of time series into a new time series. It accepts as input a sequence such as $\{x_t\}_{t=-\infty}^{\infty}$ or a group of sequences such as $(\{x_t\}_{t=-\infty}^{\infty}, \{w_t\}_{t=-\infty}^{\infty})$ and has as output a new sequence $\{y_t\}_{t=-\infty}^{\infty}$. Again, the operator is summarized by describing the value of a typical element of $\{y_t\}_{t=-\infty}^{\infty}$ in terms of the corresponding elements of $\{x_t\}_{t=-\infty}^{\infty}$.

An example of a time series operator is the multiplication operator, represented as

$$y_t = \beta x_t. \tag{2.1.1}$$

Although it is written exactly the same way as simple scalar multiplication, equation [2.1.1] is actually shorthand for an infinite sequence of multiplications, one for each date t. The operator multiplies the value x takes on at any date t by some constant β to generate the value of y for that date.

Another example of a time series operator is the addition operator:

$$y_t = x_t + w_t.$$

Here the value of y at any date t is the sum of the values that x and w take on for that date.

Since the multiplication or addition operators amount to element-by-element multiplication or addition, they obey all the standard rules of algebra. For example, if we multiply each observation of $\{x_t\}_{t=-\infty}^{\infty}$ by β and each observation of $\{w_t\}_{t=-\infty}^{\infty}$ by β and add the results,

$$\beta x_t + \beta w_t,$$

the outcome is the same as if we had first added $\{x_t\}_{t=-\infty}^{\infty}$ to $\{w_t\}_{t=-\infty}^{\infty}$ and then multiplied each element of the resulting series by β:

$$\beta(x_t + w_t).$$

A highly useful operator is the lag operator. Suppose that we start with a sequence $\{x_t\}_{t=-\infty}^{\infty}$ and generate a new sequence $\{y_t\}_{t=-\infty}^{\infty}$, where the value of y for date t is equal to the value x took on at date $t - 1$:

$$y_t = x_{t-1}. \tag{2.1.2}$$

This is described as applying the *lag operator* to $\{x_t\}_{t=-\infty}^{\infty}$. The operation is represented by the symbol L:

$$Lx_t \equiv x_{t-1}. \tag{2.1.3}$$

Consider the result of applying the lag operator twice to a series:

$$L(Lx_t) = L(x_{t-1}) = x_{t-2}.$$

Such a double application of the lag operator is indicated by "L^2":

$$L^2 x_t = x_{t-2}.$$

In general, for any integer k,

$$L^k x_t = x_{t-k}. \tag{2.1.4}$$

Notice that if we first apply the multiplication operator and then the lag operator, as in

$$x_t \to \beta x_t \to \beta x_{t-1},$$

the result will be exactly the same as if we had applied the lag operator first and then the multiplication operator:

$$x_t \to x_{t-1} \to \beta x_{t-1}.$$

Thus the lag operator and multiplication operator are commutative:

$$L(\beta x_t) = \beta \cdot L x_t.$$

Similarly, if we first add two series and then apply the lag operator to the result,

$$(x_t, w_t) \rightarrow x_t + w_t \rightarrow x_{t-1} + w_{t-1},$$

the result is the same as if we had applied the lag operator before adding:

$$(x_t, w_t) \rightarrow (x_{t-1}, w_{t-1}) \rightarrow x_{t-1} + w_{t-1}.$$

Thus, the lag operator is distributive over the addition operator:

$$L(x_t + w_t) = Lx_t + Lw_t.$$

We thus see that the lag operator follows exactly the same algebraic rules as the multiplication operator. For this reason, it is tempting to use the expression "multiply y_t by L" rather than "operate on $\{y_t\}_{t=-\infty}^{\infty}$ by L." Although the latter expression is technically more correct, this text will often use the former shorthand expression to facilitate the exposition.

Faced with a time series defined in terms of compound operators, we are free to use the standard commutative, associative, and distributive algebraic laws for multiplication and addition to express the compound operator in an alternative form. For example, the process defined by

$$y_t = (a + bL)Lx_t$$

is exactly the same as

$$y_t = (aL + bL^2)x_t = ax_{t-1} + bx_{t-2}.$$

To take another example,

$$\begin{aligned}(1 - \lambda_1 L)(1 - \lambda_2 L)x_t &= (1 - \lambda_1 L - \lambda_2 L + \lambda_1\lambda_2 L^2)x_t \\ &= (1 - [\lambda_1 + \lambda_2]L + \lambda_1\lambda_2 L^2)x_t \qquad [2.1.5] \\ &= x_t - (\lambda_1 + \lambda_2)x_{t-1} + (\lambda_1\lambda_2)x_{t-2}.\end{aligned}$$

An expression such as $(aL + bL^2)$ is referred to as a *polynomial in the lag operator*. It is algebraically similar to a simple polynomial $(az + bz^2)$ where z is a scalar. The difference is that the simple polynomial $(az + bz^2)$ refers to a particular number, whereas a polynomial in the lag operator $(aL + bL^2)$ refers to an operator that would be applied to one time series $\{x_t\}_{t=-\infty}^{\infty}$ to produce a new time series $\{y_t\}_{t=-\infty}^{\infty}$.

Notice that if $\{x_t\}_{t=-\infty}^{\infty}$ is just a series of constants,

$$x_t = c \qquad \text{for all } t,$$

then the lag operator applied to x_t produces the same series of constants:

$$Lx_t = x_{t-1} = c.$$

Thus, for example,

$$(\alpha L + \beta L^2 + \gamma L^3)c = (\alpha + \beta + \gamma) \cdot c. \qquad [2.1.6]$$

2.2. *First-Order Difference Equations*

Let us now return to the first-order difference equation analyzed in Section 1.1:

$$y_t = \phi y_{t-1} + w_t. \qquad [2.2.1]$$

Equation [2.2.1] can be rewritten using the lag operator [2.1.3] as

$$y_t = \phi L y_t + w_t.$$

This equation, in turn, can be rearranged using standard algebra,

$$y_t - \phi L y_t = w_t,$$

or

$$(1 - \phi L) y_t = w_t. \qquad [2.2.2]$$

Next consider "multiplying" both sides of [2.2.2] by the following operator:

$$(1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots + \phi^t L^t). \qquad [2.2.3]$$

The result would be

$$\begin{aligned}(1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots + \phi^t L^t)(1 - \phi L) y_t \\ = (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots + \phi^t L^t) w_t.\end{aligned} \qquad [2.2.4]$$

Expanding out the compound operator on the left side of [2.2.4] results in

$$\begin{aligned}&(1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots + \phi^t L^t)(1 - \phi L) \\ &\quad = (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots + \phi^t L^t) \\ &\qquad - (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots + \phi^t L^t)\phi L \\ &\quad = (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots + \phi^t L^t) \\ &\qquad - (\phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots + \phi^t L^t + \phi^{t+1} L^{t+1}) \\ &\quad = (1 - \phi^{t+1} L^{t+1}).\end{aligned} \qquad [2.2.5]$$

Substituting [2.2.5] into [2.2.4] yields

$$(1 - \phi^{t+1} L^{t+1}) y_t = (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots + \phi^t L^t) w_t. \qquad [2.2.6]$$

Writing [2.2.6] out explicitly using [2.1.4] produces

$$y_t - \phi^{t+1} y_{t-(t+1)} = w_t + \phi w_{t-1} + \phi^2 w_{t-2} + \phi^3 w_{t-3} + \cdots + \phi^t w_{t-t}$$

or

$$y_t = \phi^{t+1} y_{-1} + w_t + \phi w_{t-1} + \phi^2 w_{t-2} + \phi^3 w_{t-3} + \cdots + \phi^t w_0. \qquad [2.2.7]$$

Notice that equation [2.2.7] is identical to equation [1.1.7]. Applying the operator [2.2.3] is performing exactly the same set of recursive substitutions that were employed in the previous chapter to arrive at [1.1.7].

It is interesting to reflect on the nature of the operator [2.2.3] as t becomes large. We saw in [2.2.5] that

$$(1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots + \phi^t L^t)(1 - \phi L) y_t = y_t - \phi^{t+1} y_{-1}.$$

That is, $(1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots + \phi^t L^t)(1 - \phi L) y_t$ differs from y_t by the term $\phi^{t+1} y_{-1}$. If $|\phi| < 1$ and if y_{-1} is a finite number, this residual $\phi^{t+1} y_{-1}$ will become negligible as t becomes large:

$$(1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots + \phi^t L^t)(1 - \phi L) y_t \cong y_t \qquad \text{for } t \text{ large.}$$

A sequence $\{y_t\}_{t=-\infty}^{\infty}$ is said to be *bounded* if there exists a finite number $\bar{y}$ such that

$$|y_t| < \bar{y} \qquad \text{for all } t.$$

Thus, when $|\phi| < 1$ and when we are considering applying an operator to a bounded sequence, we can think of

$$(1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots + \phi^j L^j)$$

as approximating the inverse of the operator $(1 - \phi L)$, with this approximation made arbitrarily accurate by choosing j sufficiently large:

$$(1 - \phi L)^{-1} = \lim_{j\to\infty} (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots + \phi^j L^j). \qquad [2.2.8]$$

This operator $(1 - \phi L)^{-1}$ has the property

$$(1 - \phi L)^{-1}(1 - \phi L) = 1,$$

where "1" denotes the identity operator:

$$1 y_t = y_t.$$

The following chapter discusses stochastic sequences rather than the deterministic sequences studied here. There we will speak of mean square convergence and stationary stochastic processes in place of limits of bounded deterministic sequences, though the practical meaning of [2.2.8] will be little changed.

Provided that $|\phi| < 1$ and we restrict ourselves to bounded sequences or stationary stochastic processes, both sides of [2.2.2] can be "divided" by $(1 - \phi L)$ to obtain

$$y_t = (1 - \phi L)^{-1} w_t$$

or

$$y_t = w_t + \phi w_{t-1} + \phi^2 w_{t-2} + \phi^3 w_{t-3} + \cdots. \qquad [2.2.9]$$

It should be emphasized that if we were not restricted to considering bounded sequences or stationary stochastic processes $\{w_t\}_{t=-\infty}^{\infty}$ and $\{y_t\}_{t=-\infty}^{\infty}$, then expression [2.2.9] would not be a necessary implication of [2.2.1]. Equation [2.2.9] is consistent with [2.2.1], but adding a term $a_0\phi^t$,

$$y_t = a_0\phi^t + w_t + \phi w_{t-1} + \phi^2 w_{t-2} + \phi^3 w_{t-3} + \cdots, \qquad [2.2.10]$$

produces another series consistent with [2.2.1] for any constant a_0. To verify that [2.2.10] is consistent with [2.2.1], multiply [2.2.10] by $(1 - \phi L)$:

$$\begin{aligned}(1 - \phi L)y_t &= (1 - \phi L)a_0\phi^t + (1 - \phi L)(1 - \phi L)^{-1}w_t \\ &= a_0\phi^t - \phi\cdot a_0\phi^{t-1} + w_t \\ &= w_t,\end{aligned}$$

so that [2.2.10] is consistent with [2.2.1] for any constant a_0.

Although any process of the form of [2.2.10] is consistent with the difference equation [2.2.1], notice that since $|\phi| < 1$,

$$|a_0\phi^t| \to \infty \quad \text{as} \quad t \to -\infty.$$

Thus, even if $\{w_t\}_{t=-\infty}^{\infty}$ is a bounded sequence, the solution $\{y_t\}_{t=-\infty}^{\infty}$ given by [2.2.10] is unbounded unless $a_0 = 0$ in [2.2.10]. Thus, there was a particular reason for defining the operator [2.2.8] to be the inverse of $(1 - \phi L)$—namely, $(1 - \phi L)^{-1}$ defined in [2.2.8] is the unique operator satisfying

$$(1 - \phi L)^{-1}(1 - \phi L) = 1$$

that maps a bounded sequence $\{w_t\}_{t=-\infty}^{\infty}$ into a bounded sequence $\{y_t\}_{t=-\infty}^{\infty}$.

The nature of $(1 - \phi L)^{-1}$ when $|\phi| \geq 1$ will be discussed in Section 2.5.

2.3. *Second-Order Difference Equations*

Consider next a second-order difference equation:

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + w_t. \qquad [2.3.1]$$

Rewriting this in lag operator form produces

$$(1 - \phi_1 L - \phi_2 L^2)y_t = w_t. \tag{2.3.2}$$

The left side of [2.3.2] contains a second-order polynomial in the lag operator L. Suppose we factor this polynomial, that is, find numbers λ_1 and λ_2 such that

$$(1 - \phi_1 L - \phi_2 L^2) = (1 - \lambda_1 L)(1 - \lambda_2 L) = (1 - [\lambda_1 + \lambda_2]L + \lambda_1\lambda_2 L^2). \tag{2.3.3}$$

This is just the operation in [2.1.5] in reverse. Given values for ϕ_1 and ϕ_2, we seek numbers λ_1 and λ_2 with the properties that

$$\lambda_1 + \lambda_2 = \phi_1$$

and

$$\lambda_1\lambda_2 = -\phi_2.$$

For example, if $\phi_1 = 0.6$ and $\phi_2 = -0.08$, then we should choose $\lambda_1 = 0.4$ and $\lambda_2 = 0.2$:

$$(1 - 0.6L + 0.08L^2) = (1 - 0.4L)(1 - 0.2L). \tag{2.3.4}$$

It is easy enough to see that these values of λ_1 and λ_2 work for this numerical example, but how are λ_1 and λ_2 found in general? The task is to choose λ_1 and λ_2 so as to make sure that the operator on the right side of [2.3.3] is identical to that on the left side. This will be true whenever the following represent the identical functions of z:

$$(1 - \phi_1 z - \phi_2 z^2) = (1 - \lambda_1 z)(1 - \lambda_2 z). \tag{2.3.5}$$

This equation simply replaces the lag operator L in [2.3.3] with a scalar z. What is the point of doing so? With [2.3.5], we can now ask, For what values of z is the right side of [2.3.5] equal to zero? The answer is, if either $z = \lambda_1^{-1}$ or $z = \lambda_2^{-1}$, then the right side of [2.3.5] would be zero. It would not have made sense to ask an analogous question of [2.3.3]—L denotes a particular operator, not a number, and $L = \lambda_1^{-1}$ is not a sensible statement.

Why should we care that the right side of [2.3.5] is zero if $z = \lambda_1^{-1}$ or if $z = \lambda_2^{-1}$? Recall that the goal was to choose λ_1 and λ_2 so that the two sides of [2.3.5] represented the identical polynomial in z. This means that for any particular value z the two functions must produce the same number. If we find a value of z that sets the right side to zero, that same value of z must set the left side to zero as well. But the values of z that set the left side to zero,

$$(1 - \phi_1 z - \phi_2 z^2) = 0, \tag{2.3.6}$$

are given by the quadratic formula:

$$z_1 = \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_2} \tag{2.3.7}$$

$$z_2 = \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_2}. \tag{2.3.8}$$

Setting $z = z_1$ or z_2 makes the left side of [2.3.5] zero, while $z = \lambda_1^{-1}$ or λ_2^{-1} sets the right side of [2.3.5] to zero. Thus

$$\lambda_1^{-1} = z_1 \tag{2.3.9}$$

$$\lambda_2^{-1} = z_2. \tag{2.3.10}$$

Returning to the numerical example [2.3.4] in which $\phi_1 = 0.6$ and $\phi_2 = -0.08$, we would calculate

$$z_1 = \frac{0.6 - \sqrt{(0.6)^2 - 4(0.08)}}{2(0.08)} = 2.5$$
$$z_2 = \frac{0.6 + \sqrt{(0.6)^2 - 4(0.08)}}{2(0.08)} = 5.0,$$

and so

$$\lambda_1 = 1/(2.5) = 0.4$$
$$\lambda_2 = 1/(5.0) = 0.2,$$

as was found in [2.3.4].

When $\phi_1^2 + 4\phi_2 < 0$, the values z_1 and z_2 are complex conjugates, and their reciprocals λ_1 and λ_2 can be found by first writing the complex number in polar coordinate form. Specifically, write

$$z_1 = a + bi$$

as

$$z_1 = R\cdot[\cos(\theta) + i\cdot\sin(\theta)] = R\cdot e^{i\theta}.$$

Then

$$z_1^{-1} = R^{-1}\cdot e^{-i\theta} = R^{-1}\cdot[\cos(\theta) - i\cdot\sin(\theta)].$$

Actually, there is a more direct method for calculating the values of λ_1 and λ_2 from ϕ_1 and ϕ_2. Divide both sides of [2.3.5] by z^2:

$$(z^{-2} - \phi_1 z^{-1} - \phi_2) = (z^{-1} - \lambda_1)(z^{-1} - \lambda_2) \qquad [2.3.11]$$

and define λ to be the variable z^{-1}:

$$\lambda \equiv z^{-1}. \qquad [2.3.12]$$

Substituting [2.3.12] into [2.3.11] produces

$$(\lambda^2 - \phi_1\lambda - \phi_2) = (\lambda - \lambda_1)(\lambda - \lambda_2). \qquad [2.3.13]$$

Again, [2.3.13] must hold for all values of λ in order for the two sides of [2.3.5] to represent the same polynomial. The values of λ that set the right side to zero are $\lambda = \lambda_1$ and $\lambda = \lambda_2$. These same values must set the left side of [2.3.13] to zero as well:

$$(\lambda^2 - \phi_1\lambda - \phi_2) = 0. \qquad [2.3.14]$$

Thus, to calculate the values of λ_1 and λ_2 that factor the polynomial in [2.3.3], we can find the roots of [2.3.14] directly from the quadratic formula:

$$\lambda_1 = \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2} \qquad [2.3.15]$$

$$\lambda_2 = \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2}. \qquad [2.3.16]$$

For the example of [2.3.4], we would thus calculate

$$\lambda_1 = \frac{0.6 + \sqrt{(0.6)^2 - 4(0.08)}}{2} = 0.4$$
$$\lambda_2 = \frac{0.6 - \sqrt{(0.6)^2 - 4(0.08)}}{2} = 0.2.$$

It is instructive to compare these results with those in Chapter 1. There the dynamics of the second-order difference equation [2.3.1] were summarized by calculating the eigenvalues of the matrix $\mathbf{F}$ given by

$$\mathbf{F} = \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix}. \qquad [2.3.17]$$

The eigenvalues of $\mathbf{F}$ were seen to be the two values of λ that satisfy equation [1.2.13]:

$$(\lambda^2 - \phi_1\lambda - \phi_2) = 0.$$

But this is the same calculation as in [2.3.14]. This finding is summarized in the following proposition.

Proposition 2.1: *Factoring the polynomial* $(1 - \phi_1 L - \phi_2 L^2)$ *as*

$$(1 - \phi_1 L - \phi_2 L^2) = (1 - \lambda_1 L)(1 - \lambda_2 L) \qquad [2.3.18]$$

is the same calculation as finding the eigenvalues of the matrix $\mathbf{F}$ *in* [2.3.17]. *The eigenvalues* λ_1 *and* λ_2 *of* $\mathbf{F}$ *are the same as the parameters* λ_1 *and* λ_2 *in* [2.3.18], *and are given by equations* [2.3.15] *and* [2.3.16].

The correspondence between calculating the eigenvalues of a matrix and factoring a polynomial in the lag operator is very instructive. However, it introduces one minor source of possible semantic confusion about which we have to be careful. Recall from Chapter 1 that the system [2.3.1] is stable if both λ_1 and λ_2 are less than 1 in modulus and explosive if either λ_1 or λ_2 is greater than 1 in modulus. Sometimes this is described as the requirement that the roots of

$$(\lambda^2 - \phi_1\lambda - \phi_2) = 0 \qquad [2.3.19]$$

lie inside the unit circle. The possible confusion is that it is often convenient to work directly with the polynomial in the form in which it appears in [2.3.2],

$$(1 - \phi_1 z - \phi_2 z^2) = 0, \qquad [2.3.20]$$

whose roots, we have seen, are the reciprocals of those of [2.3.19]. Thus, we could say with equal accuracy that "the difference equation [2.3.1] is stable whenever the roots of [2.3.19] lie *inside* the unit circle" or that "the difference equation [2.3.1] is stable whenever the roots of [2.3.20] lie *outside* the unit circle." The two statements mean exactly the same thing. Some scholars refer simply to the "roots of the difference equation [2.3.1]," though this raises the possibility of confusion between [2.3.19] and [2.3.20]. This book will follow the convention of using the term "eigenvalues" to refer to the roots of [2.3.19]. Wherever the term "roots" is used, we will indicate explicitly the equation whose roots are being described.

From here on in this section, it is assumed that the second-order difference equation is stable, with the eigenvalues λ_1 and λ_2 distinct and both inside the unit circle. Where this is the case, the inverses

$$(1 - \lambda_1 L)^{-1} = 1 + \lambda_1^1 L + \lambda_1^2 L^2 + \lambda_1^3 L^3 + \cdots$$
$$(1 - \lambda_2 L)^{-1} = 1 + \lambda_2^1 L + \lambda_2^2 L^2 + \lambda_2^3 L^3 + \cdots$$

are well defined for bounded sequences. Write [2.3.2] in factored form:

$$(1 - \lambda_1 L)(1 - \lambda_2 L)y_t = w_t$$

and operate on both sides by $(1 - \lambda_1 L)^{-1}(1 - \lambda_2 L)^{-1}$:

$$y_t = (1 - \lambda_1 L)^{-1}(1 - \lambda_2 L)^{-1} w_t. \qquad [2.3.21]$$

Following Sargent (1987, p. 184), when $\lambda_1 \neq \lambda_2$, we can use the following operator:

$$(\lambda_1 - \lambda_2)^{-1}\left\{\frac{\lambda_1}{1 - \lambda_1 L} - \frac{\lambda_2}{1 - \lambda_2 L}\right\}. \qquad [2.3.22]$$

Notice that this is simply another way of writing the operator in [2.3.21]:

$$\begin{aligned}(\lambda_1 - \lambda_2)^{-1}\left\{\frac{\lambda_1}{1 - \lambda_1 L} - \frac{\lambda_2}{1 - \lambda_2 L}\right\} &= (\lambda_1 - \lambda_2)^{-1}\left\{\frac{\lambda_1(1 - \lambda_2 L) - \lambda_2(1 - \lambda_1 L)}{(1 - \lambda_1 L)\cdot(1 - \lambda_2 L)}\right\} \\ &= \frac{1}{(1 - \lambda_1 L)\cdot(1 - \lambda_2 L)}.\end{aligned}$$

Thus, [2.3.21] can be written as

$$\begin{aligned}y_t &= (\lambda_1 - \lambda_2)^{-1}\left\{\frac{\lambda_1}{1 - \lambda_1 L} - \frac{\lambda_2}{1 - \lambda_2 L}\right\}w_t \\ &= \left\{\frac{\lambda_1}{\lambda_1 - \lambda_2}[1 + \lambda_1 L + \lambda_1^2 L^2 + \lambda_1^3 L^3 + \cdots]\right. \\ &\qquad \left. - \frac{\lambda_2}{\lambda_1 - \lambda_2}[1 + \lambda_2 L + \lambda_2^2 L^2 + \lambda_2^3 L^3 + \cdots]\right\}w_t\end{aligned}$$

or

$$\begin{aligned}y_t = [c_1 + c_2]w_t + [c_1\lambda_1 + c_2\lambda_2]w_{t-1} + [c_1\lambda_1^2 + c_2\lambda_2^2]w_{t-2} \\ + [c_1\lambda_1^3 + c_2\lambda_2^3]w_{t-3} + \cdots,\end{aligned} \qquad [2.3.23]$$

where

$$c_1 = \lambda_1/(\lambda_1 - \lambda_2) \qquad [2.3.24]$$

$$c_2 = -\lambda_2/(\lambda_1 - \lambda_2). \qquad [2.3.25]$$

From [2.3.23] the dynamic multiplier can be read off directly as

$$\frac{\partial y_{t+j}}{\partial w_t} = c_1\lambda_1^j + c_2\lambda_2^j,$$

the same result arrived at in equations [1.2.24] and [1.2.25].

2.4. *p*th-Order Difference Equations

These techniques generalize in a straightforward way to a pth-order difference equation of the form

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + w_t. \qquad [2.4.1]$$

Write [2.4.1] in terms of lag operators as

$$(1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p)y_t = w_t. \qquad [2.4.2]$$

Factor the operator on the left side of [2.4.2] as

$$(1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p) = (1 - \lambda_1 L)(1 - \lambda_2 L)\cdots(1 - \lambda_p L). \qquad [2.4.3]$$

This is the same as finding the values of $(\lambda_1, \lambda_2, \ldots, \lambda_p)$ such that the following polynomials are the same for all z:

$$(1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p) = (1 - \lambda_1 z)(1 - \lambda_2 z)\cdots(1 - \lambda_p z).$$

As in the second-order system, we multiply both sides of this equation by z^{-p} and define $\lambda \equiv z^{-1}$:

$$\begin{aligned}(\lambda^p - \phi_1\lambda^{p-1} - \phi_2\lambda^{p-2} - \cdots - \phi_{p-1}\lambda - \phi_p)\\ = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_p).\end{aligned} \qquad [2.4.4]$$

Clearly, setting $\lambda = \lambda_i$ for $i = 1, 2, \ldots,$ or p causes the right side of [2.4.4] to equal zero. Thus the values $(\lambda_1, \lambda_2, \ldots, \lambda_p)$ must be the numbers that set the left side of expression [2.4.4] to zero as well:

$$\lambda_p - \phi_1\lambda^{p-1} - \phi_2\lambda^{p-2} - \cdots - \phi_{p-1}\lambda - \phi_p = 0. \qquad [2.4.5]$$

This expression again is identical to that given in Proposition 1.1, which characterized the eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_p)$ of the matrix $\mathbf{F}$ defined in equation [1.2.3]. Thus, Proposition 2.1 readily generalizes.

Proposition 2.2: *Factoring a pth-order polynomial in the lag operator,*

$$(1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p) = (1 - \lambda_1 L)(1 - \lambda_2 L)\cdots(1 - \lambda_p L),$$

is the same calculation as finding the eigenvalues of the matrix $\mathbf{F}$ *defined in [1.2.3]. The eigenvalues* $(\lambda_1, \lambda_2, \ldots, \lambda_p)$ *of* $\mathbf{F}$ *are the same as the parameters* $(\lambda_1, \lambda_2, \ldots, \lambda_p)$ *in [2.4.3] and are given by the solutions to equation [2.4.5].*

The difference equation [2.4.1] is stable if the eigenvalues (the roots of [2.4.5]) lie inside the unit circle, or equivalently if the roots of

$$1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p = 0 \qquad [2.4.6]$$

lie outside the unit circle.

Assuming that the eigenvalues are inside the unit circle and that we are restricting ourselves to considering bounded sequences, the inverses $(1 - \lambda_1 L)^{-1}$, $(1 - \lambda_2 L)^{-1}, \ldots, (1 - \lambda_p L)^{-1}$ all exist, permitting the difference equation

$$(1 - \lambda_1 L)(1 - \lambda_2 L)\cdots(1 - \lambda_p L)y_t = w_t$$

to be written as

$$y_t = (1 - \lambda_1 L)^{-1}(1 - \lambda_2 L)^{-1}\cdots(1 - \lambda_p L)^{-1}w_t. \qquad [2.4.7]$$

Provided further that the eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_p)$ are all distinct, the polynomial associated with the operator on the right side of [2.4.7] can again be expanded with partial fractions:

$$\begin{aligned}&\frac{1}{(1 - \lambda_1 z)(1 - \lambda_2 z)\cdots(1 - \lambda_p z)}\\ &\qquad = \frac{c_1}{(1 - \lambda_1 z)} + \frac{c_2}{(1 - \lambda_2 z)} + \cdots + \frac{c_p}{(1 - \lambda_p z)}.\end{aligned} \qquad [2.4.8]$$

Following Sargent (1987, pp. 192–93), the values of $(c_1, c_2, \ldots, c_p)$ that make [2.4.8] true can be found by multiplying both sides by $(1 - \lambda_1 z)(1 - \lambda_2 z)\cdots(1 - \lambda_p z)$:

$$\begin{aligned}1 = {} & c_1(1 - \lambda_2 z)(1 - \lambda_3 z)\cdots(1 - \lambda_p z)\\ & + c_2(1 - \lambda_1 z)(1 - \lambda_3 z)\cdots(1 - \lambda_p z) + \cdots\\ & + c_p(1 - \lambda_1 z)(1 - \lambda_2 z)\cdots(1 - \lambda_{p-1} z).\end{aligned} \qquad [2.4.9]$$

Equation [2.4.9] has to hold for all values of z. Since it is a $(p - 1)$th-order polynomial, if $(c_1, c_2, \ldots, c_p)$ are chosen so that [2.4.9] holds for p particular distinct values of z, then [2.4.9] must hold for all z. To ensure that [2.4.9] holds at $z = \lambda_1^{-1}$ requires that

$$1 = c_1(1 - \lambda_2\lambda_1^{-1})(1 - \lambda_3\lambda_1^{-1}) \cdots (1 - \lambda_p\lambda_1^{-1})$$

or

$$c_1 = \frac{\lambda_1^{p-1}}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) \cdots (\lambda_1 - \lambda_p)}. \qquad [2.4.10]$$

For [2.4.9] to hold for $z = \lambda_2^{-1}, \lambda_3^{-1}, \ldots, \lambda_p^{-1}$ requires

$$c_2 = \frac{\lambda_2^{p-1}}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3) \cdots (\lambda_2 - \lambda_p)} \qquad [2.4.11]$$

$$\vdots$$

$$c_p = \frac{\lambda_p^{p-1}}{(\lambda_p - \lambda_1)(\lambda_p - \lambda_2) \cdots (\lambda_p - \lambda_{p-1})}. \qquad [2.4.12]$$

Note again that these are identical to expression [1.2.25] in Chapter 1. Recall from the discussion there that $c_1 + c_2 + \cdots + c_p = 1$.

To conclude, [2.4.7] can be written

$$\begin{aligned} y_t &= \frac{c_1}{(1 - \lambda_1 L)} w_t + \frac{c_2}{(1 - \lambda_2 L)} w_t + \cdots + \frac{c_p}{(1 - \lambda_p L)} w_t \\ &= c_1(1 + \lambda_1 L + \lambda_1^2 L^2 + \lambda_1^3 L^3 + \cdots) w_t + c_2(1 + \lambda_2 L + \lambda_2^2 L^2 + \lambda_2^3 L^3 + \cdots) w_t \\ &\quad + \cdots + c_p(1 + \lambda_p L + \lambda_p^2 L^2 + \lambda_p^3 L^3 + \cdots) w_t \end{aligned}$$

or

$$\begin{aligned} y_t = {} & [c_1 + c_2 + \cdots + c_p] w_t + [c_1\lambda_1 + c_2\lambda_2 + \cdots + c_p\lambda_p] w_{t-1} \\ & + [c_1\lambda_1^2 + c_2\lambda_2^2 + \cdots + c_p\lambda_p^2] w_{t-2} \\ & + [c_1\lambda_1^3 + c_2\lambda_2^3 + \cdots + c_p\lambda_p^3] w_{t-3} + \cdots \end{aligned} \qquad [2.4.13]$$

where $(c_1, c_2, \ldots, c_p)$ are given by equations [2.4.10] through [2.4.12]. Again, the dynamic multiplier can be read directly off [2.4.13]:

$$\frac{\partial y_{t+j}}{\partial w_t} = [c_1\lambda_1^j + c_2\lambda_2^j + \cdots + c_p\lambda_p^j], \qquad [2.4.14]$$

reproducing the result from Chapter 1.

There is a very convenient way to calculate the effect of w on the present value of y using the lag operator representation. Write [2.4.13] as

$$y_t = \psi_0 w_t + \psi_1 w_{t-1} + \psi_2 w_{t-2} + \psi_3 w_{t-3} + \cdots \qquad [2.4.15]$$

where

$$\psi_j = [c_1\lambda_1^j + c_2\lambda_2^j + \cdots + c_p\lambda_p^j]. \qquad [2.4.16]$$

Next rewrite [2.4.15] in lag operator notation as

$$y_t = \psi(L) w_t, \qquad [2.4.17]$$

where $\psi(L)$ denotes an infinite-order polynomial in the lag operator:

$$\psi(L) = \psi_0 + \psi_1 L + \psi_2 L^2 + \psi_3 L^3 + \cdots.$$

Notice that ψ_j is the dynamic multiplier [2.4.14]. The effect of w_t on the present value of y is given by

$$\frac{\partial \sum_{j=0}^{\infty} \beta^j y_{t+j}}{\partial w_t} = \sum_{j=0}^{\infty} \beta^j \frac{\partial y_{t+j}}{\partial w_t} = \sum_{j=0}^{\infty} \beta^j \psi_j. \tag{2.4.18}$$

Thinking of $\psi(z)$ as a polynomial in a real number z,

$$\psi(z) = \psi_0 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 + \cdots,$$

it appears that the multiplier [2.4.18] is simply this polynomial evaluated at $z = \beta$:

$$\frac{\partial \sum_{j=0}^{\infty} \beta^j y_{t+j}}{\partial w_t} = \psi(\beta) = \psi_0 + \psi_1\beta + \psi_2\beta^2 + \psi_3\beta^3 + \cdots. \tag{2.4.19}$$

But comparing [2.4.17] with [2.4.7], it is apparent that

$$\psi(L) = [(1 - \lambda_1 L)(1 - \lambda_2 L) \cdots (1 - \lambda_p L)]^{-1},$$

and from [2.4.3] this means that

$$\psi(L) = [1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p]^{-1}.$$

We conclude that

$$\psi(z) = [1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p]^{-1}$$

for any value of z, so, in particular,

$$\psi(\beta) = [1 - \phi_1\beta - \phi_2\beta^2 - \cdots - \phi_p\beta^p]^{-1}. \tag{2.4.20}$$

Substituting [2.4.20] into [2.4.19] reveals that

$$\frac{\partial \sum_{j=0}^{\infty} \beta^j y_{t+j}}{\partial w_t} = \frac{1}{1 - \phi_1\beta - \phi_2\beta^2 - \cdots - \phi_p\beta^p}, \tag{2.4.21}$$

reproducing the claim in Proposition 1.3. Again, the long-run multiplier obtains as the special case of [2.4.21] with $\beta = 1$:

$$\lim_{j\to\infty}\left[\frac{\partial y_{t+j}}{\partial w_t} + \frac{\partial y_{t+j}}{\partial w_{t+1}} + \cdots + \frac{\partial y_{t+j}}{\partial w_{t+j}}\right] = \frac{1}{1 - \phi_1 - \phi_2 - \cdots - \phi_p}.$$

2.5. *Initial Conditions and Unbounded Sequences*

Section 1.2 analyzed the following problem. Given a pth-order difference equation

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + w_t, \tag{2.5.1}$$

p initial values of y,

$$y_{-1}, y_{-2}, \ldots, y_{-p}, \tag{2.5.2}$$

and a sequence of values for the input variable w,

$$\{w_0, w_1, \ldots, w_t\}, \tag{2.5.3}$$

we sought to calculate the sequence of values for the output variable y:

$$\{y_0, y_1, \ldots, y_t\}.$$

Certainly there are systems where the question is posed in precisely this form. We may know the equation of motion for the system [2.5.1] and its current state [2.5.2] and wish to characterize the values that $\{y_0, y_1, \ldots, y_t\}$ might take on for different specifications of $\{w_0, w_1, \ldots, w_t\}$.

However, there are many examples in economics and finance in which a theory specifies just the equation of motion [2.5.1] and a sequence of driving variables [2.5.3]. Clearly, these two pieces of information alone are insufficient to determine the sequence $\{y_0, y_1, \ldots, y_t\}$, and some additional theory beyond that contained in the difference equation [2.5.1] is needed to describe fully the dependence of y on w. These additional restrictions can be of interest in their own right and also help give some insight into some of the technical details of manipulating difference equations. For these reasons, this section discusses in some depth an example of the role of initial conditions and their implications for solving difference equations.

Let P_t denote the price of a stock and D_t its dividend payment. If an investor buys the stock at date t and sells it at $t + 1$, the investor will earn a yield of D_t/P_t from the dividend and a yield of $(P_{t+1} - P_t)/P_t$ in capital gains. The investor's total return (r_{t+1}) is thus

$$r_{t+1} = (P_{t+1} - P_t)/P_t + D_t/P_t.$$

A very simple model of the stock market posits that the return investors earn on stocks is constant across time periods:

$$r = (P_{t+1} - P_t)/P_t + D_t/P_t \qquad r > 0. \tag{2.5.4}$$

Equation [2.5.4] may seem too simplistic to be of much practical interest; it assumes among other things that investors have perfect foresight about future stock prices and dividends. However, a slightly more realistic model in which *expected* stock returns are constant involves a very similar set of technical issues. The advantage of the perfect-foresight model [2.5.4] is that it can be discussed using the tools already in hand to gain some further insight into using lag operators to solve difference equations.

Multiply [2.5.4] by P_t to arrive at

$$rP_t = P_{t+1} - P_t + D_t$$

or

$$P_{t+1} = (1 + r)P_t - D_t. \tag{2.5.5}$$

Equation [2.5.5] will be recognized as a first-order difference equation of the form of [1.1.1] with $y_t = P_{t+1}$, $\phi = (1 + r)$, and $w_t = -D_t$. From [1.1.7], we know that [2.5.5] implies that

$$\begin{aligned} P_{t+1} = {} & (1 + r)^{t+1}P_0 - (1 + r)^t D_0 - (1 + r)^{t-1}D_1 - (1 + r)^{t-2}D_2 \\ & - \cdots - (1 + r)D_{t-1} - D_t. \end{aligned} \tag{2.5.6}$$

If the sequence $\{D_0, D_1, \ldots, D_t\}$ *and* the value of P_0 were given, then [2.5.6] could determine the values of $\{P_1, P_2, \ldots, P_{t+1}\}$. But if only the values $\{D_0, D_1, \ldots, D_t\}$ are given, then equation [2.5.6] would not be enough to pin down $\{P_1, P_2, \ldots, P_{t+1}\}$. There are an infinite number of possible sequences $\{P_1, P_2, \ldots, P_{t+1}\}$ consistent with [2.5.5] and with a given $\{D_0, D_1, \ldots, D_t\}$. This infinite number of possibilities is indexed by the initial value P_0.

A further simplifying assumption helps clarify the nature of these different paths for $\{P_1, P_2, \ldots, P_{t+1}\}$. Suppose that dividends are constant over time:

$$D_t = D \qquad \text{for all } t.$$

Then [2.5.6] becomes

$$\begin{aligned} P_{t+1} &= (1+r)^{t+1}P_0 - [(1+r)^t + (1+r)^{t-1} \\ &\quad + \cdots + (1+r) + 1]D \\ &= (1+r)^{t+1}P_0 - \frac{1-(1+r)^{t+1}}{1-(1+r)}D \\ &= (1+r)^{t+1}[P_0 - (D/r)] + (D/r). \end{aligned} \qquad [2.5.7]$$

Consider first the solution in which $P_0 = D/r$. If the initial stock price should happen to take this value, then [2.5.7] implies that

$$P_t = D/r \qquad [2.5.8]$$

for all t. In this solution, dividends are constant at D and the stock price is constant at D/r. With no change in stock prices, investors never have any capital gains or losses, and their return is solely the dividend yield $D/P = r$. In a world with no changes in dividends this seems to be a sensible expression of the theory represented by [2.5.4]. Equation [2.5.8] is sometimes described as the "market fundamentals" solution to [2.5.4] for the case of constant dividends.

However, even with constant dividends, equation [2.5.8] is not the only result consistent with [2.5.4]. Suppose that the initial price exceeded D/r:

$$P_0 > D/r.$$

Investors seem to be valuing the stock beyond the potential of its constant dividend stream. From [2.5.7] this could be consistent with the asset pricing theory [2.5.4] provided that P_1 exceeds D/r by an even larger amount. As long as investors all believe that prices will continue to rise over time, each will earn the required return r from the realized capital gain and [2.5.4] will be satisfied. This scenario has reminded many economists of a speculative bubble in stock prices.

If such bubbles are to be ruled out, additional knowledge about the process for $\{P_t\}_{t=-\infty}^{\infty}$ is required beyond that contained in the theory of [2.5.4]. For example, we might argue that finite world resources put an upper limit on feasible stock prices, as in

$$|P_t| < \overline{P} \qquad \text{for all } t. \qquad [2.5.9]$$

Then the only sequence for $\{P_t\}_{t=-\infty}^{\infty}$ consistent with both [2.5.4] and [2.5.9] would be the market fundamentals solution [2.5.8].

Let us now relax the assumption that dividends are constant and replace it with the assumption that $\{D_t\}_{t=-\infty}^{\infty}$ is a bounded sequence. What path for $\{P_t\}_{t=-\infty}^{\infty}$ in [2.5.6] is consistent with [2.5.9] in this case? The answer can be found by returning to the difference equation [2.5.5]. We arrived at the form [2.5.6] by recursively substituting this equation backward. That is, we used the fact that [2.5.5] held for dates $t, t-1, t-2, \ldots, 0$ and recursively substituted to arrive at [2.5.6] as a logical implication of [2.5.5]. Equation [2.5.5] could equally well be solved recursively *forward*. To do so, equation [2.5.5] is written as

$$P_t = \frac{1}{1+r}[P_{t+1} + D_t]. \qquad [2.5.10]$$

An analogous equation must hold for date $t + 1$:

$$P_{t+1} = \frac{1}{1 + r}[P_{t+2} + D_{t+1}]. \qquad [2.5.11]$$

Substitute [2.5.11] into [2.5.10] to deduce

$$\begin{aligned} P_t &= \frac{1}{1 + r}\left[\frac{1}{1 + r}[P_{t+2} + D_{t+1}] + D_t\right] \\ &= \left[\frac{1}{1 + r}\right]^2 P_{t+2} + \left[\frac{1}{1 + r}\right]^2 D_{t+1} + \left[\frac{1}{1 + r}\right] D_t. \end{aligned} \qquad [2.5.12]$$

Using [2.5.10] for date $t + 2$,

$$P_{t+2} = \frac{1}{1 + r}[P_{t+3} + D_{t+2}],$$

and substituting into [2.5.12] gives

$$P_t = \left[\frac{1}{1 + r}\right]^3 P_{t+3} + \left[\frac{1}{1 + r}\right]^3 D_{t+2} + \left[\frac{1}{1 + r}\right]^2 D_{t+1} + \left[\frac{1}{1 + r}\right] D_t.$$

Continuing in this fashion T periods into the future produces

$$\begin{aligned} P_t = \left[\frac{1}{1 + r}\right]^T P_{t+T} &+ \left[\frac{1}{1 + r}\right]^T D_{t+T-1} + \left[\frac{1}{1 + r}\right]^{T-1} D_{t+T-2} \\ &+ \cdots + \left[\frac{1}{1 + r}\right]^2 D_{t+1} + \left[\frac{1}{1 + r}\right] D_t. \end{aligned} \qquad [2.5.13]$$

If the sequence $\{P_t\}_{t=-\infty}^{\infty}$ is to satisfy [2.5.9], then

$$\lim_{T\to\infty} \left[\frac{1}{1 + r}\right]^T P_{t+T} = 0.$$

If $\{D_t\}_{t=-\infty}^{\infty}$ is likewise a bounded sequence, then the following limit exists:

$$\lim_{T\to\infty} \sum_{j=0}^{T} \left[\frac{1}{1 + r}\right]^{j+1} D_{t+j}.$$

Thus, if $\{P_t\}_{t=-\infty}^{\infty}$ is to be a bounded sequence, then we can take the limit of [2.5.13] as $T \to \infty$ to conclude

$$P_t = \sum_{j=0}^{\infty} \left[\frac{1}{1 + r}\right]^{j+1} D_{t+j}, \qquad [2.5.14]$$

which is referred to as the "market fundamentals" solution of [2.5.5] for the general case of time-varying dividends. Notice that [2.5.14] produces [2.5.8] as a special case when $D_t = D$ for all t.

Describing the value of a variable at time t as a function of future realizations of another variable as in [2.5.14] may seem an artifact of assuming a perfect-foresight model of stock prices. However, an analogous set of operations turns out to be appropriate in a system similar to [2.5.4] in which expected returns are constant.[1] In such systems [2.5.14] generalizes to

$$P_t = \sum_{j=0}^{\infty} \left[\frac{1}{1 + r}\right]^{j+1} E_t D_{t+j},$$

[1]See Sargent (1987) and Whiteman (1983) for an introduction to the manipulation of difference equations involving expectations.

where E_t denotes an expectation of an unknown future quantity based on information available to investors at date t.

Expression [2.5.14] determines the particular value for the initial price P_0 that is consistent with the boundedness condition [2.5.9]. Setting $t = 0$ in [2.5.14] and substituting into [2.5.6] produces

$$P_{t+1} = (1 + r)^{t+1}\left\{\left[\frac{1}{1 + r}\right]D_0 + \left[\frac{1}{1 + r}\right]^2 D_1 + \left[\frac{1}{1 + r}\right]^3 D_2 \right.$$
$$\left. + \cdots + \left[\frac{1}{1 + r}\right]^{t+1} D_t + \left[\frac{1}{1 + r}\right]^{t+2} D_{t+1} + \cdots \right\} - (1 + r)^t D_0$$
$$- (1 + r)^{t-1}D_1 - (1 + r)^{t-2}D_2 - \cdots - (1 + r)D_{t-1} - D_t$$
$$= \left[\frac{1}{1 + r}\right]D_{t+1} + \left[\frac{1}{1 + r}\right]^2 D_{t+2} + \left[\frac{1}{1 + r}\right]^3 D_{t+3} + \cdots.$$

Thus, setting the initial condition P_0 to satisfy [2.5.14] is sufficient to ensure that it holds for all t. Choosing P_0 equal to any other value would cause the consequences of each period's dividends to accumulate over time so as to lead to a violation of [2.5.9] eventually.

It is useful to discuss these same calculations from the perspective of lag operators. In Section 2.2 the recursive substitution backward that led from [2.5.5] to [2.5.6] was represented by writing [2.5.5] in terms of lag operators as

$$[1 - (1 + r)L]P_{t+1} = -D_t \qquad [2.5.15]$$

and multiplying both sides of [2.5.15] by the following operator:

$$[1 + (1 + r)L + (1 + r)^2L^2 + \cdots + (1 + r)^tL^t]. \qquad [2.5.16]$$

If $(1 + r)$ were less than unity, it would be natural to consider the limit of [2.5.16] as $t \rightarrow \infty$:

$$[1 - (1 + r)L]^{-1} = 1 + (1 + r)L + (1 + r)^2L^2 + \cdots.$$

In the case of the theory of stock returns discussed here, however, $r > 0$ and this operator is not defined. In this case, a lag operator representation can be sought for the recursive substitution forward that led from [2.5.5] to [2.5.13]. This is accomplished using the inverse of the lag operator,

$$L^{-1}w_t = w_{t+1},$$

which extends result [2.1.4] to negative values of k. Note that L^{-1} is indeed the inverse of the operator L:

$$L^{-1}(Lw_t) = L^{-1}w_{t-1} = w_t.$$

In general,

$$L^{-k}L^j = L^{j-k},$$

with L^0 defined as the identity operator:

$$L^0w_t \equiv w_t.$$

Now consider multiplying [2.5.15] by

$$[1 + (1 + r)^{-1}L^{-1} + (1 + r)^{-2}L^{-2} + \cdots + (1 + r)^{-(T-1)}L^{-(T-1)}] \times [-(1 + r)^{-1}L^{-1}] \qquad [2.5.17]$$

to obtain

$$\begin{aligned} &[1 + (1 + r)^{-1}L^{-1} + (1 + r)^{-2}L^{-2} + \cdots + (1 + r)^{-(T-1)}L^{-(T-1)}] \\ &\times [1 - (1 + r)^{-1}L^{-1}]P_{t+1} \\ &= [1 + (1 + r)^{-1}L^{-1} + (1 + r)^{-2}L^{-2} + \cdots \\ &\quad + (1 + r)^{-(T-1)}L^{-(T-1)}] \times (1 + r)^{-1}D_{t+1} \end{aligned}$$

or

$$\begin{aligned} [1 - (1 + r)^{-T}L^{-T}]P_{t+1} = \left[\frac{1}{1 + r}\right]D_{t+1} + \left[\frac{1}{1 + r}\right]^2 D_{t+2} \\ + \left[\frac{1}{1 + r}\right]^3 D_{t+3} + \cdots + \left[\frac{1}{1 + r}\right]^T D_{t+T}, \end{aligned}$$

which is identical to [2.5.13] with t in [2.5.13] replaced with $t + 1$.

When $r > 0$ and $\{P_t\}_{t=-\infty}^{\infty}$ is a bounded sequence, the left side of the preceding equation will approach P_{t+1} as T becomes large. Thus, when $r > 0$ and $\{P_t\}_{t=-\infty}^{\infty}$ and $\{D_t\}_{t=-\infty}^{\infty}$ are bounded sequences, the limit of the operator in [2.5.17] exists and could be viewed as the inverse of the operator on the left side of [2.5.15]:

$$\begin{aligned} [1 - (1 + r)L]^{-1} = &-(1 + r)^{-1}L^{-1} \\ &\times [1 + (1 + r)^{-1}L^{-1} + (1 + r)^{-2}L^{-2} + \cdots]. \end{aligned}$$

Applying this limiting operator to [2.5.15] amounts to solving the difference equation forward as in [2.5.14] and selecting the market fundamentals solution among the set of possible time paths for $\{P_t\}_{t=-\infty}^{\infty}$ given a particular time path for dividends $\{D_t\}_{t=-\infty}^{\infty}$.

Thus, given a first-order difference equation of the form

$$(1 - \phi L)y_t = w_t, \qquad [2.5.18]$$

Sargent's (1987) advice was to solve the equation "backward" when $|\phi| < 1$ by multiplying by

$$[1 - \phi L]^{-1} = [1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots] \qquad [2.5.19]$$

and to solve the equation "forward" when $|\phi| > 1$ by multiplying by

$$\begin{aligned} [1 - \phi L]^{-1} &= \frac{-\phi^{-1}L^{-1}}{1 - \phi^{-1}L^{-1}} \\ &= -\phi^{-1}L^{-1}[1 + \phi^{-1}L^{-1} + \phi^{-2}L^{-2} + \phi^{-3}L^{-3} + \cdots]. \end{aligned} \qquad [2.5.20]$$

Defining the inverse of $[1 - \phi L]$ in this way amounts to selecting an operator $[1 - \phi L]^{-1}$ with the properties that

$$[1 - \phi L]^{-1} \times [1 - \phi L] = 1 \qquad \text{(the identity operator)}$$

and that, when it is applied to a bounded sequence $\{w_t\}_{t=-\infty}^{\infty}$,

$$[1 - \phi L]^{-1}w_t,$$

the result is another bounded sequence.

The conclusion from this discussion is that in applying an operator such as $[1 - \phi L]^{-1}$, we are implicitly imposing a boundedness assumption that rules out phenomena such as the speculative bubbles of equation [2.5.7] a priori. Where that is our intention, so much the better, though we should not apply the rules [2.5.19] or [2.5.20] without some reflection on their economic content.

Chapter 2 References

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3 Stationary ARMA Processes

This chapter introduces univariate *ARMA* processes, which provide a very useful class of models for describing the dynamics of an individual time series. The chapter begins with definitions of some of the key concepts used in time series analysis. Sections 3.2 through 3.5 then investigate the properties of various *ARMA* processes. Section 3.6 introduces the autocovariance-generating function, which is useful for analyzing the consequences of combining different time series and for an understanding of the population spectrum. The chapter concludes with a discussion of invertibility (Section 3.7), which can be important for selecting the *ARMA* representation of an observed time series that is appropriate given the uses to be made of the model.

3.1. *Expectations, Stationarity, and Ergodicity*

Expectations and Stochastic Processes

Suppose we have observed a sample of size T of some random variable Y_t:

$$\{y_1, y_2, \ldots, y_T\}. \qquad [3.1.1]$$

For example, consider a collection of T independent and identically distributed (i.i.d.) variables ε_t,

$$\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_T\}, \qquad [3.1.2]$$

with

$$\varepsilon_t \sim N(0, \sigma^2).$$

This is referred to as a sample of size T from a *Gaussian white noise* process.

The observed sample [3.1.1] represents T particular numbers, but this set of T numbers is only one possible outcome of the underlying stochastic process that generated the data. Indeed, even if we were to imagine having observed the process for an infinite period of time, arriving at the sequence

$$\{y_t\}_{t=-\infty}^{\infty} = \{\ldots, y_{-1}, y_0, y_1, y_2, \ldots, y_T, y_{T+1}, y_{T+2}, \ldots\},$$

the infinite sequence $\{y_t\}_{t=-\infty}^{\infty}$ would still be viewed as a single realization from a time series process. For example, we might set one computer to work generating an infinite sequence of i.i.d. $N(0, \sigma^2)$ variates, $\{\varepsilon_t^{(1)}\}_{t=-\infty}^{\infty}$, and a second computer generating a separate sequence, $\{\varepsilon_t^{(2)}\}_{t=-\infty}^{\infty}$. We would then view these as two independent realizations of a Gaussian white noise process.