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Introduction

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Exercise: Expectations of Multiple Random Variables

(7/9 points)

How does expected value work when we have many random variables? In this exercise, we look at some specific cases that draw out some important properties involving expected values.

Let's look at when there are two random variables X and Y with alphabets $\mathcal{X}$ and $\mathcal{Y}$ respectively. Then how expectation is defined for multiple random variables is as follows: For any function $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$,

$$\mathbb{E}[f(X,Y)] = \sum_{x\in\mathcal{X},y\in\mathcal{Y}} f(x,y)p_{X,Y}(x,y).$$

(Note that "$\sum_{x\in\mathcal{X},y\in\mathcal{Y}}$" can also be written as "$\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}}$".)

For example:

$$\mathbb{E}[X+Y] = \sum_{x\in\mathcal{X},y\in\mathcal{Y}} (x+y)p_{X,Y}(x,y).$$

Let's look at some properties of the expected value of the sum of multiple random variables.

First, as a warm-up, show that:

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y].$$

This equality is called *linearity of expectation* and it holds *regardless of whether X and Y are independent*.

We previously saw that a binomial random variable with parameters n and p can be thought of as how many heads there are in n tosses where the probability of heads is p.

A different way to view a binomial random variable is that it is the sum of n i.i.d. Bernoulli random variables each of parameter p. As a reminder, a Bernoulli random variable is 1 with probability p and 0 otherwise. Suppose that $X_1, X_2, \ldots, X_n$ are i.i.d. $\text{Bernoulli}(p)$ random variables, and $S = \sum_{i=1}^n X_i$.

- What is $\mathbb{E}[X_i]$?

Exercises due Oct 12, 2016 at 21:00 UTC

Measuring Randomness (Week 4)
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Towards Infinity in Modeling Uncertainty (Week 4)
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Homework 3 (Week 4)
Homework due Oct 12, 2016 at 21:00 UTC

p ✓ **Answer:** p

- What is $\mathbb{E}[S]$?

  In this part, please provide your answer as a mathematical formula (and not as Python code). Use ^ for exponentiation, e.g., `x^2` denotes x^2. Explicitly include multiplication using `*`, e.g. `x*y` is xy.

  n * p ✓ **Answer:** n*p

  $n \cdot p$

A widely useful random variable is the indicator random variable $\mathbb{1}(\cdot)$, defined as

$$\mathbb{1}(\mathcal{S}) = \begin{cases} 1 & \text{if } \mathcal{S} \text{ happens,} \\ 0 & \text{otherwise} \end{cases}$$

where $\mathcal{S}$ is an event.

- Can a Bernoulli random variable corresponding to a coin flip be written as an indicator random variable?

  ● Yes ✓

  ○ No

- Suppose a monkey types 1,000,000,000 characters using the 26 letters of the alphabet. The monkey selects each character independently and is equally likely to choose any of the 26 letters. What is the expected number of occurrences of the word “caesar"?

  Hint: First write up the number of occurrences of the word “caesar" as the sum of indicator random variables.

  (Please be precise with at least 3 decimal places, unless of course the answer doesn't need that many decimal places. You could also put a fraction.)

  ✗ **Answer:** 999999995/(26^6)

- Consider a standard 52-card deck that has been shuffled. What is the expected number of times for which a king in the deck is followed by another king? For example, if all four kings are next to each other, then we'd say that there are 3 kings that are followed by another king.

  (Please be precise with at least 3 decimal places, unless of course the answer doesn't need that many decimal places. You could also put a fraction.)

  ✗ **Answer:** 3/13

Finally, let's look at variance again. As a reminder

$$\textbf{var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2].$$

- If X and Y are independent, is it true that $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$?

  ● Yes ✓

  ○ No

- If X and Y are independent, is it true that $\textbf{var}(X + Y) = \textbf{var}(X) + \textbf{var}(Y)$?

  ● Yes ✓

  ○ No

- What is the variance of a $\text{Ber}(p)$ random variable?

  In this part, please provide your answer as a mathematical formula (and not as Python code). Use `^` for exponentiation, e.g., `x^2` denotes x^2. Explicitly include multiplication using `*`, e.g. `x*y` is xy.

  p * (1-p) ✓ **Answer:** p*(1-p)

  $p \cdot (1 - p)$

- Recall that a binomial random variable S that has a $\text{Binomial}(n, p)$ distribution can be written as $S = \sum_{i=1}^{n} X_i$ where the X_i's are i.i.d. $\text{Ber}(p)$ random variables. What is the variance of S?

  In this part, please provide your answer as a mathematical formula (and not as Python code). Use `^` for exponentiation, e.g., `x^2` denotes x^2. Explicitly include multiplication using `*`, e.g. `x*y` is xy.

  n * p * (1-p) ✓ **Answer:** n*p*(1-p)

  $n \cdot p \cdot (1 - p)$

Solution:

- What is $\mathbb{E}[X_i]$?

  Solution:

$$\mathbb{E}[X_i] = 1 \cdot p + 0 \cdot (1-p) = \boxed{p}.$$

- What is $\mathbb{E}[S]$?

  Solution: By linearity of expectation,

$$\mathbb{E}[S] = \mathbb{E}\Big[\sum_{i=1}^{n} X_i\Big] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} p = \boxed{np}.$$

- Can a Bernoulli random variable corresponding to a coin flip be written as an indicator random variable?

  Solution: The answer is **yes**: $\mathbb{1}(\textbf{coin turns up heads})$ is a Bernoulli random variable with parameter p given by the probability of heads.

- Suppose a monkey types 1,000,000,000 characters using the 26 letters of the alphabet. The monkey selects each character independently and is equally likely to choose any of the 26 letters. What is the expected number of occurrences of the word “caesar"?

  Hint: First write up the number of occurrences of the word “caesar" as the sum of indicator random variables.

  Solution: Let Z_i be the indicator random variable that the word “caesar" starts at position i. Note that

$$\begin{aligned} \mathbb{E}[Z_i] &= \mathbb{P}(\text{the word ``caesar'' starts at position } i) \\ \text{(by independence)} &= \mathbb{P}(i\text{-th letter is ``c''})\mathbb{P}((i+1)\text{-th letter is ``a''}) \\ &\quad \cdots \mathbb{P}((i+5)\text{ -th letter is ``r''}) \\ &= \underbrace{\frac{1}{26} \times \frac{1}{26} \times \cdots \times \frac{1}{26}}_{6 \text{ times}} \\ &= \frac{1}{(26)^6}. \end{aligned}$$

  Then the number of occurrences of “caesar" is given by $Y = \sum_{i=1}^{1{,}000{,}000{,}000-5} Z_i$, so

$$\mathbb{E}[Y] = \mathbb{E}[\sum_{i=1}^{999{,}999{,}995} Z_i] = \sum_{i=1}^{999{,}999{,}995} \mathbb{E}[Z_i] = \boxed{999,999,995 \times \frac{1}{(26)^6}}.$$

  Note that in the second equality we used the fact that expectation is linear, even though X_i's are dependent. This is because the linearity of expectation doesn't require independence.

- Consider a standard 52-card deck that has been shuffled. What is the expected number of times for which a king in the deck is followed by another king? For example, if all four kings are next to each other, then we'd say that there are 3 kings that are followed by another king.

  Solution: Let i be an index between two cards in the deck, i.e., $i = 1$ refers to the index between cards 1 and 2. Thus, the range of possible i is $i = 1, 2, \cdots, 51$. Define an indicator random variable X_i that is 1 when the cards at index i and $i + 1$ are both kings. Note that

  $$\mathbb{E}[X_i] = \mathbb{P}(i\text{-th card is a king, } (i+1)\text{-th card is a king}) = \frac{4}{52} \cdot \frac{3}{51}.$$

  Then the number of kings followed by another king is $Y = \sum_{i=1}^{51} X_i$, so $\mathbb{E}[Y] = 51 \cdot \frac{4}{52} \cdot \frac{3}{51} = \frac{12}{52} = \boxed{\frac{3}{13}}$.

Finally, let's look at variance again. As a reminder

$$\text{var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2].$$

- If X and Y are independent, is it true that $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$?

  Solution: The answer is **yes**:

  $$\begin{aligned} \mathbb{E}[XY] &= \sum_x \sum_y xy p_{X,Y}(x, y) \\ \text{(by independence)} &= \sum_x \sum_y xy p_X(x) p_Y(y) \\ &= \sum_x \sum_y (x p_x(x))(y p_Y(y)) \\ &= \Big(\sum_x x p_x(x)\Big)\Big(\sum_y (y p_Y(y))\Big) \\ &= \mathbb{E}[X]\mathbb{E}[Y]. \end{aligned}$$

- If X and Y are independent, is it true that $\text{var}(X + Y) = \text{var}(X) + \text{var}(Y)$?

  Solution: The answer is **yes**.

  Let's show this. We'll be using linearity of expectation repeatedly. To keep the notation here from getting cluttered with the squaring, let $\mu_X = \mathbb{E}[X]$ and $\mu_Y = \mathbb{E}[Y]$. First off, note that

  $$\begin{aligned} \text{var}(X) &= \mathbb{E}[(X - \mu_X)^2] \\ &= \mathbb{E}[X^2 - 2\mu_X X + \mu_X^2] \\ &= \mathbb{E}[X^2] - 2\mu_X \underbrace{\mathbb{E}[X]}_{\mu_X} + \mu_X^2 \end{aligned}$$

$$= \mathbb{E}[X^2] - \mu_X^2.$$

Similarly,

$$\mathrm{var}(Y) = \mathbb{E}[Y^2] - \mu_Y^2,$$

and

$$\mathrm{var}(X+Y) = \mathbb{E}[(X+Y)^2] - (\mu_X + \mu_Y)^2.$$

Let's simplify the right-hand side above:

$$\begin{aligned} \mathrm{var}(X+Y) &= \mathbb{E}[(X+Y)^2] - (\mu_X+\mu_Y)^2 \\ &= \mathbb{E}[X^2 + 2XY + Y^2] - \mu_X^2 - 2\mu_X\mu_Y - \mu_Y^2 \\ &= \mathbb{E}[X^2] + 2\underbrace{\mathbb{E}[XY]}_{=\mu_X\mu_Y \text{ by indep.}} + \mathbb{E}[Y^2] - \mu_X^2 - 2\mu_X\mu_Y - \mu_Y^2 \\ &= \underbrace{\mathbb{E}[X^2] - \mu_X^2}_{\mathrm{var}(X)} + \underbrace{\mathbb{E}[Y^2] - \mu_Y^2}_{\mathrm{var}(Y)}. \end{aligned}$$

- What is the variance of a $\mathrm{Ber}(p)$ random variable?

  Solution: For $X_i \sim \mathrm{Ber}(p)$, we have

$$\begin{aligned} \mathbb{E}[(X_i - \underbrace{\mathbb{E}[X_i]}_{p})^2] &= (1-p)^2 \cdot p + (0-p)^2 \cdot (1-p) \\ &= (1-2p+p^2)p + p^2(1-p) \\ &= p - 2p^2 + p^3 + p^2 - p^3 \\ &= p - p^2 \\ &= \boxed{p(1-p)}. \end{aligned}$$

- Recall that a binomial random variable S that has a $\mathrm{Binomial}(n,p)$ distribution can be written as $S = \sum_{i=1}^n X_i$ where the X_i's are i.i.d. $\mathrm{Ber}(p)$ random variables. What is the variance of S?

  Solution: Since the X_i's are independent,

$$\mathrm{var}(S) = \mathrm{var}\Big(\sum_{i=1}^n X_i\Big) = \sum_{i=1}^n \mathrm{var}(X_i) = \boxed{np(1-p)}.$$

You have used 5 of 5 submissions