DOC 221 Dinámica orbital y control de actitud

Solutions to Problems Lecture ADCS – VIII

Problem 1:

The three eigenvalues of the inertia matrix $\mathbf{J}$ can be obtained as shown in the lecture with $(\mathbf{J} - \lambda\mathbf{1})\boldsymbol{\omega} = \mathbf{0}$ and with $\det(\mathbf{J} - \lambda\mathbf{1}) = 0$. (See also Problem 2 for a step by step derivation.) The eigenvalues are 1000, 2700 and 3500. Letting (λ_1, λ_2, λ_3) = (1000, 2700, 3500), we find the inertia matrix $\mathbf{I}$ about the principal axes as

$$\mathbf{I} = \begin{bmatrix} 1500 & 0 & 0 \\ 0 & 2700 & 0 \\ 0 & 0 & 3500 \end{bmatrix} \text{ kg m}^2$$

Furthermore, rotation matrix of the corresponding principle axes A´ relative to A can be obtained as

$$\mathbf{C} = \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \end{bmatrix}$$

One can verify that $\mathbf{CJC}^T = \mathbf{I}$

$$\mathbf{CJC}^T = \mathbf{I} = \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 1500 & 0 & -1000 \\ 0 & 2700 & 0 \\ -1000 & 0 & 3000 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 1500 & 0 & 0 \\ 0 & 2700 & 0 \\ 0 & 0 & 3500 \end{bmatrix}$$

Problem 2:

The rotational kinetic energy with the form

$$2T_r = 20\omega_x^2 + 30\omega_y^2 + 15\omega_z^2 - 20\omega_x\omega_y - 30\omega_x\omega_z$$

can be written in matrix notation as follows $2T_r = \boldsymbol{\omega}^T \mathbf{J}\boldsymbol{\omega}$. The associated non-diagonal inertia matrix has the following form:

$$\mathbf{J} = \begin{bmatrix} 20 & -10 & -15 \\ -10 & 30 & 0 \\ -15 & 0 & 15 \end{bmatrix}$$

The equations $(\mathbf{J} - \lambda\mathbf{1})\boldsymbol{\omega} = \mathbf{0}$ take on the form

$$\begin{aligned} (20-\lambda)\omega_x \quad & -10\omega_y & -15\omega_z & = 0 \\ -10\omega_x \quad & +(30-\lambda)\omega_y & & = 0 \\ -15\omega_x \quad & & +(15-\lambda)\omega_z & = 0 \end{aligned}$$

and the characteristic equation becomes

$$\lambda^3 - 65\lambda^2 + 1025\lambda - 750 = 0$$

The associated roots are 39.58, 24.65, and 0.77. Take one at a time to determine the eigenvectors. When $\lambda = \lambda_1 = 39.58$ above equations become

$$\begin{aligned} -19.58\omega_x \quad & -10\omega_y & -15\omega_z & = 0 \\ -10\omega_x \quad & -9.58\omega_y & & = 0 \\ -15\omega_x \quad & & -24.58\omega_z & = 0 \end{aligned}$$

The general solution of this set is

$$\begin{aligned} \omega_x &= c_1 \\ \omega_y &= -1.04c_1 \\ \omega_z &= -0.61c_1 \end{aligned}$$

This can be written as

$$\boldsymbol{\omega} = c_1\mathbf{u}_1$$

where

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1.04 \\ -0.61 \end{bmatrix}$$

This first normalized eigenvector is

$$\mathbf{e}_1 = \frac{\mathbf{u}_1}{u_1} = \begin{bmatrix} 0.64 \\ -0.67 \\ -0.39 \end{bmatrix}$$

In a similar manner $\lambda = \lambda_2 = 24.65$ yields

$$\mathbf{e}_2 = \begin{bmatrix} 0.38 \\ 0.71 \\ -0.59 \end{bmatrix}$$

And $\lambda = \lambda_3 = 0.77$ leads to

$$\mathbf{e}_3 = \begin{bmatrix} 0.67 \\ 0.23 \\ 0.71 \end{bmatrix}$$

Rotation matrix $\mathbf{C}_{21}$ becomes

$$\begin{bmatrix} 0.64 & 0.38 & 0.67 \\ -0.67 & 0.71 & 0.23 \\ -0.39 & -0.59 & 0.71 \end{bmatrix}$$

which gives the principle axis components of angular velocity as

$$\begin{aligned} \omega_1 &= 0.64\omega_x - 0.67\omega_y - 0.39\omega_z \\ \omega_2 &= 0.38\omega_x + 0.71\omega_y - 0.59\omega_z \\ \omega_3 &= 0.67\omega_x + 0.23\omega_y + 0.71\omega_z \end{aligned}$$

The principal moments of inertia

$$I_1 = 39.58$$
$$I_2 = 24.65$$
$$I_3 = 0.77$$

and principle axes are located by the components of $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$. The rotational kinetic energy can be simply written as

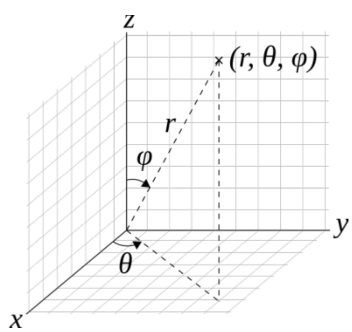
$$2T_r = 39.58\omega_1^2 + 24.65\omega_2^2 + 0.77\omega_3^2$$

Problem 3:

Because of the symmetry of the sphere, each principal moment is the same, so the moment of inertia of the sphere taken about any diameter has the same value. This means due to symmetry we have

$$I_{xx} = I_{yy} = I_{zz}$$

The density of the sphere is given by $\sigma = kr^3$.



In spherical coordinates we have:

$$x = r\sin\theta\sin\varphi$$
$$y = r\sin\theta\cos\varphi$$
$$z = r\cos\theta$$

The volume element dV in spherical coordinates is given by

$$\mathrm{d}V = r^2\sin\theta\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi$$

with $0 \le r \le a \qquad 0 \le \theta \le \pi \qquad 0 \le \varphi \le 2\pi$.

The total mass of the sphere is

$$M = \int \sigma(\rho_x, \rho_z, \rho_z)\, \mathrm{d}V = \int_V kr^3 \mathrm{d}V = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} kr^3 r^2 \sin\theta\, \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi = k\tfrac{a^6}{6}\cdot 2\cdot 2\pi = \tfrac{2\pi k a^6}{3}$$

The moment of inertia about the z-axis is given by

$$I_{zz} = \int_V (\rho_x^2 + \rho_y^2)\sigma(\rho_x, \rho_z, \rho_z)\mathrm{d}V = \int_V r^2 \sin^2\theta\, \sigma(\rho_x, \rho_z, \rho_z)\, \mathrm{d}V =$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^2 \sin^2\theta \cdot kr^3 \cdot r^2 \sin\theta\, \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi = \tfrac{2\pi k a^8}{8}\cdot\tfrac{4}{3}$$

Since $\int_0^{\pi} \sin^3\theta\, \mathrm{d}\theta = \tfrac{1}{3}\cos^3\theta - \cos\theta\,|_0^{\pi} = \tfrac{4}{3}$

The moment of inertias expressed as radius *a* and mass *M* are given by

$$I_{xx} = I_{yy} = I_{zz} = \tfrac{Ma^2}{2} .$$

Compare this with a sphere of the same radius but with uniform density for which the moment of inertias are

$$I_{xx} = I_{yy} = I_{zz} = \tfrac{2Ma^2}{5} .$$

Problem 4:

Using following equation

$$\mathbf{J} \triangleq -\int_V \boldsymbol{\rho}^{\times}\boldsymbol{\rho}^{\times}\mathrm{d}m$$
$$= \int_V \begin{bmatrix} (\rho_y^2+\rho_z^2) & -\rho_x\rho_y & -\rho_x\rho_z \\ -\rho_x\rho_y & (\rho_x^2+\rho_z^2) & -\rho_y\rho_z \\ -\rho_x\rho_z & -\rho_y\rho_z & (\rho_x^2+\rho_y^2) \end{bmatrix} \sigma(\rho_x,\rho_y,\rho_z)\mathrm{d}V$$

we have

$$J_{xx} = \int_V (\rho_y^2+\rho_z^2)dm \quad , \quad J_{yy} = \int_V (\rho_x^2+\rho_z^2)dm \quad , \quad J_{zz} = \int_V (\rho_x^2+\rho_y^2)dm$$

Therefore

$$J_{xx} - J_{yy} + J_{zz} = \int_V 2\rho_y^2 dm .$$

For any 3-dimensional body, there are points within the body with non-zero y-coordinates. Therefore, the above integral is positive, which gives the result sought after in the question.