Fun with Prime Numbers

Invitation to the Mysterious World of Mathematics

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Laws of prime numbers

- Prime numbers up to 100 are

  2 3 5 7 11 13 17 19 23 29 31 37 41 43

  47 53 59 61 67 71 73 79 83 89 97

- Do you see any laws behind them?
- When we divide a prime number P by 2, the remainder is 1 except P=2.
- What happens when we divide by 4?

Laws of prime numbers (2)

- Divide prime numbers by 4. The remainder is either 1, 2, or 3.

Remainder 1: 5 13 17 29 37 41 53 61 73 89 97
Remainder 2: 2
Remainder 3: 3 7 11 19 23 31 43 47 59 67 71 79 83

- Are there infinitely many prime numbers with remainder 1 (or 3)?

  Answer: Yes.

Dirichlet's theorem on arithmetic progressions

Reference
https://en.wikipedia.org/wiki/Peter_Gustav_Lejeune_Dirichlet

Peter Gustav Lejeune Dirichlet (1805-1859)

Theorem (Dirichlet (1837))

Let $A, B \geqq 1$ be integers such that

$$A = P_1 \times \cdots \times P_N$$

$$B = Q_1 \times \cdots \times Q_M$$

$$P_i \neq Q_j \text{ (for any } i,j).$$

(A,B are relatively prime.)

There exist infinitely many prime numbers of the form

$$P = KA + B.$$

Dirichlet's theorem on arithmetic progressions (2)

- Apply Theorem for A=4, B=1.

⇨ There exist infinitely many prime numbers with remainder 1.

- Similarly, apply Theorem for A=4, B=3.

⇨ There exist infinitely many prime numbers with remainder 3.

(We can't apply Theorem for A=4, B=2 because 4 and 2 are not relatively prime.)

Laws of prime numbers (3)

- Divide prime numbers by 4.

Remainder 1: 5 13 17 29 37 41 53 61 73 89 97
Remainder 3: 3 7 11 19 23 31 43 47 59 67 71 79 83

- Are there any laws which distinguish these two groups of prime numbers?

⇨ Fermat's theorem on sums of two squares.