such as a *barbell* portfolio of cash and a 30-year zero. However, minimizing absolute market risk is not the same as minimizing relative market risk.

This example demonstrates that duration hedging only provides a first approximation to interest-rate risk management. If the goal is to minimize tracking error relative to an index, it is essential to use a fine decomposition of the index by maturity.

11.3 MAPPING LINEAR DERIVATIVES

11.3.1 Forward Contracts

Forward and futures contracts are the simplest types of derivatives. Since their value is linear in the underlying spot rates, their risk can be constructed easily from basic building blocks. Assume, for instance, that we are dealing with a forward contract on a foreign currency. The basic valuation formula can be derived from an arbitrage argument.

To establish notations, define

S_t = spot price of one unit of the underlying cash asset
K = contracted forward price
r = domestic risk-free rate
y = income flow on the asset
τ = time to maturity.

When the asset is a foreign currency, y represents the foreign risk-free rate r^*. We will use these two notations interchangeably. For convenience, we assume that all rates are compounded continuously.

We seek to find the current value of a forward contract f_t to buy one unit of foreign currency at K after time τ. To do this, we consider the fact that investors have two alternatives that are economically equivalent: (1) Buy $e^{-y\tau}$ units of the asset at the price S_t and hold for one period, or (2) enter a forward contract to buy one unit of the asset in one period. Under alternative 1, the investment will grow, with reinvestment of dividend, to exactly one unit of the asset after one period. Under alternative 2, the contract costs f_t upfront, and we need to set aside enough cash to pay K in the future, which is $Ke^{-r\tau}$. After 1 year, the two alternatives lead to the same position, one unit of the asset. Therefore, their initial cost must be identical. This leads to the following valuation formula for outstanding forward contracts:

$$f_t = S_t e^{-y\tau} - Ke^{-r\tau} \tag{11.9}$$

Note that we can repeat the preceding reasoning to find the current forward rate F_t that would set the value of the contract to zero. Setting $K = F_t$ and $f_t = 0$ in Equation (11.9), we have

$$F_t = (S_t e^{-y\tau}) e^{r\tau} \tag{11.10}$$

This allows us to rewrite Equation (11.9) as

$$f_t = F_t e^{-r\tau} - Ke^{-r\tau} = (F_t - K)e^{-r\tau} \tag{11.11}$$

In other words, the current value of the forward contract is the present value of the difference between the current forward rate and the locked-in delivery rate. If we are long a forward contract with contracted rate K, we can liquidate the contract by entering a new contract to sell at the current rate F_t. This will lock in a profit of $(F_t - K)$, which we need to discount to the present time to find f_t.

Let us examine the risk of a 1-year forward contract to purchase 100 million euros in exchange for \$130.086 million. Table 11-6 displays pricing information for the contract (current spot, forward, and interest rates), risk, and correlations. The first step is to find the market value of the contract. We can use Equation (11.9), accounting for the fact that the quoted interest rates are discretely compounded, as

$$f_t = \$1.2877 \frac{1}{(1 + 2.2810/100)} - \$1.3009 \frac{1}{(1 + 3.3304/100)} = \$1.2589 - \$1.2589 = 0$$

TABLE 11-6

Risk and Correlations for Forward Contract Risk Factors (Monthly VAR at 95 Percent Level)

			Correlations		
Risk Factor	**Price or Rate**	**VAR (%)**	**EUR Spot**	**EUR 1Y**	**USD 1Y**
EUR spot	\$1.2877	4.5381	1	0.1289	0.0400
Long EUR bill	2.2810%	0.1396	0.1289	1	−0.0583
Short USD bill	3.3304%	0.2121	0.0400	−0.0583	1
EUR forward	\$1.3009				

Thus the initial value of the contract is zero. This value, however, may change, creating market risk.

Among the three sources of risk, the volatility of the spot contract is the highest by far, with a 4.54 percent VAR (corresponding to 1.65 standard deviations over a month for a 95 percent confidence level). This is much greater than the 0.14 percent VAR for the EUR 1-year bill or even the 0.21 percent VAR for the USD bill. Thus most of the risk of the forward contract is driven by the cash EUR position.

But risk is also affected by correlations. The positive correlation of 0.13 between the EUR spot and bill positions indicates that when the EUR goes up in value against the dollar, the value of a 1-year EUR investment is likely to appreciate. Therefore, higher values of the EUR are associated with lower EUR interest rates.

This positive correlation increases the risk of the combined position. On the other hand, the position is also short a 1-year USD bill, which is correlated with the other two legs of the transaction. The issue is, what will be the net effect on the risk of the forward contract?

VAR provides an exact answer to this question, which is displayed in Table 11-7. But first we have to compute the positions x on each of the three building blocks of the contract. By taking the partial derivative of Equation (11.9) with respect to the risk factors, we have

$$df = \frac{\partial f}{\partial S} dS + \frac{\partial f}{\partial r^*} dr^* + \frac{\partial f}{\partial r} dr = e^{-r^*\tau} dS - Se^{-r^*\tau}\tau dr^* + Ke^{-r\tau}\tau dr \qquad (11.12)$$

Here, the building blocks consist of the spot rate and interest rates. Alternatively, we can replace interest rates by the price of bills. Define these

TABLE 11-7

Computing VAR for a EUR 100 Million Forward Contract (Monthly VAR at 95 Percent Level)

Position	Present-Value Factor	Cash Flows (CF)	PV of Flows, x	Individual VAR, $\lvert x \rvert V$	Component VAR, $x\Delta$VAR
EUR spot			\$125.89	\$5.713	\$5.704
Long EUR bill	0.977698	EUR100.00	\$125.89	\$0.176	\$0.029
Short USD bill	0.967769	−\$130.09	−\$125.89	\$0.267	\$0.002
Undiversified VAR				\$6.156	
Diversified VAR					\$5.735

as $P = e^{-r\tau}$ and $P^* = e^{-r^*\tau}$. We then replace dr with dP using $dP = (-\tau)e^{-r\tau}$ dr and $dP^* = (-\tau)e^{-r^*\tau}$ dr^*. The risk of the forward contract becomes

$$df = (Se^{-r^*\tau})\frac{dS}{S} + (Se^{-r^*\tau})\frac{dP^*}{P^*} - (Ke^{-r\tau})\frac{dP}{P} \tag{11.13}$$

This shows that the forward position can be separated into three cash flows: (1) a long spot position in EUR, worth EUR 100 million = \$130.09 million in a year, or $(Se^{-r^*\tau})$ = \$125.89 million now, (2) a long position in a EUR investment, also worth \$125.89 million now, and (3) a short position in a USD investment, worth \$130.09 million in a year, or $(Ke^{-r\tau})$ = \$125.89 million now. Thus a position in the forward contract has three building blocks:

> Long forward contract = long foreign currency spot + long foreign currency bill + short U.S. dollar bill

Considering only the spot position, the VAR is \$125.89 million times the risk of 4.538 percent, which is \$5.713 million. To compute the diversified VAR, we use the risk matrix from the data in Table 11-6 and pre- and postmultiply by the vector of positions (PV of flows column in the table). The total VAR for the forward contract is \$5.735 million. This number is about the same size as that of the spot contract because exchange-rate volatility dominates the volatility of 1-year bonds.

More generally, the same methodology can be used for long-term currency swaps, which are equivalent to portfolios of forward contracts. For instance, a 10-year contract to pay dollars and receive euros is equivalent to a series of 10 forward contracts to exchange a set amount of dollars into marks. To compute the VAR, the contract must be broken down into a currency-risk component and a string of USD and EUR fixed-income components. As before, the total VAR will be driven primarily by the currency component.

11.3.2 Commodity Forwards

The valuation of forward or futures contracts on commodities is substantially more complex than for financial assets such as currencies, bonds, or stock indices. Such financial assets have a well-defined income flow y, which is the foreign interest rate, the coupon payment, or the dividend yield, respectively.

Things are not so simple for commodities, such as metals, agricultural products, or energy products. Most products do not make monetary payments but instead are consumed, thus creating an implied benefit. This flow of benefit, net of storage cost, is loosely called *convenience yield* to represent the benefit from holding the cash product. This convenience yield, however, is not tied to another financial variable, such as the foreign interest rate for currency futures. It is also highly variable, creating its own source of risk.

As a result, the risk measurement of commodity futures uses Equation (11.11) directly, where the main driver of the value of the contract is the current forward price for this commodity. Table 11-8 illustrates the term structure of volatilities for selected energy products and base metals. First, we note that monthly VAR measures are very high, reaching 29 percent for near contracts. In contrast, currency and equity market

TABLE 11-8

Risk of Commodity Contracts (Monthly VAR at 95 Percent Level)

	Energy Products			
Maturity	**Natural Gas**	**Heating Oil**	**Unleaded Gasoline**	**Crude Oil–WTI**
1 month	28.77	22.07	20.17	19.20
3 months	22.79	20.60	18.29	17.46
6 months	16.01	16.67	16.26	15.87
12 months	12.68	14.61	—	14.05
	Base Metals			
Maturity	**Aluminum**	**Copper**	**Nickel**	**Zinc**
Cash	11.34	13.09	18.97	13.49
3 months	11.01	12.34	18.41	13.18
15 months	8.99	10.51	15.44	11.95
27 months	7.27	9.57	—	11.59
	Precious Metals			
Maturity	**Gold**	**Silver**	**Platinum**	
Cash	6.18	14.97	7.70	

VARs are typically around 6 percent. Thus commodities are much more volatile than typical financial assets.

Second, we observe that volatilities decrease with maturity. The effect is strongest for less storable products such as energy products and less so for base metals. It is actually imperceptible for precious metals, which have low storage costs and no convenience yield. For financial assets, volatilities are driven primarily by spot prices, which implies basically constant volatilities across contract maturities.

Let us now say that we wish to compute the VAR for a 12-month forward position on 1 million barrels of oil priced at \$45.2 per barrel. Using a present-value factor of 0.967769, this translates into a current position of \$43,743,000.

Differentiating Equation (11.11), we have

$$df = \frac{\partial f}{\partial F} dF = e^{-r\tau} dF = (e^{-r\tau} F) \frac{dF}{F} \tag{11.14}$$

The term between parentheses therefore represents the exposure. The contract VAR is

$$\text{VAR} = \$43{,}743{,}000 \times 14.05/100 = \$6{,}146{,}000$$

In general, the contract cash flows will fall between the maturities of the risk factors, and present values must be apportioned accordingly.

11.3.3 Forward Rate Agreements

Forward rate agreements (FRAs) are forward contracts that allow users to lock in an interest rate at some future date. The buyer of an FRA locks in a borrowing rate; the seller locks in a lending rate. In other words, the "long" receives a payment if the spot rate is above the forward rate.

Define the timing of the short leg as τ_1 and of the long leg as τ_2, both expressed in years. Assume linear compounding for simplicity. The forward rate can be defined as the implied rate that equalizes the return on a τ_2-period investment with a τ_1-period investment rolled over, that is,

$$(1 + R_2\tau_2) = (1 + R_1\tau_1)\,[1 + F_{1,2}(\tau_2 - \tau_1)] \tag{11.15}$$

For instance, suppose that you sold a 6 × 12 FRA on \$100 million. This is equivalent to borrowing \$100 million for 6 months and investing the proceeds for 12 months. When the FRA expires in 6 months, assume that the prevailing 6-month spot rate is higher than the locked-in forward rate. The seller then pays the buyer the difference between the spot and forward rates applied to the principal. In effect, this payment offsets the higher return that the investor otherwise would receive, thus guaranteeing a return equal to the forward rate. Therefore, an FRA can be decomposed into two zero-coupon building blocks.

Long 6 × 12 FRA = long 6-month bill + short 12-month bill

Table 11-9 provides a worked-out example. If the 360-day spot rate is 5.8125 percent and the 180-day rate is 5.6250 percent, the forward rate must be such that

$$(1+F_{1,2}/2)=\frac{(1+5.8125/100)}{(1+5.6250/200)}$$

or F = 5.836 percent. The present value of the notional \$100 million in 6 months is x = \$100/(1+5.625/200)= \$97.264 million. This amount is invested for 12 months. In the meantime, what is the risk of this FRA?

Table 11-9 displays the computation of VAR for the FRA. The VARs of 6- and 12-month zeroes are 0.1629 and 0.4696, respectively, with a correlation of 0.8738. Applied to the principal of \$97.26 million, the individual VARs are \$0.158 million and \$0.457 million, which gives an undiversified VAR of \$0.615 million. Fortunately, the correlation substantially lowers the FRA risk. The largest amount the position can lose over a month at the 95 percent level is \$0.327 million.

TABLE 11-9

Computing the VAR of a \$100 Million FRA (Monthly VAR at 95 Percent Level)

Position	PV of Flows, x	Risk (%), V	Correlation Matrix, R		Individual VAR, $\|x\|V$	Component VAR, $x\Delta$VAR
180 days	−\$97.264	0.1629	1	0.8738	\$0.158	−\$0.116
360 days	\$97.264	0.4696	0.8738	1	\$0.457	\$0.444
Undiversified VAR					\$0.615	
Diversified VAR						\$0.327

11.3.4 Interest-Rate Swaps

Interest-rate swaps are the most actively used derivatives. They create exchanges of interest-rate flows from fixed to floating or vice versa. Swaps can be decomposed into two legs, a fixed leg and a floating leg. The fixed leg can be priced as a coupon-paying bond; the floating leg is equivalent to a floating-rate note (FRN).

To illustrate, let us compute the VAR of a \$100 million 5-year interest-rate swap. We enter a dollar swap that pays 6.195 percent annually for 5 years in exchange for floating-rate payments indexed to London Interbank Offer Rate (LIBOR). Initially, we consider a situation where the floating-rate note is about to be reset. Just before the reset period, we know that the coupon will be set at the prevailing market rate. Therefore, the note carries no market risk, and its value can be mapped on cash only. Right after the reset, however, the note becomes similar to a bill with maturity equal to the next reset period.

Interest-rate swaps can be viewed in two different ways: as (1) a combined position in a fixed-rate bond and in a floating-rate bond or (2) a portfolio of forward contracts. We first value the swap as a position in two bonds using risk data from Table 11-4. The analysis is detailed in Table 11-10.

TABLE 11-10

Computing the VAR of a \$100 Million Interest-Rate Swap (Monthly VAR at 95 Percent Level)

Term (Year)	Cash Flows Fixed	Cash Flows Float	Spot Rate	PV of Net Cash Flows	Individual VAR	Component VAR
0	\$0	+\$100		+\$100.000	\$0	\$0
1	−\$6.195	\$0	5.813%	−\$5.855	\$0.027	\$0.024
2	−\$6.195	\$0	5.929%	−\$5.521	\$0.054	\$0.053
3	−\$6.195	\$0	6.034%	−\$5.196	\$0.077	\$0.075
4	−\$6.195	\$0	6.130%	−\$4.883	\$0.096	\$0.096
5	−\$106.195	\$0	6.217%	−\$78.546	\$1.905	\$1.905
Total				\$0.000		
Undiversified VAR					\$2.160	
Diversified VAR						\$2.152

The second and third columns lay out the payments on both legs. Assuming that this is an at-the-market swap, that is, that its coupon is equal to prevailing swap rates, the short position in the fixed-rate bond is worth \$100 million. Just before reset, the long position in the FRN is also worth \$100 million, so the market value of the swap is zero. To clarify the allocation of current values, the FRN is allocated to cash, with a zero maturity. This has no risk.

The next column lists the zero-coupon swap rates for maturities going from 1 to 5 years. The fifth column reports the present value of the net cash flows, fixed minus floating. The last column presents the component VAR, which adds up to a total diversified VAR of \$2.152 million. The undiversified VAR is obtained from summing all individual VARs. As usual, the \$2.160 million value somewhat overestimates risk.

This swap can be viewed as the sum of five forward contracts, as shown in Table 11-11. The 1-year contract promises payment of \$100 million plus the coupon of 6.195 percent; discounted at the spot rate of 5.813 percent, this yields a present value of −\$100.36 million. This is in exchange for \$100 million now, which has no risk.

The next contract is a 1 × 2 forward contract that promises to pay the principal plus the fixed coupon in 2 years, or −\$106.195 million; discounted at the 2-year spot rate, this yields −\$94.64 million. This is in exchange for \$100 million in 1 year, which is also \$94.50 million when discounted at the 1-year spot rate. And so on until the fifth contract, a 4 × 5 forward contract.

Table 11-11 shows the VAR of each contract. The undiversified VAR of \$2.401 million is the result of a simple summation of the five VARs. The fully diversified VAR is \$2.152 million, exactly the same as in the preceding table. This demonstrates the equivalence of the two approaches.

Finally, we examine the change in risk after the first payment has just been set on the floating-rate leg. The FRN then becomes a 1-year bond initially valued at par but subject to fluctuations in rates. The only change in the pattern of cash flows in Table 11-10 is to add \$100 million to the position on year 1 (from −\$5.855 to \$94.145). The resulting VAR then decreases from \$2.152 million to \$1.763 million. More generally, the swap's VAR will converge to zero as the swap matures, dipping each time a coupon is set.

TABLE 11-11

An Interest-Rate Swap Viewed as Forward Contracts (Monthly VAR at 95 Percent Level)

	PV of Flows: Contract					
Term (Year)	1	1 × 2	2 × 3	3 × 4	4 × 5	VAR
1	−$100.36	$94.50				
2		−$94.64	$89.11			
3			−$89.08	$83.88		
4				−$83.70	$78.82	
5					−$78.55	
VAR	$0.471	$0.571	$0.488	$0.446	$0.425	
Undiversified VAR						$2.401
Diversified VAR						$2.152

11.4 MAPPING OPTIONS

We now consider the mapping process for nonlinear derivatives, or options. Obviously, this nonlinearity may create problems for risk measurement systems based on the delta-normal approach, which is fundamentally linear.

To simplify, consider the Black-Scholes (BS) model for European options.[2] The model assumes, in addition to perfect capital markets, that the underlying spot price follows a continuous *geometric brownian motion* with constant volatility $\sigma(dS/S)$. Based on these assumptions, the Black-Scholes (1973) model, as expanded by Merton (1973), gives the value of a European call as

$$c = c(S,K,\tau,r,r^*,\sigma) = Se^{-r^*\tau}N(d_1) - Ke^{-r\tau}N(d_2) \qquad (11.16)$$

where $N(d)$ is the cumulative normal distribution function described in Chapter 5 with arguments

$$d_1 = \frac{\ln(Se^{-r^*\tau} / Ke^{-r\tau})}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2}, \qquad d_2 = d_1 - \sigma\sqrt{\tau}$$

where K is now the *exercise price* at which the option holder can, but is not obligated to, buy the asset.

[2] For a systematic approach to pricing derivatives, see the excellent book by Hull (2005).

Changes in the value of the option can be approximated by taking partial derivatives, that is,

$$
\begin{aligned}
dc &= \frac{\partial c}{\partial S}dS + \tfrac{1}{2}\frac{\partial^2 c}{\partial S^2}dS^2 + \frac{\partial c}{\partial r^*}dr^* + \frac{\partial c}{\partial r}dr + \frac{\partial c}{\partial \sigma}d\sigma + \frac{\partial c}{\partial t}dt \\
&= \Delta dS + \tfrac{1}{2}\Gamma dS^2 + \rho^* dr^* + \rho dr + \Lambda d\sigma + \Theta dt
\end{aligned}
\qquad (11.17)
$$

The advantage of the BS model is that it leads to closed-form solutions for all these partial derivatives. Table 11-12 gives typical values for 3-month European call options with various exercise prices.

The first partial derivative, or *delta*, is particularly important. For a European call, this is

$$\Delta = e^{-r^*\tau}N(d_1) \qquad (11.18)$$

This is related to the cumulative normal density function covered in Chapter 5. Figure 11-2 displays its behavior as a function of the underlying spot price and for various maturities.

The figure shows that delta is not a constant, which may make linear methods inappropriate for measuring the risk of options. Delta increases with the underlying spot price. The relationship becomes more nonlinear

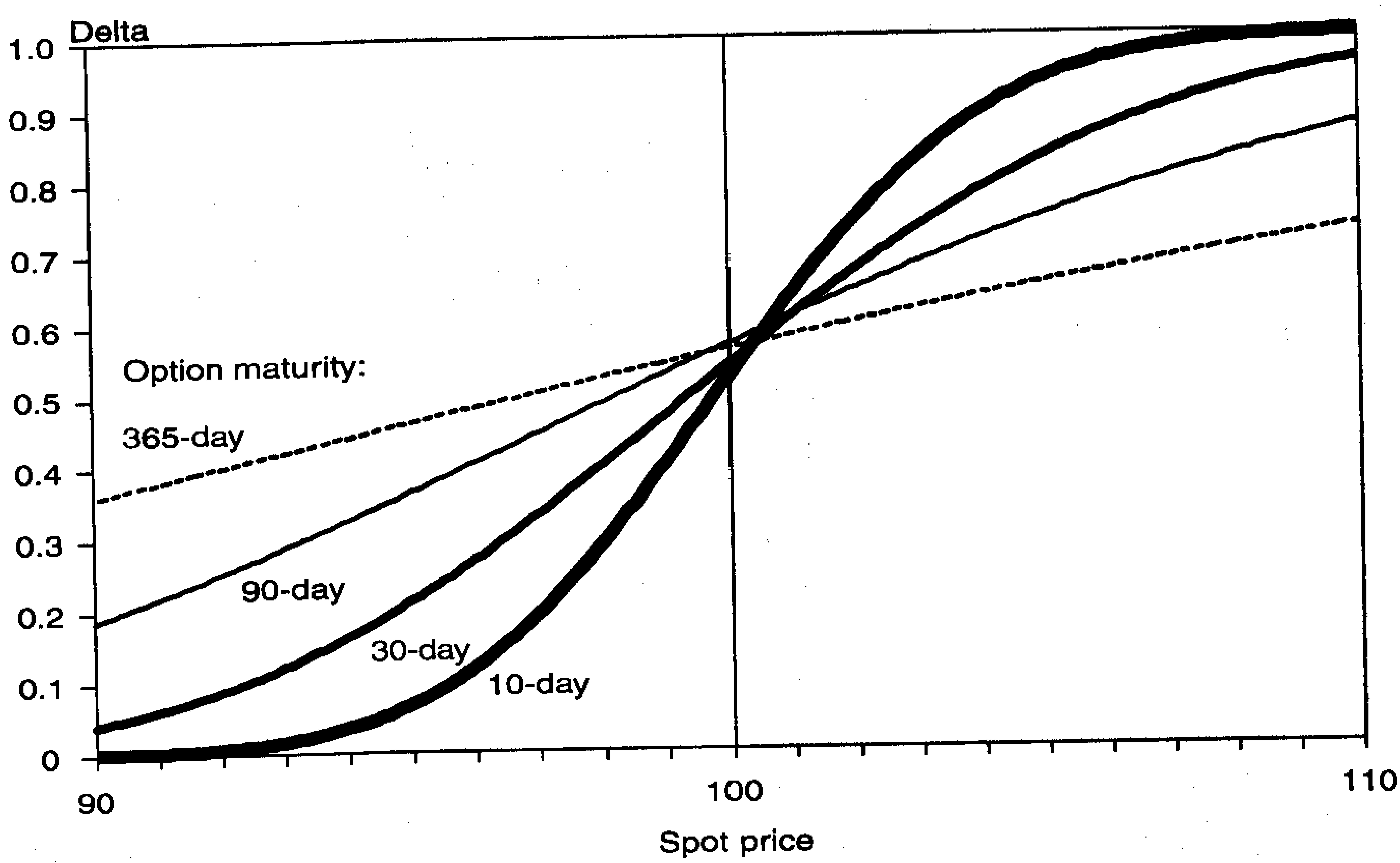
TABLE 11-12

Derivatives for a European Call

Parameters: S = \$100, σ = 20%, r = 5%, r^* = 3%, τ = 3 months					
			Exercise Price		
	Variable	**Unit**	**K = 90**	**K = 100**	**K = 110**
c		*Dollars*	11.01	4.20	1.04
		Change per			
Δ	Spot price	Dollar	0.869	0.536	0.195
Γ	Spot price	Dollar	0.020	0.039	0.028
Λ	Volatility	(% pa)	0.102	0.197	0.138
ρ	Interest rate	(% pa)	0.190	0.123	0.046
ρ^*	Asset yield	(% pa)	−0.217	−0.133	−0.049
θ	Time	Day	−0.014	−0.024	−0.016

FIGURE 11-2

Delta as a function of the risk factor.



for short-term options, for example, with an option maturity of 10 days. Linear methods approximate delta by a constant value over the risk horizon. The quality of this approximation depends on parameter values.

For instance, if the risk horizon is 1 day, the worst down move in the spot price is $-\alpha S\sigma\sqrt{T} = -1.645 \times \$100 \times 0.20\sqrt{1/252} = -\$2.08$, leading to a worst price of \$97.92. With a 90-day option, delta changes from 0.536 to 0.452 only. With such a small change, the linear effect will dominate the nonlinear effect. Thus linear approximations may be acceptable for options with long maturities when the risk horizon is short.

It is instructive to consider only the linear effects of the spot rate and two interest rates, that is,

$$
\begin{aligned}
dc &= \Delta dS + \rho^* dr^* + \rho dr \\
&= [e^{-r^*\tau} N(d_1)]dS + [-Se^{-r^*\tau}\tau N(d_1)]dr^* + [Ke^{-r\tau}\tau N(d_2)]dr \\
&= [Se^{-r^*\tau} N(d_1)]\frac{dS}{S} + [Se^{-r^*\tau} N(d_1)]\frac{dP^*}{P^*} - [Ke^{-r\tau} N(d_2)]\frac{dP}{P} \\
&= x_1\frac{dS}{S} + x_2\frac{dP^*}{P^*} + x_3\frac{dP}{P}
\end{aligned}
\tag{11.19}
$$

This formula bears a striking resemblance to that for foreign currency forwards, as in Equation (11.13). The only difference is that the position on the spot foreign currency and on the foreign currency bill $x_1 = x_2$ now involves $N(d_1)$, and the position on the dollar bill x_3 involves $N(d_2)$. In the extreme case, where the option is deep in the money, both $N(d_1)$ and $N(d_2)$ are equal to unity, and the option behaves exactly like a position in a forward contract. In this case, the BS model reduces to $c = Se^{-r^*\tau} - Ke^{-r\tau}$, which is indeed the valuation formula for a forward contract, as in Equation (11.9).

Also note that the position on the dollar bill $Ke^{-r\tau}N(d_2)$ is equivalent to $Se^{-r^*\tau}N(d_1) - c = S\Delta - c$. This shows that the call option is equivalent to a position of Δ in the underlying asset plus a short position of $(\Delta S - c)$ in a dollar bill, that is

Long option = long Δ asset + short $(\Delta S - c)$ bill

For instance, assume that the delta for an at-the-money call option on an asset worth \$100 is $\Delta = 0.536$. The option itself is worth \$4.20. This option is equivalent to a $\Delta S =$ \$53.60 position in the underlying asset financed by a loan of $\Delta S - c =$ \$53.60 − \$4.20 = \$49.40.

The next step in the risk measurement process is the aggregation of exposures across the portfolio. Thus all options on the same underlying risk factor are decomposed into their delta equivalents, which are summed across the portfolio. This generalizes to movements in the implied volatility, if necessary. The option portfolio would be characterized by its net *vega*, or Λ. This decomposition also can take into account second-order derivatives using the net *gamma*, or Γ. These exposures can be combined with simulations of the underlying risk factors to generate a risk distribution.

11.5 CONCLUSIONS

Risk measurement at financial institutions is a top-level aggregation problem involving too many positions to be modeled individually. As a result, instruments have to be mapped on a smaller set of primitive risk factors.

Choosing the appropriate set of risk factors, however, is part of the art of risk management. Too many risk factors would be unnecessary, slow, and wasteful. Too few risk factors, in contrast, could create blind spots in the risk measurement system. These issues will be discussed in Chapter 21, where we will discuss limitations of VAR.

The mapping process consists of replacing the current values of all instruments by their exposures on these risk factors. Next, exposures are aggregated across the portfolio to create a net exposure to each risk factor. The risk engine then combines these exposures with the distribution of risk factors to generate a distribution of portfolio values.

For some instruments, the allocation into general-market risk factors is exhaustive. In other words, there is no specific risk left. This is typically the case with derivatives, which are tightly priced in relation to their underlying risk factor. For others positions, such as individual stocks or corporate bonds, there remains some risk, called *specific risk*. In large, well-diversified portfolios, this remaining risk tends to wash away. Otherwise, specific risk needs to be taken into account.

APPENDIX 11.A

Assigning Weights to Vertices

The duration-mapping example in this chapter showed that, in general, cash flows fall between the selected vertices.[3] In our example, the portfolio consists of one cash flow with maturity of D_p =2.7325 years and present value of \$200 million. The question is, how should we allocate the \$200 million to the adjoining vertices in a way that best represents the risk of the original investment?

A simple method consists of allocating funds according to *duration matching*. Define x as the weight on the first vertex and D_1, D_2 as the duration of the first and second vertices. The portfolio duration D_p will be matched if

$$xD_1 + (1 - x)D_2 = D_p \tag{11.20}$$

or $x = (D_2 - D_p)/(D_2 - D_1)$. In our case, $x = (3 - 2.7325)/(3 - 2) = 0.2675$, which leads to an amount of $\$200 \times 0.2675 = \$53.49$ million on the first vertex. The balance of \$146.51 million is allocated to the 3-year vertex.

Unfortunately, this approach may not create a portfolio with the same risk as the original portfolio. The second method aims at *variance matching*. Define σ_1 and σ_2 as the respective volatilities and ρ as the correlation. The portfolio variance is

$$V(R_p) = x^2\sigma_1^2 + (1 - x)^2\sigma_2^2 + 2x(1 - x)\rho\sigma_1\sigma_2 \tag{11.21}$$

[3] See Henrard (2000) for comparisons of various cash-flow maps.

which we set equal to the variance of the zero-coupon bond falling between the two vertices. By linear interpolation of the price volatilities for 2- and 3-year zeroes, the portfolio volatility is $\sigma_p = 1.351$ percent, as we have done before. Therefore, the weight x that maintains the portfolio risk to that of the initial investment is found from solving the quadratic equation

$$(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)x^2 + 2(-\sigma_2^2 + \rho\sigma_1\sigma_2)x + (\sigma_2^2 - \sigma_p^2) = 0 \quad (11.22)$$

The solution to the equation $ax^2 + 2bx + c = 0$ is $x = (-b \pm\sqrt{b^2 - ac})/a$, which leads to the two roots $x_1 = 0.2635$ and $x_2 = 5.2168$. We choose the first root, which is between zero and unity. As shown in Table 11-13, this translates into a position of \$52.71 million on the 2-year vertex and \$147.29 million on the 3-year vertex.

In this example, the difference between the two approaches is minor. The VAR from variance matching is \$2.702 million versus \$2.698 million with duration matching. In fact, the duration approximation is exact under two conditions: (1) the correlation coefficient is unity, and (2) the volatility of each vertex is proportional to its duration ($\sigma_1 = \sigma D_1, \sigma_2 = \sigma D_2, \rho = 1$). Under these conditions, Equation (11.21) simplifies to

$$\begin{aligned} V(R_p) &= x^2\sigma^2 D_1^2 + (1-x)^2\sigma^2 D_2^2 + 2x(1-x)\sigma^2 D_1 D_2 \\ &= \sigma^2[xD_1 + (1-x)D_2]^2 \end{aligned} \quad (11.23)$$

which equals $(\sigma D_p)^2$ if $[xD_1 + (1-x)D_2] = D_p$. In other words, duration matching is perfectly appropriate under these conditions. In more general cases, especially if ρ is much lower than 1, the duration approximation will fail to provide a portfolio with the same risk as that of the original portfolio.

TABLE 11-13

Assigning Weights to Vertices

	Variance Matching				Duration Matching	
Term (Year)	**VAR (%)**	**Correlation**	**Weight**	**Amount**	**Weight**	**Amount**
2	0.9868		0.2635	\$52.71	0.2675	\$53.49
3	1.4841	0.9908	0.7365	\$147.29	0.7325	\$146.51
2.7325	1.3510					
Total			1.0000	\$200.00	1.0000	\$200.00
VAR				\$2.702		\$2.698

QUESTIONS

1. Why do risk managers use mapping instead of using historical data on each of the positions?
2. A risk manager is asked to provide a VAR measure for a portfolio of stocks, including IPOs. Discuss whether these IPO stocks should be ignored in the risk measurement process or not.
3. A portfolio of 100 fixed-income instruments is mapped to six risk factors. If the net exposures are zero, the portfolio has no risk. Discuss.
4. A portfolio of corporate bonds carries four different credit ratings and has maturities of 1, 5, and 10 years. How many general risk factors is the portfolio exposed to?
5. What is the drawback of principal mapping for bonds?
6. What is the main assumption for the duration approximation? What is the implication for the structure of a covariance matrix with different maturities as risk factors?
7. Consider a 3-year zero-coupon bond with a face value of $100 and a yield of 4% (annually compounded). Compute this bond's modified duration, dollar duration, and DVBP.
8. A portfolio manager evaluates the risk of a two-bond portfolio:

	Price	Modified Duration	Number Held
30-year bond	$100	13.84	5,000
10-year bond	$100	7.44	5,000

   We assume that specific risk is negligible and that the volatility of changes in market yields is 29 basis points. Under these conditions, what is the volatility of the portfolio value?
9. A portfolio manager enters a 10-year pay-fixed swap with notional of $100 million. The duration of the fixed leg is 7.44 years, and the floating leg is about to be reset. Assume a flat term structure and an annual volatility of yield changes of 100 basis points. What is the 95 percent VAR over the next month?
10. Now assume that the floating leg has just been reset for payment in a year. Compute the VAR.
11. Is duration hedging an appropriate way to minimize tracking error relative to an index?

12. A portfolio manager holds a \$100 million price position in 10-year Treasury notes with a daily volatility of 0.9 percent. The manager can hedge by selling 5-year T-notes with a daily volatility of 0.5 percent and correlation of 0.97. For computation of VAR, assume normal distributions and a 95 percent confidence level. Based on this information, what amount did the manager sell, and what was the resulting VAR?
13. Market risk can be defined in absolute or relative terms. Can a portfolio have a positive return yet have relative risk? Give an example.
14. A U.S. exporter sold forward Y125 million at the 7-month forward rate of Y124.27/\$. Immediately after the deal is signed, the spot rate moves from Y125 to Y130/\$. Dollar and yen rates are still at 6 and 5 percent, respectively. What is the gain/loss of the contract?
15. What features of cash and futures prices tend to make hedging possible?
16. In a foreign-currency futures contract, how is basis risk created?
17. When is basis risk greatest in general?
18. Assume that the spot rate for the euro against the U.S. dollar is \$1.05 (i.e., 1 euro buys 1.05 dollars). A U.S. bank pays 5.5 percent compounded annually for 1 year for a dollar deposit, and a German bank pays 2.5 percent compounded annually for 1 year for a euro deposit. The forward exchange rate is set on the contract at \$1.06. What is the current value of this forward contract to buy one euro 1 year from now?
19. A U.S. exporter anticipates receiving 1 million British pounds in 3 months. This is hedged with a short position in BP futures expiring in 6 months. The initial spot and futures prices are \$1.5000 and \$1.4703. At the time the hedge is lifted, the respective prices are \$1.4000 and \$1.3861. Ignoring the daily marking to market, what are the total proceeds to the exporter?
20. What is the major risk factor for a forward currency position?
21. Explain why the spot price for natural gas has, or should have, greater volatility than for gold.
22. A trader has a long position in at-the-money calls on \$1 million worth of an underlying stock with volatility of 20 percent. Roughly, what is the daily VAR at the 95 percent confidence level?
23. Discuss whether the delta VAR computed in the preceding question is likely to be more appropriate if the maturity of the option is 3 months or 5 days.

CHAPTER 12

Monte Carlo Methods

Deus ex machina.

Wall Street is often compared to a casino. The analogy is appropriate in one respect: Securities firms commonly use simulation techniques, known as *Monte Carlo methods*, to value complex derivatives and to measure risk. Simulation methods approximate the behavior of financial prices by using computer simulations to generate random price paths.

These methods are used to simulate a variety of different scenarios for the portfolio value on the target date. These scenarios can be generated in a random fashion (as in Monte Carlo simulation) or from historical data (as in historical simulation) or in other, more systematic ways. The portfolio value at risk (VAR) then can be read off directly from the distribution of simulated portfolio values.

Because of its flexibility, the simulation method is by far the most powerful approach to VAR. It potentially can account for a wide range of risks, including price risk, volatility risk, and complex interactions such as described by copulas in Chapter 8. Simulations can account for nonlinear exposures and complex pricing patterns. In principle, simulations can be extended to longer horizons, which is important for credit risk measurement and to more complex models of expected returns. Also, it can be used for operational risk measurement, as well as integrated risk management.

This approach, however, involves costly investments in intellectual and systems development. It also requires substantially more computing power than simpler methods. VAR measures using Monte Carlo methods often require hours to run. Time requirements, however, are being whittled down by advances in computers and faster valuation methods.

This chapter shows how simulation methods can be used to uncover VAR. The first section presents the rationale for Monte Carlo simulations. Section 12.2 introduces a simple case with just one random variable. Section 12.3 then discusses the tradeoff between speed and accuracy. The case with many sources of risk is discussed in Section 12.4. Next, Sections 12.5 and 12.6 turn to newer methods, such as deterministic simulations. The choice of models is reviewed in Section 12.7.

12.1 WHY MONTE CARLO SIMULATIONS?

The basic concept behind the Monte Carlo approach is to simulate repeatedly a random process for the financial variable of interest covering a wide range of possible situations. These variables are drawn from prespecified probability distributions that are assumed to be known, including the analytical function and its parameters. Thus simulations recreate the entire distribution of portfolio values, from which VAR can be derived.

Monte Carlo simulations were developed initially as a technique of statistical sampling to find solutions to integration problems, as shown in Box 12-1. For instance, take the problem of numerical integration of a

BOX 12-1

MONTE CARLO SIMULATIONS

Numerical simulations were first used by atom bomb scientists at Los Alamos in 1942 to crack problems that could not be solved by conventional means. Stanislaw Ulam, a Polish mathematician, is usually credited with inventing the Monte Carlo method while working at the Los Alamos laboratory.

While there, Ulam suggested that numerical simulations could be used to evaluate complicated mathematical integrals that arise in the theory of nuclear chain reactions. This suggestion led to the more formal development of Monte Carlo methods by John Von Neumann, Nicholas Metropolis, and others.

In his autobiography, *Adventures of a Mathematician*, Ulam recollects that the method was named in honor of his uncle, who was a gambler. The name *Monte Carlo* was derived from the name of a famous casino established in 1862 in the south of France (actually, in Monaco). What better way to evoke random draws, roulette, and games of chance?

function with many variables. A straightforward method is to perform the integration by computing the area under the curve using a number of evenly spaced samples from the function. In general, this works very well for functions of one variable. For functions with many variables, however, this method quickly becomes inefficient. With two variables, a 10×10 grid requires 100 points. With 100 variables, the grid requires 10^{100} points, which is too many to compute. This problem is called the *curse of dimensionality*.

Monte Carlo simulation instead provides an approximate solution to the problem that is much faster. Instead of systematically covering all values in the multidimensional space, it generates K random samples for the vector of variables. By the *central limit theorem*, this method generates estimates whose standard error decreases at the rate of $1/\sqrt{K}$, which does not depend on the size of the sample space. Thus the method does not suffer from the curse of dimensionality.

12.2 SIMULATIONS WITH ONE RANDOM VARIABLE

12.2.1 Simulating a Price Path

We first concentrate on a simple case with just one random variable. The first, and most crucial, step in the simulation consists of choosing a particular stochastic model for the behavior of prices. A commonly used model is the *geometric brownian motion* (GBM) *model*, which underlies much of options pricing theory. The model assumes that innovations in the asset price are uncorrelated over time and that small movements in prices can be described by

$$dS_t = \mu_t S_t dt + \sigma_t S_t dz \tag{12.1}$$

where dz is a random variable distributed normally with mean zero and variance dt. This variable drives the random shocks to the price and does not depend on past information. It is *brownian* in the sense that its variance decreases continuously with the time interval, $V(dz) = dt$. This rules out processes with sudden jumps, for instance. The process is also *geometric* because all parameters are scaled by the current price S_t.

The parameters μ_t and σ_t represent the instantaneous drift and volatility at time t, which can evolve over time. For simplicity, we will assume in what follows that these parameters are constant over time. But since μ_t

and σ_t can be functions of past variables, it would be easy to simulate time variation in the variances as in a GARCH process, for example.

In practice, the process with infinitesimally small increments dt can be approximated by discrete moves of size Δt. Define t as the present time, T as the target time, and $\tau = T - t$ as the (VAR) horizon. To generate a series of random variables S_{t+i} over the interval τ, we first chop up τ into n increments, with $\Delta t = \tau/n$.[1]

Integrating dS/S over a finite interval, we have approximately

$$\Delta S_t = S_{t-1}\,(\mu\Delta t + \sigma\epsilon\sqrt{\Delta t}) \tag{12.2}$$

where ϵ is now a standard normal random variable, that is, with mean zero and unit variance. We can verify that this process generates a mean $E(\Delta S/S) = \mu\Delta t$, which grows with time, as does the variance $V(\Delta S/S) = \sigma^2\Delta t$.

To simulate the price path for S, we start from S_t and generate a sequence of epsilons (ϵ's) *for* $i = 1, 2, \ldots, n$. Then S_{t+1} is set at $S_{t+1} = S_t + S_t(\mu\Delta t + \sigma\epsilon_1\sqrt{\Delta t})$, S_{t+2} is similarly computed from $S_{t+1} + S_{t+1}(\mu\Delta t + \sigma\epsilon_2\sqrt{\Delta t})$, and so on for future values, until the target horizon is reached, at which point the price is $S_{t+n} = S_T$.

Table 12-1 illustrates a simulation of a process with a drift μ of zero and volatility σ of 10 percent over the total interval. The initial price is \$100, and the interval is cut into 100 steps. Therefore, the local volatility is $0.10 \times \sqrt{1/100} = 0.01$.

TABLE 12-1

Simulating a Price Path

Step i	Previous Price S_{t+i-1}	Random Variable ϵ_i	Increment ΔS	Current Price S_{t+i}
1	100.00	0.199	0.00199	100.20
2	100.20	1.665	0.01665	101.87
3	101.87	−0.445	−0.00446	101.41
4	101.41	−0.667	−0.00668	100.74
⋮	⋮	⋮	⋮	⋮
100	92.47	1.153	−0.0153	91.06

[1] The choice of the number of steps should depend on the VAR horizon and the required accuracy. A smaller number of steps will be faster to implement but may not provide a good approximation of the stochastic process.

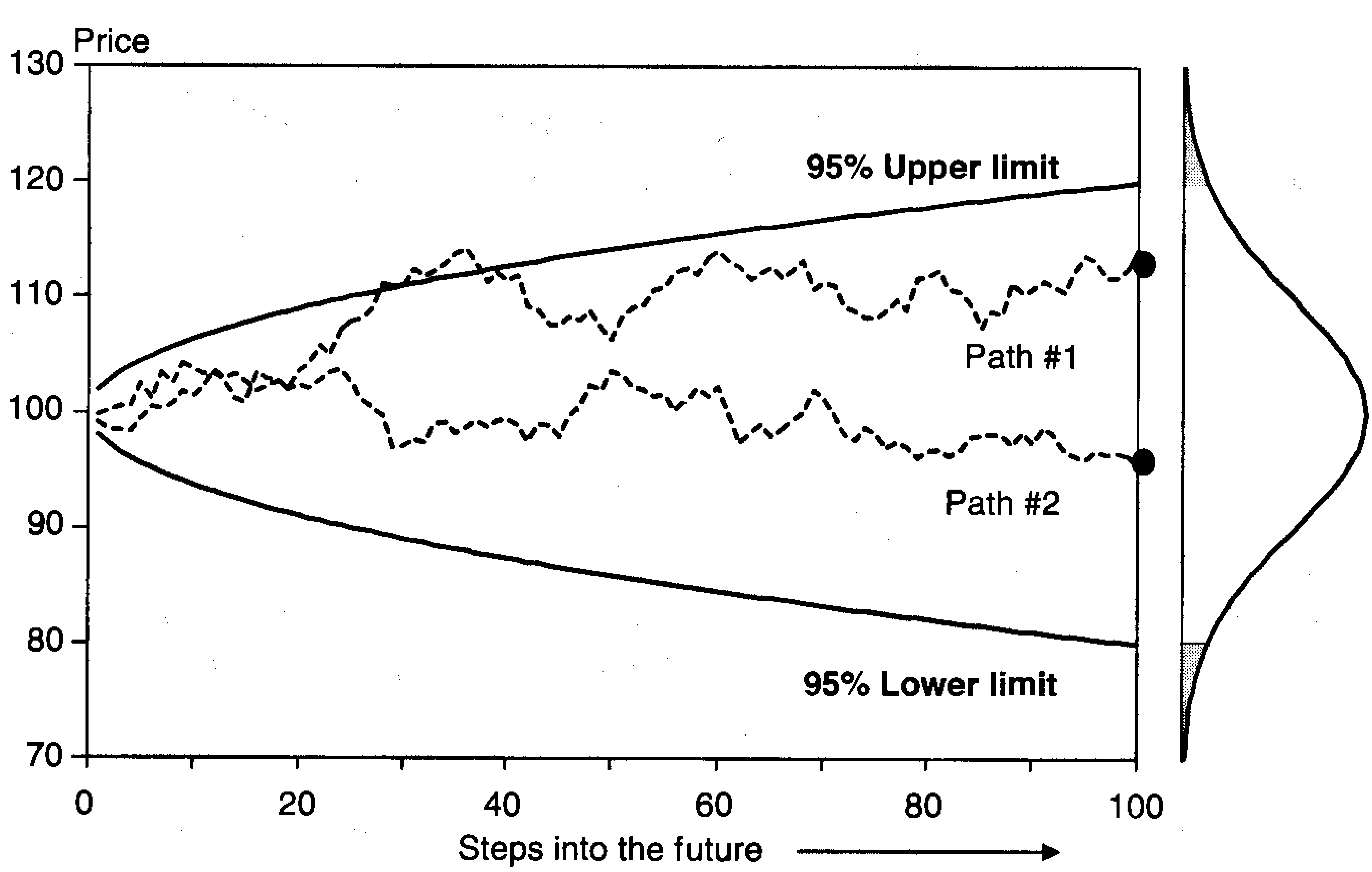
The second column starts with the initial price. The next column displays the realization of a standard normal variable. With no drift, the increment in the following column is simply ($\epsilon \times 0.01$). Finally, the last column computes the current price from the previous price and the increment. The values at each point are conditional on the simulated values at the previous point. The process is repeated until the final price of \$91.06 is reached at the 100th step.

Figure 12-1 presents two price paths, each leading to a different ending price. Given these assumptions, the ending price must follow a normal distribution with mean of \$100 and standard deviation of \$10.[2] This distribution is illustrated on the right side of the figure, along with 95 percent confidence bands, corresponding to two standard deviation intervals.

But the distribution also is known at any intermediate point. The figure displays 95 percent confidence bands that increase with the square root of time until they reach $\pm 2 \times 10$ percent. In this simple model, risk can be computed at any point up to the target horizon.

FIGURE 12-1

Simulating price paths.



[2] In fact, the ending distribution is lognormal because the price can never fall below 0.

12.2.2 Creating Random Numbers

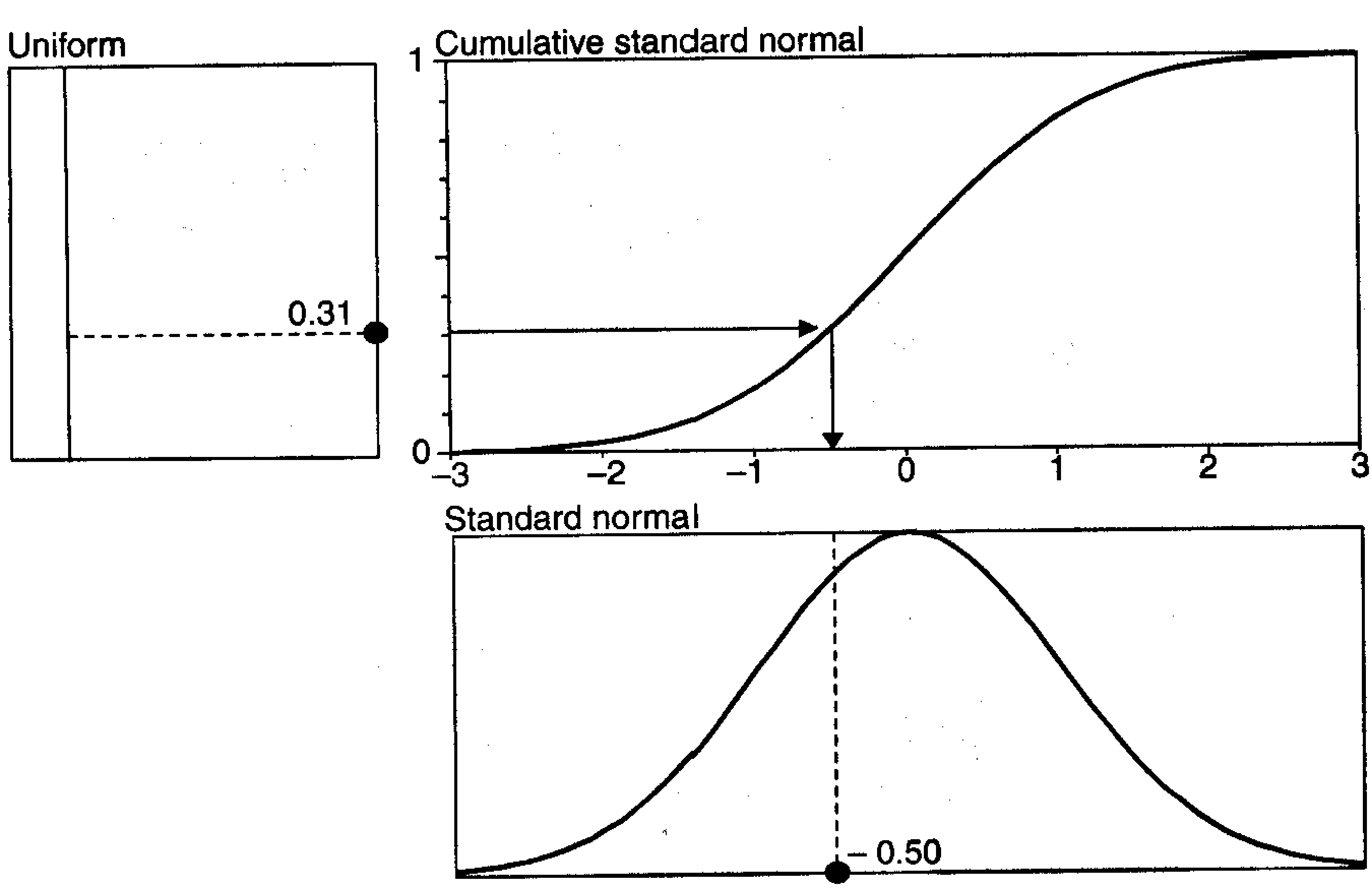
Monte Carlo simulations are based on random draws ϵ from a variable with the desired probability distribution. The numerical analysis usually proceeds in two steps.

The first building block for a random-number generator is a uniform distribution over the interval [0,1] that produces a random variable x. More properly speaking, these numbers are "pseudo" random because they are generated from an algorithm using a predefined rule. Starting from the same "seed" number, the sequence can be repeated at will.

The next step is to transform the uniform random number x into the desired distribution through the inverse cumulative probability distribution function (pdf). Take the normal distribution. By definition, the cumulative pdf $N(y)$ is always between 0 and 1. Therefore, to generate a normally distributed random variable, we compute y such that $x = N(y)$ or $y = N^{-1}(x)$.[3] More generally, any distribution function can be generated as long as the function $N(y)$ can be inverted. Figure 12-2 illustrates this procedure, called the *inverse transform method.*

FIGURE 12-2

Transformation from uniform to normal.



[3] Moro (1995) shows how to use approximations to the function N^{-1} to accelerate the speed of computation.

At this point, an important caveat is in order. It seems easy to generate variables that are purely random, but in practice, it is quite difficult. A well-designed algorithm should generate draws that "appear" independent over time. Whether this sequence is truly random is a philosophical issue that we will not address. Good random-number generators must create series that pass all conventional tests of independence. Otherwise, the characteristics of the simulated price process will not obey the underlying model.

Most operating systems, unfortunately, provide a random-number generator that is simple but inaccurate. All algorithms "cycle" after some iterations; that is, they repeat the same sequence of pseudorandom numbers. Good algorithms cycle after billions of draws; bad ones may cycle after a few thousand only.

If the cycle is too short, dependencies will be introduced in the price process solely because of the random-number generator. As a result, the range of possible portfolio values may be incomplete, thus leading to incorrect measures of VAR. This is why it is important to use a good-quality algorithm, such as those found in numerical libraries.

12.2.3 The Bootstrap

An alternative to generating random numbers from a hypothetical distribution is to sample from historical data. Thus we are agnostic about the distribution. For example, suppose that we observe a series of M returns $R = \Delta S/S$, $\{\mathrm{R}\} = (R_1 \cdots R_M)$, which can be assumed to be i.i.d. random variables drawn from an unknown distribution. The historical simulation method consists of using this series once to generate pseudoreturns. But this can be extended much further.

The bootstrap estimates this distribution by the empirical distribution of R, assigning equal probability to each realization. The method was proposed initially by Efron (1979) as a nonparametric randomization technique that draws from the observed distribution of the data to model the distribution of a statistic of interest.[4]

The procedure is carried out by sampling from $\{R\}$, with replacement, as many observations as necessary. For instance, assume that we want to generate 100 returns into the future, but we do not want to impose any assumption on the distribution of daily returns. We could project returns by randomly picking one return at a time from the sample over the

[4] The asymptotic properties of the bootstrap for commonly used statistics such as the mean, median, variance, and distribution quantiles have been studied by Bickel and Freedman (1981).

past $M = 500$ days, with replacement. Define the index choice as $m(1)$, a number between 1 and 500. The selected return then is $R_{m(1)}$, and the simulated next-day return is $S_{t+1} = S_t(1 + R_{m(1)})$. Repeating the operation for a total of 100 draws yields a total of 100 pseudovalues $S_{t+1}, \ldots, S_{t+n}$.

An essential advantage of the bootstrap is that it can include fat tails, jumps, or any departure from the normal distribution. For instance, one could include the return for the crash of October 19, 1987, which would never (or nearly never) occur under a normal distribution. The method also accounts for correlations across series because one draw consists of the simultaneous returns for N series, such as stock, bonds, and currency prices.

The bootstrap approach, it should be noted, has limitations. For small sample sizes M, the bootstrapped distribution may be a poor approximation of the actual one. Therefore, it is important to have access to sufficient data points. The other drawback of the bootstrap is that is relies heavily on the assumption that returns are independent. By resampling at random, any pattern of time variation is broken.

The bootstrap, however, can accommodate some time variation in parameters as long as we are willing to take a stand on the model. For instance, the bootstrap can be applied to the normalized residuals of a GARCH process, that is,

$$\epsilon_t = \frac{r_t}{\sigma_t}$$

where r_t is the actual return, and σ_t is the conditional standard deviation from the estimated GARCH process. To recreate pseudoreturns, one then would first sample from the historical distribution of ϵ and then reconstruct the conditional variance and pseudoreturns.

12.2.4 Computing VAR

Once a price path has been simulated, we can build the portfolio distribution at the end of the selected horizon. The simulation is carried out by the following steps:

1. Choose a stochastic process and parameters.
2. Generate a pseudosequence of variables $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$, from which prices are computed as $S_{t+1}, S_{t+2}, \ldots, S_{t+n}$.
3. Calculate the value of the asset (or portfolio) $F_{t+n} = F_T$ under this particular sequence of prices at the target horizon.
4. Repeat steps 2 and 3 as many times as necessary, say, $K = 10{,}000$.