Exercise: Bayes' Theorem for Random Variables - Medical Diagnosis, Continued

(4/4 points)

Recall the medical diagnosis setup from before, summarized in these tables:

X		Prob.
	healthy	0.999
	infected	0.001

$p_{Y\mid X}$		X: healthy	X: infected
Y	positive	0.01	0.99
	negative	0.99	0.01

Recall that Bayes' theorem is given by

$$p_{X|Y}(x \mid y) = \frac{p_X(x)p_{Y|X}(y \mid x)}{\sum_{x'} p_X(x')p_{Y|X}(y \mid x')}$$

for all values x that random variable X can take on.

Use Bayes' theorem to compute the following probabilities: (Please be precise with at least 3 decimal places, unless of course the answer doesn't need that many decimal places. You could also put a fraction.)

- $p_{X|Y}(\text{healthy} \mid \text{positive}) =$ 0.9098360655737705 ✔

  Answer: 0.9098360656

- $p_{X|Y}(\text{healthy} \mid \text{negative}) =$ 0.9999898889810116 ✔

  Answer: 0.999989889

What is the MAP estimate for X given $Y = \text{positive}$?

- (selected) healthy ✔
- infected

What is the MAP estimate for X given $Y = \text{negative}$?

- (selected) healthy ✔
- infected

Solution:

Use Bayes' theorem to compute the following probabilities:

- $$\begin{aligned} & p_{X|Y}(\text{healthy} \mid \text{positive}) \\ &= \frac{p_{X|Y}(\text{positive} \mid \text{healthy}) p_X(\text{healthy})}{p_{X|Y}(\text{positive} \mid \text{healthy}) p_X(\text{healthy}) + p_{X|Y}(\text{positive} \mid \text{infected}) p_X\text{(infect}} \\ &= \frac{0.01 \times 0.999}{0.01 \times 0.999 + 0.99 \times 0.001} \\ &\approx \boxed{0.9098360656}. \end{aligned}$$

- $$\begin{aligned} & p_{X|Y}(\text{healthy} \mid \text{negative}) \\ &= \frac{p_{X|Y}(\text{negative} \mid \text{healthy}) p_X(\text{healthy})}{p_{X|Y}(\text{negative} \mid \text{healthy}) p_X(\text{healthy}) + p_{X|Y}(\text{negative} \mid \text{infected}) p_X\text{(infec}} \\ &= \frac{0.99 \times 0.999}{0.99 \times 0.999) + 0.01 \times 0.001} \\ &\approx \boxed{0.999989889}. \end{aligned}$$

Note that if $Y = \text{positive}$, then the probability that $X = \text{infected}$ is just 1 minus the first probability computed. In this case, it is clear that **healthy** is the MAP estimate of X given $Y = \text{positive}$ because

$$\underbrace{p_{X|Y}(\text{healthy} \mid \text{positive})}_{\approx 0.9098360656} > \underbrace{p_{X|Y}(\text{infected} \mid \text{positive})}_{\approx 1 - 0.9098360656 = 0.0901639344}.$$

Similarly, if $Y = \text{negative}$, the MAP estimate for X is still **healthy** since

$$\underbrace{p_{X|Y}(\text{healthy} \mid \text{negative})}_{\approx 0.999989889} > \underbrace{p_{X|Y}(\text{infected} \mid \text{negative})}_{\approx 1-0.999989889=0.000010111}.$$

You have used 1 of 5 submissions