Exercise: Gambler's Fallacy

(4/4 points)

Suppose you have a 27-sided fair die (with faces numbered $1, 2, \ldots, 27$) that you get to roll 100 times. You win a prize if you roll 27 at least once. In this problem we look at what happens if you don't roll 27 for a while and see whether or not you're more likely to roll a 27 in your remaining rolls.

- **(a)** What is the probability that you roll 27 at least once out of the 100 rolls? (Please be precise with at least 3 decimal places, unless of course the answer doesn't need that many decimal places. You could also put a fraction.)

  0.9770407138326136 ✔ **Answer:** 1-(26/27)^100

- **(b)** Suppose you roll the die once and don't get 27. What is the probability that of the remaining 99 rolls, you will roll 27 at least once? (Please be precise with at least 3 decimal places, unless of course the answer doesn't need that many decimal places. You could also put a fraction.)

  0.9761576643646371 ✔ **Answer:** 1-(26/27)^99

- **(c)** Suppose you roll the die n times and don't get 27 any of those times. What is the probability that of the remaining $100 - n$ rolls, you will roll 27 at least once? Express your answer in terms of n.

  In this part, please provide your answer as a mathematical formula (and not as Python code). Use ^ for exponentiation, e.g., x^2 denotes x^2. Explicitly include multiplication using *, e.g. x*y is xy.

  1 - (26/27) ^ (100-n)

  ✔ **Answer:** 1-(26/27)^(100-n)

  $$1 - \left(\frac{26}{27}\right)^{100-n}$$

- **(d)** Plot the probability in part (c) as a function of n for $n = 1, 2, \ldots, 99$. Does this probability increase or decrease as n increases?

Notation Summary (Up Through Week 3)

Mini-project 1: Movie Recommendations (Week 3)
Mini-projects due Oct 12, 2016 at 21:00 UTC

- Probability decreases as n increases ✔
- Probability increases as n increases

Solution:

In this problem, it is easier to reason using the probability of not rolling any heads and then subtracting that from 1.

- **(a)** What is the probability that you roll 27 at least once out of the 100 rolls? (Please be precise with at least 3 decimal places, unless of course the answer doesn't need that many decimal places. You could also put a fraction.)

  Solution: Let $\mathcal{A}$ be the event that you roll 27 at least once in the 100 rolls. Note that $\mathcal{A}^c$ is the event that you never roll 27 in the 100 rolls. As it turns out, it will be easier to compute $\mathbb{P}(\mathcal{A}^c)$.

  We know that

  $$\mathbb{P}(\mathcal{A}) + \mathbb{P}(\mathcal{A}^c) = 1,$$

  so

  $$\mathbb{P}(\mathcal{A}) = 1 - \mathbb{P}(\mathcal{A}^c).$$

  Next, note that

  $$\begin{aligned}\mathbb{P}(\mathcal{A}^c) &= \mathbb{P}(\textbf{none of the 100 rolls is 27}) \\ &= (\mathbb{P}(\textbf{a single roll is not 27}))^{100},\end{aligned}$$

  where the last step uses independence!

  With a 27-sided fair die, the probability of not rolling the face 27 in a single roll is 26/27.

  Thus, we have

  $$\mathbb{P}(\mathcal{A}^c) = \left(\frac{26}{27}\right)^{100},$$

and so

$$\mathbb{P}(\mathcal{A}) = 1 - \mathbb{P}(\mathcal{A}^c) = \boxed{1 - \left(\frac{26}{27}\right)^{100}}.$$

- **(b)** Suppose you roll the die once and don't get 27. What is the probability that of the remaining 99 rolls, you will roll 27 at least once? (Please be precise with at least 3 decimal places, unless of course the answer doesn't need that many decimal places. You could also put a fraction.)

  Solution: Due to independence, the first roll does not affect the outcome of the remaining 99 rolls. What this means is that the solution to this part is the same as the solution to the previous part where we treat the number of rolls as 99 instead of 100. In particular, the probability that in 99 rolls, we see face 27 at least once is

$$\boxed{1 - \left(\frac{26}{27}\right)^{99}}.$$

- **(c)** Suppose you roll the die n times and don't get 27 any of those times. What is the probability that of the remaining $100 - n$ rolls, you will roll 27 at least once? Express your answer in terms of n.

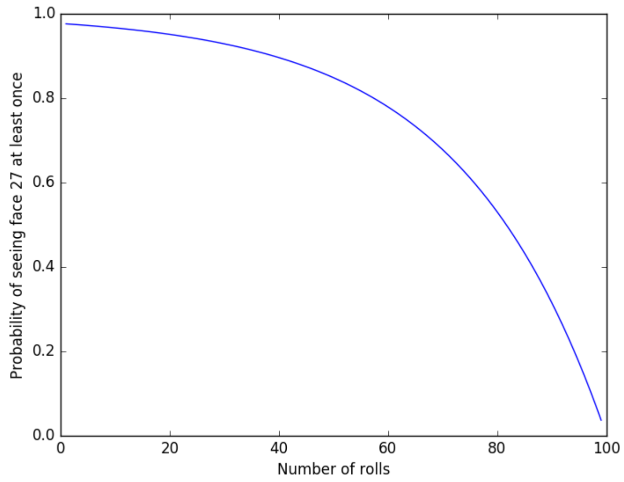
  Solution: This part generalizes the answers from parts (a) and (b). Again, by independence, the first n rolls do not affect the last $100 - n$. We can again use the solution to part (a) except instead of 100 rolls, we have $100 - n$ rolls. Thus, the probability that in $100 - n$ rolls, we see the face 27 at least once is

$$\boxed{1 - \left(\frac{26}{27}\right)^{100-n}}.$$

- **(d)** Plot the probability as a function of n for $n = 1, 2, \dots, 99$. Does this probability increase or decrease as n increases?

  We can produce a plot with the following Python code.

```
n = np.array(range(1, 100))
plt.plot(n, 1 - (26/27)**(100 - n))
plt.xlabel('Number of rolls')
plt.ylabel('Probability of seeing face 27 at least once')
```



In particular, the **probability decreases as n increases**.

Thus, in 100 rolls, if we do not see the face 27 in the first $n < 100$ rolls, then our probability of seeing 27 in the remaining $100 - n$ rolls actually decreases as a function of n — it is not the case that we should think that because there haven't been many 27's so far that there are going to be more 27's in the remaining rolls!

You have used 5 of 5 submissions