ADCS - II
Orbital Mechanics and Perturbations

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Summary of last lecture

Overview of ADCS and lecture

Spacecraft environment:

Gravity

- Newton´s Cannonball

Sun

- Electromagnetic spectrum → Planck´s distribution → Stefan Boltzmann law → Luminosity → Solar flux density
- Solar radiation pressure
- Solar wind

Earth atmosphere

- Maxwell-Boltzmann distribution (thermal vs. escape velocity)

Earth magnetic field

- Spherical harmonic expansion of magnetic potential

Outline

Summary of Kepler orbits

Orbital perturbations

- Special and general perturbation

Non-spherical Earth gravity field

- Gravity potential
- Laplace equation
- Spherical harmonic functions
- Gravity potential for ideal spherical Earth and for real Earth

Non-spherical Earth perturbation

- Periodic and Secular
- Effect on Ω (Sun-synchronous orbits)
- Effect on ω (Molniya orbits)
- No effect on a, e and i

Summary of Kepler orbits

Keplerian two-body motion

Definition of two-body problem:
Motion of two bodies due only to own mutual gravitational attraction

Assumption: two point masses (or spherical mass distribution)

Differential equation defining Kepler orbit (in inertial frame):

$$\boxed{\ddot{\vec{r}} + \frac{\mu}{|\vec{r}|^3}\vec{r} = 0} \qquad \mu = G(m_1 + m_2) \approx Gm_2$$

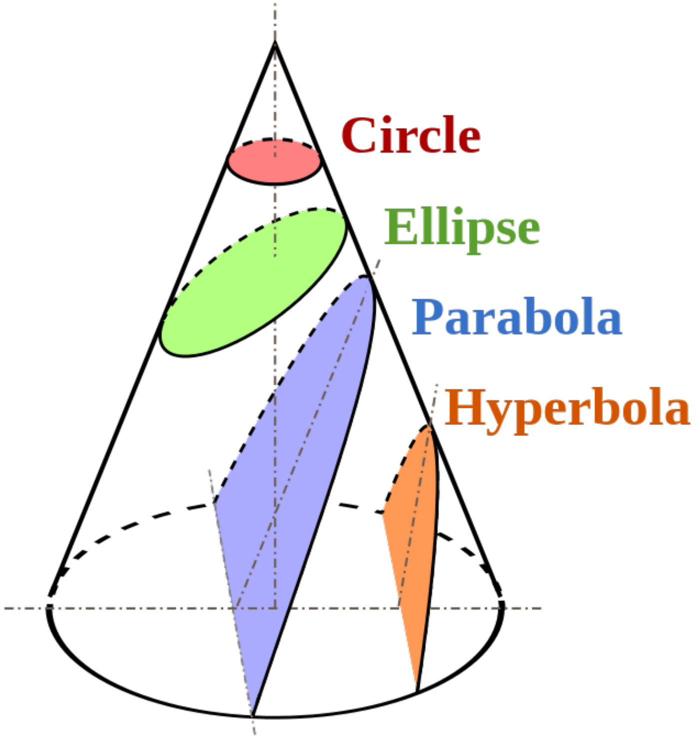
Initial conditions: $\vec{r}(0) = \vec{r}_0$

$$\dot{\vec{r}}(0) = \vec{v}(0) = \vec{v}_0$$

Reasonable model for: - spacecraft orbiting planets
- planets orbiting sun
- etc.

Shape of Kepler orbit

Most general description of shape of Kepler orbit is a conic section

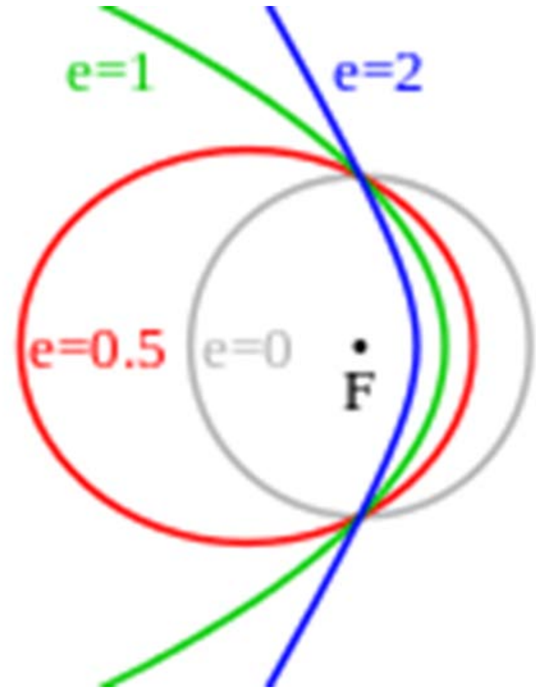


Orbital motion and equation
Polar equation of conic section

Kepler first law

Each planet moves along an elliptical orbit with Sun at one focus

$$\boxed{r(\theta) = \frac{a(1-e^2)}{1+e\cos\theta}}$$



Possible motions of 2 body system

r and Θ are polar coordinates

a = length of semi-major axis

e = eccentricity

e is constant and determined by angular momentum and energy

$e = 0$	circle
$0 < e < 1$	ellipse
$e = 1$	parabola
$e > 1$	hyperbola

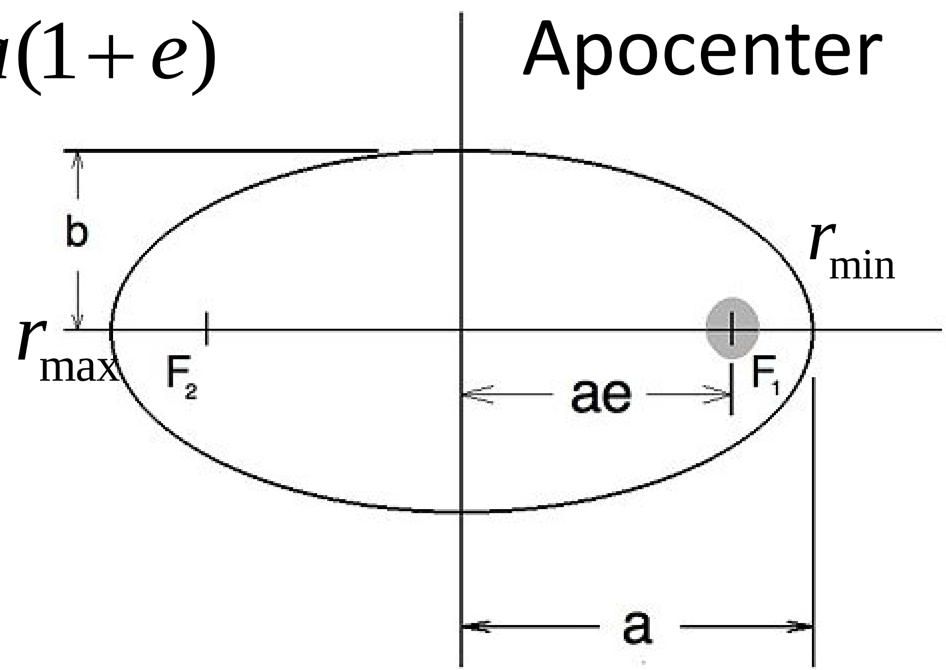
2-dimensional Kepler orbits

$$r(\theta) = \frac{a(1-e^2)}{1+e\cos\theta}$$

$$r_{\min}(\theta = 0) = \frac{a(1-e^2)}{1+e} = a(1-e)$$

$$r_{\max}(\theta = \pi) = \frac{a(1-e^2)}{1-e} = a(1+e)$$

Pericenter
Apocenter



Constants of motion of Kepler problem

Total orbital energy is conserved $E_{tot} = E_{kin} + E_{pot} = \frac{v^2}{2} - \frac{\mu}{r}$

Orbital motion lies in a fixed plane

Orbital plane is normal to angular momentum vector

Angular momentum is a constant vector

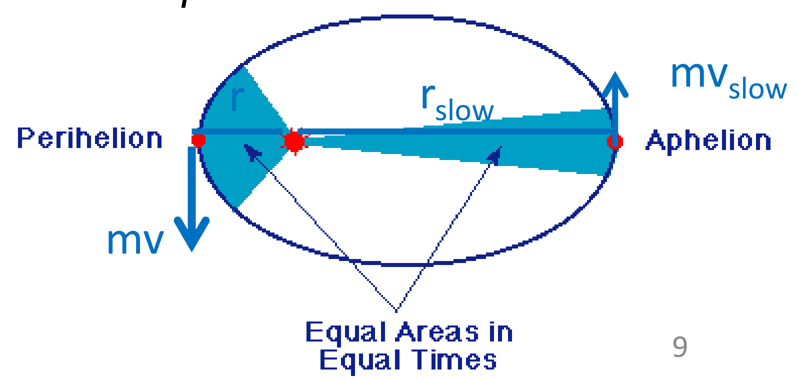
and is perpendicular to position and velocity vector $\vec{h} = \vec{r} \times m\vec{v}$

$$\dot{\vec{h}} = \dot{\vec{r}} \times m\dot{\vec{r}} + \vec{r} \times m\ddot{\vec{r}} = \vec{r} \times m\vec{r}\,\frac{GM}{r^3} = 0$$

<u>**Kepler second law**</u>

Line joining Sun and planet sweeps out equal areas in equal times

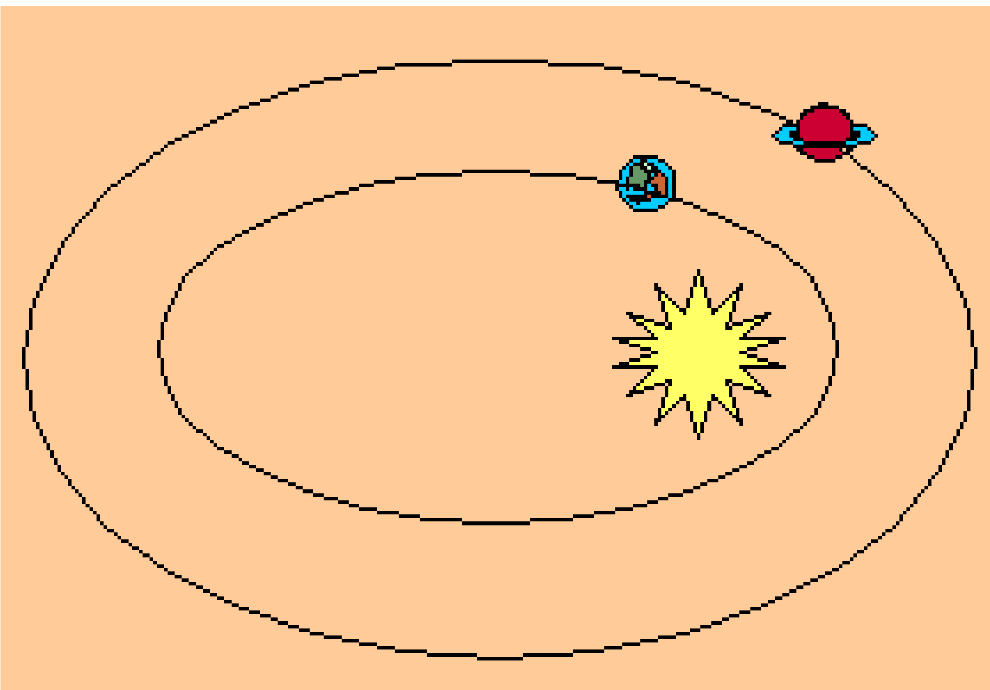
Shape of orbital motion is constant



Relationship of orbital period and distance from Sun

Kepler third law

Squares of periods of planets proportional to cubes of their mean distances from Sun



For elliptical orbit
T is orbital period

$$T = 2\pi\sqrt{\frac{a^3}{\mu}}$$

(Newton´s gravitational law)

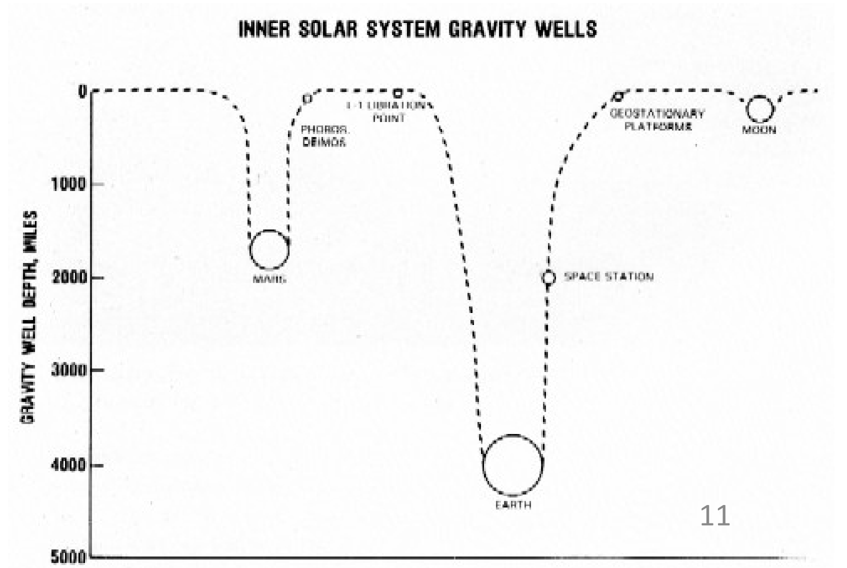
Vis-viva equation

$$\boxed{E_{tot} = E_{kin} + E_{pot} = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}}$$

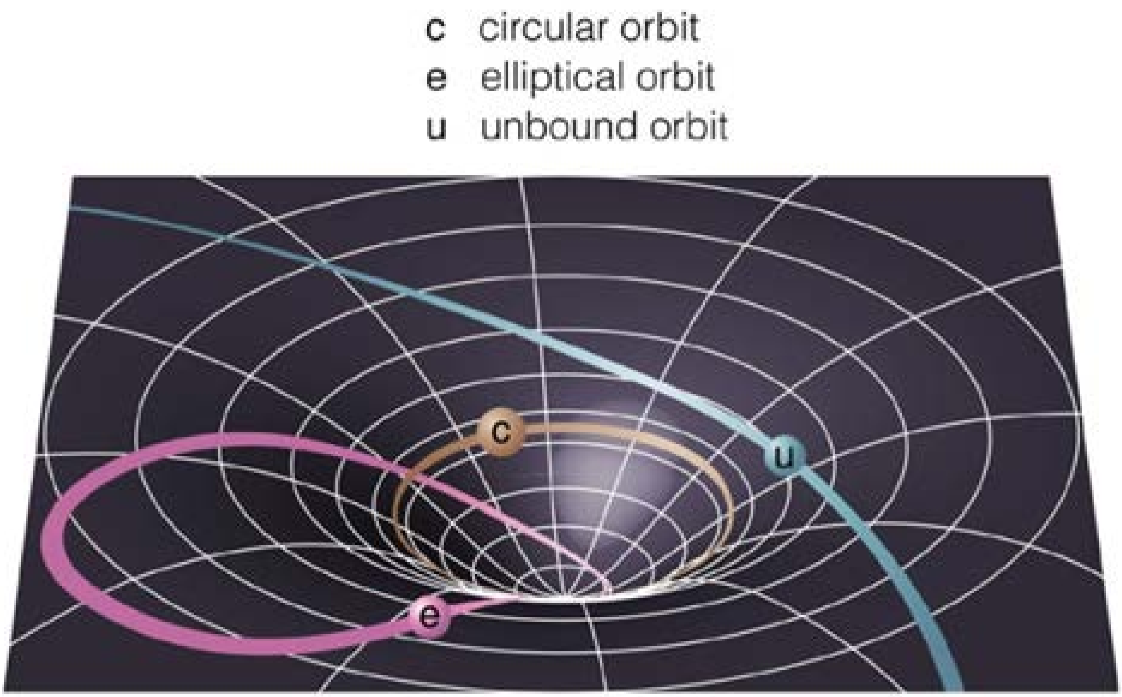
v = orbital speed of orbiting object
r = distance of orbiting body from central body
a = length of semi-major axis
μ = standard gravitational parameter

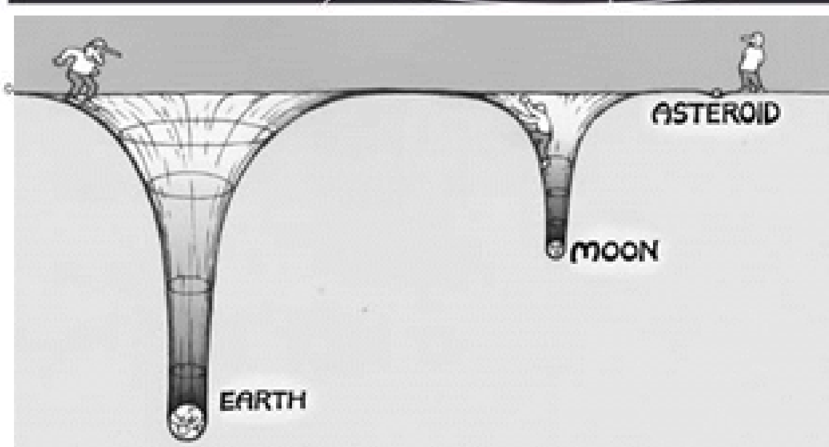
The vis-viva equation gives a relation between velocity and position (and semi-major axis)

Satellite position and speed (magnitude of velocity) determine total amount of energy of satellite



Orbits in gravity well (Gravitationsbrunnen)





Gravity well is conceptual model of gravitational field around massive body in space

Massive objects appear deep and far-reaching

Balls rolling around walls of gravity well behave like orbiting satellites

E.g.

Exact velocity →circle orbit

Faster velocity →elliptical orbit

Even faster →escape

How many variables define an orbit?

Two-body problem

$$\ddot{\vec{r}} = -\frac{\mu}{|\vec{r}|^3}\vec{r}$$

3 second order differential equations
Need 6 variables to solve
3 second order differential equations

Equivalent to

$$\dot{\vec{r}} = \vec{v} \qquad \vec{r}(0) = \vec{r}_0$$

$$\dot{\vec{v}} = -\tfrac{\mu}{r^3}\vec{r} \qquad \vec{v}(0) = \vec{v}_0$$

6 first order differential equations
Need 6 initial conditions
(position and velocity)

But are position and velocity an useful parameterization of orbit?

Position and velocity have not much information about orbit
No idea what type of orbit or what is orbit altitude

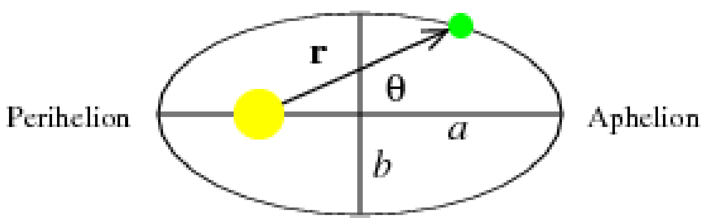
Better use other set of six variables

2- and 3-dimensional Kepler orbits

2-dimensional orbit (orbit lies in a plane)
Ellipse specified by size (*a*) and shape (*e*)
Third parameter to temporally locate satellite in orbit

- *a* = semi-major axis
- *e* = eccentricity
- *T* = time of perigee passage

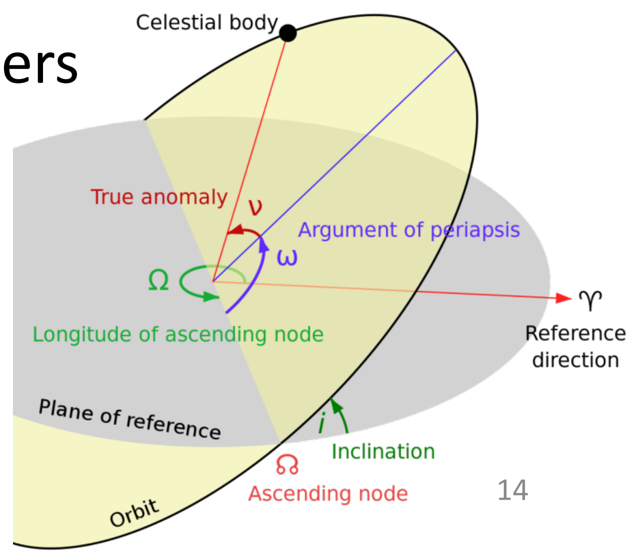
(*a, e, T*) specify motion within orbit plane



3-dimensional orbit (three additional parameters to specify orientation of orbital plane)

- i = inclination $0 < i < \pi$
- Ω = right ascension of ascending node $0 < i < 2\pi$
- ω = argument of perigee $0 < i < 2\pi$

3-1-3 Euler rotation



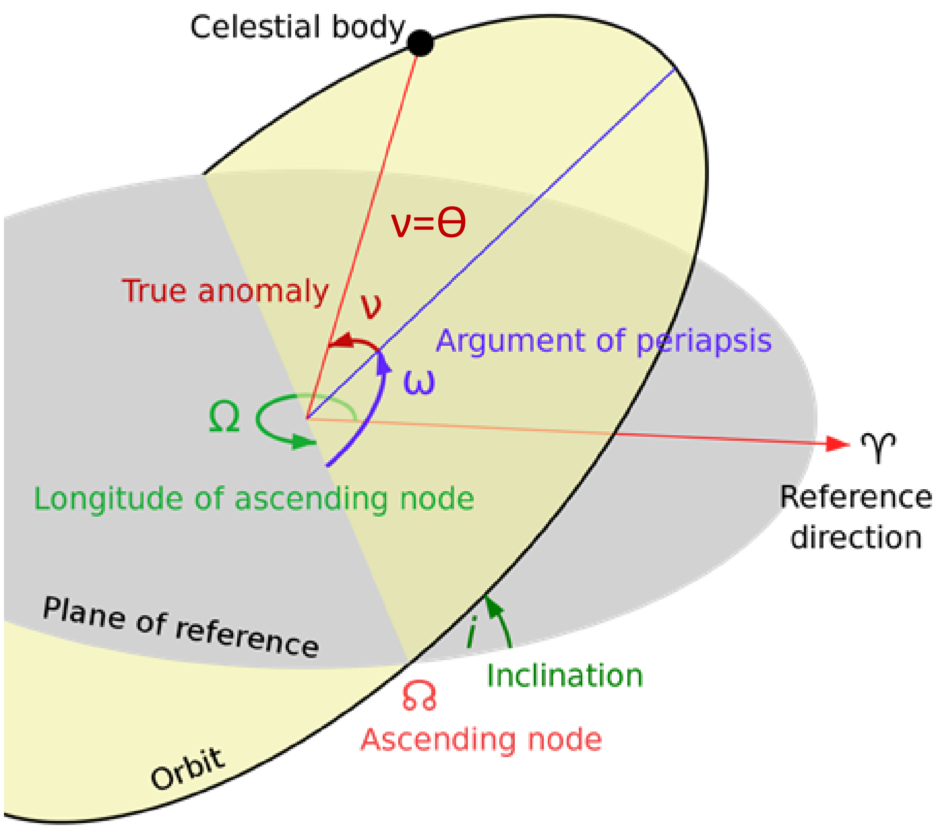
3-dimensional Kepler orbit with 6 orbital elements

Definition of ellipse
e: shape of orbit
a: size of orbit

Definition of orbital plane
i: orients orbital plane with respect to ecliptic plane
Ω: longitude of intersection of orbital and ecliptic planes

Orientation of ellipse within orbital plane
ω: orients semi-major axis with respect to ascending node

Position of object on ellipse
Θ: orients celestial body in space



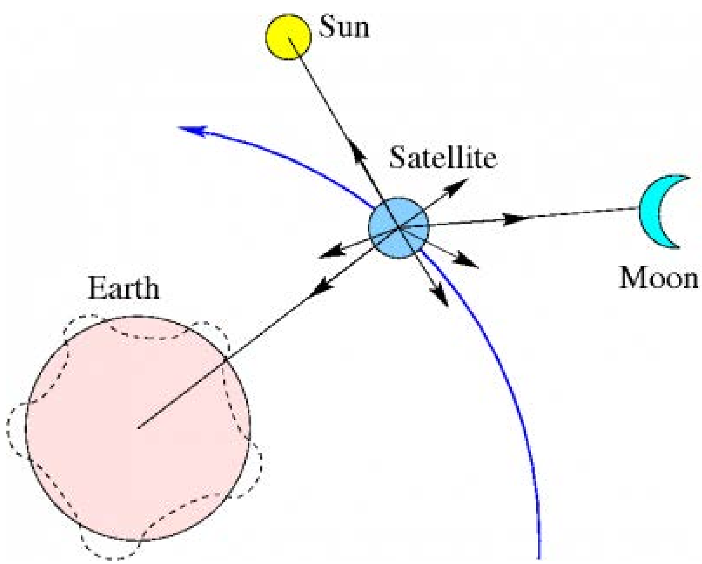
(*e,a,i,Ω,ω,Θ*) are a complete set of orbital elements. They completely specify orbit and are equivalent to knowing position and velocity for a given moment.

Question

Assume:

Circular or elliptic orbit for a satellite

Will satellite stay in Kepler orbit forever?



Orbital perturbations

Non-Kepler motion

- In practical situation satellite experiences significant perturbations
  External forces and accelerations cause change to theoretical two-body orbit

- Perturbations are sufficient to cause predictions of position of satellite based on Kepler motion to be significant error in short time

- Kepler orbits not sufficient for accurate orbit or attitude calculation, but good enough for overall mission characteristics

- Historically, many people worked on analytical expressions of these perturbations, i.e. Lagrange, Euler, Gauss, etc.

- Today: Orbits are mostly numerically integrated although some analytical studies still exist

Orbital perturbations

Perturbation of Kepler orbits due to:

Non-spherical Earth (today´s Lecture)

Earth is not point mass or perfectly spherical

Third body

Presence of other bodies (Sun, Moon) and their gravitational fields

Atmospheric drag

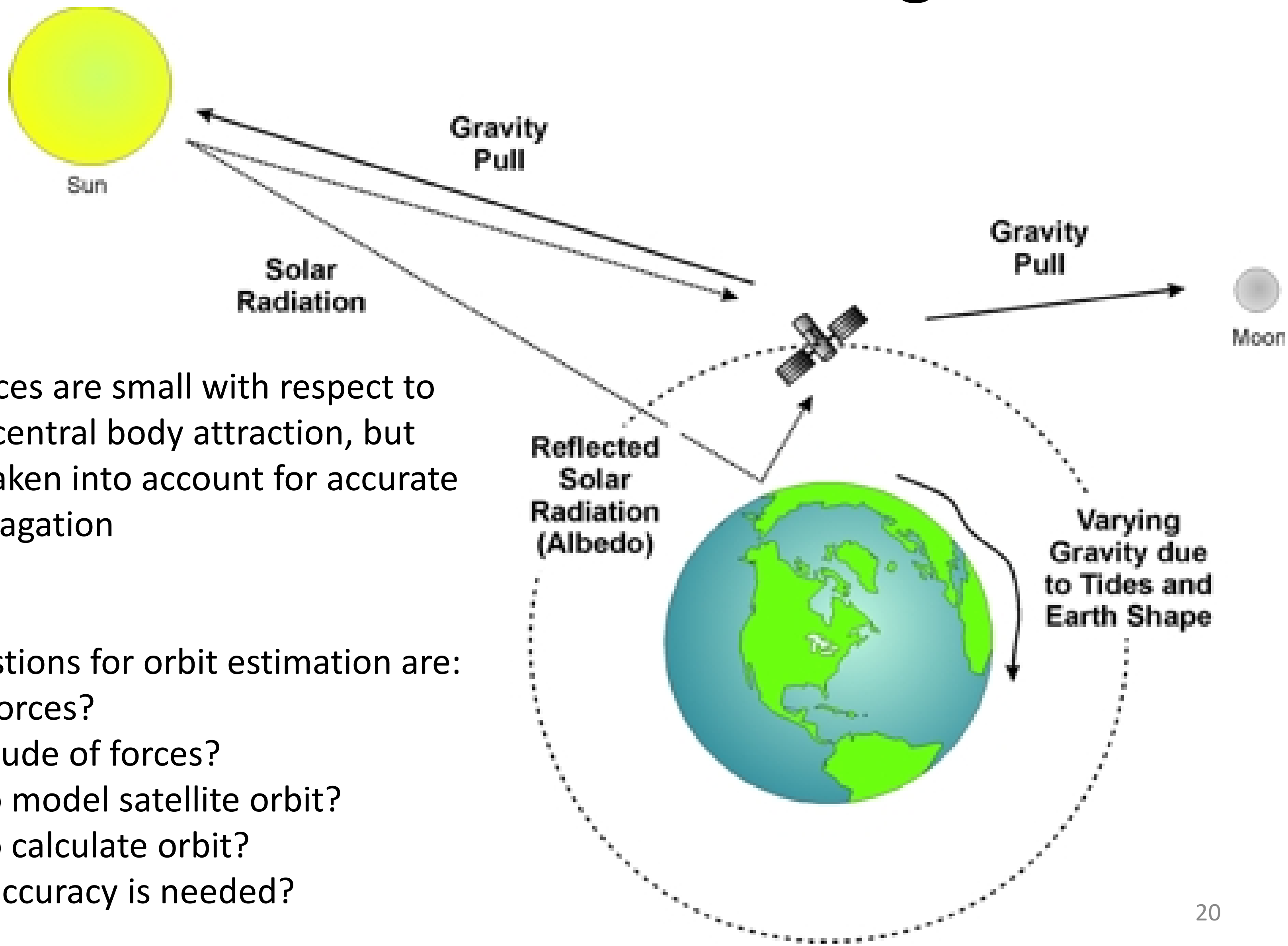
Residual atmosphere creates drag → results in gradual orbit decay

Solar radiation

Light from Sun creates pressure on spacecraft

Caused by momentum transfer from photons to spacecraft

Disturbance forces affecting orbit

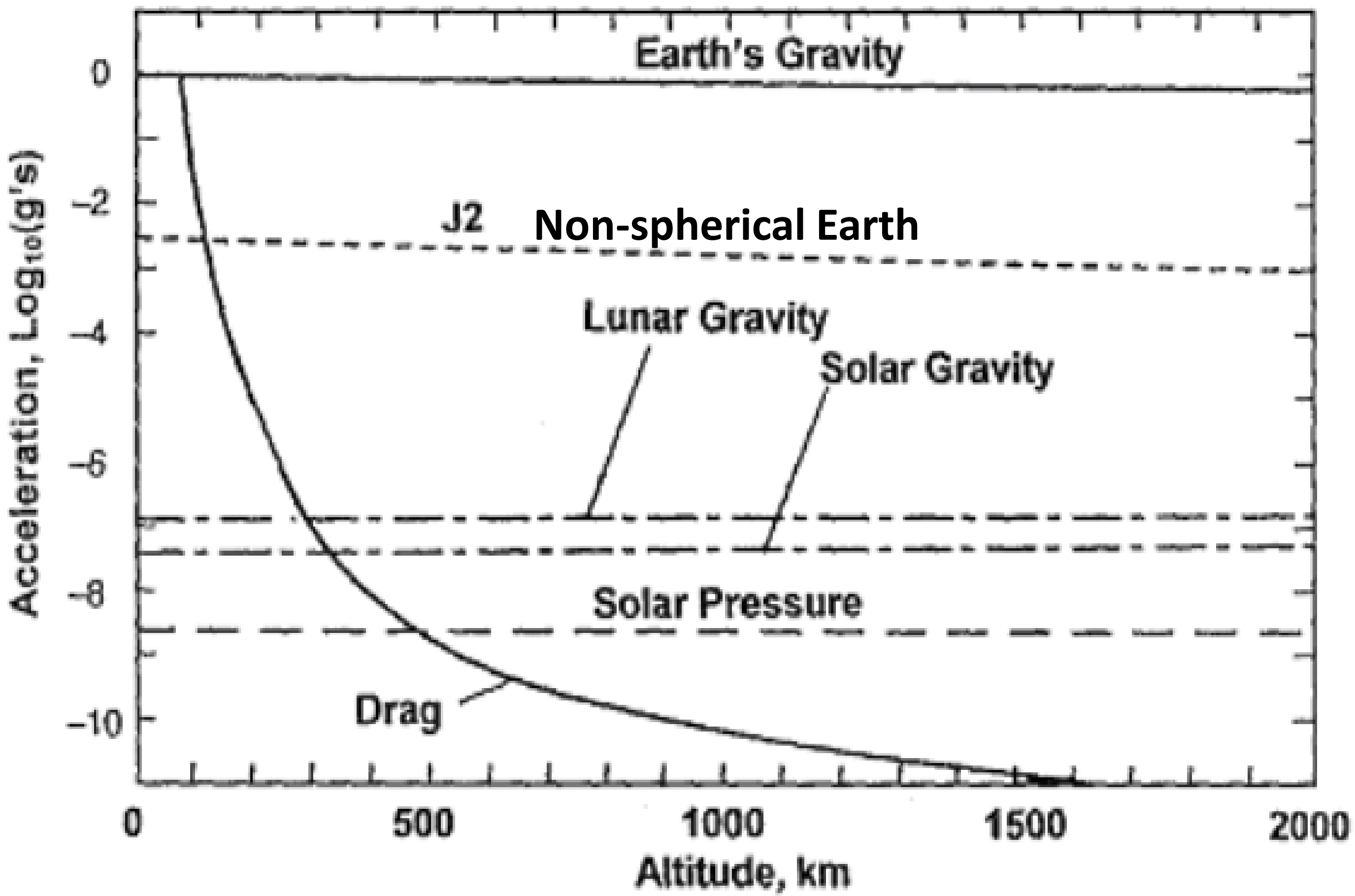


Disturbances are small with respect to spherical central body attraction, but must be taken into account for accurate orbit propagation

Main questions for orbit estimation are:

- What forces?
- Magnitude of forces?
- How to model satellite orbit?
- How to calculate orbit?
- What accuracy is needed?

Relative importance of orbit perturbation for near Earth orbit



Logarithm of forces normalized with 1 g as function of altitude
Dominant forces: Earth´s gravity field and J_2 perturbation due to non-spherical Earth
Curve for drag has large uncertainty up to one order of magnitude (due to solar activity)

For near Earth orbits

Earth gravity is dominant

Non-spherical Earth (irregularity of Earth gravity) and atmospheric drag

Source	**Acceleration** [m/s^2]
Earth gravity	10
Non-spherical Earth	$5 \cdot 10^{-2}$
Atmospheric drag	$1 \cdot 10^{-1}$ to $1 \cdot 10^{-9}$
Solar radiation	$1 \cdot 10^{-9}$
Sun, Moon (3rd body)	$1 \cdot 10^{-7}$

Perturbation and their respective importance for different altitude

300 km

Non-spherical Earth
Atmospheric drag

1000 km

Non-spherical Earth
Sun and Moon

36000 km

Non-spherical Earth
Sun and Moon
Solar radiation pressure

Perturbations to Kepler orbit

Include perturbation directly in equation of motion

$$\boxed{\ddot{\vec{r}} = -\frac{\mu}{|\vec{r}|^3}\vec{r} + \vec{a}_{pert.}}$$

$$\vec{r}(0) = \vec{r}_0$$

$$\dot{\vec{r}}(0) = \vec{v}(0) = \vec{v}_0$$

$$\vec{a}_{pert.} = \vec{a}_{J2} + \vec{a}_{drag} + \vec{a}_{Sol\,\mathrm{Pr}\,es} + \vec{a}_{Third-body}$$

Deal with perturbation techniques (two categories):

<u>Special perturbations</u>

Numerical integration of equation of motion with initial conditions
Position and velocity at requested time are computed directly from initial conditions in single step (e. g. Cowell´s Method, Encke´s Method)

<u>General perturbations</u>

Variation of orbital parameters (normally constant for Kepler orbit)
(Determine analytically variations in orbital elements from perturbation effects)

$$\left\{\frac{\mathrm{d}a}{\mathrm{d}t}, \frac{\mathrm{d}e}{\mathrm{d}t}, \frac{\mathrm{d}i}{\mathrm{d}t}, \frac{\mathrm{d}\Omega}{\mathrm{d}t}, \frac{\mathrm{d}\omega}{\mathrm{d}t}\right\}$$

Special versus general perturbation

<u>Special (numerical) perturbation</u>:

- Perform some kind of integration of $\ddot{\vec{r}} = -\frac{\mu}{|\vec{r}|^3}\vec{r} + \vec{a}_{pert.}$
- Solution only valid for one set of initial conditions
  (Need to repeat if other set of initial conditions)
- For high accuracy satellite orbits it is necessary to use numerical integration
- Method flexible and can incorporate any arbitrary perturbative acceleration

<u>General (analytical) perturbation</u>:

- Deal with variations in orbital elements due to perturbative acceleration $a_{\text{pert.}}$
- Solution valid for any set of initial conditions
- Better physical understanding of perturbing forces
- Useful for mission planning (fast answer)
- Method needs approximations in derivations and therefore not as accurate as special perturbation method

General perturbation
Variation of orbital parameters

Originally developed by Euler and improved by Lagrange and Gauss

Called variation of orbital parameters, because orbital elements (i.e. constant parameters for Kepler orbit) are changing in presence of perturbations

Variation of orbital parameters equations are system of first-order differential equations that describe rates of change of orbital elements

What are $\left\{ \frac{\mathrm{d}a}{\mathrm{d}t}, \frac{\mathrm{d}e}{\mathrm{d}t}, \frac{\mathrm{d}i}{\mathrm{d}t}, \frac{\mathrm{d}\Omega}{\mathrm{d}t}, \frac{\mathrm{d}\omega}{\mathrm{d}t} \right\}$ as a function of perturbation $a_{\text{pert.}}$?

Derivation and formula shown in books (i.e. Spacecraft Dynamics and Control)

$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = f(orbital\ \ parameter, a_{pert.})$

$\frac{\mathrm{d}\omega}{\mathrm{d}t} = f(orbital\ \ parameter, a_{pert.})$

etc.

General perturbation
Variation of orbital parameters

Express rate of change of orbit elements in terms of disturbing accelerations

Perturbation equations for right ascension of ascending node dΩ/dt:

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \sqrt{\frac{a(1-e^2)}{GM}}\,\frac{\sin(\omega+\theta)}{\sin i\,(1+e\cos\theta)}\,a_z$$

where a_z is acceleration due to pert. expressed in cylindrical coordinates

Notice that only z component of force perturbs Ω

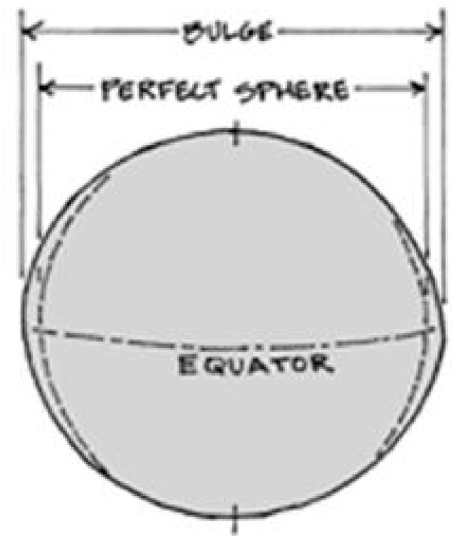
Similar formulas also for other orbital parameters:

- Semi-major axis
- Eccentricity
- Inclination
- Argument of perigee

Non-spherical Earth perturbation

Effects of non-spherical Earth

Search expression for perturbative acceleration $\boldsymbol{a}_{\text{pert.}}$ due to non-spherical Earth
Effect is very important for Earth orbiting satellites



Earth is not a sphere (due to rotational effect)
- Equatorial radius: about 6,378 km
- Polar radius: about 6,356 km (about 22 km smaller)
- Radius not constant with flattening ratio of 1/300

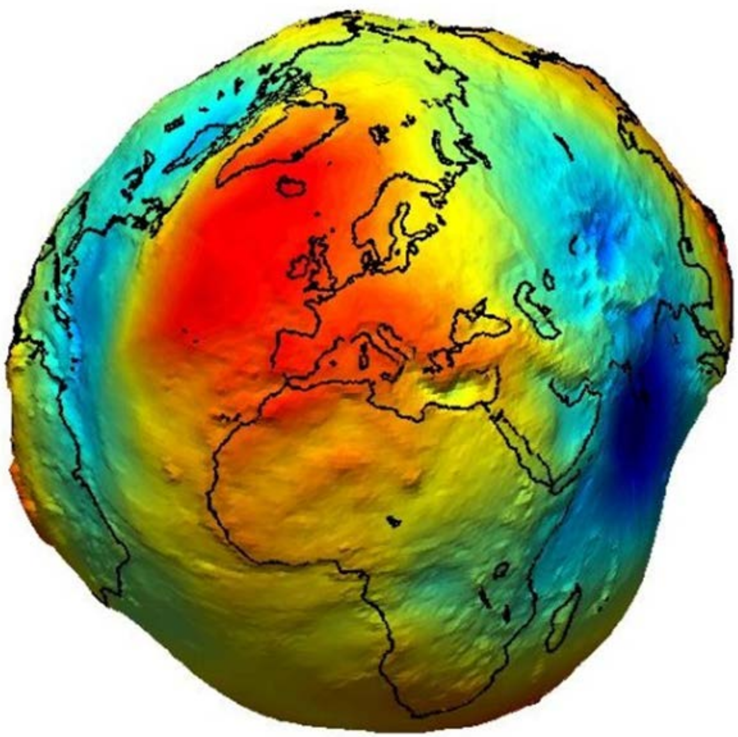
Earth mass distribution
- Earth mass distribution not uniform
- Regions of mass concentrations

Non-spherical Earth causes change from simple 1/r gravitational potential

Forces resulting from non-spherical Earth act on satellite and produce changes in Kepler´s orbital parameters

Gravitational perturbations due to non-spherical Earth

- Gravity field Earth: real Earth has highly irregular gravity field
- Gravity field is not smooth (gravity field is stronger or weaker in some area than smooth and homogenous Earth gravity field)
- Earth not sphere: force of gravity is no longer within orbital plane → Non-planar motion



Geoid:
Representations of Earth gravity field surface of equal gravitational potential of a hypothetic ocean at rest

Gravity field Earth

Newton gravitational law in vector form

$$\vec{F} = -G\frac{M_1 \cdot M_2}{|\vec{r}|^2}\frac{\vec{r}}{|\vec{r}|}$$

Acceleration due to symmetrical Earth

$$\ddot{\vec{r}} = -\frac{GM}{|r|^3}\vec{r}$$

Radial acceleration

$$\ddot{r} = -\frac{GM}{r^2}$$

Radial acceleration written as gravitational scalar potential

$$\ddot{r} = -\frac{\partial U}{\partial r} = -\frac{\partial}{\partial r}\left[-\frac{GM}{r}\right]$$

Force as negative gradient of scalar potential (Conservative forces)

$$\boxed{\vec{F} = -m\vec{\nabla}U}$$

To study gravitational attraction of non-spherical mass it is easier to work with gravitational potential than with forces

Laplace equation

- It can be shown that Gauss law for gravity is equivalent to Newton´s law
- Differential form of Gauss law becomes Poisson equation

  where ϱ is mass density $$\nabla^2 U = 4\pi G \rho$$
- Exterior to body: $$\rho = 0$$
- Laplace equation $$\nabla^2 U = 0$$
- Laplace equation gives an alternative method to calculate gravitational potential and gravitational field
- Solution of Laplace equation of spherical symmetric objects

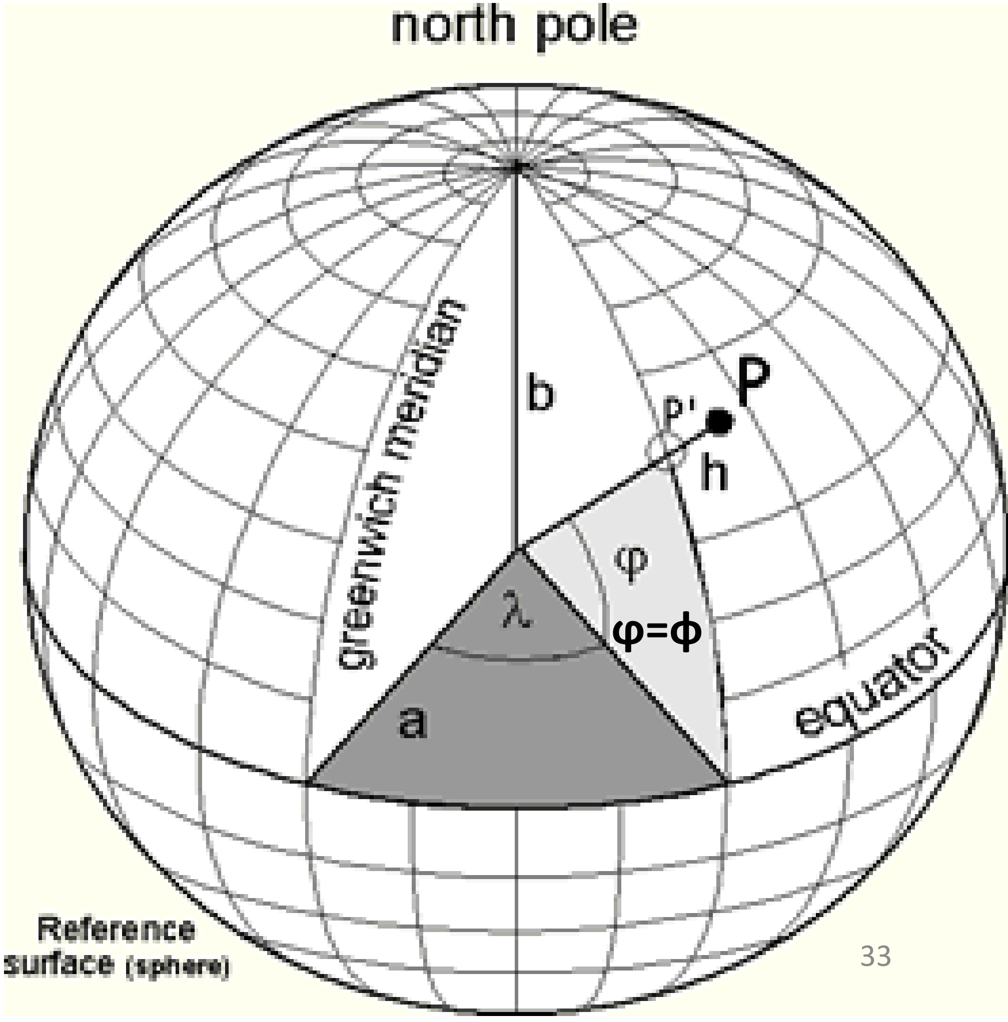
  → Spherical harmonics
- Laplace equation seen also in Lecture 1 for magnetic potential

Spherical coordinate system

Express position using spherical coordinates r, λ, ϕ

- r = radius
- ϕ = declination [0 < ϕ < π]
- λ = right ascension [0 < λ < 2π]

Have function of form U(r, λ, ϕ)



Coordinates transformations

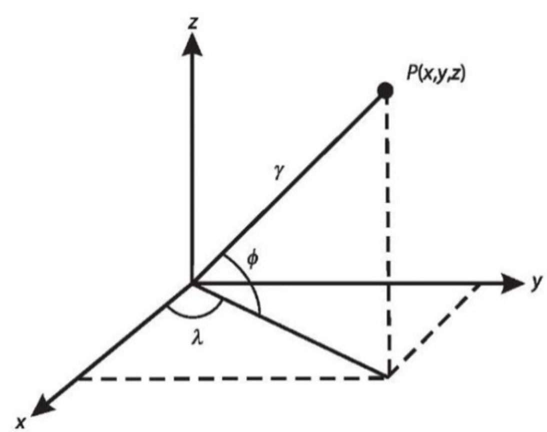
Measurement on celestial sphere best by spherical coordinate system

From spherical to Cartesian:

$$x = r\cos\phi\cos\lambda$$

$$y = r\cos\phi\sin\lambda$$

$$z = r\sin\phi$$



From Cartesian to spherical:

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\lambda = \arctan\tfrac{y}{x}$$

$$\phi = \arcsin\tfrac{z}{r}$$

Note:

For quadrant checks better use:

$$r_{xy} = \sqrt{x^2 + y^2}$$

$$\lambda = \mathrm{atan}\,2\left(\tfrac{y}{r_{xy}}, \tfrac{x}{r_{xy}}\right)$$

Spherical harmonics derivation

Laplace equation $$\nabla^2 U = 0$$

Spherical symmetry → write Laplace equation in spherical coordinates

Solution $$U(r,\lambda,\phi) = \left[Ar^n + Br^{-(n+1)} \right] Y(\lambda,\phi)$$

B = 0 for *U* interior of spherical surface
A = 0 for *U* exterior of spherical surface

Separation of variables $$Y_{nm}(\lambda,\phi) = P_n(\sin\phi)\Phi_m(\lambda)$$

$Y_{nm}(\lambda,\phi)$ are spherical harmonics and solve differential equation

P_n are Legendre polynomials
$\Phi_m(\lambda)$ has following form $$\Phi_m(\lambda) = C\cos m\lambda + S\sin m\lambda$$

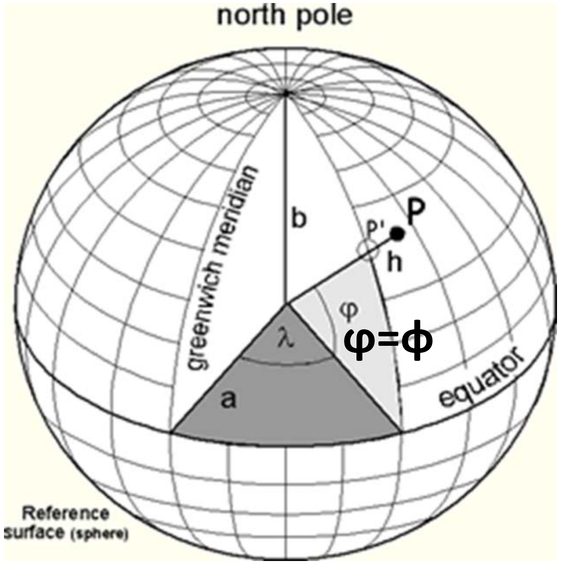
Real Earth gravity field

Highly irregular gravity field depends on distance to Earth center

$$\nabla^2 U = 0$$

Complete solution to Laplace equation given by

$$\boxed{U(r,\lambda,\phi) = -\frac{GM}{r} + U_{pert}(r,\lambda,\phi)}$$



$$U_{pert}(r,\lambda,\phi) = \frac{GM}{r}\left(\sum_{n=2}^{\infty}\left(\frac{R_e}{r}\right)^n J_n P_{n0}(\sin\phi)\right.$$

$$\left. + \sum_{n=2}^{\infty}\sum_{m=1}^{n}\left(\frac{R_e}{r}\right)^n [C_{nm}\cos m\lambda + S_{nm}\sin m\lambda] P_{nm}(\sin\phi)\right)$$

Describing potential exterior to spherical surface of R_e = radius of Earth

Note n = 1 term is absent (center of mass)

Spherical harmonics

Lets digest this formula

$$U_{pert}(r,\lambda,\phi) = \frac{GM}{r}\left(\sum_{n=2}^{\infty}\left(\frac{R_e}{r}\right)^n J_n P_{n0}(\sin\phi) + \sum_{n=2}^{\infty}\sum_{m=1}^{n}\left(\frac{R_e}{r}\right)^n [C_{nm}\cos m\lambda + S_{nm}\sin m\lambda] P_{nm}(\sin\phi)\right)$$

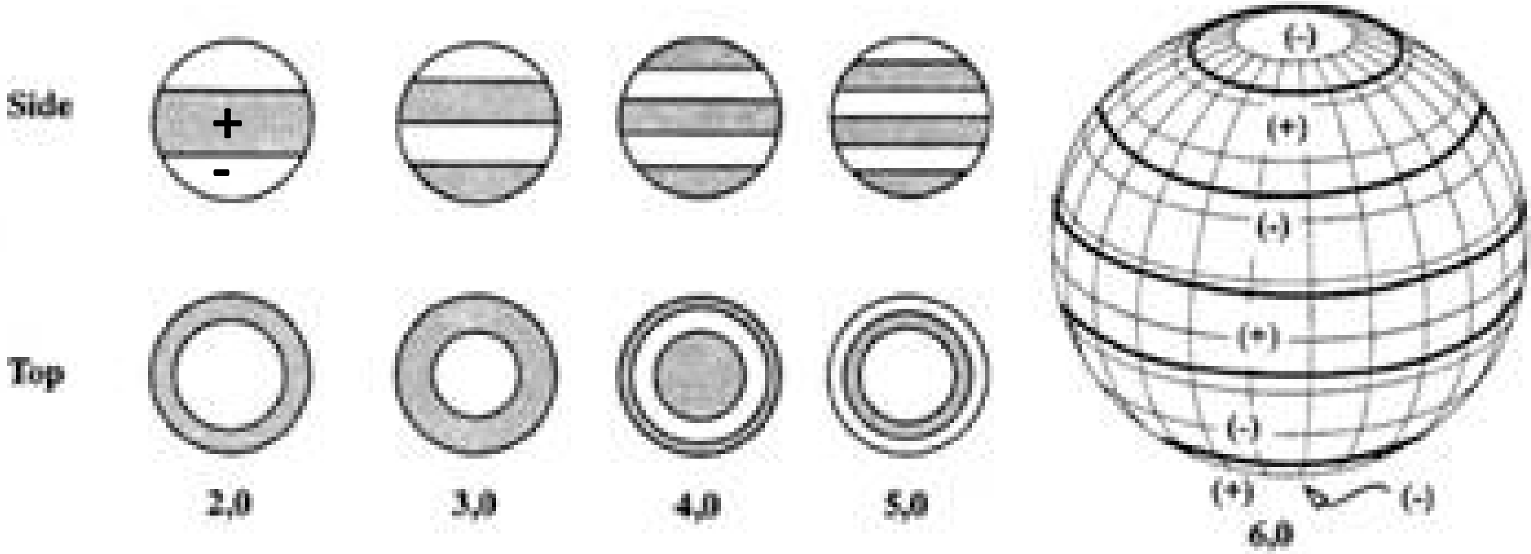
Perturbed potential has following form:

$$U_{pert}(r,\lambda,\phi) = U_{\text{zonal}}(r,\phi) + + U_{\text{sectorial}}(r,\lambda) + U_{\text{tesseral}}(r,\lambda,\phi)$$

Zonal harmonics

m=0

$$U_{\text{zonal}}(r,\phi)=\frac{GM}{r}\left[\sum_{n=2}^{\infty}\left(\frac{R_e}{r}\right)^n J_n P_{n0}(\sin\phi)\right]$$



Zonal harmonics vary only in declination (positive and negative sectors)

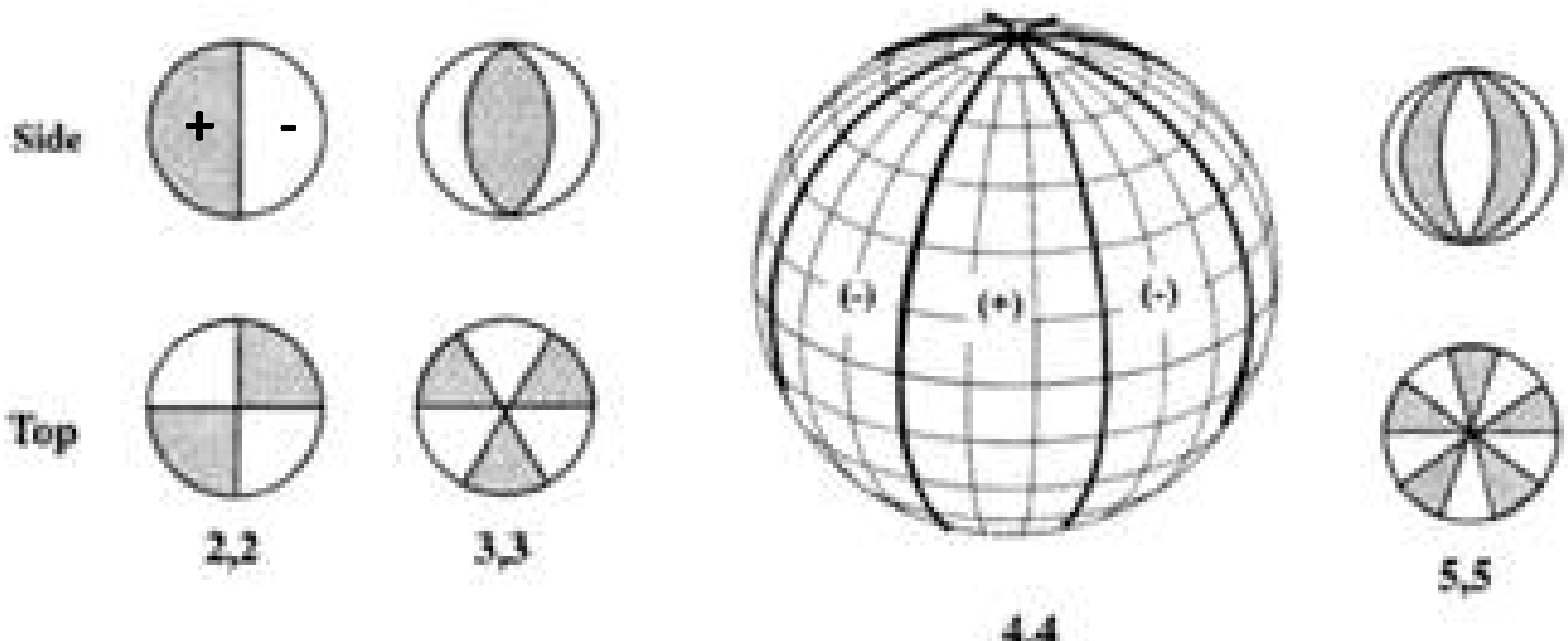
P_{n0} are associated Legendre polynomials

J_n are coefficients determined by experimental data

Sectorial harmonics

m=n

$$U_{\text{sectorial}}(r,\lambda) = \frac{GM}{r}\sum_{n=2}^{\infty}\left(\frac{R_e}{r}\right)^n \left[C_{nn}\cos n\lambda + S_{nn}\sin n\lambda\right]P_{nn}(\sin\phi)$$



Sectorial harmonics vary only right ascension

P_{nn} are Legendre polynomials which are uniformly positive, $P_{nn} \approx (\sin\phi)^n$

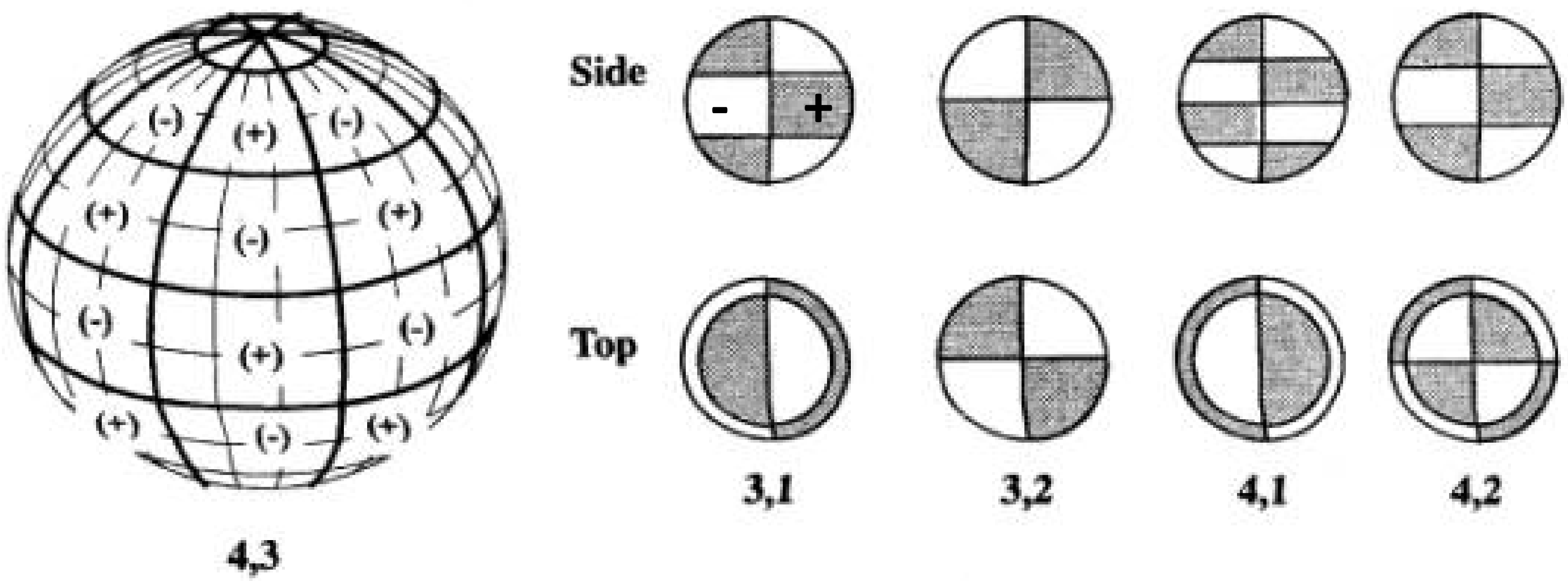
Sphere divided into melon slices

C_{nn} and S_{nn} are coefficients determined by experimental data

Tesseral harmonics

$\boxed{m \neq n}$

$$U_{\text{tesseral}}(r,\lambda,\phi) = \frac{GM}{r}\sum_{n=2}^{\infty}\sum_{m=1}^{n}\left(\frac{R_e}{r}\right)^n [C_{nm}\cos m\lambda + S_{nm}\sin m\lambda] P_{nm}(\sin\phi)$$



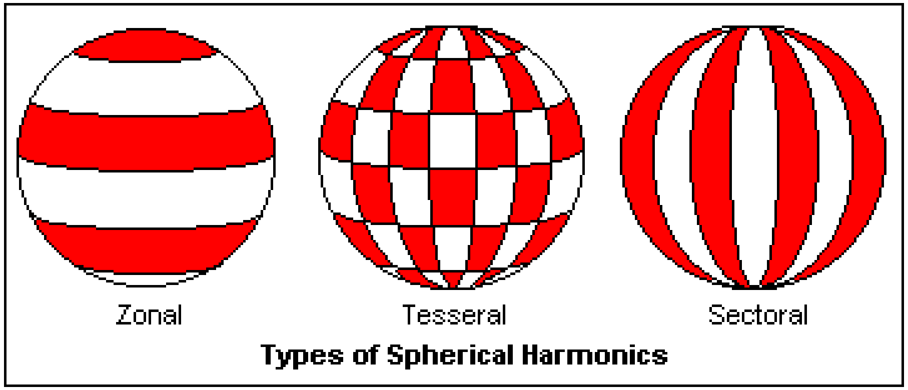
Tesseral harmonics vary in both declination and right ascension

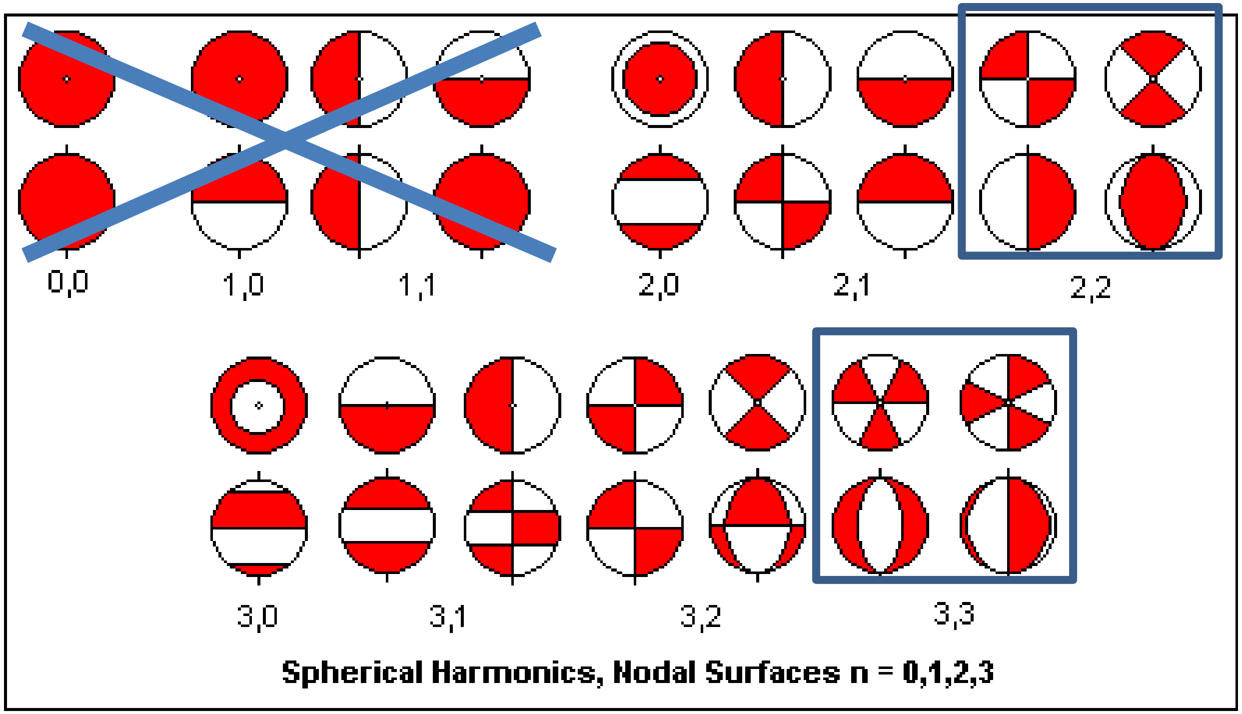
P_{nm} are associated Legendre polynomials with $P_{nm} \approx f(\cos\phi)^{n-m}(\sin\phi)^m$

Sphere divided into mosaic (Latin: tessera)

C_{nm} and S_{nm} are coefficients determined by experimental data

Visualization of spherical harmonics





Remember: Perturbation starts n ≥ 2

Real Earth gravity field

Gravity potential:

Harmonic coefficients
(Determined experimentally from satellite observation)

$$U_{pert}(r,\lambda,\phi) = \frac{GM}{r}\left(\sum_{n=2}^{\infty}\left(\frac{R_e}{r}\right)^n J_n P_{n0}(\sin\phi) + \sum_{n=2}^{\infty}\sum_{m=1}^{n}\left(\frac{R_e}{r}\right)^n [C_{nm}\cos m\lambda + S_{nm}\sin m\lambda] P_{nm}(\sin\phi) \right)$$

Associated Legendre function

$$\boxed{P_{nm}(x) = (1-x^2)^{m/2}\frac{\partial^m P_n(x)}{\partial x^m}}$$

Legendre polynomial

$$\boxed{P_n(x) = \frac{1}{(-2)^n n!}\frac{\partial^n}{\partial x^n}(1-x^2)^n}$$

Real Earth gravity field

Main term: $U_0 = -\frac{GM}{r}$

Largest term with n = 0 varies with 1/r

First correction term: $U_{J_2}(r,\phi) = \frac{GM}{r}\left(\frac{R_e}{r}\right)^2 J_2 P_{20}\left(\sin\phi\right)$

Polar flattening of Earth reflected in term that adds mass near equator

First correction term with *n* = 2 and *m* = 0 with radial dependence $1/r^3$ and angular dependence P_{20}

Higher order terms *n* > 2 used to fit real mass distribution of Earth

How to calculate $P_{20}\left(\sin\phi\right)$?

Legendre polynomial $P_{20}\left(\sin\phi\right)$

Example $$U_{J_2}(r,\phi) = \tfrac{GM}{r}\left(\tfrac{R_e}{r}\right)^2 J_2 P_{20}\left(\sin\phi\right)$$

First derive and later set x = sin(ϕ) $$P_{20}\left(\sin\phi\right) = P_{20}(x)\,|_{x=\sin\phi}$$

1. Derivate $$P_2(x) = \frac{1}{(-2)^2\,2!}\frac{\partial^2}{\partial x^2}(1-x^2)^2 = \tfrac{3}{2}x^2 - \tfrac{1}{2}$$

$$P_2(x) = P_{20}(x)$$

2. Set x = sin(ϕ) $$U_{J_2} = \tfrac{GM}{r} J_2 \left(\tfrac{R_e}{r}\right)^2 (\tfrac{3}{2}\sin^2\phi - \tfrac{1}{2})$$

Rotational symmetry

Most satellite orbits (other than geosynchronous orbits) rotational symmetry is good assumption of Earth potential

$$U(r,\phi) = -\frac{GM}{r}\left[1 - \sum_{n=2}^{\infty}\left(\frac{R_e}{r}\right)^n J_n P_{n0}(\sin\phi)\right]$$

$$J_2 = 1.083 \cdot 10^{-3}$$

$$J_3 = -2.56 \cdot 10^{-6}$$

Satellite position described in Earth-fixed (rotating) reference frame by distance *r* from center of Earth and declination ϕ from equator

Gravitational potential without J_2 correction term is accurate to 0.1%

Gravity field coefficient

Example of gravity field coefficients:

- NASA EGM96

http://cddis.gsfc.nasa.gov/926/egm96/egm96.html

- Earth Gravitational Model 2008 (EGM 2008) (n = 2160)

http://earth-info.nga.mil/GandG/wgs84/gravitymod/egm2008/index.html

Table 4.4 Magnitude of low order *J*, *C* and *S* values for Earth

		S_{21} 0
J_2 1082.7×10^{-6}	C_{21} 0	S_{22} -0.897×10^{-6}
J_3 -2.56×10^{-6}	C_{22} 1.57×10^{-6}	S_{31} 0.16×10^{-6}
J_4 -1.58×10^{-6}	C_{31} 2.10×10^{-6}	S_{32} -0.27×10^{-6}
J_5 -0.15×10^{-6}	C_{32} 0.25×10^{-6}	S_{33} 0.173×10^{-6}
J_6 0.59×10^{-6}	C_{33} 0.077×10^{-6}	

P.Fortescue

Parameters J_n, C_{nm}, S_{nm} are coefficient determined to be $O(10^{-6})$ for n>2
Except for J_2 which is about 1083×10^{-6}

Earth equatorial bulge

For simplicity, ignore all harmonics except first zonal harmonic

$$U(r,\phi) = -\frac{GM}{r}\left[1-\left(\frac{R_e}{r}\right)^2 J_2 P_{20}(\sin\phi)\right]$$

$$= -\frac{GM}{r} + \frac{GM}{r^3} R_e^{\,2} J_2 \left(\tfrac{3}{2}\sin^2\phi - \tfrac{1}{2}\right)$$

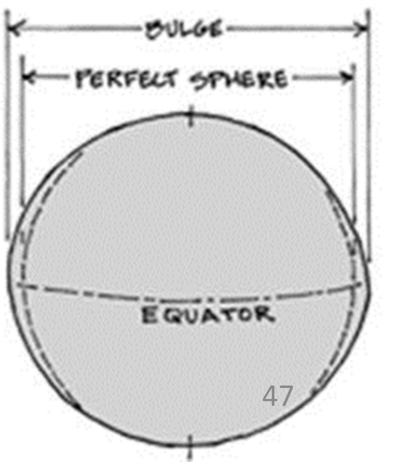
Potential consist of two terms:

- Keplerian or two-body potential
- Perturbation potential J_2

For Earth: most dominant perturbing effect is J_2 term, which results from Earth equatorial bulge

Perturbing potential includes only J_2 effect:

$$U_{J_2} = \frac{GM}{r} J_2 \left(\frac{R_e}{r}\right)^2 \left(\tfrac{3}{2}\sin^2\phi - \tfrac{1}{2}\right)$$



Perturbing acceleration due to J_2

$$U_{J_2} = \tfrac{GM}{r} J_2 \left(\tfrac{R_e}{r}\right)^2 \left(\tfrac{3}{2}\sin^2\phi - \tfrac{1}{2}\right)$$

How to go from perturbative potential to
perturbative acceleration $a_{\text{pert.}}$ due to non-spherical Earth?

Like always: Take gradient from potential

Acceleration in gravity field

Acceleration given by partial derivatives from potential in direction of interest

$$\vec{F} = -m\vec{\nabla} U$$

$$a_x = -\frac{\partial U}{\partial x}$$

$$a_y = -\frac{\partial U}{\partial y}$$

$$a_z = -\frac{\partial U}{\partial z}$$

$$a_r = -\frac{\partial U}{\partial r}$$

$$a_\phi = -\frac{1}{r}\frac{\partial U}{\partial \phi}$$

$$a_\lambda = -\frac{1}{r\cos\phi}\frac{\partial U}{\partial \lambda}$$

Be aware:
This are Earth-fixed coordinates,
(i.e. Earth Centered Earth Fixed (ECEF) frame)

Spherical Earth vs. Equatorial bulge Earth

Spherical Earth **Earth-fixed coordinate** **Equatorial bulge Earth**

$$U_{Sphere} = -\frac{GM}{r}$$

$$a_r = -\frac{\partial U}{\partial r} = \frac{GM}{r^2}$$

$$a_\phi = -\frac{1}{r}\frac{\partial U}{\partial \phi} = 0$$

$$a_\lambda = -\frac{1}{r\cos\phi}\frac{\partial U}{\partial \lambda} = 0$$

Because of symmetry
only radial acceleration

$$U(r,\theta) = U_{Sphere} + U_{J_2}$$

$$U_{J_2} = \frac{GM}{r} J_2 \left(\frac{R_e}{r}\right)^2 \left(\frac{3}{2}\sin^2\phi - \frac{1}{2}\right)$$

$$a_r = -\frac{\partial U_{J_2}}{\partial r} = 3\frac{GM}{r^4} R_e^{\,2} J_2 \left(\frac{3}{2}\sin^2\phi - \frac{1}{2}\right)$$

$$a_\phi = -\frac{1}{r}\frac{\partial U_{J_2}}{\partial \phi} = \frac{GM}{r^4} R_e^{\,2} J_2 (3\sin\phi\cos\phi)$$

$$a_\lambda = -\frac{1}{r\sin\phi}\frac{\partial U_{J_2}}{\partial \lambda} = 0$$

Perturbation of radial acceleration

Acceleration in declination (i.e. **North-South**)

No acceleration in right ascension
(i.e. **East-West**) because of symmetry

Effect of J_2 on orbital elements

Effect of J_2 on orbital elements

Seen how in first order Earth equatorial bulge produces accelerations in declination but not in right ascension in Earth-fixed coordinate system

Next goal: What is effect of irregularities in Earth gravity field on satellite orbits?

Now that one has perturbative acceleration (grad U_{J_2}), one can use general perturbation (with variation of orbital parameter equation) to get effect of J_2 on orbital elements, e.g.

$$\frac{d\Omega}{dt} = \sqrt{\frac{a(1-e^2)}{GM}} \frac{\sin(\omega+\theta)}{\sin i(1+e\cos\theta)} a_z$$

Note:

Taking gradient is not trivial because potential is in spherical coordinates and terms needed for acceleration are given in cylindrical coordinates

Effect of J_2 on orbital elements

$$U_{J_2} = \tfrac{GM}{r} J_2 \left(\tfrac{R_e}{r}\right)^2 \left(\tfrac{3}{2}\sin^2\phi - \tfrac{1}{2}\right)$$

Express equation in Earth Centered Earth fixed (ECEF) coordinates
Know $z = r \sin\phi \rightarrow \sin\phi = z/r$

$$U_{J_2} = \tfrac{GM}{r} J_2 \left(\tfrac{R_e}{r}\right)^2 \left(\tfrac{3}{2}\left(\tfrac{z}{r}\right)^2 - \tfrac{1}{2}\right)$$

Calculate perturbative acceleration taking gradient in ECEF coordinates

$$\vec{F}_{J_2} = -\nabla U_{J_2} = \tfrac{\partial U}{\partial x}\vec{x}_G + \tfrac{\partial U}{\partial y}\vec{y}_G + \tfrac{\partial U}{\partial z}\vec{z}_G$$

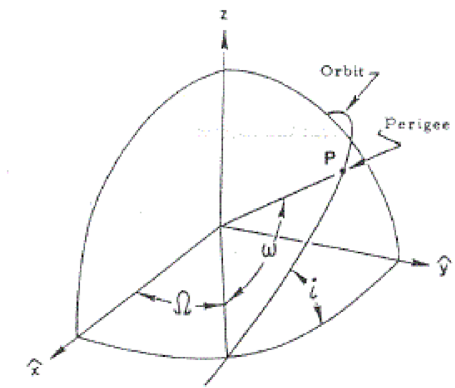
But variation of orbital parameters perturbation equations (dΩ/dt) are given cylindrical coordinates

Coordinate rotations cylindrical to ECEF transformation

Cylindrical coordinates can be reached from ECEF

via following three rotations (3-1-3 Euler rotation):

1. Rotate Ω about z
2. Rotate i about x
3. Rotate ω about z



Please look missing steps in books (i.e. Spacecraft dynamics and control)

From rotation matrices obtain (Connection of ECEF and cylinder coord.)

$$\vec{z}_G = \sin i \sin(\omega+\theta)\vec{x}_0 + \sin i \cos(\omega+\theta)\vec{y}_0 + \cos i \ \vec{z}_0$$

Obtain perturbative acceleration in cylindrical coordinates

$$\vec{F}_{J_2} = a_r(i,\omega,\theta)\vec{x}_0 + a_\theta(i,\omega,\theta)\vec{y}_0 + a_z(i,\omega,\theta)\ \vec{z}_0$$

Effect of J_2 on orbital elements

Primary effect of J_2 is on Ω and ω

Set perturbative acceleration equation (grad U_{J_2}) into expressions for

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \sqrt{\frac{a(1-e^2)}{GM}} \frac{\sin(\omega+\theta)}{\sin i(1+e\cos\theta)} a_z$$

Obtain

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \sqrt{\frac{a(1-e^2)}{GM}} \frac{\sin(\omega+\theta)}{\sin i(1+e\cos\theta)} a_z(i,\omega,\theta)$$

Which gives instantaneous rate of change

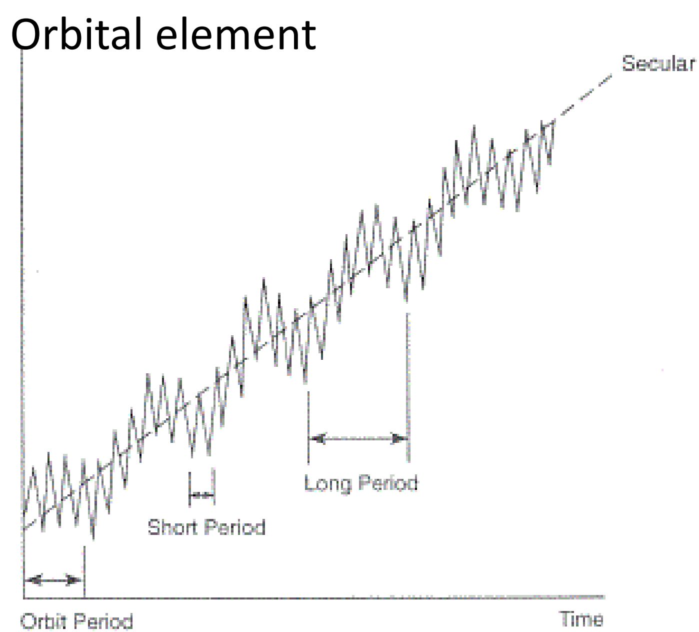
Angle ω and Θ cycle from 0 to 2π

Typically perturbed orbital element will have secular and periodic variations

Periodic and secular variation
Effect of J_2 on orbital elements

Three types of disturbances

- **Short periodic**
  Cycles every orbital period → No net change in orbital element
- **Long periodic**
  Cycles last longer than one orbital period → No net change in orbital element
- **Secular**
  Does not cycle
  Disturbances mount over time
  Long term changes in orbital elements



Secular disturbances must be corrected

Secular rate of change are
average rate of change over many orbits

Averaging J_2 perturbation on orbital element

Goal is to isolate secular contribution

Determine secular change in dΩ/dt due to perturbation by averaging over one orbit

What is average over true anomaly θ?

Use chain rule: $\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{\mathrm{d}\Omega}{\mathrm{d}\theta}\dot{\theta} \Leftrightarrow \frac{\mathrm{d}\Omega}{\mathrm{d}\theta} = \frac{1}{\dot{\theta}}\frac{\mathrm{d}\Omega}{\mathrm{d}t}$

Find average change over one orbit

$$\Delta\Omega_{2\pi} = \int_0^{2\pi} \frac{\mathrm{d}\Omega}{\mathrm{d}\theta} d\theta = ... = -\frac{3\pi J_2 R_e^2}{a^2(1-e^2)^2}\cos i$$

Note: J_2 causes periodic orbit perturbations in all orbit elements which average out over one orbit revolution

Determine secular change in Ω

Obtain secular average rate of change of Ω
by dividing with orbital period ΔT

$$\Delta T = 2\pi\sqrt{\frac{a^3}{\mu}}$$

$$\boxed{\dot{\Omega}_{avg} = \frac{\triangle\Omega_{2\pi}}{\Delta T} = -\frac{3J_2R_e^2}{2(1-e^2)^2}\sqrt{\frac{\mu}{a^7}}\cos i}$$

Secular change in Ω is called **nodal regression**

Repeat procedure for other orbital elements

$$\boxed{\begin{aligned}\dot{a}_{avg} &= 0\\ \dot{e}_{avg} &= 0\\ \dot{i}_{avg} &= 0\end{aligned}}$$

$$\boxed{\dot{\omega}_{avg} = \frac{3\pi J_2R_e^2}{4(1-e^2)^2}\sqrt{\frac{\mu}{a^7}}(5\cos^2 i - 1)}$$

Is called **apsidal rotation**

Effect of irregularity in gravity field on satellite orbit

Conclusion with first-order approximation:

Perturbation on orbital elements due to J_2

(Integrated and averaged over full orbital period 2π)

$$\dot{a}_{avg} = 0$$

$$\dot{e}_{avg} = 0$$

$$\dot{i}_{avg} = 0$$

Earth equatorial bulge has **no effect** on:

- shape and size of orbit (*a* and *e*)
- inclination of orbital plane (*i*)

$$\dot{\Omega}_{avg} = -\frac{3J_2R_e^2}{2(1-e^2)^2}\sqrt{\frac{\mu}{a^7}}\cos i$$

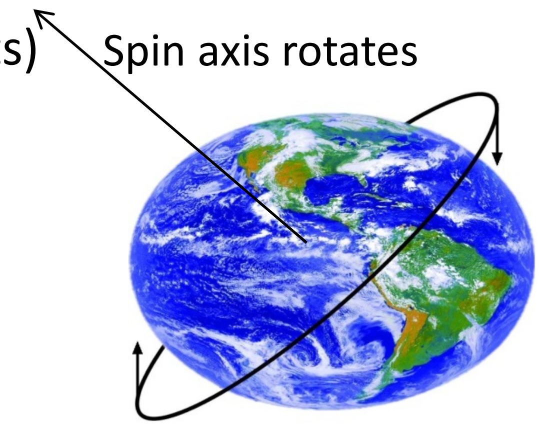
$$\dot{\omega}_{avg} = \frac{3\pi J_2R_e^2}{4(1-e^2)^2}\sqrt{\frac{\mu}{a^7}}(5\cos^2 i - 1)$$

Only effect on *Ω* and *ω*:

- Orbital plane turns about Earth spin axes through d*Ω*/d*t*
- Orbit perigee turns through d*ω*/d*t*

Physical interpretation of Earth equatorial bulge effect

- Earth equatorial bulge (Equator has extra mass elements) means that force of gravity is no longer within orbital plane
  → non-planar motion
- Perturbed orbit before and after passing equator is different compared to unperturbed orbit
- Extra mass acts as additional force that pulls satellite back to equatorial plane
  → tries to align orbital plane with equator
- Angular momentum conserved
  → orbit behaves like gyroscope
- Orbit has precessional motion around Earth rotation axes (no change in inclination angle)
- What happens if orbit inclination is 0 or 90 degrees?



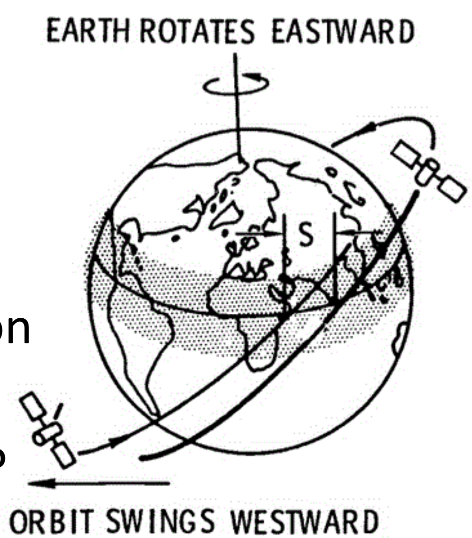


Fig. 10.1 The gravitation pull of the Earth's equatorial bulge causes the orbital plane of an eastbound satellite to swing westward.

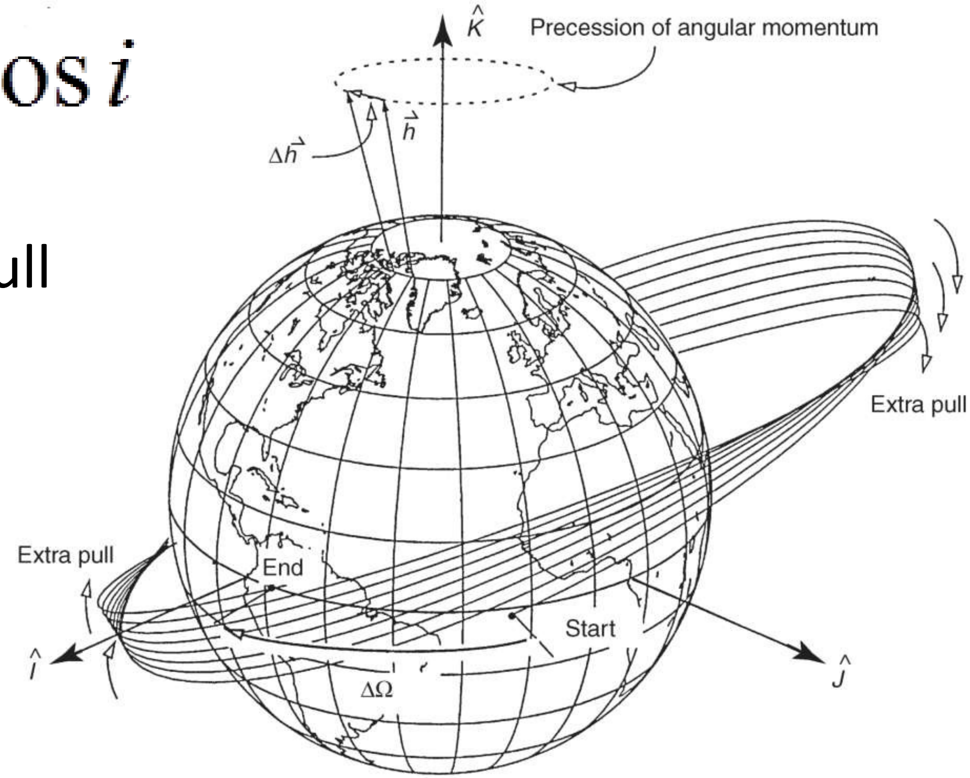
Secular effect: J_2 nodal regression rate

Nodal regression due to secular effects of J_2

$$\dot{\Omega}_{avg} = -\frac{3J_2R_e^2}{2(1-e^2)^2}\sqrt{\frac{\mu}{a^7}}\cos i$$

Equatorial bulge produces extra pull in equatorial plane

- Creates torque on angular momentum vector
- Like gravity torque causes angular momentum to precess
- Depends on inclination, altitude and eccentricity



Secular effects and Earth satellite application

In first order Earth equatorial bulge affects only Ω and ω
→ they both precess

$$\dot{\Omega}_{avg} = -\frac{3J_2R_e^2}{2(1-e^2)^2}\sqrt{\frac{\mu}{a^7}}\cos i$$

Nodal regression rate
Rate of change of right ascension of ascending node

$$\dot{\omega}_{avg} = \frac{3\pi J_2R_e^2}{4(1-e^2)^2}\sqrt{\frac{\mu}{a^7}}(5\cos^2 i - 1)$$

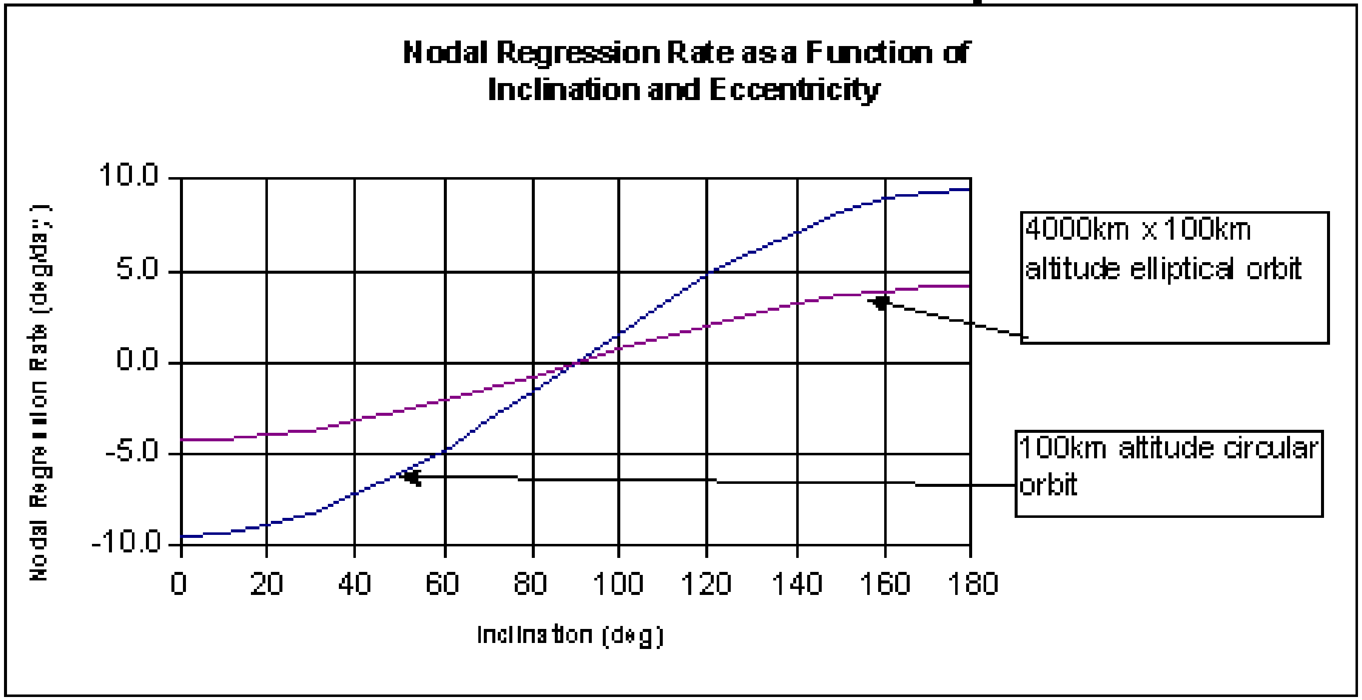
Apsidal rotation rate
Rate of change of argument of perigee

Applications:

Polar orbit
Sun-synchronous orbit
Molniya orbits

Polar orbit
Polar orbit (i = 90º) equatorial extra mass has no effect
Precession remains zero and orbital plan keeps orientation
Extra mass for polar orbit has symmetric distributions

Nodal regression due to Earth´s equatorial bulge



Nodal regression rate as function of inclination and orbital altitude

$$\dot{\Omega}_{avg} = -\frac{3J_2R_e^2}{2(1-e^2)^2}\sqrt{\frac{\mu}{a^7}}\cos i$$

Nodal regression rate for various orbit inclination

For polar orbits i = 90^o nodal regression is zero

For i=0^o nodal regression is maximal

Nodes move:
Westward if orbit inclination is < 90 degrees (posigrade orbit)
Eastward if orbit inclination is > 90 degrees (retrograde orbit)

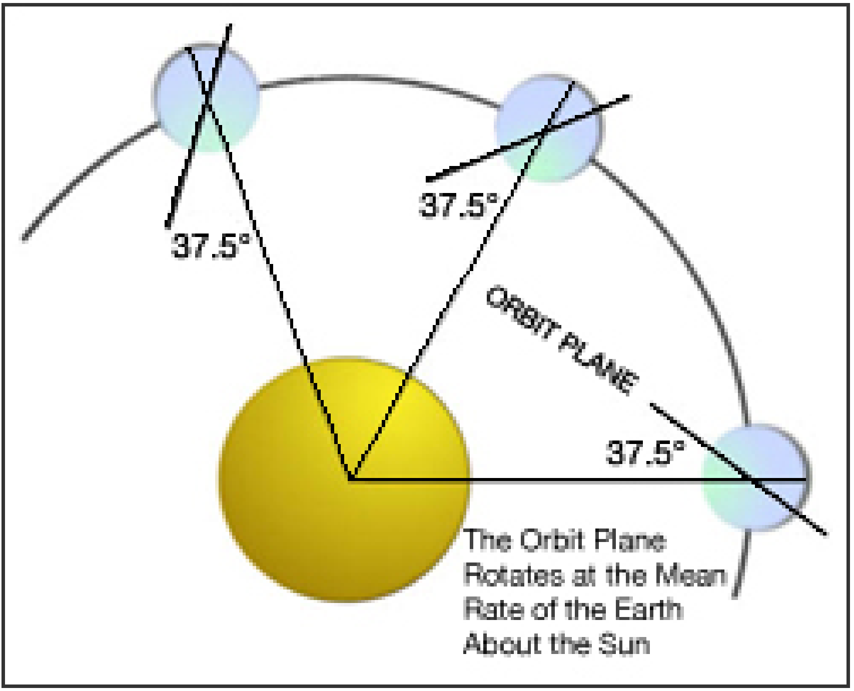
Higher the altitude → smaller J_2 effect and if satellite in polar orbit (center of graph), no J_2 effect
Greatest effect occurs at low altitudes with low inclinations (up to 9 degrees per day)

Application of J_2: Sun-synchronous orbit

Application of J_2 effect:

- Take advantage of orbit perturbation caused by Earth's non-spherical gravitational field
- Choose correct inclination and altitude of satellite orbit
- Nodal regression rate can be made to precess at same rate as Earth revolves around Sun (orbit plane rotates at same rate as Earth revolves around Sun)
- Direction of satellite orbit follows direction towards Sun
- Satellite has same pattern of light conditions throughout an orbit
- Satellite sees any given part of planet under nearly same condition of daylight or darkness day after day
- No need to move solar panels (dawn/dusk orbits only)
- Good for meteorological and Earth remote sensing

Sun-synchronous orbit



Sun-synchronous orbit is not fixed in space

Orbital plane must rotate in inertial space with angular velocity of Earth when orbiting around Sun

Orbit plane rotates once every year

Orbit must move about 1° per day to compensate for the Earth's revolution around Sun

Earth makes one revolution (360°) around Sun/year
Calculate: 360/365.25 days= 0.9855° per solar day
→ this is required rate of precession of *Ω*
d*Ω*/d*t* = 0.9855°/day=1.99 · 10^{-7} rad/s

$$\dot{\Omega}_{avg} = -\frac{3J_2R_e^2}{2(1-e^2)^2}\sqrt{\frac{\mu}{a^7}}\cos i = 1.99\cdot 10^{-7}\ \frac{\text{rad}}{\text{s}}$$

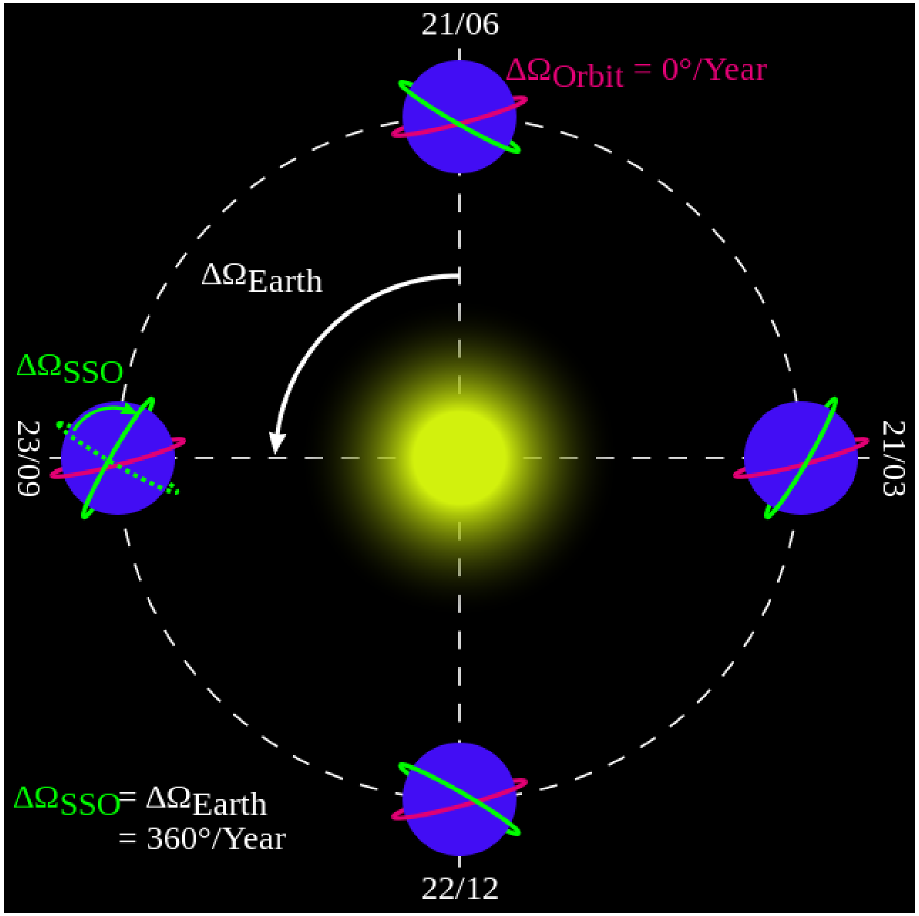
Sun-synchronous orbits

$$\dot{\Omega}_{avg} = -\frac{3J_2R_e^2}{2(1-e^2)^2}\sqrt{\frac{\mu}{a^7}}\cos i$$

Choose combination of:
semi-major axis *a*
eccentricity *e*
inclination *i*
such that:

$$\dot{\Omega}_{avg} = 360^0 / year$$

Orbit plane rotates at same rate that Earth revolves around Sun

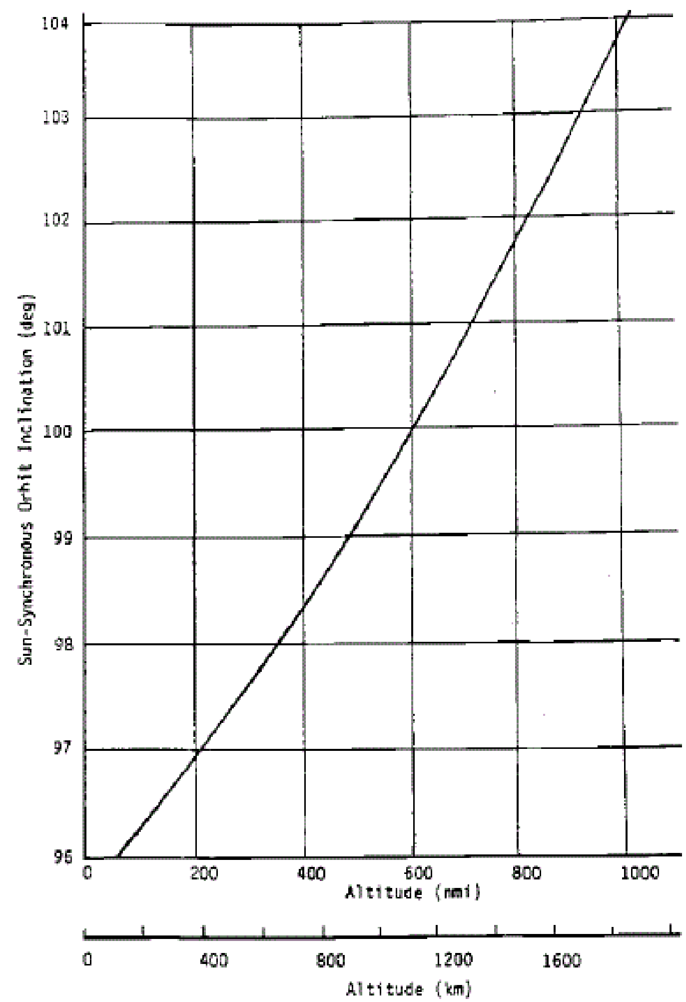


Orientation of Sun-synchronous orbit (green)
Non-sun-synchronous orbit (magenta)

Sun-synchronous orbits

Be aware:
Sun-synchronous orbits depend on altitude

$$\dot{\Omega}_{avg} = -\frac{3J_2R_e^2}{2(1-e^2)^2}\sqrt{\frac{\mu}{a^7}}\cos i$$



Example: Sun-synchronous orbit

Problem:

Design a Sun-synchronous orbit with $r_p = R_e$+695 km and $r_a = R_e$+705 km

Example: Sun-synchronous orbit

Problem:

Design a Sun-synchronous orbit with $r_p = R_e$+695 km and $r_a = R_e$+705 km

Solution:

Desired inclination for a sun-synchronous orbit is given by

$$\dot{\Omega}_{avg} = -\frac{3J_2R_e^2}{2(1-e^2)^2}\sqrt{\frac{\mu}{a^7}}\cos i = 1.99\cdot 10^{-7}\ \frac{\text{rad}}{\text{s}}$$

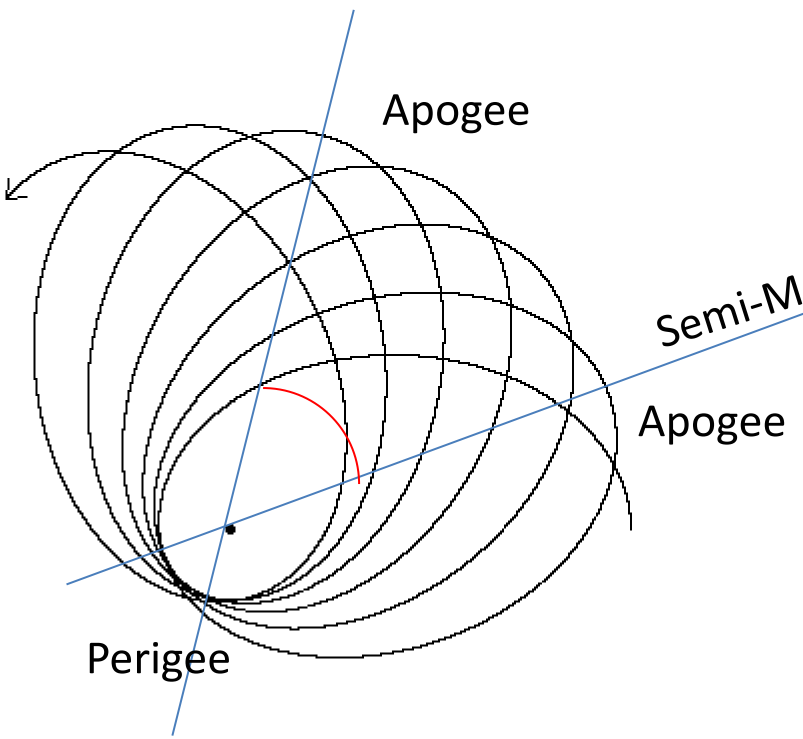
For this orbit $a = R_e$ + 700 km = 7078 km and eccentricity is

$$e = 1 - \frac{r_p}{a} = 0.00071$$

Thus with J_2 = 0.001082 and μ= 398600 km^3 s^{-2} → cos i = - 0.151

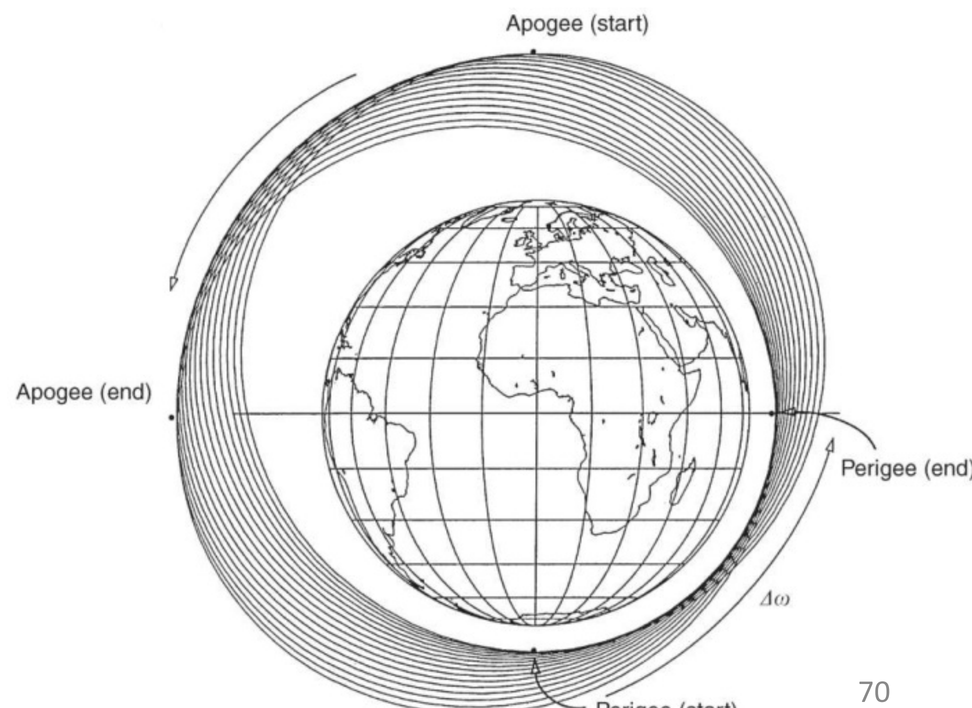
→ Required inclination is i = 98.7°

Secular effect: J_2 apsidal rotation rate

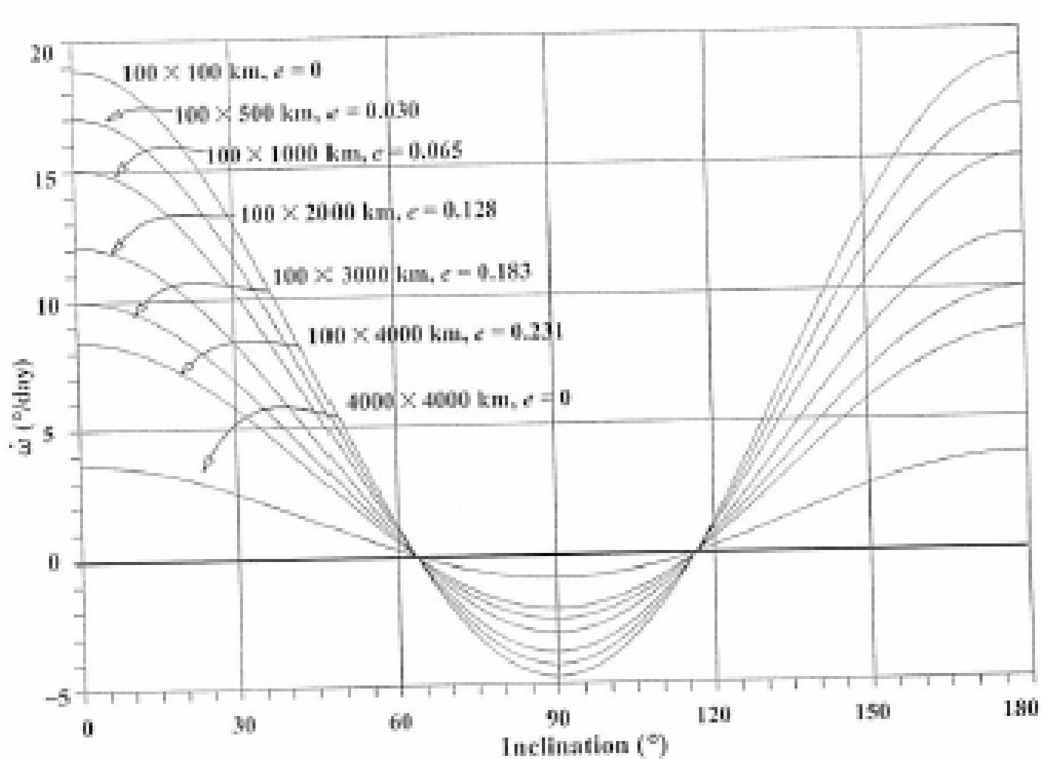


Similar to nodal regression

$$\dot{\omega}_{avg} = \frac{3\pi J_2 R_e^2}{4(1-e^2)^2}\sqrt{\frac{\mu}{a^7}}(5\cos^2 i - 1)$$



Secular effect: J_2 apsidal rotation rate



Daily apsidal regression as function of inclination

Apsidal rotation rate can be large

Perigee and apogee move forward or backward depending on inclination

0< i < 63.4 or 116.6 < i <180 → dω/dt > 0
Perigee advances in direction of motion of satellite

Molniya Orbit

If $$5\cos^2 i - 1 = 0$$

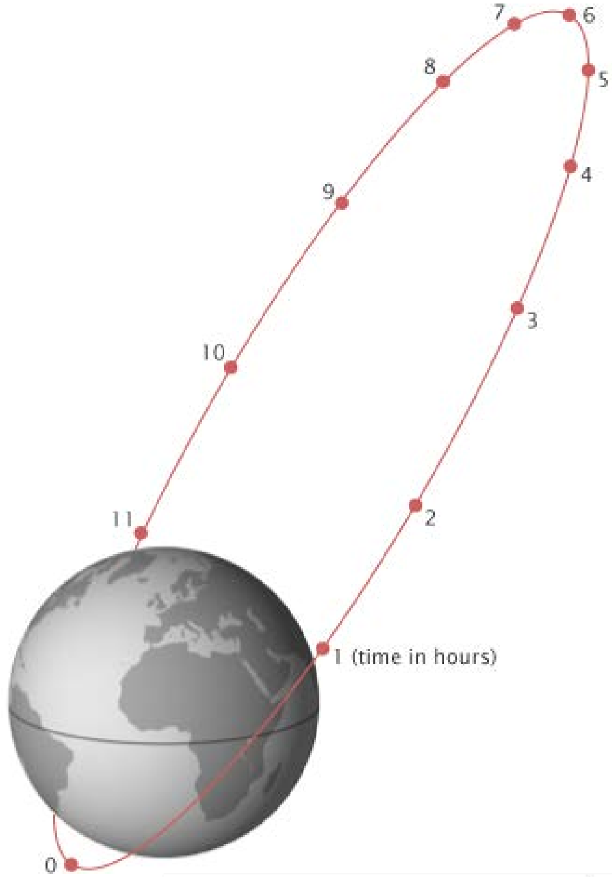
→dω/dt = 0 (perigee advance is zero)

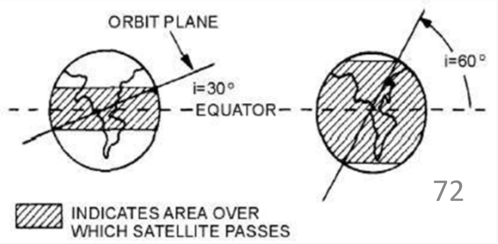
For i = 63.4° or i = 116.6°
Apsidal rotation rate is zero
(does not move)

Use J_2 effect: Molniya orbit

- Perigee advance is zero $d\omega/dt = 0$
- If $5\cos^2 i - 1 = 0$

  $i = 63.4$ or $i = 116.6$ degrees
- Highly eccentric orbits
- By arresting perigee advance perigee occurs over same declination each orbit
- Orbit with 12 hours every second apogee occurs over same point of Earth
- Covers high declinations and polar regions very well





Example: Molniya orbit

Problem:

Molniya orbits are usually designed so that perigee always occurs over same declination. Design a Molniya orbit with a period of 12 hours and which precesses at $d\Omega/dt = -0.2^{o}$ / day

Example: Molniya orbit

Problem:

Molniya orbits are usually designed so that perigee always occurs over same declination. Design a Molniya orbit with a period of 12 hours and which precesses at $d\Omega/dt$ = -0.2° / day

Solution:

First use period to solve for *a*, from equation $\Delta T = 2\pi\sqrt{\frac{a^3}{\mu}}$

Molniya orbit: $d\omega/dt = 0$ if i = 63.4° or i = 116.6°

To achieve $d\Omega/dt$ = -0.2°/day use:

$$\dot{\Omega}_{avg} = -\frac{3J_2R_e^2}{2(1-e^2)^2}\sqrt{\frac{\mu}{a^7}}\cos i$$

($d\Omega/dt$ = -0.2°/day = $-4\cdot10^{-8}$ rad/s and a = 26600 km)

Since *a* and *i* already fixed, choose $e \rightarrow e = 0.78$

Summary

Summary of Kepler orbits

Orbital perturbations

- Special and general perturbation

Account for gravitational perturbation to non-spherical Earth

- Gravity potential
- Laplace equation
- Spherical harmonic functions
- J_2 perturbation
  - Effect on Ω (Sun-synchronous orbits)
  - Effect on ω (Molniya orbits)
  - In first-order no effect on a, e and i