Exercise: Bernoulli and Binomial Random Variables

(5/5 points)

This problem introduces two of the most common random variables that people use in probabilistic models: the Bernoulli random variable, and the Binomial random variable. We have actually already encountered these albeit with a disguise! A Bernoulli random variable is like a biased coin flip. A Binomial random variable is like counting the number of heads for n of these biased coin flips.

These two distributions appear all the time in many, many application domains that use inference! We introduce them now to equip you with some vocabulary and also to let you see our first example of a random variable whose probability table can be described by only a few numbers even if the number of entries in the table can be much larger!

As mentioned, a Bernoulli random variable is like a biased coin flip where probability of heads is p. In particular, a Bernoulli random variables is 1 with probability p, and 0 with probability $1-p$. If a random variable X has this particular distribution, then we write $X \sim \text{Bernoulli}(p)$, where "$\sim$" can be read as "is distributed as" or "has distribution". Some people like to abbreviate $\text{Bernoulli}(p)$ by writing $\text{Bern}(p)$, $\text{Ber}(p)$, or even just $B(p)$.

A Binomial random variable can be thought of as n independent coin flips, each with probability p of heads. For a random variable S that has this Binomial distribution with parameters n and p, we denote it as $S \sim \text{Binomial}(n,p)$, read as "$S$ is distributed as Binomial with parameters n and p". Some people might also abbreviate and instead of writing $\text{Binomial}(n,p)$, they write $\text{Binom}(n,p)$ or $\text{Bin}(n,p)$.

- **(a)** True or false: If $Y \sim \text{Binomial}(1,p)$, then Y is a Bernoulli random variable.

  - (•) True ✔
  - ( ) False

- **(b)** Let's say we have a coin that turns up heads with probability 0.6. We flip this coin 10 times. What is the probability of seeing the sequence HTHTTTTTHH, where H denotes heads and T denotes tails (so we have heads in the first toss, tails in the second, heads in the third, etc)? (Please be precise with at least 3 decimal places, unless of course the answer doesn't need that many decimal places. You could also put a fraction.)

  0.0005

  ✔ **Answer:** 0.6^4 * 0.4^6

- **(c)** In the previous part, there were 4 heads and 6 tails. Did the ordering of them matter? In other words, would your answer to the previous part be the same if, for example, instead we saw the sequence HHHHTTTTTT (or any other permutation of 4 heads and 6 tails)?

  ◉ The probability stays the same so long as we have 4 heads and 6 tails. ✔

  ○ The probability is different depending on the ordering of heads and tails.

- **(d)** From the previous two parts, what we were analyzing was the same as the random variable $S \sim \text{Binomial}(10, 0.6)$. Note that $S = 4$ refers to the event that we see exactly 4 heads. Note that HTHTTTTTHH and HHHHTTTTTT are different outcomes of the underlying experiment of coin flipping. How many ways are there to see 4 heads in 10 tosses? **(Please provide the exact answer.)**

  210

  ✔ **Answer:** 210

- **(e)** Using your answers to parts (b) through (d), what is the probability that $S = 4$? (Please be precise with at least 3 decimal places, unless of course the answer doesn't need that many decimal places. You could also put a fraction.)

  0.111476736

  ✔ **Answer:** 210 * 0.6^4 * 0.4^6

In general, for a random variable $S \sim \text{Binomial}(n, p)$, the reasoning used in parts (b) through (e) could be used to obtain the probability that $S = s$ for any $s \in \{0, 1, 2, \ldots, n\}$. Importantly, what this means is that by just specifying two numbers n and p, we know the full probability table for random variable S, which has $n + 1$ entries! This is an example of where we could have many probability table entries yet we can fully specify the entire table using fewer numbers than the number of entries in the table.

Please be sure to look at the solution to this problem after you have finished it to see the general equation for what the probability table entry $p_S(s)$ is, and also why the probability table entries sum to 1.

Solution:

- **(a)** True or false: If $Y \sim \textbf{Binomial}(1, p)$, then Y is a Bernoulli random variable.

  Solution: The answer is **true**. When there's only a single flip, counting the number if heads yields precisely the Bernoulli distribution. In particular, we have $Y \sim \textbf{Bernoulli}(p)$.

- **(b)** Let's say we have a coin that turns up heads with probability 0.6. We flip this coin 10 times. What is the probability of seeing the sequence HTHTTTTTHH, where H denotes heads and T denotes tails (so we have heads in the first toss, tails in the second, heads in the third, etc)? (Please be precise with at least 3 decimal places, unless of course the answer doesn't need that many decimal places. You could also put a fraction.)

  By independence, the sequence HTHTTTTTHH has probability

  $$\begin{aligned} &0.6 \times 0.4 \times 0.6 \times 0.4 \times 0.4 \times 0.4 \times 0.4 \times 0.4 \times 0.6 \times 0.6 \\ &= \boxed{0.6^4 \times 0.4^6}. \end{aligned}$$

  Note that in general, if there were instead n coin flips where heads appears with probability p, then we would have $p^{\text{number of heads}}(1-p)^{n-\text{number of heads}}$.

- **(c)** In the previous part, there were 4 heads and 6 tails. Did the ordering of them matter? In other words, would your answer to the previous part be the same if, for example, instead we saw the sequence HHHHTTTTTT (or any other permutation of 4 heads and 6 tails)?

  Solution: As our solution to part (b) shows, the ordering of heads/tails does not matter.

- **(d)** From the previous two parts, what we were analyzing was the same as the random variable $S \sim \textbf{Binomial}(10, 0.6)$. Note that $S = 4$ refers to the event that we see exactly 4 heads. Note that HTHTTTTTHH and HHHHTTTTTT are different outcomes of the underlying experiment of coin flipping. How many ways are there to see 4 heads in 10 tosses?

Solution: The number of ways to see 4 heads in 10 tosses is precisely the number of ways to choose 4 items out of 10, given by the choose operator: $\binom{10}{4} = \frac{10!}{4!6!} = \boxed{210}$.

In general, the number of ways to see s heads in n tosses is $\binom{n}{s}$.

- **(e)** Using your answers to parts (b) through (d), what is the probability that $S = 4$?

  Solution: There are $\binom{10}{4} = 210$ ways to get 4 heads out of 10, and each way is equally likely with probability given by $0.6^4 \times 0.4^6$, so summing up the probabilities across all these ways, we have $\boxed{210 \times 0.6^4 \times 0.4^6}$.

  In general, for random variable $S \sim \text{Binomial}(n, p)$, the probability that $S = s$ for $s \in \{0, 1, \dots, n\}$ is given by

  $$p_S(s) = \binom{n}{s} p^s (1-p)^{n-s}.$$

  It might not be a priori obvious why this probability table's entries should sum to 1.

  To show this result, we can use the Binomial Theorem, which says that for any two numbers x and y, and any nonnegative integer n,

  $$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

  Plugging in $x = p$ and $y = 1 - p$, we see that the right-hand side directly corresponds to summing across all the entries of probability table p_S, and the left-hand side is $(p + (1-p))^n = 1^n = 1$.

You have used 3 of 5 submissions