LASSO, Ridge, and Elastic Net

Summary

For all examples:

- true model is $\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$
- where $\boldsymbol{\epsilon} \sim N_n(\mathbf{0}, \boldsymbol{I})$

Example 1

Small signal. Lots of noise.

- $\boldsymbol{\beta} = (\underbrace{1, \ldots, 1}_{15}, \underbrace{0, \ldots, 0}_{4085})^T$
- $p = 5000 > n = 1000$
- Uncorrelated predictors:
  - $\boldsymbol{X}_i \overset{\text{iid}}{\sim} N(\mathbf{0}, \boldsymbol{I})$

Generate Data

```
library(MASS)  # Package needed to generate correlated precictors
library(glmnet)  # Package to fit ridge/lasso/elastic net models
```

```
## Loading required package: Matrix
## Loaded glmnet 1.9-8
```

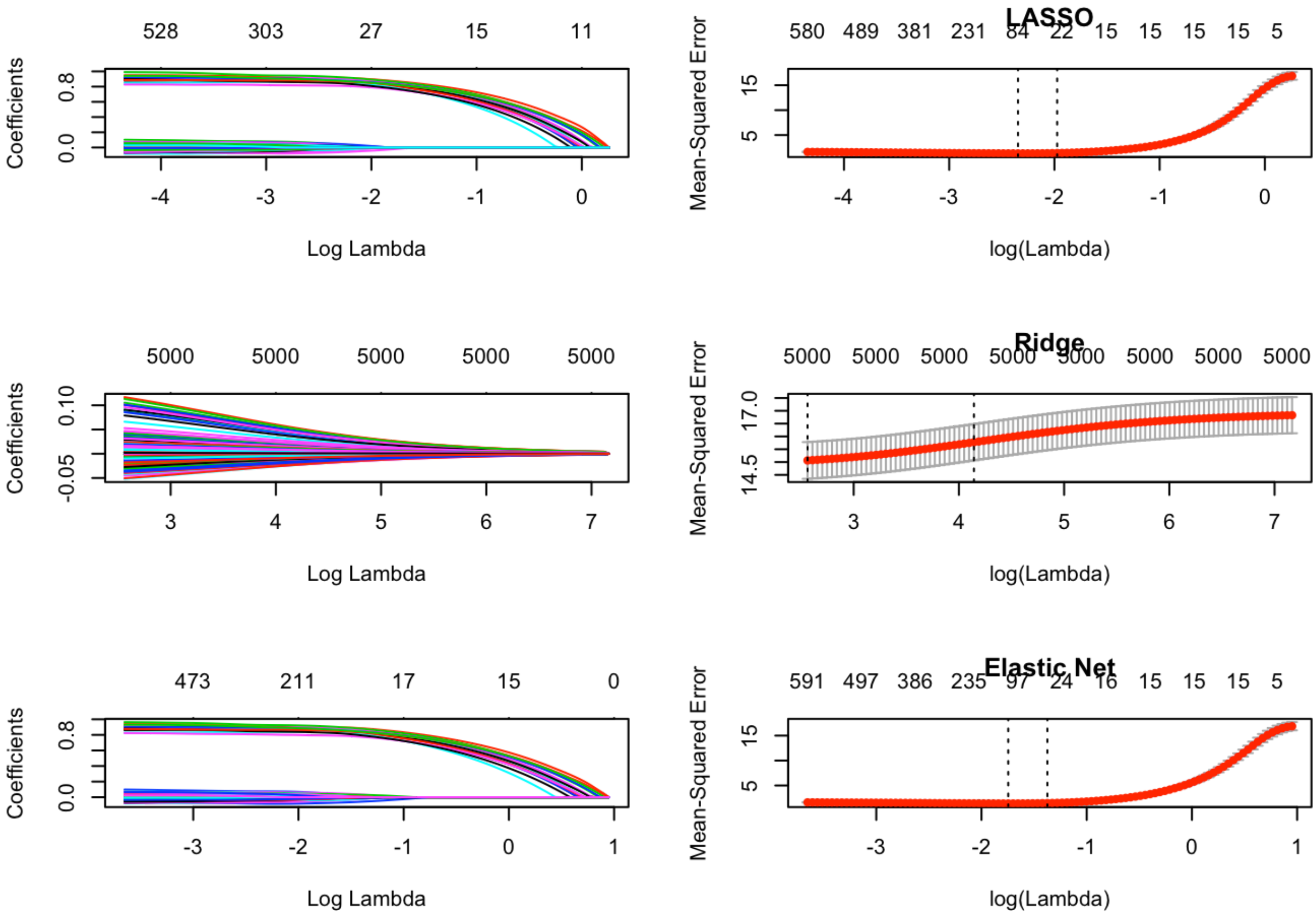
```r
# Generate data
set.seed(19875)  # Set seed for reproducibility
n <- 1000  # Number of observations
p <- 5000  # Number of predictors included in model
real_p <- 15  # Number of true predictors
x <- matrix(rnorm(n*p), nrow=n, ncol=p)
y <- apply(x[,1:real_p], 1, sum) + rnorm(n)

# Split data into train (2/3) and test (1/3) sets
train_rows <- sample(1:n, .66*n)
x.train <- x[train_rows, ]
x.test <- x[-train_rows, ]

y.train <- y[train_rows]
y.test <- y[-train_rows]
```

Fit models

```r
# Fit models
# (For plots on left):
fit.lasso <- glmnet(x.train, y.train, family="gaussian", alpha=1)
fit.ridge <- glmnet(x.train, y.train, family="gaussian", alpha=0)
fit.elnet <- glmnet(x.train, y.train, family="gaussian", alpha=.5)


# 10-fold Cross validation for each alpha = 0, 0.1, ... , 0.9, 1.0
# (For plots on Right)
for (i in 0:10) {
    assign(paste("fit", i, sep=""), cv.glmnet(x.train, y.train, type.measure="mse",
                                              alpha=i/10,family="gaussian"))
}
```

Plot solution path and cross-validated MSE as function of λ.

```r
# Plot solution paths:
par(mfrow=c(3,2))
# For plotting options, type '?plot.glmnet' in R console
plot(fit.lasso, xvar="lambda")
plot(fit10, main="LASSO")

plot(fit.ridge, xvar="lambda")
plot(fit0, main="Ridge")

plot(fit.elnet, xvar="lambda")
plot(fit5, main="Elastic Net")
```



MSE on test set

```
yhat0 <- predict(fit0, s=fit0$lambda.1se, newx=x.test)
yhat1 <- predict(fit1, s=fit1$lambda.1se, newx=x.test)
yhat2 <- predict(fit2, s=fit2$lambda.1se, newx=x.test)
yhat3 <- predict(fit3, s=fit3$lambda.1se, newx=x.test)
yhat4 <- predict(fit4, s=fit4$lambda.1se, newx=x.test)
yhat5 <- predict(fit5, s=fit5$lambda.1se, newx=x.test)
yhat6 <- predict(fit6, s=fit6$lambda.1se, newx=x.test)
yhat7 <- predict(fit7, s=fit7$lambda.1se, newx=x.test)
yhat8 <- predict(fit8, s=fit8$lambda.1se, newx=x.test)
yhat9 <- predict(fit9, s=fit9$lambda.1se, newx=x.test)
yhat10 <- predict(fit10, s=fit10$lambda.1se, newx=x.test)

mse0 <- mean((y.test - yhat0)^2)
mse1 <- mean((y.test - yhat1)^2)
mse2 <- mean((y.test - yhat2)^2)
mse3 <- mean((y.test - yhat3)^2)
mse4 <- mean((y.test - yhat4)^2)
mse5 <- mean((y.test - yhat5)^2)
mse6 <- mean((y.test - yhat6)^2)
mse7 <- mean((y.test - yhat7)^2)
mse8 <- mean((y.test - yhat8)^2)
mse9 <- mean((y.test - yhat9)^2)
mse10 <- mean((y.test - yhat10)^2)
```

$\boldsymbol{\alpha}$	**MSE**
$\alpha = 0$ (Ridge)	16.1313
$\alpha = 0.2$	1.8063
$\alpha = 0.4$	1.4351
$\alpha = 0.6$	1.4146
$\alpha = 0.8$	1.4427
$\alpha = 1$ (LASSO)	1.3759

LASSO is the winner! LASSO is good at picking up a small signal through lots of noise.

Example 2

Big signal and big noise.

- $\boldsymbol{\beta} = (\underbrace{1, \ldots, 1}_{1500}, \underbrace{0, \ldots, 0}_{3500})^T$
- $p = 5000 > n = 1000$
- Uncorrelated predictors:
  - $\boldsymbol{X}_i \overset{\text{iid}}{\sim} N(\mathbf{0}, \boldsymbol{I})$
- Note that LASSO can pick at max 1000 predictors

Generate Data

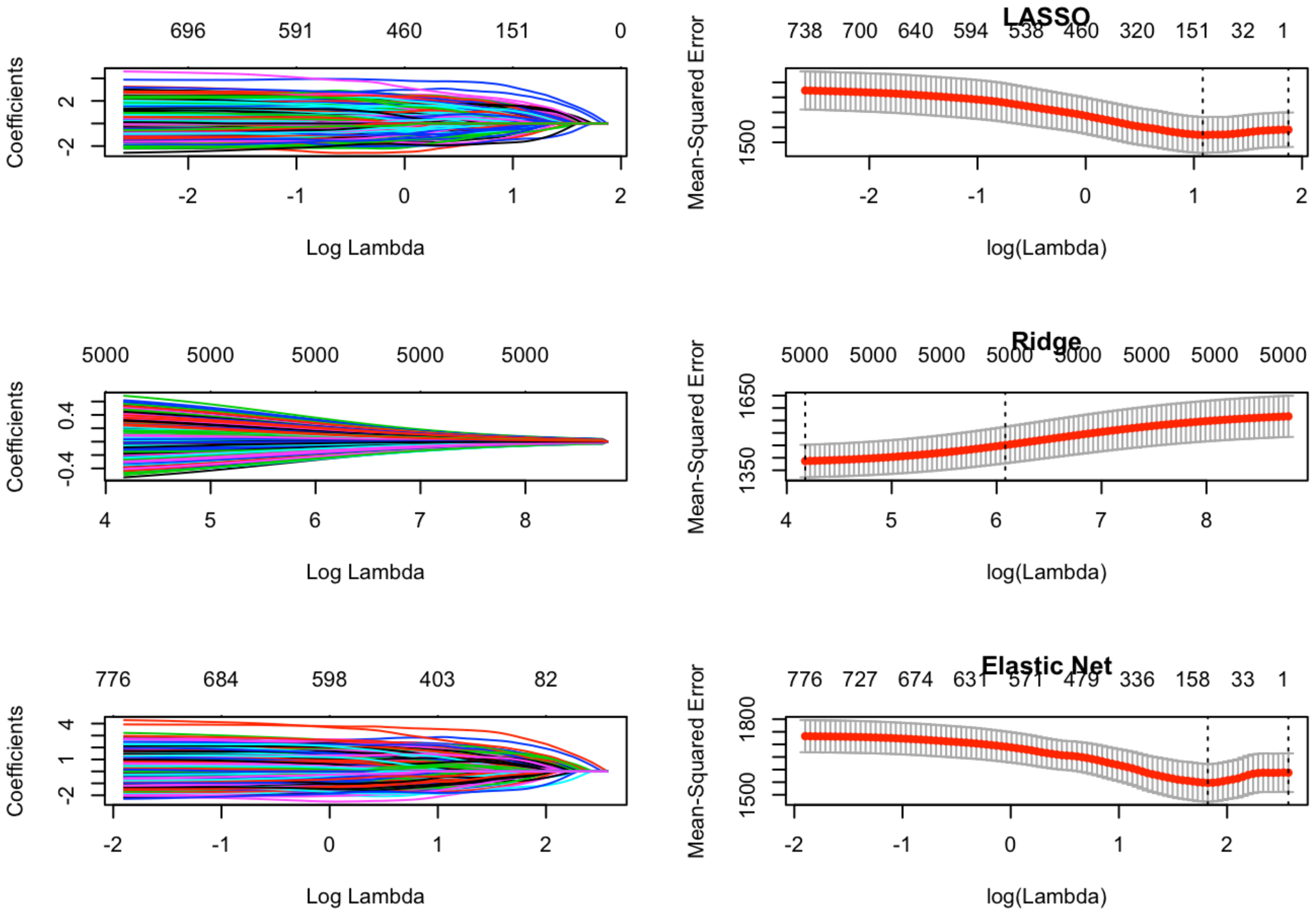
```r
library(MASS)  # Package needed to generate correlated precictors
library(glmnet)  # Package to fit ridge/lasso/elastic net models

# Generate data
set.seed(19874)
n <- 1000    # Number of observations
p <- 5000     # Number of predictors included in model
real_p <- 1500  # Number of true predictors
x <- matrix(rnorm(n*p), nrow=n, ncol=p)
y <- apply(x[,1:real_p], 1, sum) + rnorm(n)

# Split data into train and test sets
train_rows <- sample(1:n, .66*n)
x.train <- x[train_rows, ]
x.test <- x[-train_rows, ]

y.train <- y[train_rows]
y.test <- y[-train_rows]
```

Fit Models

```r
# Fit models:
fit.lasso <- glmnet(x.train, y.train, family="gaussian", alpha=1)
fit.ridge <- glmnet(x.train, y.train, family="gaussian", alpha=0)
fit.elnet <- glmnet(x.train, y.train, family="gaussian", alpha=.5)



# 10-fold Cross validation for each alpha = 0, 0.1, ... , 0.9, 1.0
fit.lasso.cv <- cv.glmnet(x.train, y.train, type.measure="mse", alpha=1,
                          family="gaussian")
fit.ridge.cv <- cv.glmnet(x.train, y.train, type.measure="mse", alpha=0,
                          family="gaussian")
fit.elnet.cv <- cv.glmnet(x.train, y.train, type.measure="mse", alpha=.5,
                          family="gaussian")

for (i in 0:10) {
    assign(paste("fit", i, sep=""), cv.glmnet(x.train, y.train, type.measure="mse",
                                              alpha=i/10,family="gaussian"))
}
```

Plot solution path and cross-validated MSE as function of λ.

```
# Plot solution paths:
par(mfrow=c(3,2))
# For plotting options, type '?plot.glmnet' in R console
plot(fit.lasso, xvar="lambda")
plot(fit10, main="LASSO")

plot(fit.ridge, xvar="lambda")
plot(fit0, main="Ridge")

plot(fit.elnet, xvar="lambda")
plot(fit5, main="Elastic Net")
```



MSE on test set

```
yhat0 <- predict(fit0, s=fit0$lambda.1se, newx=x.test)
yhat1 <- predict(fit1, s=fit1$lambda.1se, newx=x.test)
yhat2 <- predict(fit2, s=fit2$lambda.1se, newx=x.test)
yhat3 <- predict(fit3, s=fit3$lambda.1se, newx=x.test)
yhat4 <- predict(fit4, s=fit4$lambda.1se, newx=x.test)
yhat5 <- predict(fit5, s=fit5$lambda.1se, newx=x.test)
yhat6 <- predict(fit6, s=fit6$lambda.1se, newx=x.test)
yhat7 <- predict(fit7, s=fit7$lambda.1se, newx=x.test)
yhat8 <- predict(fit8, s=fit8$lambda.1se, newx=x.test)
yhat9 <- predict(fit9, s=fit9$lambda.1se, newx=x.test)
yhat10 <- predict(fit10, s=fit10$lambda.1se, newx=x.test)

mse0 <- mean((y.test - yhat0)^2)
mse1 <- mean((y.test - yhat1)^2)
mse2 <- mean((y.test - yhat2)^2)
mse3 <- mean((y.test - yhat3)^2)
mse4 <- mean((y.test - yhat4)^2)
mse5 <- mean((y.test - yhat5)^2)
mse6 <- mean((y.test - yhat6)^2)
mse7 <- mean((y.test - yhat7)^2)
mse8 <- mean((y.test - yhat8)^2)
mse9 <- mean((y.test - yhat9)^2)
mse10 <- mean((y.test - yhat10)^2)
```

$\boldsymbol{\alpha}$	**MSE**
$\alpha = 0$ (Ridge)	1490.7833
$\alpha = 0.2$	1637.0238
$\alpha = 0.4$	1641.1414
$\alpha = 0.6$	1633.5475
$\alpha = 0.8$	1641.1414
$\alpha = 1$ (LASSO)	1641.1414

Ridge is the winner! Ridge in general is good at prediction, but is not very interpretable.

Example 3

Varying signals. High correlation between predictors

- $\boldsymbol{\beta} = (10, 10, 5, 5, \underbrace{1, \ldots, 1}_{10}, \underbrace{0, \ldots, 0}_{36})^T$
- $p = 50$
- $n = 100$
- Correlated predictors: $Cov(\boldsymbol{X})_{ij} = (0.7)^{|i-j|}$

Generate Data

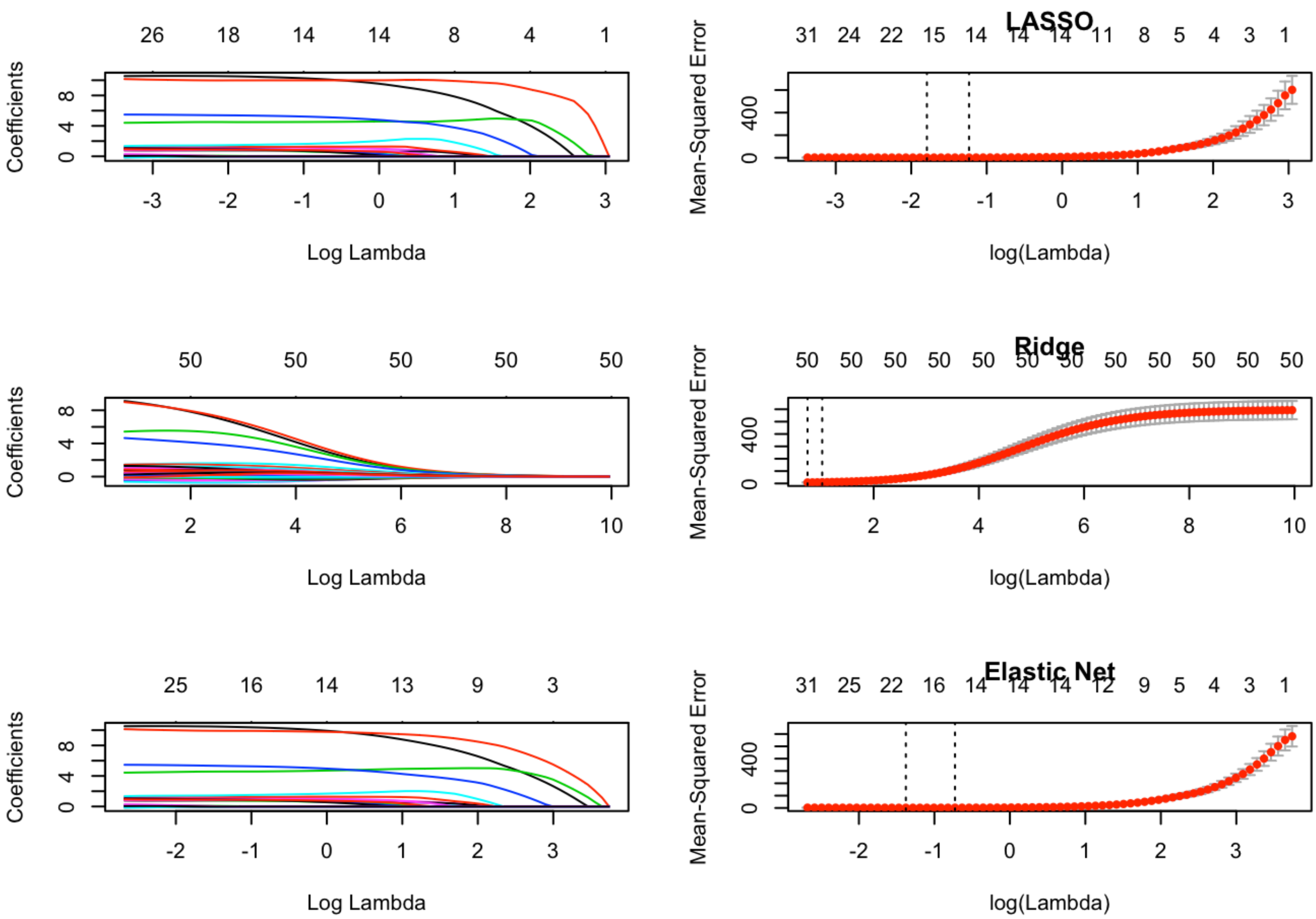
```r
# Generate data
set.seed(19873)
n <- 100    # Number of observations
p <- 50     # Number of predictors included in model
CovMatrix <- outer(1:p, 1:p, function(x,y) {.7^abs(x-y)})
x <- mvrnorm(n, rep(0,p), CovMatrix)
y <- 10 * apply(x[, 1:2], 1, sum) +
  5 * apply(x[, 3:4], 1, sum) +
  apply(x[, 5:14], 1, sum) +
  rnorm(n)

# Split data into train and test sets
train_rows <- sample(1:n, .66*n)
x.train <- x[train_rows, ]
x.test <- x[-train_rows, ]

y.train <- y[train_rows]
y.test <- y[-train_rows]
```

Fit models

```r
# Fit models:
fit.lasso <- glmnet(x.train, y.train, family="gaussian", alpha=1)
fit.ridge <- glmnet(x.train, y.train, family="gaussian", alpha=0)
fit.elnet <- glmnet(x.train, y.train, family="gaussian", alpha=.5)



# 10-fold Cross validation for each alpha = 0, 0.1, ... , 0.9, 1.0
fit.lasso.cv <- cv.glmnet(x.train, y.train, type.measure="mse", alpha=1,
                          family="gaussian")
fit.ridge.cv <- cv.glmnet(x.train, y.train, type.measure="mse", alpha=0,
                          family="gaussian")
fit.elnet.cv <- cv.glmnet(x.train, y.train, type.measure="mse", alpha=.5,
                          family="gaussian")

for (i in 0:10) {
    assign(paste("fit", i, sep=""), cv.glmnet(x.train, y.train, type.measure="mse",
                                              alpha=i/10,family="gaussian"))
}
```

Plot solution path and cross-validated MSE as function of λ

```
# Plot solution paths:
par(mfrow=c(3,2))
# For plotting options, type '?plot.glmnet' in R console
plot(fit.lasso, xvar="lambda")
plot(fit10, main="LASSO")

plot(fit.ridge, xvar="lambda")
plot(fit0, main="Ridge")

plot(fit.elnet, xvar="lambda")
plot(fit5, main="Elastic Net")
```



MSE on test set

```
yhat0 <- predict(fit0, s=fit0$lambda.1se, newx=x.test)
yhat1 <- predict(fit1, s=fit1$lambda.1se, newx=x.test)
yhat2 <- predict(fit2, s=fit2$lambda.1se, newx=x.test)
yhat3 <- predict(fit3, s=fit3$lambda.1se, newx=x.test)
yhat4 <- predict(fit4, s=fit4$lambda.1se, newx=x.test)
yhat5 <- predict(fit5, s=fit5$lambda.1se, newx=x.test)
yhat6 <- predict(fit6, s=fit6$lambda.1se, newx=x.test)
yhat7 <- predict(fit7, s=fit7$lambda.1se, newx=x.test)
yhat8 <- predict(fit8, s=fit8$lambda.1se, newx=x.test)
yhat9 <- predict(fit9, s=fit9$lambda.1se, newx=x.test)
yhat10 <- predict(fit10, s=fit10$lambda.1se, newx=x.test)

mse0 <- mean((y.test - yhat0)^2)
mse1 <- mean((y.test - yhat1)^2)
mse2 <- mean((y.test - yhat2)^2)
mse3 <- mean((y.test - yhat3)^2)
mse4 <- mean((y.test - yhat4)^2)
mse5 <- mean((y.test - yhat5)^2)
mse6 <- mean((y.test - yhat6)^2)
mse7 <- mean((y.test - yhat7)^2)
mse8 <- mean((y.test - yhat8)^2)
mse9 <- mean((y.test - yhat9)^2)
mse10 <- mean((y.test - yhat10)^2)
```

$\boldsymbol{\alpha}$	**MSE**
$\alpha = 0$ (Ridge)	6.7541
$\alpha = 0.2$	1.2461
$\alpha = 0.4$	1.4948
$\alpha = 0.6$	1.4219
$\alpha = 0.8$	1.4985
$\alpha = 1$ (LASSO)	1.7152

Elastic Net is the winner! It’s interesting to note the best solution is “close” to Ridge, but Ridge ($\alpha = 0$) in fact performs the worst.