# Chapter 11

# The Dual Space and Duality

In this chapter all vector spaces are defined over an arbitrary field $K$. For the sake of concreteness, the reader may safely assume that $K = \mathbb{R}$.

## 11.1 The Dual Space $E^*$ and Linear Forms

In Section 3.9 we defined linear forms, the dual space $E^* = \mathrm{Hom}(E, K)$ of a vector space $E$, and showed the existence of dual bases for vector spaces of finite dimension.

In this chapter we take a deeper look at the connection between a space $E$ and its dual space $E^*$. As we will see shortly, every linear map $f\colon E \to F$ gives rise to a linear map $f^\top\colon F^* \to E^*$, and it turns out that in a suitable basis, the matrix of $f^\top$ is the transpose of the matrix of $f$. *Thus, the notion of dual space provides a conceptual explanation of the phenomena associated with transposition.*

*But it does more, because it allows us to view a linear equation as an element of the dual space $E^*$, and thus to view subspaces of $E$ as solutions of sets of linear equations and vice-versa.* The relationship between subspaces and sets of linear forms is the essence of *duality*, a term which is often used loosely, but can be made precise as a bijection between the set of subspaces of a given vector space $E$ and the set of subspaces of its dual $E^*$. In this correspondence, a subspace $V$ of $E$ yields the subspace $V^0$ of $E^*$ consisting of all linear forms that vanish on $V$ (that is, have the value zero for all input in $V$).

Consider the following set of two "linear equations" in $\mathbb{R}^3$,

$$\begin{aligned} x - y + z &= 0 \\ x - y - z &= 0, \end{aligned}$$

and let us find out what is their set $V$ of common solutions $(x, y, z) \in \mathbb{R}^3$. By subtracting the second equation from the first, we get $2z = 0$, and by adding the two equations, we find that $2(x - y) = 0$, so the set $V$ of solutions is given by

$$\begin{aligned} y &= x \\ z &= 0. \end{aligned}$$

This is a one dimensional subspace of $\mathbb{R}^3$. Geometrically, this is the line of equation $y = x$ in the plane $z = 0$ as illustrated by Figure 11.1.

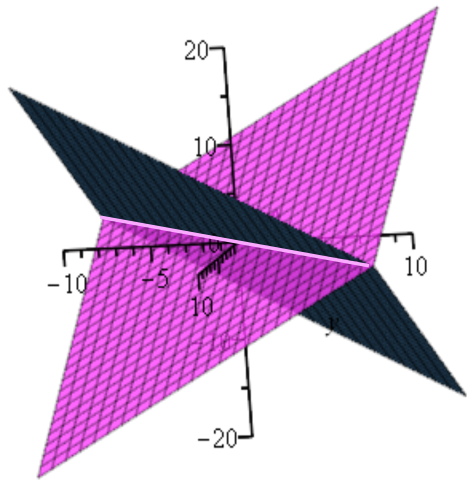


Figure 11.1: The intersection of the magenta plane $x - y + z = 0$ with the blue-gray plane $x - y - z = 0$ is the pink line $y = x$.

Now why did we say that the above equations are linear? Because as functions of $(x, y, z)$, both maps $f_1 : (x, y, z) \mapsto x - y + z$ and $f_2 : (x, y, z) \mapsto x - y - z$ are linear. The set of all such linear functions from $\mathbb{R}^3$ to $\mathbb{R}$ is a vector space; we used this fact to form linear combinations of the "equations" $f_1$ and $f_2$. Observe that the dimension of the subspace $V$ is 1. The ambient space has dimension $n = 3$ and there are two "independent" equations $f_1, f_2$, so it appears that the dimension $\dim(V)$ of the subspace $V$ defined by $m$ independent equations is

$$\dim(V) = n - m,$$

which is indeed a general fact (proven in Theorem 11.4).

More generally, in $\mathbb{R}^n$, a linear equation is determined by an $n$-tuple $(a_1, \ldots, a_n) \in \mathbb{R}^n$, and the solutions of this linear equation are given by the $n$-tuples $(x_1, \ldots, x_n) \in \mathbb{R}^n$ such that

$$a_1 x_1 + \cdots + a_n x_n = 0;$$

these solutions constitute the kernel of the linear map $(x_1, \ldots, x_n) \mapsto a_1 x_1 + \cdots + a_n x_n$. The above considerations assume that we are working in the canonical basis $(e_1, \ldots, e_n)$ of

$\mathbb{R}^n$, but we can define "linear equations" independently of bases and in any dimension, by viewing them as elements of the vector space $\mathrm{Hom}(E, K)$ of linear maps from $E$ to the field $K$.

**Definition 11.1.** Given a vector space $E$, the vector space $\mathrm{Hom}(E, K)$ of linear maps from $E$ to the field $K$ is called the *dual space (or dual)* of $E$. The space $\mathrm{Hom}(E, K)$ is also denoted by $E^*$, and the linear maps in $E^*$ are called *the linear forms*, or *covectors*. The dual space $E^{**}$ of the space $E^*$ is called the *bidual* of $E$.

As a matter of notation, linear forms $f \colon E \to K$ will also be denoted by starred symbol, such as $u^*$, $x^*$, *etc.*

Given a vector space $E$ and any basis $(u_i)_{i\in I}$ for $E$, we can associate to each $u_i$ a linear form $u_i^* \in E^*$, and the $u_i^*$ have some remarkable properties.

**Definition 11.2.** Given a vector space $E$ and any basis $(u_i)_{i\in I}$ for $E$, by Proposition 3.18, for every $i \in I$, there is a unique linear form $u_i^*$ such that

$$u_i^*(u_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

for every $j \in I$. The linear form $u_i^*$ is called the *coordinate form* of index $i$ w.r.t. the basis $(u_i)_{i\in I}$.

The reason for the terminology *coordinate form* was explained in Section 3.9.

We proved in Theorem 3.23 that if $(u_1, \ldots, u_n)$ is a basis of $E$, then $(u_1^*, \ldots, u_n^*)$ is a basis of $E^*$ called the *dual basis*.

If $(u_1, \ldots, u_n)$ is a basis of $\mathbb{R}^n$ (more generally $K^n$), it is possible to find explicitly the dual basis $(u_1^*, \ldots, u_n^*)$, where each $u_i^*$ is represented by a row vector.

**Example 11.1.** For example, consider the columns of the Bézier matrix

$$B_4 = \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In other words, we have the basis

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad u_2 = \begin{pmatrix} -3 \\ 3 \\ 0 \\ 0 \end{pmatrix} \qquad u_3 = \begin{pmatrix} 3 \\ -6 \\ 3 \\ 0 \end{pmatrix} \qquad u_4 = \begin{pmatrix} -1 \\ 3 \\ -3 \\ 1 \end{pmatrix}.$$

Since the form $u_1^*$ is defined by the conditions $u_1^*(u_1) = 1, u_1^*(u_2) = 0, u_1^*(u_3) = 0, u_1^*(u_4) = 0$, it is represented by a row vector $(\lambda_1 \;\; \lambda_2 \;\; \lambda_3 \;\; \lambda_4)$ such that

$$\begin{pmatrix}\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4\end{pmatrix}\begin{pmatrix}1 & -3 & 3 & -1\\ 0 & 3 & -6 & 3\\ 0 & 0 & 3 & -3\\ 0 & 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}.$$

This implies that $u_1^*$ is the first row of the inverse of $B_4$. Since

$$B_4^{-1} = \begin{pmatrix}1 & 1 & 1 & 1\\ 0 & 1/3 & 2/3 & 1\\ 0 & 0 & 1/3 & 1\\ 0 & 0 & 0 & 1\end{pmatrix},$$

the linear forms $(u_1^*, u_2^*, u_3^*, u_4^*)$ correspond to the rows of $B_4^{-1}$. In particular, $u_1^*$ is represented by $(1 \;\; 1 \;\; 1 \;\; 1)$.

The above method works for any $n$. Given any basis $(u_1, \ldots, u_n)$ of $\mathbb{R}^n$, if $P$ is the $n \times n$ matrix whose $j$th column is $u_j$, then the dual form $u_i^*$ is given by the $i$th row of the matrix $P^{-1}$.

When $E$ is of finite dimension $n$ and $(u_1, \ldots, u_n)$ is a basis of $E$, by Theorem 11.4 (1), the family $(u_1^*, \ldots, u_n^*)$ is a basis of the dual space $E^*$. Let us see how the coordinates of a linear form $\varphi^* \in E^*$ over the dual basis $(u_1^*, \ldots, u_n^*)$ vary under a change of basis.

Let $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ be two bases of $E$, and let $P = (a_{i\,j})$ be the change of basis matrix from $(u_1, \ldots, u_n)$ to $(v_1, \ldots, v_n)$, so that

$$v_j = \sum_{i=1}^{n} a_{i\,j} u_i,$$

and let $P^{-1} = (b_{i\,j})$ be the inverse of $P$, so that

$$u_i = \sum_{j=1}^{n} b_{j\,i} v_j.$$

For fixed $j$, where $1 \leq j \leq n$, we want to find scalars $(c_i)_{i=1}^n$ such that

$$v_j^* = c_1 u_1^* + c_2 u_2^* + \cdots + c_n u_n^*.$$

To find each $c_i$, we evaluate the above expression at $u_i$. Since $u_i^*(u_j) = \delta_{i\,j}$ and $v_i^*(v_j) = \delta_{i\,j}$, we get

$$\begin{aligned} v_j^*(u_i) &= (c_1 u_1^* + c_2 u_2^* + \cdots + c_n u_n^*)(u_i) = c_i \\ v_j^*(u_i) &= v_j^*(\sum_{k=1}^{n} b_{k\,i} v_k) = b_{j\,i}, \end{aligned}$$

and thus
$$v_j^* = \sum_{i=1}^n b_{j\,i} u_i^*.$$
Similar calculations show that
$$u_i^* = \sum_{j=1}^n a_{i\,j} v_j^*.$$
This means that the change of basis from the dual basis $(u_1^*, \ldots, u_n^*)$ to the dual basis $(v_1^*, \ldots, v_n^*)$ is $(P^{-1})^\top$. Since
$$\varphi^* = \sum_{i=1}^n \varphi_i u_i^* = \sum_{i=1}^n \varphi_i \sum_{j=1}^n a_{ij} v_j^* = \sum_{j=1}^n \left( \sum_{i=1}^n a_{ij}\varphi_i \right) v_j = \sum_{i=1}^n \varphi_i' v_i^*,$$
we get
$$\varphi_j' = \sum_{i=1}^n a_{i\,j} \varphi_i,$$
so the new coordinates $\varphi_j'$ are expressed in terms of the old coordinates $\varphi_i$ using the matrix $P^\top$. If we use the row vectors $(\varphi_1, \ldots, \varphi_n)$ and $(\varphi_1', \ldots, \varphi_n')$, we have
$$(\varphi_1', \ldots, \varphi_n') = (\varphi_1, \ldots, \varphi_n) P.$$
These facts are summarized in the following proposition.

**Proposition 11.1.** *Let $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ be two bases of $E$, and let $P = (a_{i\,j})$ be the change of basis matrix from $(u_1, \ldots, u_n)$ to $(v_1, \ldots, v_n)$, so that*
$$v_j = \sum_{i=1}^n a_{i\,j} u_i.$$
*Then the change of basis from the dual basis $(u_1^*, \ldots, u_n^*)$ to the dual basis $(v_1^*, \ldots, v_n^*)$ is $(P^{-1})^\top$, and for any linear form $\varphi$, the new coordinates $\varphi_j'$ of $\varphi$ are expressed in terms of the old coordinates $\varphi_i$ of $\varphi$ using the matrix $P^\top$; that is,*
$$(\varphi_1', \ldots, \varphi_n') = (\varphi_1, \ldots, \varphi_n) P.$$

To best understand the preceding paragraph, recall Example 3.1, in which $E = \mathbb{R}^2$, $u_1 = (1, 0)$, $u_2 = (0, 1)$, and $v_1 = (1, 1)$, $v_2 = (-1, 1)$. Then $P$, the change of basis matrix from $(u_1, u_2)$ to $(v_1, v_2)$, is given by
$$P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$
with $(v_1, v_2) = (u_1, u_2)P$, and $(u_1, u_2) = (v_1, v_2)P^{-1}$, where
$$P^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

Let $(u_1^*, u_2^*)$ be the dual basis for $(u_1, u_2)$ and $(v_1^*, v_2^*)$ be the dual basis for $(v_1, v_2)$. We claim that

$$(v_1^*, v_2^*) = (u_1^*, u_2^*) \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix} = (u_1^*, u_2^*)(P^{-1})^\top,$$

Indeed, since $v_1^* = c_1 u_1^* + c_2 u_2^*$ and $v_2^* = C_1 u_1^* + C_2 u_2^*$ we find that

$$\begin{aligned} c_1 &= v_1^*(u_1) = v_1^*(1/2 v_1 - 1/2 v_2) = 1/2 & c_2 &= v_1^*(u_2) = v_1^*(1/2 v_1 + 1/2 v_2) = 1/2 \\ C_1 &= v_2^*(u_1) = v_2^*(1/2 v_1 - 1/2 v_2) = -1/2 & C_2 &= v_2^*(u_2) = v_1^*(1/2 v_1 + 1/2 v_2) = 1/2. \end{aligned}$$

Furthermore, since $(u_1^*, u_2^*) = (v_1^*, v_2^*)P^\top$ (since $(v_1^*, v_2^*) = (u_1^*, u_2^*)(P^\top)^{-1}$), we find that

$$\varphi^* = \varphi_1 u_1^* + \varphi_2 u_2^* = \varphi_1(v_1^* - v_2^*) + \varphi_(v_1^* + v_2^*) = (\varphi_1 + \varphi_2)v_1^* + (-\varphi_1 + \varphi_2)v_2^* = \varphi_1' v_1^* + \varphi_2' v_2^*$$

Hence

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} \varphi_1' \\ \varphi_2' \end{pmatrix},$$

where

$$P^\top = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Comparing with the change of basis

$$v_j = \sum_{i=1}^n a_{i\,j} u_i,$$

we note that this time, the coordinates $(\varphi_i)$ of the linear form $\varphi^*$ change in the *same direction* as the change of basis. For this reason, we say that the coordinates of linear forms are *covariant*. By abuse of language, it is often said that linear forms are *covariant*, which explains why the term *covector* is also used for a linear form.

Observe that if $(e_1, \ldots, e_n)$ is a basis of the vector space $E$, then, as a linear map from $E$ to $K$, every linear form $f \in E^*$ is represented by a $1 \times n$ matrix, that is, by a *row vector*

$$(\lambda_1 \ \cdots \ \lambda_n),$$

with respect to the basis $(e_1, \ldots, e_n)$ of $E$, and 1 of $K$, where $f(e_i) = \lambda_i$. A vector $u = \sum_{i=1}^n u_i e_i \in E$ is represented by a $n \times 1$ matrix, that is, by a *column vector*

$$\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix},$$

and the action of $f$ on $u$, namely $f(u)$, is represented by the matrix product

$$\begin{pmatrix} \lambda_1 & \cdots & \lambda_n \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \lambda_1 u_1 + \cdots + \lambda_n u_n.$$

On the other hand, with respect to the dual basis $(e_1^*, \ldots, e_n^*)$ of $E^*$, the linear form $f$ is represented by the column vector

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}.$$

**Remark:** In many texts using tensors, vectors are often indexed with lower indices. If so, it is more convenient to write the coordinates of a vector $x$ over the basis $(u_1, \ldots, u_n)$ as $(x^i)$, using an upper index, so that

$$x = \sum_{i=1}^n x^i u_i,$$

and in a change of basis, we have

$$v_j = \sum_{i=1}^n a_j^i u_i$$

and

$$x^i = \sum_{j=1}^n a_j^i x'^j.$$

Dually, linear forms are indexed with upper indices. Then it is more convenient to write the coordinates of a covector $\varphi^*$ over the dual basis $(u^{*1}, \ldots, u^{*n})$ as $(\varphi_i)$, using a lower index, so that

$$\varphi^* = \sum_{i=1}^n \varphi_i u^{*i}$$

and in a change of basis, we have

$$u^{*i} = \sum_{j=1}^n a_j^i v^{*j}$$

and

$$\varphi'_j = \sum_{i=1}^n a_j^i \varphi_i.$$

With these conventions, the index of summation appears once in upper position and once in lower position, and the summation sign can be safely omitted, a trick due to *Einstein*. For example, we can write

$$\varphi'_j = a_j^i \varphi_i$$

as an abbreviation for

$$\varphi'_j = \sum_{i=1}^n a_j^i \varphi_i.$$

For another example of the use of Einstein's notation, if the vectors $(v_1, \ldots, v_n)$ are linear combinations of the vectors $(u_1, \ldots, u_n)$, with

$$v_i = \sum_{j=1}^{n} a_{ij} u_j, \quad 1 \leq i \leq n,$$

then the above equations are written as

$$v_i = a_i^j u_j, \quad 1 \leq i \leq n.$$

Thus, in Einstein's notation, the $n \times n$ matrix $(a_{ij})$ is denoted by $(a_i^j)$, a $(1,1)$-*tensor*.

Beware that some authors view a matrix as a mapping between *coordinates*, in which case the matrix $(a_{ij})$ is denoted by $(a_j^i)$.

## 11.2 Pairing and Duality Between $E$ and $E^*$

Given a linear form $u^* \in E^*$ and a vector $v \in E$, the result $u^*(v)$ of applying $u^*$ to $v$ is also denoted by $\langle u^*, v\rangle$. This defines a binary operation $\langle -, -\rangle \colon E^* \times E \to K$ satisfying the following properties:

$$\begin{aligned}
\langle u_1^* + u_2^*, v\rangle &= \langle u_1^*, v\rangle + \langle u_2^*, v\rangle \\
\langle u^*, v_1 + v_2\rangle &= \langle u^*, v_1\rangle + \langle u^*, v_2\rangle \\
\langle \lambda u^*, v\rangle &= \lambda\langle u^*, v\rangle \\
\langle u^*, \lambda v\rangle &= \lambda\langle u^*, v\rangle.
\end{aligned}$$

The above identities mean that $\langle -, -\rangle$ is a *bilinear map*, since it is linear in each argument. It is often called the *canonical pairing* between $E^*$ and $E$. In view of the above identities, given any fixed vector $v \in E$, the map $\mathrm{eval}_v \colon E^* \to K$ (*evaluation at* $v$) defined such that

$$\mathrm{eval}_v(u^*) = \langle u^*, v\rangle = u^*(v) \quad \text{for every } u^* \in E^*$$

is a linear map from $E^*$ to $K$, that is, $\mathrm{eval}_v$ is a linear form in $E^{**}$. Again, from the above identities, the map $\mathrm{eval}_E \colon E \to E^{**}$, defined such that

$$\mathrm{eval}_E(v) = \mathrm{eval}_v \quad \text{for every } v \in E,$$

is a linear map. Observe that

$$\mathrm{eval}_E(v)(u^*) = \mathrm{eval}_v(u^*) = \langle u^*, v\rangle = u^*(v), \quad \text{for all } v \in E \text{ and all } u^* \in E^*.$$

We shall see that the map $\mathrm{eval}_E$ is injective, and that it is an isomorphism when $E$ has finite dimension.

We now formalize the notion of the set $V^0$ of linear equations vanishing on all vectors in a given subspace $V \subseteq E$, and the notion of the set $U^0$ of common solutions of a given set $U \subseteq E^*$ of linear equations. The duality theorem (Theorem 11.4) shows that the dimensions of $V$ and $V^0$, and the dimensions of $U$ and $U^0$, are related in a crucial way. It also shows that, in finite dimension, the maps $V \mapsto V^0$ and $U \mapsto U^0$ are inverse bijections from subspaces of $E$ to subspaces of $E^*$.

**Definition 11.3.** Given a vector space $E$ and its dual $E^*$, we say that a vector $v \in E$ and a linear form $u^* \in E^*$ are *orthogonal* iff $\langle u^*, v\rangle = 0$. Given a subspace $V$ of $E$ and a subspace $U$ of $E^*$, we say that $V$ *and* $U$ *are orthogonal* iff $\langle u^*, v\rangle = 0$ for every $u^* \in U$ and every $v \in V$. Given a subset $V$ of $E$ (resp. a subset $U$ of $E^*$), the *orthogonal* $V^0$ *of* $V$ is the subspace $V^0$ of $E^*$ defined such that

$$V^0 = \{u^* \in E^* \mid \langle u^*, v\rangle = 0, \text{ for every } v \in V\}$$

(resp. the *orthogonal* $U^0$ *of* $U$ is the subspace $U^0$ of $E$ defined such that

$$U^0 = \{v \in E \mid \langle u^*, v\rangle = 0, \text{ for every } u^* \in U\}).$$

The subspace $V^0 \subseteq E^*$ is also called the *annihilator* of $V$. The subspace $U^0 \subseteq E$ annihilated by $U \subseteq E^*$ does not have a special name. It seems reasonable to call it the *linear subspace (or linear variety) defined by* $U$.

Informally, $V^0$ is the *set of linear equations that vanish on* $V$, and $U^0$ is *the set of common zeros of all linear equations in* $U$. We can also define $V^0$ by

$$V^0 = \{u^* \in E^* \mid V \subseteq \operatorname{Ker} u^*\}$$

and $U^0$ by

$$U^0 = \bigcap_{u^* \in U} \operatorname{Ker} u^*.$$

Observe that $E^0 = \{0\} = (0)$, and $\{0\}^0 = E^*$.

**Proposition 11.2.** *If $V_1 \subseteq V_2 \subseteq E$, then $V_2^0 \subseteq V_1^0 \subseteq E^*$, and if $U_1 \subseteq U_2 \subseteq E^*$, then $U_2^0 \subseteq U_1^0 \subseteq E$. See Figure 11.2.*

*Proof.* Indeed, if $V_1 \subseteq V_2 \subseteq E$, then for any $f^* \in V_2^0$ we have $f^*(v) = 0$ for all $v \in V_2$, and thus $f^*(v) = 0$ for all $v \in V_1$, so $f^* \in V_1^0$. Similarly, if $U_1 \subseteq U_2 \subseteq E^*$, then for any $v \in U_2^0$, we have $f^*(v) = 0$ for all $f^* \in U_2$, so $f^*(v) = 0$ for all $f^* \in U_1$, which means that $v \in U_1^0$. ☐

Here are some examples.

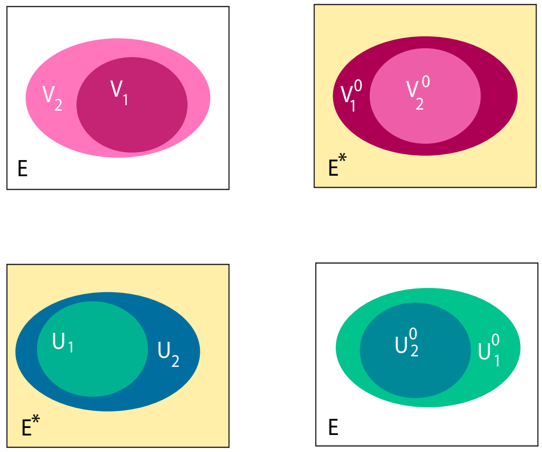


Figure 11.2: The top pair of figures schematically illustrates the relation if $V_1 \subseteq V_2 \subseteq E$, then $V_2^0 \subseteq V_1^0 \subseteq E^*$, while the bottom pair of figures illustrates the relationship if $U_1 \subseteq U_2 \subseteq E^*$, then $U_2^0 \subseteq U_1^0 \subseteq E$.

**Example 11.2.** Let $E = \mathrm{M}_2(\mathbb{R})$, the space of real $2 \times 2$ matrices, and let $V$ be the subspace of $\mathrm{M}_2(\mathbb{R})$ spanned by the matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

We check immediately that the subspace $V$ consists of all matrices of the form

$$\begin{pmatrix} b & a \\ a & c \end{pmatrix},$$

that is, all symmetric matrices. The matrices

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

in $V$ satisfy the equation

$$a_{12} - a_{21} = 0,$$

and all scalar multiples of these equations, so $V^0$ is the subspace of $E^*$ spanned by the linear form given by $u^*(a_{11}, a_{12}, a_{21}, a_{22}) = a_{12} - a_{21}$. By the duality theorem (Theorem 11.4) we have

$$\dim(V^0) = \dim(E) - \dim(V) = 4 - 3 = 1.$$

**Example 11.3.** The above example generalizes to $E = \mathrm{M}_n(\mathbb{R})$ for any $n \geq 1$, but this time, consider the space $U$ of linear forms asserting that a matrix $A$ is symmetric; these are the linear forms spanned by the $n(n-1)/2$ equations

$$a_{ij} - a_{ji} = 0, \quad 1 \leq i < j \leq n;$$

Note there are no constraints on diagonal entries, and half of the equations

$$a_{ij} - a_{ji} = 0, \quad 1 \leq i \neq j \leq n$$

are redundant. It is easy to check that the equations (linear forms) for which $i < j$ are linearly independent. To be more precise, let $U$ be the space of linear forms in $E^*$ spanned by the linear forms

$$u_{ij}^*(a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{n1}, \ldots, a_{nn}) = a_{ij} - a_{ji}, \quad 1 \leq i < j \leq n.$$

The dimension of $U$ is $n(n-1)/2$. Then the set $U^0$ of common solutions of these equations is the space $\mathbf{S}(n)$ of symmetric matrices. By the duality theorem (Theorem 11.4), this space has dimension

$$\frac{n(n+1)}{2} = n^2 - \frac{n(n-1)}{2}.$$

We leave it as an exercise to find a basis of $\mathbf{S}(n)$.

**Example 11.4.** If $E = \mathrm{M}_n(\mathbb{R})$, consider the subspace $U$ of linear forms in $E^*$ spanned by the linear forms

$$\begin{aligned} u_{ij}^*(a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{n1}, \ldots, a_{nn}) &= a_{ij} + a_{ji}, \quad 1 \leq i < j \leq n \\ u_{ii}^*(a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{n1}, \ldots, a_{nn}) &= a_{ii}, \quad 1 \leq i \leq n. \end{aligned}$$

It is easy to see that these linear forms are linearly independent, so $\dim(U) = n(n+1)/2$. The space $U^0$ of matrices $A \in \mathrm{M}_n(\mathbb{R})$ satifying all of the above equations is clearly the space $\mathbf{Skew}(n)$ of skew-symmetric matrices. By the duality theorem (Theorem 11.4), the dimension of $U^0$ is

$$\frac{n(n-1)}{2} = n^2 - \frac{n(n+1)}{2}.$$

We leave it as an exercise to find a basis of $\mathbf{Skew}(n)$.

**Example 11.5.** For yet another example with $E = \mathrm{M}_n(\mathbb{R})$, for any $A \in \mathrm{M}_n(\mathbb{R})$, consider the linear form in $E^*$ given by

$$\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn},$$

called the *trace* of $A$. The subspace $U^0$ of $E$ consisting of all matrices $A$ such that $\mathrm{tr}(A) = 0$ is a space of dimension $n^2 - 1$. We leave it as an exercise to find a basis of this space.

The dimension equations

$$\begin{aligned} \dim(V) + \dim(V^0) &= \dim(E) \\ \dim(U) + \dim(U^0) &= \dim(E) \end{aligned}$$

are always true (if $E$ is finite-dimensional). This is part of the duality theorem (Theorem 11.4).

**Remark:** In contrast with the previous examples, given a matrix $A \in \mathrm{M}_n(\mathbb{R})$, the equations asserting that $A^\top A = I$ are not linear constraints. For example, for $n = 2$, we have

$$\begin{aligned} a_{11}^2 + a_{21}^2 &= 1 \\ a_{21}^2 + a_{22}^2 &= 1 \\ a_{11}a_{12} + a_{21}a_{22} &= 0. \end{aligned}$$

**Remarks:**

(1) The notation $V^0$ (resp. $U^0$) for the orthogonal of a subspace $V$ of $E$ (resp. a subspace $U$ of $E^*$) is not universal. Other authors use the notation $V^\perp$ (resp. $U^\perp$). However, the notation $V^\perp$ is also used to denote the orthogonal complement of a subspace $V$ with respect to an inner product on a space $E$, in which case $V^\perp$ is a subspace of $E$ and not a subspace of $E^*$ (see Chapter 12). To avoid confusion, we prefer using the notation $V^0$.

(2) Since linear forms can be viewed as linear equations (at least in finite dimension), given a subspace (or even a subset) $U$ of $E^*$, we can define the set $\mathcal{Z}(U)$ of *common zeros* of the equations in $U$ by

$$\mathcal{Z}(U) = \{v \in E \mid u^*(v) = 0, \text{ for all } u^* \in U\}.$$

Of course $\mathcal{Z}(U) = U^0$, but the notion $\mathcal{Z}(U)$ can be generalized to more general kinds of equations, namely polynomial equations. In this more general setting, $U$ is a set of *polynomials* in $n$ variables with coefficients in a field $K$ (where $n = \dim(E)$). Sets of the form $\mathcal{Z}(U)$ are called *algebraic varieties*. Linear forms correspond to the special case where homogeneous polynomials of degree 1 are considered.

If $V$ is a subset of $E$, it is natural to associate with $V$ the *set of polynomials in* $K[X_1, \ldots, X_n]$ *that vanish on* $V$. This set, usually denoted $\mathcal{I}(V)$, has some special properties that make it an *ideal*. If $V$ is a linear subspace of $E$, it is natural to restrict our attention to the space $V^0$ of linear forms that vanish on $V$, and in this case we identify $\mathcal{I}(V)$ and $V^0$ (although technically, $\mathcal{I}(V)$ is no longer an ideal).

For any arbitrary set of polynomials $U \subseteq K[X_1, \ldots, X_n]$ (resp. subset $V \subseteq E$), the relationship between $\mathcal{I}(\mathcal{Z}(U))$ and $U$ (resp. $\mathcal{Z}(\mathcal{I}(V))$ and $V$) is generally not simple, even though we always have

$$U \subseteq \mathcal{I}(\mathcal{Z}(U)) \quad \text{(resp.} \quad V \subseteq \mathcal{Z}(\mathcal{I}(V))).$$

However, when the field $K$ is algebraically closed, then $\mathcal{I}(\mathcal{Z}(U))$ is equal to the *radical* of the ideal $U$, a famous result due to Hilbert known as the *Nullstellensatz* (see Lang [109] or Dummit and Foote [54]). The study of algebraic varieties is the main subject of *algebraic geometry*, a beautiful but formidable subject. For a taste of algebraic geometry, see Lang [109] or Dummit and Foote [54].

The duality theorem (Theorem 11.4) shows that the situation is much simpler if we restrict our attention to linear subspaces; in this case

$$U = \mathcal{I}(\mathcal{Z}(U)) \quad \text{and} \quad V = \mathcal{Z}(\mathcal{I}(V)).$$

**Proposition 11.3.** *We have $V \subseteq V^{00}$ for every subspace $V$ of $E$, and $U \subseteq U^{00}$ for every subspace $U$ of $E^*$.*

*Proof.* Indeed, for any $v \in V$, to show that $v \in V^{00}$ we need to prove that $u^*(v) = 0$ for all $u^* \in V^0$. However, $V^0$ consists of all linear forms $u^*$ such that $u^*(y) = 0$ for *all* $y \in V$; in particular, for a fixed $v \in V$, we have $u^*(v) = 0$ for all $u^* \in V^0$, as required.

Similarly, for any $u^* \in U$, to show that $u^* \in U^{00}$ we need to prove that $u^*(v) = 0$ for all $v \in U^0$. However, $U^0$ consists of all vectors $v$ such that $f^*(v) = 0$ for *all* $f^* \in U$; in particular, for a fixed $u^* \in U$, we have $u^*(v) = 0$ for all $v \in U^0$, as required. ☐

We will see shortly that in finite dimension, we have $V = V^{00}$ and $U = U^{00}$.

However, even though $V = V^{00}$ is always true, when $E$ is of infinite dimension, it is not always true that $U = U^{00}$.

Given a vector space $E$ and a subspace $U$ of $E$, by Theorem 3.7, every basis $(u_i)_{i\in I}$ of $U$ can be extended to a basis $(u_j)_{j\in I\cup J}$ of $E$, where $I \cap J = \emptyset$.

## 11.3 The Duality Theorem and Some Consequences

We have the following important theorem adapted from E. Artin [6] (Chapter 1).

**Theorem 11.4.** *(Duality theorem) Let $E$ be a vector space. The following properties hold:*

*(a) For every basis $(u_i)_{i\in I}$ of $E$, the family $(u_i^*)_{i\in I}$ of coordinate forms is linearly independent.*

*(b) For every subspace $V$ of $E$, we have $V^{00} = V$.*

*(c) For every subspace $V$ of finite codimension $m$ of $E$, for every subspace $W$ of $E$ such that $E = V \oplus W$ (where $W$ is of finite dimension $m$), for every basis $(u_i)_{i\in I}$ of $E$ such that $(u_1, \ldots, u_m)$ is a basis of $W$, the family $(u_1^*, \ldots, u_m^*)$ is a basis of the orthogonal $V^0$ of $V$ in $E^*$, so that*

$$\dim(V^0) = \operatorname{codim}(V).$$

*Furthermore, we have $V^{00} = V$.*

(d) *For every subspace $U$ of finite dimension $m$ of $E^*$, the orthogonal $U^0$ of $U$ in $E$ is of finite codimension $m$, so that*

$$\mathrm{codim}(U^0) = \dim(U).$$

*Furthermore, $U^{00} = U$.*

*Proof.* (a) Assume that

$$\sum_{i \in I} \lambda_i u_i^* = 0,$$

for a family $(\lambda_i)_{i\in I}$ (of scalars in $K$). Since $(\lambda_i)_{i\in I}$ has finite support, there is a finite subset $J$ of $I$ such that $\lambda_i = 0$ for all $i \in I - J$, and we have

$$\sum_{j \in J} \lambda_j u_j^* = 0.$$

Applying the linear form $\sum_{j\in J} \lambda_j u_j^*$ to each $u_j$ ($j \in J$), by Definition 11.2, since $u_i^*(u_j) = 1$ if $i = j$ and 0 otherwise, we get $\lambda_j = 0$ for all $j \in J$, that is $\lambda_i = 0$ for all $i \in I$ (by definition of $J$ as the support). Thus, $(u_i^*)_{i\in I}$ is linearly independent.

(b) Clearly, we have $V \subseteq V^{00}$. If $V \neq V^{00}$, then let $(u_i)_{i\in I\cup J}$ be a basis of $V^{00}$ such that $(u_i)_{i\in I}$ is a basis of $V$ (where $I \cap J = \emptyset$). Since $V \neq V^{00}$, $u_{j_0} \in V^{00}$ for some $j_0 \in J$ (and thus, $j_0 \notin I$). Since $u_{j_0} \in V^{00}$, $u_{j_0}$ is orthogonal to every linear form in $V^0$. Now, we have $u_{j_0}^*(u_i) = 0$ for all $i \in I$, and thus $u_{j_0}^* \in V^0$. However, $u_{j_0}^*(u_{j_0}) = 1$, contradicting the fact that $u_{j_0}$ is orthogonal to every linear form in $V^0$. Thus, $V = V^{00}$.

(c) Let $J = I - \{1, \ldots, m\}$. Every linear form $f^* \in V^0$ is orthogonal to every $u_j$, for $j \in J$, and thus, $f^*(u_j) = 0$, for all $j \in J$. For such a linear form $f^* \in V^0$, let

$$g^* = f^*(u_1)u_1^* + \cdots + f^*(u_m)u_m^*.$$

We have $g^*(u_i) = f^*(u_i)$, for every $i$, $1 \leq i \leq m$. Furthermore, by definition, $g^*$ vanishes on all $u_j$, where $j \in J$. Thus, $f^*$ and $g^*$ agree on the basis $(u_i)_{i\in I}$ of $E$, and so, $g^* = f^*$. This shows that $(u_1^*, \ldots, u_m^*)$ generates $V^0$, and since it is also a linearly independent family, $(u_1^*, \ldots, u_m^*)$ is a basis of $V^0$. It is then obvious that $\dim(V^0) = \mathrm{codim}(V)$, and by part (b), we have $V^{00} = V$.

(d) Let $(u_1^*, \ldots, u_m^*)$ be a basis of $U$. Note that the map $h\colon E \to K^m$ defined such that

$$h(v) = (u_1^*(v), \ldots, u_m^*(v))$$

for every $v \in E$, is a linear map, and that its kernel $\mathrm{Ker}\, h$ is precisely $U^0$. Then, by Proposition 6.16,

$$E \approx \mathrm{Ker}\,(h) \oplus \mathrm{Im}\, h = U^0 \oplus \mathrm{Im}\, h,$$

and since $\dim(\mathrm{Im}\, h) \leq m$, we deduce that $U^0$ is a subspace of $E$ of finite codimension at most $m$, and by (c), we have $\dim(U^{00}) = \mathrm{codim}(U^0) \leq m = \dim(U)$. However, it is clear that $U \subseteq U^{00}$, which implies $\dim(U) \leq \dim(U^{00})$, and so $\dim(U^{00}) = \dim(U) = m$, and we must have $U = U^{00}$. $\square$

Part (a) of Theorem 11.4 shows that

$$\dim(E) \leq \dim(E^*).$$

When $E$ is of finite dimension $n$ and $(u_1, \ldots, u_n)$ is a basis of $E$, by part (c), the family $(u_1^*, \ldots, u_n^*)$ is a basis of the dual space $E^*$, called the *dual basis* of $(u_1, \ldots, u_n)$. This fact was also proven directly in Theorem 3.23.

Define the function $\mathcal{E}$ ($\mathcal{E}$ for equations) from subspaces of $E$ to subspaces of $E^*$ and the function $\mathcal{Z}$ ($\mathcal{Z}$ for zeros) from subspaces of $E^*$ to subspaces of $E$ by

$$\begin{aligned} \mathcal{E}(V) &= V^0, \quad V \subseteq E \\ \mathcal{Z}(U) &= U^0, \quad U \subseteq E^*. \end{aligned}$$

By Parts (c) and (d) of Theorem 11.4,

$$\begin{aligned} (\mathcal{Z} \circ \mathcal{E})(V) &= V^{00} = V \\ (\mathcal{E} \circ \mathcal{Z})(U) &= U^{00} = U, \end{aligned}$$

so $\mathcal{Z} \circ \mathcal{E} = \mathrm{id}$ and $\mathcal{E} \circ \mathcal{Z} = \mathrm{id}$, and the maps $\mathcal{E}$ and $\mathcal{Z}$ are inverse bijections. These maps set up a *duality* between subspaces of $E$ and subspaces of $E^*$. In particular, every subspace $V \subseteq E$ of dimension $m$ is the set of common zeros of the space of linear forms (equations) $V^0$, which has dimension $n - m$. This confirms the claim we made about the dimension of the subpsace defined by a set of linear equations.

One should be careful that this bijection does not extend to subspaces of $E^*$ of infinite dimension.

When $E$ is of infinite dimension, for every basis $(u_i)_{i \in I}$ of $E$, the family $(u_i^*)_{i \in I}$ of coordinate forms is never a basis of $E^*$. It is linearly independent, but it is "too small" to generate $E^*$. For example, if $E = \mathbb{R}^{(\mathbb{N})}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$, the map $f \colon E \to \mathbb{R}$ that sums the nonzero coordinates of a vector in $E$ is a linear form, but it is easy to see that it cannot be expressed as a linear combination of coordinate forms. As a consequence, when $E$ is of infinite dimension, $E$ and $E^*$ are not isomorphic.

We now discuss some applications of the duality theorem.

**Problem 1** . Suppose that $V$ is a subspace of $\mathbb{R}^n$ of dimension $m$ and that $(v_1, \ldots, v_m)$ is a basis of $V$. The problem is to find a basis of $V^0$.

We first extend $(v_1, \ldots, v_m)$ to a basis $(v_1, \ldots, v_n)$ of $\mathbb{R}^n$, and then by part (c) of Theorem 11.4, we know that $(v_{m+1}^*, \ldots, v_n^*)$ is a basis of $V^0$.

**Example 11.6.** For example, suppose that $V$ is the subspace of $\mathbb{R}^4$ spanned by the two linearly independent vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix},$$

the first two vectors of the Haar basis in $\mathbb{R}^4$. The four columns of the Haar matrix

$$W = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{pmatrix}$$

form a basis of $\mathbb{R}^4$, and the inverse of $W$ is given by

$$W^{-1} = \begin{pmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & -1/4 & -1/4 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \end{pmatrix}.$$

Since the dual basis $(v_1^*, v_2^*, v_3^*, v_4^*)$ is given by the rows of $W^{-1}$, the last two rows of $W^{-1}$,

$$\begin{pmatrix} 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \end{pmatrix},$$

form a basis of $V^0$. We also obtain a basis by rescaling by the factor $1/2$, so the linear forms given by the row vectors

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

form a basis of $V^0$, the space of linear forms (linear equations) that vanish on the subspace $V$.

The method that we described to find $V^0$ requires first extending a basis of $V$ and then inverting a matrix, but there is a more direct method. Indeed, let $A$ be the $n \times m$ matrix whose columns are the basis vectors $(v_1, \ldots, v_m)$ of $V$. Then a linear form $u$ represented by a row vector belongs to $V^0$ iff $uv_i = 0$ for $i = 1, \ldots, m$ iff

$$uA = 0$$

iff

$$A^\top u^\top = 0.$$

Therefore, all we need to do is to find a basis of the nullspace of $A^\top$. This can be done quite effectively using the reduction of a matrix to reduced row echelon form (rref); see Section 8.10.

**Example 11.7.** For example, if we reconsider the previous example, $A^\top u^\top = 0$ becomes

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since the rref of $A^\top$ is

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

the above system is equivalent to

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} u_1 + u_2 \\ u_3 + u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where the free variables are associated with $u_2$ and $u_4$. Thus to determine a basis for the kernel of $A^\top$, we set $u_2 = 1, u_4 = 0$ and $u_2 = 0, u_4 = 1$ and obtain a basis for $V^0$ as

$$\begin{pmatrix} 1 & -1 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 1 & -1 \end{pmatrix}.$$

**Problem 2** . Let us now consider the problem of finding a basis of the hyperplane $H$ in $\mathbb{R}^n$ defined by the equation

$$c_1x_1 + \cdots + c_nx_n = 0.$$

More precisely, if $u^*(x_1, \ldots, x_n)$ is the linear form in $(\mathbb{R}^n)^*$ given by $u^*(x_1, \ldots, x_n) = c_1x_1 + \cdots + c_nx_n$, then the hyperplane $H$ is the kernel of $u^*$. Of course we assume that some $c_j$ is nonzero, in which case the linear form $u^*$ spans a one-dimensional subspace $U$ of $(\mathbb{R}^n)^*$, and $U^0 = H$ has dimension $n - 1$.

Since $u^*$ is not the linear form which is identically zero, there is a smallest positive index $j \leq n$ such that $c_j \neq 0$, so our linear form is really $u^*(x_1, \ldots, x_n) = c_jx_j + \cdots + c_nx_n$. We claim that the following $n - 1$ vectors (in $\mathbb{R}^n$) form a basis of $H$:

$$\begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ j-1 \\ j \\ j+1 \\ j+2 \\ \vdots \\ n \end{array}
\begin{array}{c}
\begin{array}{cccccccc} 1 & 2 & \ldots & j-1 & j & j+1 & \ldots & n-1 \end{array} \\
\begin{pmatrix}
1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & -c_{j+1}/c_j & -c_{j+2}/c_j & \ldots & -c_n/c_j \\
0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1
\end{pmatrix}
\end{array}.$$

Observe that the $(n-1) \times (n-1)$ matrix obtained by deleting row $j$ is the identity matrix, so the columns of the above matrix are linearly independent. A simple calculation also shows that the linear form $u^*(x_1, \ldots, x_n) = c_jx_j + \cdots + c_nx_n$ vanishes on every column of the above matrix. For a concrete example in $\mathbb{R}^6$, if $u^*(x_1, \ldots, x_6) = x_3 + 2x_4 + 3x_5 + 4x_6$, we obtain the basis for the hyperplane $H$ of equation

$$x_3 + 2x_4 + 3x_5 + 4x_6 = 0$$

given by the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & -3 & -4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

**Problem 3** . Conversely, given a hyperplane $H$ in $\mathbb{R}^n$ given as the span of $n-1$ linearly vectors $(u_1, \ldots, u_{n-1})$, it is possible using determinants to find a linear form $(\lambda_1, \ldots, \lambda_n)$ that vanishes on $H$.

In the case $n = 3$, we are looking for a row vector $(\lambda_1, \lambda_2, \lambda_3)$ such that if

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

are two linearly independent vectors, then

$$\begin{pmatrix} u_1 & u_2 & u_2 \\ v_1 & v_2 & v_2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and the cross-product $u \times v$ of $u$ and $v$ given by

$$u \times v = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$$

is a solution. In other words, the equation of the plane spanned by $u$ and $v$ is

$$(u_2 v_3 - u_3 v_2)x + (u_3 v_1 - u_1 v_3)y + (u_1 v_2 - u_2 v_1)z = 0.$$

**Problem 4** . Here is another example illustrating the power of Theorem 11.4. Let $E = \mathrm{M}_n(\mathbb{R})$, and consider the equations asserting that the sum of the entries in every row of a matrix $A \in \mathrm{M}_n(\mathbb{R})$ is equal to the same number. We have $n-1$ equations

$$\sum_{j=1}^{n} (a_{ij} - a_{i+1j}) = 0, \quad 1 \leq i \leq n-1,$$

and it is easy to see that they are linearly independent. Therefore, the space $U$ of linear forms in $E^*$ spanned by the above linear forms (equations) has dimension $n-1$, and the space $U^0$ of matrices satisfying all these equations has dimension $n^2-n+1$. It is not so obvious to find a basis for this space.

We will now pin down the relationship between a vector space $E$ and its bidual $E^{**}$.

## 11.4 The Bidual and Canonical Pairings

**Proposition 11.5.** *Let $E$ be a vector space. The following properties hold:*

*(a) The linear map $\mathrm{eval}_E\colon E \to E^{**}$ defined such that*

$$\mathrm{eval}_E(v) = \mathrm{eval}_v \quad \textit{for all } v \in E,$$

*that is, $\mathrm{eval}_E(v)(u^*) = \langle u^*, v\rangle = u^*(v)$ for every $u^* \in E^*$, is injective.*

*(b) When $E$ is of finite dimension $n$, the linear map $\mathrm{eval}_E\colon E \to E^{**}$ is an isomorphism (called the canonical isomorphism).*

*Proof.* (a) Let $(u_i)_{i\in I}$ be a basis of $E$, and let $v = \sum_{i\in I} v_i u_i$. If $\mathrm{eval}_E(v) = 0$, then in particular $\mathrm{eval}_E(v)(u_i^*) = 0$ for all $u_i^*$, and since

$$\mathrm{eval}_E(v)(u_i^*) = \langle u_i^*, v\rangle = v_i,$$

we have $v_i = 0$ for all $i \in I$, that is, $v = 0$, showing that $\mathrm{eval}_E\colon E \to E^{**}$ is injective.

If $E$ is of finite dimension $n$, by Theorem 11.4, for every basis $(u_1, \ldots, u_n)$, the family $(u_1^*, \ldots, u_n^*)$ is a basis of the dual space $E^*$, and thus the family $(u_1^{**}, \ldots, u_n^{**})$ is a basis of the bidual $E^{**}$. This shows that $\dim(E) = \dim(E^{**}) = n$, and since by Part (a), we know that $\mathrm{eval}_E\colon E \to E^{**}$ is injective, in fact, $\mathrm{eval}_E\colon E \to E^{**}$ is bijective (by Proposition 6.19). $\square$

When a vector space $E$ has infinite dimension, $E$ and its bidual $E^{**}$ are never isomorphic.

When $E$ is of finite dimension and $(u_1, \ldots, u_n)$ is a basis of $E$, in view of the canonical isomorphism $\mathrm{eval}_E\colon E \to E^{**}$, the basis $(u_1^{**}, \ldots, u_n^{**})$ of the bidual is *identified* with $(u_1, \ldots, u_n)$.

Proposition 11.5 can be reformulated very fruitfully in terms of pairings, a remarkably useful concept discovered by Pontrjagin in 1931 (adapted from E. Artin [6], Chapter 1). Given two vector spaces $E$ and $F$ over a field $K$, we say that a function $\varphi\colon E \times F \to K$ is *bilinear* if for every $v \in V$, the map $u \mapsto \varphi(u, v)$ (from $E$ to $K$) is linear, and for every $u \in E$, the map $v \mapsto \varphi(u, v)$ (from $F$ to $K$) is linear.

**Definition 11.4.** Given two vector spaces $E$ and $F$ over $K$, a *pairing between $E$ and $F$* is a bilinear map $\varphi\colon E \times F \to K$. Such a pairing is *nondegenerate* iff

(1) for every $u \in E$, if $\varphi(u, v) = 0$ for all $v \in F$, then $u = 0$, and

(2) for every $v \in F$, if $\varphi(u, v) = 0$ for all $u \in E$, then $v = 0$.

A pairing $\varphi\colon E \times F \to K$ is often denoted by $\langle -, - \rangle\colon E \times F \to K$. For example, the map $\langle -, - \rangle\colon E^* \times E \to K$ defined earlier is a nondegenerate pairing (use the proof of (a) in Proposition 11.5). If $E = F$ and $K = \mathbb{R}$, any inner product on $E$ is a nondegenerate pairing (because an inner product is positive definite); see Chapter 12. Other interesting nondegenerate pairings arise in exterior algebra and differential geometry.

Given a pairing $\varphi\colon E \times F \to K$, we can define two maps $l_\varphi\colon E \to F^*$ and $r_\varphi\colon F \to E^*$ as follows: For every $u \in E$, we define the linear form $l_\varphi(u)$ in $F^*$ such that

$$l_\varphi(u)(y) = \varphi(u, y) \quad \text{for every } y \in F,$$

and for every $v \in F$, we define the linear form $r_\varphi(v)$ in $E^*$ such that

$$r_\varphi(v)(x) = \varphi(x, v) \quad \text{for every } x \in E.$$

We have the following useful proposition.

**Proposition 11.6.** *Given two vector spaces $E$ and $F$ over $K$, for every nondegenerate pairing $\varphi\colon E \times F \to K$ between $E$ and $F$, the maps $l_\varphi\colon E \to F^*$ and $r_\varphi\colon F \to E^*$ are linear and injective. Furthermore, if $E$ and $F$ have finite dimension, then this dimension is the same and $l_\varphi\colon E \to F^*$ and $r_\varphi\colon F \to E^*$ are bijections.*

*Proof.* The maps $l_\varphi\colon E \to F^*$ and $r_\varphi\colon F \to E^*$ are linear because a pairing is bilinear. If $l_\varphi(u) = 0$ (the null form), then

$$l_\varphi(u)(v) = \varphi(u, v) = 0 \quad \text{for every } v \in F,$$

and since $\varphi$ is nondegenerate, $u = 0$. Thus, $l_\varphi\colon E \to F^*$ is injective. Similarly, $r_\varphi\colon F \to E^*$ is injective. When $F$ has finite dimension $n$, we have seen that $F$ and $F^*$ have the same dimension. Since $l_\varphi\colon E \to F^*$ is injective, we have $m = \dim(E) \leq \dim(F) = n$. The same argument applies to $E$, and thus $n = \dim(F) \leq \dim(E) = m$. But then, $\dim(E) = \dim(F)$, and $l_\varphi\colon E \to F^*$ and $r_\varphi\colon F \to E^*$ are bijections. $\square$

When $E$ has finite dimension, the nondegenerate pairing $\langle -, - \rangle\colon E^* \times E \to K$ yields another proof of the existence of a natural isomorphism between $E$ and $E^{**}$. When $E = F$, the nondegenerate pairing induced by an inner product on $E$ yields a natural isomorphism between $E$ and $E^*$ (see Section 12.2).

We now show the relationship between hyperplanes and linear forms.

## 11.5 Hyperplanes and Linear Forms

Actually Proposition 11.7 below follows from Parts (c) and (d) of Theorem 11.4, but we feel that it is also interesting to give a more direct proof.

**Proposition 11.7.** *Let $E$ be a vector space. The following properties hold:*

(a) *Given any nonnull linear form $f^* \in E^*$, its kernel $H = \mathrm{Ker}\, f^*$ is a hyperplane.*

(b) *For any hyperplane $H$ in $E$, there is a (nonnull) linear form $f^* \in E^*$ such that $H = \mathrm{Ker}\, f^*$.*

(c) *Given any hyperplane $H$ in $E$ and any (nonnull) linear form $f^* \in E^*$ such that $H = \mathrm{Ker}\, f^*$, for every linear form $g^* \in E^*$, $H = \mathrm{Ker}\, g^*$ iff $g^* = \lambda f^*$ for some $\lambda \neq 0$ in $K$.*

*Proof.* (a) If $f^* \in E^*$ is nonnull, there is some vector $v_0 \in E$ such that $f^*(v_0) \neq 0$. Let $H = \mathrm{Ker}\, f^*$. For every $v \in E$, we have

$$f^*\left(v - \frac{f^*(v)}{f^*(v_0)} v_0\right) = f^*(v) - \frac{f^*(v)}{f^*(v_0)} f^*(v_0) = f^*(v) - f^*(v) = 0.$$

Thus,

$$v - \frac{f^*(v)}{f^*(v_0)} v_0 = h \in H,$$

and

$$v = h + \frac{f^*(v)}{f^*(v_0)} v_0,$$

that is, $E = H + Kv_0$. Also since $f^*(v_0) \neq 0$, we have $v_0 \notin H$, that is, $H \cap Kv_0 = 0$. Thus, $E = H \oplus Kv_0$, and $H$ is a hyperplane.

(b) If $H$ is a hyperplane, $E = H \oplus Kv_0$ for some $v_0 \notin H$. Then every $v \in E$ can be written in a unique way as $v = h + \lambda v_0$. Thus there is a well-defined function $f^* \colon E \to K$, such that, $f^*(v) = \lambda$, for every $v = h + \lambda v_0$. We leave as a simple exercise the verification that $f^*$ is a linear form. Since $f^*(v_0) = 1$, the linear form $f^*$ is nonnull. Also, by definition, it is clear that $\lambda = 0$ iff $v \in H$, that is, $\mathrm{Ker}\, f^* = H$.

(c) Let $H$ be a hyperplane in $E$, and let $f^* \in E^*$ be any (nonnull) linear form such that $H = \mathrm{Ker}\, f^*$. Clearly, if $g^* = \lambda f^*$ for some $\lambda \neq 0$, then $H = \mathrm{Ker}\, g^*$. Conversely, assume that $H = \mathrm{Ker}\, g^*$ for some nonnull linear form $g^*$. From (a), we have $E = H \oplus Kv_0$, for some $v_0$ such that $f^*(v_0) \neq 0$ and $g^*(v_0) \neq 0$. Then observe that

$$g^* - \frac{g^*(v_0)}{f^*(v_0)} f^*$$

is a linear form that vanishes on $H$, since both $f^*$ and $g^*$ vanish on $H$, but also vanishes on $Kv_0$. Thus, $g^* = \lambda f^*$, with

$$\lambda = \frac{g^*(v_0)}{f^*(v_0)}.$$

$\square$

We leave as an exercise the fact that every subspace $V \neq E$ of a vector space $E$ is the intersection of all hyperplanes that contain $V$. We now consider the notion of transpose of a linear map and of a matrix.

## 11.6 Transpose of a Linear Map and of a Matrix

Given a linear map $f \colon E \to F$, it is possible to define a map $f^\top \colon F^* \to E^*$ which has some interesting properties.

**Definition 11.5.** Given a linear map $f \colon E \to F$, the *transpose $f^\top \colon F^* \to E^*$ of $f$* is the linear map defined such that

$$f^\top(v^*) = v^* \circ f, \quad \text{for every } v^* \in F^*,$$

as shown in the diagram below:

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[dr, "f^\top(v^*)"'] & F \arrow[d, "v^*"] \\ & K. \end{tikzcd}$$

Equivalently, the linear map $f^\top \colon F^* \to E^*$ is defined such that

$$\langle v^*, f(u) \rangle = \langle f^\top(v^*), u \rangle, \tag{$*$}$$

for all $u \in E$ and all $v^* \in F^*$.

It is easy to verify that the following properties hold:

$$\begin{aligned} (f+g)^\top &= f^\top + g^\top \\ (g \circ f)^\top &= f^\top \circ g^\top \\ \mathrm{id}_E^\top &= \mathrm{id}_{E^*}. \end{aligned}$$

Note the reversal of composition on the right-hand side of $(g \circ f)^\top = f^\top \circ g^\top$.

The equation $(g \circ f)^\top = f^\top \circ g^\top$ implies the following useful proposition.

**Proposition 11.8.** *If $f \colon E \to F$ is any linear map, then the following properties hold:*

*(1) If $f$ is injective, then $f^\top$ is surjective.*

*(2) If $f$ is surjective, then $f^\top$ is injective.*

*Proof.* If $f\colon E \to F$ is injective, then it has a retraction $r\colon F \to E$ such that $r \circ f = \mathrm{id}_E$, and if $f\colon E \to F$ is surjective, then it has a section $s\colon F \to E$ such that $f \circ s = \mathrm{id}_F$. Now if $f\colon E \to F$ is injective, then we have

$$(r \circ f)^\top = f^\top \circ r^\top = \mathrm{id}_{E^*},$$

which implies that $f^\top$ is surjective, and if $f$ is surjective, then we have

$$(f \circ s)^\top = s^\top \circ f^\top = \mathrm{id}_{F^*},$$

which implies that $f^\top$ is injective. ☐

The following proposition gives a natural interpretation of the dual $(E/U)^*$ of a quotient space $E/U$.

**Proposition 11.9.** *For any subspace $U$ of a vector space $E$, if $p\colon E \to E/U$ is the canonical surjection onto $E/U$, then $p^\top$ is injective and*

$$\mathrm{Im}(p^\top) = U^0 = (\mathrm{Ker}\,(p))^0.$$

*Therefore, $p^\top$ is a linear isomorphism between $(E/U)^*$ and $U^0$.*

*Proof.* Since $p$ is surjective, by Proposition 11.8, the map $p^\top$ is injective. Obviously, $U = \mathrm{Ker}\,(p)$. Observe that $\mathrm{Im}(p^\top)$ consists of all linear forms $\psi \in E^*$ such that $\psi = \varphi \circ p$ for some $\varphi \in (E/U)^*$, and since $\mathrm{Ker}\,(p) = U$, we have $U \subseteq \mathrm{Ker}\,(\psi)$. Conversely for any linear form $\psi \in E^*$, if $U \subseteq \mathrm{Ker}\,(\psi)$, then $\psi$ factors through $E/U$ as $\psi = \overline{\psi} \circ p$ as shown in the following commutative diagram

$$\begin{tikzcd} E \arrow[r, "p"] \arrow[dr, "\psi"'] & E/U \arrow[d, "\overline{\psi}"] \\ & K, \end{tikzcd}$$

where $\overline{\psi}\colon E/U \to K$ is given by

$$\overline{\psi}(\overline{v}) = \psi(v), \quad v \in E,$$

where $\overline{v} \in E/U$ denotes the equivalence class of $v \in E$. The map $\overline{\psi}$ does not depend on the representative chosen in the equivalence class $\overline{v}$, since if $\overline{v'} = \overline{v}$, that is $v' - v = u \in U$, then $\psi(v') = \psi(v + u) = \psi(v) + \psi(u) = \psi(v) + 0 = \psi(v)$. Therefore, we have

$$\begin{aligned} \mathrm{Im}(p^\top) &= \{\varphi \circ p \mid \varphi \in (E/U)^*\} \\ &= \{\psi\colon E \to K \mid U \subseteq \mathrm{Ker}\,(\psi)\} \\ &= U^0, \end{aligned}$$

which proves our result. ☐

Proposition 11.9 yields another proof of part (b) of the duality theorem (theorem 11.4) that does not involve the existence of bases (in infinite dimension).

**Proposition 11.10.** *For any vector space $E$ and any subspace $V$ of $E$, we have $V^{00} = V$.*

*Proof.* We begin by observing that $V^0 = V^{000}$. This is because, for any subspace $U$ of $E^*$, we have $U \subseteq U^{00}$, so $V^0 \subseteq V^{000}$. Furthermore, $V \subseteq V^{00}$ holds, and for any two subspaces $M, N$ of $E$, if $M \subseteq N$ then $N^0 \subseteq N^0$, so we get $V^{000} \subseteq V^0$. Write $V_1 = V^{00}$, so that $V_1^0 = V^{000} = V^0$. We wish to prove that $V_1 = V$.

Since $V \subseteq V_1 = V^{00}$, the canonical projection $p_1 \colon E \to E/V_1$ factors as $p_1 = f \circ p$ as in the diagram below,

$$\begin{tikzcd} E \arrow[r, "p"] \arrow[dr, "p_1"'] & E/V \arrow[d, "f"] \\ & E/V_1 \end{tikzcd}$$

where $p \colon E \to E/V$ is the canonical projection onto $E/V$ and $f \colon E/V \to E/V_1$ is the quotient map induced by $p_1$, with $f(\overline{u}_{E/V}) = p_1(u) = \overline{u}_{E/V_1}$, for all $u \in E$ (since $V \subseteq V_1$, if $u - u' = v \in V$, then $u - u' = v \in V_1$, so $p_1(u) = p_1(u')$). Since $p_1$ is surjective, so is $f$. We wish to prove that $f$ is actually an isomorphism, and for this, it is enough to show that $f$ is injective. By transposing all the maps, we get the commutative diagram

$$\begin{tikzcd} E^* & (E/V)^* \arrow[l, "p^\top"'] \\ & (E/V_1)^*, \arrow[ul, "p_1^\top"] \arrow[u, "f^\top"'] \end{tikzcd}$$

but by Proposition 11.9, the maps $p^\top \colon (E/V)^* \to V^0$ and $p_1^\top \colon (E/V_1)^* \to V_1^0$ are isomorphism, and since $V^0 = V_1^0$, we have the following diagram where both $p^\top$ and $p_1^\top$ are isomorphisms:

$$\begin{tikzcd} V^0 & (E/V)^* \arrow[l, "p^\top"'] \\ & (E/V_1)^*. \arrow[ul, "p_1^\top"] \arrow[u, "f^\top"'] \end{tikzcd}$$

Therefore, $f^\top = (p^\top)^{-1} \circ p_1^\top$ is an isomorphism. We claim that this implies that $f$ is injective.

If $f$ is not injective, then there is some $x \in E/V$ such that $x \neq 0$ and $f(x) = 0$, so for every $\varphi \in (E/V_1)^*$, we have $f^\top(\varphi)(x) = \varphi(f(x)) = 0$. However, there is linear form $\psi \in (E/V)^*$ such that $\psi(x) = 1$, so $\psi \neq f^\top(\varphi)$ for all $\varphi \in (E/V_1)^*$, contradicting the fact that $f^\top$ is surjective. To find such a linear form $\psi$, pick any supplement $W$ of $Kx$ in $E/V$, so that $E/V = Kx \oplus W$ ($W$ is a hyperplane in $E/V$ not containing $x$), and define $\psi$ to be zero on $W$ and 1 on $x$.[1] Therefore, $f$ is injective, and since we already know that it is surjective, it is bijective. This means that the canonical map $f\colon E/V \to E/V_1$ with $V \subseteq V_1$ is an isomorphism, which implies that $V = V_1 = V^{00}$ (otherwise, if $v \in V_1 - V$, then $p_1(v) = 0$, so $f(p(v)) = p_1(v) = 0$, but $p(v) \neq 0$ since $v \notin V$, and $f$ is not injective). ☐

The following proposition shows the relationship between orthogonality and transposition.

**Proposition 11.11.** *Given a linear map $f\colon E \to F$, for any subspace $V$ of $E$, we have*

$$f(V)^0 = (f^\top)^{-1}(V^0) = \{w^* \in F^* \mid f^\top(w^*) \in V^0\}.$$

*As a consequence,*

$$\mathrm{Ker}\, f^\top = (\mathrm{Im}\, f)^0.$$

*We also have*

$$\mathrm{Ker}\, f = (\mathrm{Im}\, f^\top)^0.$$

*Proof.* We have

$$\langle w^*, f(v)\rangle = \langle f^\top(w^*), v\rangle,$$

for all $v \in E$ and all $w^* \in F^*$, and thus, we have $\langle w^*, f(v)\rangle = 0$ for every $v \in V$, i.e. $w^* \in f(V)^0$ iff $\langle f^\top(w^*), v\rangle = 0$ for every $v \in V$ iff $f^\top(w^*) \in V^0$, i.e. $w^* \in (f^\top)^{-1}(V^0)$, proving that

$$f(V)^0 = (f^\top)^{-1}(V^0).$$

Since we already observed that $E^0 = (0)$, letting $V = E$ in the above identity we obtain that

$$\mathrm{Ker}\, f^\top = (\mathrm{Im}\, f)^0.$$

From the equation

$$\langle w^*, f(v)\rangle = \langle f^\top(w^*), v\rangle,$$

we deduce that $v \in (\mathrm{Im}\, f^\top)^0$ iff $\langle f^\top(w^*), v\rangle = 0$ for all $w^* \in F^*$ iff $\langle w^*, f(v)\rangle = 0$ for all $w^* \in F^*$. Assume that $v \in (\mathrm{Im}\, f^\top)^0$. If we pick a basis $(w_i)_{i\in I}$ of $F$, then we have the linear forms $w_i^*\colon F \to K$ such that $w_i^*(w_j) = \delta_{ij}$, and since we must have $\langle w_i^*, f(v)\rangle = 0$ for all $i \in I$ and $(w_i)_{i\in I}$ is a basis of $F$, we conclude that $f(v) = 0$, and thus $v \in \mathrm{Ker}\, f$ (this is because $\langle w_i^*, f(v)\rangle$ is the coefficient of $f(v)$ associated with the basis vector $w_i$). Conversely, if $v \in \mathrm{Ker}\, f$, then $\langle w^*, f(v)\rangle = 0$ for all $w^* \in F^*$, so we conclude that $v \in (\mathrm{Im}\, f^\top)^0$. Therefore, $v \in (\mathrm{Im}\, f^\top)^0$ iff $v \in \mathrm{Ker}\, f$; that is,

$$\mathrm{Ker}\, f = (\mathrm{Im}\, f^\top)^0,$$

as claimed. ☐

[1] Using Zorn's lemma, we pick $W$ maximal among all subspaces of $E/V$ such that $Kx \cap W = (0)$; then, $E/V = Kx \oplus W$.

The following theorem shows the relationship between the rank of $f$ and the rank of $f^\top$.

**Theorem 11.12.** *Given a linear map $f\colon E \to F$, the following properties hold.*

*(a) The dual $(\operatorname{Im} f)^*$ of $\operatorname{Im} f$ is isomorphic to $\operatorname{Im} f^\top = f^\top(F^*)$; that is,*

$$(\operatorname{Im} f)^* \cong \operatorname{Im} f^\top.$$

*(b) If $F$ is finite dimensional, then $\operatorname{rk}(f) = \operatorname{rk}(f^\top)$.*

*Proof.* (a) Consider the linear maps

$$E \xrightarrow{p} \operatorname{Im} f \xrightarrow{j} F,$$

where $E \xrightarrow{p} \operatorname{Im} f$ is the surjective map induced by $E \xrightarrow{f} F$, and $\operatorname{Im} f \xrightarrow{j} F$ is the injective inclusion map of $\operatorname{Im} f$ into $F$. By definition, $f = j \circ p$. To simplify the notation, let $I = \operatorname{Im} f$. By Proposition 11.8, since $E \xrightarrow{p} I$ is surjective, $I^* \xrightarrow{p^\top} E^*$ is injective, and since $\operatorname{Im} f \xrightarrow{j} F$ is injective, $F^* \xrightarrow{j^\top} I^*$ is surjective. Since $f = j \circ p$, we also have

$$f^\top = (j \circ p)^\top = p^\top \circ j^\top,$$

and since $F^* \xrightarrow{j^\top} I^*$ is surjective, and $I^* \xrightarrow{p^\top} E^*$ is injective, we have an isomorphism between $(\operatorname{Im} f)^*$ and $f^\top(F^*)$.

(b) We already noted that Part (a) of Theorem 11.4 shows that $\dim(F) = \dim(F^*)$, for every vector space $F$ of finite dimension. Consequently, $\dim(\operatorname{Im} f) = \dim((\operatorname{Im} f)^*)$, and thus, by Part (a) we have $\operatorname{rk}(f) = \operatorname{rk}(f^\top)$.

**Remark:** When both $E$ and $F$ are finite-dimensional, there is also a simple proof of (b) that doesn't use the result of Part (a). By Theorem 11.4(c)

$$\dim(\operatorname{Im} f) + \dim((\operatorname{Im} f)^0) = \dim(F),$$

and by Theorem 6.16

$$\dim(\operatorname{Ker} f^\top) + \dim(\operatorname{Im} f^\top) = \dim(F^*).$$

Furthermore, by Proposition 11.11, we have

$$\operatorname{Ker} f^\top = (\operatorname{Im} f)^0,$$

and since $F$ is finite-dimensional $\dim(F) = \dim(F^*)$, so we deduce

$$\dim(\operatorname{Im} f) + \dim((\operatorname{Im} f)^0) = \dim((\operatorname{Im} f)^0) + \dim(\operatorname{Im} f^\top),$$

which yields $\dim(\operatorname{Im} f) = \dim(\operatorname{Im} f^\top)$; that is, $\operatorname{rk}(f) = \operatorname{rk}(f^\top)$. $\square$

**Proposition 11.13.** *If $f\colon E \to F$ is any linear map, then the following identities hold:*

$$\begin{aligned} \operatorname{Im} f^\top &= (\operatorname{Ker}(f))^0 \\ \operatorname{Ker}(f^\top) &= (\operatorname{Im} f)^0 \\ \operatorname{Im} f &= (\operatorname{Ker}(f^\top))^0 \\ \operatorname{Ker}(f) &= (\operatorname{Im} f^\top)^0. \end{aligned}$$

*Proof.* The equation $\operatorname{Ker}(f^\top) = (\operatorname{Im} f)^0$ has already been proven in Proposition 11.11.

By the duality theorem $(\operatorname{Ker}(f))^{00} = \operatorname{Ker}(f)$, so from $\operatorname{Im} f^\top = (\operatorname{Ker}(f))^0$ we get $\operatorname{Ker}(f) = (\operatorname{Im} f^\top)^0$. Similarly, $(\operatorname{Im} f)^{00} = \operatorname{Im} f$, so from $\operatorname{Ker}(f^\top) = (\operatorname{Im} f)^0$ we get $\operatorname{Im} f = (\operatorname{Ker}(f^\top))^0$. Therefore, what is left to be proven is that $\operatorname{Im} f^\top = (\operatorname{Ker}(f))^0$.

Let $p\colon E \to E/\operatorname{Ker}(f)$ be the canonical surjection, $\overline{f}\colon E/\operatorname{Ker}(f) \to \operatorname{Im} f$ be the isomorphism induced by $f$, and $j\colon \operatorname{Im} f \to F$ be the inclusion map. Then, we have

$$f = j \circ \overline{f} \circ p,$$

which implies that

$$f^\top = p^\top \circ \overline{f}^\top \circ j^\top.$$

Since $p$ is surjective, $p^\top$ is injective, since $j$ is injective, $j^\top$ is surjective, and since $\overline{f}$ is bijective, $\overline{f}^\top$ is also bijective. It follows that $(E/\operatorname{Ker}(f))^* = \operatorname{Im}(\overline{f}^\top \circ j^\top)$, and we have

$$\operatorname{Im} f^\top = \operatorname{Im} p^\top.$$

Since $p\colon E \to E/\operatorname{Ker}(f)$ is the canonical surjection, by Proposition 11.9 applied to $U = \operatorname{Ker}(f)$, we get

$$\operatorname{Im} f^\top = \operatorname{Im} p^\top = (\operatorname{Ker}(f))^0,$$

as claimed. $\square$

In summary, the equation

$$\operatorname{Im} f^\top = (\operatorname{Ker}(f))^0$$

applies in any dimension, and it implies that

$$\operatorname{Ker}(f) = (\operatorname{Im} f^\top)^0.$$

The following proposition shows the relationship between the matrix representing a linear map $f\colon E \to F$ and the matrix representing its transpose $f^\top\colon F^* \to E^*$.

**Proposition 11.14.** *Let $E$ and $F$ be two vector spaces, and let $(u_1, \ldots, u_n)$ be a basis for $E$ and $(v_1, \ldots, v_m)$ be a basis for $F$. Given any linear map $f\colon E \to F$, if $M(f)$ is the $m \times n$-matrix representing $f$ w.r.t. the bases $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_m)$, then the $n \times m$-matrix $M(f^\top)$ representing $f^\top\colon F^* \to E^*$ w.r.t. the dual bases $(v_1^*, \ldots, v_m^*)$ and $(u_1^*, \ldots, u_n^*)$ is the transpose $M(f)^\top$ of $M(f)$.*

*Proof.* Recall that the entry $a_{i\,j}$ in row $i$ and column $j$ of $M(f)$ is the $i$-th coordinate of $f(u_j)$ over the basis $(v_1, \ldots, v_m)$. By definition of $v_i^*$, we have $\langle v_i^*, f(u_j)\rangle = a_{i\,j}$. The entry $a_{j\,i}^\top$ in row $j$ and column $i$ of $M(f^\top)$ is the $j$-th coordinate of

$$f^\top(v_i^*) = a_{1\,i}^\top u_1^* + \cdots + a_{j\,i}^\top u_j^* + \cdots + a_{n\,i}^\top u_n^*$$

over the basis $(u_1^*, \ldots, u_n^*)$, which is just $a_{j\,i}^\top = f^\top(v_i^*)(u_j) = \langle f^\top(v_i^*), u_j\rangle$. Since

$$\langle v_i^*, f(u_j)\rangle = \langle f^\top(v_i^*), u_j\rangle,$$

we have $a_{i\,j} = a_{j\,i}^\top$, proving that $M(f^\top) = M(f)^\top$. □

We now can give a very short proof of the fact that the rank of a matrix is equal to the rank of its transpose.

**Proposition 11.15.** *Given an $m \times n$ matrix $A$ over a field $K$, we have* $\mathrm{rk}(A) = \mathrm{rk}(A^\top)$.

*Proof.* The matrix $A$ corresponds to a linear map $f\colon K^n \to K^m$, and by Theorem 11.12, $\mathrm{rk}(f) = \mathrm{rk}(f^\top)$. By Proposition 11.14, the linear map $f^\top$ corresponds to $A^\top$. Since $\mathrm{rk}(A) = \mathrm{rk}(f)$, and $\mathrm{rk}(A^\top) = \mathrm{rk}(f^\top)$, we conclude that $\mathrm{rk}(A) = \mathrm{rk}(A^\top)$. □

Thus, given an $m \times n$-matrix $A$, the maximum number of linearly independent columns is equal to the maximum number of linearly independent rows. There are other ways of proving this fact that do not involve the dual space, but instead some elementary transformations on rows and columns.

Proposition 11.15 immediately yields the following criterion for determining the rank of a matrix:

**Proposition 11.16.** *Given any $m \times n$ matrix $A$ over a field $K$ (typically $K = \mathbb{R}$ or $K = \mathbb{C}$), the rank of $A$ is the maximum natural number $r$ such that there is an invertible $r \times r$ submatrix of $A$ obtained by selecting $r$ rows and $r$ columns of $A$.*

For example, the $3 \times 2$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

has rank 2 iff one of the three $2 \times 2$ matrices

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{pmatrix} \quad \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

is invertible.

If we combine Proposition 7.11 with Proposition 11.16, we obtain the following criterion for finding the rank of a matrix.

**Proposition 11.17.** *Given any $m \times n$ matrix $A$ over a field $K$ (typically $K = \mathbb{R}$ or $K = \mathbb{C}$), the rank of $A$ is the maximum natural number $r$ such that there is an $r \times r$ submatrix $B$ of $A$ obtained by selecting $r$ rows and $r$ columns of $A$, such that* $\det(B) \neq 0$.

This is not a very efficient way of finding the rank of a matrix. We will see that there are better ways using various decompositions such as LU, QR, or SVD.

## 11.7 Properties of the Double Transpose

First we have the following property showing the naturality of the eval map.

**Proposition 11.18.** *For any linear map $f\colon E \to F$, we have*

$$f^{\top\top} \circ \mathrm{eval}_E = \mathrm{eval}_F \circ f,$$

*or equivalently the following diagram commutes:*

$$\begin{tikzcd}
E^{**} \arrow[r, "f^{\top\top}"] & F^{**} \\
E \arrow[u, "\mathrm{eval}_E"] \arrow[r, "f"'] & F. \arrow[u, "\mathrm{eval}_F"']
\end{tikzcd}$$

*Proof.* For every $u \in E$ and every $\varphi \in F^*$, we have

$$\begin{aligned}
(f^{\top\top} \circ \mathrm{eval}_E)(u)(\varphi) &= \langle f^{\top\top}(\mathrm{eval}_E(u)), \varphi\rangle \\
&= \langle \mathrm{eval}_E(u), f^{\top}(\varphi)\rangle \\
&= \langle f^{\top}(\varphi), u\rangle \\
&= \langle \varphi, f(u)\rangle \\
&= \langle \mathrm{eval}_F(f(u)), \varphi\rangle \\
&= \langle (\mathrm{eval}_F \circ f)(u), \varphi\rangle \\
&= (\mathrm{eval}_F \circ f)(u)(\varphi),
\end{aligned}$$

which proves that $f^{\top\top} \circ \mathrm{eval}_E = \mathrm{eval}_F \circ f$, as claimed. ∎

If $E$ and $F$ are finite-dimensional, then $\mathrm{eval}_E$ and $\mathrm{eval}_F$ are isomorphisms, so Proposition 11.18 shows that

$$f^{\top\top} = \mathrm{eval}_F \circ f \circ \mathrm{eval}_E^{-1}. \tag{$*$}$$

The above equation is often interpreted as follows: if we identify $E$ with its bidual $E^{**}$ and $F$ with its bidual $F^{**}$, then $f^{\top\top} = f$. This is an abuse of notation; the rigorous statement is $(*)$.

As a corollary of Proposition 11.18, we obtain the following result.

**Proposition 11.19.** *If* $\dim(E)$ *is finite, then we have*

$$\mathrm{Ker}\,(f^{\top\top}) = \mathrm{eval}_E(\mathrm{Ker}\,(f)).$$

*Proof.* Indeed, if $E$ is finite-dimensional, the map $\mathrm{eval}_E \colon E \to E^{**}$ is an isomorphism, so every $\varphi \in E^{**}$ is of the form $\varphi = \mathrm{eval}_E(u)$ for some $u \in E$, the map $\mathrm{eval}_F \colon F \to F^{**}$ is injective, and we have

$$\begin{aligned}
f^{\top\top}(\varphi) = 0 \quad & \text{iff} \quad f^{\top\top}(\mathrm{eval}_E(u)) = 0 \\
& \text{iff} \quad \mathrm{eval}_F(f(u)) = 0 \\
& \text{iff} \quad f(u) = 0 \\
& \text{iff} \quad u \in \mathrm{Ker}\,(f) \\
& \text{iff} \quad \varphi \in \mathrm{eval}_E(\mathrm{Ker}\,(f)),
\end{aligned}$$

which proves that $\mathrm{Ker}\,(f^{\top\top}) = \mathrm{eval}_E(\mathrm{Ker}\,(f))$. ∎

**Remarks:** If $\dim(E)$ is finite, following an argument of Dan Guralnik, the fact that $\mathrm{rk}(f) = \mathrm{rk}(f^{\top})$ can be proven using Proposition 11.19.

*Proof.* We know from Proposition 11.11 applied to $f^{\top} \colon F^* \to E^*$ that

$$\mathrm{Ker}\,(f^{\top\top}) = (\mathrm{Im}\, f^{\top})^0,$$

and we showed in Proposition 11.19 that

$$\mathrm{Ker}\,(f^{\top\top}) = \mathrm{eval}_E(\mathrm{Ker}\,(f)).$$

It follows (since $\mathrm{eval}_E$ is an isomorphism) that

$$\dim((\mathrm{Im}\, f^{\top})^0) = \dim(\mathrm{Ker}\,(f^{\top\top})) = \dim(\mathrm{Ker}\,(f)) = \dim(E) - \dim(\mathrm{Im}\, f),$$

and since

$$\dim(\mathrm{Im}\, f^{\top}) + \dim((\mathrm{Im}\, f^{\top})^0) = \dim(E),$$

we get

$$\dim(\mathrm{Im}\, f^{\top}) = \dim(\mathrm{Im}\, f).$$ ∎

As indicated by Dan Guralnik, if $\dim(E)$ is finite, the above result can be used to prove the following result.

**Proposition 11.20.** *If* $\dim(E)$ *is finite, then for any linear map* $f \colon E \to F$, *we have*

$$\mathrm{Im}\, f^{\top} = (\mathrm{Ker}\,(f))^0.$$

*Proof.* From

$$\langle f^\top(\varphi), u\rangle = \langle \varphi, f(u)\rangle$$

for all $\varphi \in F^*$ and all $u \in E$, we see that if $u \in \mathrm{Ker}\,(f)$, then $\langle f^\top(\varphi), u\rangle = \langle \varphi, 0\rangle = 0$, which means that $f^\top(\varphi) \in (\mathrm{Ker}\,(f))^0$, and thus, $\mathrm{Im}\, f^\top \subseteq (\mathrm{Ker}\,(f))^0$. For the converse, since $\dim(E)$ is finite, we have

$$\dim((\mathrm{Ker}\,(f))^0) = \dim(E) - \dim(\mathrm{Ker}\,(f)) = \dim(\mathrm{Im}\, f),$$

but we just proved that $\dim(\mathrm{Im}\, f^\top) = \dim(\mathrm{Im}\, f)$, so we get

$$\dim((\mathrm{Ker}\,(f))^0) = \dim(\mathrm{Im}\, f^\top),$$

and since $\mathrm{Im}\, f^\top \subseteq (\mathrm{Ker}\,(f))^0$, we obtain

$$\mathrm{Im}\, f^\top = (\mathrm{Ker}\,(f))^0,$$

as claimed. ☐

**Remarks:**

1. By the duality theorem, since $(\mathrm{Ker}\,(f))^{00} = \mathrm{Ker}\,(f)$, the above equation yields another proof of the fact that
$$\mathrm{Ker}\,(f) = (\mathrm{Im}\, f^\top)^0,$$
when $E$ is finite-dimensional.

2. The equation
$$\mathrm{Im}\, f^\top = (\mathrm{Ker}\,(f))^0$$
is actually valid even if when $E$ if infinite-dimensional, but we will not prove this here.

## 11.8 The Four Fundamental Subspaces

Given a linear map $f\colon E \to F$ (where $E$ and $F$ are finite-dimensional), Proposition 11.11 revealed that the four spaces

$$\mathrm{Im}\, f,\ \mathrm{Im}\, f^\top,\ \mathrm{Ker}\, f,\ \mathrm{Ker}\, f^\top$$

play a special role. They are often called the *fundamental subspaces* associated with $f$. These spaces are related in an intimate manner, since Proposition 11.11 shows that

$$\begin{aligned}\mathrm{Ker}\, f &= (\mathrm{Im}\, f^\top)^0\\ \mathrm{Ker}\, f^\top &= (\mathrm{Im}\, f)^0,\end{aligned}$$

and Theorem 11.12 shows that

$$\mathrm{rk}(f) = \mathrm{rk}(f^\top).$$

It is instructive to translate these relations in terms of matrices (actually, certain linear algebra books make a big deal about this!). If $\dim(E) = n$ and $\dim(F) = m$, given any basis $(u_1, \ldots, u_n)$ of $E$ and a basis $(v_1, \ldots, v_m)$ of $F$, we know that $f$ is represented by an $m \times n$ matrix $A = (a_{i\,j})$, where the $j$th column of $A$ is equal to $f(u_j)$ over the basis $(v_1, \ldots, v_m)$. Furthermore, the transpose map $f^\top$ is represented by the $n \times m$ matrix $A^\top$ (with respect to the dual bases). Consequently, the four fundamental spaces

$$\mathrm{Im}\, f,\ \mathrm{Im}\, f^\top,\ \mathrm{Ker}\, f,\ \mathrm{Ker}\, f^\top$$

correspond to

(1) The *column space* of $A$, denoted by $\mathrm{Im}\, A$ or $\mathcal{R}(A)$; this is the subspace of $\mathbb{R}^m$ spanned by the columns of $A$, which corresponds to the image $\mathrm{Im}\, f$ of $f$.

(2) The *kernel* or *nullspace* of $A$, denoted by $\mathrm{Ker}\, A$ or $\mathcal{N}(A)$; this is the subspace of $\mathbb{R}^n$ consisting of all vectors $x \in \mathbb{R}^n$ such that $Ax = 0$.

(3) The *row space* of $A$, denoted by $\mathrm{Im}\, A^\top$ or $\mathcal{R}(A^\top)$; this is the subspace of $\mathbb{R}^n$ spanned by the rows of $A$, or equivalently, spanned by the columns of $A^\top$, which corresponds to the image $\mathrm{Im}\, f^\top$ of $f^\top$.

(4) The *left kernel* or *left nullspace* of $A$ denoted by $\mathrm{Ker}\, A^\top$ or $\mathcal{N}(A^\top)$; this is the kernel (nullspace) of $A^\top$, the subspace of $\mathbb{R}^m$ consisting of all vectors $y \in \mathbb{R}^m$ such that $A^\top y = 0$, or equivalently, $y^\top A = 0$.

Recall that the dimension $r$ of $\mathrm{Im}\, f$, which is also equal to the dimension of the column space $\mathrm{Im}\, A = \mathcal{R}(A)$, is the *rank* of $A$ (and $f$). Then, some our previous results can be reformulated as follows:

1. The column space $\mathcal{R}(A)$ of $A$ has dimension $r$.
2. The nullspace $\mathcal{N}(A)$ of $A$ has dimension $n - r$.
3. The row space $\mathcal{R}(A^\top)$ has dimension $r$.
4. The left nullspace $\mathcal{N}(A^\top)$ of $A$ has dimension $m - r$.

The above statements constitute what Strang calls the *Fundamental Theorem of Linear Algebra, Part I* (see Strang [170]).

The two statements

$$\begin{aligned}\mathrm{Ker}\, f &= (\mathrm{Im}\, f^\top)^0\\ \mathrm{Ker}\, f^\top &= (\mathrm{Im}\, f)^0\end{aligned}$$

translate to

(1) The nullspace of $A$ is the orthogonal of the row space of $A$.

(2) The left nullspace of $A$ is the orthogonal of the column space of $A$.

The above statements constitute what Strang calls the *Fundamental Theorem of Linear Algebra, Part II* (see Strang [170]).

Since vectors are represented by column vectors and linear forms by row vectors (over a basis in $E$ or $F$), a vector $x \in \mathbb{R}^n$ is orthogonal to a linear form $y$ iff

$$yx = 0.$$

Then, a vector $x \in \mathbb{R}^n$ is orthogonal to the row space of $A$ iff $x$ is orthogonal to every row of $A$, namely $Ax = 0$, which is equivalent to the fact that $x$ belong to the nullspace of $A$. Similarly, the column vector $y \in \mathbb{R}^m$ (representing a linear form over the dual basis of $F^*$) belongs to the nullspace of $A^\top$ iff $A^\top y = 0$, iff $y^\top A = 0$, which means that the linear form given by $y^\top$ (over the basis in $F$) is orthogonal to the column space of $A$.

Since (2) is equivalent to the fact that the column space of $A$ is equal to the orthogonal of the left nullspace of $A$, we get the following criterion for the solvability of an equation of the form $Ax = b$:

*The equation $Ax = b$ has a solution iff for all $y \in \mathbb{R}^m$, if $A^\top y = 0$, then $y^\top b = 0$.*

Indeed, the condition on the right-hand side says that $b$ is orthogonal to the left nullspace of $A$; that is, $b$ belongs to the column space of $A$.

This criterion can be cheaper to check that checking directly that $b$ is spanned by the columns of $A$. For example, if we consider the system

$$\begin{aligned} x_1 - x_2 &= b_1 \\ x_2 - x_3 &= b_2 \\ x_3 - x_1 &= b_3 \end{aligned}$$

which, in matrix form, is written $Ax = b$ as below:

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},$$

we see that the rows of the matrix $A$ add up to 0. In fact, it is easy to convince ourselves that the left nullspace of $A$ is spanned by $y = (1, 1, 1)$, and so the system is solvable iff $y^\top b = 0$, namely

$$b_1 + b_2 + b_3 = 0.$$

Note that the above criterion can also be stated negatively as follows:

*The equation $Ax = b$ has no solution iff there is some $y \in \mathbb{R}^m$ such that $A^\top y = 0$ and $y^\top b \neq 0$.*

Since $A^\top y = 0$ iff $y^\top A = 0$, we can view $y^\top$ as a row vector representing a linear form, and $y^\top A = 0$ asserts that the linear form $y^\top$ vanishes on the columns $A^1, \ldots, A^n$ of $A$ but does not vanish on $b$. Since the linear form $y^\top$ defines the hyperplane $H$ of equation $y^\top z = 0$ (with $z \in \mathbb{R}^m$), geometrically the equation $Ax = b$ has no solution iff there is a hyperplane $H$ containing $A^1, \ldots, A^n$ and not containing $b$.

## 11.9 Summary

The main concepts and results of this chapter are listed below:

- The *dual space* $E^*$ and *linear forms* (*covector*). The *bidual* $E^{**}$.
- The *bilinear pairing* $\langle -, - \rangle \colon E^* \times E \to K$ (the *canonical pairing*).
- *Evaluation at* $v$: $\mathrm{eval}_v \colon E^* \to K$.
- The map $\mathrm{eval}_E \colon E \to E^{**}$.
- *Othogonality* between a subspace $V$ of $E$ and a subspace $U$ of $E^*$; the *orthogonal* $V^0$ and the *orthogonal* $U^0$.
- *Coordinate forms*.
- The *Duality theorem* (Theorem 11.4).
- The *dual basis* of a basis.
- The isomorphism $\mathrm{eval}_E \colon E \to E^{**}$ when $\dim(E)$ is finite.
- *Pairing* between two vector spaces; *nondegenerate pairing*; Proposition 11.6.
- Hyperplanes and linear forms.
- The *transpose* $f^\top \colon F^* \to E^*$ of a linear map $f \colon E \to F$.
- The fundamental identities:

  $$\mathrm{Ker}\, f^\top = (\mathrm{Im}\, f)^0 \quad \text{and} \quad \mathrm{Ker}\, f = (\mathrm{Im}\, f^\top)^0$$

  (Proposition 11.11).
- If $F$ is finite-dimensional, then

  $$\mathrm{rk}(f) = \mathrm{rk}(f^\top).$$

  (Theorem 11.12).

- The matrix of the transpose map $f^\top$ is equal to the transpose of the matrix of the map $f$ (Proposition 11.14).
- For any $m \times n$ matrix $A$,
$$\mathrm{rk}(A) = \mathrm{rk}(A^\top).$$
- Characterization of the rank of a matrix in terms of a maximal invertible submatrix (Proposition 11.16).
- The *four fundamental subspaces*:
$$\mathrm{Im}\, f,\ \mathrm{Im}\, f^\top,\ \mathrm{Ker}\, f,\ \mathrm{Ker}\, f^\top.$$
- The *column space*, the *nullspace*, the *row space*, and the *left nullspace* (of a matrix).
- Criterion for the solvability of an equation of the form $Ax = b$ in terms of the left nullspace.

## 11.10 Problems

**Problem 11.1.** Prove the following properties of transposition:

$$\begin{aligned}(f+g)^\top &= f^\top + g^\top \\ (g \circ f)^\top &= f^\top \circ g^\top \\ \mathrm{id}_E^\top &= \mathrm{id}_{E^*}.\end{aligned}$$

**Problem 11.2.** Let $(u_1, \ldots, u_{n-1})$ be $n-1$ linearly independent vectors $u_i \in \mathbb{C}^n$. Prove that the hyperlane $H$ spanned by $(u_1, \ldots, u_{n-1})$ is the nullspace of the linear form

$$x \mapsto \det(u_1, \ldots, u_{n-1}, x), \quad x \in \mathbb{C}^n.$$

Prove that if $A$ is the $n \times n$ matrix whose columns are $(u_1, \ldots, u_{n-1}, x)$, and if $c_i = (-1)^{i+n} \det(A_{in})$ is the cofactor of $a_{in} = x_i$ for $i = 1, \ldots, n$, then $H$ is defined by the equation

$$c_1 x_1 + \cdots + c_n x_n = 0.$$

**Problem 11.3.** (1) Let $\varphi \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the map defined by

$$\varphi((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = x_1 y_1 + \cdots + x_n y_n.$$

Prove that $\varphi$ is a bilinear nondegenerate pairing. Deduce that $(\mathbb{R}^n)^*$ is isomorphic to $\mathbb{R}^n$.

Prove that $\varphi(x, x) = 0$ iff $x = 0$.