Another method to increase the power of the test would be to increase the number of observations. With $T = 1000$, for instance, we would choose a cutoff of 14 exceptions, for a type 1 error rate of 13.4 percent and a type 2 error rate of 0.03 percent, which is now very small. Increasing the number of observations drastically improves the test.

6.2.3 Conditional Coverage Models

So far the framework focuses on *unconditional coverage* because it ignores conditioning, or time variation in the data. The observed exceptions, however, could cluster or "bunch" closely in time, which also should invalidate the model.

With a 95 percent VAR confidence level, we would expect to have about 13 exceptions every year. In theory, these occurrences should be evenly spread over time. If, instead, we observed that 10 of these exceptions occurred over the last 2 weeks, this should raise a red flag. The market, for instance, could experience increased volatility that is not captured by VAR. Or traders could have moved into unusual positions or risk "holes." Whatever the explanation, a verification system should be designed to measure proper *conditional coverage*, that is, conditional on current conditions. Management then can take the appropriate action.

Such a test has been developed by Christoffersen (1998), who extends the LR_{uc} statistic to specify that the deviations must be serially independent. The test is set up as follows: Each day we set a deviation indicator to 0 if VAR is not exceeded and to 1 otherwise. We then define T_{ij} as the number of days in which state j occurred in one day while it was at i the previous day and π_i as the probability of observing an exception conditional on state i the previous day. Table 6-5 shows how to construct a table of conditional exceptions.

If today's occurrence of an exception is independent of what happened the previous day, the entries in the second and third columns should be identical. The relevant test statistic is

$$\begin{aligned} LR_{\text{ind}} = & -2 \ln [(1 - \pi)^{(T_{00} + T_{10})} \pi^{(T_{01} + T_{11})}] + \\ & 2 \ln [(1 - \pi_0)^{T_{00}} \pi_0^{T_{01}} (1 - \pi_1)^{T_{10}} \pi_1^{T_{11}}] \end{aligned} \tag{6.4}$$

Here, the first term represents the maximized likelihood under the hypothesis that exceptions are independent across days, or $\pi = \pi_0 = \pi_1 = (T_{01} + T_{11})/T$. The second term is the maximized likelihood for the observed data.

TABLE 6-5

Building an Exception Table: Expected Number of Exceptions

	Conditional Day Before		
	No Exception	Exception	Unconditional
Current day			
No exception	$T_{00} = T_0\,(1 - \pi_0)$	$T_{10} = T_1\,(1 - \pi_1)$	$T(1 - \pi)$
Exception	$T_{01} = T_0\,(\pi_0)$	$T_{11} = T_1\,(\pi_1)$	$T(\pi)$
Total	T_0	T_1	$T = T_0 + T_1$

The combined test statistic for conditional coverage then is

$$\mathrm{LR}_{cc} = \mathrm{LR}_{uc} + \mathrm{LR}_{\mathrm{ind}} \tag{6.5}$$

Each component is independently distributed as $\chi^2(1)$ asymptotically. The sum is distributed as $\chi^2(2)$. Thus we would reject at the 95 percent test confidence level if LR > 5.991. We would reject independence alone if $\mathrm{LR}_{\mathrm{ind}} > 3.841$.

As an example, assume that JPM observed the following pattern of exceptions during 1998. Of 252 days, we have 20 exceptions, which is a fraction of $\pi = 7.9$ percent. Of these, 6 exceptions occurred following an exception the previous day. Alternatively, 14 exceptions occurred when there was none the previous day. This defines conditional probability ratios of $\pi_0 = 14/232 = 6.0$ percent and $\pi_1 = 6/20 = 30.0$ percent. We seem to have a much higher probability of having an exception following another one. Setting these numbers into Equation (6.4), we find $\mathrm{LR}_{\mathrm{ind}} = 9.53$. Because this is higher than the cutoff value of 3.84, we reject independence. Exceptions do seem to cluster abnormally. As a result, the risk manager may want to explore models that allow for time variation in risk, as developed in Chapter 9.

6.2.4 Extensions

We have seen that the standard exception tests often lack power, especially when the VAR confidence level is high and when the number of observations is low. This has led to a search for improved tests.

	Conditional		
	Day Before		
	No Exception	Exception	Unconditional
Current day			
No exception	218	14	232
Exception	14	6	20
Total	232	20	252

The problem, however, is that statistical decision theory has shown that this exception test is the most powerful among its class. More effective tests would have to focus on a different hypothesis or use more information.

For example, Crnkovic and Drachman (1996) developed a test focusing on the entire probability distribution, based on the *Kuiper statistic*. This test is still nonparametric but is more powerful. However, it uses other information than the VAR forecast at a given confidence level. Another approach is to focus on the time period between exceptions, called *duration*. Christoffersen and Pelletier (2004) show that duration-based tests can be more powerful than the standard test when risk is time-varying.

Finally, backtests could use parametric information instead. If the VAR is obtained from a multiple of the standard deviation, the risk manager could test the fit between the realized and forecast volatility. This would lead to more powerful tests because more information is used. Another useful avenue would be to backtest the portfolio components as well. From the viewpoint of the regulator, however, the only information provided is the daily VAR, which explains why exception tests are used most commonly nowadays.

6.3 APPLICATIONS

Berkowitz and O'Brien (2002) provide the first empirical study of the accuracy of internal VAR models, using data reported to U.S. regulators. They describe the distributions of P&L, which are compared with the VAR forecasts. Generally, the P&L distributions are symmetric, although they display fatter tails than the normal. Stahl et al. (2006) also report that, although the components of a trading portfolio could be strongly nonnormal, aggregation to the highest level of a bank typically produces symmetric distributions that resemble the normal.

FIGURE 6-4

Bank VAR and trading profits.

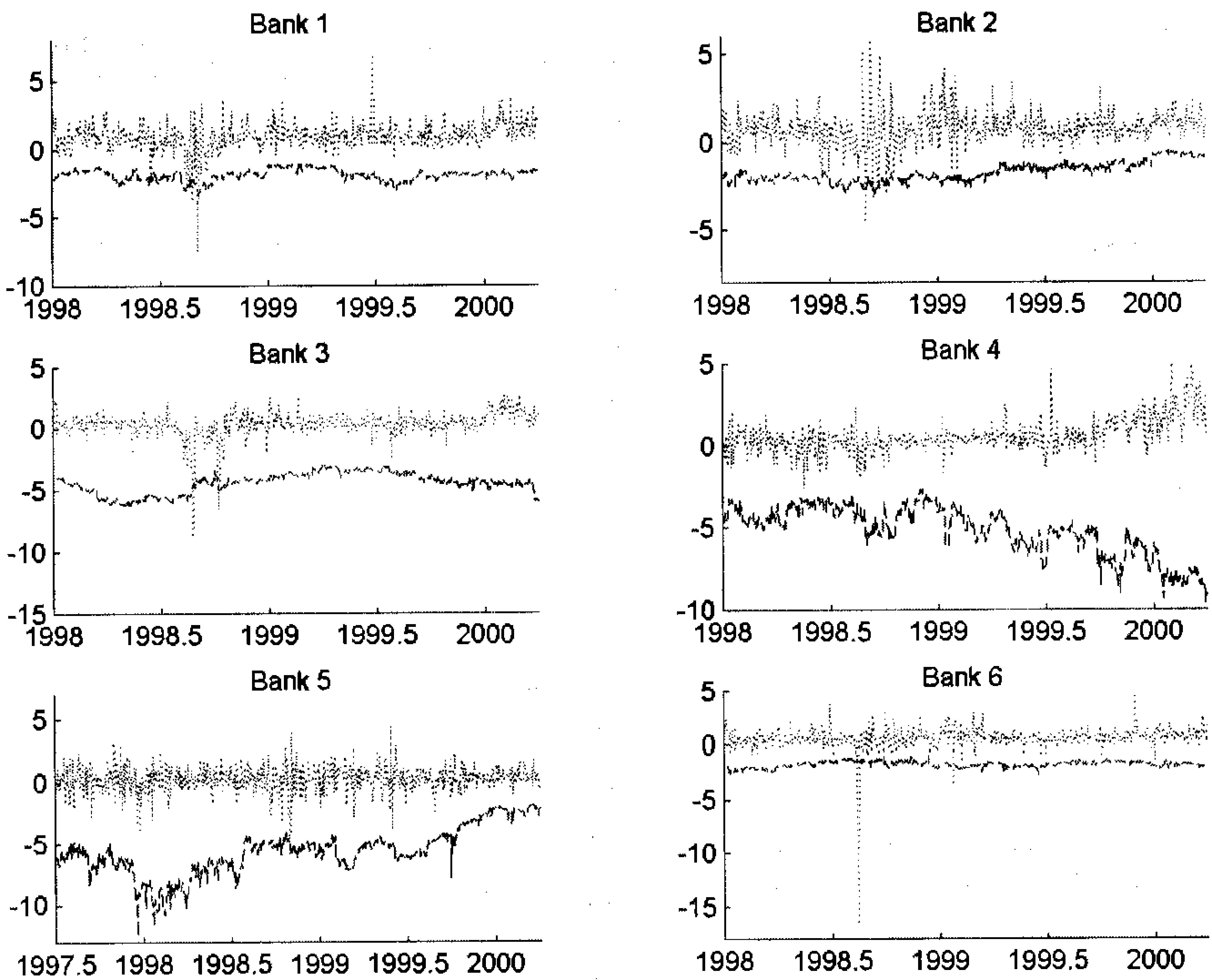


Figure 6-4 plots the time series of P&L along with the daily VAR (the lower lines) for a sample of six U.S. commercial banks. With approximately 600 observations, we should observe on average 6 violations, given a VAR confidence level of 99 percent.

It is striking to see the abnormally small number of exceptions, even though the sample includes the turbulent 1998 period. Bank 4, for example, has zero exceptions over this sample. Its VAR is several times greater than the magnitude of extreme fluctuations in its P&L. Indeed, for banks 3 to 6, the average VAR is at least 60 percent higher than the actual 99th percentile of the P&L distribution. Thus banks report VAR measures that are *conservative,* or too large relative to their actual risks. These results are surprising because they imply that the banks' VAR and hence their market-risk charges are too high. Banks therefore allocate too much regulatory capital to their trading activities. Box 6-2 describes a potential explanation, which is simplistic.

BOX 6-2

NO EXCEPTIONS

The CEO of a large bank receives a daily report of the bank's VAR and P&L. Whenever there is an exception, the CEO calls in the risk officer for an explanation.

Initially, the risk officer explained that a 99 percent VAR confidence level implies an average of 2 to 3 exceptions per year. The CEO is never quite satisfied, however. Later, tired of going "upstairs," the risk officer simply increases the confidence level to cut down on the number of exceptions.

Annual reports suggest that this is frequently the case. Financial institutions routinely produce plots of P&L that show no violation of their 99 percent confidence VAR over long periods, proclaiming that this supports their risk model.

Perhaps these observations could be explained by the use of actual instead of hypothetical returns.[4] Or maybe the models are too simple, for example failing to account for diversification effects. Yet another explanation is that capital requirements are currently not binding. The amount of economic capital U.S. banks currently hold is in excess of their regulatory capital. As a result, banks may prefer to report high VAR numbers to avoid the possibility of regulatory intrusion. Still, these practices impoverish the informational content of VAR numbers.

6.4 CONCLUSIONS

Model verification is an integral component of the risk management process. Backtesting VAR numbers provides valuable feedback to users about the accuracy of their models. The procedure also can be used to search for possible improvements.

Due thought should be given to the choice of VAR quantitative parameters for backtesting purposes. First, the horizon should be as short as possible in order to increase the number of observations and to mitigate

[4] Including fees increases the P&L, reducing the number of violations. Using hypothetical income, as currently prescribed in the European Union, could reduce this effect. Jaschke, Stahl, and Stehle (2003) compare the VARs for 13 German banks and find that VAR measures are, on average, less conservative than for U.S. banks. Even so, VAR forecasts are still too high.

the effect of changes in the portfolio composition. Second, the confidence level should not be too high because this decreases the effectiveness, or power, of the statistical tests.

Verification tests usually are based on "exception" counts, defined as the number of exceedences of the VAR measure. The goal is to check if this count is in line with the selected VAR confidence level. The method also can be modified to pick up bunching of deviations.

Backtesting involves balancing two types of errors: rejecting a correct model versus accepting an incorrect model. Ideally, one would want a framework that has very high power, or high probability of rejecting an incorrect model. The problem is that the power of exception-based tests is low. The current framework could be improved by choosing a lower VAR confidence level or by increasing the number of data observations.

Adding to these statistical difficulties, we have to recognize other practical problems. Trading portfolios do change over the horizon. Models do evolve over time as risk managers improve their risk modeling techniques. All this may cause further structural instability.

Despite all these issues, backtesting has become a central component of risk management systems. The methodology allows risk managers to improve their models constantly. Perhaps most important, backtesting should ensure that risk models do not go astray.

QUESTIONS

1. Define backtesting and exceptions.
2. Assume that a bank's backtests fail using the actual P&L return but not using the hypothetical return. Should the risk manager reexamine the risk model?
3. How is *type 1 error* different from *type 2 error* for a decision rule? Explain the meaning of these errors for backtesting the trading book of a bank. Can both errors be avoided?
4. For a fixed type 1 error rate, how can a test minimize the probability of a type 2 error?
5. Say that a bank reports 9 exceptions to its 99 percent daily VAR over the last year (252 days). Give two interpretations of this observation.
6. A bank reports 9 exceptions to its 99 percent VAR over the last year (252 days). Using the normal approximation to the binomial distribution, compute the z-statistic, and discuss whether the results would justify rejecting the model.

7. Backtesting is usually conducted on a short horizon, such as daily returns. Explain why.
8. A commercial bank subject to the Basel market-risk charge reports 4 exceptions over the last year. What is the multiplier k? Repeat with 10 exceptions.
9. Why is it useful to consider not only unconditional coverage but also conditional coverage?
10. A bank reports 6 exceptions to its 99 percent VAR over the last year (252 days), including 4 that follow another day of exception. Compute the likelihood-ratio tests, and discuss whether unconditional and conditional coverage is rejected.
11. The Berkowitz and O'Brien study indicates that bank are *conservative*, that is, generate VAR forecasts that are too large in relation to actual risks. What could explain this observation?

CHAPTER 7

Portfolio Risk: Analytical Methods

Trust not all your goods to one ship.

—*Erasmus*

The preceding chapters have focused on single financial instruments. Absent any insight into the future, prudent investors should diversify across sources of financial risk. This was the message of portfolio analysis laid out by Harry Markowitz in 1952. Thus the concept of value at risk (VAR), or portfolio risk, is not new. What is new is the systematic application of VAR to many sources of financial risk, or portfolio risk. VAR explicitly accounts for leverage and portfolio diversification and provides a simple, single measure of risk based on current positions.

As will be seen in Chapter 10, there are many approaches to measuring VAR. The shortest road assumes that asset payoffs are linear (or delta) functions of normally distributed risk factors. Indeed, the *delta-normal method* is a direct application of traditional portfolio analysis based on variances and covariances, which is why it is sometimes called the *covariance matrix approach.*

This approach is *analytical* because VAR is derived from closed-form solutions. The analytical method developed in this chapter is very useful because it creates a more intuitive understanding of the drivers of risk within a portfolio. It also lends itself to a simple decomposition of the portfolio VAR.

This chapter shows how to measure and manage portfolio VAR. Section 7.1 details the construction of VAR using information on positions and the covariance matrix of its constituent components.

The fact that portfolio risk is not cumulative provides great diversification benefits. To manage risk, however, we also need to understand what will reduce it. Section 7.2 provides a detailed analysis of VAR tools that are essential to control portfolio risk. These include marginal VAR, incremental VAR, and component VAR. These VAR tools allow users to identify the asset that contributes most to their total risk, to pick the best hedge, to rank trades, or in general, to select the asset that provides the best risk-return tradeoff. Section 7.3 presents a fully worked out example of VAR computations for a global equity portfolio and for Barings' fatal positions.

The advantage of analytical models is that they provide closed-form solutions that help our intuition. The methods presented here, however, are quite general. Section 7.4 shows how to build these VAR tools in a nonparametric environment. This applies to simulations, for example.

Finally, Section 7.5 takes us toward portfolio optimization, which should be the ultimate purpose of VAR. We first show how the passive measurement of risk can be extended to the management of risk, in particular, risk minimization. We then integrate risk with expected returns and show how VAR tools can be use to move the portfolio toward the best combination of risk and return.

7.1 PORTFOLIO VAR

A portfolio can be characterized by positions on a certain number of constituent assets, expressed in the base currency, say, dollars. If the positions are fixed over the selected horizon, the portfolio rate of return is a *linear* combination of the returns on underlying assets, where the weights are given by the relative amounts invested at the beginning of the period. Therefore, the VAR of a portfolio can be constructed from a combination of the risks of underlying securities.

Define the portfolio rate of return from t to $t + 1$ as

$$R_{p,t+1} = \sum_{i=1}^{N} w_i R_{i,t+1} \tag{7.1}$$

where N is the number of assets, $R_{i,t+1}$ is the rate of return on asset i, and w_i is the weight. The *rate of return* is defined as the change in the dollar value, or dollar return, scaled by the initial investment. This is a unitless measure.

Weights are constructed to sum to unity by scaling the dollar positions in each asset W_i by the portfolio total market value W. This immediately rules out portfolios that have zero net investment $W = 0$, such as some derivatives positions. But we could have positive and negative weights w_i, including values much larger than 1, as with a highly leveraged hedge fund. If the net portfolio value is zero, we could use another measure, such as the sum of the gross positions or absolute value of all dollar positions W^*. All weights then would be defined in relation to this benchmark. Alternatively, we could express returns in dollar terms, defining a dollar amount invested in asset i as $W_i = w_i W$. We will be using x as representing the vector of dollar amount invested in each asset so as to avoid confusion with the total dollar amount W.

It is important to note that in traditional mean-variance analysis, each constituent asset is a security. In contrast, VAR defines the component as a *risk factor* and w_i as the linear exposure to this risk factor. We shall see in Chapter 11 how to choose the risk factors and how to map securities into exposures on these risk factors. Whether dealing with assets or risk factors, the mathematics of portfolio VAR are equivalent, however.

To shorten notation, the portfolio return can be written using *matrix notation*, replacing a string of numbers by a single vector:

$$R_p = w_1 R_1 + w_2 R_2 + \cdots + w_N R_N = [w_1 w_2 \cdots w_N] \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_N \end{bmatrix} = w'R \tag{7.2}$$

where w' represents the transposed vector (i.e., horizontal) of weights, and R is the vertical vector containing individual asset returns. Appendix 7.A explains the rules for matrix multiplication.

By extension of the formulas in Chapter 4, the portfolio expected return is

$$E(R_p) = \mu_p = \sum_{i=1}^{N} w_i \mu_i \tag{7.3}$$

and the variance is

$$V(R_p) = \sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} w_i w_j \sigma_{ij} = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2 \sum_{i=1}^{N} \sum_{j<i}^{N} w_i w_j \sigma_{ij} \tag{7.4}$$

This sum accounts not only for the risk of the individual securities σ_i^2 but also for all covariances, which add up to a total of $N(N-1)/2$ different terms.

As the number of assets increases, it becomes difficult to keep track of all covariance terms, which is why it is more convenient to use matrix notation. The variance can be written as

$$\sigma_p^2 = [w_1 \cdots w_N] \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \dots & \sigma_{1N} \\ \vdots & & & & \\ \sigma_{N1} & \sigma_{N2} & \sigma_{N3} & \dots & \sigma_N^2 \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix}$$

Defining Σ as the covariance matrix, the variance of the portfolio rate of return can be written more compactly as

$$\sigma_p^2 = w' \Sigma w \tag{7.5}$$

where w are weights, which have no units. This also can be written in terms of dollar exposures x as

$$\sigma_p^2 W^2 = x' \Sigma x \tag{7.6}$$

So far nothing has been said about the distribution of the portfolio return. Ultimately, we would like to translate the portfolio variance into a VAR measure. To do so, we need to know the distribution of the portfolio return. In the delta-normal model, all individual security returns are assumed normally distributed. This is particularly convenient because the portfolio return, a linear combination of jointly normal random variables, is also normally distributed. If so, we can translate the confidence level c into a standard normal deviate α such that the probability of observing a loss worse than $-\alpha$ is c. Defining W as the initial portfolio value, the portfolio VAR is

$$\text{Portfolio VAR} = \text{VAR}_p = \alpha \sigma_p W = \alpha \sqrt{x' \Sigma x} \tag{7.7}$$

Diversified VAR The portfolio VAR, taking into account diversification benefits between components.

At this point, we also can define the individual risk of each component as

$$\text{VAR}_i = \alpha \sigma_i \mid W_i \mid = \alpha \sigma_i \mid w_i \mid W \tag{7.8}$$

Note that we took the absolute value of the weight w_i because it can be negative, whereas the risk measure must be positive.

Individual VAR The VAR of one component taken in isolation.

Equation (7.4) shows that the portfolio VAR depends on variances, covariances, and the number of assets. Covariance is a measure of the extent to which two variables move linearly together. If two variables are independent, their covariance is equal to zero. A positive covariance means that the two variables tend to move in the same direction; a negative covariance means that they tend to move in opposite directions. The magnitude of covariance, however, depends on the variances of the individual components and is not easily interpreted. The *correlation coefficient* is a more convenient, scale-free measure of linear dependence:

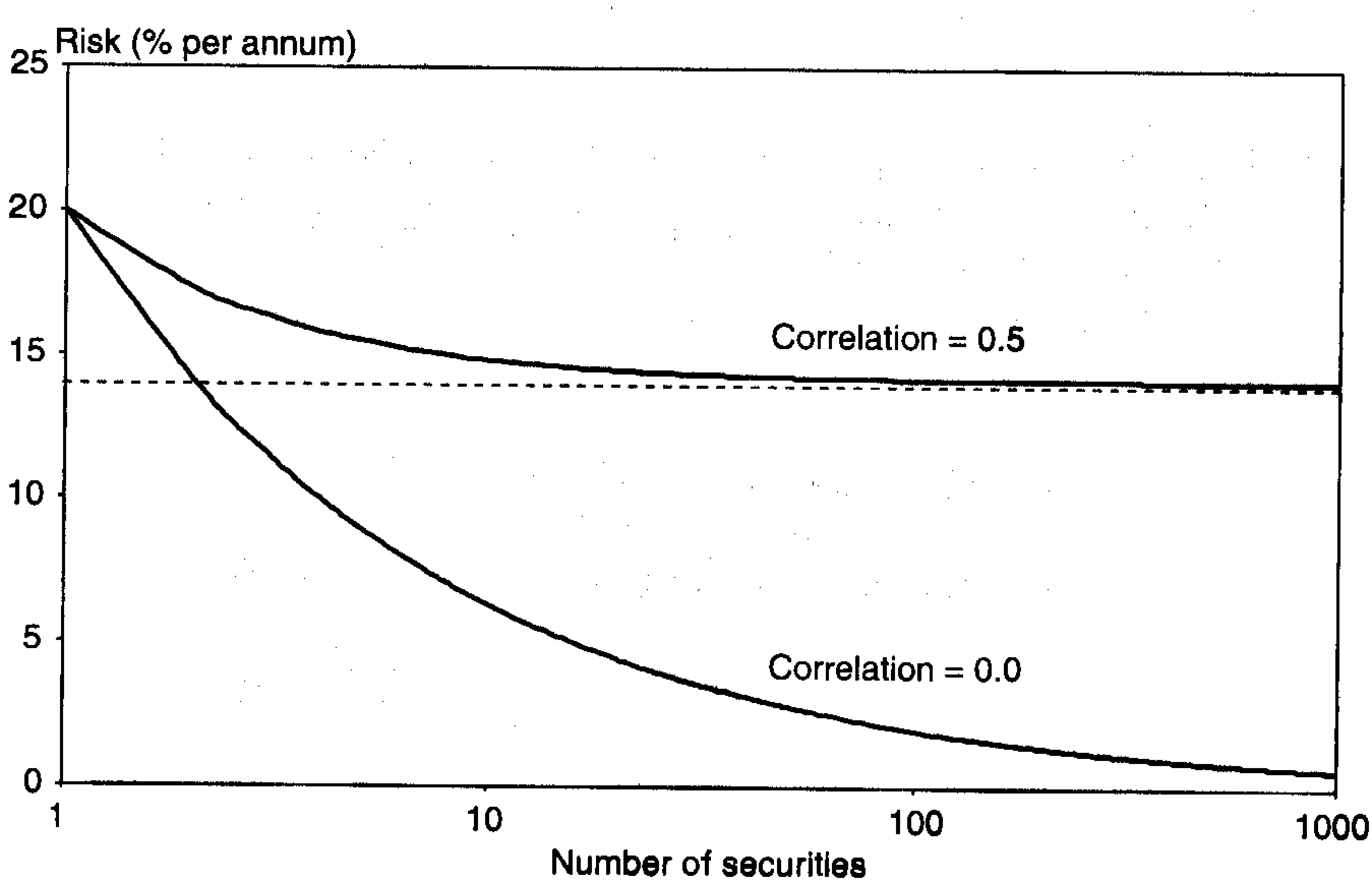
$$\rho_{12} = \sigma_{12}/(\sigma_1\sigma_2) \tag{7.9}$$

The correlation coefficient ρ always lies between -1 and $+1$. When equal to unity, the two variables are said to be *perfectly correlated.* When 0, the variables are *uncorrelated.*

Lower portfolio risk can be achieved through low correlations or a large number of assets. To see the effect of N, assume that all assets have the same risk and that all correlations are the same, that equal weight is put on each asset. Figure 7-1 shows how portfolio risk decreases with the number of assets.

FIGURE 7-1

Risk and number of securities.



Start with the risk of one security, which is assumed to be 20 percent. When ρ is equal to zero, the risk of a 10-asset portfolio drops to 6.3 percent; increasing N to 100 drops the risk even further to 2.0 percent. Risk tends asymptotically to zero. More generally, portfolio risk is

$$\sigma_p = \sigma\sqrt{\frac{1}{N}+\left(1-\frac{1}{N}\right)\rho} \tag{7.10}$$

which tends to $\sigma\sqrt{\rho}$ as N increases. Thus, when $\rho = 0.5$, risk decreases rapidly from 20 to 14.8 percent as N goes to 10 and afterward converges more slowly toward its minimum value of 14.1 percent.

Low correlations thus help to diversify portfolio risk. Take a simple example with two assets only. The "diversified" portfolio variance is

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2 \tag{7.11}$$

The portfolio VAR is then

$$\text{VAR}_p = \alpha\sigma_p W = \alpha\sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2}\,W \tag{7.12}$$

This can be related to the individual VAR as defined in Equation (7.8).

When the correlation ρ is zero, the portfolio VAR reduces to

$$\text{VAR}_p = \sqrt{\alpha^2w_1^2W^2\sigma_1^2 + \alpha^2w_2^2W^2\sigma_2^2} = \sqrt{\text{VAR}_1^2 + \text{VAR}_2^2} \tag{7.13}$$

The portfolio risk must be lower than the sum of the individual VARs: $\text{VAR}_p < \text{VAR}_1 + \text{VAR}_2$. This reflects the fact that with assets that move independently, a portfolio will be less risky than either asset. Thus VAR is a *coherent* risk measure for normal and, more generally, elliptical distributions (See section 5.1.4).

When the correlation is exactly unity and w_1 and w_2 are both positive, Equation (7.12) reduces to

$$\text{VAR}_p = \sqrt{\text{VAR}_1^2 + \text{VAR}_2^2 + 2\text{VAR}_1\times\text{VAR}_2} = \text{VAR}_1 + \text{VAR}_2 \tag{7.14}$$

In other words, the portfolio VAR is equal to the sum of the individual VAR measures if the two assets are perfectly correlated. In general, though, this will not be the case because correlations typically are imperfect. The benefit from diversification can be measured by the difference between the *diversified* VAR and the *undiversified* VAR, which typically is shown in VAR reporting systems.

Undiversified VAR The sum of individual VARs, or the portfolio VAR when there is no short position and all correlations are unity.

This interpretation differs when short sales are allowed. Suppose that the portfolio is long asset 1 but short asset 2 (w_1 is positive, and w_2 is negative). This could represent a hedge fund that has \$1 in capital and a \$1 billion long position in corporate bonds and a \$1 billion short position in Treasury bonds, the rationale for the position being that corporate yields are slightly higher than Treasury yields. If the correlation is exactly unity, the fund has no risk because any loss in one asset will be offset by a matching gain in the other. The portfolio VAR then is zero.

Instead, the risk will be greatest if the correlation is -1, in which case losses in one asset will be amplified by the other. Here, the *undiversified* VAR can be interpreted as the portfolio VAR when the correlation attains its worst value, which is -1. Therefore, the undiversified VAR provides an upper bound on the portfolio VAR should correlations prove unstable and all move at the same time in the wrong direction. It provides an absolute worst-case scenario for the portfolio at hand.

Example

Consider a portfolio with two foreign currencies, the Canadian dollar (CAD) and the euro (EUR). Assume that these two currencies are uncorrelated and have a volatility against the dollar of 5 and 12 percent, respectively. The first step is to mark to market the positions in the base currency. The portfolio has US\$2 million invested in the CAD and US\$1 million in the EUR. We seek to find the portfolio VAR at the 95 percent confidence level.

First, we will compute the variance of the portfolio dollar return. Define x as the dollar amounts allocated to each risk factor, in millions. Compute the product

$$\Sigma x = \begin{bmatrix} 0.05^2 & 0 \\ 0 & 0.12^2 \end{bmatrix} \begin{bmatrix} \$2 \\ \$1 \end{bmatrix} = \begin{bmatrix} 0.05^2 \times \$2 + 0 \times \$1 \\ 0 \times \$2 + 0.12^2 \times \$1 \end{bmatrix} = \begin{bmatrix} \$0.0050 \\ \$0.0144 \end{bmatrix}$$

The portfolio variance then is (in dollar units)

$$\sigma_p^2 W^2 = x'(\Sigma x) = [\$2\ \$1] \begin{bmatrix} \$0.0050 \\ \$0.0144 \end{bmatrix} = 0.0100 + 0.0144 = 0.0244$$

The dollar volatility is $\sqrt{0.0244}$ = \$0.156205 million. Using α = 1.65, we find VAR_p = 1.65 × 156,205 = \$257,738.

Next, the individual (undiversified) VAR is found simply as $\text{VAR}_i = \alpha\sigma_i x_i$, that is,

$$\begin{bmatrix}\text{VAR}_1\\ \text{VAR}_2\end{bmatrix} = \begin{bmatrix}1.65\times 0.05\times \$2 \text{ million}\\ 1.65\times 0.12\times \$1 \text{ million}\end{bmatrix} = \begin{bmatrix}\$165{,}000\\ \$198{,}000\end{bmatrix}$$

Note that these numbers sum to an undiversified VAR of \$363,000, which is greater than the portfolio VAR of \$257,738 owing to diversification effects.

7.2 VAR TOOLS

Initially, VAR was developed as a methodology to measure portfolio risk. There is much more to VAR than simply reporting a single number, however. Over time, risk managers have discovered that they could use the VAR process for active risk management. A typical question may be, "Which position should I alter to modify my VAR most effectively?" Such information is quite useful because portfolios typically are traded incrementally owing to transaction costs. This is the purpose of VAR tools, which include marginal, incremental, and component VAR.

7.2.1 Marginal VAR

To measure the effect of changing positions on portfolio risk, individual VARs are not sufficient. Volatility measures the uncertainty in the return of an asset, taken in isolation. When this asset belongs to a portfolio, however, what matters is the contribution to portfolio risk.

We start from the existing portfolio, which is made up of N securities, numbered as $j = 1, \ldots, N$. A new portfolio is obtained by adding one unit of security i. To assess the impact of this trade, we measure its "marginal" contribution to risk by increasing w by a small amount or differentiating Equation (7.4) with respect to w_i, that is,

$$\frac{\partial \sigma_p^2}{\partial w_i} = 2w_i\sigma_i^2 + 2\sum_{j=1,j\neq i}^{N} w_j\sigma_{ij} = 2\operatorname{cov}(R_i, w_iR_i + \sum_{j\neq i}^{N} w_jR_j) = 2\operatorname{cov}(R_i, R_p) \quad (7.15)$$

Instead of the derivative of the variance, we need that of the volatility. Noting that $\partial\sigma_p^2/\partial w_i = 2\sigma_p\partial\sigma_p/\partial w_i$, the sensitivity of the portfolio volatility to a change in the weight is then

$$\frac{\partial \sigma_p}{\partial w_i} = \frac{\mathrm{cov}(R_i, R_p)}{\sigma_p} \tag{7.16}$$

Converting into a VAR number, we find an expression for the *marginal VAR*, which is a vector with component

$$\Delta \mathrm{VAR}_i = \frac{\partial \mathrm{VAR}}{\partial x_i} = \frac{\partial \mathrm{VAR}}{\partial w_i W} = \alpha \frac{\partial \sigma_p}{\partial w_i} = \alpha \frac{\mathrm{cov}(R_i, R_p)}{\sigma_p} \tag{7.17}$$

Since this was defined as a ratio of the dollar amounts, this marginal VAR measure is unitless.

> **Marginal VAR** The change in portfolio VAR resulting from taking an additional dollar of exposure to a given component. It is also the partial (or linear) derivative with respect to the component position.

This marginal VAR is closely related to the *beta*, defined as

$$\beta_i = \frac{\mathrm{cov}(R_i, R_p)}{\sigma_p^2} = \frac{\sigma_{ip}}{\sigma_p^2} = \frac{\rho_{ip}\sigma_i\sigma_p}{\sigma_p^2} = \rho_{ip}\frac{\sigma_i}{\sigma_p} \tag{7.18}$$

which measures the contribution of one security to total portfolio risk. Beta is also called the *systematic risk* of security i vis-à-vis portfolio p and can be measured from the slope coefficient in a regression of R_i on R_p, that is,

$$R_{i,t} = \alpha_i + \beta_i R_{p,t} + \epsilon_{i,t} \qquad t = 1, \ldots, T \tag{7.19}$$

Using matrix notation, we can write the vector β, including all assets, as

$$\beta = \frac{\Sigma w}{(w'\Sigma w)}$$

Note that we already computed the vector Σw as an intermediate step in the calculation of VAR. Therefore, β and the marginal VAR can be derived easily once VAR has been calculated.

Beta risk is the basis for capital asset pricing model (CAPM) developed by Sharpe (1964). According to the CAPM, well-diversified investors only need to be compensated for the systematic risk of securities relative to the market. In other words, the risk premium on all assets should depend on beta only. Whether this is an appropriate description of capital markets has been the subject of much of finance research in the last decades. Even though this proposition is still debated hotly, the fact remains that systematic risk is a useful statistical measure of marginal portfolio risk.

To summarize, the relationship between the ΔVAR and β is

$$\Delta\text{VAR}_i = \frac{\partial\text{VAR}}{\partial x_i} = \alpha(\beta_i \times \sigma_p) = \frac{\text{VAR}}{W} \times \beta_i \tag{7.20}$$

The marginal VAR can be used for a variety of risk management purposes. Suppose that an investor wants to lower the portfolio VAR and has the choice to reduce all positions by a fixed amount, say, \$100,000. The investor should rank all marginal VAR numbers and pick the asset with the largest ΔVAR because it will have the greatest hedging effect.

7.2.2 Incremental VAR

This methodology can be extended to evaluate the total impact of a proposed trade on portfolio p. The new trade is represented by position a, which is a vector of additional exposures to our risk factors, measured in dollars.

Ideally, we should measure the portfolio VAR at the initial position VAR_p and then again at the new position VAR_{p+a}. The incremental VAR then is obtained, as described in Figure 7-2, as

$$\text{Incremental VAR} = \text{VAR}_{p+a} - \text{VAR}_p \tag{7.21}$$

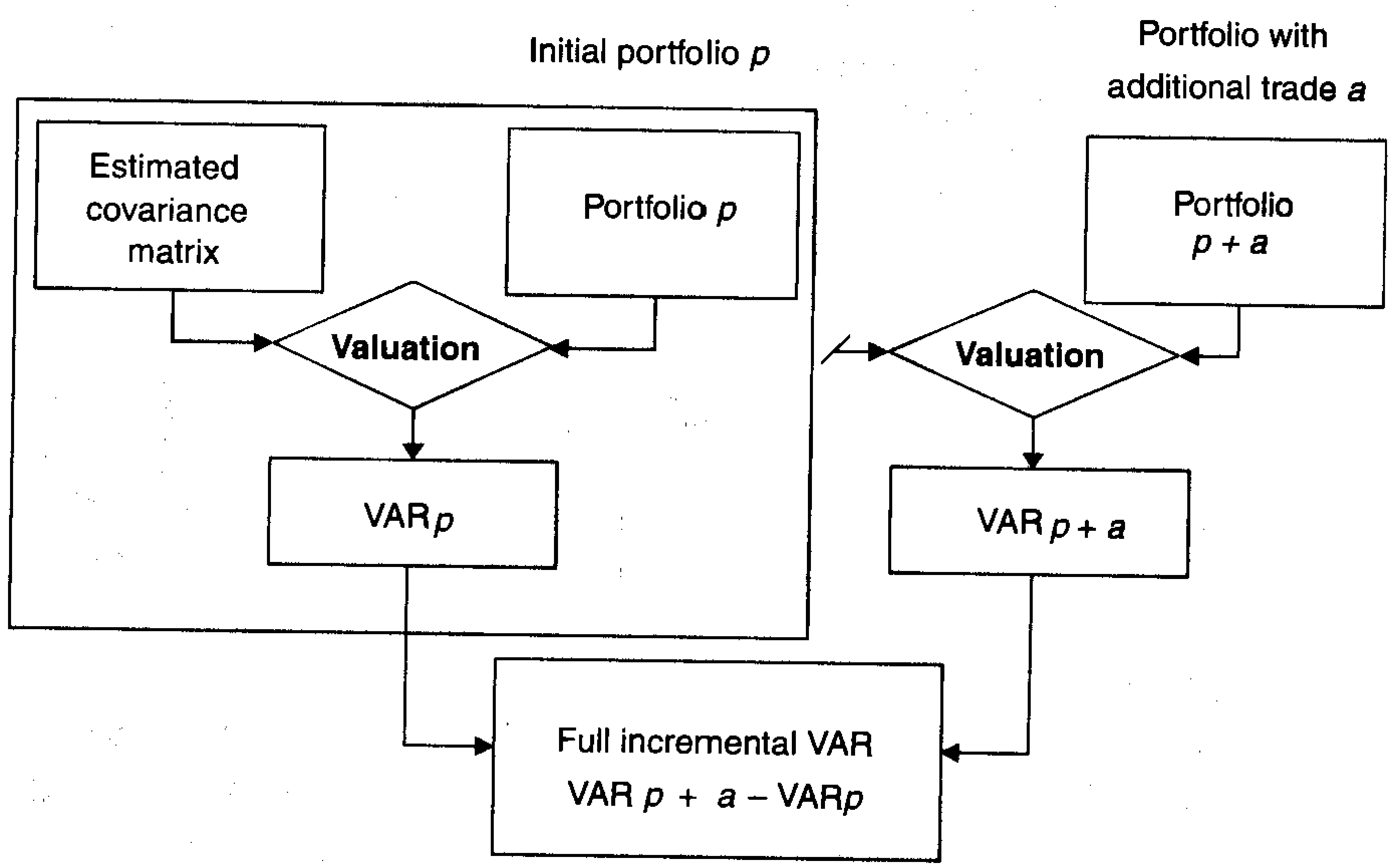
This "before and after" comparison is quite informative. If VAR is decreased, the new trade is risk-reducing or is a hedge; otherwise, the new trade is risk-increasing. Note that a may represent a change in a single component or a more complex trade with changes in multiple components. Hence, in general, a represents a vector of new positions.

Incremental VAR The change in VAR owing to a new position. It differs from the marginal VAR in that the amount added or subtracted can be large, in which case VAR changes in a nonlinear fashion.

The main drawback of this approach is that it requires a full revaluation of the portfolio VAR with the new trade. This can be quite time-consuming for large portfolios. Suppose, for instance, that an institution has 100,000 trades on its books and that it takes 10 minutes to do a VAR calculation. The bank has measured its VAR at some point during the day.

FIGURE 7-2

The impact of a proposed trade with full revaluation.



Then a client comes with a proposed trade. Evaluating the effect of this trade on the bank's portfolio again would require 10 minutes using the incremental-VAR approach. Most likely, this will be too long to wait to take action. If we are willing to accept an approximation, however, we can take a shortcut.[1]

Expanding VAR_{p+a} in series around the original point,

$$\mathrm{VAR}_{p+a} = \mathrm{VAR}_p + (\Delta\mathrm{VAR})' \times a + \cdots \tag{7.22}$$

where we ignored second-order terms if the deviations a are small. Hence the incremental VAR can be reported as, approximately,

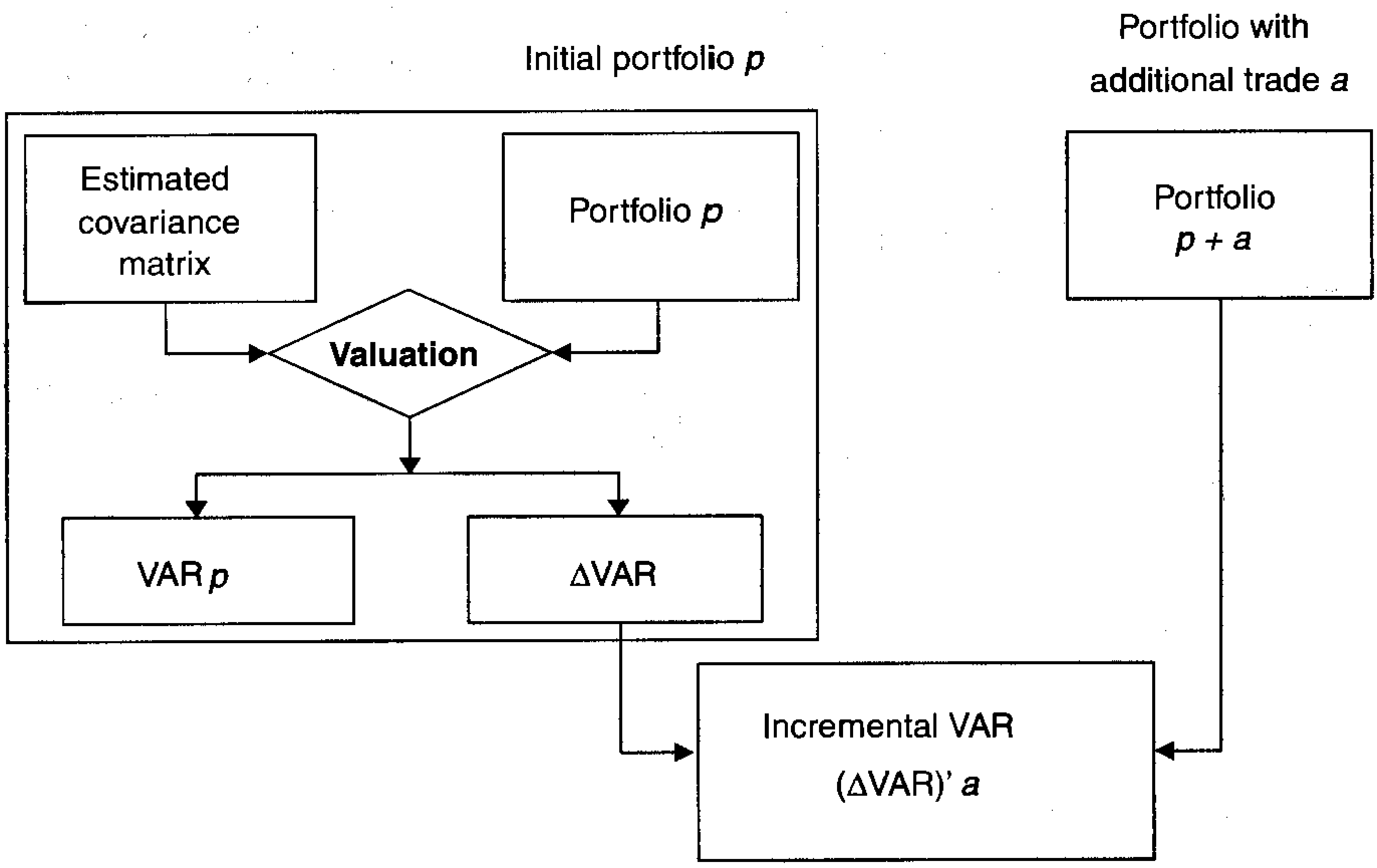
$$\text{Incremental VAR} \approx (\Delta\mathrm{VAR})' \times a \tag{7.23}$$

This measure is much faster to implement because the ΔVAR vector is a by-product of the initial VAR_p computation. The new process is described in Figure 7-3.

[1] See also Garman (1996 and 1997).

FIGURE 7-3

The impact of a proposed trade with marginal VAR.



Here we are trading off faster computation time against accuracy. How much of an improvement is this shortcut relative to the full incremental VAR method? The shortcut will be especially useful for large portfolios where a full revaluation requires a large number of computations. Indeed, the number of operations increases with the square of the number of risk factors. In addition, the shortcut will prove to be a good approximation for large portfolios where a proposed trade is likely to be small relative to the outstanding portfolio. Thus the simplified VAR method allows real-time trading limits.

The incremental VAR method applies to the general case where a trade involves a set of new exposures on the risk factors. Consider instead the particular case where a new trade involves a position in one risk factor only (or asset). The portfolio value changes from the old value of W to the new value of $W_{p+a}=W+a$, where a is the amount invested in asset i. We can write the variance of the dollar returns on the new portfolio as

$$\sigma^2_{p+a} W^2_{p+a} = \sigma^2_p W^2 + 2aW\sigma_{ip} + a^2\sigma^2_i \qquad (7.24)$$