9 Linear Systems of Simultaneous Equations

The previous chapter described a number of possible departures from the ideal regression model arising from errors that are non-Gaussian, heteroskedastic, or autocorrelated. We saw that while these factors can make a difference for the small-sample validity of t and F tests, under any of Assumptions 8.1 through 8.6, the *OLS* estimator $\mathbf{b}_T$ is either unbiased or consistent. This is because all these cases retained the crucial assumption that u_t, the error term for observation t, is uncorrelated with $\mathbf{x}_t$, the explanatory variables for that observation. Unfortunately, this critical assumption is unlikely to be satisfied in many important applications.

Section 9.1 discusses why this assumption often fails to hold, by examining a concrete example of *simultaneous equations bias*. Subsequent sections discuss a variety of techniques for dealing with this problem. These results will be used in the structural interpretation of vector autoregressions in Chapter 11 and for understanding generalized method of moments estimation in Chapter 14.

9.1. Simultaneous Equations Bias

To illustrate the difficulties with endogenous regressors, consider an investigation of the public's demand for oranges. Let p_t denote the log of the price of oranges in a particular year and q_t^d the log of the quantity the public is willing to buy. To keep the example very simple, suppose that price and quantity are covariance-stationary and that each is measured as deviations from its population mean. The demand curve is presumed to take the form

$$q_t^d = \beta p_t + \varepsilon_t^d, \tag{9.1.1}$$

with $\beta < 0$; a higher price reduces the quantity that the public is willing to buy. Here ε_t^d represents factors that influence demand other than price. These are assumed to be independent and identically distributed with mean zero and variance σ_d^2.

The price also influences the supply of oranges brought to the market,

$$q_t^s = \gamma p_t + \varepsilon_t^s, \tag{9.1.2}$$

where $\gamma > 0$ and ε_t^s represents factors that influence supply other than price. These omitted factors are again assumed to be i.i.d. with mean zero and variance σ_s^2, with the supply disturbance ε_t^s uncorrelated with the demand disturbance ε_t^d.

Equation [9.1.1] describes the behavior of buyers of oranges, and equation [9.1.2] describes the behavior of sellers. Market equilibrium requires $q_t^d = q_t^s$, or

$$\beta p_t + \varepsilon_t^d = \gamma p_t + \varepsilon_t^s.$$

Rearranging,

$$p_t = \frac{\varepsilon_t^d - \varepsilon_t^s}{\gamma - \beta}. \qquad [9.1.3]$$

Substituting this back into [9.1.2],

$$q_t = \gamma \frac{\varepsilon_t^d - \varepsilon_t^s}{\gamma - \beta} + \varepsilon_t^s = \frac{\gamma}{\gamma - \beta}\varepsilon_t^d - \frac{\beta}{\gamma - \beta}\varepsilon_t^s. \qquad [9.1.4]$$

Consider the consequences of trying to estimate [9.1.1] by *OLS*. A regression of quantity on price will produce the estimate

$$b_T = \frac{(1/T)\sum_{t=1}^{T} p_t q_t}{(1/T)\sum_{t=1}^{T} p_t^2}. \qquad [9.1.5]$$

Substituting [9.1.3] and [9.1.4] into the numerator in [9.1.5] results in

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T} p_t q_t &= \frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{\gamma - \beta}\varepsilon_t^d - \frac{1}{\gamma - \beta}\varepsilon_t^s\right]\left[\frac{\gamma}{\gamma - \beta}\varepsilon_t^d - \frac{\beta}{\gamma - \beta}\varepsilon_t^s\right] \\
&= \frac{1}{T}\sum_{t=1}^{T}\left[\frac{\gamma}{(\gamma - \beta)^2}(\varepsilon_t^d)^2 + \frac{\beta}{(\gamma - \beta)^2}(\varepsilon_t^s)^2 - \frac{\gamma + \beta}{(\gamma - \beta)^2}\varepsilon_t^d\varepsilon_t^s\right] \\
&\xrightarrow{p} \frac{\gamma\sigma_d^2 + \beta\sigma_s^2}{(\gamma - \beta)^2}.
\end{aligned}$$

Similarly, for the denominator,

$$\frac{1}{T}\sum_{t=1}^{T} p_t^2 = \frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{\gamma - \beta}\varepsilon_t^d - \frac{1}{\gamma - \beta}\varepsilon_t^s\right]^2 \xrightarrow{p} \frac{\sigma_d^2 + \sigma_s^2}{(\gamma - \beta)^2}.$$

Hence,

$$b_T \xrightarrow{p} \left[\frac{\sigma_d^2 + \sigma_s^2}{(\gamma - \beta)^2}\right]^{-1}\left[\frac{\gamma\sigma_d^2 + \beta\sigma_s^2}{(\gamma - \beta)^2}\right] = \frac{\gamma\sigma_d^2 + \beta\sigma_s^2}{\sigma_d^2 + \sigma_s^2}. \qquad [9.1.6]$$

OLS regression thus gives not the demand elasticity β but rather an average of β and the supply elasticity γ, with weights depending on the sizes of the variances σ_d^2 and σ_s^2. If the error in the demand curve is negligible ($\sigma_d^2 \to 0$) or if the error term in the supply curve has a big enough variance ($\sigma_s^2 \to \infty$), then [9.1.6] indicates that *OLS* would give a consistent estimate of the demand elasticity β. On the other hand, if $\sigma_d^2 \to \infty$ or $\sigma_s^2 \to 0$, then *OLS* gives a consistent estimate of the supply elasticity γ. In the cases in between, one economist might believe the regression was estimating the demand curve [9.1.1] and a second economist might perform the same regression calling it the supply curve [9.1.2]. The actual *OLS* estimates would represent a mixture of both. This phenomenon is known as *simultaneous equations bias*.

Figure 9.1 depicts the problem graphically.[1] At any date in the sample, there is some demand curve (determined by the value of ε_t^d) and a supply curve (determined by ε_t^s), with the observation on (p_t, q_t) given by the intersection of these two curves. For example, date 1 may have been associated with a small negative shock to demand, producing the curve D_1, and a large positive shock to supply, producing S_1. The date 1 observation will then be (p_1, q_1). Date 2 might have seen

[1]Economists usually display these figures with the axes reversed from those displayed in Figure 9.1.

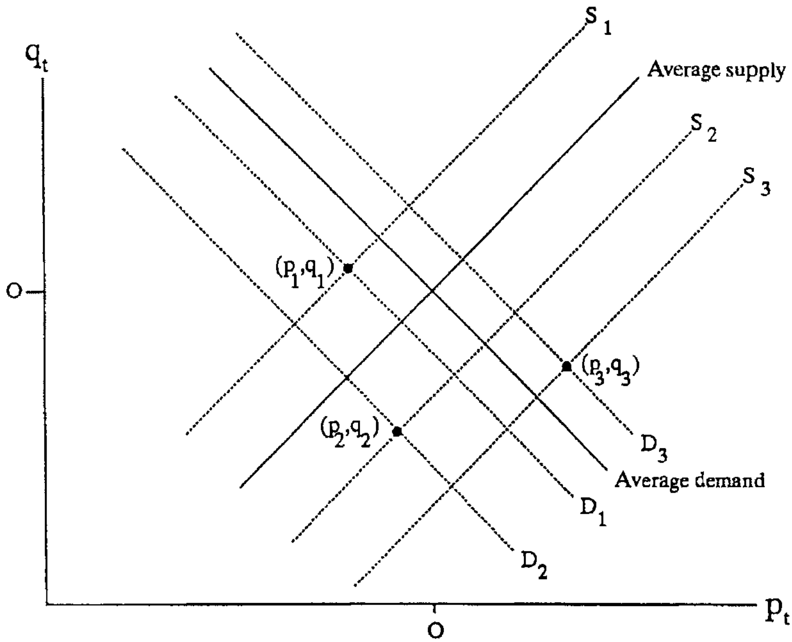

FIGURE 9.1 Observations on price and quantity implied by disturbances to both supply functions and demand functions.

a bigger negative shock to demand and a negative shock to supply, while date 3 as drawn reflects a modest positive shock to demand and a large negative shock to supply. *OLS* tries to fit a line through the scatter of points $\{p_t, q_t\}_{t=1}^{T}$.

If the shocks are known to be due to the supply curve and not the demand curve, then the scatter of points will trace out the demand curve, as in Figure 9.2. If the shocks are due to the demand curve rather than the supply curve, the scatter will trace out the supply curve, as in Figure 9.3.

The problem of simultaneous equations bias is extremely widespread in the social sciences. It is rare that the relation that we would like to estimate is the only possible reason why there might be a correlation among a group of variables.

Consistent Estimation of the Demand Elasticity

The above analysis suggests that consistent estimates of the demand elasticity might be obtained if we could find a variable that shifts the supply curve but not the demand curve. For example, let w_t represent the number of days of below-freezing temperatures in Florida during year t. Recalling that the supply disturbance ε_t^s was defined as factors influencing supply other than price, w_t seems likely to be an important component of ε_t^s. Define h to be the coefficient from a linear projection of ε_t^s on w_t, and write

$$\varepsilon_t^s = hw_t + u_t^s. \qquad [9.1.7]$$

Thus, u_t^s is uncorrelated with w_t, by the definition of h. Although Florida weather is likely to influence the supply of oranges, it is natural to assume that weather

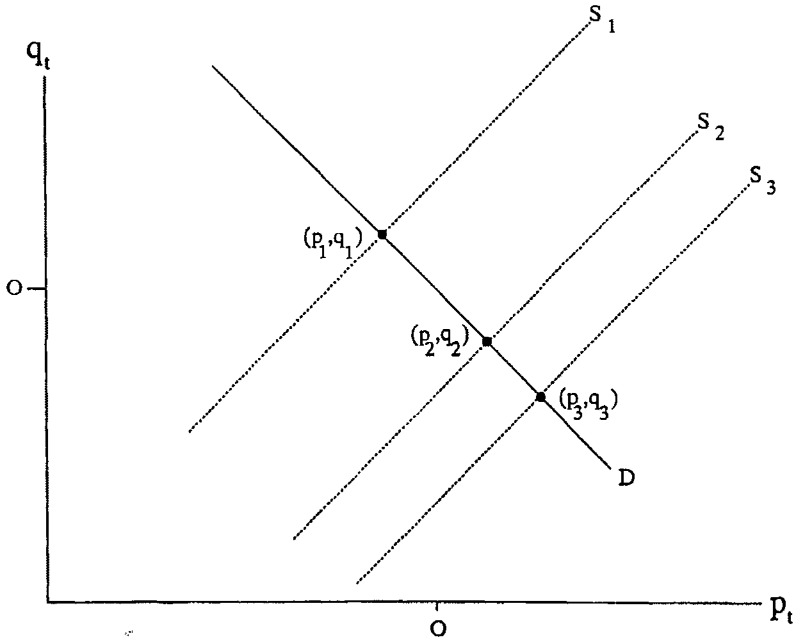

FIGURE 9.2 Observations on price and quantity implied by disturbances to supply function only.

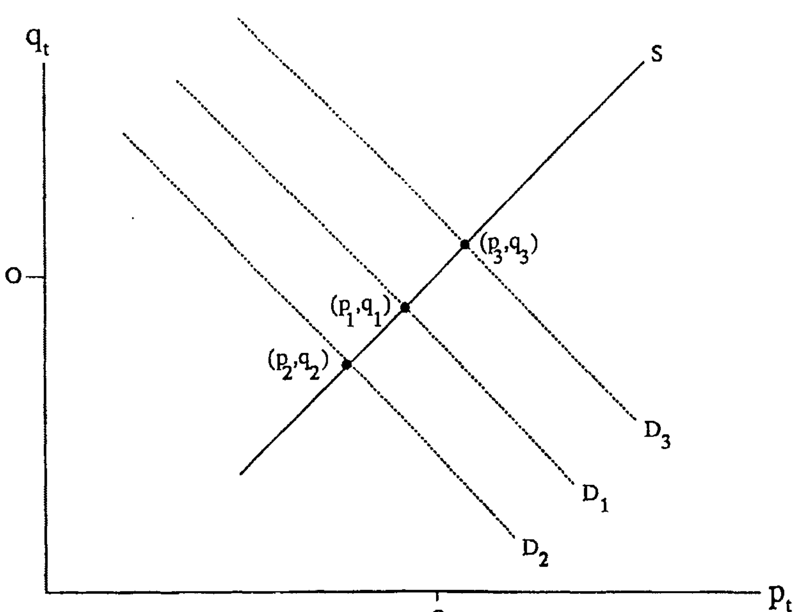

FIGURE 9.3 Observations on price and quantity implied by disturbances to demand function only.

matters for the public's demand for oranges only through its effect on the price. Under this assumption, both w_t and u_t^s are uncorrelated with ε_t^d. Changes in price that can be attributed to the weather represent supply shifts and not demand shifts.

Define p_t^* to be the linear projection of p_t on w_t. Substituting [9.1.7] into [9.1.3],

$$p_t = \frac{\varepsilon_t^d - hw_t - u_t^s}{\gamma - \beta}, \qquad [9.1.8]$$

and thus

$$p_t^* = \frac{-h}{\gamma - \beta} w_t, \qquad [9.1.9]$$

since ε_t^d and u_t^s are uncorrelated with w_t. Equation [9.1.8] can thus be written

$$p_t = p_t^* + \frac{\varepsilon_t^d - u_t^s}{\gamma - \beta},$$

and substituting this into [9.1.1],

$$q_t = \beta\left\{p_t^* + \frac{\varepsilon_t^d - u_t^s}{\gamma - \beta}\right\} + \varepsilon_t^d = \beta p_t^* + v_t, \qquad [9.1.10]$$

where

$$v_t \equiv \frac{-\beta u_t^s}{\gamma - \beta} + \frac{\gamma \varepsilon_t^d}{\gamma - \beta}.$$

Since u_t^s and ε_t^d are both uncorrelated with w_t, it follows that v_t is uncorrelated with p_t^*. Hence, if [9.1.10] were estimated by ordinary least squares, the result would be a consistent estimate of β:

$$\begin{aligned}
\hat{\beta}_T^* &= \frac{(1/T)\sum_{t=1}^T p_t^* q_t}{(1/T)\sum_{t=1}^T [p_t^*]^2} \\
&= \frac{(1/T)\sum_{t=1}^T p_t^*(\beta p_t^* + v_t)}{(1/T)\sum_{t=1}^T [p_t^*]^2} \\
&= \beta + \frac{(1/T)\sum_{t=1}^T p_t^* v_t}{(1/T)\sum_{t=1}^T [p_t^*]^2} \\
&\xrightarrow{p} \beta.
\end{aligned} \qquad [9.1.11]$$

The suggestion is thus to regress quantity on that component of price that is induced by the weather, that is, regress quantity on the linear projection of price on the weather.

In practice, we will not know the values of the population parameters h, γ, and β necessary to construct p_t^* in [9.1.9]. However, the linear projection p_t^* can be consistently estimated by the fitted value for observation t from an *OLS* regression of p on w,

$$\hat{p}_t = \hat{\delta}_T w_t, \qquad [9.1.12]$$

where

$$\hat{\delta}_T = \frac{(1/T)\sum_{t=1}^{T} w_t p_t}{(1/T)\sum_{t=1}^{T} w_t^2}.$$

The estimator [9.1.11] with p_t^* replaced by $\hat{p}_t$ is known as the *two-stage least squares* ($2SLS$) coefficient estimator:

$$\hat{\beta}_{2SLS} = \frac{(1/T)\sum_{t=1}^{T} \hat{p}_t q_t}{(1/T)\sum_{t=1}^{T} [\hat{p}_t]^2}. \qquad [9.1.13]$$

Like $\hat{\beta}_T^*$, the $2SLS$ estimator is consistent, as will be shown in the following section.

9.2. *Instrumental Variables and Two-Stage Least Squares*

General Description of Two-Stage Least Squares

A generalization of the previous example is as follows. Suppose the objective is to estimate the vector $\boldsymbol{\beta}$ in the regression model

$$y_t = \boldsymbol{\beta}'\mathbf{z}_t + u_t, \qquad [9.2.1]$$

where $\mathbf{z}_t$ is a $(k \times 1)$ vector of explanatory variables. Some subset $n \leq k$ of the variables in $\mathbf{z}_t$ are thought to be endogenous, that is, correlated with u_t. The remaining $k - n$ variables in $\mathbf{z}_t$ are said to be *predetermined*, meaning that they are uncorrelated with u_t. Estimation of $\boldsymbol{\beta}$ requires variables known as *instruments*. To be a valid instrument, a variable must be correlated with an endogenous explanatory variable in $\mathbf{z}_t$ but uncorrelated with the regression disturbance u_t. In the supply-and-demand example, the weather variable w_t served as an instrument for price. At least one valid instrument must be found for each endogenous explanatory variable.

Collect the predetermined explanatory variables together with the instruments in an $(r \times 1)$ vector $\mathbf{x}_t$. For example, to estimate the demand curve, there were no predetermined explanatory variables in equation [9.1.1] and only a single instrument; hence, $r = 1$, and $\mathbf{x}_t$ would be the scalar w_t. As a second example, suppose that the equation to be estimated is

$$y_t = \beta_1 + \beta_2 z_{2t} + \beta_3 z_{3t} + \beta_4 z_{4t} + \beta_5 z_{5t} + u_t.$$

In this example, z_{4t} and z_{5t} are endogenous (meaning that they are correlated with u_t), z_{2t} and z_{3t} are predetermined (uncorrelated with u_t), and ξ_{1t}, ξ_{2t}, and ξ_{3t} are valid instruments (correlated with z_{4t} and z_{5t} but uncorrelated with u_t). Then $r = 6$ and $\mathbf{x}_t' = (1, z_{2t}, z_{3t}, \xi_{1t}, \xi_{2t}, \xi_{3t})$. The requirement that there be at least as many instruments as endogenous explanatory variables implies that $r \geq k$.

Consider an OLS regression of z_{it} (the ith explanatory variable in [9.2.1]) on $\mathbf{x}_t$:

$$z_{it} = \boldsymbol{\delta}_i'\mathbf{x}_t + e_{it}. \qquad [9.2.2]$$

The fitted values for the regression are given by

$$\hat{z}_{it} = \hat{\boldsymbol{\delta}}_i'\mathbf{x}_t, \qquad [9.2.3]$$

where

$$\hat{\boldsymbol{\delta}}_i = \left[\sum_{t=1}^{T} \mathbf{x}_t\mathbf{x}_t'\right]^{-1}\left[\sum_{t=1}^{T} \mathbf{x}_t z_{it}\right].$$

If z_{it} is one of the predetermined variables, then z_{it} is one of the elements of $\mathbf{x}_t$ and equation [9.2.3] simplifies to

$$\hat{z}_{it} = z_{it}.$$

This is because when the dependent variable (z_{it}) is included in the regressors ($\mathbf{x}_t$), a unit coefficient on z_{it} and zero coefficients on the other variables produce a perfect fit and thus minimize the residual sum of squares.

Collect the equations in [9.2.3] for $i = 1, 2, \ldots, k$ in a $(k \times 1)$ vector equation

$$\hat{\mathbf{z}}_t = \hat{\boldsymbol{\delta}}'\mathbf{x}_t, \qquad [9.2.4]$$

where the $(k \times r)$ matrix $\hat{\boldsymbol{\delta}}'$ is given by

$$\hat{\boldsymbol{\delta}}' = \begin{bmatrix} \hat{\boldsymbol{\delta}}_1' \\ \hat{\boldsymbol{\delta}}_2' \\ \vdots \\ \hat{\boldsymbol{\delta}}_k' \end{bmatrix} = \left[\sum_{t=1}^{T} \mathbf{z}_t\mathbf{x}_t'\right]\left[\sum_{t=1}^{T} \mathbf{x}_t\mathbf{x}_t'\right]^{-1}. \qquad [9.2.5]$$

The *two-stage least squares* (*2SLS*) estimate of $\boldsymbol{\beta}$ is found from an *OLS* regression of y_t on $\hat{\mathbf{z}}_t$:

$$\hat{\boldsymbol{\beta}}_{2SLS} = \left[\sum_{t=1}^{T} \hat{\mathbf{z}}_t\hat{\mathbf{z}}_t'\right]^{-1}\left[\sum_{t=1}^{T} \hat{\mathbf{z}}_t y_t\right]. \qquad [9.2.6]$$

An alternative way of writing [9.2.6] is sometimes useful. Let $\hat{e}_{it}$ denote the sample residual from *OLS* estimation of [9.2.2]; that is, let

$$z_{it} = \hat{\boldsymbol{\delta}}_i'\mathbf{x}_t + \hat{e}_{it} = \hat{z}_{it} + \hat{e}_{it}. \qquad [9.2.7]$$

OLS causes this residual to be orthogonal to $\mathbf{x}_t$:

$$\sum_{t=1}^{T} \mathbf{x}_t\hat{e}_{it} = \mathbf{0},$$

meaning that the residual is orthogonal to $\hat{z}_{jt}$:

$$\sum_{t=1}^{T} \hat{z}_{jt}\hat{e}_{it} = \hat{\boldsymbol{\delta}}_j' \sum_{t=1}^{T} \mathbf{x}_t\hat{e}_{it} = 0.$$

Hence, if [9.2.7] is multiplied by $\hat{z}_{jt}$ and summed over t, the result is

$$\sum_{t=1}^{T} \hat{z}_{jt}z_{it} = \sum_{t=1}^{T} \hat{z}_{jt}(\hat{z}_{it} + \hat{e}_{it}) = \sum_{t=1}^{T} \hat{z}_{jt}\hat{z}_{it}$$

for all i and j. This means that

$$\sum_{t=1}^{T} \hat{\mathbf{z}}_t\mathbf{z}_t' = \sum_{t=1}^{T} \hat{\mathbf{z}}_t\hat{\mathbf{z}}_t',$$

so that the 2*SLS* estimator [9.2.6] can equivalently be written as

$$\hat{\boldsymbol{\beta}}_{2SLS} = \left[\sum_{t=1}^{T} \hat{\mathbf{z}}_t\mathbf{z}_t'\right]^{-1}\left[\sum_{t=1}^{T} \hat{\mathbf{z}}_t y_t\right]. \qquad [9.2.8]$$

Consistency of 2SLS Estimator

Substituting [9.2.1] into [9.2.8],

$$\begin{aligned}\hat{\boldsymbol{\beta}}_{2SLS,T} &= \left[\sum_{t=1}^{T} \hat{\mathbf{z}}_t \mathbf{z}_t'\right]^{-1}\left[\sum_{t=1}^{T} \hat{\mathbf{z}}_t(\mathbf{z}_t'\boldsymbol{\beta} + u_t)\right] \\ &= \boldsymbol{\beta} + \left[\sum_{t=1}^{T} \hat{\mathbf{z}}_t \mathbf{z}_t'\right]^{-1}\left[\sum_{t=1}^{T} \hat{\mathbf{z}}_t u_t\right],\end{aligned} \tag{9.2.9}$$

where the subscript T has been added to keep explicit track of the sample size T on which estimation is based. It follows from [9.2.9] that

$$\hat{\boldsymbol{\beta}}_{2SLS,T} - \boldsymbol{\beta} = \left[(1/T)\sum_{t=1}^{T} \hat{\mathbf{z}}_t \mathbf{z}_t'\right]^{-1}\left[(1/T)\sum_{t=1}^{T} \hat{\mathbf{z}}_t u_t\right]. \tag{9.2.10}$$

Consistency of the $2SLS$ estimator can then be shown as follows. First note from [9.2.4] and [9.2.5] that

$$\begin{aligned}(1/T)\sum_{t=1}^{T} \hat{\mathbf{z}}_t \mathbf{z}_t' &= \hat{\boldsymbol{\delta}}_T'(1/T)\sum_{t=1}^{T} \mathbf{x}_t \mathbf{z}_t' \\ &= \left[(1/T)\sum_{t=1}^{T} \mathbf{z}_t \mathbf{x}_t'\right]\left[(1/T)\sum_{t=1}^{T} \mathbf{x}_t \mathbf{x}_t'\right]^{-1}\left[(1/T)\sum_{t=1}^{T} \mathbf{x}_t \mathbf{z}_t'\right].\end{aligned} \tag{9.2.11}$$

Assuming that the process $(\mathbf{z}_t, \mathbf{x}_t)$ is covariance-stationary and ergodic for second moments,

$$(1/T)\sum_{t=1}^{T} \hat{\mathbf{z}}_t \mathbf{z}_t' \xrightarrow{p} \mathbf{Q}, \tag{9.2.12}$$

where

$$\mathbf{Q} = [E(\mathbf{z}_t\mathbf{x}_t')][E(\mathbf{x}_t\mathbf{x}_t')]^{-1}[E(\mathbf{x}_t\mathbf{z}_t')]. \tag{9.2.13}$$

Turning next to the second term in [9.2.10],

$$\left[(1/T)\sum_{t=1}^{T} \hat{\mathbf{z}}_t u_t\right] = \hat{\boldsymbol{\delta}}_T'(1/T)\sum_{t=1}^{T} \mathbf{x}_t u_t.$$

Again, ergodicity for second moments implies from [9.2.5] that

$$\hat{\boldsymbol{\delta}}_T' \xrightarrow{p} [E(\mathbf{z}_t\mathbf{x}_t')][E(\mathbf{x}_t\mathbf{x}_t')]^{-1}, \tag{9.2.14}$$

while the law of large numbers will typically ensure that

$$(1/T)\sum_{t=1}^{T} \mathbf{x}_t u_t \xrightarrow{p} E(\mathbf{x}_t u_t) = \mathbf{0},$$

under the assumed absence of correlation between $\mathbf{x}_t$ and u_t. Hence,

$$\left[(1/T)\sum_{t=1}^{T} \hat{\mathbf{z}}_t u_t\right] \xrightarrow{p} \mathbf{0}. \tag{9.2.15}$$

Substituting [9.2.12] and [9.2.15] into [9.2.10], it follows that

$$\hat{\boldsymbol{\beta}}_{2SLS,T} - \boldsymbol{\beta} \xrightarrow{p} \mathbf{Q}^{-1}\cdot\mathbf{0} = \mathbf{0}.$$

Hence, the $2SLS$ estimator is consistent as long as the matrix $\mathbf{Q}$ in [9.2.13] is nonsingular.

Notice that if none of the predetermined variables is correlated with z_{it}, then the ith row of $E(\mathbf{z}_t\mathbf{x}_t')$ contains all zeros and the corresponding row of $\mathbf{Q}$ in [9.2.13] contains all zeros, in which case $2SLS$ is not consistent. Alternatively, if z_{it} is correlated with $\mathbf{x}_t$ only through, say, the first element x_{1t} and z_{jt} is also correlated with $\mathbf{x}_t$ only through x_{1t}, then subtracting some multiple of the ith row of $\mathbf{Q}$ from the jth row produces a row of zeros, and $\mathbf{Q}$ again is not invertible. In general, consistency of the $2SLS$ estimator requires the rows of $E(\mathbf{z}_t\mathbf{x}_t')$ to be linearly independent. This essentially amounts to the requirement that there be a way of assigning instruments to endogenous variables such that each endogenous variable has an instrument associated with it, with no instrument counted twice for this purpose.

Asymptotic Distribution of 2SLS Estimator

Equation [9.2.10] implies that

$$\sqrt{T}(\hat{\boldsymbol{\beta}}_{2SLS,T} - \boldsymbol{\beta}) = \left[(1/T)\sum_{t=1}^{T}\hat{\mathbf{z}}_t\mathbf{z}_t'\right]^{-1}\left[(1/\sqrt{T})\sum_{t=1}^{T}\hat{\mathbf{z}}_t u_t\right], \qquad [9.2.16]$$

where

$$\left[(1/\sqrt{T})\sum_{t=1}^{T}\hat{\mathbf{z}}_t u_t\right] = \hat{\boldsymbol{\delta}}_T'\,(1/\sqrt{T})\sum_{t=1}^{T}\mathbf{x}_t u_t.$$

Hence, from [9.2.12] and [9.2.14],

$$\sqrt{T}(\hat{\boldsymbol{\beta}}_{2SLS,T} - \boldsymbol{\beta}) \xrightarrow{p} \mathbf{Q}^{-1}\cdot[E(\mathbf{z}_t\mathbf{x}_t')][E(\mathbf{x}_t\mathbf{x}_t')]^{-1}\left((1/\sqrt{T})\sum_{t=1}^{T}\mathbf{x}_t u_t\right). \qquad [9.2.17]$$

Suppose that $\mathbf{x}_t$ is covariance-stationary and that $\{u_t\}$ is an i.i.d. sequence with mean zero and variance σ^2 with u_t independent of $\mathbf{x}_s$ for all $s \leq t$. Then $\{\mathbf{x}_t u_t\}$ is a martingale difference sequence with variance-covariance matrix given by $\sigma^2\cdot E(\mathbf{x}_t\mathbf{x}_t')$. If u_t and $\mathbf{x}_t$ have finite fourth moments, then we can expect from Proposition 7.9 that

$$\left((1/\sqrt{T})\sum_{t=1}^{T}\mathbf{x}_t u_t\right) \xrightarrow{L} N(\mathbf{0},\, \sigma^2\cdot E(\mathbf{x}_t\mathbf{x}_t')). \qquad [9.2.18]$$

Thus, [9.2.17] implies that

$$\sqrt{T}(\hat{\boldsymbol{\beta}}_{2SLS,T} - \boldsymbol{\beta}) \xrightarrow{L} N(\mathbf{0}, \mathbf{V}), \qquad [9.2.19]$$

where

$$\begin{aligned}\mathbf{V} &= \mathbf{Q}^{-1}[E(\mathbf{z}_t\mathbf{x}_t')][E(\mathbf{x}_t\mathbf{x}_t')]^{-1}[\sigma^2\cdot E(\mathbf{x}_t\mathbf{x}_t')][E(\mathbf{x}_t\mathbf{x}_t')]^{-1}[E(\mathbf{x}_t\mathbf{z}_t')]\mathbf{Q}^{-1}\\ &= \sigma^2\mathbf{Q}^{-1}\cdot\mathbf{Q}\cdot\mathbf{Q}^{-1}\\ &= \sigma^2\mathbf{Q}^{-1}\end{aligned} \qquad [9.2.20]$$

for $\mathbf{Q}$ given in [9.2.13]. Hence,

$$\hat{\boldsymbol{\beta}}_{2SLS,T} \approx N(\boldsymbol{\beta},\, (1/T)\sigma^2\mathbf{Q}^{-1}). \qquad [9.2.21]$$

Since $\hat{\boldsymbol\beta}_{2SLS}$ is a consistent estimate of $\boldsymbol\beta$, clearly a consistent estimate of the population residual for observation t is afforded by

$$\hat{u}_{t,T} \equiv y_t - \mathbf{z}_t'\hat{\boldsymbol\beta}_{2SLS,T} \xrightarrow{p} u_t. \qquad [9.2.22]$$

Similarly, it is straightforward to show that σ^2 can be consistently estimated by

$$\hat{\sigma}_T^2 = (1/T)\sum_{t=1}^{T}(y_t - \mathbf{z}_t'\hat{\boldsymbol\beta}_{2SLS,T})^2 \qquad [9.2.23]$$

(see Exercise 9.1). Note well that although $\hat{\boldsymbol\beta}_{2SLS}$ can be calculated from an *OLS* regression of y_t on $\hat{\mathbf{z}}_t$, the estimates $\hat{u}_t$ and $\hat{\sigma}^2$ in [9.2.22] and [9.2.23] are not based on the residuals from this regression:

$$\hat{u}_t \neq y_t - \hat{\mathbf{z}}_t'\hat{\boldsymbol\beta}_{2SLS}$$

$$\hat{\sigma}^2 \neq (1/T)\sum_{t=1}^{T}(y_t - \hat{\mathbf{z}}_t'\hat{\boldsymbol\beta}_{2SLS})^2.$$

The correct estimates [9.2.22] and [9.2.23] use the actual explanatory variables $\mathbf{z}_t$ rather than the fitted values $\hat{\mathbf{z}}_t$.

A consistent estimate of $\mathbf{Q}$ is provided by [9.2.11]:

$$\begin{aligned}\hat{\mathbf{Q}}_T &= (1/T)\sum_{t=1}^{T}\hat{\mathbf{z}}_t\hat{\mathbf{z}}_t' \\ &= \left[(1/T)\sum_{t=1}^{T}\mathbf{z}_t\mathbf{x}_t'\right]\left[(1/T)\sum_{t=1}^{T}\mathbf{x}_t\mathbf{x}_t'\right]^{-1}\left[(1/T)\sum_{t=1}^{T}\mathbf{x}_t\mathbf{z}_t'\right].\end{aligned} \qquad [9.2.24]$$

Substituting [9.2.23] and [9.2.24] into [9.2.21], the estimated variance-covariance matrix of the *2SLS* estimator is

$$\begin{aligned}\hat{\mathbf{V}}_T/T &= \hat{\sigma}_T^2\cdot(1/T)\cdot\left[(1/T)\sum_{t=1}^{T}\hat{\mathbf{z}}_t\hat{\mathbf{z}}_t'\right]^{-1} \\ &= \hat{\sigma}_T^2\cdot\left\{\left[\sum_{t=1}^{T}\mathbf{z}_t\mathbf{x}_t'\right]\left[\sum_{t=1}^{T}\mathbf{x}_t\mathbf{x}_t'\right]^{-1}\left[\sum_{t=1}^{T}\mathbf{x}_t\mathbf{z}_t'\right]\right\}^{-1}.\end{aligned} \qquad [9.2.25]$$

A test of the null hypothesis $\mathbf{R}\boldsymbol\beta = \mathbf{r}$ can thus be based on

$$(\mathbf{R}\hat{\boldsymbol\beta}_{2SLS,T} - \mathbf{r})'[\mathbf{R}(\hat{\mathbf{V}}_T/T)\mathbf{R}']^{-1}(\mathbf{R}\hat{\boldsymbol\beta}_{2SLS,T} - \mathbf{r}), \qquad [9.2.26]$$

which, under the null hypothesis, has an asymptotic distribution that is χ^2 with degrees of freedom given by m, where m represents the number of restrictions or the number of rows of $\mathbf{R}$.

Heteroskedasticity- and autocorrelation-consistent standard errors for *2SLS* estimation will be discussed in Chapter 14.

Instrumental Variable Estimation

Substituting [9.2.4] and [9.2.5] into [9.2.8], the *2SLS* estimator can be written as

$$\begin{aligned}\hat{\boldsymbol\beta}_{2SLS} &= \left[\sum_{t=1}^{T}\hat{\boldsymbol\delta}'\mathbf{x}_t\mathbf{z}_t'\right]^{-1}\left[\sum_{t=1}^{T}\hat{\boldsymbol\delta}'\mathbf{x}_t y_t\right] \\ &= \left\{\left[\sum_{t=1}^{T}\mathbf{z}_t\mathbf{x}_t'\right]\left[\sum_{t=1}^{T}\mathbf{x}_t\mathbf{x}_t'\right]^{-1}\left[\sum_{t=1}^{T}\mathbf{x}_t\mathbf{z}_t'\right]\right\}^{-1}\left\{\left[\sum_{t=1}^{T}\mathbf{z}_t\mathbf{x}_t'\right]\left[\sum_{t=1}^{T}\mathbf{x}_t\mathbf{x}_t'\right]^{-1}\left[\sum_{t=1}^{T}\mathbf{x}_t y_t\right]\right\}.\end{aligned}$$

[9.2.27]

Consider the special case in which the number of instruments is exactly equal to the number of endogenous explanatory variables, so that $r = k$, as was the case for estimation of the demand curve in Section 9.1. Then $\Sigma_{t=1}^{T} \mathbf{z}_t \mathbf{x}_t'$ is a $(k \times k)$ matrix and [9.2.27] becomes

$$\begin{aligned}\hat{\boldsymbol{\beta}}_{IV} &= \left\{\left[\sum_{t=1}^{T} \mathbf{x}_t \mathbf{z}_t'\right]^{-1}\left[\sum_{t=1}^{T} \mathbf{x}_t \mathbf{x}_t'\right]\left[\sum_{t=1}^{T} \mathbf{z}_t \mathbf{x}_t'\right]^{-1}\right\} \\ &\quad \times \left\{\left[\sum_{t=1}^{T} \mathbf{z}_t \mathbf{x}_t'\right]\left[\sum_{t=1}^{T} \mathbf{x}_t \mathbf{x}_t'\right]^{-1}\left[\sum_{t=1}^{T} \mathbf{x}_t y_t\right]\right\} \\ &= \left[\sum_{t=1}^{T} \mathbf{x}_t \mathbf{z}_t'\right]^{-1}\left[\sum_{t=1}^{T} \mathbf{x}_t y_t\right].\end{aligned} \qquad [9.2.28]$$

Expression [9.2.28] is known as the *instrumental variable* (*IV*) estimator.

A key property of the *IV* estimator can be seen by premultiplying both sides of [9.2.28] by $\Sigma_{t=1}^{T} \mathbf{x}_t \mathbf{z}_t'$:

$$\sum_{t=1}^{T} \mathbf{x}_t \mathbf{z}_t' \hat{\boldsymbol{\beta}}_{IV} = \sum_{t=1}^{T} \mathbf{x}_t y_t,$$

implying that

$$\sum_{t=1}^{T} \mathbf{x}_t (y_t - \mathbf{z}_t' \hat{\boldsymbol{\beta}}_{IV}) = \mathbf{0}. \qquad [9.2.29]$$

Thus, the *IV* sample residual $(y_t - \mathbf{z}_t' \hat{\boldsymbol{\beta}}_{IV})$ has the property that it is orthogonal to the instruments $\mathbf{x}_t$, in contrast to the *OLS* sample residual $(y_t - \mathbf{z}_t' \mathbf{b})$, which is orthogonal to the explanatory variables $\mathbf{z}_t$. The *IV* estimator is preferred to *OLS* because the population residual of the equation we are trying to estimate (u_t) is correlated with $\mathbf{z}_t$ but uncorrelated with $\mathbf{x}_t$.

Since the *IV* estimator is a special case of *2SLS*, it shares the consistency property of the *2SLS* estimator. Its estimated variance with i.i.d. residuals can be calculated from [9.2.25]:

$$\hat{\sigma}_T^2 \left[\sum_{t=1}^{T} \mathbf{x}_t \mathbf{z}_t'\right]^{-1}\left[\sum_{t=1}^{T} \mathbf{x}_t \mathbf{x}_t'\right]\left[\sum_{t=1}^{T} \mathbf{z}_t \mathbf{x}_t'\right]^{-1}. \qquad [9.2.30]$$

9.3. *Identification*

We noted in the supply-and-demand example in Section 9.1 that the demand elasticity β could not be estimated consistently by an *OLS* regression of quantity on price. Indeed, in the absence of a valid instrument such as w_t, the demand elasticity cannot be estimated by any method! To see this, recall that the system as written in [9.1.1] and [9.1.2] implied the expressions [9.1.4] and [9.1.3]:

$$q_t = \frac{\gamma}{\gamma - \beta} \varepsilon_t^d - \frac{\beta}{\gamma - \beta} \varepsilon_t^s$$

$$p_t = \frac{\varepsilon_t^d - \varepsilon_t^s}{\gamma - \beta}.$$

If ε_t^d and ε_t^s are i.i.d. Gaussian, then these equations imply that the vector $(q_t, p_t)'$ is Gaussian with mean zero and variance-covariance matrix

$$\boldsymbol{\Omega} \equiv [1/(\gamma - \beta)^2] \cdot \begin{bmatrix} \gamma^2\sigma_d^2 + \beta^2\sigma_s^2 & \gamma\sigma_d^2 + \beta\sigma_s^2 \\ \gamma\sigma_d^2 + \beta\sigma_s^2 & \sigma_d^2 + \sigma_s^2 \end{bmatrix}.$$

This matrix is completely described by three magnitudes, these being the variances of q and p along with their covariance. Given a large enough sample, the values of these three magnitudes can be inferred with considerable confidence, but that is all that can be inferred, because these magnitudes can completely specify the process that generated the data under the maintained assumption of zero-mean i.i.d. Gaussian observations. There is no way to uncover the four parameters of the structural model $(\beta, \gamma, \sigma_d^2, \sigma_s^2)$ from these three magnitudes. For example, the values $(\beta, \gamma, \sigma_d^2, \sigma_s^2) = (1, 2, 3, 4)$ imply exactly the same observable properties for the data as would $(\beta, \gamma, \sigma_d^2, \sigma_s^2) = (2, 1, 4, 3)$.

If two different values for a parameter vector $\boldsymbol{\theta}$ imply the same probability distribution for the observed data, then the vector $\boldsymbol{\theta}$ is said to be *unidentified.*

When a third Gaussian white noise variable w_t is added to the set of observations, three additional magnitudes are available to characterize the process for observables, these being the variance of w, the covariance between w and p, and the covariance between w and q. If the new variable w enters both the demand and the supply equation, then three new parameters would be required to estimate the structural model—the parameter that summarizes the effect of w on demand, the parameter that summarizes its effect on supply, and the variance of w. With three more estimable magnitudes but three more parameters to estimate, we would be stuck with the same problem, having no basis for estimation of β.

Consistent estimation of the demand elasticity was achieved by using two-stage least squares because it was assumed that w appeared in the supply equation but was excluded from the demand equation. This is known as achieving identification through *exclusion restrictions.*

We showed in Section 9.2 that the parameters of an equation could be estimated (and thus must be identified) if (1) the number of instruments for that equation is at least as great as the number of endogenous explanatory variables for that equation and (2) the rows of $E(\mathbf{z}_t\mathbf{x}_t')$ are linearly independent. The first condition is known as the *order* condition for identification, and the second is known as the *rank* condition.

The rank condition for identification can be summarized more explicitly by specifying a complete system of equations for all of the endogenous variables. Let $\mathbf{y}_t$ denote an $(n \times 1)$ vector containing all of the endogenous variables in the system, and let $\mathbf{x}_t$ denote an $(m \times 1)$ vector containing all of the predetermined variables. Suppose that the system consists of n equations written as

$$\mathbf{B}\mathbf{y}_t + \boldsymbol{\Gamma}\mathbf{x}_t = \mathbf{u}_t, \qquad [9.3.1]$$

where $\mathbf{B}$ and $\boldsymbol{\Gamma}$ are $(n \times n)$ and $(n \times m)$ matrices of coefficients, respectively, and $\mathbf{u}_t$ is an $(n \times 1)$ vector of disturbances. The statement that $\mathbf{x}_t$ is predetermined is taken to mean that $E(\mathbf{x}_t\mathbf{u}_t') = \mathbf{0}$. For example, the demand and supply equations considered in Section 9.1 were

$$q_t = \beta p_t + u_t^d \qquad \text{(demand)} \qquad [9.3.2]$$

$$q_t = \gamma p_t + h w_t + u_t^s \qquad \text{(supply).} \qquad [9.3.3]$$

For this system, there are $n = 2$ endogenous variables, with $\mathbf{y}_t = (q_t, p_t)'$; and $m = 1$ predetermined variable, so that $\mathbf{x}_t = w_t$. This system can be written in the form of [9.3.1] as

$$\begin{bmatrix} 1 & -\beta \\ 1 & -\gamma \end{bmatrix}\begin{bmatrix} q_t \\ p_t \end{bmatrix} + \begin{bmatrix} 0 \\ -h \end{bmatrix} w_t = \begin{bmatrix} u_t^d \\ u_t^s \end{bmatrix}. \qquad [9.3.4]$$

Suppose we are interested in the equation represented by the first row of the vector system of equations in [9.3.1]. Let y_{0t} be the dependent variable in the first equation, and let $\mathbf{y}_{1t}$ denote an $(n_1 \times 1)$ vector consisting of those endogenous variables that appear in the first equation as explanatory variables. Similarly, let $\mathbf{x}_{1t}$ denote an $(m_1 \times 1)$ vector consisting of those predetermined variables that appear in the first equation as explanatory variables. Then the first equation in the system is

$$y_{0t} + \mathbf{B}_{01}\mathbf{y}_{1t} + \mathbf{\Gamma}_{01}\mathbf{x}_{1t} = u_{0t},$$

where $\mathbf{B}_{01}$ is a $(1 \times n_1)$ vector and $\mathbf{\Gamma}_{01}$ is a $(1 \times m_1)$ vector. Let $\mathbf{y}_{2t}$ denote an $(n_2 \times 1)$ vector consisting of those endogenous variables that do not appear in the first equation; thus, $\mathbf{y}_t' = (y_{0t}, \mathbf{y}_{1t}', \mathbf{y}_{2t}')$ and $1 + n_1 + n_2 = n$. Similarly, let $\mathbf{x}_{2t}$ denote an $(m_2 \times 1)$ vector consisting of those predetermined variables that do not appear in the first equation, so that $\mathbf{x}_t' = (\mathbf{x}_{1t}', \mathbf{x}_{2t}')$ and $m_1 + m_2 = m$. Then the system in [9.3.1] can be written in partitioned form as

$$\begin{bmatrix} 1 & \mathbf{B}_{01} & \mathbf{0}' \\ \mathbf{B}_{10} & \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{20} & \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix}\begin{bmatrix} y_{0t} \\ \mathbf{y}_{1t} \\ \mathbf{y}_{2t} \end{bmatrix} + \begin{bmatrix} \mathbf{\Gamma}_{01} & \mathbf{0}' \\ \mathbf{\Gamma}_{11} & \mathbf{\Gamma}_{12} \\ \mathbf{\Gamma}_{21} & \mathbf{\Gamma}_{22} \end{bmatrix}\begin{bmatrix} \mathbf{x}_{1t} \\ \mathbf{x}_{2t} \end{bmatrix} = \begin{bmatrix} u_{0t} \\ \mathbf{u}_{1t} \\ \mathbf{u}_{2t} \end{bmatrix}. \qquad [9.3.5]$$

Here, for example, $\mathbf{B}_{12}$ is an $(n_1 \times n_2)$ matrix consisting of rows 2 through $(n_1 + 1)$ and columns $(n_1 + 2)$ through n of the matrix $\mathbf{B}$.

An alternative useful representation of the system is obtained by moving $\mathbf{\Gamma x}_t$ to the right side of [9.3.1] and premultiplying both sides by $\mathbf{B}^{-1}$:

$$\mathbf{y}_t = -\mathbf{B}^{-1}\mathbf{\Gamma}\mathbf{x}_t + \mathbf{B}^{-1}\mathbf{u}_t = \mathbf{\Pi}'\mathbf{x}_t + \mathbf{v}_t, \qquad [9.3.6]$$

where

$$\mathbf{\Pi}' = -\mathbf{B}^{-1}\mathbf{\Gamma} \qquad [9.3.7]$$

$$\mathbf{v}_t = \mathbf{B}^{-1}\mathbf{u}_t. \qquad [9.3.8]$$

Expression [9.3.6] is known as the *reduced-form* representation of the structural system [9.3.1]. In the reduced-form representation, each endogenous variable is expressed solely as a function of predetermined variables. For the example of [9.3.4], the reduced form is

$$\begin{aligned} \begin{bmatrix} q_t \\ p_t \end{bmatrix} &= -\begin{bmatrix} 1 & -\beta \\ 1 & -\gamma \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ -h \end{bmatrix} w_t + \begin{bmatrix} 1 & -\beta \\ 1 & -\gamma \end{bmatrix}^{-1}\begin{bmatrix} u_t^d \\ u_t^s \end{bmatrix} \\ &= [1/(\beta - \gamma)]\begin{bmatrix} -\gamma & \beta \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ h \end{bmatrix} w_t \\ &\quad + [1/(\beta - \gamma)]\begin{bmatrix} -\gamma & \beta \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_t^d \\ u_t^s \end{bmatrix} \\ &= [1/(\beta - \gamma)]\begin{bmatrix} \beta h \\ h \end{bmatrix} w_t + [1/(\beta - \gamma)]\begin{bmatrix} -\gamma u_t^d + \beta u_t^s \\ -u_t^d + u_t^s \end{bmatrix}. \end{aligned} \qquad [9.3.9]$$

The reduced form for a general system can be written in partitioned form as

$$\begin{bmatrix} y_{0t} \\ \mathbf{y}_{1t} \\ \mathbf{y}_{2t} \end{bmatrix} = \begin{bmatrix} \mathbf{\Pi}_{01} & \mathbf{\Pi}_{02} \\ \mathbf{\Pi}_{11} & \mathbf{\Pi}_{12} \\ \mathbf{\Pi}_{21} & \mathbf{\Pi}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_{1t} \\ \mathbf{x}_{2t} \end{bmatrix} + \begin{bmatrix} v_{0t} \\ \mathbf{v}_{1t} \\ \mathbf{v}_{2t} \end{bmatrix}, \qquad [9.3.10]$$

where, for example, $\mathbf{\Pi}_{12}$ denotes an $(n_1 \times m_2)$ matrix consisting of rows 2 through $(n_1 + 1)$ and columns $(m_1 + 1)$ through m of the matrix $\mathbf{\Pi}'$.

To apply the rank condition for identification of the first equation stated earlier, we would form the matrix of cross products between the explanatory variables in the first equation ($\mathbf{x}_{1t}$ and $\mathbf{y}_{1t}$) and the predetermined variables for the whole system ($\mathbf{x}_{1t}$ and $\mathbf{x}_{2t}$):

$$\mathbf{M} = \begin{bmatrix} E(\mathbf{x}_{1t}\mathbf{x}'_{1t}) & E(\mathbf{x}_{1t}\mathbf{x}'_{2t}) \\ E(\mathbf{y}_{1t}\mathbf{x}'_{1t}) & E(\mathbf{y}_{1t}\mathbf{x}'_{2t}) \end{bmatrix}. \qquad [9.3.11]$$

In the earlier notation, the explanatory variables for the first equation consist of $\mathbf{z}_t = (\mathbf{x}'_{1t}, \mathbf{y}'_{1t})'$, while the predetermined variables for the system as a whole consist of $\mathbf{x}_t = (\mathbf{x}'_{1t}, \mathbf{x}'_{2t})'$. Thus, the rank condition, which required the rows of $E(\mathbf{z}_t\mathbf{x}'_t)$ to be linearly independent, amounts to the requirement that the rows of the $[(m_1 + n_1) \times m]$ matrix $\mathbf{M}$ in [9.3.11] be linearly independent. The rank condition can equivalently be stated in terms of the structural parameter matrices $\mathbf{B}$ and $\mathbf{\Gamma}$ or the reduced-form parameter matrix $\mathbf{\Pi}$. The following proposition is adapted from Fisher (1966) and is proved in Appendix 9.A at the end of this chapter.

Proposition 9.1: *If the matrix* $\mathbf{B}$ *in [9.3.1] and the matrix of second moments of the predetermined variables* $E(\mathbf{x}_t\mathbf{x}'_t)$ *are both nonsingular, then the following conditions are equivalent:*

(a) *The rows of the* $[(m_1 + n_1) \times m]$ *matrix* $\mathbf{M}$ *in [9.3.11] are linearly independent.*

(b) *The rows of the* $[(n_1 + n_2) \times (m_2 + n_2)]$ *matrix*

$$\begin{bmatrix} \mathbf{\Gamma}_{12} & \mathbf{B}_{12} \\ \mathbf{\Gamma}_{22} & \mathbf{B}_{22} \end{bmatrix} \qquad [9.3.12]$$

are linearly independent.

(c) *The rows of the* $(n_1 \times m_2)$ *matrix* $\mathbf{\Pi}_{12}$ *are linearly independent.*

For example, for the system in [9.3.4], no endogenous variables are excluded from the first equation, and so $y_{0t} = q_t$, $\mathbf{y}_{1t} = p_t$, and $\mathbf{y}_{2t}$ contains no elements. No predetermined variables appear in the first equation, and so $\mathbf{x}_{1t}$ contains no elements and $\mathbf{x}_{2t} = w_t$. The matrix in [9.3.12] is then just given by the parameter $\mathbf{\Gamma}_{12}$. This represents the coefficient on $\mathbf{x}_{2t}$ in the equation describing $\mathbf{y}_{1t}$ and is equal to the scalar parameter $-h$. Result (b) of Proposition 9.1 thus states that the first equation is identified provided that $h \neq 0$. The value of $\mathbf{\Pi}_{12}$ can be read directly off the coefficient on w_t in the second row of [9.3.9] and turns out to be given by $h/(\beta - \gamma)$. Since $\mathbf{B}$ is assumed to be nonsingular, $(\beta - \gamma)$ is nonzero, and so $\mathbf{\Gamma}_{12}$ is zero if and only if $\mathbf{\Pi}_{12}$ is zero.

Achieving Identification Through Covariance Restrictions

Another way in which parameters can be identified is through restrictions on the covariances of the errors of the structural equations. For example, consider again the supply and demand model, [9.3.2] and [9.3.3]. We saw that the demand elasticity β was identified by the exclusion of w_t from the demand equation. Consider now estimation of the supply elasticity γ.

Suppose first that we somehow knew the value of the demand elasticity β with certainty. Then the error in the demand equation could be constructed from

$$u_t^d = q_t - \beta p_t.$$

Notice that u_t^d would then be a valid instrument for the supply equation [9.3.3], since u_t^d is correlated with the endogenous explanatory variable for that equation (p_t) but u_t^d is uncorrelated with the error for that equation (u_t^s). Since w_t is also uncorrelated with the error u_t^s, it follows that the parameters of the supply equation could be estimated consistently by instrumental variable estimation with $\mathbf{x}_t = (u_t^d, w_t)'$:

$$\begin{bmatrix} \hat{\gamma}_T^* \\ \hat{h}_T^* \end{bmatrix} = \begin{bmatrix} \Sigma u_t^d p_t & \Sigma u_t^d w_t \\ \Sigma w_t p_t & \Sigma w_t^2 \end{bmatrix}^{-1} \begin{bmatrix} \Sigma u_t^d q_t \\ \Sigma w_t q_t \end{bmatrix} \xrightarrow{p} \begin{bmatrix} \gamma \\ h \end{bmatrix}, \qquad [9.3.13]$$

where Σ indicates summation over $t = 1, 2, \ldots, T$.

Although in practice we do not know the true value of β, it can be estimated consistently by *IV* estimation of [9.3.2] with w_t as an instrument:

$$\hat{\beta} = (\Sigma w_t p_t)^{-1}(\Sigma w_t q_t).$$

Then the residual u_t^d can be estimated consistently with $\hat{u}_t^d = q_t - \hat{\beta} p_t$. Consider, therefore, the estimator [9.3.13] with the population residual u_t^d replaced by the *IV* sample residual:

$$\begin{bmatrix} \hat{\gamma}_T \\ \hat{h}_T \end{bmatrix} = \begin{bmatrix} \Sigma \hat{u}_t^d p_t & \Sigma \hat{u}_t^d w_t \\ \Sigma w_t p_t & \Sigma w_t^2 \end{bmatrix}^{-1} \begin{bmatrix} \Sigma \hat{u}_t^d q_t \\ \Sigma w_t q_t \end{bmatrix}. \qquad [9.3.14]$$

It is straightforward to use the fact that $\hat{\beta} \xrightarrow{p} \beta$ to deduce that the difference between the estimators in [9.3.14] and [9.3.13] converges in probability to zero. Hence, the estimator [9.3.14] is also consistent.

Two assumptions allowed the parameters of the supply equation (γ and h) to be estimated. First, an exclusion restriction allowed β to be estimated consistently. Second, a restriction on the covariance between u_t^d and u_t^s was necessary. If u_t^d were correlated with u_t^s, then u_t^d would not be a valid instrument for the supply equation and the estimator [9.3.13] would not be consistent.

Other Approaches to Identification

A good deal more can be said about identification. For example, parameters can also be identified through the imposition of certain restrictions on parameters such as $\beta_1 + \beta_2 = 1$. Useful references include Fisher (1966), Rothenberg (1971), and Hausman and Taylor (1983).

9.4. *Full-Information Maximum Likelihood Estimation*

Up to this point we have considered estimation of a single equation of the form $y_t = \boldsymbol{\beta}'\mathbf{z}_t + u_t$. A more general approach is to specify a similar equation for every endogenous variable in the system, calculate the joint density of the vector of all of the endogenous variables conditional on the predetermined variables, and maximize the joint likelihood function. This is known as *full-information maximum likelihood* estimation, or *FIML*.

For illustration, suppose in [9.3.1] that the $(n \times 1)$ vector of structural disturbances $\mathbf{u}_t$ for date t is distributed $N(\mathbf{0}, \mathbf{D})$. Assume, further, that $\mathbf{u}_t$ is independent of $\mathbf{u}_\tau$ for $t \neq \tau$ and that $\mathbf{u}_t$ is independent of $\mathbf{x}_\tau$ for all t and τ. Then the reduced-form disturbance $\mathbf{v}_t = \mathbf{B}^{-1}\mathbf{u}_t$ is distributed $N\big(\mathbf{0}, \mathbf{B}^{-1}\mathbf{D}(\mathbf{B}^{-1})'\big)$, and the reduced-form representation [9.3.6] implies that

$$\mathbf{y}_t|\mathbf{x}_t \sim N\Big(\mathbf{\Pi}'\mathbf{x}_t, \mathbf{B}^{-1}\mathbf{D}(\mathbf{B}^{-1})'\Big) = N\Big(-\mathbf{B}^{-1}\mathbf{\Gamma}\mathbf{x}_t, \mathbf{B}^{-1}\mathbf{D}(\mathbf{B}^{-1})'\Big).$$

The conditional log likelihood can then be found from

$$\begin{aligned} &\mathscr{L}(\mathbf{B}, \mathbf{\Gamma}, \mathbf{D}) \\ &\quad = \sum_{t=1}^{T} \log f(\mathbf{y}_t|\mathbf{x}_t; \mathbf{B}, \mathbf{\Gamma}, \mathbf{D}) \\ &\quad = -(Tn/2)\log(2\pi) - (T/2)\log\left|\mathbf{B}^{-1}\mathbf{D}(\mathbf{B}^{-1})'\right| \\ &\qquad - (1/2)\sum_{t=1}^{T} [\mathbf{y}_t + \mathbf{B}^{-1}\mathbf{\Gamma}\mathbf{x}_t]'[\mathbf{B}^{-1}\mathbf{D}(\mathbf{B}^{-1})']^{-1}[\mathbf{y}_t + \mathbf{B}^{-1}\mathbf{\Gamma}\mathbf{x}_t]. \end{aligned} \qquad [9.4.1]$$

But

$$\begin{aligned} &[\mathbf{y}_t + \mathbf{B}^{-1}\mathbf{\Gamma}\mathbf{x}_t]'[\mathbf{B}^{-1}\mathbf{D}(\mathbf{B}^{-1})']^{-1}[\mathbf{y}_t + \mathbf{B}^{-1}\mathbf{\Gamma}\mathbf{x}_t] \\ &\qquad = [\mathbf{y}_t + \mathbf{B}^{-1}\mathbf{\Gamma}\mathbf{x}_t]'[\mathbf{B}'\mathbf{D}^{-1}\mathbf{B}][\mathbf{y}_t + \mathbf{B}^{-1}\mathbf{\Gamma}\mathbf{x}_t] \\ &\qquad = [\mathbf{B}(\mathbf{y}_t + \mathbf{B}^{-1}\mathbf{\Gamma}\mathbf{x}_t)]'\mathbf{D}^{-1}[\mathbf{B}(\mathbf{y}_t + \mathbf{B}^{-1}\mathbf{\Gamma}\mathbf{x}_t)] \\ &\qquad = [\mathbf{B}\mathbf{y}_t + \mathbf{\Gamma}\mathbf{x}_t]'\mathbf{D}^{-1}[\mathbf{B}\mathbf{y}_t + \mathbf{\Gamma}\mathbf{x}_t]. \end{aligned} \qquad [9.4.2]$$

Furthermore,

$$\begin{aligned} \left|\mathbf{B}^{-1}\mathbf{D}(\mathbf{B}^{-1})'\right| &= |\mathbf{B}^{-1}|\cdot|\mathbf{D}|\cdot|\mathbf{B}^{-1}| \\ &= |\mathbf{D}|/|\mathbf{B}|^2. \end{aligned} \qquad [9.4.3]$$

Substituting [9.4.2] and [9.4.3] into [9.4.1],

$$\begin{aligned} \mathscr{L}(\mathbf{B}, \mathbf{\Gamma}, \mathbf{D}) = &-(Tn/2)\log(2\pi) + (T/2)\log|\mathbf{B}|^2 \\ &- (T/2)\log|\mathbf{D}| - (1/2)\sum_{t=1}^{T} [\mathbf{B}\mathbf{y}_t + \mathbf{\Gamma}\mathbf{x}_t]'\mathbf{D}^{-1}[\mathbf{B}\mathbf{y}_t + \mathbf{\Gamma}\mathbf{x}_t]. \end{aligned} \qquad [9.4.4]$$

The *FIML* estimates are then the values of $\mathbf{B}$, $\mathbf{\Gamma}$, and $\mathbf{D}$ for which [9.4.4] is maximized.

For example, for the system of [9.3.4], the *FIML* estimates of β, γ, h, σ_d^2, and σ_s^2 are found by maximizing

$$\begin{aligned} &\mathscr{L}(\beta, \gamma, h, \sigma_d^2, \sigma_s^2) \\ &\quad = -T\log(2\pi) + \frac{T}{2}\log\begin{vmatrix} 1 & -\beta \\ 1 & -\gamma \end{vmatrix}^2 - \frac{T}{2}\log\begin{vmatrix} \sigma_d^2 & 0 \\ 0 & \sigma_s^2 \end{vmatrix} \\ &\qquad - \frac{1}{2}\sum_{t=1}^{T}\left\{[q_t - \beta p_t \quad q_t - \gamma p_t - hw_t]\begin{bmatrix} \sigma_d^2 & 0 \\ 0 & \sigma_s^2 \end{bmatrix}^{-1}\begin{bmatrix} q_t - \beta p_t \\ q_t - \gamma p_t - hw_t \end{bmatrix}\right\} \\ &\quad = -T\log(2\pi) + T\log(\gamma - \beta) - (T/2)\log(\sigma_d^2) \\ &\qquad - (T/2)\log(\sigma_s^2) - (1/2)\sum_{t=1}^{T}(q_t - \beta p_t)^2/\sigma_d^2 \\ &\qquad - (1/2)\sum_{t=1}^{T}(q_t - \gamma p_t - hw_t)^2/\sigma_s^2. \end{aligned} \qquad [9.4.5]$$

The first-order conditions for maximization are

$$\frac{\partial \mathcal{L}}{\partial \beta} = -\frac{T}{\gamma - \beta} + \frac{\sum_{t=1}^{T} (q_t - \beta p_t) p_t}{\sigma_d^2} = 0 \qquad [9.4.6]$$

$$\frac{\partial \mathcal{L}}{\partial \gamma} = \frac{T}{\gamma - \beta} + \frac{\sum_{t=1}^{T} (q_t - \gamma p_t - h w_t) p_t}{\sigma_s^2} = 0 \qquad [9.4.7]$$

$$\frac{\partial \mathcal{L}}{\partial h} = \frac{\sum_{t=1}^{T} (q_t - \gamma p_t - h w_t) w_t}{\sigma_s^2} = 0 \qquad [9.4.8]$$

$$\frac{\partial \mathcal{L}}{\partial \sigma_d^2} = -\frac{T}{2\sigma_d^2} + \frac{\sum_{t=1}^{T} (q_t - \beta p_t)^2}{2\sigma_d^4} = 0 \qquad [9.4.9]$$

$$\frac{\partial \mathcal{L}}{\partial \sigma_s^2} = -\frac{T}{2\sigma_s^2} + \frac{\sum_{t=1}^{T} (q_t - \gamma p_t - h w_t)^2}{2\sigma_s^4} = 0. \qquad [9.4.10]$$

The last two equations characterize the maximum likelihood estimates of the variances as the average squared residuals:

$$\hat{\sigma}_d^2 = (1/T) \sum_{t=1}^{T} (q_t - \hat{\beta} p_t)^2 \qquad [9.4.11]$$

$$\hat{\sigma}_s^2 = (1/T) \sum_{t=1}^{T} (q_t - \hat{\gamma} p_t - \hat{h} w_t)^2. \qquad [9.4.12]$$

Multiplying equation [9.4.7] by $(\beta - \gamma)/T$ results in

$$\begin{aligned} 0 &= -1 + \sum_{t=1}^{T} (q_t - \gamma p_t - h w_t)(\beta p_t - \gamma p_t)/(T\sigma_s^2) \\ &= -1 + \sum_{t=1}^{T} (q_t - \gamma p_t - h w_t)(\beta p_t - q_t + q_t - \gamma p_t)/(T\sigma_s^2). \end{aligned} \qquad [9.4.13]$$

If [9.4.8] is multiplied by h/T and subtracted from [9.4.13], the result is

$$\begin{aligned} 0 &= -1 + \sum_{t=1}^{T} (q_t - \gamma p_t - h w_t)(\beta p_t - q_t + q_t - \gamma p_t - h w_t)/(T\sigma_s^2) \\ &= -1 + \sum_{t=1}^{T} (q_t - \gamma p_t - h w_t)(\beta p_t - q_t)/(T\sigma_s^2) \\ &\quad + \sum_{t=1}^{T} (q_t - \gamma p_t - h w_t)^2/(T\sigma_s^2) \\ &= -1 - \sum_{t=1}^{T} (q_t - \gamma p_t - h w_t)(q_t - \beta p_t)/(T\sigma_s^2) + 1, \end{aligned}$$

by virtue of [9.4.12]. Hence, the *MLE*s satisfy

$$\sum_{t=1}^{T} (q_t - \hat{\gamma} p_t - \hat{h} w_t)(q_t - \hat{\beta} p_t) = 0. \qquad [9.4.14]$$

Similarly, multiplying [9.4.6] by $(\gamma - \beta)/T$,

$$\begin{aligned} 0 &= -1 + \sum_{t=1}^{T} (q_t - \beta p_t)(\gamma p_t - q_t + q_t - \beta p_t)/(T\sigma_d^2) \\ &= -1 - \sum_{t=1}^{T} (q_t - \beta p_t)(q_t - \gamma p_t)/(T\sigma_d^2) + \sum_{t=1}^{T} (q_t - \beta p_t)^2/(T\sigma_d^2). \end{aligned}$$

Using [9.4.11],

$$\sum_{t=1}^{T} (q_t - \hat{\beta} p_t)(q_t - \hat{\gamma} p_t) = 0. \tag{9.4.15}$$

Subtracting [9.4.14] from [9.4.15],

$$0 = \sum_{t=1}^{T} (q_t - \hat{\beta} p_t)[(q_t - \hat{\gamma} p_t) - (q_t - \hat{\gamma} p_t - \hat{h} w_t)] = \hat{h} \sum_{t=1}^{T} (q_t - \hat{\beta} p_t) w_t.$$

Assuming that $\hat{h} \neq 0$, the *FIML* estimate of β thus satisfies

$$\sum_{t=1}^{T} (q_t - \hat{\beta} p_t) w_t = 0;$$

that is, the demand elasticity is chosen so as to make the estimated residual for the demand equation orthogonal to w_t. Hence, the instrumental variable estimator $\hat{\beta}_{IV}$ turns out also to be the *FIML* estimator. Equations [9.4.8] and [9.4.14] assert that the parameters for the supply equation (γ and h) are chosen so as to make the residual for that equation orthogonal to w_t and to the demand residual $\hat{u}_t^d = q_t - \hat{\beta} p_t$. Hence, the *FIML* estimates for these parameters are the same as the instrumental-variable estimates suggested in [9.3.14].

For this example, two-stage least squares, instrumental variable estimation, and full-information maximum likelihood all produced the identical estimates. This is because the model is *just identified*. A model is said to be just identified if for any admissible value for the parameters of the reduced-form representation there exists a unique value for the structural parameters that implies those reduced-form parameters. A model is said to be *overidentified* if some admissible values for the reduced-form parameters are ruled out by the structural restrictions. In an overidentified model, *IV*, *2SLS*, and *FIML* estimation are not equivalent, and *FIML* typically produces the most efficient estimates.

For a general overidentified simultaneous equation system with no restrictions on the variance-covariance matrix, the *FIML* estimates can be calculated by iterating on a procedure known as *three-stage least squares*; see, for example, Maddala (1977, pp. 482–90). Rothenberg and Ruud (1990) discussed *FIML* estimation in the presence of covariance restrictions. *FIML* estimation of dynamic time series models will be discussed further in Chapter 11.

9.5 *Estimation Based on the Reduced Form*

If a system is just identified as in [9.3.2] and [9.3.3] with u_t^d uncorrelated with u_t^s, one approach is to maximize the likelihood function with respect to the reduced-form parameters. The values of the structural parameters associated with these values for the reduced-form parameters are the same as the *FIML* estimates in a just-identified model.

The log likelihood [9.4.1] can be expressed in terms of the reduced-form parameters $\mathbf{\Pi}$ and $\mathbf{\Omega}$ as

$$
\begin{aligned}
\mathscr{L}(\mathbf{\Pi}, \mathbf{\Omega}) &= \sum_{t=1}^{T} \log f(\mathbf{y}_t|\mathbf{x}_t; \mathbf{\Pi}, \mathbf{\Omega}) \\
&= -(Tn/2)\log(2\pi) - (T/2)\log|\mathbf{\Omega}| \qquad [9.5.1] \\
&\quad - (1/2)\sum_{t=1}^{T} [\mathbf{y}_t - \mathbf{\Pi}'\mathbf{x}_t]'\mathbf{\Omega}^{-1}[\mathbf{y}_t - \mathbf{\Pi}'\mathbf{x}_t],
\end{aligned}
$$

where $\mathbf{\Omega} = E(\mathbf{v}_t\mathbf{v}_t') = \mathbf{B}^{-1}\mathbf{D}(\mathbf{B}^{-1})'$. The value of $\mathbf{\Pi}$ that maximizes [9.5.1] will be shown in Chapter 11 to be given by

$$\hat{\mathbf{\Pi}}' = \left[\sum_{t=1}^{T} \mathbf{y}_t\mathbf{x}_t'\right]\left[\sum_{t=1}^{T} \mathbf{x}_t\mathbf{x}_t'\right]^{-1};$$

in other words, the ith row of $\hat{\mathbf{\Pi}}'$ is obtained from an *OLS* regression of the ith endogenous variable on all of the predetermined variables:

$$\hat{\boldsymbol{\pi}}_i' = \left[\sum_{t=1}^{T} y_{it}\mathbf{x}_t'\right]\left[\sum_{t=1}^{T} \mathbf{x}_t\mathbf{x}_t'\right]^{-1}.$$

The *MLE* of $\mathbf{\Omega}$ turns out to be

$$\hat{\mathbf{\Omega}} = (1/T)\left[\sum_{t=1}^{T} (\mathbf{y}_t - \hat{\mathbf{\Pi}}'\mathbf{x}_t)(\mathbf{y}_t - \hat{\mathbf{\Pi}}'\mathbf{x}_t)'\right].$$

For a just-identified model, the *FIML* estimates are the values of $(\mathbf{B}, \mathbf{\Gamma}, \mathbf{D})$ for which $\hat{\mathbf{\Pi}}' = -\mathbf{B}^{-1}\mathbf{\Gamma}$ and $\hat{\mathbf{\Omega}} = \mathbf{B}^{-1}\mathbf{D}(\mathbf{B}^{-1})'$.

We now show that the estimates of $\mathbf{B}$, $\mathbf{\Gamma}$, and $\mathbf{D}$ inferred in this fashion from the reduced-form parameters for the just-identified supply-and-demand example are the same as the *FIML* estimates. The estimate $\hat{\pi}_1$ is found by *OLS* regression of q_t on w_t, while $\hat{\pi}_2$ is the coefficient from an *OLS* regression of p_t on w_t. These estimates satisfy

$$\sum_{t=1}^{T} (q_t - \hat{\pi}_1 w_t)w_t = 0 \qquad [9.5.2]$$

$$\sum_{t=1}^{T} (p_t - \hat{\pi}_2 w_t)w_t = 0 \qquad [9.5.3]$$

and

$$\begin{bmatrix} \hat{\Omega}_{11} & \hat{\Omega}_{12} \\ \hat{\Omega}_{21} & \hat{\Omega}_{22} \end{bmatrix} = (1/T)\begin{bmatrix} \Sigma(q_t - \hat{\pi}_1 w_t)^2 & \Sigma(q_t - \hat{\pi}_1 w_t)(p_t - \hat{\pi}_2 w_t) \\ \Sigma(p_t - \hat{\pi}_2 w_t)(q_t - \hat{\pi}_1 w_t) & \Sigma(p_t - \hat{\pi}_2 w_t)^2 \end{bmatrix}. \qquad [9.5.4]$$

The structural estimates satisfy $\mathbf{B}\hat{\mathbf{\Pi}}' = -\mathbf{\Gamma}$ or

$$\begin{bmatrix} 1 & -\beta \\ 1 & -\gamma \end{bmatrix}\begin{bmatrix} \hat{\pi}_1 \\ \hat{\pi}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ h \end{bmatrix}. \qquad [9.5.5]$$

Multiplying [9.5.3] by β and subtracting the result from [9.5.2] produces

$$
\begin{aligned}
0 &= \sum_{t=1}^{T} (q_t - \hat{\pi}_1 w_t - \beta p_t + \beta\hat{\pi}_2 w_t) w_t \\
&= \sum_{t=1}^{T} (q_t - \beta p_t) w_t - \sum_{t=1}^{T} (\hat{\pi}_1 - \beta\hat{\pi}_2) w_t^2 \\
&= \sum_{t=1}^{T} (q_t - \beta p_t) w_t,
\end{aligned}
$$

by virtue of the first row of [9.5.5]. Thus, the estimate of β inferred from the reduced-form parameters is the same as the *IV* or *FIML* estimate derived earlier. Similarly, multiplying [9.5.3] by γ and subtracting the result from [9.5.2] gives

$$
\begin{aligned}
0 &= \sum_{t=1}^{T} (q_t - \hat{\pi}_1 w_t - \gamma p_t + \gamma\hat{\pi}_2 w_t) w_t \\
&= \sum_{t=1}^{T} [q_t - \gamma p_t - (\hat{\pi}_1 - \gamma\hat{\pi}_2) w_t] w_t \\
&= \sum_{t=1}^{T} [q_t - \gamma p_t - h w_t] w_t,
\end{aligned}
$$

by virtue of the second row of [9.5.5], reproducing the first-order condition [9.4.8] for *FIML*. Finally, we need to solve $\mathbf{D} = \mathbf{B}\hat{\mathbf{\Omega}}\mathbf{B}'$ for $\mathbf{D}$ and γ (the remaining element of $\mathbf{B}$). These equations are

$$
\begin{aligned}
&\begin{bmatrix} \sigma_d^2 & 0 \\ 0 & \sigma_s^2 \end{bmatrix} \\
&\quad = \begin{bmatrix} 1 & -\beta \\ 1 & -\gamma \end{bmatrix} \begin{bmatrix} \hat{\Omega}_{11} & \hat{\Omega}_{12} \\ \hat{\Omega}_{21} & \hat{\Omega}_{22} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -\beta & -\gamma \end{bmatrix} \\
&\quad = \frac{1}{T} \sum_{t=1}^{T} \left\{ \begin{bmatrix} 1 & -\beta \\ 1 & -\gamma \end{bmatrix} \begin{bmatrix} q_t - \hat{\pi}_1 w_t \\ p_t - \hat{\pi}_2 w_t \end{bmatrix} [q_t - \hat{\pi}_1 w_t \quad p_t - \hat{\pi}_2 w_t] \begin{bmatrix} 1 & 1 \\ -\beta & -\gamma \end{bmatrix} \right\} \\
&\quad = \frac{1}{T} \sum_{t=1}^{T} \left\{ \begin{bmatrix} q_t - \beta p_t - (\hat{\pi}_1 - \beta\hat{\pi}_2) w_t \\ q_t - \gamma p_t - (\hat{\pi}_1 - \gamma\hat{\pi}_2) w_t \end{bmatrix} \begin{bmatrix} q_t - \beta p_t - (\hat{\pi}_1 - \beta\hat{\pi}_2) w_t \\ q_t - \gamma p_t - (\hat{\pi}_1 - \gamma\hat{\pi}_2) w_t \end{bmatrix}' \right\} \\
&\quad = \frac{1}{T} \sum_{t=1}^{T} \left\{ \begin{bmatrix} q_t - \beta p_t \\ q_t - \gamma p_t - h w_t \end{bmatrix} [q_t - \beta p_t \quad q_t - \gamma p_t - h w_t] \right\}.
\end{aligned}
$$

The diagonal elements of this matrix system of equations reproduce the earlier formulas for the *FIML* estimates of the variance parameters, while the off-diagonal element reproduces the result [9.4.14].

9.6. *Overview of Simultaneous Equations Bias*

The problem of simultaneous equations bias is extremely widespread in the social sciences. It is rare that the relation that we are interested in estimating is the only possible reason why the dependent and explanatory variables might be correlated. For example, consider trying to estimate the effect of military service on an individual's subsequent income. This parameter cannot be estimated by a regression of income on a measure of military service and other observed variables. The error term in such a regression represents other characteristics of the individual that influence income, and these omitted factors are also likely to have influenced the individual's military participation. As another example, consider trying to estimate the success of long prison sentences in deterring crime. This cannot be estimated by regressing the crime rate in a state on the average prison term in that state, because some states may have adopted stiffer prison sentences in response to higher crime. The error term in the regression, which represents other factors that influence crime, is thus likely also to be correlated with the explanatory variable. Regardless of whether the researcher is interested in the factors that determine military service or prison terms or has any theory about them, simultaneous equations bias must be recognized and dealt with.

Furthermore, it is not enough to find an instrument $\mathbf{x}_t$ that is uncorrelated with the residual u_t. In order to satisfy the rank condition, the instrument $\mathbf{x}_t$ must be correlated with the endogenous explanatory variables $\mathbf{z}_t$. The calculations by Nelson and Startz (1990) suggest that very poor estimates can result if $\mathbf{x}_t$ is only weakly correlated with $\mathbf{z}_t$.

Finding valid instruments is often extremely difficult and requires careful thought and a bit of good luck. For the question about military service, Angrist (1990) found an ingenious instrument for military service based on the institutional details of the draft in the United States during the Vietnam War. The likelihood that an individual was drafted into military service was determined by a lottery based on birthdays. Thus, an individual's birthday during the year would be correlated with military service but presumably uncorrelated with other factors influencing income. Unfortunately, it is unusual to be able to find such a compelling instrument for many questions that one would like to ask of the data.

APPENDIX 9.A. *Proofs of Chapter 9 Proposition*

■ **Proof of Proposition 9.1.** We first show that (a) implies (c). The middle block of [9.3.10] states that

$$\mathbf{y}_{1t} = \mathbf{\Pi}_{11}\mathbf{x}_{1t} + \mathbf{\Pi}_{12}\mathbf{x}_{2t} + \mathbf{v}_{1t}.$$

Hence,

$$\begin{aligned}\mathbf{M} &= E\left\{\begin{bmatrix}\mathbf{x}_{1t}\\ \mathbf{y}_{1t}\end{bmatrix}[\mathbf{x}_{1t}' \quad \mathbf{x}_{2t}']\right\}\\ &= E\left\{\begin{bmatrix}\mathbf{I}_{m_1} & \mathbf{0}\\ \mathbf{\Pi}_{11} & \mathbf{\Pi}_{12}\end{bmatrix}\begin{bmatrix}\mathbf{x}_{1t}\\ \mathbf{x}_{2t}\end{bmatrix}[\mathbf{x}_{1t}' \quad \mathbf{x}_{2t}'] + \begin{bmatrix}\mathbf{0}\\ \mathbf{v}_{1t}\end{bmatrix}[\mathbf{x}_{1t}' \quad \mathbf{x}_{2t}']\right\}\\ &= \begin{bmatrix}\mathbf{I}_{m_1} & \mathbf{0}\\ \mathbf{\Pi}_{11} & \mathbf{\Pi}_{12}\end{bmatrix}E(\mathbf{x}_t\mathbf{x}_t'),\end{aligned} \qquad [9.A.1]$$

since $\mathbf{x}_t$ is uncorrelated with $\mathbf{u}_t$ and thus uncorrelated with $\mathbf{v}_t$.

Suppose that the rows of $\mathbf{M}$ are linearly independent. This means that $[\boldsymbol{\lambda}' \quad \boldsymbol{\mu}']\mathbf{M} \neq \mathbf{0}'$ for any $(m_1 \times 1)$ vector $\boldsymbol{\lambda}$ and any $(n_1 \times 1)$ vector $\boldsymbol{\mu}$ that are not both zero. In particular, $[-\boldsymbol{\mu}'\mathbf{\Pi}_{11} \quad \boldsymbol{\mu}']\mathbf{M} \neq \mathbf{0}'$. But from the right side of [9.A.1], this implies that

$$[-\boldsymbol{\mu}'\mathbf{\Pi}_{11} \quad \boldsymbol{\mu}']\begin{bmatrix}\mathbf{I}_{m_1} & \mathbf{0}\\ \mathbf{\Pi}_{11} & \mathbf{\Pi}_{12}\end{bmatrix}E(\mathbf{x}_t\mathbf{x}_t') = [\mathbf{0}' \quad \boldsymbol{\mu}'\mathbf{\Pi}_{12}]E(\mathbf{x}_t\mathbf{x}_t') \neq \mathbf{0}'$$

for any nonzero $(n_1 \times 1)$ vector $\boldsymbol{\mu}$. But this could be true only if $\boldsymbol{\mu}'\mathbf{\Pi}_{12} \neq \mathbf{0}'$. Hence, if the rows of $\mathbf{M}$ are linearly independent, then the rows of $\mathbf{\Pi}_{12}$ are also linearly independent.

To prove that (c) implies (a), premultiply both sides of [9.A.1] by any nonzero vector $[\boldsymbol{\lambda}' \quad \boldsymbol{\mu}']$. The right side becomes

$$[\boldsymbol{\lambda}' \quad \boldsymbol{\mu}']\begin{bmatrix} \mathbf{I}_{m_1} & \mathbf{0} \\ \boldsymbol{\Pi}_{11} & \boldsymbol{\Pi}_{12} \end{bmatrix} E(\mathbf{x}_t\mathbf{x}_t') = [(\boldsymbol{\lambda}' + \boldsymbol{\mu}'\boldsymbol{\Pi}_{11}) \quad \boldsymbol{\mu}'\boldsymbol{\Pi}_{12}]E(\mathbf{x}_t\mathbf{x}_t') \equiv \boldsymbol{\eta}'E(\mathbf{x}_t\mathbf{x}_t'),$$

where $\boldsymbol{\eta}' \equiv [(\boldsymbol{\lambda}' + \boldsymbol{\mu}'\boldsymbol{\Pi}_{11}) \quad \boldsymbol{\mu}'\boldsymbol{\Pi}_{12}]$. If the rows of $\boldsymbol{\Pi}_{12}$ are linearly independent, then $\boldsymbol{\eta}'$ cannot be the zero vector unless both $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ are zero. To see this, note that if $\boldsymbol{\mu}$ is nonzero, then $\boldsymbol{\mu}'\boldsymbol{\Pi}_{12}$ cannot be the zero vector, while if $\boldsymbol{\mu} = \mathbf{0}$, then $\boldsymbol{\eta}$ will be zero only if $\boldsymbol{\lambda}$ is also the zero vector. Furthermore, since $E(\mathbf{x}_t\mathbf{x}_t')$ is nonsingular, a nonzero $\boldsymbol{\eta}$ means that $\boldsymbol{\eta}'E(\mathbf{x}_t\mathbf{x}_t')$ cannot be the zero vector. Thus, if the right side of [9.A.1] is premultiplied by any nonzero vector $(\boldsymbol{\lambda}', \boldsymbol{\mu}')$, the result is not zero. The same must be true of the left side: $[\boldsymbol{\lambda}' \quad \boldsymbol{\mu}']\mathbf{M} \neq \mathbf{0}'$ for any nonzero $(\boldsymbol{\lambda}', \boldsymbol{\mu}')$, establishing that linear independence of the rows of $\boldsymbol{\Pi}_{12}$ implies linear independence of the rows of $\mathbf{M}$.

To see that (b) implies (c), write [9.3.7] as

$$\begin{bmatrix} \boldsymbol{\Pi}_{01} & \boldsymbol{\Pi}_{02} \\ \boldsymbol{\Pi}_{11} & \boldsymbol{\Pi}_{12} \\ \boldsymbol{\Pi}_{21} & \boldsymbol{\Pi}_{22} \end{bmatrix} = -\mathbf{B}^{-1}\begin{bmatrix} \boldsymbol{\Gamma}_{01} & \mathbf{0}' \\ \boldsymbol{\Gamma}_{11} & \boldsymbol{\Gamma}_{12} \\ \boldsymbol{\Gamma}_{21} & \boldsymbol{\Gamma}_{22} \end{bmatrix}. \qquad [9.A.2]$$

We also have the identity

$$\begin{bmatrix} 1 & \mathbf{0}' & \mathbf{0}' \\ \mathbf{0} & \mathbf{I}_{n_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{n_2} \end{bmatrix} = \mathbf{B}^{-1}\begin{bmatrix} 1 & \mathbf{B}_{01} & \mathbf{0}' \\ \mathbf{B}_{10} & \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{20} & \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix}. \qquad [9.A.3]$$

The system of equations represented by the second block column of [9.A.2] and the third block column of [9.A.3] can be collected as

$$\begin{bmatrix} \boldsymbol{\Pi}_{02} & \mathbf{0}' \\ \boldsymbol{\Pi}_{12} & \mathbf{0} \\ \boldsymbol{\Pi}_{22} & \mathbf{I}_{n_2} \end{bmatrix} = \mathbf{B}^{-1}\begin{bmatrix} \mathbf{0}' & \mathbf{0}' \\ -\boldsymbol{\Gamma}_{12} & \mathbf{B}_{12} \\ -\boldsymbol{\Gamma}_{22} & \mathbf{B}_{22} \end{bmatrix}. \qquad [9.A.4]$$

If both sides of [9.A.4] are premultiplied by the row vector $[0 \quad \boldsymbol{\mu}_1' \quad \mathbf{0}']$ where $\boldsymbol{\mu}_1$ is any $(n_1 \times 1)$ vector, the result is

$$\begin{aligned} [\boldsymbol{\mu}_1'\boldsymbol{\Pi}_{12} \quad \mathbf{0}'] &= [0 \quad \boldsymbol{\mu}_1' \quad \mathbf{0}']\mathbf{B}^{-1}\begin{bmatrix} \mathbf{0}' & \mathbf{0}' \\ -\boldsymbol{\Gamma}_{12} & \mathbf{B}_{12} \\ -\boldsymbol{\Gamma}_{22} & \mathbf{B}_{22} \end{bmatrix} \\ &= [\lambda_0 \quad \boldsymbol{\lambda}_1' \quad \boldsymbol{\lambda}_2']\begin{bmatrix} \mathbf{0}' & \mathbf{0}' \\ -\boldsymbol{\Gamma}_{12} & \mathbf{B}_{12} \\ -\boldsymbol{\Gamma}_{22} & \mathbf{B}_{22} \end{bmatrix} \\ &= [\boldsymbol{\lambda}_1' \quad \boldsymbol{\lambda}_2']\begin{bmatrix} -\boldsymbol{\Gamma}_{12} & \mathbf{B}_{12} \\ -\boldsymbol{\Gamma}_{22} & \mathbf{B}_{22} \end{bmatrix}, \end{aligned} \qquad [9.A.5]$$

where

$$[\lambda_0 \quad \boldsymbol{\lambda}_1' \quad \boldsymbol{\lambda}_2'] \equiv [0 \quad \boldsymbol{\mu}_1' \quad \mathbf{0}']\mathbf{B}^{-1},$$

implying

$$[0 \quad \boldsymbol{\mu}_1' \quad \mathbf{0}'] = [\lambda_0 \quad \boldsymbol{\lambda}_1' \quad \boldsymbol{\lambda}_2']\mathbf{B}. \qquad [9.A.6]$$

Suppose that the rows of the matrix $\begin{bmatrix} \boldsymbol{\Gamma}_{12} & \mathbf{B}_{12} \\ \boldsymbol{\Gamma}_{22} & \mathbf{B}_{22} \end{bmatrix}$ are linearly independent. Then the only values for $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ for which the right side of [9.A.5] can be zero are $\boldsymbol{\lambda}_1 = \mathbf{0}$ and $\boldsymbol{\lambda}_2 = \mathbf{0}$. Substituting these values into [9.A.6], the only value of $\boldsymbol{\mu}_1$ for which the left side of [9.A.5] can be zero must satisfy

$$\begin{aligned} [0 \quad \boldsymbol{\mu}_1' \quad \mathbf{0}'] &= [\lambda_0 \quad \mathbf{0}' \quad \mathbf{0}']\mathbf{B} \\ &= [\lambda_0 \quad \lambda_0\mathbf{B}_{01} \quad \mathbf{0}']. \end{aligned}$$

Matching the first elements in these vectors implies $\lambda_0 = 0$, and thus matching the second elements requires $\boldsymbol{\mu}_1 = \mathbf{0}$. Thus, if condition (b) is satisfied, then the only value of $\boldsymbol{\mu}_1$ for which the left side of [9.A.5] can be zero is $\boldsymbol{\mu}_1 = \mathbf{0}$, establishing that the rows of $\boldsymbol{\Pi}_{12}$ are linearly independent. Hence, condition (c) is satisfied whenever (b) holds.

Conversely, to see that (c) implies (b), let $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ denote any $(n_1 \times 1)$ and $(n_2 \times 1)$ vectors, and premultiply both sides of [9.A.4] by the row vector $[0 \quad \boldsymbol{\lambda}_1' \quad \boldsymbol{\lambda}_2']\mathbf{B}$:

$$[0 \quad \boldsymbol{\lambda}_1' \quad \boldsymbol{\lambda}_2']\mathbf{B}\begin{bmatrix} \boldsymbol{\Pi}_{02} & \mathbf{0}' \\ \boldsymbol{\Pi}_{12} & \mathbf{0} \\ \boldsymbol{\Pi}_{22} & \mathbf{I}_{n_2} \end{bmatrix} = [0 \quad \boldsymbol{\lambda}_1' \quad \boldsymbol{\lambda}_2']\begin{bmatrix} \mathbf{0}' & \mathbf{0}' \\ -\boldsymbol{\Gamma}_{12} & \mathbf{B}_{12} \\ -\boldsymbol{\Gamma}_{22} & \mathbf{B}_{22} \end{bmatrix}$$

or

$$[\mu_0 \quad \boldsymbol{\mu}_1' \quad \boldsymbol{\mu}_2']\begin{bmatrix} \boldsymbol{\Pi}_{02} & \mathbf{0}' \\ \boldsymbol{\Pi}_{12} & \mathbf{0} \\ \boldsymbol{\Pi}_{22} & \mathbf{I}_{n_2} \end{bmatrix} = [\boldsymbol{\lambda}_1' \quad \boldsymbol{\lambda}_2']\begin{bmatrix} -\boldsymbol{\Gamma}_{12} & \mathbf{B}_{12} \\ -\boldsymbol{\Gamma}_{22} & \mathbf{B}_{22} \end{bmatrix} \qquad [9.A.7]$$

where

$$[\mu_0 \quad \boldsymbol{\mu}_1' \quad \boldsymbol{\mu}_2'] \equiv [0 \quad \boldsymbol{\lambda}_1' \quad \boldsymbol{\lambda}_2']\mathbf{B}. \qquad [9.A.8]$$

Premultiplying both sides of equation [9.A.4] by $\mathbf{B}$ implies that

$$\begin{bmatrix} 1 & \mathbf{B}_{01} & \mathbf{0}' \\ \mathbf{B}_{10} & \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{20} & \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix}\begin{bmatrix} \boldsymbol{\Pi}_{02} & \mathbf{0}' \\ \boldsymbol{\Pi}_{12} & \mathbf{0} \\ \boldsymbol{\Pi}_{22} & \mathbf{I}_{n_2} \end{bmatrix} = \begin{bmatrix} \mathbf{0}' & \mathbf{0}' \\ -\boldsymbol{\Gamma}_{12} & \mathbf{B}_{12} \\ -\boldsymbol{\Gamma}_{22} & \mathbf{B}_{22} \end{bmatrix}.$$

The upper left element of this matrix system asserts that

$$\boldsymbol{\Pi}_{02} + \mathbf{B}_{01}\boldsymbol{\Pi}_{12} = \mathbf{0}'. \qquad [9.A.9]$$

Substituting [9.A.9] into [9.A.7],

$$[\mu_0 \quad \boldsymbol{\mu}_1' \quad \boldsymbol{\mu}_2']\begin{bmatrix} -\mathbf{B}_{01}\boldsymbol{\Pi}_{12} & \mathbf{0}' \\ \boldsymbol{\Pi}_{12} & \mathbf{0} \\ \boldsymbol{\Pi}_{22} & \mathbf{I}_{n_2} \end{bmatrix} = [\boldsymbol{\lambda}_1' \quad \boldsymbol{\lambda}_2']\begin{bmatrix} -\boldsymbol{\Gamma}_{12} & \mathbf{B}_{12} \\ -\boldsymbol{\Gamma}_{22} & \mathbf{B}_{22} \end{bmatrix}. \qquad [9.A.10]$$

In order for the left side of [9.A.10] to be zero, it must be the case that $\boldsymbol{\mu}_2 = \mathbf{0}$ and that

$$-\mu_0\mathbf{B}_{01}\boldsymbol{\Pi}_{12} + \boldsymbol{\mu}_1'\boldsymbol{\Pi}_{12} = (\boldsymbol{\mu}_1' - \mu_0\mathbf{B}_{01})\boldsymbol{\Pi}_{12} = \mathbf{0}'. \qquad [9.A.11]$$

But if the rows of $\boldsymbol{\Pi}_{12}$ are linearly independent, [9.A.11] can be zero only if

$$\boldsymbol{\mu}_1' = \mu_0\mathbf{B}_{01}. \qquad [9.A.12]$$

Substituting these results into [9.A.8], it follows that [9.A.10] can be zero only if

$$\begin{aligned}
[0 \quad \boldsymbol{\lambda}_1' \quad \boldsymbol{\lambda}_2']\mathbf{B} &= [\mu_0 \quad \mu_0\mathbf{B}_{01} \quad \mathbf{0}'] \\
&= [\mu_0 \quad \mathbf{0}' \quad \mathbf{0}']\begin{bmatrix} 1 & \mathbf{B}_{01} & \mathbf{0}' \\ \mathbf{B}_{10} & \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{20} & \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} \qquad [9.A.13] \\
&= [\mu_0 \quad \mathbf{0}' \quad \mathbf{0}']\mathbf{B}.
\end{aligned}$$

Since $\mathbf{B}$ is nonsingular, both sides of [9.A.13] can be postmultiplied by $\mathbf{B}^{-1}$ to deduce that [9.A.10] can be zero only if

$$[0 \quad \boldsymbol{\lambda}_1' \quad \boldsymbol{\lambda}_2'] = [\mu_0 \quad \mathbf{0}' \quad \mathbf{0}'].$$

Thus, the right side of [9.A.10] can be zero only if $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ are both zero, establishing that the rows of the matrix in [9.3.12] must be linearly independent. ■

Chapter 9 Exercise

9.1. Verify that [9.2.23] gives a consistent estimate of σ^2.

Chapter 9 References

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10 Covariance-Stationary Vector Processes

This is the first of two chapters introducing vector time series. Chapter 10 is devoted to the theory of multivariate dynamic systems, while Chapter 11 focuses on empirical issues of estimating and interpreting vector autoregressions. Only the first section of Chapter 10 is necessary for understanding the material in Chapter 11.

Section 10.1 introduces some of the key ideas in vector time series analysis. Section 10.2 develops some convergence results that are useful for deriving the asymptotic properties of certain statistics and for characterizing the consequences of multivariate filters. Section 10.3 introduces the autocovariance-generating function for vector processes, which is used to analyze the multivariate spectrum in Section 10.4. Section 10.5 develops a multivariate generalization of Proposition 7.5, describing the asymptotic properties of the sample mean of a serially correlated vector process. These last results are useful for deriving autocorrelation- and heteroskedasticity-consistent estimators for *OLS*, for understanding the properties of generalized method of moments estimators discussed in Chapter 14, and for deriving some of the tests for unit roots discussed in Chapter 17.

10.1. *Introduction to Vector Autoregressions*

Chapter 3 proposed modeling a scalar time series y_t in terms of an autoregression:

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t, \qquad [10.1.1]$$

where

$$E(\varepsilon_t) = 0 \qquad [10.1.2]$$

$$E(\varepsilon_t \varepsilon_\tau) = \begin{cases} \sigma^2 & \text{for } t = \tau \\ 0 & \text{otherwise.} \end{cases} \qquad [10.1.3]$$

Note that we will continue to use the convention introduced in Chapter 8 of using lowercase letters to denote either a random variable or its realization. This chapter describes the dynamic interactions among a set of variables collected in an $(n \times 1)$ vector $\mathbf{y}_t$. For example, the first element of $\mathbf{y}_t$ (denoted y_{1t}) might represent the level of GNP in year t, the second element (y_{2t}) the interest rate paid on Treasury bills in year t, and so on. A *pth-order vector autoregression*, denoted *VAR*(p), is a vector generalization of [10.1.1] through [10.1.3]:

$$\mathbf{y}_t = \mathbf{c} + \mathbf{\Phi}_1 \mathbf{y}_{t-1} + \mathbf{\Phi}_2 \mathbf{y}_{t-2} + \cdots + \mathbf{\Phi}_p \mathbf{y}_{t-p} + \boldsymbol{\varepsilon}_t. \qquad [10.1.4]$$

Here $\mathbf{c}$ denotes an $(n \times 1)$ vector of constants and $\mathbf{\Phi}_j$ an $(n \times n)$ matrix of autoregressive coefficients for $j = 1, 2, \ldots, p$. The $(n \times 1)$ vector $\boldsymbol{\varepsilon}_t$ is a vector generalization of white noise:

$$E(\varepsilon_t) = \mathbf{0} \qquad [10.1.5]$$

$$E(\varepsilon_t\varepsilon_\tau') = \begin{cases} \mathbf{\Omega} & \text{for } t = \tau \\ \mathbf{0} & \text{otherwise,} \end{cases} \qquad [10.1.6]$$

with $\mathbf{\Omega}$ an $(n \times n)$ symmetric positive definite matrix.

Let c_i denote the ith element of the vector $\mathbf{c}$ and let $\phi_{ij}^{(1)}$ denote the row i, column j element of the matrix $\mathbf{\Phi}_1$. Then the first row of the vector system in [10.1.4] specifies that

$$\begin{aligned} y_{1t} = c_1 &+ \phi_{11}^{(1)}y_{1,t-1} + \phi_{12}^{(1)}y_{2,t-1} + \cdots + \phi_{1n}^{(1)}y_{n,t-1} \\ &+ \phi_{11}^{(2)}y_{1,t-2} + \phi_{12}^{(2)}y_{2,t-2} + \cdots + \phi_{1n}^{(2)}y_{n,t-2} \\ &+ \cdots + \phi_{11}^{(p)}y_{1,t-p} + \phi_{12}^{(p)}y_{2,t-p} + \cdots + \phi_{1n}^{(p)}y_{n,t-p} + \varepsilon_{1t}. \end{aligned} \qquad [10.1.7]$$

Thus, a vector autoregression is a system in which each variable is regressed on a constant and p of its own lags as well as on p lags of each of the other variables in the *VAR*. Note that each regression has the same explanatory variables.

Using lag operator notation, [10.1.4] can be written in the form

$$[\mathbf{I}_n - \mathbf{\Phi}_1 L - \mathbf{\Phi}_2 L^2 - \cdots - \mathbf{\Phi}_p L^p]\mathbf{y}_t = \mathbf{c} + \varepsilon_t$$

or

$$\mathbf{\Phi}(L)\mathbf{y}_t = \mathbf{c} + \varepsilon_t.$$

Here $\mathbf{\Phi}(L)$ indicates an $(n \times n)$ matrix polynomial in the lag operator L. The row i, column j element of $\mathbf{\Phi}(L)$ is a scalar polynomial in L:

$$\mathbf{\Phi}(L) = [\delta_{ij} - \phi_{ij}^{(1)}L^1 - \phi_{ij}^{(2)}L^2 - \cdots - \phi_{ij}^{(p)}L^p],$$

where δ_{ij} is unity if $i = j$ and zero otherwise.

A vector process $\mathbf{y}_t$ is said to be covariance-stationary if its first and second moments ($E[\mathbf{y}_t]$ and $E[\mathbf{y}_t\mathbf{y}_{t-j}']$, respectively) are independent of the date t. If the process is covariance-stationary, we can take expectations of both sides of [10.1.4] to calculate the mean $\boldsymbol{\mu}$ of the process:

$$\boldsymbol{\mu} = \mathbf{c} + \mathbf{\Phi}_1\boldsymbol{\mu} + \mathbf{\Phi}_2\boldsymbol{\mu} + \cdots + \mathbf{\Phi}_p\boldsymbol{\mu},$$

or

$$\boldsymbol{\mu} = (\mathbf{I}_n - \mathbf{\Phi}_1 - \mathbf{\Phi}_2 - \cdots - \mathbf{\Phi}_p)^{-1}\mathbf{c}.$$

Equation [10.1.4] can then be written in terms of deviations from the mean as

$$\begin{aligned} (\mathbf{y}_t - \boldsymbol{\mu}) = \mathbf{\Phi}_1(\mathbf{y}_{t-1} - \boldsymbol{\mu}) \\ + \mathbf{\Phi}_2(\mathbf{y}_{t-2} - \boldsymbol{\mu}) + \cdots + \mathbf{\Phi}_p(\mathbf{y}_{t-p} - \boldsymbol{\mu}) + \varepsilon_t. \end{aligned} \qquad [10.1.8]$$

Rewriting a VAR(p) *as a* VAR(*1*)

As in the case of the univariate $AR(p)$ process, it is helpful to rewrite [10.1.8] in terms of a $VAR(1)$ process. Toward this end, define

$$\underset{(np \times 1)}{\boldsymbol{\xi}_t} \equiv \begin{bmatrix} \mathbf{y}_t - \boldsymbol{\mu} \\ \mathbf{y}_{t-1} - \boldsymbol{\mu} \\ \vdots \\ \mathbf{y}_{t-p+1} - \boldsymbol{\mu} \end{bmatrix} \qquad [10.1.9]$$

$$\underset{(np \times np)}{\mathbf{F}} \equiv \begin{bmatrix} \boldsymbol{\Phi}_1 & \boldsymbol{\Phi}_2 & \boldsymbol{\Phi}_3 & \cdots & \boldsymbol{\Phi}_{p-1} & \boldsymbol{\Phi}_p \\ \mathbf{I}_n & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_n & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I}_n & \mathbf{0} \end{bmatrix} \qquad [10.1.10]$$

$$\underset{(np \times 1)}{\mathbf{v}_t} \equiv \begin{bmatrix} \boldsymbol{\varepsilon}_t \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}.$$

The $VAR(p)$ in [10.1.8] can then be rewritten as the following $VAR(1)$:

$$\boldsymbol{\xi}_t = \mathbf{F}\boldsymbol{\xi}_{t-1} + \mathbf{v}_t, \qquad [10.1.11]$$

where

$$E(\mathbf{v}_t\mathbf{v}_\tau') = \begin{cases} \mathbf{Q} & \text{for } t = \tau \\ \mathbf{0} & \text{otherwise} \end{cases}$$

and

$$\underset{(np \times np)}{\mathbf{Q}} \equiv \begin{bmatrix} \boldsymbol{\Omega} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}$$

Conditions for Stationarity

Equation [10.1.11] implies that

$$\boldsymbol{\xi}_{t+s} = \mathbf{v}_{t+s} + \mathbf{F}\mathbf{v}_{t+s-1} + \mathbf{F}^2\mathbf{v}_{t+s-2} + \cdots + \mathbf{F}^{s-1}\mathbf{v}_{t+1} + \mathbf{F}^s\boldsymbol{\xi}_t. \qquad [10.1.12]$$

In order for the process to be covariance-stationary, the consequences of any given $\boldsymbol{\varepsilon}_t$ must eventually die out. If the eigenvalues of $\mathbf{F}$ all lie inside the unit circle, then the VAR turns out to be covariance-stationary.

The following result generalizes Proposition 1.1 from Chapter 1 (for a proof see Appendix 10.A at the end of this chapter).

Proposition 10.1: *The eigenvalues of the matrix* $\mathbf{F}$ *in [10.1.10] satisfy*

$$|\mathbf{I}_n\lambda^p - \boldsymbol{\Phi}_1\lambda^{p-1} - \boldsymbol{\Phi}_2\lambda^{p-2} - \cdots - \boldsymbol{\Phi}_p| = 0. \qquad [10.1.13]$$

Hence, a $VAR(p)$ is covariance-stationary as long as $|\lambda| < 1$ for all values of λ satisfying [10.1.13]. Equivalently, the VAR is covariance-stationary if all values of z satisfying

$$|\mathbf{I}_n - \boldsymbol{\Phi}_1 z - \boldsymbol{\Phi}_2 z^2 - \cdots - \boldsymbol{\Phi}_p z^p| = 0$$

lie outside the unit circle.

Vector MA(∞) Representation

The first n rows of the vector system represented in [10.1.12] constitute a vector generalization of equation [4.2.20]:

$$\begin{aligned} \mathbf{y}_{t+s} &= \boldsymbol{\mu} + \boldsymbol{\varepsilon}_{t+s} + \boldsymbol{\Psi}_1\boldsymbol{\varepsilon}_{t+s-1} + \boldsymbol{\Psi}_2\boldsymbol{\varepsilon}_{t+s-2} + \cdots + \boldsymbol{\Psi}_{s-1}\boldsymbol{\varepsilon}_{t+1} \\ &\quad + \mathbf{F}_{11}^{(s)}(\mathbf{y}_t - \boldsymbol{\mu}) + \mathbf{F}_{12}^{(s)}(\mathbf{y}_{t-1} - \boldsymbol{\mu}) + \cdots + \mathbf{F}_{1p}^{(s)}(\mathbf{y}_{t-p+1} - \boldsymbol{\mu}). \end{aligned} \quad [10.1.14]$$

Here $\boldsymbol{\Psi}_j = \mathbf{F}_{11}^{(j)}$ and $\mathbf{F}_{11}^{(j)}$ denotes the upper left block of $\mathbf{F}^j$, where $\mathbf{F}^j$ is the matrix $\mathbf{F}$ raised to the jth power—that is, the $(n \times n)$ matrix $\mathbf{F}_{11}^{(j)}$ indicates rows 1 through n and columns 1 through n of the $(np \times np)$ matrix $\mathbf{F}^j$. Similarly, $\mathbf{F}_{12}^{(j)}$ denotes the block of $\mathbf{F}^j$ consisting of rows 1 through n and columns $(n + 1)$ through $2n$, while $\mathbf{F}_{1p}^{(j)}$ denotes rows 1 through n and columns $[n(p - 1) + 1]$ through np of $\mathbf{F}^j$.

If the eigenvalues of $\mathbf{F}$ all lie inside the unit circle, then $\mathbf{F}^s \to \mathbf{0}$ as $s \to \infty$ and $\mathbf{y}_t$ can be expressed as a convergent sum of the history of $\boldsymbol{\varepsilon}$:

$$\mathbf{y}_t = \boldsymbol{\mu} + \boldsymbol{\varepsilon}_t + \boldsymbol{\Psi}_1\boldsymbol{\varepsilon}_{t-1} + \boldsymbol{\Psi}_2\boldsymbol{\varepsilon}_{t-2} + \boldsymbol{\Psi}_3\boldsymbol{\varepsilon}_{t-3} + \cdots \equiv \boldsymbol{\mu} + \boldsymbol{\Psi}(L)\boldsymbol{\varepsilon}_t, \quad [10.1.15]$$

which is a vector $MA(\infty)$ representation.

Note that $\mathbf{y}_{t-j}$ is a linear function of $\boldsymbol{\varepsilon}_{t-j}, \boldsymbol{\varepsilon}_{t-j-1}, \ldots$, each of which is uncorrelated with $\boldsymbol{\varepsilon}_{t+1}$ for $j = 0, 1, \ldots$. It follows that $\boldsymbol{\varepsilon}_{t+1}$ is uncorrelated with $\mathbf{y}_{t-j}$ for any $j \geq 0$. Thus, the linear forecast of $\mathbf{y}_{t+1}$ on the basis of $\mathbf{y}_t, \mathbf{y}_{t-1}, \ldots$ is given by

$$\hat{\mathbf{y}}_{t+1|t} = \boldsymbol{\mu} + \boldsymbol{\Phi}_1(\mathbf{y}_t - \boldsymbol{\mu}) + \boldsymbol{\Phi}_2(\mathbf{y}_{t-1} - \boldsymbol{\mu}) + \cdots + \boldsymbol{\Phi}_p(\mathbf{y}_{t-p+1} - \boldsymbol{\mu}),$$

and $\boldsymbol{\varepsilon}_{t+1}$ can be interpreted as the fundamental innovation for $\mathbf{y}_{t+1}$, that is, the error in forecasting $\mathbf{y}_{t+1}$ on the basis of a linear function of a constant and $\mathbf{y}_t, \mathbf{y}_{t-1}, \ldots$. More generally, it follows from [10.1.14] that a forecast of $\mathbf{y}_{t+s}$ on the basis of $\mathbf{y}_t, \mathbf{y}_{t-1}, \ldots$ will take the form

$$\begin{aligned} \hat{\mathbf{y}}_{t+s|t} &= \boldsymbol{\mu} + \mathbf{F}_{11}^{(s)}(\mathbf{y}_t - \boldsymbol{\mu}) + \mathbf{F}_{12}^{(s)}(\mathbf{y}_{t-1} - \boldsymbol{\mu}) \\ &\quad + \cdots + \mathbf{F}_{1p}^{(s)}(\mathbf{y}_{t-p+1} - \boldsymbol{\mu}). \end{aligned} \quad [10.1.16]$$

The moving average matrices $\boldsymbol{\Psi}_j$ could equivalently be calculated as follows. The operators $\boldsymbol{\Phi}(L)$ and $\boldsymbol{\Psi}(L)$ are related by

$$\boldsymbol{\Psi}(L) = [\boldsymbol{\Phi}(L)]^{-1},$$

requiring

$$[\mathbf{I}_n - \boldsymbol{\Phi}_1 L - \boldsymbol{\Phi}_2 L^2 - \cdots - \boldsymbol{\Phi}_p L^p][\mathbf{I}_n + \boldsymbol{\Psi}_1 L + \boldsymbol{\Psi}_2 L^2 + \cdots] = \mathbf{I}_n.$$

Setting the coefficient on L^1 equal to the zero matrix, as in Exercise 3.3 of Chapter 3, produces

$$\boldsymbol{\Psi}_1 - \boldsymbol{\Phi}_1 = \mathbf{0}. \quad [10.1.17]$$

Similarly, setting the coefficient on L^2 equal to zero gives

$$\boldsymbol{\Psi}_2 = \boldsymbol{\Phi}_1\boldsymbol{\Psi}_1 + \boldsymbol{\Phi}_2, \quad [10.1.18]$$

and in general for L^s,

$$\boldsymbol{\Psi}_s = \boldsymbol{\Phi}_1\boldsymbol{\Psi}_{s-1} + \boldsymbol{\Phi}_2\boldsymbol{\Psi}_{s-2} + \cdots + \boldsymbol{\Phi}_p\boldsymbol{\Psi}_{s-p} \quad \text{for } s = 1, 2, \ldots, \quad [10.1.19]$$

with $\boldsymbol{\Psi}_0 = \mathbf{I}_n$ and $\boldsymbol{\Psi}_s = \mathbf{0}$ for $s < 0$.

Note that the innovation in the $MA(\infty)$ representation [10.1.15] is $\boldsymbol{\varepsilon}_t$, the fundamental innovation for $\mathbf{y}$. There are alternative moving average representations based on vector white noise processes other than $\boldsymbol{\varepsilon}_t$. Let $\mathbf{H}$ denote a nonsingular

$(n \times n)$ matrix, and define

$$\mathbf{u}_t \equiv \mathbf{H}\boldsymbol{\varepsilon}_t. \qquad [10.1.20]$$

Then certainly $\mathbf{u}_t$ is white noise. Moreover, from [10.1.15] we could write

$$\begin{aligned} \mathbf{y}_t &= \boldsymbol{\mu} + \mathbf{H}^{-1}\mathbf{H}\boldsymbol{\varepsilon}_t + \boldsymbol{\Psi}_1\mathbf{H}^{-1}\mathbf{H}\boldsymbol{\varepsilon}_{t-1} + \boldsymbol{\Psi}_2\mathbf{H}^{-1}\mathbf{H}\boldsymbol{\varepsilon}_{t-2} \\ &\quad + \boldsymbol{\Psi}_3\mathbf{H}^{-1}\mathbf{H}\boldsymbol{\varepsilon}_{t-3} + \cdots \qquad [10.1.21] \\ &= \boldsymbol{\mu} + \mathbf{J}_0\mathbf{u}_t + \mathbf{J}_1\mathbf{u}_{t-1} + \mathbf{J}_2\mathbf{u}_{t-2} + \mathbf{J}_3\mathbf{u}_{t-3} + \cdots, \end{aligned}$$

where

$$\mathbf{J}_s \equiv \boldsymbol{\Psi}_s\mathbf{H}^{-1}.$$

For example, $\mathbf{H}$ could be any matrix that diagonalizes $\boldsymbol{\Omega}$, the variance-covariance matrix of $\boldsymbol{\varepsilon}_t$:

$$\mathbf{H}\boldsymbol{\Omega}\mathbf{H}' = \mathbf{D},$$

with $\mathbf{D}$ a diagonal matrix. For such a choice of $\mathbf{H}$, the elements of $\mathbf{u}_t$ are uncorrelated with one another:

$$E(\mathbf{u}_t\mathbf{u}_t') = E(\mathbf{H}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\mathbf{H}') = \mathbf{D}.$$

Thus, it is always possible to write a stationary $VAR(p)$ process as a convergent infinite moving average of a white noise vector $\mathbf{u}_t$ whose elements are mutually uncorrelated.

There is one important difference between the $MA(\infty)$ representations [10.1.15] and [10.1.21], however. In [10.1.15], the leading MA parameter matrix ($\boldsymbol{\Psi}_0$) is the identity matrix, whereas in [10.1.21] the leading MA parameter matrix ($\mathbf{J}_0$) is not the identity matrix. To obtain the MA representation for the fundamental innovations, we must impose the normalization $\boldsymbol{\Psi}_0 = \mathbf{I}_n$.

Assumptions Implicit in a VAR

For a covariance-stationary process, the parameters $\mathbf{c}$ and $\boldsymbol{\Phi}_1, \ldots, \boldsymbol{\Phi}_p$ in equation [10.1.4] could be defined as the coefficients of the projection of $\mathbf{y}_t$ on a constant and $\mathbf{y}_{t-1}, \ldots, \mathbf{y}_{t-p}$. Thus, $\boldsymbol{\varepsilon}_t$ is uncorrelated with $\mathbf{y}_{t-1}, \ldots, \mathbf{y}_{t-p}$ by the definition of $\boldsymbol{\Phi}_1, \ldots, \boldsymbol{\Phi}_p$. The parameters of a vector autoregression can accordingly be estimated consistently with n *OLS* regressions of the form of [10.1.7]. The additional assumption implicit in a *VAR* is that the $\boldsymbol{\varepsilon}_t$ defined by this projection is further uncorrelated with $\mathbf{y}_{t-p-1}, \mathbf{y}_{t-p-2}, \ldots$. The assumption that $\mathbf{y}_t$ follows a vector autoregression is basically the assumption that p lags are sufficient to summarize all of the dynamic correlations between elements of $\mathbf{y}$.

10.2. *Autocovariances and Convergence Results for Vector Processes*

The j*th Autocovariance Matrix*

For a covariance-stationary n-dimensional vector process, the jth autocovariance is defined to be the following $(n \times n)$ matrix:

$$\boldsymbol{\Gamma}_j = E[(\mathbf{y}_t - \boldsymbol{\mu})(\mathbf{y}_{t-j} - \boldsymbol{\mu})']. \qquad [10.2.1]$$

Note that although $\gamma_j = \gamma_{-j}$ for a scalar process, the same is not true of a vector process:

$$\boldsymbol{\Gamma}_j \neq \boldsymbol{\Gamma}_{-j}.$$

For example, the (1, 2) element of $\boldsymbol{\Gamma}_j$ gives the covariance between y_{1t} and $y_{2,t-j}$. The (1, 2) element of $\boldsymbol{\Gamma}_{-j}$ gives the covariance between y_{1t} and $y_{2,t+j}$. There is no reason that these should be related—the response of y_1 to previous movements in y_2 could be completely different from the response of y_2 to previous movements in y_1.

Instead, the correct relation is

$$\boldsymbol{\Gamma}_j' = \boldsymbol{\Gamma}_{-j}. \qquad [10.2.2]$$

To derive [10.2.2], notice that covariance-stationarity would mean that t in [10.2.1] could be replaced with $t + j$:

$$\boldsymbol{\Gamma}_j = E[(\mathbf{y}_{t+j} - \boldsymbol{\mu})(\mathbf{y}_{(t+j)-j} - \boldsymbol{\mu})'] = E[(\mathbf{y}_{t+j} - \boldsymbol{\mu})(\mathbf{y}_t - \boldsymbol{\mu})'].$$

Taking transposes,

$$\boldsymbol{\Gamma}_j' = E[(\mathbf{y}_t - \boldsymbol{\mu})(\mathbf{y}_{t+j} - \boldsymbol{\mu})'] = \boldsymbol{\Gamma}_{-j},$$

as claimed.

Vector MA(q) *Process*

A *vector moving average process* of order q takes the form

$$\mathbf{y}_t = \boldsymbol{\mu} + \boldsymbol{\varepsilon}_t + \boldsymbol{\Theta}_1\boldsymbol{\varepsilon}_{t-1} + \boldsymbol{\Theta}_2\boldsymbol{\varepsilon}_{t-2} + \cdots + \boldsymbol{\Theta}_q\boldsymbol{\varepsilon}_{t-q}, \qquad [10.2.3]$$

where $\boldsymbol{\varepsilon}_t$ is a vector white noise process satisfying [10.1.5] and [10.1.6] and $\boldsymbol{\Theta}_j$ denotes an $(n \times n)$ matrix of *MA* coefficients for $j = 1, 2, \ldots, q$. The mean of $\mathbf{y}_t$ is $\boldsymbol{\mu}$, and the variance is

$$\begin{aligned}\boldsymbol{\Gamma}_0 &= E[(\mathbf{y}_t - \boldsymbol{\mu})(\mathbf{y}_t - \boldsymbol{\mu})'] \\ &= E[\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'] + \boldsymbol{\Theta}_1 E[\boldsymbol{\varepsilon}_{t-1}\boldsymbol{\varepsilon}_{t-1}']\boldsymbol{\Theta}_1' + \boldsymbol{\Theta}_2 E[\boldsymbol{\varepsilon}_{t-2}\boldsymbol{\varepsilon}_{t-2}']\boldsymbol{\Theta}_2' \\ &\quad + \cdots + \boldsymbol{\Theta}_q E[\boldsymbol{\varepsilon}_{t-q}\boldsymbol{\varepsilon}_{t-q}']\boldsymbol{\Theta}_q' \\ &= \boldsymbol{\Omega} + \boldsymbol{\Theta}_1\boldsymbol{\Omega}\boldsymbol{\Theta}_1' + \boldsymbol{\Theta}_2\boldsymbol{\Omega}\boldsymbol{\Theta}_2' + \cdots + \boldsymbol{\Theta}_q\boldsymbol{\Omega}\boldsymbol{\Theta}_q',\end{aligned} \qquad [10.2.4]$$

with autocovariances

$$\boldsymbol{\Gamma}_j = \begin{cases} \boldsymbol{\Theta}_j\boldsymbol{\Omega} + \boldsymbol{\Theta}_{j+1}\boldsymbol{\Omega}\boldsymbol{\Theta}_1' + \boldsymbol{\Theta}_{j+2}\boldsymbol{\Omega}\boldsymbol{\Theta}_2' + \cdots + \boldsymbol{\Theta}_q\boldsymbol{\Omega}\boldsymbol{\Theta}_{q-j}' \\ \qquad\qquad \text{for } j = 1, 2, \ldots, q \\ \boldsymbol{\Omega}\boldsymbol{\Theta}_{-j}' + \boldsymbol{\Theta}_1\boldsymbol{\Omega}\boldsymbol{\Theta}_{-j+1}' + \boldsymbol{\Theta}_2\boldsymbol{\Omega}\boldsymbol{\Theta}_{-j+2}' + \cdots + \boldsymbol{\Theta}_{q+j}\boldsymbol{\Omega}\boldsymbol{\Theta}_q' \\ \qquad\qquad \text{for } j = -1, -2, \ldots, -q \\ \mathbf{0} \qquad\qquad \text{for } |j| > q, \end{cases} \qquad [10.2.5]$$

where $\boldsymbol{\Theta}_0 = \mathbf{I}_n$. Thus, any vector $MA(q)$ process is covariance-stationary.

Vector MA(∞) *Process*

The vector $MA(\infty)$ process is written

$$\mathbf{y}_t = \boldsymbol{\mu} + \boldsymbol{\varepsilon}_t + \boldsymbol{\Psi}_1\boldsymbol{\varepsilon}_{t-1} + \boldsymbol{\Psi}_2\boldsymbol{\varepsilon}_{t-2} + \cdots \qquad [10.2.6]$$

for $\boldsymbol{\varepsilon}_t$ again satisfying [10.1.5] and [10.1.6].

A sequence of scalars $\{h_s\}_{s=-\infty}^{\infty}$ was said to be absolutely summable if $\Sigma_{s=-\infty}^{\infty} |h_s| < \infty$. For $\mathbf{H}_s$ an $(n \times m)$ matrix, the sequence of matrices $\{\mathbf{H}_s\}_{s=-\infty}^{\infty}$ is absolutely summable if each of its elements forms an absolutely summable scalar sequence. For example, if $\psi_{ij}^{(s)}$ denotes the row i, column j element of the moving average parameter matrix $\mathbf{\Psi}_s$ associated with lag s, then the sequence $\{\mathbf{\Psi}_s\}_{s=0}^{\infty}$ is absolutely summable if

$$\sum_{s=0}^{\infty} |\psi_{ij}^{(s)}| < \infty \qquad \text{for } i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, n. \qquad [10.2.7]$$

Many of the results for scalar $MA(\infty)$ processes with absolutely summable coefficients go through for vector processes as well. This is summarized by the following theorem, proved in Appendix 10.A to this chapter.

Proposition 10.2: *Let $\mathbf{y}_t$ be an $(n \times 1)$ vector satisfying*

$$\mathbf{y}_t = \boldsymbol{\mu} + \sum_{k=0}^{\infty} \mathbf{\Psi}_k \boldsymbol{\varepsilon}_{t-k},$$

where $\boldsymbol{\varepsilon}_t$ is vector white noise satisfying [10.1.5] and [10.1.6] and $\{\mathbf{\Psi}_k\}_{k=0}^{\infty}$ is absolutely summable. Let y_{it} denote the ith element of $\mathbf{y}_t$, and let μ_i denote the ith element of $\boldsymbol{\mu}$. Then

(a) the autocovariance between the ith variable at time t and the jth variable s periods earlier, $E(y_{it} - \mu_i)(y_{j,t-s} - \mu_j)$, exists and is given by the row i, column j element of

$$\mathbf{\Gamma}_s = \sum_{v=0}^{\infty} \mathbf{\Psi}_{s+v} \mathbf{\Omega} \mathbf{\Psi}_v' \qquad \textit{for } s = 0, 1, 2, \ldots;$$

(b) the sequence of matrices $\{\mathbf{\Gamma}_s\}_{s=0}^{\infty}$ is absolutely summable.

If, furthermore, $\{\boldsymbol{\varepsilon}_t\}_{t=-\infty}^{\infty}$ is an i.i.d. sequence with $E|\varepsilon_{i_1,t}\varepsilon_{i_2,t}\varepsilon_{i_3,t}\varepsilon_{i_4,t}| < \infty$ for $i_1, i_2, i_3, i_4 = 1, 2, \ldots, n$, then also,

(c) $E|y_{i_1,t_1}y_{i_2,t_2}y_{i_3,t_3}y_{i_4,t_4}| < \infty$ for $i_1, i_2, i_3, i_4 = 1, 2, \ldots, n$ and for all t_1, t_2, t_3, t_4;

(d) $(1/T)\sum_{t=1}^{T} y_{it}y_{j,t-s} \xrightarrow{p} E(y_{it}y_{j,t-s})$ for $i, j = 1, 2, \ldots, n$ and for all s.

Result (a) implies that the second moments of an $MA(\infty)$ vector process with absolutely summable coefficients can be found by taking the limit of [10.2.5] as $q \to \infty$. Result (b) is a convergence condition on these moments that will turn out to ensure that the vector process is ergodic for the mean (see Proposition 10.5 later in this chapter). Result (c) says that $\mathbf{y}_t$ has bounded fourth moments, while result (d) establishes that $\mathbf{y}_t$ is ergodic for second moments.

Note that the vector $MA(\infty)$ representation of a stationary vector autoregression calculated from [10.1.4] satisfies the absolute summability condition. To see this, recall from [10.1.14] that $\mathbf{\Psi}_s$ is a block of the matrix $\mathbf{F}^s$. If $\mathbf{F}$ has np distinct eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_{np})$, then any element of $\mathbf{\Psi}_s$ can be written as a weighted average of these eigenvalues as in equation [1.2.20]:

$$\psi_{ij}^{(s)} = c_1(i,j)\cdot\lambda_1^s + c_2(i,j)\cdot\lambda_2^s + \cdots + c_{np}(i,j)\cdot\lambda_{np}^s,$$

where $c_v(i, j)$ denotes a constant that depends on v, i, and j but not s. Absolute summability [10.2.7] then follows from the same arguments as in Exercise 3.5.

Multivariate Filters

Suppose that the $(n \times 1)$ vector $\mathbf{y}_t$ follows an $MA(\infty)$ process:

$$\mathbf{y}_t = \boldsymbol{\mu}_Y + \boldsymbol{\Psi}(L)\boldsymbol{\varepsilon}_t, \qquad [10.2.8]$$

with $\{\boldsymbol{\Psi}_k\}_{k=0}^{\infty}$ absolutely summable. Let $\{\mathbf{H}_k\}_{k=-\infty}^{\infty}$ be an absolutely summable sequence of $(r \times n)$ matrices and suppose that an $(r \times 1)$ vector $\mathbf{x}_t$ is related to $\mathbf{y}_t$ according to

$$\mathbf{x}_t = \mathbf{H}(L)\mathbf{y}_t = \sum_{k=-\infty}^{\infty} \mathbf{H}_k \mathbf{y}_{t-k}. \qquad [10.2.9]$$

That is,

$$\begin{aligned} \mathbf{x}_t &= \mathbf{H}(L)[\boldsymbol{\mu}_Y + \boldsymbol{\Psi}(L)\boldsymbol{\varepsilon}_t] \\ &= \mathbf{H}(1)\boldsymbol{\mu}_Y + \mathbf{H}(L)\boldsymbol{\Psi}(L)\boldsymbol{\varepsilon}_t \\ &= \boldsymbol{\mu}_X + \mathbf{B}(L)\boldsymbol{\varepsilon}_t, \end{aligned} \qquad [10.2.10]$$

where $\boldsymbol{\mu}_X \equiv \mathbf{H}(1)\boldsymbol{\mu}_Y$ and $\mathbf{B}(L)$ is the compound operator given by

$$\mathbf{B}(L) = \sum_{k=-\infty}^{\infty} \mathbf{B}_k L^k = \mathbf{H}(L)\boldsymbol{\Psi}(L). \qquad [10.2.11]$$

The following proposition establishes that $\mathbf{x}_t$ follows an absolutely summable two-sided $MA(\infty)$ process.

Proposition 10.3: *Let $\{\boldsymbol{\Psi}_k\}_{k=0}^{\infty}$ be an absolutely summable sequence of $(n \times n)$ matrices and let $\{\mathbf{H}_k\}_{k=-\infty}^{\infty}$ be an absolutely summable sequence of $(r \times n)$ matrices. Then the sequence of matrices $\{\mathbf{B}_k\}_{k=-\infty}^{\infty}$ associated with the operator $\mathbf{B}(L) = \mathbf{H}(L)\boldsymbol{\Psi}(L)$ is absolutely summable.*

If $\{\boldsymbol{\varepsilon}_t\}$ in [10.2.8] is i.i.d. with finite fourth moments, then $\{\mathbf{x}_t\}$ in [10.2.9] has finite fourth moments and is ergodic for second moments.

Vector Autoregression

Next we derive expressions for the second moments for $\mathbf{y}_t$ following a $VAR(p)$. Let $\boldsymbol{\xi}_t$ be as defined in equation [10.1.9]. Assuming that $\boldsymbol{\xi}$ and $\mathbf{y}$ are covariance-stationary, let $\boldsymbol{\Sigma}$ denote the variance of $\boldsymbol{\xi}$,

$$\begin{aligned} \boldsymbol{\Sigma} &= E(\boldsymbol{\xi}_t\boldsymbol{\xi}_t') \\ &= E\left\{\begin{bmatrix} \mathbf{y}_t - \boldsymbol{\mu} \\ \mathbf{y}_{t-1} - \boldsymbol{\mu} \\ \vdots \\ \mathbf{y}_{t-p+1} - \boldsymbol{\mu} \end{bmatrix} \times [(\mathbf{y}_t - \boldsymbol{\mu})' \quad (\mathbf{y}_{t-1} - \boldsymbol{\mu})' \quad \cdots \quad (\mathbf{y}_{t-p+1} - \boldsymbol{\mu})']\right\} \\ &= \begin{bmatrix} \boldsymbol{\Gamma}_0 & \boldsymbol{\Gamma}_1 & \cdots & \boldsymbol{\Gamma}_{p-1} \\ \boldsymbol{\Gamma}_1' & \boldsymbol{\Gamma}_0 & \cdots & \boldsymbol{\Gamma}_{p-2} \\ \vdots & \vdots & \cdots & \vdots \\ \boldsymbol{\Gamma}_{p-1}' & \boldsymbol{\Gamma}_{p-2}' & \cdots & \boldsymbol{\Gamma}_0 \end{bmatrix}, \end{aligned} \qquad [10.2.12]$$

where Γ_j denotes the jth autocovariance of the original process y. Postmultiplying [10.1.11] by its own transpose and taking expectations gives

$$E[\xi_t\xi_t'] = E[(F\xi_{t-1} + v_t)(F\xi_{t-1} + v_t)'] = FE(\xi_{t-1}\xi_{t-1}')F' + E(v_tv_t'),$$

or

$$\Sigma = F\Sigma F' + Q. \qquad [10.2.13]$$

A closed-form solution to [10.2.13] can be obtained in terms of the *vec* operator. If $\mathbf{A}$ is an $(m \times n)$ matrix, then vec($\mathbf{A}$) is an $(mn \times 1)$ column vector, obtained by stacking the columns of $\mathbf{A}$, one below the other, with the columns ordered from left to right. For example, if

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix},$$

then

$$\text{vec}(\mathbf{A}) = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}. \qquad [10.2.14]$$

Appendix 10.A establishes the following useful result.

Proposition 10.4: Let $\mathbf{A}$, $\mathbf{B}$, *and* $\mathbf{C}$ *be matrices whose dimensions are such that the product* $\mathbf{ABC}$ *exists. Then*

$$\text{vec}(\mathbf{ABC}) = (\mathbf{C}' \otimes \mathbf{A})\cdot\text{vec}(\mathbf{B}) \qquad [10.2.15]$$

where the symbol $\otimes$ *denotes the Kronecker product.*

Thus, if the vec operator is applied to both sides of [10.2.13], the result is

$$\text{vec}(\Sigma) = (\mathbf{F} \otimes \mathbf{F})\cdot\text{vec}(\Sigma) + \text{vec}(\mathbf{Q}) = \mathcal{A}\,\text{vec}(\Sigma) + \text{vec}(\mathbf{Q}), \qquad [10.2.16]$$

where

$$\mathcal{A} \equiv (\mathbf{F} \otimes \mathbf{F}). \qquad [10.2.17]$$

Let $r = np$, so that $\mathbf{F}$ is an $(r \times r)$ matrix and $\mathcal{A}$ is an $(r^2 \times r^2)$ matrix. Equation [10.2.16] has the solution

$$\text{vec}(\Sigma) = [\mathbf{I}_{r^2} - \mathcal{A}]^{-1}\,\text{vec}(\mathbf{Q}), \qquad [10.2.18]$$

provided that the matrix $[\mathbf{I}_{r^2} - \mathcal{A}]$ is nonsingular. This will be true as long as unity is not an eigenvalue of $\mathcal{A}$. But recall that the eigenvalues of $\mathbf{F} \otimes \mathbf{F}$ are all of the form $\lambda_i\lambda_j$, where λ_i and λ_j are eigenvalues of $\mathbf{F}$. Since $|\lambda_i| < 1$ for all i, it follows that all eigenvalues of $\mathcal{A}$ are inside the unit circle, meaning that $[\mathbf{I}_{r^2} - \mathcal{A}]$ is indeed nonsingular.

The first p autocovariance matrices of a $VAR(p)$ process can be calculated by substituting [10.2.12] into [10.2.18]:

$$\text{vec}\begin{bmatrix} \Gamma_0 & \Gamma_1 & \cdots & \Gamma_{p-1} \\ \Gamma_1' & \Gamma_0 & \cdots & \Gamma_{p-2} \\ \vdots & \vdots & \cdots & \vdots \\ \Gamma_{p-1}' & \Gamma_{p-2}' & \cdots & \Gamma_0 \end{bmatrix} = [\mathbf{I}_{r^2} - \mathcal{A}]^{-1}\,\text{vec}(\mathbf{Q}). \qquad [10.2.19]$$

The jth autocovariance of $\boldsymbol{\xi}$ (denoted $\boldsymbol{\Sigma}_j$) can be found by postmultiplying [10.1.11] by $\boldsymbol{\xi}_{t-j}'$ and taking expectations:

$$E(\boldsymbol{\xi}_t\boldsymbol{\xi}_{t-j}') = \mathbf{F}\cdot E(\boldsymbol{\xi}_{t-1}\boldsymbol{\xi}_{t-j}') + E(\mathbf{v}_t\boldsymbol{\xi}_{t-j}').$$

Thus,

$$\boldsymbol{\Sigma}_j = \mathbf{F}\boldsymbol{\Sigma}_{j-1} \qquad \text{for } j = 1, 2, \ldots, \qquad [10.2.20]$$

or

$$\boldsymbol{\Sigma}_j = \mathbf{F}^j\boldsymbol{\Sigma} \qquad \text{for } j = 1, 2, \ldots. \qquad [10.2.21]$$

The jth autocovariance Γ_j of the original process $\mathbf{y}_t$ is given by the first n rows and n columns of [10.2.20]:

$$\Gamma_j = \Phi_1\Gamma_{j-1} + \Phi_2\Gamma_{j-2} + \cdots + \Phi_p\Gamma_{j-p} \qquad \text{for } j = p, p+1, p+2, \ldots. \qquad [10.2.22]$$

10.3. *The Autocovariance-Generating Function for Vector Processes*

Definition of Autocovariance-Generating Function for Vector Processes

Recall that for a covariance-stationary univariate process y_t with absolutely summable autocovariances, the (scalar-valued) autocovariance-generating function $g_Y(z)$ is defined as

$$g_Y(z) \equiv \sum_{j=-\infty}^{\infty} \gamma_j z^j$$

with

$$\gamma_j \equiv E[(y_t - \mu)(y_{t-j} - \mu)]$$

and z a complex scalar. For a covariance-stationary vector process $\mathbf{y}_t$ with an absolutely summable sequence of autocovariance matrices, the analogous matrix-valued autocovariance-generating function $\mathbf{G}_\mathbf{Y}(z)$ is defined as

$$\mathbf{G}_\mathbf{Y}(z) \equiv \sum_{j=-\infty}^{\infty} \Gamma_j z^j, \qquad [10.3.1]$$

where

$$\Gamma_j \equiv E[(\mathbf{y}_t - \boldsymbol{\mu})(\mathbf{y}_{t-j} - \boldsymbol{\mu})']$$

and z is again a complex scalar.

Autocovariance-Generating Function for a Vector Moving Average Process

For example, for the vector white noise process $\boldsymbol{\varepsilon}_t$ characterized by [10.1.5] and [10.1.6], the autocovariance-generating function is

$$\mathbf{G}_{\boldsymbol{\varepsilon}}(z) = \boldsymbol{\Omega}. \tag{10.3.2}$$

For the vector $MA(q)$ process of [10.2.3], the univariate expression [3.6.3] for the autocovariance-generating function generalizes to

$$\begin{aligned}\mathbf{G}_{\mathbf{Y}}(z) = {} & (\mathbf{I}_n + \boldsymbol{\Theta}_1 z + \boldsymbol{\Theta}_2 z^2 + \cdots + \boldsymbol{\Theta}_q z^q)\boldsymbol{\Omega} \\ & \times (\mathbf{I}_n + \boldsymbol{\Theta}_1' z^{-1} + \boldsymbol{\Theta}_2' z^{-2} + \cdots + \boldsymbol{\Theta}_q' z^{-q}).\end{aligned} \tag{10.3.3}$$

This can be verified by noting that the coefficient on z^j in [10.3.3] is equal to $\boldsymbol{\Gamma}_j$ as given in [10.2.5].

For an $MA(\infty)$ process of the form

$$\mathbf{y}_t = \boldsymbol{\mu} + \boldsymbol{\Psi}_0\boldsymbol{\varepsilon}_t + \boldsymbol{\Psi}_1\boldsymbol{\varepsilon}_{t-1} + \boldsymbol{\Psi}_2\boldsymbol{\varepsilon}_{t-2} + \cdots = \boldsymbol{\mu} + \boldsymbol{\Psi}(L)\boldsymbol{\varepsilon}_t,$$

with $\{\boldsymbol{\Psi}_k\}_{k=0}^{\infty}$ absolutely summable, [10.3.3] generalizes to

$$\mathbf{G}_{\mathbf{Y}}(z) = [\boldsymbol{\Psi}(z)]\boldsymbol{\Omega}[\boldsymbol{\Psi}(z^{-1})]'. \tag{10.3.4}$$

Autocovariance-Generating Function for a Vector Autoregression

Consider the $VAR(1)$ process $\boldsymbol{\xi}_t = \mathbf{F}\boldsymbol{\xi}_{t-1} + \mathbf{v}_t$ with eigenvalues of $\mathbf{F}$ inside the unit circle and with $\boldsymbol{\xi}_t$ an $(r \times 1)$ vector and $E(\mathbf{v}_t\mathbf{v}_t') = \mathbf{Q}$. Equation [10.3.4] implies that the autocovariance-generating function can be expressed as

$$\begin{aligned}\mathbf{G}_{\boldsymbol{\xi}}(z) &= [\mathbf{I}_r - \mathbf{F}z]^{-1}\mathbf{Q}[\mathbf{I}_r - \mathbf{F}'z^{-1}]^{-1} \\ &= [\mathbf{I}_r + \mathbf{F}z + \mathbf{F}^2z^2 + \mathbf{F}^3z^3 + \cdots]\mathbf{Q} \\ &\quad \times [\mathbf{I}_r + (\mathbf{F}')z^{-1} + (\mathbf{F}')^2z^{-2} + (\mathbf{F}')^3z^{-3} + \cdots].\end{aligned} \tag{10.3.5}$$

Transformations of Vector Processes

The autocovariance-generating function of the sum of two univariate processes that are uncorrelated with each other is equal to the sum of their individual autocovariance-generating functions (equation [4.7.19]). This result readily generalizes to the vector case:

$$\begin{aligned}\mathbf{G}_{\mathbf{X}+\mathbf{W}}(z) &= \sum_{j=-\infty}^{\infty} E[(\mathbf{x}_t + \mathbf{w}_t - \boldsymbol{\mu}_{\mathbf{X}} - \boldsymbol{\mu}_{\mathbf{W}}) \\ &\qquad \times (\mathbf{x}_{t-j} + \mathbf{w}_{t-j} - \boldsymbol{\mu}_{\mathbf{X}} - \boldsymbol{\mu}_{\mathbf{W}})']z^j \\ &= \sum_{j=-\infty}^{\infty} E[(\mathbf{x}_t - \boldsymbol{\mu}_{\mathbf{X}})(\mathbf{x}_{t-j} - \boldsymbol{\mu}_{\mathbf{X}})'z^j] \\ &\qquad + \sum_{j=-\infty}^{\infty} E[(\mathbf{w}_t - \boldsymbol{\mu}_{\mathbf{W}})(\mathbf{w}_{t-j} - \boldsymbol{\mu}_{\mathbf{W}})'z^j] \\ &= \mathbf{G}_{\mathbf{X}}(z) + \mathbf{G}_{\mathbf{W}}(z).\end{aligned}$$