in this approach.

FIGURE 10-9

Delta-normal method.



10.3.2 Advantages

The delta-normal method is particularly *easy* to implement because it involves a simple matrix multiplication. It is also *computationally fast*, even with a very large number of assets, because it replaces each position by its linear exposure. As a result, it can be run in *real time*, or during the day as positions change.

As a parametric approach, VAR is easily *amenable to analysis* because measures of marginal and incremental risk are a by-product of the VAR computation. This is useful to manage the portfolio risk.

10.3.3 Drawbacks

The delta-normal method has a number of drawbacks, however. A first problem is the existence of *fat tails* in the distribution of returns on most financial assets. These fat tails are particularly worrisome precisely because VAR attempts to capture the behavior of the portfolio return in the left tail. In this situation, a model based on a normal distribution would underestimate the proportion of outliers and hence the true VAR. A simple ad hoc adjustment consists of increasing the parameter α to compensate.

This problem depends on the choice of the confidence level. Typically, there is not much bias from using a normal distribution at the 95 percent confidence level. The underestimation increases, however, for higher confidence levels.

Another problem is that the method is inadequate for *nonlinear instruments*, such as options and mortgages. As we have seen in the preceding section, asymmetries in the distribution of options are not captured by the delta-normal VAR.

For simple portfolios, however, the delta-normal method may be adequate. At the highest level of financial institutions, asymmetries tend to wash away, as predicted by the central limit theorem. For more complex portfolios, however, the delta-normal method generally is not sufficient.

10.4 HISTORICAL SIMULATION METHOD

10.4.1 Implementation

The historical simulation (HS) approach is a nonparametric method that makes no specific assumption about the distribution of risk factors. It consists of going back in time and replaying the tape of history on the current positions. Positions can be priced using full or local valuation.

In the most simple case, this method applies current weights to a time series of historical asset returns, that is,

$$R_{p,k} = \sum_{i=1}^{N} w_{i,t} R_{i,k} \quad k = 1, \ldots, t \tag{10.15}$$

Note that the weights w_t are kept at their current values. This return does not represent an actual portfolio but rather reconstructs the history of a hypothetical portfolio using the current position. The approach is sometimes called *bootstrapping* because it uses the actual distribution of recent historical data without replacement. Each scenario k is drawn from the history of t observations.

More generally, the method can use *full valuation*, employing hypothetical values for the risk factors, which are obtained from applying historical changes in prices to the current level of prices, that is,

$$S^*_{i,k} = S_{i,0} + \Delta S_{i,k} \quad i = 1, \ldots, N \tag{10.16}$$

A new portfolio value $V^*_{p,k}$ then is computed from the full set of hypothetical prices, perhaps incorporating nonlinear relationships $V^*_k = V(S^*_{i,k})$. Note that to capture *vega risk*, owing to changing volatilities, the set of risk factors can incorporate implied volatility measures. This creates the hypothetical return corresponding to simulation k, that is,

$$R_{p,k} = \frac{V^*_k - V_0}{V_0} \tag{10.17}$$

VAR then is obtained from the entire distribution of hypothetical returns, where each historical scenario is assigned the same weight of $(1/t)$. Figure 10-10 details the process. Because the approach does not assume a parametric distribution for the risk factors, it is called *nonparametric*.

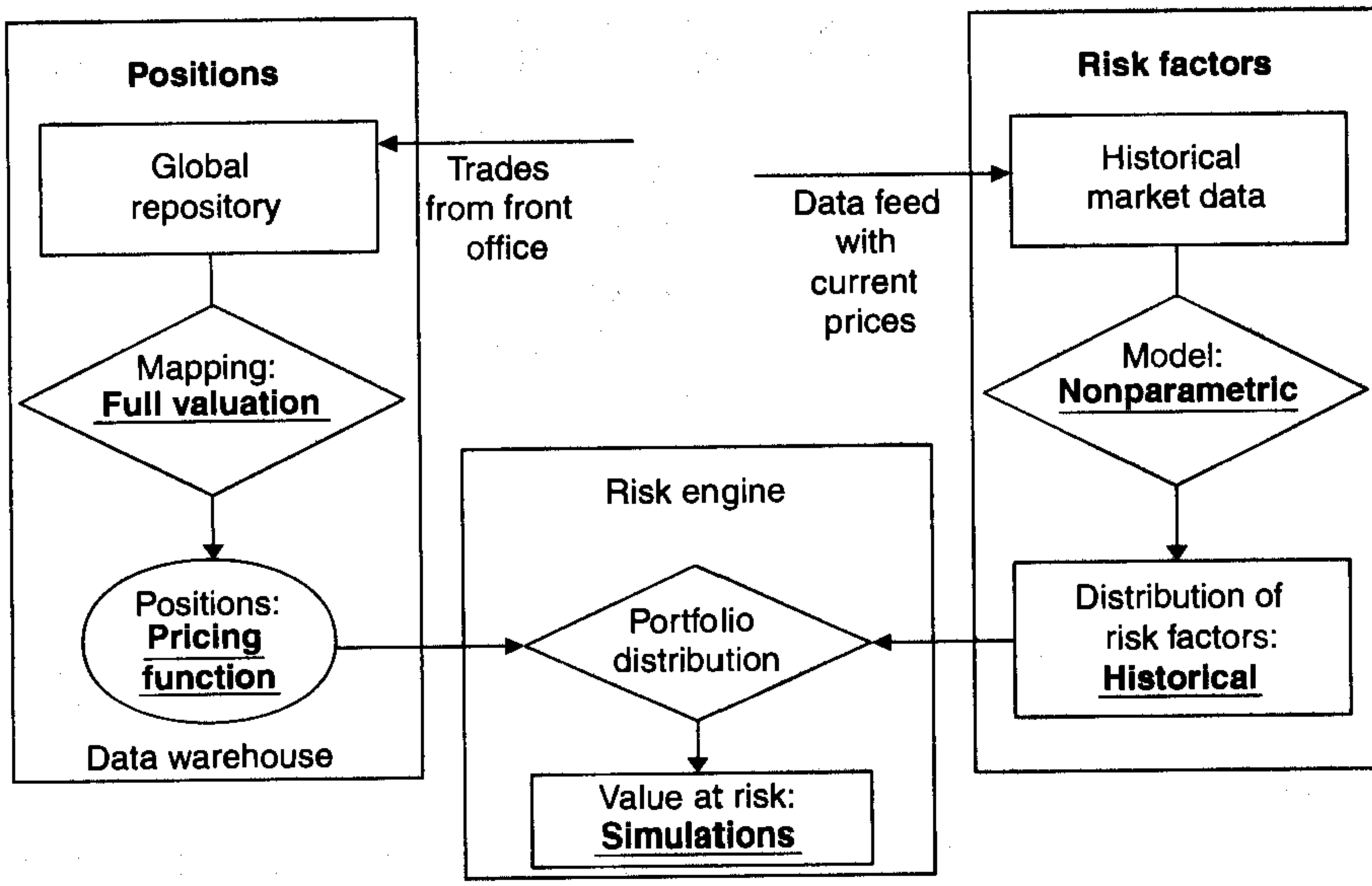
10.4.2 Advantages

This method is relatively *simple to implement* if historical data on risk factors have been collected in-house for daily marking to market. The same data can be stored for later reuse in estimating VAR.

Historical simulation also short circuits the need to estimate a covariance matrix. This simplifies the computations in cases of portfolios with a large number of assets and short sample periods. All that is needed is the time series of the aggregate portfolio value.

FIGURE 10-10

Historical simulation method.



Perhaps most important, historical simulation accounts for *fat tails* to the extent that they are present in the historical data. The method does not require distributional assumptions and therefore is *robust*. Historical simulation can be implemented using *full valuation*. Thus the method can capture gamma and vega risk.

The method also deals directly with the *choice of horizon* for measuring VAR. Returns simply are measured over intervals that correspond to the length of the horizon. For instance, to obtain a monthly VAR, the user would reconstruct historical monthly portfolio returns over, say, the last 5 years.

Historical simulation is also *intuitive*. VAR corresponds to a large loss sustained over a recent period. Hence users can go back in time and explain the circumstances behind the VAR measure.

10.4.3 Drawbacks

On the other hand, the historical simulation method has a number of drawbacks. Only *one sample path* is used. The assumption is that the past represents the immediate future fairly. If the window omits important events, the tails will not be well represented. Vice versa, the sample may contain events that will not reappear in the future.

Next, the *sampling variation* of the historical simulation VAR is greater than for a parametric method. As was pointed out in Chapter 5, there is substantial estimation error in the sample quantile, especially with short sample sizes and high confidence levels. For instance, a 99 percent daily VAR estimated over a window of 100 days only produces one observation in the tail on average, which necessarily leads to an imprecise VAR measure. Thus long sample paths are required to obtain meaningful quantiles. The dilemma is that this may involve observations that are no longer relevant. In practice, most banks use periods between 250 and 750 days, which is taken as a reasonable tradeoff between precision and nonstationarity.

Finally, the method assumes that the distribution is stationary over the selected window. In practice, there may be significant and predictable time variation in risk. This can be taken into account with the following steps. First, we fit a time-series model for the volatility of the series R_t; assume that the volatility forecast is σ_t for each day. The residual then is measured as $\epsilon_t = R_t/\sigma_t$. Second, we bootstrap the scaled residuals from the selected window. Third, we apply these residuals to tomorrow's volatility forecast σ_{t+1}. This is essentially a historical simulation on the ϵ's, which then are multiplied by the current volatility forecast. This method is called *filtered simulation*.[3]

10.5 MONTE CARLO SIMULATION METHOD

10.5.1 Implementation

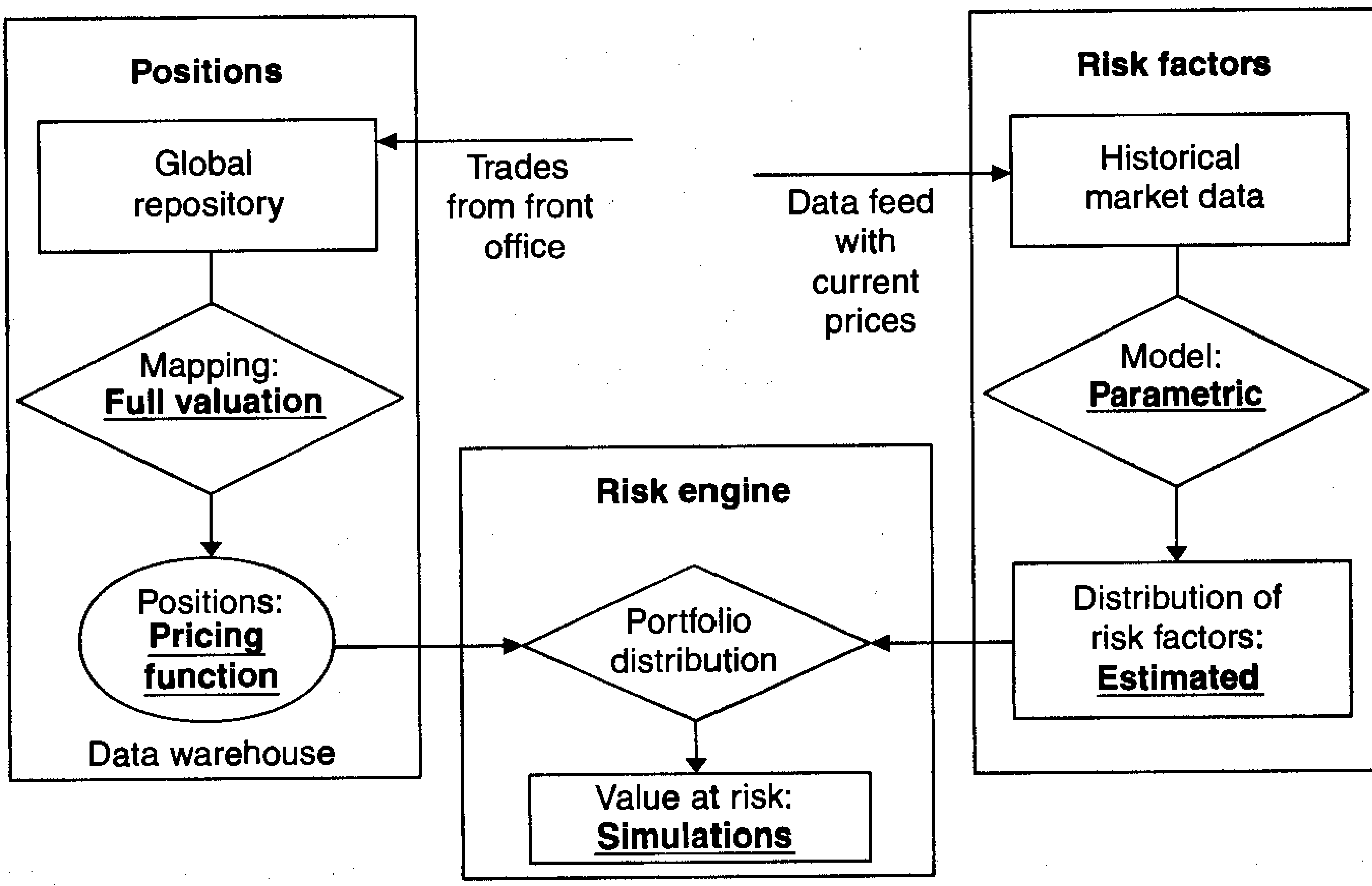
The Monte Carlo (MC) simulation approach is a parametric method that generates random movements in risk factors from estimated parametric distributions. Positions can be priced using full valuation.

The methodology behind MC simulation will be developed in more detail in Chapter 12. In brief, the method proceeds in two steps. First, the risk manager specifies a parametric stochastic process for all risk factors.

[3] For applications, see Hull and White (1998). Another issue is that the method puts the same weight on all observations in the window, including old data points. To alleviate this problem, Boudoukh et al. (1998) propose a scheme whereby each observation R_k is assigned a weight w_k that declines as it ages. The distribution then is obtained from ranking the R_k observations and cumulating the associated weights to find the selected confidence level.

FIGURE 10-11

Monte-Carlo method.



Parameters such as risk and correlations can be derived from historical or options data. Second, fictitious price paths are simulated for all the risk factors. At each horizon considered, the portfolio is marked to market using full valuation as in the historical simulation method, that is, $V_k^* = V(S_{i,k}^*)$. Each of these "pseudo" realizations then is used to compile a distribution of returns, from which a VAR figure can be measured. The method is summarized in Figure 10-11.

The Monte Carlo method thus is similar to the historical simulation method, except that the hypothetical changes in prices ΔS_i for asset i in Equation (10.16) are created by random draws from a prespecified stochastic process instead of sampled from historical data.

10.5.2 Advantages

Monte Carlo analysis is by far the most *powerful method* to compute VAR. For the risk factors, it is flexible enough to incorporate time variation in volatility or in expected returns, *fat tails*, and extreme scenarios. For the instruments in the portfolios, it can account for *nonlinear price exposure*, vega risk, and complex pricing models.

MC simulation can incorporate the *passage of time*, which will create structural changes in the portfolio. This includes the time decay of options; the daily settlement of fixed, floating, or contractually specified cash flows; and the effect of prespecified trading or hedging strategies. These effects are especially important as the time horizon lengthens, which is the case for the measurement of credit risk.

10.5.3 Drawbacks

The biggest drawback to this method is its *computational time*. If 1000 sample paths are generated with a portfolio of 1000 assets, the total number of valuations amounts to 1 million. In addition, if the valuation of assets on the target date involves a simulation, the method requires a "simulation within a simulation." This quickly becomes too onerous to implement on a frequent basis.

This method is the most *expensive to implement* in terms of systems infrastructure and especially intellectual development. MC simulation needs powerful computer systems. It also requires substantial investment in human capital if developed from scratch. Perhaps, then, it should be purchased from outside vendors. On the other hand, when the institution already has in place a system to model complex structures using simulations, implementing MC simulation is less costly because the required expertise is in place. Also, these are situations where proper risk management of complex positions is absolutely necessary.

Another potential weakness of the method is *model risk*. MC relies on specific stochastic processes for the underlying risk factors, which could be wrong. To check if the results are robust to changes in the model, simulation results should be complemented by some sensitivity analysis. Otherwise, the approach is like a black box that provides no intuition for the results.

Finally, VAR estimates from MC simulation are subject to *sampling variation*, which is due to the limited number of replications. Consider, for instance, a case where the risk factors are jointly normal and all payoffs linear. The delta-normal method then will provide the correct measure of VAR in one easy step. MC simulations based on the same covariance matrix will only give an approximation, albeit increasingly good as the number of replications increases.

Overall, this method probably is the most comprehensive approach to measuring market risk if the modeling is done correctly. This is the only method that can handle credit risks. A full chapter will be devoted to the implementation of Monte Carlo simulation methods (see Chapter 12).

10.6 EMPIRICAL COMPARISONS

It is instructive to compare the VAR numbers obtained from these three methods. Hendricks (1996), for instance, calculated 1-day VARs for randomly selected foreign-currency portfolios using a delta-normal method based on fixed windows of equal weights and exponential weights as well as a historical simulation method.

Table 10-2 summarizes the results, which are compared in terms of percentage of outcomes falling within the VAR forecast. This is also one minus the fraction of exceptions. At the 95 percent confidence level, all methods give a coverage that is very close to the ideal number. At the 99 percent confidence level, however, the delta-normal methods seem to underestimate VAR slightly. The historical-simulation method with windows of 1 year or more seem well calibrated.

TABLE 10-2

Empirical Comparison of VAR Methods: Fraction of Outcomes Covered

Method	95% VAR	99% VAR
Delta-normal		
Equal weights over		
50 days	95.1%	98.4%
250 days	95.3%	98.4%
1250 days	95.4%	98.5%
Delta-normal		
Exponential weights		
$\lambda = 0.94$	94.7%	98.2%
$\lambda = 0.97$	95.0%	98.4%
$\lambda = 0.99$	95.4%	98.5%
Historical simulation		
Equal weights over		
125 days	94.4%	98.3%
250 days	94.9%	98.8%
1250 days	95.1%	99.0%

Hendricks also indicates that the delta-normal VAR measures should be increased by about 9 to 15 percent to achieve correct coverage. In other words, the fat tails in the data could be modeled by choosing a distribution with a greater α parameter. A student t distribution with 4 to 6 degrees of freedom, for example, would be appropriate.

This empirical analysis, however, examined positions with linear risk profiles. The delta-normal methods could prove less accurate with options positions, although it should be much faster. Pritsker (1997) examines the tradeoff between speed and accuracy for a portfolio of options.

Table 10-3 reports the accuracy of various methods, measured as the mean absolute percentage error in VAR, as well as their computational times. The table shows that the delta method, as expected, has the highest average absolute error, at 5.34 percent of the true VAR. It is also by far the fastest method, with an execution time of 0.08 second. At the other end, the most accurate method is the full Monte Carlo, which comes arbitrarily close to the true VAR, but with an average run time of 66 seconds. In between, the delta-gamma-delta, delta-gamma-Monte-Carlo, and grid Monte Carlo methods offer a tradeoff between accuracy and speed.

An interesting but still unresolved issue is, how would these approximation work in the context of large, diversified bank portfolios? There is very little evidence on this point. The industry initially seemed to prefer the analytical covariance approach owing to its simplicity. With the rapidly decreasing cost of computing power, however, there is now a marked trend toward the generalized use of historical simulation methods.

TABLE 10-3

Accuracy and Speed VAR Methods: 99 Percent VAR for Option Portfolios

Method	Accuracy, Mean Absolute Error in VAR, %	Speed, Computation Time, seconds
Delta	5.34	0.08
Delta-gamma-delta	4.72	1.17
Delta-gamma-MC	3.08	3.88
Grid Monte Carlo	3.07	32.19
Full Monte Carlo	0	66.27

10.7 SUMMARY

A number of different methods are available to measure VAR. At the most fundamental level, they separate into local valuation and full valuation. This separation reflects a tradeoff between speed of computation and accuracy of valuation.

Among local-valuation models, delta-normal models use a combination of the delta or linear exposures with the covariance matrix. Among full-valuation models, historical simulation is the easiest to implement. It uses the recent history of the risk factors to generate hypothetical scenarios, to which full valuation is applied. Finally, the most complete model but also the most difficult to implement is the Monte Carlo simulation approach. This imposes a particular stochastic process for the risk factors, from which various sample paths are simulated. Full valuation for each sample path generates a distribution of portfolio values.

Table 10-4 describes the pros and cons of each method. The choice of the method largely depends on the composition of the portfolio. For

TABLE 10-4

Comparison of Approaches to VAR

Features	Delta-Normal Simulation	Historical Simulation	Monte Carlo Simulation
Positions			
Valuation	Linear	Full	Full
Distribution			
Shape	Normal	Actual	General
Time varying	Yes	Possible	Yes
Implied data	Possible	No	Possible
Extreme events	Low probability	In recent data	Possible
Use correlations	Yes	Yes	Yes
VAR precision	Excellent	Poor with short window	Good with many iterations
Implementation			
Ease of computation	Yes	Yes	No
Pricing accuracy	Depends on portfolio	Yes	Yes
Communicability	Easy	Easy	Difficult
VAR analysis	Easy	More difficult	More difficult
Major pitfalls	Nonlinearities, fat tails	Time variation in risk, unusual events	Model risk

portfolios with no options and whose distributions are close to the normal, the delta-normal method may well be the best choice. VAR will be relatively easy to compute, fast, and accurate. In addition, it is not too prone to model risk owing to faulty assumptions or computations. The resulting VAR is easy to explain to management and to the public. Because the method is analytical, it provides tools for decomposing VAR into marginal and component measures. For portfolios with options positions, however, the method may not be appropriate. Instead, users should turn to a full-valuation method.

The second method, historical simulation, is also relatively easy to implement and uses full valuation of all securities. However, the method relies on a narrow window only and creates substantial imprecision in VAR numbers.

In theory, the Monte Carlo approach can alleviate all these difficulties. It can incorporate nonlinear positions, nonnormal distributions, and even user-defined scenarios. The price to pay for this flexibility, however, is heavy. Computer and data requirements are a quantum step above the other two approaches, model risk looms large, and VAR loses its intuitive appeal. As the price of computing power continues to fall, however, this method is bound to take on increasing importance.

In practice, all these methods are used. Initially, banks used the delta-normal method because of its simplicity. By now, many institutions are using historical simulation over a window of 1 to 4 years, duly supplemented by stress tests to help minimize the possibility of blinds spots in the risk management system.

APPENDIX 10.A

Analytical Second-Order Approximations

This appendix discusses analytical methods to provide approximations to VAR when the value function can be described by the Taylor expansion, that is,

$$dV = \Delta dS + \tfrac{1}{2}\Gamma dS^2 + \cdots \tag{10.18}$$

In a multivariate framework, the Taylor expansion is

$$dV(S) = \Delta' dS + \tfrac{1}{2}(dS)'\Gamma(dS) + \cdots \tag{10.19}$$

where dS is now a vector of N changes in market prices, Δ a vector of N deltas, and Γ an N by N symmetric matrix of gammas with respect to the various risk factors.

Various approaches can be used to derive analytical approximations for the VAR quantile. A simple method is the *delta-gamma-delta approach.* Taking the variance of both sides of the quadratic approximation, we obtain

$$\sigma^2(dV) = \Delta^2\sigma^2(dS) + (\tfrac{1}{2}\Gamma)^2\sigma^2(dS^2) + 2(\Delta\tfrac{1}{2}\Gamma)\mathrm{cov}(dS, dS^2) \tag{10.20}$$

If the variable dS is normally distributed, all its odd moments are zero, and the last term in the equation vanishes. Under the same assumption, one can show that $V(dS^2) = 2V(dS)^2$, and the variance simplifies to

$$\sigma^2(dV) = \Delta^2\sigma^2(dS) + \tfrac{1}{2}[\Gamma\sigma^2(dS)]^2 \tag{10.21}$$

Assume now that the variables dS and dS^2 are jointly normally distributed. Then dV is normally distributed, with VAR given by

$$\mathrm{VAR} = \alpha\sqrt{(\Delta S\sigma)^2 + \tfrac{1}{2}(\Gamma S^2\sigma^2)^2} \tag{10.22}$$

This is, of course, only an approximation. Even if dS were normal, its square dS^2 could not possibly be normally distributed. Rather, it is a chi-squared variable.

A further improvement can be obtained by accounting for the skewness coefficient ξ, as defined in Chapter 4.[4] The corrected VAR, using the *Cornish-Fisher expansion*, then is obtained by replacing α in Equation (10.22) by

$$\alpha' = \alpha - \tfrac{1}{6}(\alpha^2 - 1)\xi$$

There is no correction under a normal distribution, for which skewness is zero. When there is negative skewness (i.e., a long left tail), VAR is increased.

As an application, let us examine the risk of Leeson's short straddle. First, let us examine the delta-gamma-delta approximation. The total gamma of the position is the exposure times the sum of gamma for a call and put, or \$0.175 million $\times$ 0.000422 = \$0.0000739 million. Over a 1-month horizon, the standard deviation of the Nikkei is $\sigma S = 19{,}000 \times 20 \text{ percent } \sqrt{12} = 1089$.

Ignoring the time drift, the VAR is, from Equation (10.22), in millions,

$$\text{VAR} = \alpha\sqrt{\tfrac{1}{2}[\Gamma(\sigma S)^2]^2} = 1.65\sqrt{\tfrac{1}{2}(\$0.0000739 \times 1089^2)^2} = 1.65 \times \$62 = \$102$$

This is substantially better than the delta-normal VAR of zero, which could have fooled us into believing that the position was riskless.

Using the *Cornish-Fisher expansion* and a skewness coefficient of -2.83, we obtain a correction factor of $\alpha' = 1.65 - \tfrac{1}{6}(1.65^2 - 1)\,(-2.83) = 2.45$. The refined VAR measure then is 2.45 $\times$ \$62 = \$152 million, much closer to the true value of \$138 million.

Other methods have been proposed to measure VAR using the quadratic Equation (10.18). For instance, the random variable resulting from the quadratic form can be defined by its *characteristic function*. For any random variable X, this function is

$$\Psi(t) = E(e^{itX}) \tag{10.24}$$

[4] Skewness can be computed analytically as $\xi = [E(dV^3) - 3E(dV^2)E(dV) + 2E(dV)^3]/\sigma^3(dV)$ using the third moment of dV, which is $E(dV^3) = (9/2)\Delta^2\Gamma S^4\sigma^4 + (15/8)\Gamma^3 S^6\sigma^6$.

where $i = \sqrt{-1}$ is the imaginary number. This function can be computed from the combination of normal random variables in the quadratic form and then inverted to give the cumulative distribution function, as in Rouvinez (1997). Another approach uses saddlepoint approximations and is presented by Feuerverger and Wong (2000).

QUESTIONS

1. Discuss the basic tradeoffs between cost, speed, and accuracy when choosing a VAR method. Which method is the fastest? Which has highest accuracy?
2. As a risk manager, you are asked to design a system to measure the risk of a complex interest-rate option book. Which method should you choose?
3. As a risk manager, you are asked to design a system to measure the risk of a portfolio of forward contracts on foreign currencies. The trader insists that he should have intraday VAR measures at the 95 percent confidence level. Which method should you choose?
4. Is the delta-normal valuation method more appropriate for a short horizon or a long horizon?
5. Are local linear valuation methods appropriate if the portfolio contains complex options?
6. Is the following statement true? Explain why or why not. "Computing VAR with a full-valuation method is feasible with two-function valuations when the payoff function is monotonic."
7. A risk manager must measure the risk of a long call option on one share of General Electric. The worst move, up or down, for the stock is \$10.66 at the 95 percent confidence level. The initial value of the call is \$3.89, with a delta of 0.54. The call is revalued at \$11.71 and \$0.38 for the worst up and down stock-price moves. What is the option's VAR?
8. What are the drawbacks of Monte Carlo simulation?
9. A risk manager has implemented a GARCH model that indicates that the current volatility is twice the historical average over the last 4 years used for historical simulation. How can this information be taken into account?
10. You are a hedge-fund manager with large positions in hedged convertible bonds. What method would you choose for VAR?

11. What is the major shortcoming of the historical simulation method?
12. "Because the historical simulation is a nonparametric method, it makes no assumption about the distribution of risk factors." Is this correct?
13. Can we convert the daily VAR value for an options portfolio to a weekly value using the square-root-of-time rule?
14. Assume that we sample from a multivariate normal distribution with fixed covariance matrix. Under what conditions will Monte Carlo VAR approach the delta-normal VAR?
15. How does the grid Monte Carlo method reduce the number of full valuations?
16. A risk manager computes from Monte Carlo simulation a VAR of $15 million using 1000 replications. The manager estimates the standard error of VAR is $3 million, which is too high. What is the standard error of VAR if the number of replications is increased from 1000 to 10,000?
17. Consider the risk of a long call on an asset with a notional amount of $1 million. The VAR of the underlying asset is 8 percent. If the option is a short-term at-the-money option, what is the linear VAR of the option? How would gamma affect this VAR?
18. "Positive gamma decreases VAR." Explain.
19. Suppose that an investor is long a call option on a stock index futures contract. Each option gives the right to purchase one unit of the index, which is priced at S = $400. The delta is 0.569, and the gamma is 0.010. If the volatility of the rate of return on the index is 30 percent, what is the option's VAR over the next 2 weeks at the 95 percent confidence level?
20. What is the most commonly used VAR method?

CHAPTER 11

VAR Mapping

> The second [principle], to divide each of the difficulties under examination into as many parts as possible, and as might be necessary for its adequate solution.
>
> —*René Descartes*

Whichever value-at-risk (VAR) method is used, the risk measurement process needs to simplify the portfolio by *mapping* the positions on the selected risk factors. Mapping is the process by which the current values of the portfolio positions are replaced by exposures on the risk factors.

Mapping arises because of the fundamental nature of VAR, which is portfolio measurement at the highest level. As a result, this is usually a very large-scale aggregation problem. It would be too complex and time-consuming to model all positions individually as risk factors. Furthermore, this is unnecessary because many positions are driven by the same set of risk factors and can be aggregated into a small set of exposures without loss of risk information. Once a portfolio has been mapped on the risk factors, any of the three VAR methods can be used to build the distribution of profits and losses.

This chapter illustrates the mapping process for major financial instruments. Section 11.1 first reviews the basic principles behind mapping for VAR. We then proceed to illustrate cases where instruments are broken down into their constituent components. We will see that the mapping process is instructive because it reveals useful insights into the risk drivers of derivatives. Section 11.2 deals with fixed-income securities, and Section 11.3 with linear derivatives. We cover the most important instruments, forward contracts, forward rate agreements, and interest-rate swaps. Section 11.4 then describes nonlinear derivatives, or options.

11.1 MAPPING FOR RISK MEASUREMENT

11.1.1 Why Mapping?

The essence of VAR is aggregation at the highest level. This generally involves a very large number of positions, including bonds, stocks, currencies, commodities, and their derivatives. As a result, it would be impractical to consider each position separately (see Box 11-1). Too many computations would be required, and the time needed to measure risk would slow to a crawl.

Fortunately, mapping provides a shortcut. Many positions can be simplified to a smaller number of positions on an set of elementary, or *primitive*, risk factors. Consider, for instance, a trader's desk with thousands of open dollar/euro forward contracts. The positions may differ owing to different maturities and delivery prices. It is unnecessary, however, to model all these positions individually. Basically, the positions are exposed to a single major risk factor, which is the dollar/euro spot exchange rate. Thus they could be summarized by a single aggregate exposure on this risk factor. Such aggregation, of course, is not appropriate for the pricing of the portfolio. For risk measurement purposes, however, it is perfectly acceptable. This is why risk management methods can differ from pricing methods.

Mapping is also the only solution when the characteristics of the instrument change over time. The risk profile of bonds, for instance, changes as they age. One cannot use the history of prices on a bond directly. Instead, the bond must be mapped on yields that best represent its current profile. Similarly, the risk profile of options changes very quickly. Options must be mapped on their primary risk factors. Mapping provides a way to tackle these practical problems.

11.1.2 Mapping as a Solution to Data Problems

Mapping is also required in many common situations. Often a complete history of all securities may not exist or may not be relevant. Consider a

BOX 11-1

WHY MAPPING?

"J.P. Morgan Chase's VAR calculation is highly granular, comprising more than 2.1 million positions and 240,000 pricing series (e.g., securities prices, interest rates, foreign exchange rates)." (Annual report, 2004) mutual fund with a strategy of investing in *initial public offerings* (IPOs) of common stock. By definition, these stocks have no history. They certainly cannot be ignored in the risk system, however. The risk manager would have to replace these positions by exposures on similar risk factors already in the system.

Another common problem with global markets is the time at which prices are recorded. Consider, for instance, a portfolio or mutual funds invested in international stocks. As much as 15 hours can elapse from the time the market closes in Tokyo at 1:00 A.M. EST (3:00 P.M. in Japan) to the time it closes in the United States at 4:00 P.M. As a result, prices from the Tokyo close ignore intervening information and are said to be *stale*. This led to the mutual-fund scandal of 2003, which is described in Box 11-2.

BOX 11-2

MARKET TIMING AND STALE PRICES

In September 2003, New York Attorney General Eliot Spitzer accused a number of investment companies of allowing *market timing* into their funds. Market timing is a short-term trading strategy of buying and selling the same funds.

Consider, for example, our portfolio of Japanese and U.S. stocks, for which prices are set in different time zones. The problem is that U.S. investors can trade up to the close of the U.S. market. *Market timers* could take advantage of this discrepancy by rapid trading. For instance, if the U.S. market moves up following good news, it is likely the Japanese market will move up as well the following day. Market timers would buy the fund at the stale price and resell it the next day.

Such trading, however, creates transactions costs that are borne by the other investors in the fund. As a result, fund companies usually state in their prospectus that this practice is not allowed. In practice, Eliot Spitzer found out that many mutual-fund companies had encouraged market timers, which he argued was fraudulent. Eventually, a number of funds settled by paying more than $2 billion.

This practice can be stopped in a number of ways. Many mutual funds now impose short-term redemption fees, which make market timing uneconomical. Alternatively, the cutoff time for placing trades can be moved earlier.

For risk managers, stale prices cause problems. Because returns are not synchronous, daily correlations across markets are too low, which will affect the measurement of portfolio risk.

One possible solution is mapping. For instance, prices at the close of the U.S. market can be estimated from a regression of Japanese returns on U.S. returns and using the forecast value conditional on the latest U.S. information. Alternatively, correlations can be measured from returns taken over longer time intervals, such as weekly. In practice, the risk manager needs to make sure that the data-collection process will lead to meaningful risk estimates.

11.1.3 The Mapping Process

Figure 11-1 illustrates a simple mapping process, where six instruments are mapped on three risk factors. The first step in the analysis is marking all positions to market in current dollars or whatever reference currency is used. The market value for each instrument then is allocated to the three risk factors.

FIGURE 11-1

Mapping instruments on risk factors.

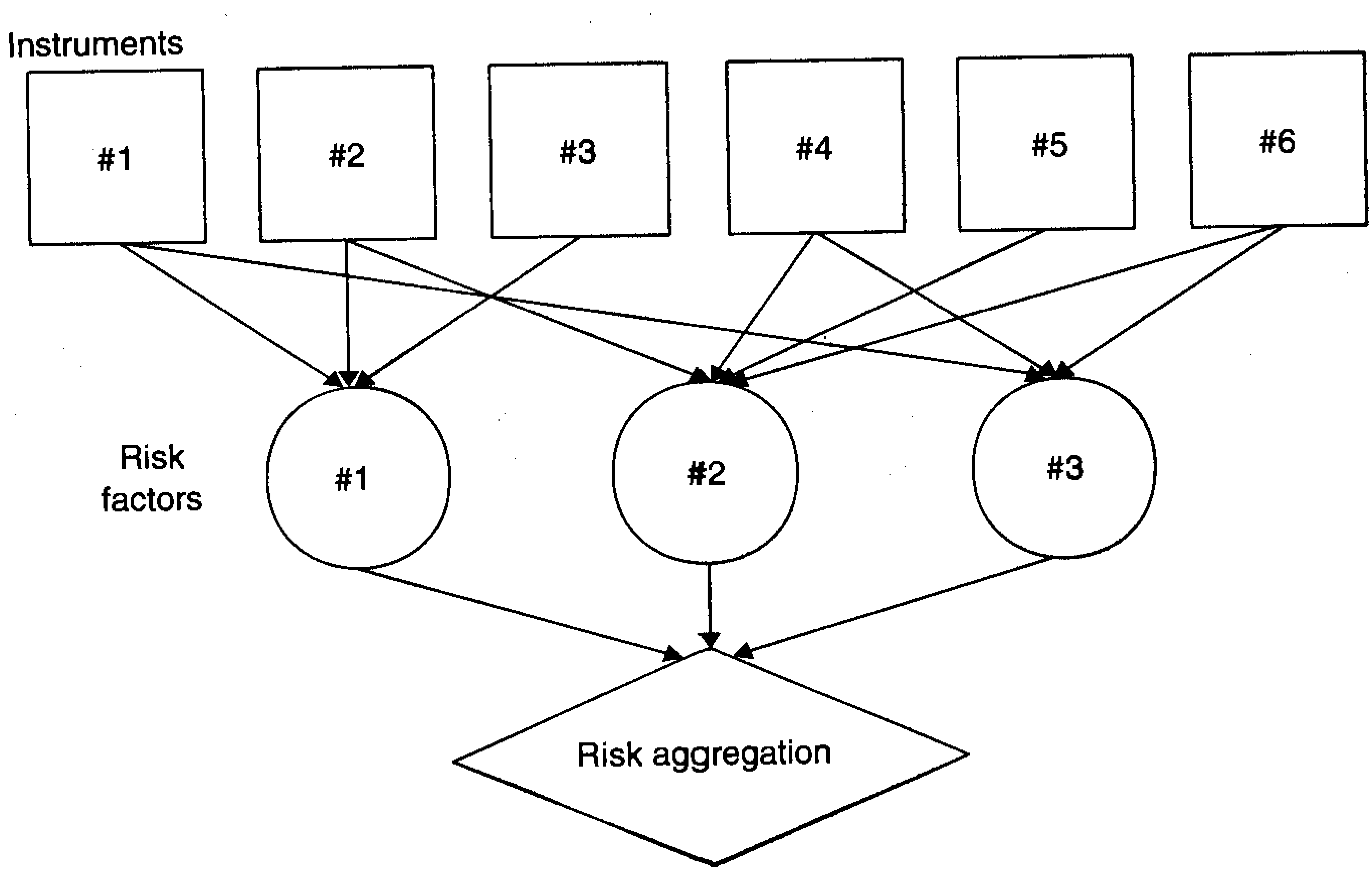


TABLE 11-1

Mapping Exposures

		Exposure on Risk Factor		
	Market Value	1	2	3
Instrument 1	V_1	x_{11}	x_{12}	x_{13}
Instrument 2	V_2	x_{21}	x_{22}	x_{23}
⋮	⋮	⋮	⋮	⋮
Instrument 6	V_6	x_{61}	x_{62}	x_{63}
Total portfolio	V	$x_1=\Sigma_{i=1}^6 x_{i1}$	$x_2=\Sigma_{i=1}^6 x_{i2}$	$x_3=\Sigma_{i=1}^6 x_{i3}$

Table 11-1 shows that the first instrument has a market value of V_1, which is allocated to three exposures, x_{11}, x_{12}, and x_{13}. If the current market value is not fully allocated to the risk factors, it must mean that the remainder is allocated to cash, which is not a risk factor because it has no risk.

Next, the system allocates the position for instrument 2 and so on. At the end of the process, positions are summed for each risk factor. For the first risk factor, the dollar exposure is $x_1=\Sigma_{i=1}^6 x_{i1}$. This creates a vector x of three exposures that can be fed into the risk measurement system.

Mapping can be of two kinds. The first provides an exact allocation of exposures on the risk factors. This is obtained for derivatives, for instance, when the price is an exact function of the risk factors. As we shall see in the rest of this chapter, the partial derivatives of the price function generate *analytical* measures of exposures on the risk factors.

Alternatively, exposures may have to be *estimated*. This occurs, for instance, when a stock is replaced by a position in the stock index. The exposure then is estimated by the slope coefficient from a regression of the stock return on the index return.

11.1.4 General and Specific Risk

This brings us to the issue of the choice of the set of primitive risk factors. This choice should reflect the tradeoff between better quality of the approximation and faster processing. More factors lead to tighter risk measurement but also require more time devoted to the modeling process and risk computation.

The choice of primitive risk factors also influences the size of specific risks. *Specific risk* can be defined as risk that is due to issuer-specific price movements, after accounting for general market factors. Hence the definition of specific risk depends on that of general market risk. The Basel rules have a separate charge for specific risk.[1]

To illustrate this decomposition, consider a portfolio of N stocks. We are mapping each stock on a position in the stock market index, which is our primitive risk factor. The return on a stock R_i is regressed on the return on the stock market index R_m, that is,

$$R_i = \alpha_i + \beta_i R_m + \epsilon_i \tag{11.1}$$

which gives the exposure β_i. In what follows, ignore α, which does not contribute to risk. We assume that the specific risk owing to ϵ is not correlated across stocks or with the market. The relative weight of each stock in the portfolio is given by w_i. Thus the portfolio return is

$$R_p = \sum_{i=1}^{N} w_i R_i = \sum_{i=1}^{N} w_i \beta_i R_m + \sum_{i=1}^{N} w_i \epsilon_i \tag{11.2}$$

These exposures are aggregated across all the stocks in the portfolio. This gives

$$\beta_p = \sum_{i=1}^{N} w_i \beta_i \tag{11.3}$$

If the portfolio value is W, the mapping on the index is $x = W\beta_p$.

Next, we decompose the variance of R_p in Equation (11.2) and find

$$V(R_p) = (\beta_p^2)V(R_m) + \sum_{i=1}^{N} w_i^2 \sigma_{\epsilon i}^2 \tag{11.4}$$

The first component is the general market risk. The second component is the aggregate of specific risk for the entire portfolio. This decomposition shows that with more detail on the primitive or general-market risk factors, there will be less specific risk for a fixed amount of total risk $V(R_p)$.

As another example, consider a corporate bond portfolio. Bond positions describe the distribution of money flows over time by their amount, timing, and credit quality of the issuer. This creates a continuum of risk factors, going from overnight to long maturities for various credit risks.

[1] Typically, the charge is 4 percent of the position value for equities and unrated debt, assuming that the banks' models do not incorporate specific risks. See Chapter 3.

In practice, we have to restrict the number of risk factors to a small set. For some portfolios, one risk factor may be sufficient. For others, 15 maturities may be necessary. For portfolios with options, we need to model movements not only in yields but also in their implied volatilities.

Our primitive risk factors could be movements in a set of J government bond yields z_j and in a set of K credit spreads s_k sorted by credit rating. We model the movement in each corporate bond yield dy_i by a movement in z at the closest maturity and in s for the same credit rating. The remaining component is ϵ_i.

The movement in value W then is

$$dW = \sum_{i=1}^{N} \text{DVBP}_i dy_i = \sum_{j=1}^{J} \text{DVBP}_j dz_j + \sum_{k=1}^{K} \text{DVBP}_k ds_k + \sum_{i=1}^{N} \text{DVBP}_i d\epsilon_i \tag{11.5}$$

where DVBP is the total dollar value of a basis point for the associated risk factor. The values for DVBP_j then represent the summation of the DVBP across all individual bonds for each maturity.

This leads to a total risk decomposition of

$$V(dW) = \text{ general risk} + \sum_{i=1}^{N} \text{DVBP}_i^2 V(d\epsilon_i) \tag{11.6}$$

A greater number of general risk factors should create less residual risk. Even so, we need to ascertain the size of the second, specific risk term. In practice, there may not be sufficient history to measure the specific risk of individual bonds, which is why it is often assumed that all issuers within the same risk class have the same risk.

11.2 MAPPING FIXED-INCOME PORTFOLIOS

11.2.1 Mapping Approaches

Once the risk factors have been selected, the question is how to map the portfolio positions into exposures on these risk factors. We can distinguish three mapping systems for fixed-income portfolios: principal, duration, and cash flows. With *principal mapping*, one risk factor is chosen that corresponds to the average portfolio maturity. With *duration mapping*, one risk factor is chosen that corresponds to the portfolio duration. With *cash-flow mapping*, the portfolio cash flows are grouped into maturity buckets. Mapping should preserve the market value of the position. Ideally, it also should preserve its market risk.