1 Difference Equations

1.1. *First-Order Difference Equations*

This book is concerned with the dynamic consequences of events over time. Let's say we are studying a variable whose value at date t is denoted y_t. Suppose we are given a dynamic equation relating the value y takes on at date t to another variable w_t and to the value y took on in the previous period:

$$y_t = \phi y_{t-1} + w_t. \qquad [1.1.1]$$

Equation [1.1.1] is a *linear first-order difference equation.* A *difference equation* is an expression relating a variable y_t to its previous values. This is a *first-order* difference equation because only the first lag of the variable (y_{t-1}) appears in the equation. Note that it expresses y_t as a linear function of y_{t-1} and w_t.

An example of [1.1.1] is Goldfeld's (1973) estimated money demand function for the United States. Goldfeld's model related the log of the real money holdings of the public (m_t) to the log of aggregate real income (I_t), the log of the interest rate on bank accounts (r_{bt}), and the log of the interest rate on commercial paper (r_{ct}):

$$m_t = 0.27 + 0.72m_{t-1} + 0.19I_t - 0.045r_{bt} - 0.019r_{ct}. \qquad [1.1.2]$$

This is a special case of [1.1.1] with $y_t = m_t$, $\phi = 0.72$, and

$$w_t = 0.27 + 0.19I_t - 0.045r_{bt} - 0.019r_{ct}.$$

For purposes of analyzing the dynamics of such a system, it simplifies the algebra a little to summarize the effects of all the input variables (I_t, r_{bt}, and r_{ct}) in terms of a scalar w_t as here.

In Chapter 3 the input variable w_t will be regarded as a random variable, and the implications of [1.1.1] for the statistical properties of the output series y_t will be explored. In preparation for this discussion, it is necessary first to understand the mechanics of difference equations. For the discussion in Chapters 1 and 2, the values for the input variable $\{w_1, w_2, \ldots\}$ will simply be regarded as a sequence of deterministic numbers. Our goal is to answer the following question: If a dynamic system is described by [1.1.1], what are the effects on y of changes in the value of w?

Solving a Difference Equation by Recursive Substitution

The presumption is that the dynamic equation [1.1.1] governs the behavior of y for all dates t. Thus, for each date we have an equation relating the value of

y for that date to its previous value and the current value of w:

Date	*Equation*	
0	$y_0 = \phi y_{-1} + w_0$	[1.1.3]
1	$y_1 = \phi y_0 + w_1$	[1.1.4]
2	$y_2 = \phi y_1 + w_2$	[1.1.5]
$\vdots$	$\vdots$	
t	$y_t = \phi y_{t-1} + w_t.$	[1.1.6]

If we know the starting value of y for date $t = -1$ and the value of w for dates $t = 0, 1, 2, \ldots$, then it is possible to simulate this dynamic system to find the value of y for any date. For example, if we know the value of y for $t = -1$ and the value of w for $t = 0$, we can calculate the value of y for $t = 0$ directly from [1.1.3]. Given this value of y_0 and the value of w for $t = 1$, we can calculate the value of y for $t = 1$ from [1.1.4]:

$$y_1 = \phi y_0 + w_1 = \phi(\phi y_{-1} + w_0) + w_1,$$

or

$$y_1 = \phi^2 y_{-1} + \phi w_0 + w_1.$$

Given this value of y_1 and the value of w for $t = 2$, we can calculate the value of y for $t = 2$ from [1.1.5]:

$$y_2 = \phi y_1 + w_2 = \phi(\phi^2 y_{-1} + \phi w_0 + w_1) + w_2,$$

or

$$y_2 = \phi^3 y_{-1} + \phi^2 w_0 + \phi w_1 + w_2.$$

Continuing recursively in this fashion, the value that y takes on at date t can be described as a function of its initial value y_{-1} and the history of w between date 0 and date t:

$$y_t = \phi^{t+1} y_{-1} + \phi^t w_0 + \phi^{t-1} w_1 + \phi^{t-2} w_2 + \cdots + \phi w_{t-1} + w_t. \quad [1.1.7]$$

This procedure is known as solving the difference equation [1.1.1] by *recursive substitution*.

Dynamic Multipliers

Note that [1.1.7] expresses y_t as a linear function of the initial value y_{-1} and the historical values of w. This makes it very easy to calculate the effect of w_0 on y_t. If w_0 were to change with y_{-1} and $w_1, w_2, \ldots, w_t$ taken as unaffected, the effect on y_t would be given by

$$\frac{\partial y_t}{\partial w_0} = \phi^t. \quad [1.1.8]$$

Note that the calculations would be exactly the same if the dynamic simulation were started at date t (taking y_{t-1} as given); then y_{t+j} could be described as a function of y_{t-1} and $w_t, w_{t+1}, \ldots, w_{t+j}$:

$$y_{t+j} = \phi^{j+1}y_{t-1} + \phi^j w_t + \phi^{j-1}w_{t+1} + \phi^{j-2}w_{t+2} + \cdots + \phi w_{t+j-1} + w_{t+j}. \qquad [1.1.9]$$

The effect of w_t on y_{t+j} is given by

$$\frac{\partial y_{t+j}}{\partial w_t} = \phi^j. \qquad [1.1.10]$$

Thus the *dynamic multiplier* [1.1.10] depends only on j, the length of time separating the disturbance to the input (w_t) and the observed value of the output (y_{t+j}). The multiplier does not depend on t; that is, it does not depend on the dates of the observations themselves. This is true of any linear difference equation.

As an example of calculating a dynamic multiplier, consider again Goldfeld's money demand specification [1.1.2]. Suppose we want to know what will happen to money demand two quarters from now if current income I_t were to increase by one unit today with future income I_{t+1} and I_{t+2} unaffected:

$$\frac{\partial m_{t+2}}{\partial I_t} = \frac{\partial m_{t+2}}{\partial w_t} \times \frac{\partial w_t}{\partial I_t} = \phi^2 \times \frac{\partial w_t}{\partial I_t}.$$

From [1.1.2], a one-unit increase in I_t will increase w_t by 0.19 units, meaning that $\partial w_t/\partial I_t = 0.19$. Since $\phi = 0.72$, we calculate

$$\frac{\partial m_{t+2}}{\partial I_t} = (0.72)^2(0.19) = 0.098.$$

Because I_t is the log of income, an increase in I_t of 0.01 units corresponds to a 1% increase in income. An increase in m_t of $(0.01)\cdot(0.098) \cong 0.001$ corresponds to a 0.1% increase in money holdings. Thus the public would be expected to increase its money holdings by a little less than 0.1% two quarters following a 1% increase in income.

Different values of ϕ in [1.1.1] can produce a variety of dynamic responses of y to w. If $0 < \phi < 1$, the multiplier $\partial y_{t+j}/\partial w_t$ in [1.1.10] decays geometrically toward zero. Panel (a) of Figure 1.1 plots ϕ^j as a function of j for $\phi = 0.8$. If $-1 < \phi < 0$, the multiplier $\partial y_{t+j}/\partial w_t$ will alternate in sign as in panel (b). In this case an increase in w_t will cause y_t to be higher, y_{t+1} to be lower, y_{t+2} to be higher, and so on. Again the absolute value of the effect decays geometrically toward zero. If $\phi > 1$, the dynamic multiplier increases exponentially over time as in panel (c). A given increase in w_t has a larger effect the farther into the future one goes. For $\phi < -1$, the system [1.1.1] exhibits explosive oscillation as in panel (d).

Thus, if $|\phi| < 1$, the system is stable; the consequences of a given change in w_t will eventually die out. If $|\phi| > 1$, the system is explosive. An interesting possibility is the borderline case, $\phi = 1$. In this case, the solution [1.1.9] becomes

$$y_{t+j} = y_{t-1} + w_t + w_{t+1} + w_{t+2} + \cdots + w_{t+j-1} + w_{t+j}. \qquad [1.1.11]$$

Here the output variable y is the sum of the historical inputs w. A one-unit increase in w will cause a permanent one-unit increase in y:

$$\frac{\partial y_{t+j}}{\partial w_t} = 1 \qquad \text{for } j = 0, 1, \ldots.$$

We might also be interested in the effect of w on the present value of the stream of future realizations of y. For a given stream of future values $y_t, y_{t+1},$

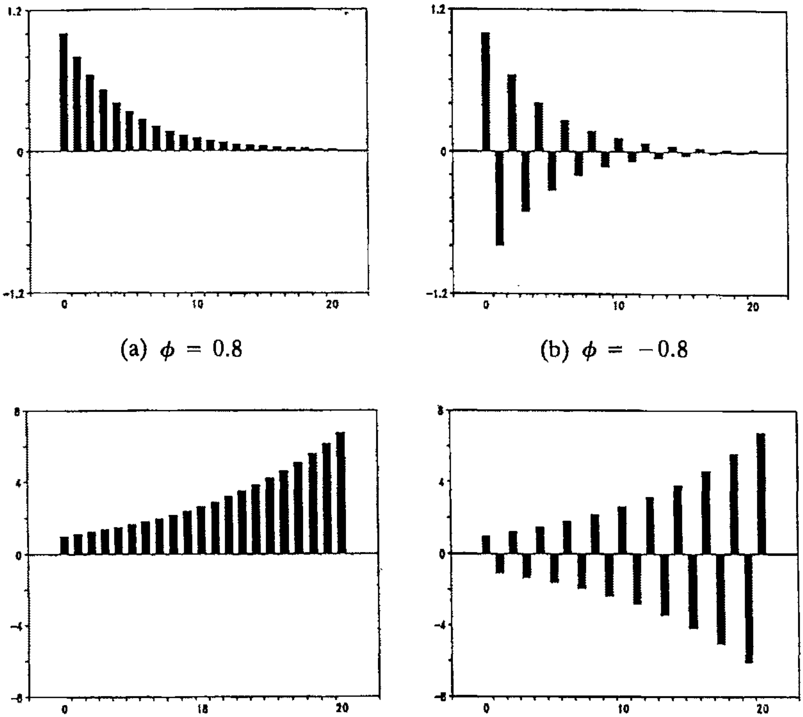


FIGURE 1.1 Dynamic multiplier for first-order difference equation for different values of ϕ (plot of $\partial y_{t+j}/\partial w_t = \phi^j$ as a function of the lag j).

$y_{t+2}, \ldots$ and a constant interest rate[1] $r > 0$, the *present value* of the stream at time t is given by

$$y_t + \frac{y_{t+1}}{1 + r} + \frac{y_{t+2}}{(1 + r)^2} + \frac{y_{t+3}}{(1 + r)^3} + \cdots. \qquad [1.1.12]$$

Let β denote the discount factor:

$$\beta \equiv 1/(1 + r).$$

Note that $0 < \beta < 1$. Then the present value [1.1.12] can be written as

$$\sum_{j=0}^{\infty} \beta^j y_{t+j}. \qquad [1.1.13]$$

Consider what would happen if there were a one-unit increase in w_t with $w_{t+1}, w_{t+2}, \ldots$ unaffected. The consequences of this change for the present value of y are found by differentiating [1.1.13] with respect to w_t and then using [1.1.10]

[1]The interest rate is measured here as a fraction of 1; thus $r = 0.1$ corresponds to a 10% interest rate.

to evaluate each derivative:

$$\sum_{j=0}^{\infty} \beta^j \frac{\partial y_{t+j}}{\partial w_t} = \sum_{j=0}^{\infty} \beta^j \phi^j = 1/(1 - \beta\phi), \qquad [1.1.14]$$

provided that $|\beta\phi| < 1$.

In calculating the dynamic multipliers [1.1.10] or [1.1.14], we were asking what would happen if w_t were to increase by one unit with $w_{t+1}, w_{t+2}, \ldots, w_{t+j}$ unaffected. We were thus finding the effect of a purely transitory change in w. Panel (a) of Figure 1.2 shows the time path of w associated with this question, and panel (b) shows the implied path for y. Because the dynamic multiplier [1.1.10] calculates the response of y to a single impulse in w, it is also referred to as the *impulse-response function*.

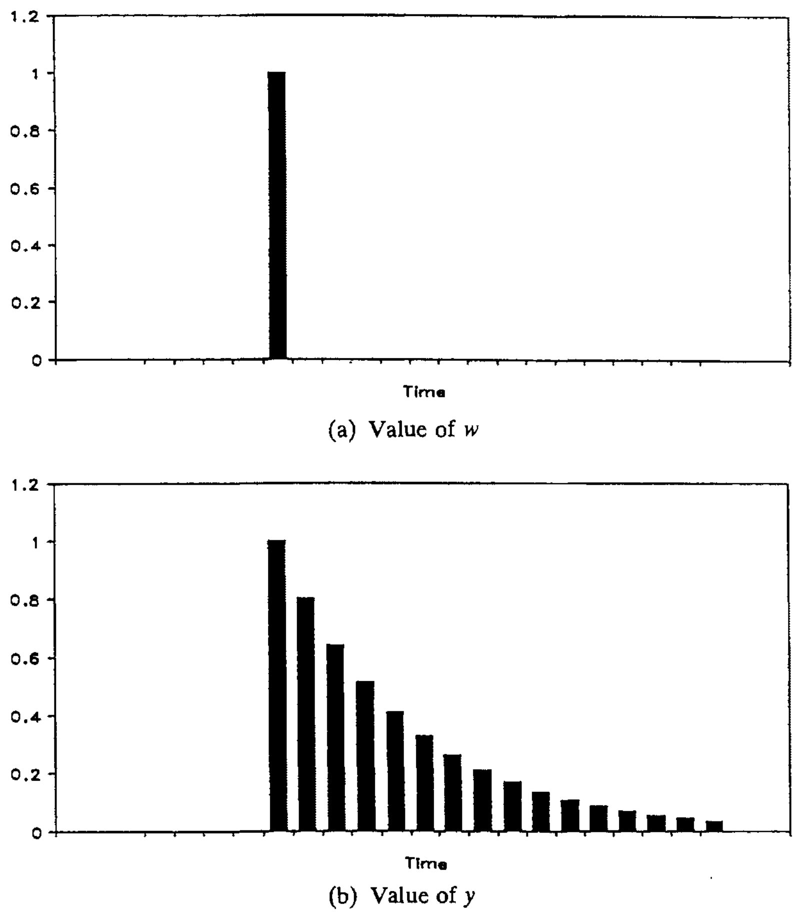


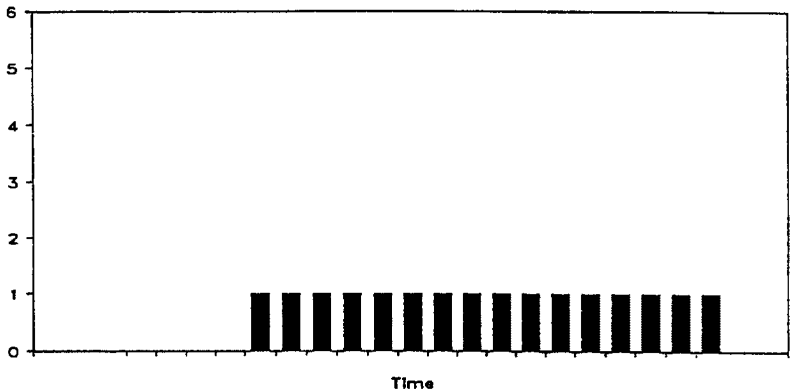
FIGURE 1.2 Paths of input variable (w_t) and output variable (y_t) assumed for dynamic multiplier and present-value calculations.

Sometimes we might instead be interested in the consequences of a permanent change in w. A permanent change in w means that $w_t, w_{t+1}, \ldots,$ and w_{t+j} would all increase by one unit, as in Figure 1.3. From formula [1.1.10], the effect on y_{t+j} of a permanent change in w beginning in period t is given by

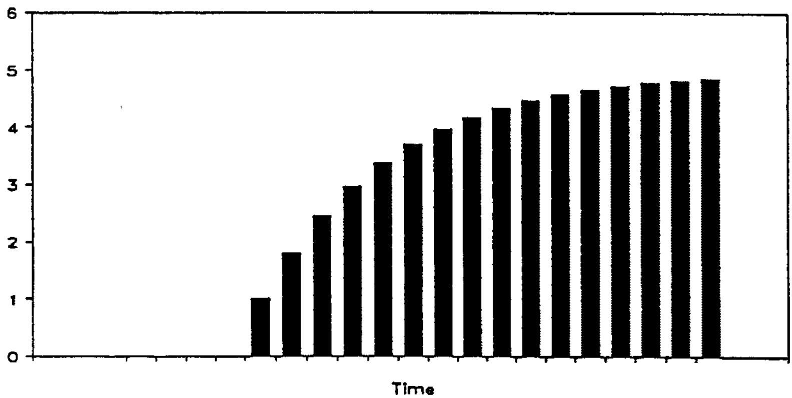
$$\frac{\partial y_{t+j}}{\partial w_t} + \frac{\partial y_{t+j}}{\partial w_{t+1}} + \frac{\partial y_{t+j}}{\partial w_{t+2}} + \cdots + \frac{\partial y_{t+j}}{\partial w_{t+j}} = \phi^j + \phi^{j-1} + \phi^{j-2} + \cdots + \phi + 1.$$

When $|\phi| < 1$, the limit of this expression as j goes to infinity is sometimes described as the "long-run" effect of w on y:

$$\begin{aligned} \lim_{j\to\infty}\left[\frac{\partial y_{t+j}}{\partial w_t} + \frac{\partial y_{t+j}}{\partial w_{t+1}} + \frac{\partial y_{t+j}}{\partial w_{t+2}} + \cdots + \frac{\partial y_{t+j}}{\partial w_{t+j}}\right] &= 1 + \phi + \phi^2 + \cdots \\ &= 1/(1 - \phi). \end{aligned} \qquad [1.1.15]$$



(a) Value of w



(b) Value of y

FIGURE 1.3 Paths of input variable (w_t) and output variable (y_t) assumed for long-run effect calculations.

For example, the long-run income elasticity of money demand in the system [1.1.2] is given by

$$\frac{0.19}{1 - 0.72} = 0.68.$$

A permanent 1% increase in income will eventually lead to a 0.68% increase in money demand.

Another related question concerns the cumulative consequences for y of a one-time change in w. Here we consider a transitory disturbance to w as in panel (a) of Figure 1.2, but wish to calculate the sum of the consequences for all future values of y. Another way to think of this is as the effect on the present value of y [1.1.13] with the discount rate $\beta = 1$. Setting $\beta = 1$ in [1.1.14] shows this cumulative effect to be equal to

$$\sum_{j=0}^{\infty} \frac{\partial y_{t+j}}{\partial w_t} = 1/(1 - \phi), \qquad [1.1.16]$$

provided that $|\phi| < 1$. Note that the cumulative effect on y of a transitory change in w (expression [1.1.16]) is the same as the long-run effect on y of a permanent change in w (expression [1.1.15]).

1.2. *p*th-Order Difference Equations

Let us now generalize the dynamic system [1.1.1] by allowing the value of y at date t to depend on p of its own lags along with the current value of the input variable w_t:

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + w_t. \qquad [1.2.1]$$

Equation [1.2.1] is a linear pth-order difference equation.

It is often convenient to rewrite the pth-order difference equation [1.2.1] in the scalar y_t as a first-order difference equation in a vector $\boldsymbol{\xi}_t$. Define the $(p \times 1)$ vector $\boldsymbol{\xi}_t$ by

$$\boldsymbol{\xi}_t \equiv \begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \end{bmatrix}. \qquad [1.2.2]$$

That is, the first element of the vector $\boldsymbol{\xi}$ at date t is the value y took on at date t. The second element of $\boldsymbol{\xi}_t$ is the value y took on at date $t - 1$, and so on. Define the $(p \times p)$ matrix $\mathbf{F}$ by

$$\mathbf{F} \equiv \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \qquad [1.2.3]$$

For example, for $p = 4$, $\mathbf{F}$ refers to the following 4×4 matrix:

$$\mathbf{F} = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

For $p = 1$ (the first-order difference equation [1.1.1]), $\mathbf{F}$ is just the scalar ϕ. Finally, define the $(p \times 1)$ vector $\mathbf{v}_t$ by

$$\mathbf{v}_t \equiv \begin{bmatrix} w_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \qquad [1.2.4]$$

Consider the following first-order vector difference equation:

$$\boldsymbol{\xi}_t = \mathbf{F}\boldsymbol{\xi}_{t-1} + \mathbf{v}_t, \qquad [1.2.5]$$

or

$$\begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ y_{t-3} \\ \vdots \\ y_{t-p} \end{bmatrix} + \begin{bmatrix} w_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

This is a system of p equations. The first equation in this system is identical to equation [1.2.1]. The second equation is simply the identity

$$y_{t-1} = y_{t-1},$$

owing to the fact that the second element of $\boldsymbol{\xi}_t$ is the same as the first element of $\boldsymbol{\xi}_{t-1}$. The third equation in [1.2.5] states that $y_{t-2} = y_{t-2}$; the pth equation states that $y_{t-p+1} = y_{t-p+1}$.

Thus, the first-order vector system [1.2.5] is simply an alternative representation of the pth-order scalar system [1.2.1]. The advantage of rewriting the pth-order system [1.2.1] in the form of a first-order system [1.2.5] is that first-order systems are often easier to work with than pth-order systems.

A dynamic multiplier for [1.2.5] can be found in exactly the same way as was done for the first-order scalar system of Section 1.1. If we knew the value of the vector $\boldsymbol{\xi}$ for date $t = -1$ and of $\mathbf{v}$ for date $t = 0$, we could find the value of $\boldsymbol{\xi}$ for date 0 from

$$\boldsymbol{\xi}_0 = \mathbf{F}\boldsymbol{\xi}_{-1} + \mathbf{v}_0.$$

The value of $\boldsymbol{\xi}$ for date 1 is

$$\boldsymbol{\xi}_1 = \mathbf{F}\boldsymbol{\xi}_0 + \mathbf{v}_1 = \mathbf{F}(\mathbf{F}\boldsymbol{\xi}_{-1} + \mathbf{v}_0) + \mathbf{v}_1 = \mathbf{F}^2\boldsymbol{\xi}_{-1} + \mathbf{F}\mathbf{v}_0 + \mathbf{v}_1.$$

Proceeding recursively in this fashion produces a generalization of [1.1.7]:

$$\boldsymbol{\xi}_t = \mathbf{F}^{t+1}\boldsymbol{\xi}_{-1} + \mathbf{F}^t\mathbf{v}_0 + \mathbf{F}^{t-1}\mathbf{v}_1 + \mathbf{F}^{t-2}\mathbf{v}_2 + \cdots + \mathbf{F}\mathbf{v}_{t-1} + \mathbf{v}_t. \qquad [1.2.6]$$

Writing this out in terms of the definitions of ξ and v,

$$\begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \end{bmatrix} = \mathbf{F}^{t+1}\begin{bmatrix} y_{-1} \\ y_{-2} \\ y_{-3} \\ \vdots \\ y_{-p} \end{bmatrix} + \mathbf{F}^{t}\begin{bmatrix} w_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \mathbf{F}^{t-1}\begin{bmatrix} w_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots$$
$$+ \mathbf{F}^{1}\begin{bmatrix} w_{t-1} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} w_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \qquad [1.2.7]$$

Consider the first equation of this system, which characterizes the value of y_t. Let $f_{11}^{(t)}$ denote the (1, 1) element of $\mathbf{F}^t$, $f_{12}^{(t)}$ the (1, 2) element of $\mathbf{F}^t$, and so on. Then the first equation of [1.2.7] states that

$$\begin{aligned} y_t = f_{11}^{(t+1)}y_{-1} + f_{12}^{(t+1)}y_{-2} + \cdots + f_{1p}^{(t+1)}y_{-p} + f_{11}^{(t)}w_0 \\ + f_{11}^{(t-1)}w_1 + \cdots + f_{11}^{(1)}w_{t-1} + w_t. \end{aligned} \qquad [1.2.8]$$

This describes the value of y at date t as a linear function of p initial values of y $(y_{-1}, y_{-2}, \ldots, y_{-p})$ and the history of the input variable w since time 0 $(w_0, w_1, \ldots, w_t)$. Note that whereas only one initial value for y (the value y_{-1}) was needed in the case of a first-order difference equation, p initial values for y (the values $y_{-1}, y_{-2}, \ldots, y_{-p}$) are needed in the case of a pth-order difference equation.

The obvious generalization of [1.1.9] is

$$\begin{aligned} \xi_{t+j} = \mathbf{F}^{j+1}\xi_{t-1} + \mathbf{F}^{j}\mathbf{v}_t + \mathbf{F}^{j-1}\mathbf{v}_{t+1} + \mathbf{F}^{j-2}\mathbf{v}_{t+2} + \cdots \\ + \mathbf{F}\mathbf{v}_{t+j-1} + \mathbf{v}_{t+j} \end{aligned} \qquad [1.2.9]$$

from which

$$\begin{aligned} y_{t+j} = f_{11}^{(j+1)}y_{t-1} + f_{12}^{(j+1)}y_{t-2} + \cdots + f_{1p}^{(j+1)}y_{t-p} + f_{11}^{(j)}w_t \\ + f_{11}^{(j-1)}w_{t+1} + f_{11}^{(j-2)}w_{t+2} + \cdots + f_{11}^{(1)}w_{t+j-1} + w_{t+j}. \end{aligned} \qquad [1.2.10]$$

Thus, for a pth-order difference equation, the dynamic multiplier is given by

$$\frac{\partial y_{t+j}}{\partial w_t} = f_{11}^{(j)} \qquad [1.2.11]$$

where $f_{11}^{(j)}$ denotes the (1, 1) element of $\mathbf{F}^j$. For $j = 1$, this is simply the (1, 1) element of $\mathbf{F}$, or the parameter ϕ_1. Thus, for any pth-order system, the effect on y_{t+1} of a one-unit increase in w_t is given by the coefficient relating y_t to y_{t-1} in equation [1.2.1]:

$$\frac{\partial y_{t+1}}{\partial w_t} = \phi_1.$$

Direct multiplication of [1.2.3] reveals that the (1, 1) element of $\mathbf{F}^2$ is $(\phi_1^2 + \phi_2)$, so

$$\frac{\partial y_{t+2}}{\partial w_t} = \phi_1^2 + \phi_2$$

in a pth-order system.

For larger values of j, an easy way to obtain a numerical value for the dynamic multiplier $\partial y_{t+j}/\partial w_t$ is to simulate the system. This is done as follows. Set $y_{-1} = y_{-2} = \cdots = y_{-p} = 0$, $w_0 = 1$, and set the value of w for all other dates to 0. Then use [1.2.1] to calculate the value of y_t for $t = 0$ (namely, $y_0 = 1$). Next substitute this value along with $y_{t-1}, y_{t-2}, \ldots, y_{t-p+1}$ back into [1.2.1] to calculate y_{t+1}, and continue recursively in this fashion. The value of y at step t gives the effect of a one-unit change in w_0 on y_t.

Although numerical simulation may be adequate for many circumstances, it is also useful to have a simple analytical characterization of $\partial y_{t+j}/\partial w_t$, which, we know from [1.2.11], is given by the (1, 1) element of $\mathbf{F}^j$. This is fairly easy to obtain in terms of the eigenvalues of the matrix $\mathbf{F}$. Recall that the eigenvalues of a matrix $\mathbf{F}$ are those numbers λ for which

$$|\mathbf{F} - \lambda \mathbf{I}_p| = 0. \qquad [1.2.12]$$

For example, for $p = 2$ the eigenvalues are the solutions to

$$\left| \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

or

$$\begin{vmatrix} (\phi_1 - \lambda) & \phi_2 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \phi_1\lambda - \phi_2 = 0. \qquad [1.2.13]$$

The two eigenvalues of $\mathbf{F}$ for a second-order difference equation are thus given by

$$\lambda_1 = \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2} \qquad [1.2.14]$$

$$\lambda_2 = \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2}. \qquad [1.2.15]$$

For a general pth-order system, the determinant in [1.2.12] is a pth-order polynomial in λ whose p solutions characterize the p eigenvalues of $\mathbf{F}$. This polynomial turns out to take a very similar form to [1.2.13]. The following result is proved in Appendix 1.A at the end of this chapter.

Proposition 1.1: *The eigenvalues of the matrix* $\mathbf{F}$ *defined in equation [1.2.3] are the values of* λ *that satisfy*

$$\lambda^p - \phi_1\lambda^{p-1} - \phi_2\lambda^{p-2} - \cdots - \phi_{p-1}\lambda - \phi_p = 0. \qquad [1.2.16]$$

Once we know the eigenvalues, it is straightforward to characterize the dynamic behavior of the system. First we consider the case when the eigenvalues of $\mathbf{F}$ are distinct; for example, we require that λ_1 and λ_2 in [1.2.14] and [1.2.15] be different numbers.

General Solution of a pth-Order Difference Equation with Distinct Eigenvalues

Recall[2] that if the eigenvalues of a $(p \times p)$ matrix $\mathbf{F}$ are distinct, there exists a nonsingular $(p \times p)$ matrix $\mathbf{T}$ such that

$$\mathbf{F} = \mathbf{T}\mathbf{\Lambda}\mathbf{T}^{-1} \tag{1.2.17}$$

where $\mathbf{\Lambda}$ is a $(p \times p)$ matrix with the eigenvalues of $\mathbf{F}$ along the principal diagonal and zeros elsewhere:

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_p \end{bmatrix}. \tag{1.2.18}$$

This enables us to characterize the dynamic multiplier (the (1, 1) element of $\mathbf{F}^j$ in [1.2.11]) very easily. For example, from [1.2.17] we can write $\mathbf{F}^2$ as

$$\begin{aligned} \mathbf{F}^2 &= \mathbf{T}\mathbf{\Lambda}\mathbf{T}^{-1} \times \mathbf{T}\mathbf{\Lambda}\mathbf{T}^{-1} \\ &= \mathbf{T} \times \mathbf{\Lambda} \times (\mathbf{T}^{-1}\mathbf{T}) \times \mathbf{\Lambda} \times \mathbf{T}^{-1} \\ &= \mathbf{T} \times \mathbf{\Lambda} \times \mathbf{I}_p \times \mathbf{\Lambda} \times \mathbf{T}^{-1} \\ &= \mathbf{T}\mathbf{\Lambda}^2\mathbf{T}^{-1}. \end{aligned}$$

The diagonal structure of $\mathbf{\Lambda}$ implies that $\mathbf{\Lambda}^2$ is also a diagonal matrix whose elements are the squares of the eigenvalues of $\mathbf{F}$:

$$\mathbf{\Lambda}^2 = \begin{bmatrix} \lambda_1^2 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_p^2 \end{bmatrix}.$$

More generally, we can characterize $\mathbf{F}^j$ in terms of the eigenvalues of $\mathbf{F}$ as

$$\begin{aligned} \mathbf{F}^j &= \underbrace{\mathbf{T}\mathbf{\Lambda}\mathbf{T}^{-1} \times \mathbf{T}\mathbf{\Lambda}\mathbf{T}^{-1} \times \cdots \times \mathbf{T}\mathbf{\Lambda}\mathbf{T}^{-1}}_{j \text{ terms}} \\ &= \mathbf{T} \times \mathbf{\Lambda} \times (\mathbf{T}^{-1}\mathbf{T}) \times \mathbf{\Lambda} \times (\mathbf{T}^{-1}\mathbf{T}) \times \cdots \times \mathbf{\Lambda} \times \mathbf{T}^{-1}, \end{aligned}$$

which simplifies to

$$\mathbf{F}^j = \mathbf{T}\mathbf{\Lambda}^j\mathbf{T}^{-1} \tag{1.2.19}$$

where

$$\mathbf{\Lambda}^j = \begin{bmatrix} \lambda_1^j & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^j & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_p^j \end{bmatrix}.$$

[2] See equation [A.4.24] in the Mathematical Review (Appendix A) at the end of the book.

Let t_{ij} denote the row i, column j element of $\mathbf{T}$ and let t^{ij} denote the row i, column j element of $\mathbf{T}^{-1}$. Equation [1.2.19] written out explicitly becomes

$$\mathbf{F}^j = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1p} \\ t_{21} & t_{22} & \cdots & t_{2p} \\ \vdots & \vdots & \cdots & \vdots \\ t_{p1} & t_{p2} & \cdots & t_{pp} \end{bmatrix} \begin{bmatrix} \lambda_1^j & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^j & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_p^j \end{bmatrix} \begin{bmatrix} t^{11} & t^{12} & \cdots & t^{1p} \\ t^{21} & t^{22} & \cdots & t^{2p} \\ \vdots & \vdots & \cdots & \vdots \\ t^{p1} & t^{p2} & \cdots & t^{pp} \end{bmatrix}$$

$$= \begin{bmatrix} t_{11}\lambda_1^j & t_{12}\lambda_2^j & \cdots & t_{1p}\lambda_p^j \\ t_{21}\lambda_1^j & t_{22}\lambda_2^j & \cdots & t_{2p}\lambda_p^j \\ \vdots & \vdots & \cdots & \vdots \\ t_{p1}\lambda_1^j & t_{p2}\lambda_2^j & \cdots & t_{pp}\lambda_p^j \end{bmatrix} \begin{bmatrix} t^{11} & t^{12} & \cdots & t^{1p} \\ t^{21} & t^{22} & \cdots & t^{2p} \\ \vdots & \vdots & \cdots & \vdots \\ t^{p1} & t^{p2} & \cdots & t^{pp} \end{bmatrix}$$

from which the (1, 1) element of $\mathbf{F}^j$ is given by

$$f_{11}^{(j)} = [t_{11}t^{11}]\lambda_1^j + [t_{12}t^{21}]\lambda_2^j + \cdots + [t_{1p}t^{p1}]\lambda_p^j$$

or

$$f_{11}^{(j)} = c_1\lambda_1^j + c_2\lambda_2^j + \cdots + c_p\lambda_p^j \qquad [1.2.20]$$

where

$$c_i = [t_{1i}t^{i1}]. \qquad [1.2.21]$$

Note that the sum of the c_i terms has the following interpretation:

$$c_1 + c_2 + \cdots + c_p = [t_{11}t^{11}] + [t_{12}t^{21}] + \cdots + [t_{1p}t^{p1}], \qquad [1.2.22]$$

which is the (1, 1) element of $\mathbf{T}\cdot\mathbf{T}^{-1}$. Since $\mathbf{T}\cdot\mathbf{T}^{-1}$ is just the $(p \times p)$ identity matrix, [1.2.22] implies that the c_i terms sum to unity:

$$c_1 + c_2 + \cdots + c_p = 1. \qquad [1.2.23]$$

Substituting [1.2.20] into [1.2.11] gives the form of the dynamic multiplier for a pth-order difference equation:

$$\frac{\partial y_{t+j}}{\partial w_t} = c_1\lambda_1^j + c_2\lambda_2^j + \cdots + c_p\lambda_p^j. \qquad [1.2.24]$$

Equation [1.2.24] characterizes the dynamic multiplier as a weighted average of each of the p eigenvalues raised to the jth power.

The following result provides a closed-form expression for the constants $(c_1, c_2, \ldots, c_p)$.

Proposition 1.2: *If the eigenvalues* $(\lambda_1, \lambda_2, \ldots, \lambda_p)$ *of the matrix* $\mathbf{F}$ *in [1.2.3] are distinct, then the magnitude* c_i *in [1.2.21] can be written*

$$c_i = \frac{\lambda_i^{p-1}}{\prod_{\substack{k=1 \\ k \neq i}}^{p} (\lambda_i - \lambda_k)}. \qquad [1.2.25]$$

To summarize, the pth-order difference equation [1.2.1] implies that

$$\begin{aligned} y_{t+j} = {} & f_{11}^{(j+1)}y_{t-1} + f_{12}^{(j+1)}y_{t-2} + \cdots + f_{1p}^{(j+1)}y_{t-p} \\ & + w_{t+j} + \psi_1 w_{t+j-1} + \psi_2 w_{t+j-2} + \cdots + \psi_{j-1}w_{t+1} + \psi_j w_t. \end{aligned} \qquad [1.2.26]$$

The dynamic multiplier

$$\frac{\partial y_{t+j}}{\partial w_t} = \psi_j \qquad [1.2.27]$$

is given by the (1, 1) element of $\mathbf{F}^j$:

$$\psi_j = f_{11}^{(j)}. \qquad [1.2.28]$$

A closed-form expression for ψ_j can be obtained by finding the eigenvalues of $\mathbf{F}$, or the values of λ satisfying [1.2.16]. Denoting these p values by $(\lambda_1, \lambda_2, \ldots, \lambda_p)$ and assuming them to be distinct, the dynamic multiplier is given by

$$\psi_j = c_1\lambda_1^j + c_2\lambda_2^j + \cdots + c_p\lambda_p^j \qquad [1.2.29]$$

where $(c_1, c_2, \ldots, c_p)$ is a set of constants summing to unity given by expression [1.2.25].

For a first-order system ($p = 1$), this rule would have us solve [1.2.16],

$$\lambda - \phi_1 = 0,$$

which has the single solution

$$\lambda_1 = \phi_1. \qquad [1.2.30]$$

According to [1.2.29], the dynamic multiplier is given by

$$\frac{\partial y_{t+j}}{\partial w_t} = c_1\lambda_1^j. \qquad [1.2.31]$$

From [1.2.23], $c_1 = 1$. Substituting this and [1.2.30] into [1.2.31] gives

$$\frac{\partial y_{t+j}}{\partial w_t} = \phi_1^j,$$

or the same result found in Section 1.1.

For higher-order systems, [1.2.29] allows a variety of more complicated dynamics. Suppose first that all the eigenvalues of $\mathbf{F}$ (or solutions to [1.2.16]) are real. This would be the case, for example, if $p = 2$ and $\phi_1^2 + 4\phi_2 > 0$ in the solutions [1.2.14] and [1.2.15] for the second-order system. If, furthermore, all of the eigenvalues are less than 1 in absolute value, then the system is stable, and its dynamics are represented as a weighted average of decaying exponentials or decaying exponentials oscillating in sign. For example, consider the following second-order difference equation:

$$y_t = 0.6y_{t-1} + 0.2y_{t-2} + w_t.$$

From equations [1.2.14] and [1.2.15], the eigenvalues of this system are given by

$$\lambda_1 = \frac{0.6 + \sqrt{(0.6)^2 + 4(0.2)}}{2} = 0.84$$

$$\lambda_2 = \frac{0.6 - \sqrt{(0.6)^2 + 4(0.2)}}{2} = -0.24.$$

From [1.2.25], we have

$$c_1 = \lambda_1/(\lambda_1 - \lambda_2) = 0.778$$

$$c_2 = \lambda_2/(\lambda_2 - \lambda_1) = 0.222.$$

The dynamic multiplier for this system,

$$\frac{\partial y_{t+j}}{\partial w_t} = c_1\lambda_1^j + c_2\lambda_2^j,$$

is plotted as a function of j in panel (a) of Figure 1.4.[3] Note that as j becomes larger, the pattern is dominated by the larger eigenvalue (λ_1), approximating a simple geometric decay at rate λ_1.

If the eigenvalues (the solutions to [1.2.16]) are real but at least one is greater than unity in absolute value, the system is explosive. If λ_1 denotes the eigenvalue that is largest in absolute value, the dynamic multiplier is eventually dominated by an exponential function of that eigenvalue:

$$\lim_{j\to\infty} \frac{\partial y_{t+j}}{\partial w_t} \cdot \frac{1}{\lambda_1^j} = c_1.$$

Other interesting possibilities arise if some of the eigenvalues are complex. Whenever this is the case, they appear as complex conjugates. For example, if $p = 2$ and $\phi_1^2 + 4\phi_2 < 0$, then the solutions λ_1 and λ_2 in [1.2.14] and [1.2.15] are complex conjugates. Suppose that λ_1 and λ_2 are complex conjugates, written as

$$\lambda_1 = a + bi \qquad [1.2.32]$$

$$\lambda_2 = a - bi. \qquad [1.2.33]$$

For the $p = 2$ case of [1.2.14] and [1.2.15], we would have

$$a = \phi_1/2 \qquad [1.2.34]$$

$$b = (1/2)\sqrt{-\phi_1^2 - 4\phi_2}. \qquad [1.2.35]$$

Our goal is to characterize the contribution to the dynamic multiplier $c_1\lambda_1^j$ when λ_1 is a complex number as in [1.2.32]. Recall that to raise a complex number to a power, we rewrite [1.2.32] in polar coordinate form:

$$\lambda_1 = R\cdot[\cos(\theta) + i\cdot\sin(\theta)], \qquad [1.2.36]$$

where θ and R are defined in terms of a and b by the following equations:

$$R = \sqrt{a^2 + b^2}$$

$$\cos(\theta) = a/R$$

$$\sin(\theta) = b/R.$$

Note that R is equal to the modulus of the complex number λ_1.

The eigenvalue λ_1 in [1.2.36] can be written as[4]

$$\lambda_1 = R[e^{i\theta}],$$

and so

$$\lambda_1^j = R^j[e^{i\theta j}] = R^j[\cos(\theta j) + i\cdot\sin(\theta j)]. \qquad [1.2.37]$$

Analogously, if λ_2 is the complex conjugate of λ_1, then

$$\lambda_2 = R[\cos(\theta) - i\cdot\sin(\theta)],$$

which can be written[5]

$$\lambda_2 = R[e^{-i\theta}].$$

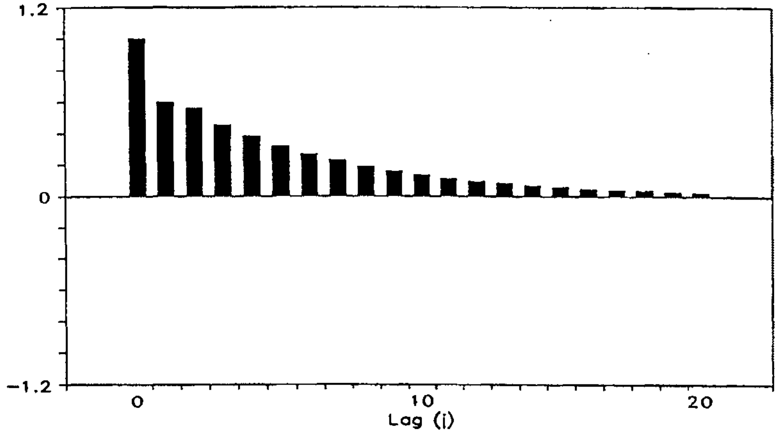
Thus

$$\lambda_2^j = R^j[e^{-i\theta j}] = R^j[\cos(\theta j) - i\cdot\sin(\theta j)]. \qquad [1.2.38]$$

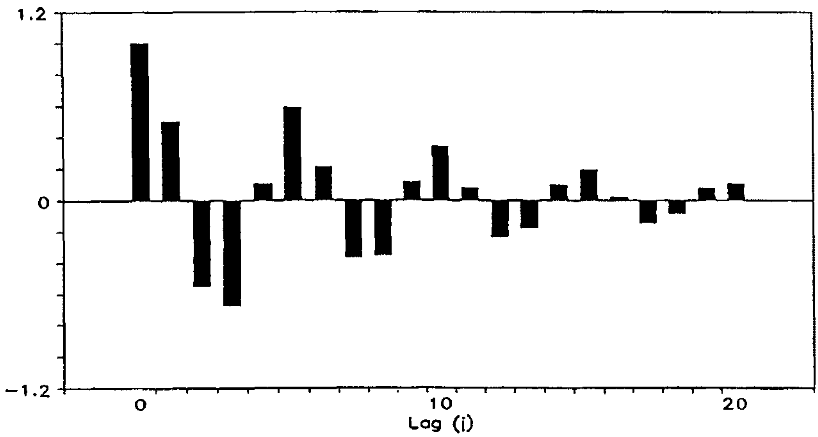
[3]Again, if one's purpose is solely to generate a numerical plot as in Figure 1.4, the easiest approach is numerical simulation of the system.

[4]See equation [A.3.25] in the Mathematical Review (Appendix A) at the end of the book.

[5]See equation [A.3.26].



(a) $\phi_1 = 0.6$, $\phi_2 = 0.2$



(b) $\phi_1 = 0.5$, $\phi_2 = -0.8$

FIGURE 1.4 Dynamic multiplier for second-order difference equation for different values of ϕ_1 and ϕ_2 (plot of $\partial y_{t+j}/\partial w_t$ as a function of the lag j).

Substituting [1.2.37] and [1.2.38] into [1.2.29] gives the contribution of the complex conjugates to the dynamic multiplier $\partial y_{t+j}/\partial w_t$:

$$\begin{aligned} c_1\lambda_1^j + c_2\lambda_2^j &= c_1R^j[\cos(\theta j) + i\cdot\sin(\theta j)] + c_2R^j[\cos(\theta j) - i\cdot\sin(\theta j)] \\ &= [c_1 + c_2]\cdot R^j\cdot\cos(\theta j) + i\cdot[c_1 - c_2]\cdot R^j\cdot\sin(\theta j). \end{aligned} \quad [1.2.39]$$

The appearance of the imaginary number i in [1.2.39] may seem a little troubling. After all, this calculation was intended to give the effect of a change in the real-valued variable w_t on the real-valued variable y_{t+j} as predicted by the real-valued system [1.2.1], and it would be odd indeed if the correct answer involved the imaginary number i! Fortunately, it turns out from [1.2.25] that if λ_1 and λ_2 are complex conjugates, then c_1 and c_2 are complex conjugates; that is, they can be written as

$$c_1 = \alpha + \beta i$$
$$c_2 = \alpha - \beta i$$

for some real numbers α and β. Substituting these expressions into [1.2.39] yields

$$\begin{aligned} c_1\lambda_1^j + c_2\lambda_2^j &= [(\alpha + \beta i) + (\alpha - \beta i)]\cdot R^j\cos(\theta j) + i\cdot[(\alpha + \beta i) - (\alpha - \beta i)]\cdot R^j\sin(\theta j) \\ &= [2\alpha]\cdot R^j\cos(\theta j) + i\cdot[2\beta i]\cdot R^j\sin(\theta j) \\ &= 2\alpha R^j\cos(\theta j) - 2\beta R^j\sin(\theta j), \end{aligned}$$

which is strictly real.

Thus, when some of the eigenvalues are complex, they contribute terms proportional to $R^j\cos(\theta j)$ and $R^j\sin(\theta j)$ to the dynamic multiplier $\partial y_{t+j}/\partial w_t$. Note that if $R = 1$—that is, if the complex eigenvalues have unit modulus—the multipliers are periodic sine and cosine functions of j. A given increase in w_t increases y_{t+j} for some ranges of j and decreases y_{t+j} over other ranges, with the impulse never dying out as $j \to \infty$. If the complex eigenvalues are less than 1 in modulus ($R < 1$), the impulse again follows a sinusoidal pattern though its amplitude decays at the rate R^j. If the complex eigenvalues are greater than 1 in modulus ($R > 1$), the amplitude of the sinusoids explodes at the rate R^j.

For an example of dynamic behavior characterized by decaying sinusoids, consider the second-order system

$$y_t = 0.5y_{t-1} - 0.8y_{t-2} + w_t.$$

The eigenvalues for this system are given from [1.2.14] and [1.2.15]:

$$\lambda_1 = \frac{0.5 + \sqrt{(0.5)^2 - 4(0.8)}}{2} = 0.25 + 0.86i$$
$$\lambda_2 = \frac{0.5 - \sqrt{(0.5)^2 - 4(0.8)}}{2} = 0.25 - 0.86i,$$

with modulus

$$R = \sqrt{(0.25)^2 + (0.86)^2} = 0.9.$$

Since $R < 1$, the dynamic multiplier follows a pattern of damped oscillation plotted in panel (b) of Figure 1.4. The frequency[6] of these oscillations is given by the parameter θ in [1.2.39], which was defined implicitly by

$$\cos(\theta) = a/R = (0.25)/(0.9) = 0.28$$

or

$$\theta = 1.29.$$

The cycles associated with the dynamic multiplier function [1.2.39] thus have a period of

$$\frac{2\pi}{\theta} = \frac{(2)(3.14159)}{1.29} = 4.9;$$

that is, the peaks in the pattern in panel (b) of Figure 1.4 appear about five periods apart.

[6]See Section A.1 of the Mathematical Review (Appendix A) at the end of the book for a discussion of the frequency and period of a sinusoidal function.

Solution of a Second-Order Difference Equation with Distinct Eigenvalues

The second-order difference equation ($p = 2$) comes up sufficiently often that it is useful to summarize the properties of the solution as a general function of ϕ_1 and ϕ_2, which we now do.[7]

The eigenvalues λ_1 and λ_2 in [1.2.14] and [1.2.15] are complex whenever

$$\phi_1^2 + 4\phi_2 < 0,$$

or whenever (ϕ_1, ϕ_2) lies below the parabola indicated in Figure 1.5. For the case of complex eigenvalues, the modulus R satisfies

$$R^2 = a^2 + b^2,$$

or, from [1.2.34] and [1.2.35],

$$R^2 = (\phi_1/2)^2 - (\phi_1^2 + 4\phi_2)/4 = -\phi_2.$$

Thus, a system with complex eigenvalues is explosive whenever $\phi_2 < -1$. Also, when the eigenvalues are complex, the frequency of oscillations is given by

$$\theta = \cos^{-1}(a/R) = \cos^{-1}[\phi_1/(2\sqrt{-\phi_2})],$$

where "$\cos^{-1}(x)$" denotes the inverse of the cosine function, or the radian measure of an angle whose cosine is x.

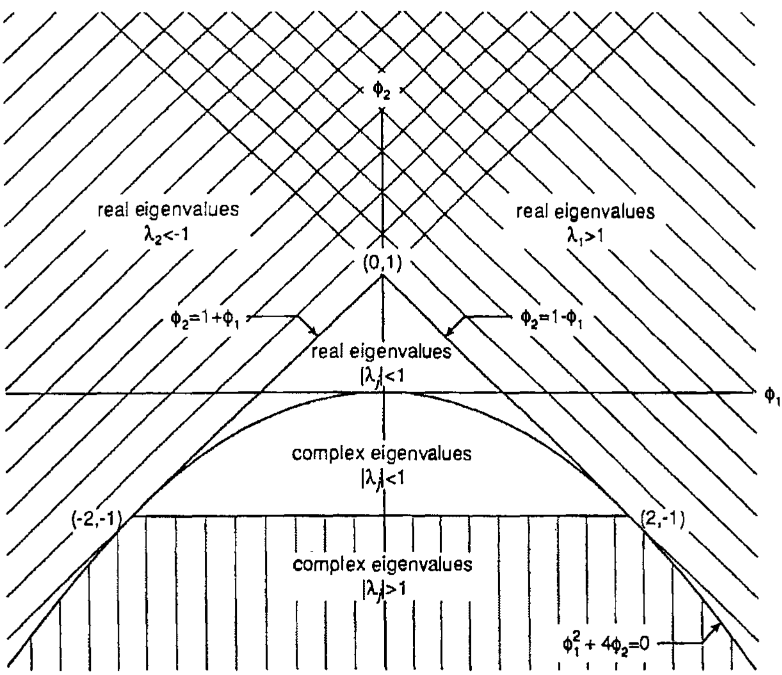


FIGURE 1.5 Summary of dynamics for a second-order difference equation.

[7]This discussion closely follows Sargent (1987, pp. 188–89).

For the case of real eigenvalues, the arithmetically larger eigenvalue (λ_1) will be greater than unity whenever

$$\frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2} > 1$$

or

$$\sqrt{\phi_1^2 + 4\phi_2} > 2 - \phi_1.$$

Assuming that λ_1 is real, the left side of this expression is a positive number and the inequality would be satisfied for any value of $\phi_1 > 2$. If, on the other hand, $\phi_1 < 2$, we can square both sides to conclude that λ_1 will exceed unity whenever

$$\phi_1^2 + 4\phi_2 > 4 - 4\phi_1 + \phi_1^2$$

or

$$\phi_2 > 1 - \phi_1.$$

Thus, in the real region, λ_1 will be greater than unity either if $\phi_1 > 2$ or if (ϕ_1, ϕ_2) lies northeast of the line $\phi_2 = 1 - \phi_1$ in Figure 1.5. Similarly, with real eigenvalues, the arithmetically smaller eigenvalue (λ_2) will be less than -1 whenever

$$\frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2} < -1$$

$$-\sqrt{\phi_1^2 + 4\phi_2} < -2 - \phi_1$$

$$\sqrt{\phi_1^2 + 4\phi_2} > 2 + \phi_1.$$

Again, if $\phi_1 < -2$, this must be satisfied, and in the case when $\phi_1 > -2$, we can square both sides:

$$\phi_1^2 + 4\phi_2 > 4 + 4\phi_1 + \phi_1^2$$

$$\phi_2 > 1 + \phi_1.$$

Thus, in the real region, λ_2 will be less than -1 if either $\phi_1 < -2$ or (ϕ_1, ϕ_2) lies to the northwest of the line $\phi_2 = 1 + \phi_1$ in Figure 1.5.

The system is thus stable whenever (ϕ_1, ϕ_2) lies within the triangular region of Figure 1.5.

General Solution of a pth-Order Difference Equation with Repeated Eigenvalues

In the more general case of a difference equation for which $\mathbf{F}$ has repeated eigenvalues and $s < p$ linearly independent eigenvectors, result [1.2.17] is generalized by using the Jordan decomposition,

$$\mathbf{F} = \mathbf{MJM}^{-1} \qquad [1.2.40]$$

where $\mathbf{M}$ is a $(p \times p)$ matrix and $\mathbf{J}$ takes the form

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{J}_s \end{bmatrix}$$

with

$$\mathbf{J}_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_i \end{bmatrix} \qquad [1.2.41]$$

for λ_i an eigenvalue of $\mathbf{F}$. If [1.2.17] is replaced by [1.2.40], then equation [1.2.19] generalizes to

$$\mathbf{F}^j = \mathbf{M}\mathbf{J}^j\mathbf{M}^{-1} \qquad [1.2.42]$$

where

$$\mathbf{J}^j = \begin{bmatrix} \mathbf{J}_1^j & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2^j & \cdots & \mathbf{0} \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{J}_s^j \end{bmatrix}.$$

Moreover, from [1.2.41], if $\mathbf{J}_i$ is of dimension $(n_i \times n_i)$, then[8]

$$\mathbf{J}_i^j = \begin{bmatrix} \lambda_i^j & \binom{j}{1}\lambda_i^{j-1} & \binom{j}{2}\lambda_i^{j-2} & \cdots & \binom{j}{n_i-1}\lambda_i^{j-n_i+1} \\ 0 & \lambda_i^j & \binom{j}{1}\lambda_i^{j-1} & \cdots & \binom{j}{n_i-2}\lambda_i^{j-n_i+2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i^j \end{bmatrix} \qquad [1.2.43]$$

where

$$\binom{j}{n} \equiv \begin{cases} \dfrac{j(j-1)(j-2)\cdots(j-n+1)}{n(n-1)\cdots 3\cdot 2\cdot 1} & \text{for } j \geq n \\ 0 & \text{otherwise.} \end{cases}$$

Equation [1.2.43] may be verified by induction by multiplying [1.2.41] by [1.2.43] and noticing that $\binom{j}{n} + \binom{j}{n-1} = \binom{j+1}{n}$.

For example, consider again the second-order difference equation, this time with repeated roots. Then

$$\mathbf{F}^j = \mathbf{M}\begin{bmatrix} \lambda^j & j\lambda^{j-1} \\ 0 & \lambda^j \end{bmatrix}\mathbf{M}^{-1},$$

so that the dynamic multiplier takes the form

$$\frac{\partial y_{t+j}}{\partial w_t} = f_{11}^{(j)} = k_1\lambda^j + k_2 j\lambda^{j-1}.$$

Long-Run and Present-Value Calculations

If the eigenvalues are all less than 1 in modulus, then $\mathbf{F}^j$ in [1.2.9] goes to zero as j becomes large. If all values of w and y are taken to be bounded, we can

[8]This expression is taken from Chiang (1980, p. 444).

think of a "solution" of y_t in terms of the infinite history of w,

$$y_t = w_t + \psi_1 w_{t-1} + \psi_2 w_{t-2} + \psi_3 w_{t-3} + \cdots, \tag{1.2.44}$$

where ψ_j is given by the (1, 1) element of $\mathbf{F}^j$ and takes the particular form of [1.2.29] in the case of distinct eigenvalues.

It is also straightforward to calculate the effect on the present value of y of a transitory increase in w. This is simplest to find if we first consider the slightly more general problem of the hypothetical consequences of a change in any element of the vector $\mathbf{v}_t$ on any element of $\boldsymbol{\xi}_{t+j}$ in a general system of the form of [1.2.5]. The answer to this more general problem can be inferred immediately from [1.2.9]:

$$\frac{\partial \boldsymbol{\xi}_{t+j}}{\partial \mathbf{v}_t'} = \mathbf{F}^j. \tag{1.2.45}$$

The true dynamic multiplier of interest, $\partial y_{t+j}/\partial w_t$, is just the (1, 1) element of the $(p \times p)$ matrix in [1.2.45]. The effect on the present value of $\boldsymbol{\xi}$ of a change in v is given by

$$\frac{\partial \sum_{j=0}^{\infty} \beta^j \boldsymbol{\xi}_{t+j}}{\partial \mathbf{v}_t'} = \sum_{j=0}^{\infty} \beta^j \mathbf{F}^j = (\mathbf{I}_p - \beta\mathbf{F})^{-1}, \tag{1.2.46}$$

provided that the eigenvalues of $\mathbf{F}$ are all less than β^{-1} in modulus. The effect on the present value of y of a change in w,

$$\frac{\partial \sum_{j=0}^{\infty} \beta^j y_{t+j}}{\partial w_t},$$

is thus the (1, 1) element of the $(p \times p)$ matrix in [1.2.46]. This value is given by the following proposition.

Proposition 1.3: If the eigenvalues of the $(p \times p)$ matrix $\mathbf{F}$ defined in [1.2.3] are all less than β^{-1} in modulus, then the matrix $(\mathbf{I}_p - \beta\mathbf{F})^{-1}$ exists and the effect of w on the present value of y is given by its (1, 1) element:

$$1/(1 - \phi_1\beta - \phi_2\beta^2 - \cdots - \phi_{p-1}\beta^{p-1} - \phi_p\beta^p).$$

Note that Proposition 1.3 includes the earlier result for a first-order system (equation [1.1.14]) as a special case.

The cumulative effect of a one-time change in w_t on $y_t, y_{t+1}, \ldots$ can be considered a special case of Proposition 1.3 with no discounting. Setting $\beta = 1$ in Proposition 1.3 shows that, provided the eigenvalues of $\mathbf{F}$ are all less than 1 in modulus, the cumulative effect of a one-time change in w on y is given by

$$\sum_{j=0}^{\infty} \frac{\partial y_{t+j}}{\partial w_t} = 1/(1 - \phi_1 - \phi_2 - \cdots - \phi_p). \tag{1.2.47}$$

Notice again that [1.2.47] can alternatively be interpreted as giving the eventual long-run effect on y of a permanent change in w:

$$\lim_{j\to\infty} \frac{\partial y_{t+j}}{\partial w_t} + \frac{\partial y_{t+j}}{\partial w_{t+1}} + \frac{\partial y_{t+j}}{\partial w_{t+2}} + \cdots + \frac{\partial y_{t+j}}{\partial w_{t+j}} = 1/(1 - \phi_1 - \phi_2 - \cdots - \phi_p).$$

APPENDIX 1.A. *Proofs of Chapter 1 Propositions*

■ **Proof of Proposition 1.1.** The eigenvalues of $\mathbf{F}$ satisfy

$$|\mathbf{F} - \lambda \mathbf{I}_p| = 0. \qquad [1.A.1]$$

For the matrix $\mathbf{F}$ defined in equation [1.2.3], this determinant would be

$$\left| \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix} \right|$$

$$= \begin{vmatrix} (\phi_1 - \lambda) & \phi_2 & \phi_3 & \cdots & \phi_{p-1} & \phi_p \\ 1 & -\lambda & 0 & \cdots & 0 & 0 \\ 0 & 1 & -\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\lambda \end{vmatrix}. \qquad [1.A.2]$$

Recall that if we multiply a column of a matrix by a constant and add the result to another column, the determinant of the matrix is unchanged. If we multiply the pth column of the matrix in [1.A.2] by $(1/\lambda)$ and add the result to the $(p-1)$th column, the result is a matrix with the same determinant as that in [1.A.2]:

$$|\mathbf{F} - \lambda \mathbf{I}_p| = \begin{vmatrix} \phi_1 - \lambda & \phi_2 & \phi_3 & \cdots & \phi_{p-2} & \phi_{p-1} + (\phi_p/\lambda) & \phi_p \\ 1 & -\lambda & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -\lambda & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\lambda & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & -\lambda \end{vmatrix}.$$

Next, multiply the $(p-1)$th column by $(1/\lambda)$ and add the result to the $(p-2)$th column:

$$|\mathbf{F} - \lambda \mathbf{I}_p| = \begin{vmatrix} \phi_1 - \lambda & \phi_2 & \phi_3 & \cdots & \phi_{p-2} + \phi_{p-1}/\lambda + \phi_p/\lambda^2 & \phi_{p-1} + \phi_p/\lambda & \phi_p \\ 1 & -\lambda & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -\lambda & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -\lambda & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & -\lambda \end{vmatrix}.$$

Continuing in this fashion shows [1.A.1] to be equivalent to the determinant of the following upper triangular matrix:

$$|\mathbf{F} - \lambda \mathbf{I}_p| = \begin{vmatrix} \phi_1 - \lambda + \phi_2/\lambda + \phi_3/\lambda^2 + \cdots + \phi_p/\lambda^{p-1} & \phi_2 + \phi_3/\lambda + \phi_4/\lambda^2 + \cdots + \phi_p/\lambda^{p-2} & \cdots & \phi_{p-1} + \phi_p/\lambda & \phi_p \\ 0 & -\lambda & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & -\lambda & 0 \\ 0 & 0 & \cdots & 0 & -\lambda \end{vmatrix}.$$

But the determinant of an upper triangular matrix is simply the product of the terms along the principal diagonal:

$$\begin{aligned} |\mathbf{F} - \lambda \mathbf{I}_p| &= [\phi_1 - \lambda + \phi_2/\lambda + \phi_3/\lambda^2 + \cdots + \phi_p/\lambda^{p-1}] \cdot [-\lambda]^{p-1} \\ &= (-1)^p \cdot [\lambda^p - \phi_1\lambda^{p-1} - \phi_2\lambda^{p-2} - \cdots - \phi_p]. \end{aligned} \qquad [1.A.3]$$

The eigenvalues of **F** are thus the values of λ for which [1.A.3] is zero, or for which

$$\lambda^p - \phi_1\lambda^{p-1} - \phi_2\lambda^{p-2} - \cdots - \phi_p = 0,$$

as asserted in Proposition 1.1. ■

■ **Proof of Proposition 1.2.** Assuming that the eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_p)$ are distinct, the matrix **T** in equation [1.2.17] can be constructed from the eigenvectors of **F**. Let $\mathbf{t}_i$ denote the following $(p \times 1)$ vector,

$$\mathbf{t}_i = \begin{bmatrix} \lambda_i^{p-1} \\ \lambda_i^{p-2} \\ \lambda_i^{p-3} \\ \vdots \\ \lambda_i^1 \\ 1 \end{bmatrix}, \qquad [1.A.4]$$

where λ_i denotes the ith eigenvalue of **F**. Notice

$$\mathbf{F}\mathbf{t}_i = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_i^{p-1} \\ \lambda_i^{p-2} \\ \lambda_i^{p-3} \\ \vdots \\ \lambda_i^1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \phi_1\lambda_i^{p-1} + \phi_2\lambda_i^{p-2} + \phi_3\lambda_i^{p-3} + \cdots + \phi_{p-1}\lambda_i + \phi_p \\ \lambda_i^{p-1} \\ \lambda_i^{p-2} \\ \vdots \\ \lambda_i^2 \\ \lambda_i \end{bmatrix}. \qquad [1.A.5]$$

Since λ_i is an eigenvalue of **F**, it satisfies [1.2.16]:

$$\lambda_i^p - \phi_1\lambda_i^{p-1} - \phi_2\lambda_i^{p-2} - \cdots - \phi_{p-1}\lambda_i - \phi_p = 0. \qquad [1.A.6]$$

Substituting [1.A.6] into [1.A.5] reveals

$$\mathbf{F}\mathbf{t}_i = \begin{bmatrix} \lambda_i^p \\ \lambda_i^{p-1} \\ \lambda_i^{p-2} \\ \vdots \\ \lambda_i^2 \\ \lambda_i \end{bmatrix} = \lambda_i \begin{bmatrix} \lambda_i^{p-1} \\ \lambda_i^{p-2} \\ \lambda_i^{p-3} \\ \vdots \\ \lambda_i^1 \\ 1 \end{bmatrix}$$

or

$$\mathbf{F}\mathbf{t}_i = \lambda_i\mathbf{t}_i. \qquad [1.A.7]$$

Thus $\mathbf{t}_i$ is an eigenvector of **F** associated with the eigenvalue λ_i.

We can calculate the matrix **T** by combining the eigenvectors $(\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_p)$ into a $(p \times p)$ matrix

$$\mathbf{T} = [\mathbf{t}_1 \quad \mathbf{t}_2 \quad \cdots \quad \mathbf{t}_p]. \qquad [1.A.8]$$

To calculate the particular values for c_i in equation [1.2.21], recall that $\mathbf{T}^{-1}$ is characterized by

$$\mathbf{T}\mathbf{T}^{-1} = \mathbf{I}_p, \qquad [1.A.9]$$

where T is given by [1.A.4] and [1.A.8]. Writing out the first column of the matrix system of equations [1.A.9] explicitly, we have

$$\begin{bmatrix} \lambda_1^{p-1} & \lambda_2^{p-1} & \cdots & \lambda_p^{p-1} \\ \lambda_1^{p-2} & \lambda_2^{p-2} & \cdots & \lambda_p^{p-2} \\ \lambda_1^{p-3} & \lambda_2^{p-3} & \cdots & \lambda_p^{p-3} \\ \vdots & \vdots & \cdots & \vdots \\ \lambda_1^1 & \lambda_2^1 & \cdots & \lambda_p^1 \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} t^{11} \\ t^{21} \\ t^{31} \\ \vdots \\ t^{p-1,1} \\ t^{p1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.$$

This gives a system of p linear equations in the p unknowns $(t^{11}, t^{21}, \ldots, t^{p1})$. Provided that the λ_i are all distinct, the solution can be shown to be[9]

$$t^{11} = \frac{1}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)\cdots(\lambda_1 - \lambda_p)}$$

$$t^{21} = \frac{1}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)\cdots(\lambda_2 - \lambda_p)}$$

$$\vdots$$

$$t^{p1} = \frac{1}{(\lambda_p - \lambda_1)(\lambda_p - \lambda_2)\cdots(\lambda_p - \lambda_{p-1})}.$$

Substituting these values into [1.2.21] gives equation [1.2.25]. ■

■ **Proof of Proposition 1.3.** The first claim in this proposition is that if the eigenvalues of $\mathbf{F}$ are less than β^{-1} in modulus, then the inverse of $(\mathbf{I}_p - \beta\mathbf{F})$ exists. Suppose the inverse of $(\mathbf{I}_p - \beta\mathbf{F})$ did not exist. Then the determinant $|\mathbf{I}_p - \beta\mathbf{F}|$ would have to be zero. But

$$|\mathbf{I}_p - \beta\mathbf{F}| = |-\beta \cdot [\mathbf{F} - \beta^{-1}\mathbf{I}_p]| = (-\beta)^p|\mathbf{F} - \beta^{-1}\mathbf{I}_p|,$$

so that $|\mathbf{F} - \beta^{-1}\mathbf{I}_p|$ would have to be zero whenever the inverse of $(\mathbf{I}_p - \beta\mathbf{F})$ fails to exist. But this would mean that β^{-1} is an eigenvalue of $\mathbf{F}$, which is ruled out by the assumption that all eigenvalues of $\mathbf{F}$ are strictly less than β^{-1} in modulus. Thus, the matrix $\mathbf{I}_p - \beta\mathbf{F}$ must be nonsingular.

Since $[\mathbf{I}_p - \beta\mathbf{F}]^{-1}$ exists, it satisfies the equation

$$[\mathbf{I}_p - \beta\mathbf{F}]^{-1}[\mathbf{I}_p - \beta\mathbf{F}] = \mathbf{I}_p. \qquad [1.A.10]$$

Let x_{ij} denote the row i, column j element of $[\mathbf{I}_p - \beta\mathbf{F}]^{-1}$, and write [1.A.10] as

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & \cdots & \vdots \\ x_{p1} & x_{p2} & \cdots & x_{pp} \end{bmatrix} \begin{bmatrix} 1 - \beta\phi_1 & -\beta\phi_2 & \cdots & -\beta\phi_{p-1} & -\beta\phi_p \\ -\beta & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & -\beta & 1 \end{bmatrix} \qquad [1.A.11]$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

The task is then to find the (1, 1) element of $[\mathbf{I}_p - \beta\mathbf{F}]^{-1}$, that is, to find the value of x_{11}. To do this we need only consider the first row of equations in [1.A.11]:

$$[x_{11} \;\; x_{12} \;\; \cdots \;\; x_{1p}] \begin{bmatrix} 1 - \beta\phi_1 & -\beta\phi_2 & \cdots & -\beta\phi_{p-1} & -\beta\phi_p \\ -\beta & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & -\beta & 1 \end{bmatrix}$$

$$= [1 \;\; 0 \;\; \cdots \;\; 0 \;\; 0]. \qquad [1.A.12]$$

[9]See Lemma 2 of Chiang (1980, p. 144).

Consider postmultiplying this system of equations by a matrix with 1s along the principal diagonal, β in the row p, column $p - 1$ position, and 0s elsewhere:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & \beta & 1 \end{bmatrix}.$$

The effect of this operation is to multiply the pth column of a matrix by β and add the result to the $(p - 1)$th column:

$$[x_{11} \quad x_{12} \cdots x_{1p}] \begin{bmatrix} 1 - \beta\phi_1 & -\beta\phi_2 & \cdots & -\beta\phi_{p-1} - \beta^2\phi_p & -\beta\phi_p \\ -\beta & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} = [1 \quad 0 \cdots 0 \quad 0].$$

Next multiply the $(p - 1)$th column by β and add the result to the $(p - 2)$th column. Proceeding in this fashion, we arrive at

$$[x_{11} \quad x_{12} \quad \cdots \quad x_{1p}] \times$$
$$\begin{bmatrix} 1-\beta\phi_1-\beta^2\phi_2-\cdots-\beta^{p-1}\phi_{p-1}-\beta^p\phi_p & -\beta\phi_2-\beta^2\phi_3-\cdots-\beta^{p-1}\phi_p & \cdots & -\beta\phi_{p-1}-\beta^2\phi_p & -\beta\phi_p \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$
$$= [1 \quad 0 \quad \cdots \quad 0 \quad 0]. \qquad [1.A.13]$$

The first equation in [1.A.13] states that

$$x_{11} \cdot (1 - \beta\phi_1 - \beta^2\phi_2 - \cdots - \beta^{p-1}\phi_{p-1} - \beta^p\phi_p) = 1$$

or

$$x_{11} = 1/(1 - \beta\phi_1 - \beta^2\phi_2 - \cdots - \beta^p\phi_p),$$

as claimed in Proposition 1.3. ■

Chapter 1 References

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2 Lag Operators

2.1. *Introduction*

The previous chapter analyzed the dynamics of linear difference equations using matrix algebra. This chapter develops some of the same results using time series operators. We begin with some introductory remarks on some useful time series operators.

A time series is a collection of observations indexed by the date of each observation. Usually we have collected data beginning at some particular date (say, $t = 1$) and ending at another (say, $t = T$):

$$(y_1, y_2, \ldots, y_T).$$

We often imagine that we could have obtained earlier observations (y_0, y_{-1}, $y_{-2}, \ldots$) or later observations ($y_{T+1}, y_{T+2}, \ldots$) had the process been observed for more time. The observed sample ($y_1, y_2, \ldots, y_T$) could then be viewed as a finite segment of a doubly infinite sequence, denoted $\{y_t\}_{t=-\infty}^{\infty}$:

$$\{y_t\}_{t=-\infty}^{\infty} = \{\ldots, y_{-1}, y_0, \underbrace{y_1, y_2, \ldots, y_T}_{\text{observed sample}}, y_{T+1}, y_{T+2}, \ldots\}.$$

Typically, a time series $\{y_t\}_{t=-\infty}^{\infty}$ is identified by describing the tth element. For example, a *time trend* is a series whose value at date t is simply the date of the observation:

$$y_t = t.$$

We could also consider a time series in which each element is equal to a constant c, regardless of the date of the observation t:

$$y_t = c.$$

Another important time series is a *Gaussian white noise process*, denoted

$$y_t = \varepsilon_t,$$

where $\{\varepsilon_t\}_{t=-\infty}^{\infty}$ is a sequence of independent random variables each of which has a $N(0, \sigma^2)$ distribution.

We are used to thinking of a function such as $y = f(x)$ or $y = g(x, w)$ as an operation that accepts as input a number (x) or group of numbers (x, w) and produces the output (y). A time series *operator* transforms one time series or group