DOC 221 Dinámica orbital y control de actitud

Solutions to Problems Lecture ADCS - VI

Problem 1:

(a) The $\boldsymbol{a}_i$ vectors have to fulfill following conditions:

$$\vec{a}_1 \cdot \vec{a}_1 = \vec{a}_2 \cdot \vec{a}_2 = \vec{a}_3 \cdot \vec{a}_3 = 1 \quad \text{and} \quad \vec{a}_1 \cdot \vec{a}_2 = \vec{a}_2 \cdot \vec{a}_3 = \vec{a}_1 \cdot \vec{a}_3 = 0$$

Analog for the $\mathbf{b}_i$ vectors.

(b) To find the direction cosine matrix $\mathbf{C}_{ab}$ (respectively $\mathbf{C}_{ai}$ or $\mathbf{C}_{bi}$) we make us of the following:

$$\text{Matrix} \quad \mathbf{C}_{ab} \quad \text{has matrix elements } C_{ij} = \cos\alpha_{ij} = \vec{a}_i \cdot \vec{b}_j$$
$$\text{Matrix} \quad \mathbf{C}_{ai} \quad \text{has matrix elements } C_{ij} = \cos\alpha_{ij} = \vec{a}_i \cdot \vec{i}_j$$
$$\text{Matrix} \quad \mathbf{C}_{bi} \quad \text{has matrix elements } C_{ij} = \cos\alpha_{ij} = \vec{b}_i \cdot \vec{i}_j$$

Because all base vectors ($\mathbf{a}_i$ and $\mathbf{b}_i$) have unit length and are expressed in terms of the frame I vector components, the scalar product of the corresponding vectors will provide the needed direction cosines.

The rotation matrix for problem (b) is $\mathbf{C}_{ab} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \end{bmatrix}$

(c) The rotation matrix is $\mathbf{C}_{ai} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \end{bmatrix}$

(d) The rotation matrix is $\mathbf{C}_{bi} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

(e) Compute $\mathbf{C}_{ab} = \mathbf{C}_{ai}\,(\mathbf{C}_{bi})^T$

$$\mathbf{C}_{ab} = \mathbf{C}_{ai}(\mathbf{C}_{bi})^T = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(f) $\mathbf{C}_{ab}(\mathbf{C}_{ab})^T = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(g) Take for example $\mathbf{A} = \mathbf{C}_{ai}$ and $\mathbf{B} = (\mathbf{C}_{bi})^T \rightarrow \mathbf{AB} \neq \mathbf{BA}$.

(h) No, because $\mathbf{CC}^T \neq \mathbf{1}$.

Problem 2:

$\mathbf{a} \times \mathbf{b} = \mathbf{a}^{\times}\mathbf{b}$ follows using the cross vector product definition

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

To show $\mathbf{a}^{\times}\mathbf{b} = -\mathbf{b}^{\times}\mathbf{a}$ compute it element wise

$$\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -a_2 b_3 + a_3 b_2 \\ -a_3 b_1 + a_1 b_3 \\ -a_1 b_2 + a_2 b_1 \end{bmatrix}$$

Finally compute $\mathbf{a}^{\times}\mathbf{a} = \mathbf{0}$ also element wise

$$\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_2 a_3 - a_3 a_2 \\ a_3 a_1 - a_1 a_3 \\ a_1 a_2 - a_2 a_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Problem 3:

(a) The rotational transformation is given by

$$\mathbf{C}_{21} = \mathbf{C}_3(\theta_3)\mathbf{C}_2(\theta_2)\mathbf{C}_3(\theta_1)$$
$$= \begin{bmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} c_3c_2c_1 - s_3s_1 & c_3c_2s_1 + s_3c_1 & -c_3s_2 \\ -s_3c_2c_1 - c_3s_1 & -s_3c_2s_1 + c_3c_1 & s_3s_2 \\ s_2c_1 & s_2s_1 & c_2 \end{bmatrix}$$

where $c_x = \cos\Theta_x$ and $s_x = \sin\Theta_x$.

(b) Where

$$\theta_1 = \tan^{-1}(\tfrac{C_{32}}{C_{31}})$$
$$\theta_2 = \cos^{-1}(C_{33})$$
$$\theta_3 = -\tan^{-1}(\tfrac{C_{23}}{C_{13}})$$

(c) For the 3-2-3 sequence, we have the angular velocity vector given by

$$\vec{\omega}^{2/1} = \vec{\omega}^{2/1''} + \vec{\omega}^{1''/1'} + \vec{\omega}^{1'/1} = \dot{\theta}_3\vec{b}_1 + \dot{\theta}_2\vec{a}_2'' + \dot{\theta}_1\vec{a}_3'$$

Write angular velocity vector in frame 2

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{bmatrix} + \mathrm{C}_3(\theta_3)\begin{bmatrix} 0 \\ \dot{\theta}_2 \\ 0 \end{bmatrix} + \mathrm{C}_3(\theta_3)\mathrm{C}_2(\theta_2)\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{bmatrix} =$$
$$= \begin{bmatrix} -\cos\theta_3\sin\theta_2 & \sin\theta_3 & 0 \\ \sin\theta_3\sin\theta_2 & \cos\theta_3 & 0 \\ \cos\theta_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

The inverse relationship is found by inverting the matrix

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} -\cos\theta_3\sin\theta_2 & \sin\theta_3 & 0 \\ \sin\theta_3\sin\theta_2 & \cos\theta_3 & 0 \\ \cos\theta_2 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

To check if the inversion matrix is the one given in the problems, we compute the following:

$$\begin{bmatrix} -\cos\theta_3\sin\theta_2 & \sin\theta_3 & 0 \\ \sin\theta_3\sin\theta_2 & \cos\theta_3 & 0 \\ \cos\theta_2 & 0 & 1 \end{bmatrix}^{-1} = \frac{1}{\sin\theta_2}\begin{bmatrix} -\cos\theta_3 & \sin\theta_3 & 0 \\ \sin\theta_3\sin\theta_2 & \cos\theta_3\sin\theta_2 & 0 \\ \cos\theta_3\cos\theta_2 & -\sin\theta_3\cos\theta_2 & \sin\theta_2 \end{bmatrix}$$

$$\begin{bmatrix} -\cos\theta_3\sin\theta_2 & \sin\theta_3 & 0 \\ \sin\theta_3\sin\theta_2 & \cos\theta_3 & 0 \\ \cos\theta_2 & 0 & 1 \end{bmatrix}\frac{1}{\sin\theta_2}\begin{bmatrix} -\cos\theta_3 & \sin\theta_3 & 0 \\ \sin\theta_3\sin\theta_2 & \cos\theta_3\sin\theta_2 & 0 \\ \cos\theta_3\cos\theta_2 & -\sin\theta_3\cos\theta_2 & \sin\theta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(d) From the kinematic relationship obtained in part (c), it is clear that the kinematics are not defined when $\sin\Theta_2 = 0$, which occurs when $\Theta_2 = 0, \pi, -\pi$. This is the singularity of the 3-2-3 Euler rotation sequence. If $\mathbf{C}_2(\Theta_2 = 0)$, then we cannot distinguish the first rotation axis from the third one.

Problem 4:

(a) Use the 3-2-1 Equation from Lecture ADCS-VI page 35. The direction cosine matrix for the 3-2-1 Euler angles is

$$\begin{aligned}\mathbf{C}_{21}(\theta_1,\theta_2,\theta_3) &= \mathbf{C}_1(\theta_1)\mathbf{C}_2(\theta_2)\mathbf{C}_3(\theta_3) \\ &= \begin{bmatrix} c_2c_3 & c_2s_3 & -s_2 \\ s_1s_2c_3 - c_1s_3 & s_1s_2s_3 + c_1c_3 & s_1c_2 \\ c_1s_2c_3 + s_1s_3 & c_1s_2s_3 - s_1c_3 & c_1c_2 \end{bmatrix}\end{aligned}$$

$$\mathbf{C}_{21} = \begin{bmatrix} 0.892539 & 0.157379 & -0.42618 \\ -0.275451 & 0.932257 & -0.234570 \\ 0.357073 & 0.325773 & 0.875426 \end{bmatrix}$$

(b) The principle Euler eigenaxis rotation angle ϕ is given by

$$\phi = \cos^{-1}\left(\tfrac{1}{2}[C_{11} + C_{22} + C_{33} - 1]\right) = 31.7762^\circ$$

(c) The principle Euler rotation eigenaxis is then given by

$$\vec{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \tfrac{1}{2\sin\phi}\begin{bmatrix} C_{23} - C_{32} \\ C_{31} - C_{13} \\ C_{12} - C_{21} \end{bmatrix} = \frac{1}{2\sin(31.7762^\circ)}\begin{bmatrix} -0.234570 - 0.325773 \\ 0.357073 + 0.422618 \\ 0.157379 + 0.275451 \end{bmatrix} = \begin{bmatrix} -0.532035 \\ 0.740302 \\ 0.410964 \end{bmatrix}$$

Check the result $\mathbf{C}_{21}\mathbf{e} = \mathbf{e}$:

$$\begin{bmatrix} 0.892539 & 0.157379 & -0.422618 \\ -0.275451 & 0.932257 & -0.234570 \\ 0.357073 & 0.325773 & 0.875426 \end{bmatrix}\begin{bmatrix} -0.532035 \\ 0.740302 \\ 0.410964 \end{bmatrix} = \begin{bmatrix} -0.532035 \\ 0.740302 \\ 0.410964 \end{bmatrix}$$

(d) Compute first q_4 term with the positive sign. The unit quaternion **q** is then given by

$$q_4 = \tfrac{1}{2}(1 + C_{11} + C_{22} + C_{33})^{\frac{1}{2}} = 0.936117$$

$$\vec{q} = \tfrac{1}{4q_4}\begin{bmatrix} C_{23} - C_{32} \\ C_{31} - C_{13} \\ C_{12} - C_{21} \end{bmatrix} = \begin{bmatrix} 0.309976 \\ -0.144544 \\ -0.0818996 \end{bmatrix}$$

$$\mathbf{q} = \begin{bmatrix} 0.309976 \\ -0.144544 \\ -0.08189 \\ 0.936117 \end{bmatrix}$$

(e) Check if

$$|\mathbf{q}| = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2} = 1$$

Otherwise make **(f)** unit quaternion using

$$\frac{\mathbf{q}}{|\mathbf{q}|}$$

Problem 5:

Using the 3-2-1 Equation from Lecture VI page 35

$$\mathbf{C}_{21}(\theta_1,\theta_2,\theta_3) = \mathbf{C}_1(\theta_1)\mathbf{C}_2(\theta_2)\mathbf{C}_3(\theta_3)$$
$$= \begin{bmatrix} c_2c_3 & c_2s_3 & -s_2 \\ s_1s_2c_3 - c_1s_3 & s_1s_2s_3 + c_1c_3 & s_1c_2 \\ c_1s_2c_3 + s_1s_3 & c_1s_2s_3 - s_1c_3 & c_1c_2 \end{bmatrix}$$

We can now write the orientation matrix $\mathbf{C}_{AI}$ and $\mathbf{C}_{BI}$:

$$\mathbf{C}_{AI} = \begin{bmatrix} 0.612372 & 0.353553 & 0.707107 \\ -0.780330 & 0.126826 & 0.612372 \\ 0.126826 & -0.926777 & 0.353553 \end{bmatrix}$$
$$\mathbf{C}_{BI} = \begin{bmatrix} 0.892539 & 0.157379 & -0.422618 \\ -0.275451 & 0.932257 & -0.234570 \\ 0.357073 & 0.325773 & 0.875426 \end{bmatrix}$$

The direction cosine matrix $\mathbf{C}_{AB}$ that describes the attitude of B relative to A is computed as $\mathbf{C}_{AB} = \mathbf{C}_{AI}\mathbf{C}_{BI}^T$.

$$\mathbf{C}_{AB} = \mathbf{C}_{AI}\left(\mathbf{C}_{BI}\right)^T = \begin{bmatrix} 0.303372 & -0.004942 & 0.952859 \\ -0.935315 & 0.189534 & 0.298769 \\ -0.182075 & -0.982862 & 0.052877 \end{bmatrix}$$

Using the above transformation $\mathbf{C}_{21}$, the relative 3-2-1 Euler angles are

$$\theta_1 = \tan^{-1}(\tfrac{C_{23}}{C_{33}}) = 79.96^o$$
$$\theta_2 = -\sin^{-1}(C_{13}) = -72.33^o$$
$$\theta_3 = \tan^{-1}(\tfrac{C_{12}}{C_{11}}) = -0.93^o$$

Problem 6:

The rotation matrix for a given principle Euler eigenaxis rotation is given by

$$\mathbf{C}(\vec{e},\phi) = \cos\phi\mathbf{1} + (1-\cos\phi)\vec{e}\vec{e}^T - \sin\phi\mathbf{e}^\times$$

$$\mathbf{1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \vec{e}\vec{e}^T = \begin{bmatrix} e_1e_1 & e_1e_2 & e_1e_3 \\ e_2e_1 & e_2e_2 & e_2e_3 \\ e_3e_1 & e_2e_3 & e_3e_3 \end{bmatrix} \quad \mathbf{e}^\times = \begin{bmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{bmatrix}$$

Compute the matrix for the given values in problem 6

$$\mathbf{C} = \begin{bmatrix} 0.80474 & 0.50588 & -0.31062 \\ -0.31062 & 0.80474 & 0.50588 \\ 0.50588 & -0.31062 & 0.80474 \end{bmatrix}$$

The corresponding 3-2-1 Euler angles are

$$\theta_1 = \tan^{-1}(\tfrac{C_{23}}{C_{33}}) = 32.2^o$$
$$\theta_2 = -\sin^{-1}(C_{13}) = 18.1^o$$
$$\theta_3 = \tan^{-1}(\tfrac{C_{12}}{C_{11}}) = 32.2^o$$