An interesting question for portfolio managers is to find the size of the new trade that leads to the lowest portfolio risk. Differentiating with respect to a,

$$\frac{\partial \sigma_{p+a}^2 W_{p+a}^2}{\partial a} = 2W\sigma_{ip} + 2a\sigma_i^2 \tag{7.25}$$

which attains a zero value for

$$a^* = -W\frac{\sigma_{ip}}{\sigma_i^2} = -W\beta_i\frac{\sigma_p^2}{\sigma_i^2} \tag{7.26}$$

This is the variance-minimizing position, also known as best hedge.

> **Best hedge** Additional amount to invest in an asset so as to minimize the risk of the total portfolio.

Example (continued)

Going back to the previous two-currency example, we are now considering increasing the CAD position by US$10,000.

First, we use the marginal-VAR method. We note that β can be obtained from a previous intermediate step. Because we used dollar amounts, this should be adjusted so that β is unitless, that is,

$$\beta = \frac{\Sigma w}{w'\Sigma w} = W \times \frac{\Sigma x}{x'\Sigma x}$$

We have

$$\beta = \$3 \times \begin{bmatrix} \$0.0050 \\ \$0.0144 \end{bmatrix} / (\$0.156^2) = \$3 \times \begin{bmatrix} 0.205 \\ 0.590 \end{bmatrix} = \begin{bmatrix} 0.615 \\ 1.770 \end{bmatrix}$$

The marginal VAR is now

$$\Delta\text{VAR} = \alpha\frac{\text{cov}(R,R_p)}{\sigma_p} = 1.65 \times \begin{bmatrix} \$0.0050 \\ \$0.0144 \end{bmatrix} / \$0.156 = \begin{bmatrix} 0.0528 \\ 0.1521 \end{bmatrix}$$

As we increase the first position by $10,000, the incremental VAR is

$$(\Delta\text{VAR})' \times a = [0.0528\ 0.1521]\begin{bmatrix} \$10{,}000 \\ 0 \end{bmatrix} = 0.0528 \times \$10{,}000 + 0.1521 \times 0 = \$528$$

Next, we compare this with the incremental VAR obtained from a full revaluation of the portfolio risk. Adding $0.01 million to the first position, we find

$$\sigma_{p+a}^2 W_{p+a}^2 = [\$2.01\ \$1] \begin{bmatrix} 0.05^2 & 0 \\ 0 & 0.12^2 \end{bmatrix} \begin{bmatrix} \$2.01 \\ \$1 \end{bmatrix}$$

which gives $\text{VAR}_{p+a} = \$258{,}267$. Relative to the initial $\text{VAR}_p = \$257{,}738$, the exact increment is \$529. Note how close the ΔVAR approximation of \$528 comes to the true value. The linear approximation is excellent because the change in the position is very small.

7.2.3 Component VAR

In order to manage risk, it would be extremely useful to have a *risk decomposition* of the current portfolio. This is not straightforward because the portfolio volatility is a highly nonlinear function of its components. Taking all individual VARs, adding them up, and computing their percentage, for instance, is not useful because it completely ignores diversification effects. Instead, what we need is an additive decomposition of VAR that recognizes the power of diversification.

This is why we turn to marginal VAR as a tool to help us measure the contribution of each asset to the existing portfolio risk. Multiply the marginal VAR by the current dollar position in asset or risk factor i, that is,

$$\text{Component VAR}_i = (\Delta\text{VAR}_i) \times w_i W = \frac{\text{VAR}\beta_i}{W} \times w_i W = \text{VAR}\beta_i w_i \quad (7.27)$$

Thus the component VAR indicates how the portfolio VAR would change approximately if the component was deleted from the portfolio. We should note, however, that the quality of this linear approximation improves when the VAR components are small. Hence this decomposition is more useful with large portfolios, which tend to have many small positions.

We now show that these component VARs precisely add up to the total portfolio VAR. The sum is

$$\text{CVAR}_1 + \text{CVAR}_2 + \cdots + \text{CVAR}_N = \text{VAR}\left(\sum_{i=1}^{N} w_i\beta_i\right) = \text{VAR} \quad (7.28)$$

because the term between parentheses is simply the beta of the portfolio with itself, which is unity.[2] Thus we established that these *component* VAR measures add up to the total VAR. We have an additive measure of portfolio risk

[2] This can be proved by expanding the portfolio variance into $\sigma_p^2 = w_1 \operatorname{cov}(R_1, R_p) + w_2 \operatorname{cov}(R_2, R_p) + \cdots = w_1(\beta_1\sigma_p^2) + w_2(\beta_2\sigma_p^2) + \cdots = \sigma_p^2(\Sigma_{i=1}^{N} w_i\beta_i)$. Therefore, the term between parentheses must be equal to 1.

that reflects correlations. Components with a negative sign act as a hedge against the remainder of the portfolio. In contrast, components with a positive sign increase the risk of the portfolio.

> **Component VAR** A partition of the portfolio VAR that indicates how much the portfolio VAR would change approximately if the given component was deleted. By construction, component VARs sum to the portfolio VAR.

The component VAR can be simplified further. Taking into account the fact that β_i is equal to the correlation ρ_i times σ_i divided by the portfolio σ_p, we can write

$$\text{CVAR}_i = \text{VAR}w_i\beta_i = (\alpha\sigma_p\, W)w_i\beta_i = (\alpha\sigma_i w_i W)\rho_i = \text{VAR}_i\rho_i \tag{7.29}$$

This conveniently transforms the individual VAR into its contribution to the total portfolio simply by multiplying it by the correlation coefficient.

Finally, we can normalize by the total portfolio VAR and report

$$\text{Percent contribution to VAR of component } i = \frac{\text{CVAR}_i}{\text{VAR}} = w_i\beta_i \tag{7.30}$$

VAR systems can provide a breakdown of the contribution to risk using any desired criterion. For large portfolios, component VAR may be shown by type of currency, by type of asset class, by geographic location, or by business unit. Such detail is invaluable for *drill-down exercises*, which enable users to control their VAR.

Example (continued)

Continuing with the previous two-currency example, we find the component VAR for the portfolio using $\text{CVAR}_i = \Delta\text{VAR}_i\, x_i$, that is,

$$\begin{bmatrix}\text{CVAR}_1\\ \text{CVAR}_2\end{bmatrix} = \begin{bmatrix}0.0528\times\$2\text{ million}\\ 0.1521\times\$1\text{ million}\end{bmatrix} = \begin{bmatrix}\$105{,}630\\ \$152{,}108\end{bmatrix} = \text{VAR}\times\begin{bmatrix}41.0\%\\ 59.0\%\end{bmatrix}$$

We verify that these two components indeed sum to the total VAR of \$257,738. The largest component is due to the EUR, which has the highest volatility. Both numbers are positive, indicating that neither position serves as a net hedge for the portfolio. Note that the percentage contribution to VAR also could have been obtained as

$$\begin{bmatrix}\text{CVAR}_1/\text{VAR}\\ \text{CVAR}_2/\text{VAR}\end{bmatrix} = \begin{bmatrix}w_1\beta_1\\ w_2\beta_2\end{bmatrix} = \begin{bmatrix}0.667\times0.615\\ 0.333\times1.770\end{bmatrix} = \begin{bmatrix}41.0\%\\ 59.0\%\end{bmatrix}$$

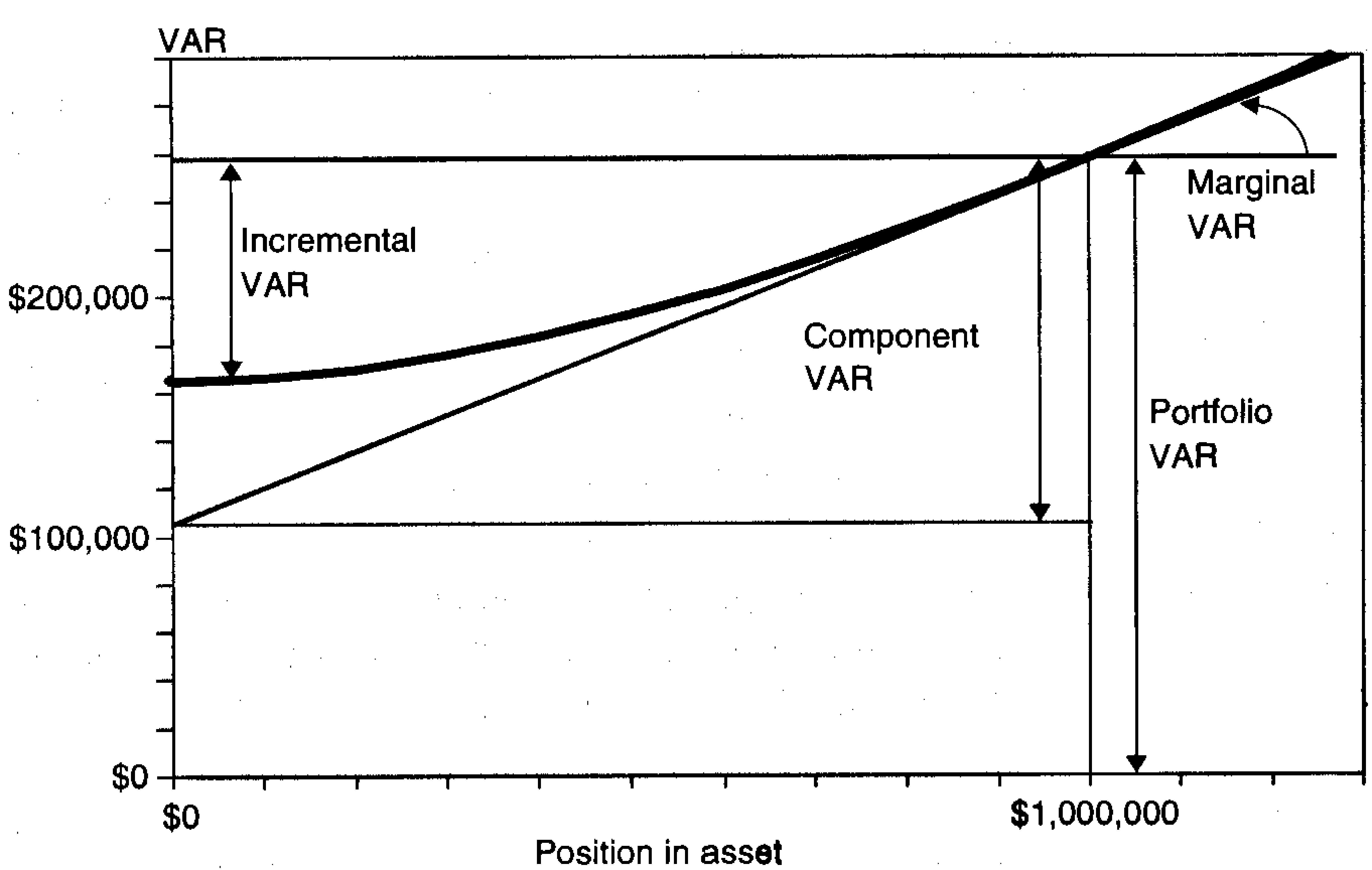
Next, we can compute the change in the VAR if the euro position is set to zero and compare with the preceding result. Since the portfolio has only two assets, the new VAR without the EUR position is simply the VAR of the CAD component, $VAR_1 = \$165{,}000$. The incremental VAR of the EUR position is ($257,738 − $165,000) = $92,738. The component VAR of $152,108 is higher, although of the same order of magnitude. The approximation is not as good as before because there are only two assets in the portfolio, which individually account for a large proportion of the total VAR. We would expect a better approximation if the VAR components are small relative to the total VAR.

7.2.4 Summary

Figure 7-4 presents a graphic summary of VAR tools for our two-currency portfolio. The graph plots the portfolio VAR as a function of the amount invested in this asset, the euro. At the current position of $1 million, the portfolio VAR is $257,738.

FIGURE 7-4

VAR decomposition.



The marginal VAR is the change in VAR owing to an addition of \$1 in EUR, or 0.0528; this represents the slope of the straight line that is tangent to the VAR curve at the current value.

The incremental VAR is the change in VAR owing to the deletion of the euro position, which is \$92,738 and is measured along the curve. This is approximated by the component VAR, which is simply the marginal VAR times the current position of \$1 million, or \$152,108. The latter is measured along the straight line that is tangent to the VAR curve. The graph illustrates that the component VAR is only an approximation of the incremental VAR. These component VAR measures add up to the total portfolio VAR, which gives a quick decomposition of the total risk.

The graph also shows that the best hedge is a net zero position in the euro. Indeed, the VAR function attains a minimum when the position in the euro is zero.

The results are summarized in Table 7-1. This report gives not only the portfolio VAR but also a wealth of information for risk managers. For instance, the marginal VAR column can be used to determine how to reduce risk. Since the marginal VAR for the EUR is three times as large as that for the CAD, cutting the position in the EUR will be much more effective than cutting the CAD position by the same amount.

7.3 EXAMPLES

This section provides a number of applications of VAR measures. The first example illustrates a risk report for a global equity portfolio. The second shows how VAR could have been used to dissect the Barings portfolio.

TABLE 7-1

VAR Decomposition for Sample Portfolio

Currency	Current Position, x_i or $w_i W$	Individual VAR, $VAR_i = \alpha\sigma_i w_i W$	Marginal VAR, $\Delta VAR_i = VAR\beta_i / W$	Component VAR, $CVAR_i = \Delta VAR_i x_i$	Percent Contribution, $CVAR_i / VAR$
CAD	\$2 million	\$165,000	0.0528	\$105,630	41.0%
EUR	\$1 million	\$198,000	0.1521	\$152,108	59.0%
Total	\$3 million				
Undiversified VAR		\$363,000			
Diversified VAR				\$257,738	100.0%

7.3.1 A Global Portfolio Equity Report

To further illustrate the use of our VAR tools, Table 7-2 displays a risk management report for a global equity portfolio. Here, risk is measured in relative terms, that is, relative to the benchmark portfolio. The current portfolio has an annualized tracking error volatility σ_p, of 1.82 percent per annum. This number can be translated easily into a VAR number using VAR $= \alpha\sigma_p W$. Hence we can deal with VAR or more directly with σ_p.

Positions are reported as deviations in percent from the benchmark in the second column. Since the weights of the benchmark and of the current portfolio must sum to one, the deviations must sum to zero. Traditional portfolio reporting systems only provide information about current positions for the portfolio. The position, data, however, could be used to provide detailed information about risk.

The next columns report the individual risk, marginal risk, and percentage contribution to total risk. Positions contributing to more than 5 percent of the total are called *Hot Spots*.[3] The table shows that two countries, Japan and Brazil, account for more than 50 percent of the risk. This is an important but not intuitive result because the positions in these markets, displayed in the first column, are not the largest in terms of weights.

TABLE 7-2

Global Equity Portfolio Report

Country	Current Position (%) w_i	Individual Risk $w_i\sigma_i$	Marginal Risk β_i	Percent Contribution to Risk $w_i\beta_i$	Best Hedge (%)	Volatility at Best Hedge
Japan	4.5	0.96%	0.068	**31.2**	−4.93	1.48%
Brazil	2.0	1.02%	0.118	**22.9**	−1.50	1.66%
U.S.	−7.0	0.89%	−0.019	**13.6**	3.80	1.75%
Thailand	2.0	0.55%	0.052	**10.2**	−2.30	1.71%
U.K.	−6.0	0.46%	0.035	**7.0**	2.10	1.80%
Italy	2.0	0.79%	−0.011	**6.8**	−2.18	1.75%
Germany	2.0	0.35%	0.019	3.7	−2.06	1.79%
France	−3.5	0.57%	−0.009	3.4	1.18	1.81%
Switzerland	2.5	0.39%	0.011	2.6	−1.45	1.81%
Canada	4.0	0.49%	0.001	1.5	−0.11	1.82%
South Africa	−1.0	0.20%	0.008	−0.7	−0.65	1.82%
Australia	−1.5	0.24%	0.014	−2.0	−1.89	1.80%
Total	0.0			100.0		
Undiversified risk		6.91%				
Diversified risk	1.82%					

Source: Adapted from Litterman (1996).

[3] Hot Spots is a trademark of Goldman Sachs.

In fact, the United States and United Kingdom, which have the largest deviations from the index, contribute to only 20 percent of the risk. The contributions of Japan and Brazil are high because of their high volatility and correlations with the portfolio.

To control risk, we turn to the "Best Hedge" column. The table shows that the 4.5 percent overweight position in Japan should be decreased to lower risk. The optimal change is a decrease of 4.93 percent, after which the new volatility will have decreased from the original value of 1.82 to 1.48 percent. In contrast, the 4.0 percent overweight position in Canada has little impact on the portfolio risk.

This type of report is invaluable to control risk. In the end, of course, portfolio managers add value by judicious bets on markets, currencies, or securities. Such VAR tools are useful, however, because analysts now can balance their return forecasts against risk explicitly.

7.3.2 Barings: An Example in Risks

Barings' collapse provides an interesting application of the VAR methodology. Leeson was reported to be long about \$7.7 billion worth of Japanese stock index (Nikkei) futures and short \$16 billion worth of Japanese government bond (JGB) futures. Unfortunately, official reports to Barings showed "nil" risk because the positions were fraudulent.

If a proper VAR system had been in place, the parent company could have answered the following questions: What was Leeson's actual VAR? Which component contributed most to VAR? Were the positions hedging each other or adding to the risk?

The top panel of Table 7-3 displays monthly volatility measures and correlations for positions in the 10-year zero JGB and the Nikkei Index. The correlation between Japanese stocks and bonds is negative, indicating that increases in stock prices are associated with decreases in bond prices or increases in interest rates. The next column displays positions that are reported in millions of dollar equivalents.

To compute the VAR, we first construct the covariance matrix Σ from the correlations. Next, we compute the vector Σx, which is in the first column of the bottom panel. For instance, the -2.82 entry is found from $\sigma_1^2 x_1 + \sigma_{12} x_2 = 0.000139 \times (-\$16{,}000) + (-0.000078) \times \$7700 = -2.82$. The next column reports $x_1(\Sigma x)_1$ and $x_2(\Sigma x)_2$, which sum to the total portfolio variance of 256,193.8, for a portfolio volatility of $\sqrt{256{,}194} = \$506$ million. At the 95 percent confidence level, Barings' VAR was $1.65 \times \$506$, or \$835 million.

TABLE 7-3

Barings' Risks

	Risk % σ	Correlation Matrix R		Covariance Matrix Σ		Positions ($ millions) x	Individual VAR $\alpha\sigma x$
10-year JGB	1.18	1	−0.114	0.000139	−0.000078	($16,000)	$310.88
Nikkei	5.83	−0.114	1	−0.000078	0.003397	$7,700	$740.51
Total						$8,300	$1051.39

	Total VAR Computation		Marginal VAR β_i for $1 million			
Asset i	$(\Sigma x)_i$	$x_i'(\Sigma x)_i$	$(\Sigma x)_i/\sigma_p^2$	β_iVAR	Component VAR $\beta_i x_i$VAR	Percent Contribution
10-yr JGB	−2.82	45138.8	−0.0000110	($0.00920)	$147.15	17.6%
Nikkei	27.41	211055.1	0.0001070	$0.08935	$688.01	82.4%
Total		256193.8			$835.16	100.0%
Risk = σ_p		506.16				
VAR = $\alpha\sigma_p$		$835.16				

This represents the worst monthly loss at the 95 percent confidence level under normal market conditions. In fact, Leeson's total loss was reported at $1.3 billion, which is comparable to the VAR reported here. The difference is because the position was changed over the course of the 2 months, there were other positions (such as short options), and also bad luck. In particular, on January 23, 1995, one week after the Kobe earthquake, the Nikkei Index lost 6.4 percent. Based on a monthly volatility of 5.83 percent, the daily VAR of Japanese stocks at the 95 percent confidence level should be 2.5 percent. Therefore, this was a very unusual move—even though we expect to exceed VAR in 5 percent of situations.

The marginal risk of each leg is also revealing. With a negative correlation between bonds and stocks, a hedged position typically would be long the two assets. Instead, Leeson was short the bond market, which market observers were at a loss to explain. A trader said, "This does not work as a hedge. It would have to be the other way round."[4] Thus Leeson was increasing his risk from the two legs of the position.

[4] *Financial Times*, March 1, 1995.

This is formalized in the table, which displays the marginal VAR computation. The β column is obtained by dividing each element of Σx by $x'\Sigma x$, for instance, −2.82 by 256,194 to obtain −0.000011. Multiplying by the VAR, we obtain the marginal change in VAR from increasing the bond position by \$1 million, which is −\$0.00920 million. Similarly, increasing the stock position by \$1 million increased the VAR by \$0.08935.

Overall, the component VAR owing to the total bond position is \$147.15 million; that owing to the stock position is \$688.01 million. By construction, these two numbers add up to the total VAR of \$835.16 million. This analysis shows that most of the risk was due to the Nikkei exposure and that the bond position, instead of hedging, made things even worse. As Box 7-1 shows, however, Leeson was able to hide his positions from the bank's VAR system.

BOX 7-1

BARINGS' RISK MISMANAGEMENT

The Barings case is a case in point of lack of trader controls. A good risk management system might have raised the alarm early and possibly avoided most of the \$1.3 billion loss.

Barings had installed in London a credit-risk management system in the 1980s. The bank was installing a market-risk management system in its London offices. The system, developed by California-based Infinity Financial Technology, has the capability to price derivatives and to support VAR reports. Barings' technology, however, was far more advanced in London than in its foreign branches. Big systems are expensive to install and support for small operations, which is why the bank relied heavily on local management.

The damning factor in the Barings affair was Leeson's joint responsibility for front- and back-office functions, which allowed him to hide trading losses. In July 1992, he created a special "error" account, numbered 88888, that was hidden from the trade file, price file, and London gross file. Losing trades and unmatched trades were parked in this account. Daily reports to Barings' Asset and Liability Committee showed Leeson's trading positions on the Nikkei 225 as fully matched. Reports to London therefore showed no risk. Had Barings used internal audits to provide independent checks on inputs, the company might have survived.

7.4 VAR TOOLS FOR GENERAL DISTRIBUTIONS

So far we have derived analytical expressions for these VAR tools assuming a normal distribution. These results can be generalized. In Equation (7.1), the portfolio return is a function of the positions on the individual components $R_p = f(w_1, \ldots, w_N)$. Multiplying all positions by a constant k will enlarge the portfolio return by the same amount, that is,

$$kR_p = f(kw_1, \ldots, kw_N) \tag{7.31}$$

Such function is said to be *homogeneous of degree one*, in which case we can apply *Euler's theorem*, which states that

$$R_p = f(w_1, \ldots, w_N) = \sum_{i=1}^{N} \frac{\partial f}{\partial w_i} w_i \tag{7.32}$$

The portfolio VAR is simply a realization of a large dollar loss. Setting R_p to the portfolio VAR gives:

$$\text{VAR} = \sum_{i=1}^{N} \frac{\partial \text{VAR}}{\partial w_i} \times w_i = \sum_{i=1}^{N} \frac{\partial \text{VAR}}{\partial x_i} \times x_i = \sum_{i=1}^{N} (\Delta\text{VAR}_i) \times x_i \tag{7.33}$$

This shows that the decomposition in Equation (7.28) is totally general. With a normal distribution, the marginal VAR is $\Delta\text{VAR}_i = \beta_i\,(\alpha\,\sigma_p)$, which is proportional to β_i. This analytical result also holds for *elliptical distributions*. In these cases, marginal VAR can be estimated using the sample beta coefficient, which uses all the sample information, such as the portfolio standard deviation, and as a result should be precisely measured.

Consider now another situation where the risk manager has generated a distribution of returns $R_{p,1}, \ldots, R_{p,T}$, and cannot to approximate it by an elliptical distribution perhaps because of an irregular shape owing to option positions. VAR is estimated from the observation R_p^*. One can show that applying Euler's theorem gives

$$R_p^* = \sum_{i=1}^{N} E(R_i \mid R_p = R_p^*) w_i \tag{7.34}$$

where the $E(\cdot)$ term is the expectation of the risk factor conditional on the portfolio having a return equal to VAR.[5] Thus CVAR_i could be estimated from the decomposition of R^* into the realized value of each component.

[5] For proofs, see Tasche (2000) or Hallerbach (2003).

Such estimates, however, are less reliable because they are based on one data point only. Another solution is to examine a window of observations around R^* and to average the realized values of each component over this window.

7.5 FROM VAR TO PORTFOLIO MANAGEMENT

7.5.1 From Risk Measurement to Risk Management

Marginal VAR and component VAR are useful tools, best suited to small changes in the portfolio. This can help the portfolio manager to decrease the risk of the portfolio. Positions should be cut first where the marginal VAR is the greatest, keeping portfolio constraints satisfied. For example, if the portfolio needs to be fully invested, some other position, with the lowest marginal VAR, should be added to make up for the first change.

This process can be repeated up to the point where the portfolio risk has reached a global minimum. At this point, all the marginal VARs, or the portfolio betas, must be equal:

$$\Delta \text{VAR}_i = \frac{\text{VAR}}{W} \times \beta_i = \text{constant} \tag{7.35}$$

Table 7-4 illustrates this process with the previous two-currency portfolio. The original position of \$2 million in CAD and \$1 million in EUR created a VAR of \$257,738, or portfolio volatility of 15.62 percent. The marginal VAR is 0.1521 for the EUR, which is higher than for the CAD.

TABLE 7-4

Risk-Minimizing Position

Asset	Original Position, w_i	Marginal VAR, ΔVAR_i	Final Position, w_i	Marginal VAR, ΔVAR_i	Beta β_i
CAD	66.67%	0.0528	85.21%	0.0762	1.000
EUR	33.33%	0.1521	14.79%	0.0762	1.000
Total	100.00%		100.00%		
Diversified VAR	\$257,738		\$228,462		
Standard deviation	15.62%		13.85%		

As a result, the EUR position should be cut first while adding to the CAD position. The table shows the final risk-minimizing position. The weight on the EUR has decreased from 33.33 to 14.79 percent. The portfolio volatility has been lowered from 15.62 to 13.85 percent, which is a substantial drop. We also verify that the betas of all positions are equal when risk is minimized.

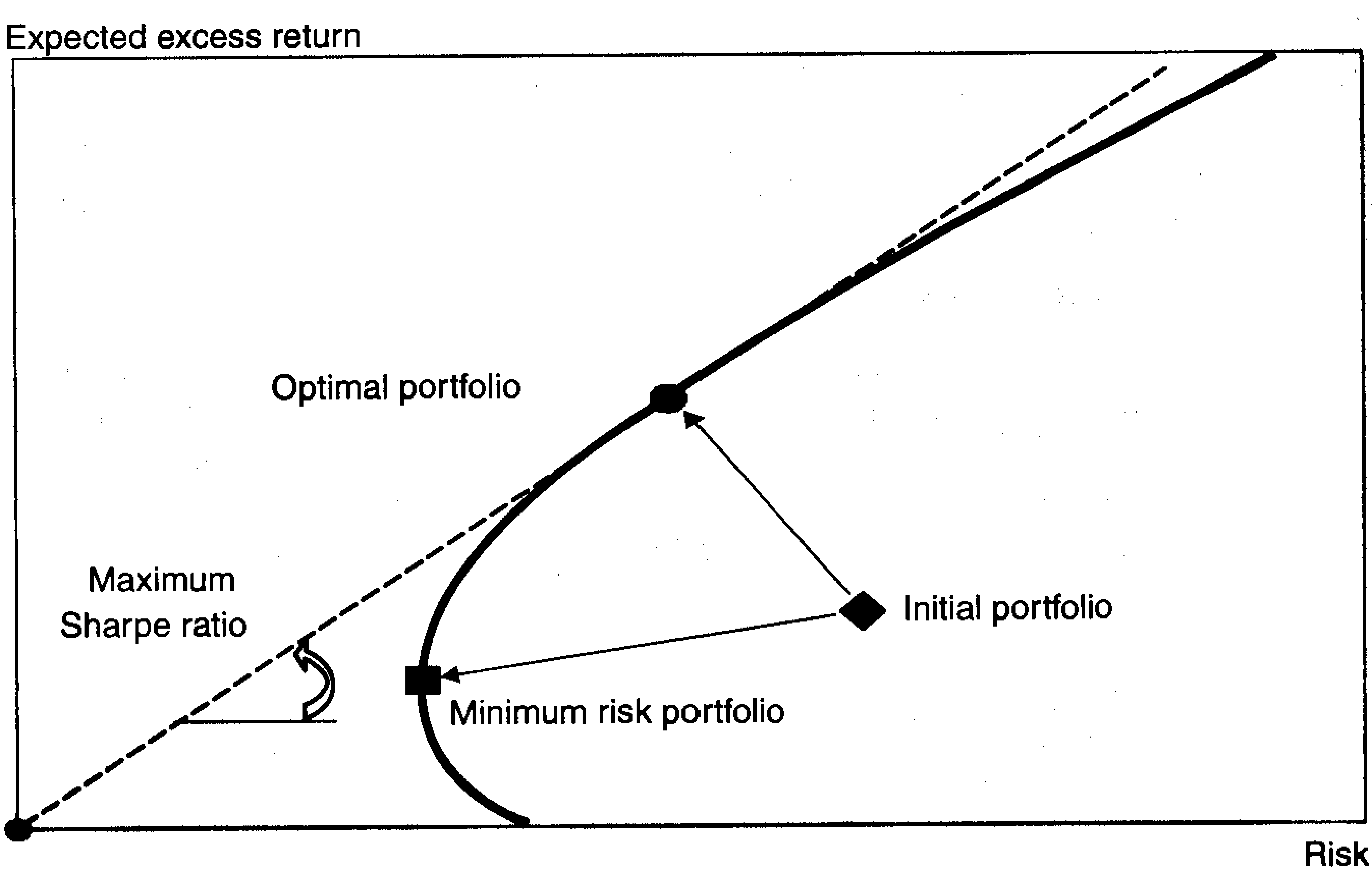
7.5.2 From Risk Management to Portfolio Management

The next step is to consider the portfolio expected return as well as its risk. Indeed, the role of the *portfolio manager* is to choose a portfolio that represents the best combination of expected return and risk. Thus we are moving from *risk management* to *portfolio management*. We will consider each portfolio in a graph that plots its expected return against its risk, as shown in Figure 7-5.

Define E_p as the expected return on the portfolio. This is a linear combination of the expected returns on the component positions, that is,

FIGURE 7-5

From VAR to portfolio management.



$$E_p = \sum_{i=1}^{N} w_i E_i \tag{7.36}$$

For simplicity, all returns are defined in excess of the risk-free rate. In the figure, this translates all the points down by the same amount so that the risk-free asset is at the origin.

We then can define the best portfolio combinations as the portfolios that minimize risk for varying levels of expected return. This defines the *efficient frontier*, which is shown as a solid line in Figure 7-5.

Suppose now that the objective function is to maximize the ratio of expected return to risk. This *Sharpe ratio* is

$$\mathrm{SR}_p = \frac{E_p}{\sigma_p} \tag{7.37}$$

More generally, this could be written with VAR in the denominator.

How do we move from the current position to this optimal portfolio? The preceding section showed how to move the portfolio from its original position to the *global minimum-risk* portfolio. This portfolio, however, does not take expected returns into account.

We now wish to increase the portfolio expected return as well, moving to the portfolio with the highest Sharpe ratio. This portfolio is on the efficient set and maximizes the slope of the tangent from the origin. We call this portfolio the *optimal portfolio*. At this point, the ratio of all expected returns to marginal VARs must be equal. This also can be written in terms of the excess expected return for each asset divided by its beta relative to the optimized portfolio. At the optimum,

$$\frac{E_i}{\Delta\mathrm{VAR}_i} = \frac{E_i}{\beta_i} = \text{constant} \tag{7.38}$$

Note that this is simply a restatement of the *capital asset pricing model*, which states that the market portfolio must be mean-variance efficient. Roll (1977) showed that the efficiency of any portfolio implies that the expected return on any component asset must be proportional to its beta relative to this portfolio, that is,

$$E_i = E_m \beta_i \tag{7.39}$$

Thus, for each asset, the ratio between the excess return E_i and the beta must be constant.

TABLE 7-5

Risk and Return–Optimizing Position

Asset	Expected Return E_i	Original Position w_i	Beta β_i	Ratio E_i/β_i	Final Position w_i	Beta β_i	Ratio E_i/β_i
CAD	8.00%	66.67%	0.615	0.1301	90.21%	1.038	0.0771
EUR	5.00%	33.33%	1.770	0.0282	9.79%	0.649	0.0771
Total		100.00%			100.00%		
Diversified VAR		$257,738			$230,720		
Standard deviation		15.62%			13.98%		
Expected return		7.00%			7.71%		
Sharpe ratio		0.448			0.551		

Table 7-5 shows our two-currency portfolio, for which we assumed that $E_1=8$ percent and $E_2=5$ percent. The original position has a Sharpe ratio of 0.448. The ratio of E_i/β_i is 0.1301 for CAD, which is greater than the 0.0282 value for EUR. This implies that the CAD position should be increased to improve portfolio performance. Indeed, at the optimum, the CAD weight has increased from 66.67 to 90.21 percent. The portfolio Sharpe ratio has increased substantially from 0.448 to 0.551. We verify that the ratios E_i/β_i are identical for the two assets at the optimum. The same values of 0.0771 indicate that there is no reason to deviate from the final allocation.

7.6 CONCLUSIONS

This chapter has shown how to measure and manage risk using analytical methods based on the standard deviation. Such methods apply when risk factors have distributions that are jointly normal or, more generally, elliptical.

Analytical methods are particularly convenient because they lead to closed-form solutions that are easy to interpret. This is akin to the Black-Scholes model, an analytical model to price options. This model is used widely because it yields powerful insights that can be applied to all options, including those that are computed using numerical methods. Thus the VAR tools developed here for parametric VAR also can be used with nonparametric, simulation-based VAR models.

We have seen that the VAR approach is much richer than the computation of a single risk measure. It provides a framework for managing risk using VAR tools such as marginal VAR and component VAR. These measures can be used to analyze the effect of marginal changes in portfolio composition.

A typical situation is that of a bank trader who has to evaluate whether a proposed trade with a client will increase or decrease the risk of the existing portfolio. Marginal VAR provides useful information to control the risk profile throughout the day. If the trade is risk-decreasing, then the trader should adjust the bid-offer spread to increase the probability that the client will do the trade. On the other hand, a trade that increases risk should be discouraged.

At the end, however, risk is only one component of the portfolio management process. Expected returns must be considered as well. The role of the portfolio manager is to balance increasing risk against increasing expected returns.

This is where VAR methods prove their usefulness. Combining expected profits into a portfolio is an intuitive process because expected returns are additive. In contrast, risk is not additive and is a complicated function of the portfolio positions and risk-factor characteristics. This explains why the battery of VAR tools is useful to manage portfolios better.

APPENDIX 7.A

Matrix Multiplication

This appendix reviews the algebra for matrix multiplication. Suppose that we have two matrices A and B that we wish to multiply to obtain the new matrix C. Their dimensions are $(n \times m)$ for A, or n rows and m columns, and $(m \times p)$ for B.

Note that for the matrix multiplication, the number of columns of A (m) must exactly match the number of rows for B. The dimensions of the resulting matrix C will be $(n \times p)$. Also note that the order of the multiplication matters. The multiplication of B times A is not conformable unless n also happens to be equal to p.

The matrix A can be written in terms of its components a_{ij}, where the first index i denotes the row and the second j denotes the column:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

For simplicity, consider now the case where the matrices are of dimension (2×3) and (3×2), that is,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

$$C = AB = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

To multiply the matrix A by B, we compute each element by taking each row of A and multiplying by the desired column of B. For instance, element c_{ij} would be obtained by multiplying each element of the ith row of A individually by each element of the jth column of B and summing over all these.

For instance, c_{11} is obtained by taking

$$c_{11} = [a_{11} \quad a_{12} \quad a_{13}] \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$$

This gives

$$C = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{11}b_{21} + a_{22}b_{21} + a_{23}b_{31} & a_{11}b_{22} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}$$

QUESTIONS

1. What is the interpretation of the marginal VAR for an asset?
2. All else equal, will portfolio risk decrease or increase under the following scenarios? (a) Correlations increase. (b) Volatilities increase. (c) The number of assets increases. (d) Assets move more closely together.
3. Assuming normal distributions, relate the risk of a portfolio invested (long) in two assets with correlation of 1 with the risks of the two assets.
4. Assuming normal distributions, relate the risk of a portfolio of two assets (long one asset and short the other) with correlation of −1 with the risks of the two assets.
5. Assume a portfolio is equally invested in N assets that have the same volatility of 10 percent and equal pairwise correlation. If the average correlation is 0.2, as N grows large, the portfolio volatility will tend to what number?
6. VAR is claimed not to be a *coherent risk measure*. Explain the meaning of this term and whether this criticism applies to normal distributions.
7. Given the following risk report, which asset serves as a hedge?

	Position	Marginal VAR	Component VAR
Asset 1	$2	100	$200
Asset 2	$1	−100	−$100

8. What is the relationship between marginal VAR and incremental VAR?
9. On average, what is the relationship between component VAR and individual VAR for a particular position?
10. How can we derive component VAR directly from marginal VAR?
11. A portfolio manager takes active positions relative to the benchmark. The manager considers changing one of the positions by a fixed amount. To reduce risk, should the manager focus on individual VAR, marginal VAR, or component VAR?
12. Define the *best hedge*.
13. If the risk of a portfolio of stocks has been minimized, do you expect the individual VAR/marginal VAR/component VAR to be zero/the same?
14. If a portfolio of stocks has been optimized to have the highest Sharpe ratio, do you expect the individual VAR/marginal VAR/component VAR to be the same/proportional to expected returns?
15. An investor holds a position that includes $100,000 invested in a 10-year Canadian government bond futures contract (CGB) and $100,000 invested in a Canadian stock index futures contract (SXF). Their annual volatility is 5 and 20 percent, respectively, with a correlation of −0.50. Assume that returns are normally distributed. VAR should be measured over 1 year at the 95 percent confidence level using the 1.645 quantile. Answer the following questions:
    (a) What are the diversified VAR and undiversified VAR?
    (b) What is the marginal and component VAR of CGB and SXF, respectively?
    (c) What is the incremental VAR from setting CGB to zero?

CHAPTER 8

Multivariate Models

Model: A simplified description of a system or process . . . that assists calculations and predictions.

—*Oxford English Dictionary*

Perhaps the defining characteristic of value-at-risk (VAR) systems is large-scale aggregation. VAR models attempt to measure the total financial risk of an institution. The scale of the problem requires the application of multivariate models to simplify the system. In many cases, it would be too difficult, and unnecessary, to model all positions individually as risk factors. Many positions are driven by the same set of risk factors and can be aggregated into a smaller set of exposures without loss of risk information.

Chapter 7 discussed the simple case where the number of positions is the same as the number of risk factors. Thus, if we had N assets, we would use N risk factors whose joint movement is described by an N by N covariance matrix. In general, however, we will choose fewer risk factors than the number of assets. This chapter provides tools for this simplification.

The fact that VAR is a large-scale portfolio aggregation has important consequences that too often are ignored. With large portfolios, the total risk depends heavily on correlations, even more so than on volatilities. Thus it is important to devote resources to model comovements between risk factors. The key challenge for the risk manager is to build a risk measurement system based on a parsimonious specification that provides a good approximation of the portfolio risk.

Multivariate models are most useful in situations where the risk manager requires internally consistent risk estimates for a portfolio of assets.

This is required, for instance, when the history of the current portfolio does not provide sufficient information to build a distribution of values. This is the case, for example, for distributions involving credit losses, such as those for collateralized debt obligations. Even when such a distribution exists, the multivariate approach is useful because it does not require reestimating the model for portfolios that differ from the current positions. Finally, multivariate models provide much better understanding of the structural drivers of losses by explicitly modeling joint movements in the risk factors.

Section 8.1 explains why the covariance matrix needs simplification. Factor models provide guidance for deciding how many risk factors are appropriate and are presented in Section 8.2. As we will see, an important role for the risk manager is to decide on the risk-factor structure. Using too many risk factors is unwieldy. Using too few, however, may create risk holes. This choice should be guided by the type of portfolio and trading strategy. Section 8.3 then discusses how to build joint distributions of the risk factors using a recently developed methodology called *copulas*. This allows more realistic modeling of the risk factors, in particular situations where markets experience extreme losses, as unfortunately is sometimes the case.

8.1. WHY SIMPLIFY THE COVARIANCE MATRIX

Chapter 7 examined the simple case where the number of assets N is the same as the number of considered risk factors. Their joint movement then is described by the covariance matrix Σ. This assumes that all the risk factors provide useful information. In practice, this may not be the case.

Examination of the covariance matrix can help us to simplify the risk structure. Correlations, or covariances, are essential driving forces behind portfolio risk. When the number of assets N is large, however, measurement of the covariance matrix becomes increasingly difficult. The covariance matrix has two dimensions, and the number of entries increases with the square of N. With 10 assets, for instance, we need to estimate $N \times (N + 1)/2 = 10 \times 11/2 = 55$ different variance and covariance terms. With 100 assets, this number climbs to 5050.

For large portfolios, this causes real problems. Correlations may not be estimated imprecisely. As a result, we could even have situations where the calculated portfolio variance is not positive, which makes no economic sense.

Define the portfolio weights as w. In practice, the covariance matrix is estimated from historical data. In the simplest method, VAR is derived from the portfolio variance, computed as

$$\sigma_p^2 = w'\Sigma w \tag{8.1}$$

The question is, Is the number resulting from this product guaranteed to be always positive? Unfortunately, not always. For this to be the case, we need the matrix Σ to be *positive definite* (abstracting from the obvious case where all elements of w are zero).

Negative values can happen, for instance, when the number of historical observations T is less than the number of assets N. In other words, if a portfolio consists of 100 assets, there must be at least 100 historical observations to ensure that the portfolio variance will be positive. This is also an issue when the covariance matrix is estimated with decaying weights, as in the GARCH method explained in Chapter 9. If the weights decay too quickly, the number of effective observations can be less than the number of assets, rendering the covariance matrix nonpositive definite.

Problems also occur when the series are linearly correlated. This happens, for example, when two assets are identical ($\rho = 1$). In this situation, a portfolio consisting of \$1 on the first asset and $-\$1$ on the second will have exactly zero risk. In practice, this problem is more likely to occur with a large number of assets that are highly correlated, such as zero-coupon bonds or currencies fixed to each other.

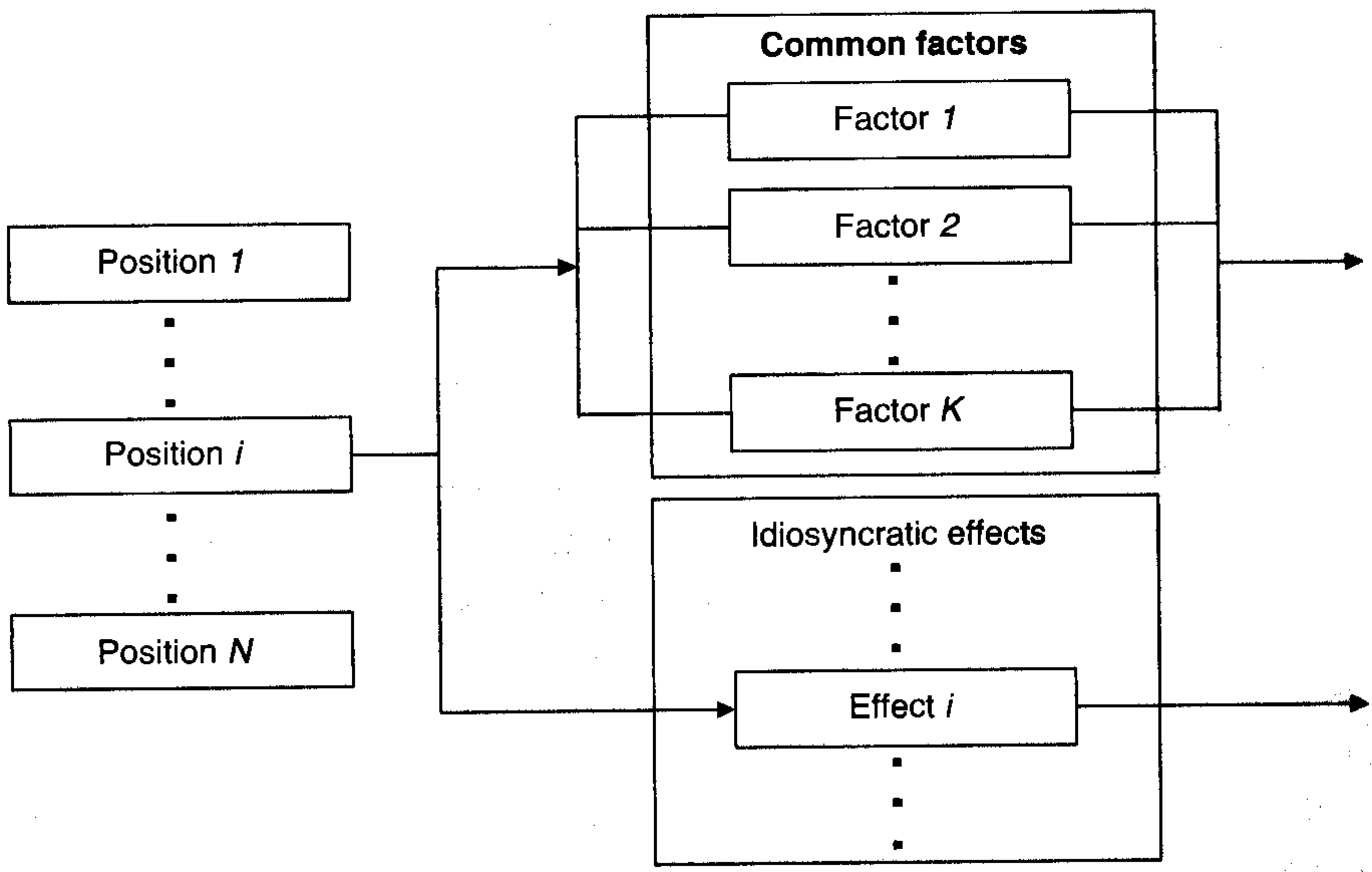
Most of the time this problem will not show up, and the portfolio variance will be positive. This may not be the case, however, if the portfolio has been *optimized* on the basis of the covariance matrix itself. Such optimization is particularly dangerous because it can create positions that are very large yet apparently offset each other with little total risk.

Such situations do arise in practice, however. As we shall see in Chapter 21, it largely explains the failure of the hedge fund Long-Term Capital Management. In practice, simple rules of thumb can help. If users notice that VAR measures appear abnormally low in relation to positions, they should check whether small changes in correlations lead to large changes in their VARs.

Alternatively, the risk structure can be simplified. Figure 8-1 shows how movements in N asset values can be decomposed into a small number of common risk factors K and asset-specific or idiosyncratic effects that are uncorrelated with each other. As we shall see, this structure reduces the number of required parameters substantially and is more robust than is using

FIGURE 8-1

Simplifying the risk structure.



a full covariance matrix. In addition, it lends itself better to an economic intuition, which helps to understand the results.

The framework described in Figure 8-1 can be extended to idiosyncratic effects that are correlated or have a more complex joint distribution, which can be modeled using the copula approach. It is also very flexible because it allows time variation in the comovements of the common factors.

8.2 FACTOR STRUCTURES

8.2.1 Simplifications

These issues become more troublesome as the number of assets increases. Assume that we want to select stocks from the entire universe of listed equities. These number more than 38,000. It is impossible to construct a covariance matrix for these assets that is positive definite.

This problem can be alleviated by the use of simpler structures for the covariance matrix. One example would be to have the same correlation coefficient across all pairs of assets. In this case, the sample is said to be *homogeneous*. The Basel II rules are based on such a model, with a correlation coefficient of 0.20. This may be too simplistic, however, because it does not allow much differentiation between risk factors.

8.2.2 Diagonal Model

Another simple model is the *diagonal model,* originally proposed by Sharpe in the context of stock portfolios. The assumption is that the common movement in all assets is due to one common factor only, the stock market index, for example. The return on a stock R_i is regressed on the return on the stock market index R_m, giving an unexplained residual ϵ_i. Formally, the model is

$$R_i = \alpha_i + \beta_i R_m + \epsilon_i \tag{8.2}$$

with assumptions

$$E(\epsilon_i) = 0 \qquad E(\epsilon_i R_m) = 0 \qquad E(\epsilon_i \epsilon_j) = 0 \tag{8.3}$$

where β_i is the exposure, or *loading*, on the market factor. For stocks, *beta* is also called *systematic risk* when the factor is the stock market index. The fixed intercept α_i can be ignored in what follows because it is not random and hence does not contribute to risk. Finally, define the variances as $\sigma_i^2 = V(R_i^2)$, $\sigma_m^2 = V(R_m^2)$, and $V(\epsilon_i^2) = \sigma_{\epsilon,i}^2$. In Equation (8.2), the $\beta_i R_m$ term is called *general market risk*, and the second term ϵ_i, *specific risk*.

There are two key assumptions in Equation (8.3). First, the errors are uncorrelated with the common factor by construction. We have $\text{cov}(\epsilon_i, R_m) = E(\epsilon_i R_m) - E(\epsilon_i) E(R_m) = 0$. Second, the errors are uncorrelated across each other because $E(\epsilon_i \epsilon_j) = 0$.

The return on asset i is driven by the market return R_m and an idiosyncratic term ϵ_i, which is not correlated with the market or across assets. As a result, the variance of stock i's return can be decomposed as

$$\begin{aligned}\sigma_i^2 = V(\beta_i R_m + \epsilon_i) &= \beta_i^2 \sigma_m^2 + 2\,\text{cov}(\beta_i R_m, \epsilon_i) + V(\epsilon_i) \\ &= \beta_i^2 \sigma_m^2 + \sigma^2{}_{\epsilon,i}\end{aligned} \tag{8.4}$$

because R_m and ϵ_i are uncorrelated. The covariance between two assets is

$$\sigma_{i,j} = \text{cov}(\beta_i R_m + \epsilon_i, \beta_j R_m + \epsilon_j) = \beta_i \beta_j \sigma_m^2 \tag{8.5}$$

which is solely due to the common factor because all the other terms are zero owing to Equation (8.3).

As a result, we can construct the full covariance matrix as