ADCS – VI
Attitude Kinematics

Salvatore Mangano

MÁSTER UNIVERSITARIO EN SISTEMAS ESPACIALES
POR LA UNIVERSIDAD POLITÉCNICA DE MADRID

POLITÉCNICA

Outline

Reference frames and vectors

Rotation matrix (orientation of reference frames)

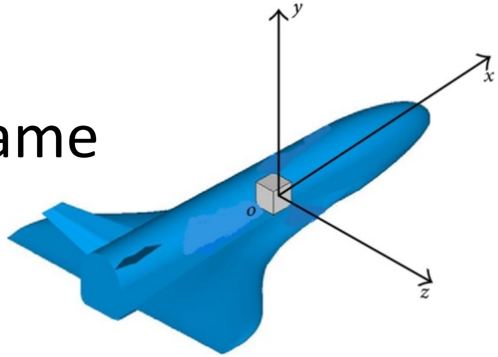
- Direction Cosine Matrix
- Euler angles
- Euler eigenaxis rotation
- Euler parameters / Quaternion

Angular velocity

- Transport theorem
- Angular acceleration

Kinematic differential equation

Introduction



Attitude coordinates are set of coordinates that describe either rigid body frame or reference frame

Infinite number of coordinate choices exist

Good choice in attitude coordinates can help to simplify mathematics of problem solving process

Bad choice in attitude coordinates can introduce singularities in attitude description, as well as highly nonlinear mathematics

Kinematics and dynamics

Kinematics

Kinematics is study of particle trajectory

Kinematic quantities are: mass, position, velocity, acceleration

No forces and torques are involved

In today´s Lecture ADCS - VI

Dynamics

Dynamics relates forces and torques to motion of objects

Lecture ADCS - VIII

Vectors

A vector is an abstract mathematical object with two properties: direction and length

Vectors used in this course are for example: position, velocity, acceleration, force, momentum, torque, angular velocity, etc.

Vectors can be expressed in any reference frame

Vectors exist even without reference frame

What is reference frame?

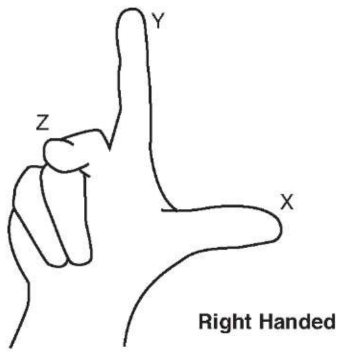
Reference frame

Reference frame is a set of

three mutually perpendicular (orthogonal) unit vectors

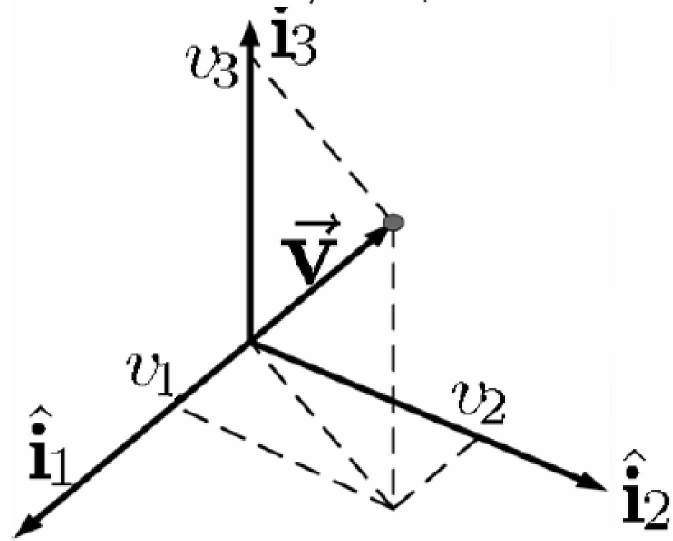
Right-handed reference frame is a set of

right-handed triad of orthonormal vectors



Vector can be expressed as

linear combination of unit vectors



Possible to have many different

reference frame at same time

Important to be clear about which reference frame is used

Examples of reference frame

Reference frames are given by

three mutually perpendicular (orthogonal) unit vectors

$\left\{\vec{i}_1, \vec{i}_2, \vec{i}_3\right\}$ (e.g. inertial frame)

$\left\{\vec{b}_1, \vec{b}_2, \vec{b}_3\right\}$ (e.g. body frame)

three basis vectors

$$\vec{i}_1 \cdot \vec{i}_1 = \vec{i}_2 \cdot \vec{i}_2 = \vec{i}_3 \cdot \vec{i}_3 = 1$$

unit vectors

$$\vec{i}_1 \cdot \vec{i}_2 = \vec{i}_2 \cdot \vec{i}_3 = \vec{i}_3 \cdot \vec{i}_1 = 0$$

orthogonal vectors

Arbitrary vector

$$\vec{r} = r_1\vec{i}_1 + r_2\vec{i}_2 + r_3\vec{i}_3$$

Arbitrary vector
A vector has always same magnitude and direction in all reference frames, but has different vector components in different frames. Components transform according to a rotation from one reference frame to an other reference frame.

$$\vec{r} = [\vec{i}_1 \quad \vec{i}_2 \quad \vec{i}_3]\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

Vector written in matrix notation
$\vec{r}$ is vector whose components in frame $\{\vec{i}_1, \vec{i}_2, \vec{i}_3\}$ are elements of 3 x 1 column matrix **r**

$$[\vec{i}_1 \quad \vec{i}_2 \quad \vec{i}_3]$$

Unconventional matrix with vectors $\{\vec{i}_1, \vec{i}_2, \vec{i}_3\}$ as its elements (called vectrix)

$$\mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

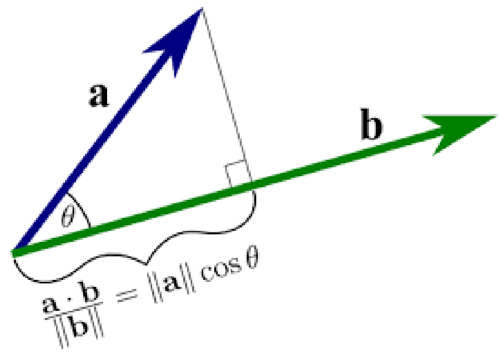
3 x 1 column matrix r
Representation of vector $\vec{r}$ with respect to basis vectors $\{\vec{i}_1, \vec{i}_2, \vec{i}_3\}$

Scalar product

Commutative
Changing order
→no influence

$$\vec{x} \cdot \vec{y} = |\vec{x}||\vec{y}|\cos\theta$$

$$\vec{x} \cdot \vec{x} = |\vec{x}|^2$$



For basis vectors in given reference frame

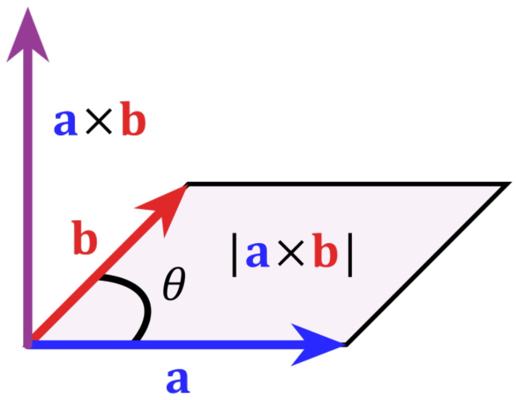
$$\begin{bmatrix} \vec{i}_1 \\ \vec{i}_2 \\ \vec{i}_3 \end{bmatrix} \cdot [\vec{i}_1 \quad \vec{i}_2 \quad \vec{i}_3] = \begin{bmatrix} \vec{i}_1 \cdot \vec{i}_1 & \vec{i}_1 \cdot \vec{i}_2 & \vec{i}_1 \cdot \vec{i}_3 \\ \vec{i}_2 \cdot \vec{i}_1 & \vec{i}_2 \cdot \vec{i}_2 & \vec{i}_2 \cdot \vec{i}_3 \\ \vec{i}_3 \cdot \vec{i}_1 & \vec{i}_3 \cdot \vec{i}_2 & \vec{i}_3 \cdot \vec{i}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} := \mathbf{1}$$

Cross product

Changing order
→reverses direction

$$\vec{x} \times \vec{y} = |\vec{x}||\vec{y}|\sin\theta\vec{n}$$

$$\vec{x} \times \vec{y} = -\vec{y} \times \vec{x}$$



Cross product written in matrix form

$$\vec{x} \times \vec{y} = [\vec{i}_1 \quad \vec{i}_2 \quad \vec{i}_3] \begin{bmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{bmatrix} = [\vec{i}_1 \quad \vec{i}_2 \quad \vec{i}_3] \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\mathbf{x}^{\times} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

Cross-product-equivalent matrix

Skew symmetry

Notation $[\ ]^{x}$ defines a skew-symmetric 3 x 3 matrix whose 3 elements are components of 3 x 1 matrix [ ]

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \mathbf{x}^{\times} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

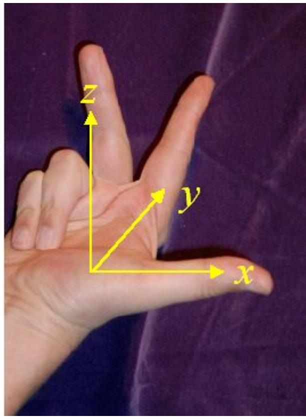
Skew-symmetric matrix has following properties:

$$\mathbf{x}^{\times}\mathbf{x} = 0$$

$$\mathbf{x}^{\times}\mathbf{y} = -\mathbf{y}^{\times}\mathbf{x}$$

$$(\mathbf{x}^{\times})^{T} = -\mathbf{x}^{\times}$$

Right-handed



Three-dimensional coordinate system in which axes satisfy right-hand rule is called right-handed coordinate system

Right-hand rule determines orientation of cross product $\vec{i}_j \times \vec{i}_k$

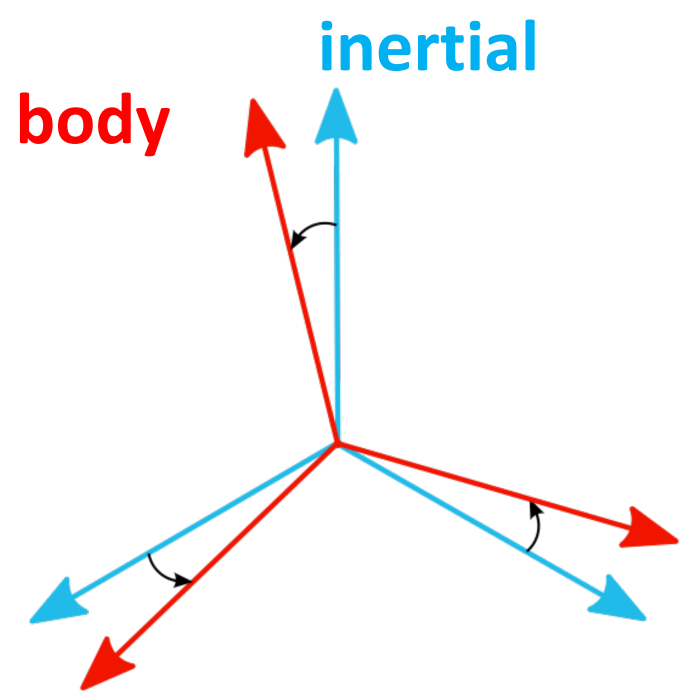
$$\vec{i}_1 \times \vec{i}_1 = \vec{0} \qquad \vec{i}_1 \times \vec{i}_2 = \vec{i}_3 \qquad \vec{i}_1 \times \vec{i}_3 = -\vec{i}_2$$
$$\vec{i}_2 \times \vec{i}_1 = -\vec{i}_3 \qquad \vec{i}_2 \times \vec{i}_2 = 0 \qquad \vec{i}_2 \times \vec{i}_3 = \vec{i}_1$$
$$\vec{i}_3 \times \vec{i}_1 = \vec{i}_2 \qquad \vec{i}_3 \times \vec{i}_2 = -\vec{i}_1 \qquad \vec{i}_3 \times \vec{i}_3 = \vec{0}$$

For basis vectors in given reference frame can be written as

$$\begin{bmatrix} \vec{i}_1 \\ \vec{i}_2 \\ \vec{i}_3 \end{bmatrix} \times [\vec{i}_1 \quad \vec{i}_2 \quad \vec{i}_3] = \begin{bmatrix} \vec{i}_1 \times \vec{i}_1 & \vec{i}_1 \times \vec{i}_2 & \vec{i}_1 \times \vec{i}_3 \\ \vec{i}_2 \times \vec{i}_1 & \vec{i}_2 \times \vec{i}_2 & \vec{i}_2 \times \vec{i}_3 \\ \vec{i}_3 \times \vec{i}_1 & \vec{i}_3 \times \vec{i}_2 & \vec{i}_3 \times \vec{i}_3 \end{bmatrix} = \begin{bmatrix} 0 & \vec{i}_3 & -\vec{i}_2 \\ -\vec{i}_3 & 0 & \vec{i}_1 \\ \vec{i}_2 & -\vec{i}_1 & 0 \end{bmatrix}$$

Rotation matrix

Typical attitude problem:
Suppose to have two reference frame (e.g. body and inertial)
What is the relationship between reference frames ?



Why working with rotations is complicated?

Rotations do not commute

Visualize sphere less familiar than plane

For computation represent rotation with numbers in R^n

Many different representation of rotations,
because of different requirements and applications

Directional Cosine Matrix

Direction cosines specify orientation of one Cartesian set of axes relative to another set with common origin

Coordinate frames

Suppose know components of vector **r** in body frame
Want to know components of vector **r** in inertial frame

$$\vec{r} = r_{b1}\vec{b}_1 + r_{b2}\vec{b}_2 + r_{b3}\vec{b}_3 \qquad \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$$

$$\vec{r} = r_{i1}\vec{i}_1 + r_{i2}\vec{i}_2 + r_{i3}\vec{i}_3 \qquad \{\vec{i}_1, \vec{i}_2, \vec{i}_3\}$$

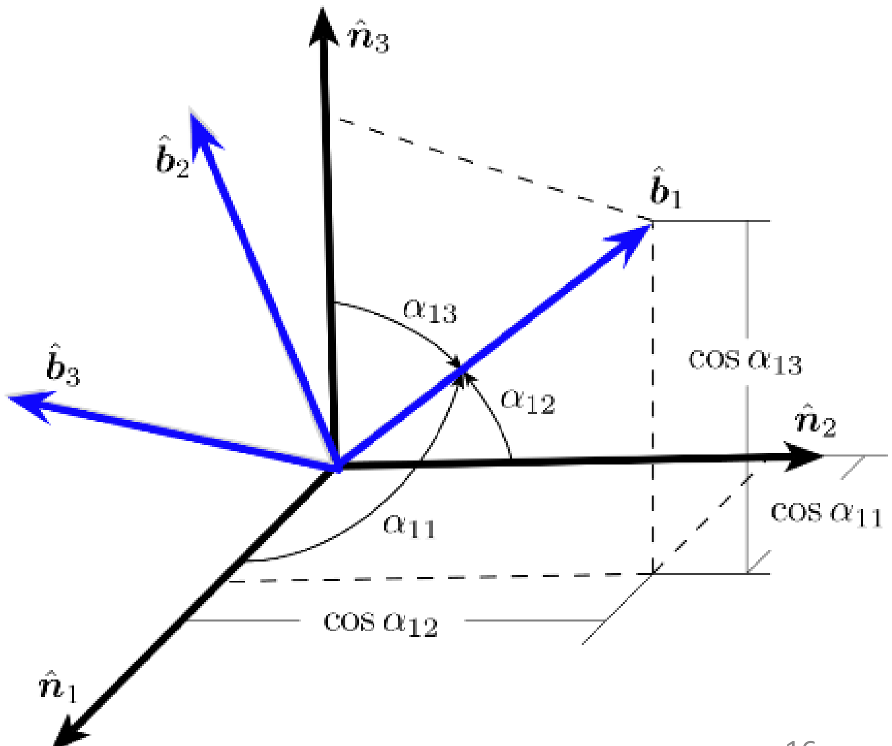
Basis vectors $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ are expressed in term of basis vectors $\{\vec{i}_1, \vec{i}_2, \vec{i}_3\}$

$$\vec{b}_1 = C_{11}\vec{i}_1 + C_{12}\vec{i}_2 + C_{13}\vec{i}_3$$

$$\vec{b}_2 = C_{21}\vec{i}_1 + C_{22}\vec{i}_2 + C_{23}\vec{i}_3$$

$$\vec{b}_3 = C_{31}\vec{i}_1 + C_{32}\vec{i}_2 + C_{33}\vec{i}_3$$

With $C_{ij} \equiv \vec{b}_i \cdot \vec{i}_j = \cos\alpha_{ij}$ cosine angle between two vectors



Frame base vectors are related
by 3 linear equation
→ 3 x 3 rotation matrix

Direction cosine matrix (DCM)

Two reference frames (e.g. body and inertial)
What is the relationship between reference frames **in matrix form**?
Express b frame with i frame

$$\begin{bmatrix} \vec{b}_1 \\ \vec{b}_1 \\ \vec{b}_1 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} \vec{i}_1 \\ \vec{i}_2 \\ \vec{i}_3 \end{bmatrix}$$

$$\mathbf{C}^{b/i} = \begin{bmatrix} \vec{b}_1 \cdot \vec{i}_1 & \vec{b}_1 \cdot \vec{i}_2 & \vec{b}_1 \cdot \vec{i}_3 \\ \vec{b}_2 \cdot \vec{i}_1 & \vec{b}_2 \cdot \vec{i}_2 & \vec{b}_2 \cdot \vec{i}_3 \\ \vec{b}_3 \cdot \vec{i}_1 & \vec{b}_3 \cdot \vec{i}_2 & \vec{b}_3 \cdot \vec{i}_3 \end{bmatrix} = \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} \cdot \begin{bmatrix} \vec{i}_1 & \vec{i}_2 & \vec{i}_3 \end{bmatrix}$$

Directional cosine matrix
with components

$$C_{ij} \equiv \vec{b}_i \cdot \vec{i}_j = \cos\alpha_{ij}$$ use scalar product

Analog
Express i frame with b frame

$$C_{ji} = \vec{i}_j \cdot \vec{b}_i = \cos\alpha_{ji}$$

$$C_{ji}^T = C_{ij}$$

Directional cosine matrix (rotation matrix) describes orientation of one reference frame with respect to another

Direction cosine matrix for components

Suppose know components of vector **r** in body frame

Want to know components of vector **r** in inertial frame

$$\vec{r} = r_{i1}\vec{i}_1 + r_{i2}\vec{i}_2 + r_{i3}\vec{i}_3$$
$$= r_{b1}\vec{b}_1 + r_{b2}\vec{b}_2 + r_{b3}\vec{b}_3$$

Components can be written

$$r_{b1} = \vec{r}\cdot\vec{b}_1 = (r_{i1}\vec{i}_1 + r_{i2}\vec{i}_2 + r_{i3}\vec{i}_3)\cdot\vec{b}_1$$
$$r_{b2} = \vec{r}\cdot\vec{b}_2 = (r_{i1}\vec{i}_1 + r_{i2}\vec{i}_2 + r_{i3}\vec{i}_3)\cdot\vec{b}_2$$
$$r_{b3} = \vec{r}\cdot\vec{b}_3 = (r_{i1}\vec{i}_1 + r_{i2}\vec{i}_2 + r_{i3}\vec{i}_3)\cdot\vec{b}_3$$

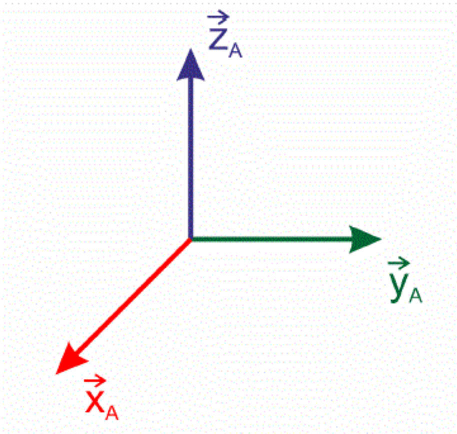
Which can be written in matrix form

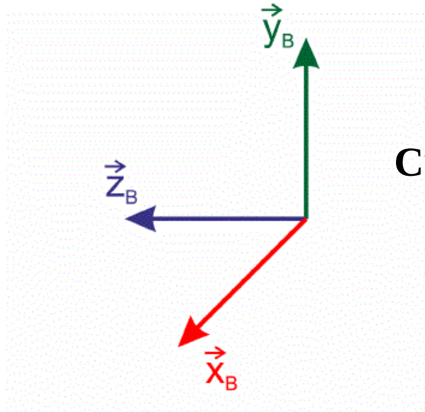
$$\begin{bmatrix} r_{b1} \\ r_{b2} \\ r_{b3} \end{bmatrix} = \begin{bmatrix} \vec{b}_1\cdot\vec{i}_1 & \vec{b}_1\cdot\vec{i}_2 & \vec{b}_1\cdot\vec{i}_3 \\ \vec{b}_2\cdot\vec{i}_1 & \vec{b}_2\cdot\vec{i}_2 & \vec{b}_2\cdot\vec{i}_3 \\ \vec{b}_3\cdot\vec{i}_1 & \vec{b}_3\cdot\vec{i}_2 & \vec{b}_3\cdot\vec{i}_3 \end{bmatrix} \begin{bmatrix} r_{i1} \\ r_{i2} \\ r_{i3} \end{bmatrix}$$

Means components of vector **r** are transformed to basis **b** from basis **i** using direction cosine matrix C_{ij} defined in previous slide for transformation of orthogonal basis vectors

Example: Direction cosine matrix

Define two coordinate systems A and B





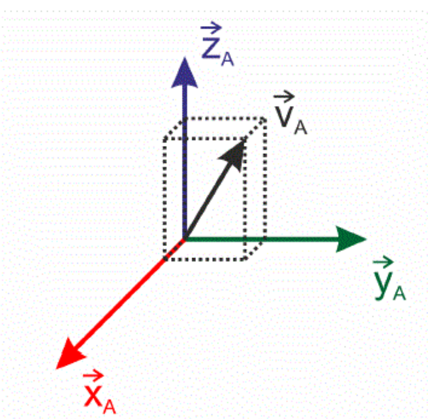
Determine DCM according definition

$$C_{ji} = \vec{i}_j \cdot \vec{b}_i = \cos\alpha_{ji}$$

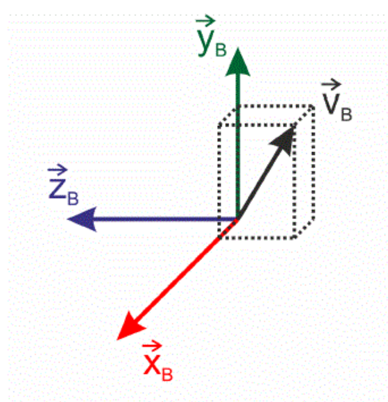
$$\mathbf{C}^{b/a} = \begin{bmatrix} \vec{b}_1\cdot\vec{i}_1 & \vec{b}_1\cdot\vec{i}_2 & \vec{b}_1\cdot\vec{i}_3 \\ \vec{b}_2\cdot\vec{i}_1 & \vec{b}_2\cdot\vec{i}_2 & \vec{b}_2\cdot\vec{i}_3 \\ \vec{b}_3\cdot\vec{i}_1 & \vec{b}_3\cdot\vec{i}_2 & \vec{b}_3\cdot\vec{i}_3 \end{bmatrix} = \begin{bmatrix} \vec{x}_B\cdot\vec{x}_A & \vec{x}_B\cdot\vec{y}_A & \vec{x}_B\cdot\vec{z}_A \\ \vec{y}_B\cdot\vec{x}_A & \vec{y}_B\cdot\vec{y}_A & \vec{y}_B\cdot\vec{z}_A \\ \vec{z}_B\cdot\vec{x}_A & \vec{z}_B\cdot\vec{y}_A & \vec{z}_B\cdot\vec{z}_A \end{bmatrix}$$

$$= \begin{bmatrix} \cos 0 & \cos 90 & \cos 90 \\ \cos 90 & \cos 90 & \cos 0 \\ \cos 90 & \cos 180 & \cos 90 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

Vector $\mathbf{v}_A$=(1, 2, 3)



Vector $\mathbf{v}_B$=?



Transform vector $\mathbf{v}_A$ from system A to system B via DCM and get result $\mathbf{v}_B$

$$\vec{v}_B = \mathbf{C}^{b/a}\vec{v}_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

Taken from: http://www.tu-berlin.de/fileadmin/fg169/miscellaneous/Quaternions.pdf

Matrix inverse

Transform from frame **b** to frame **i** and back to frame **b**

Analog for frame **i**

$$\left\{\vec{b}\right\} = \mathbf{C}\mathbf{C}^T\left\{\vec{b}\right\} \quad \longrightarrow \quad \mathbf{C}\mathbf{C}^T = \mathbf{1}$$

$$\left\{\vec{i}\right\} = \mathbf{C}^{\mathbf{T}}\mathbf{C}\left\{\vec{i}\right\} \quad \longrightarrow \quad \mathbf{C}^{\mathbf{T}}\mathbf{C} = \mathbf{1}$$

Therefore, inverse of directional cosine matrix is its transpose

$$\boxed{\mathbf{C}^{-1} = \mathbf{C}^T}$$

True for all orthogonal matrix

Rotation of components of vector

Arbitrary vector written in two reference frames with two different basis vectors

$$\vec{r} = r_{i1}\vec{i}_1 + r_{i2}\vec{i}_2 + r_{i3}\vec{i}_3$$
$$= r_{b1}\vec{b}_1 + r_{b2}\vec{b}_2 + r_{b3}\vec{b}_3$$

Components of vector can be written as:

$$r_{b1} = \vec{r}\cdot\vec{b}_1 = (r_{i1}\vec{i}_1 + r_{i2}\vec{i}_2 + r_{i3}\vec{i}_3)\cdot\vec{b}_1$$
$$r_{b2} = \vec{r}\cdot\vec{b}_2 = (r_{i1}\vec{i}_1 + r_{i2}\vec{i}_2 + r_{i3}\vec{i}_3)\cdot\vec{b}_2$$
$$r_{b3} = \vec{r}\cdot\vec{b}_3 = (r_{i1}\vec{i}_1 + r_{i2}\vec{i}_2 + r_{i3}\vec{i}_3)\cdot\vec{b}_3$$

Written in matrix form:

$$\begin{bmatrix} r_{b1} \\ r_{b2} \\ r_{b3} \end{bmatrix} = \begin{bmatrix} \vec{b}_1\cdot\vec{i}_1 & \vec{b}_1\cdot\vec{i}_2 & \vec{b}_1\cdot\vec{i}_3 \\ \vec{b}_2\cdot\vec{i}_1 & \vec{b}_2\cdot\vec{i}_2 & \vec{b}_2\cdot\vec{i}_3 \\ \vec{b}_3\cdot\vec{i}_1 & \vec{b}_3\cdot\vec{i}_2 & \vec{b}_3\cdot\vec{i}_3 \end{bmatrix} \begin{bmatrix} r_{i1} \\ r_{i2} \\ r_{i3} \end{bmatrix}$$

Vector **r** is not transformed

Both sides represent same vector

Components of vector **r** are transformed from frame **i** to frame **b** using directional cosine matrix

Why has rotation nine numbers ?

Rotation matrix has nine parameters

$$\mathbf{C} \equiv \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

But spatial rotations have only three degrees of freedom

Leaving six numbers which should have a constraint

Exist six constrains on nine parameters:

$$\begin{Bmatrix} \vec{i}_1 \cdot \vec{i}_1 = \vec{i}_2 \cdot \vec{i}_2 = \vec{i}_3 \cdot \vec{i}_3 = 1 \\ \vec{i}_1 \cdot \vec{i}_2 = \vec{i}_2 \cdot \vec{i}_3 = \vec{i}_3 \cdot \vec{i}_1 = 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \vec{i}_1 \cdot \vec{i}_1 = \vec{i}_2 \cdot \vec{i}_2 = \vec{i}_3 \cdot \vec{i}_3 = 1 \\ \vec{i}_1 \times \vec{i}_2 = \vec{i}_3 \end{Bmatrix}$$

Successive rotations (Three coordinate frame transformation)

Assume three coordinate frames $\left\{\vec{b}\right\}$ $\left\{\vec{i}\right\}$ $\left\{\vec{a}\right\}$

Components of vector **r** in these frames are: $\mathbf{r}_b$ $\mathbf{r}_i$ $\mathbf{r}_a$

Let frame b and frame i be related through: $\mathbf{r}_b = \mathbf{C}^{b/i}\mathbf{r}_i$

Let frame i and frame a be related through: $\mathbf{r}_i = \mathbf{C}^{i/a}\mathbf{r}_a$

Let frame b and frame a be related through: $\mathbf{r}_b = \mathbf{C}^{b/a}\mathbf{r}_a$

These three frames can be also related through: $\mathbf{r}_b = \mathbf{C}^{b/i}\mathbf{r}_i = \mathbf{C}^{b/i}\mathbf{C}^{i/a}\mathbf{r}_a$

Therefore:

$$\boxed{\mathbf{C}^{b/a} = \mathbf{C}^{b/i}\mathbf{C}^{i/a}}$$

Rotation matrix properties

Special properties of rotation matrices

- Orthogonal and orthonormal

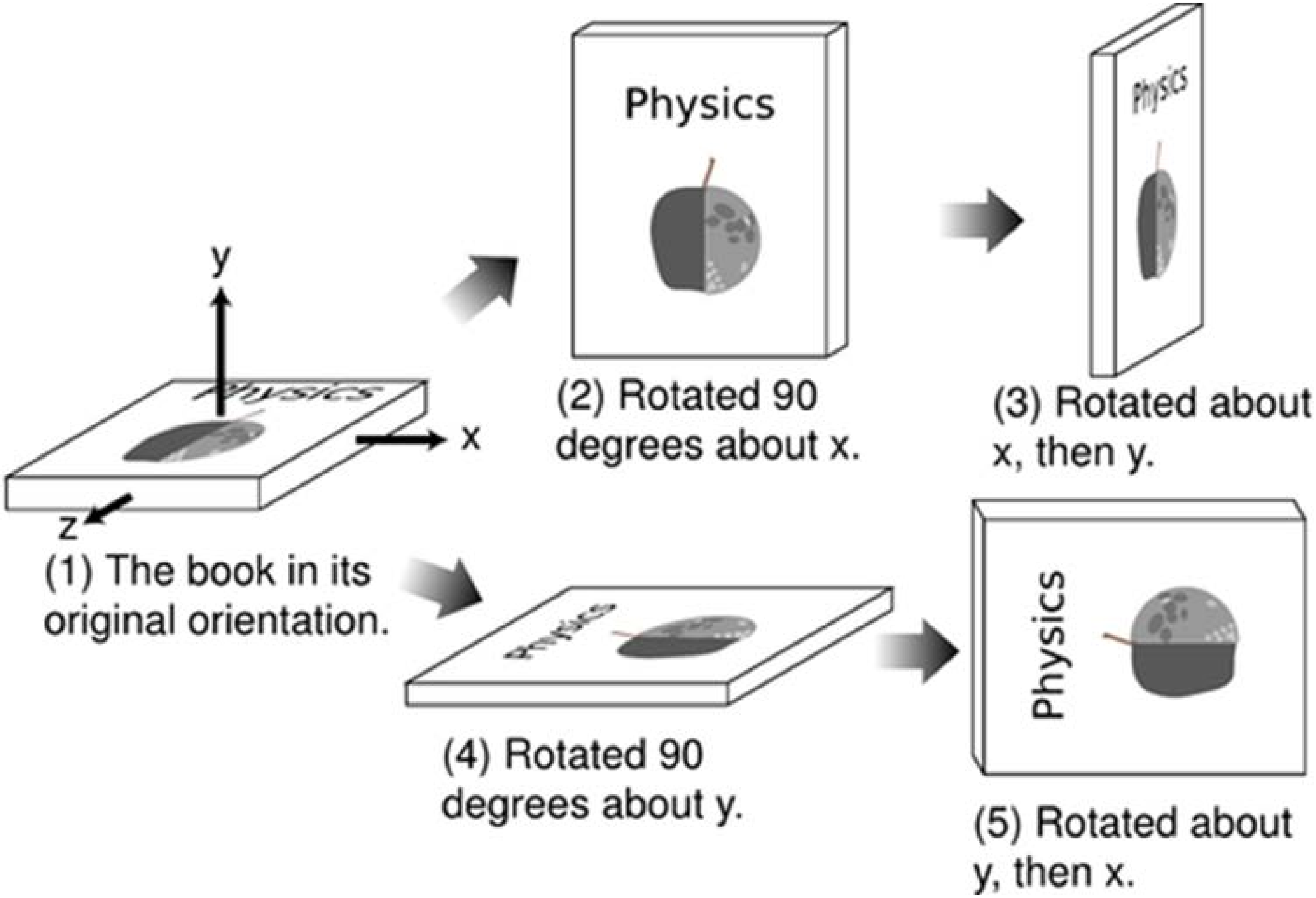
$$\boxed{\mathbf{C}\mathbf{C}^T = \mathbf{1} = \mathbf{C}^T\mathbf{C}} \qquad \boxed{\mathbf{C}^{-1} = \mathbf{C}^T}$$

- (Orthogonal if $\mathbf{C}\mathbf{C}^T$ is diagonal matrix)
- Inverse = Transpose
- det(C)=+1 or -1
- det(C)=+1 for right-handed
- Always non-singular
- Not commutative

$$\boxed{\mathbf{C}_A\mathbf{C}_B \neq \mathbf{C}_B\mathbf{C}_A}$$

Rotation order important

Not commutative: $\boxed{\mathbf{C}_A\mathbf{C}_B \neq \mathbf{C}_B\mathbf{C}_A}$



Elementary rotation matrices

Three elementary rotations about first, second and third axes of given reference frame described by following rotation matrices:

$$\mathbf{C}_3 = \begin{bmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Each rotation is through an angle about a specific coordinate axes

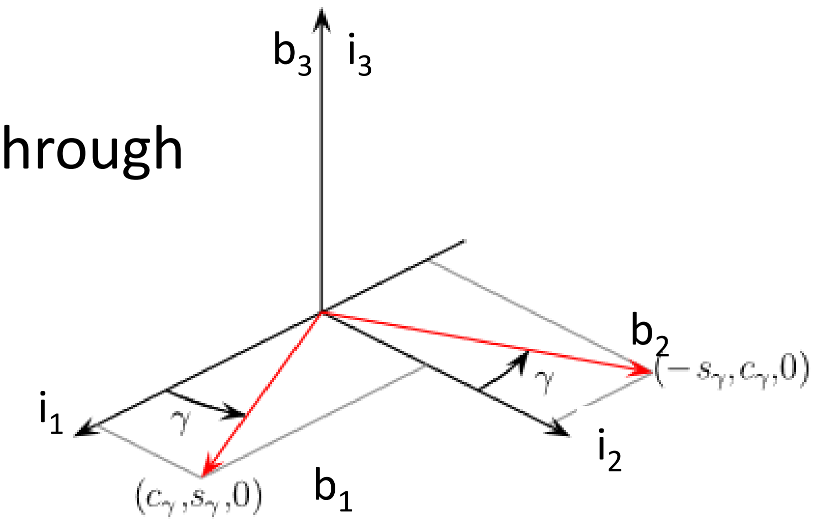
$$\mathbf{C}_2 = \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix}$$

More complex rotation matrices are constructed by combinations of elementary rotation matrices

$$\mathbf{C}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

Example of elementary rotation matrix

Consider 3-axis rotation using ϴ$_3$=ϒ



Inertial and body frame axis related through

$$\vec{b}_1 = \cos\gamma \vec{i}_1 + \sin\gamma \vec{i}_2$$

$$\vec{b}_2 = -\sin\gamma \vec{i}_1 + \cos\gamma \vec{i}_2$$

$$\vec{b}_3 = \vec{i}_3$$

This allows to write

$$\begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{i}_1 \\ \vec{i}_2 \\ \vec{i}_3 \end{bmatrix}$$

Leonhard Euler
(1707 – 1783)

Euler angles

Euler angles correspond to an orthogonal transformation via three successive rotations performed in specific sequence

Euler angles

Makes clear why transformation between reference frames are called rotation matrix

Most common set of attitude coordinates

Describes orientation between two frames using three sequential rotations

Order of rotation is important

Simple to visualize for small rotations

Euler angles

Any two independent orthonormal coordinate frames may be related by a sequence of three rotations, where no two successive rotations may be about the same axis.

Short: Any rotation may be described by only three parameters (using three angles).

Possible to bring a rigid body into an arbitrary orientation by performing three successive rotations:

1. The first rotation is about any axis
2. The second rotation is about any axis not used for first rotation
3. The third rotation is about any axis not used for second rotation

12 possible combinations

six three axis: 1-2-3, 1-3-2, 2-1-3, 2-3-1, 3-1-2, 3-2-1

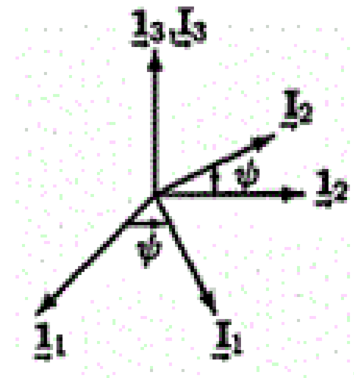
six two axis: 1-2-1, 1-3-1, 2-1-2, 2-3-2, 3-1-3, 3-2-3

Sequence is important because you may end up pointing at different direction depending on the sequence (not commutative)

3-1-3 Euler angles rotation sequence

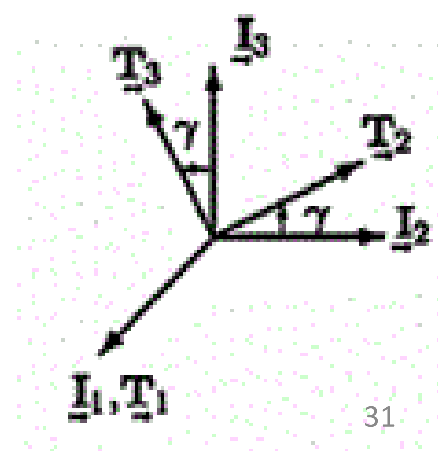
Rotate system around 3-axis

$$\mathbf{C}_3(\psi) = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Then rotate system around 1-axis

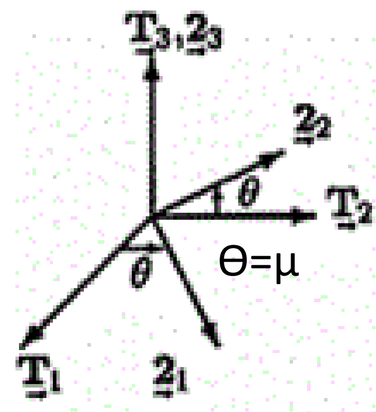
$$\mathbf{C}_1(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & \sin\gamma \\ 0 & -\sin\gamma & \cos\gamma \end{bmatrix}$$



3-1-3 Euler angles rotation sequence

Finally, rotate system around 3-axis

$$\mathbf{C}_3(\mu) = \begin{bmatrix} \cos\mu & \sin\mu & 0 \\ -\sin\mu & \cos\mu & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Complete transformation expressed as product of successive matrices

$$\boxed{\mathbf{C}_{tot}(\mu,\gamma,\psi) = \mathbf{C}_3(\mu)\mathbf{C}_1(\gamma)\mathbf{C}_3(\psi)}$$

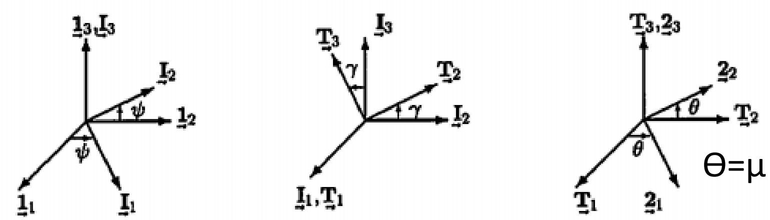
Total rotation is rotation matrix from frame 1 to frame 2

$$\mathbf{C}_{tot}(\mu,\gamma,\psi) = \mathbf{C}_{21}(\mu,\gamma,\psi)$$

3-1-3 Euler angles rotation sequence

3-1-3 transformation sequence

1. Rotation ψ about original 3-axis
2. Rotation ϒ about intermediate 1-axis
3. Rotation μ about transformed 3-axis



(In classical mechanics angles to describe orientation of a rigid body are called: ψ = precession angle, ϒ = nutation angle, μ = spin angle)

s = sin, c = cos

$$\mathbf{C}_{21}(\mu,\gamma,\psi)=\mathbf{C}_3(\mu)\mathbf{C}_1(\gamma)\mathbf{C}_3(\psi)$$

$$=\begin{bmatrix} c_\mu c_\psi - s_\mu s_\psi c_\gamma & c_\mu s_\psi + s_\mu c_\psi c_\gamma & s_\mu s_\gamma \\ -s_\mu c_\psi - c_\mu s_\psi c_\gamma & -s_\mu s_\psi + c_\mu c_\psi c_\gamma & c_\mu s_\gamma \\ s_\psi s_\gamma & -c_\psi s_\gamma & c_\gamma \end{bmatrix}$$

Singularity if ϒ=0 → angles ψ and μ are not uniquely determined

3-1-3 Euler angles rotation sequence

https://www.youtube.com/watch?v=tmtGEHTBSdQ

3-1-3 rotation sequence

https://www.youtube.com/watch?v=UpSMNYTVqQI

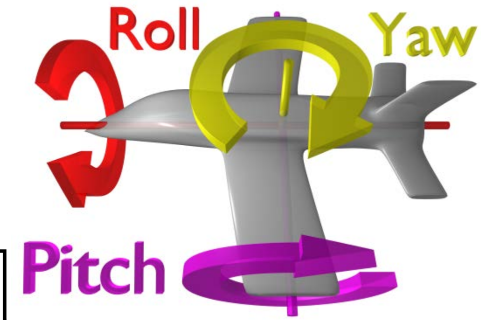
3-2-1 rotation sequence

3-2-1 Euler angles rotation sequence

3-2-1 transformation sequence

1. Rotation Θ_3 about original 3-axis (yaw rotation)
2. Rotation Θ_2 about intermediate 2-axis (pitch rotation)
3. Rotation Θ_1 about transformed 1-axis (roll rotation)

Used commonly in aerospace applications



$$\mathbf{C}(\theta_1,\theta_2,\theta_3) = \mathbf{C}_1(\theta_1)\mathbf{C}_2(\theta_2)\mathbf{C}_3(\theta_3)$$
$$= \begin{bmatrix} c_2c_3 & c_2s_3 & -s_2 \\ s_1s_2c_3 - c_1s_3 & s_1s_2s_3 + c_1c_3 & s_1c_2 \\ c_1s_2c_3 + s_1s_3 & c_1s_2s_3 - s_1c_3 & c_1c_2 \end{bmatrix}$$

Singularity when $\Theta_2=\pi/2$

Singularity when $\Theta_2 = \pi/2$

With $\Theta_2 = \pi/2$ in previous matrix → $c_2 = 0$ and $s_2 = 1$ and

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\mathbf{C}(\theta_1, \frac{\pi}{2}, \theta_3) = \begin{bmatrix} 0 & 0 & -1 \\ \sin(\theta_1 - \theta_3) & \cos(\theta_1 - \theta_3) & 0 \\ \cos(\theta_1 - \theta_3) & -\sin(\theta_1 - \theta_3) & 0 \end{bmatrix}$$

In this case Θ_1 and Θ_3 are associated with same rotation

Main problem computationally is existence of singularities

Problem can be avoided by an application of Euler´s theorem (quaternion)

Singularities in Euler rotations

Each set of Euler angles has singularity where two angles are not uniquely defined

It is always second angle which defines singularity

- two axis: 2^{nd} angle is 0 or π
  (3-1-3 with ϒ = 0 or π)
- three axis: 2^{nd} angle is +π/2 or –π/2
  (3-2-1 with Θ = π/2)

Independent which sequence of Euler rotation you choose after rotation of π/2 always encounter a singularity

→ Euler angles have singularity problems for large reorientations

Infinitesimal rotation

Consider 3-2-1 transformation with $|\theta_1|, |\theta_2|, |\theta_3| \ll 1$

and $c_i \cong 1, s_i \cong \theta_i, \theta_i\theta_j \cong 0$

$$\mathbf{C} = \begin{bmatrix} 1 & \theta_3 & -\theta_2 \\ -\theta_3 & 1 & \theta_1 \\ \theta_2 & -\theta_1 & 1 \end{bmatrix}$$

$$= 1 - \boldsymbol{\theta}^{\times} \qquad \boldsymbol{\theta} = [\theta_1 \ \theta_2 \ \theta_3]$$

Order of rotation for infinitesimal rotation are not dependent

Check the result for 3-1-3 transformation?

$$\begin{bmatrix} 1 & \psi + \mu & 0 \\ -\psi - \mu & 1 & \gamma \\ 0 & -\gamma & 1 \end{bmatrix}$$

From 3-1-3 rotation matrix to Euler angles

Know Euler angles find 3-1-3 matrix

$$\mathbf{C}_{21}(\mu,\gamma,\psi)=\mathbf{C}_3(\mu)\mathbf{C}_1(\gamma)\mathbf{C}_3(\psi)$$

$$=\begin{bmatrix} c_\mu c_\psi - s_\mu s_\psi c_\gamma & c_\mu s_\psi + s_\mu c_\psi c_\gamma & s_\mu s_\gamma \\ -s_\mu c_\psi - c_\mu s_\psi c_\gamma & -s_\mu s_\psi + c_\mu c_\psi c_\gamma & c_\mu s_\gamma \\ s_\psi s_\gamma & -c_\psi s_\gamma & c_\gamma \end{bmatrix}$$

Know 3-1-3 matrix find Euler angles (Look at matrix)

$$\mu=\tan^{-1}(\tfrac{C_{13}}{C_{23}})$$

$$\gamma=\cos^{-1}(C_{33})$$

$$\psi=-\tan^{-1}(\tfrac{C_{31}}{C_{32}})$$

Quadrants must be checked with inverse tangent function

From 3-2-1 rotation matrix to Euler angles

Know Euler angles find 3-2-1 matrix

$$\mathbf{C}(\theta_1,\theta_2,\theta_3) = \mathbf{C}_1(\theta_1)\mathbf{C}_2(\theta_2)\mathbf{C}_3(\theta_3)$$

$$= \begin{bmatrix} c_2 c_3 & c_2 s_3 & -s_2 \\ s_1 s_2 c_3 - c_1 s_3 & s_1 s_2 s_3 + c_1 c_3 & s_1 c_2 \\ c_1 s_2 c_3 + s_1 s_3 & c_1 s_2 s_3 - s_1 c_3 & c_1 c_2 \end{bmatrix}$$

Know 3-2-1 matrix find Euler angles

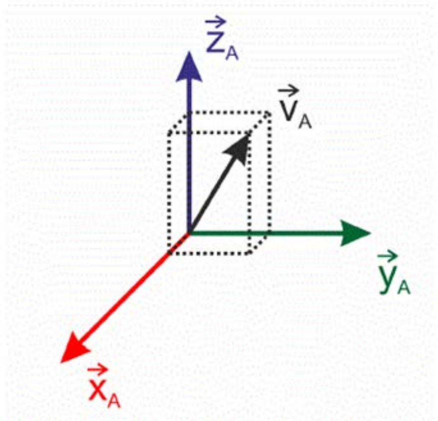
$$\theta_1 = \tan^{-1}\left(\frac{C_{23}}{C_{33}}\right)$$

$$\theta_2 = -\sin^{-1}(C_{13})$$

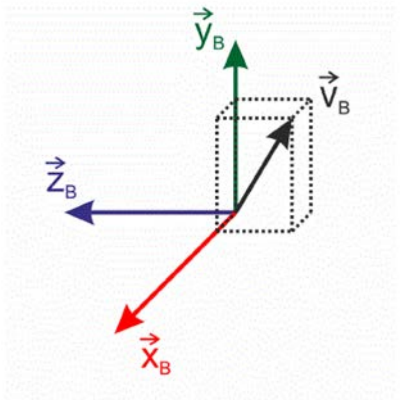
$$\theta_3 = \tan^{-1}\left(\frac{C_{12}}{C_{11}}\right)$$

Example: Euler angles

Vector $\mathbf{v}_A = (1, 2, 3)$



Vector $\mathbf{v}_B = ?$



Do vector transformation from frame A to frame B using Euler angles

$$\vec{v}_B = \mathbf{C}^{b/a}_{Euler} \vec{v}_A$$

Exist twelve different transformation sequences

In this case there is only one single rotation around x-axis

$$\mathbf{C}^{b/a}_{Euler} = \mathbf{C}^{b/a}_{1-2-3} = \mathbf{C}^{b/a}_{1}(\theta_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

For $\Theta = 90^o$ vector $\mathbf{v}_A$ transforms into $\mathbf{v}_B$ as follows

$$\vec{v}_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{bmatrix} \vec{v}_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

Information about attitude coordinates

A minimum of three coordinates is required to describe relative angular displacement between two reference frames

Any minimal set of three coordinates will contain at least one geometrical orientation where coordinates are singular, namely at least two coordinates are undefined or not unique

At or near such geometric singularity, corresponding kinematic differential equation are also singular (see later)

Geometric singularities and associated numerical difficulties can be avoided by introducing redundant sets of four or more coordinates

Euler eigenaxis rotation

Euler´s eigenaxis rotation theorem

Theorem: (Euler´s eigenaxis rotation)

Most general displacement of a rigid body with one point fixed is a rotation about a fixed axis through that point.

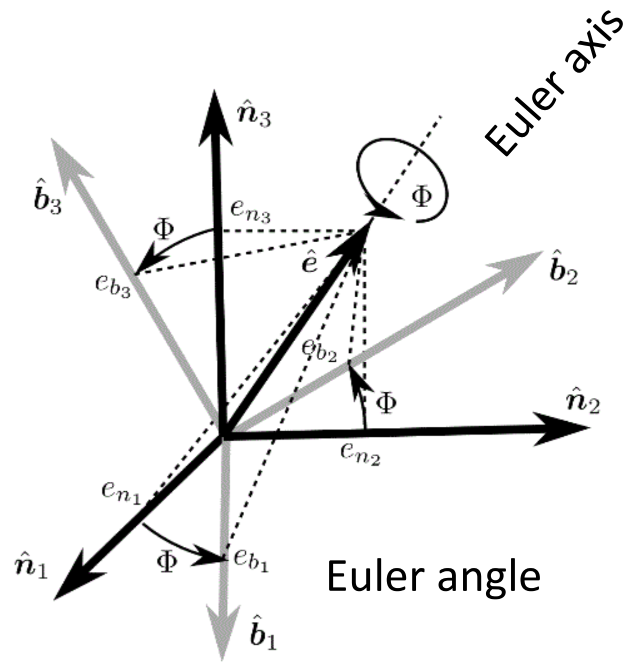


Illustration of eigenaxis rotation theorem

Euler eigenaxis rotation

Axis ***e*** is principle eigenaxis with eigenvalue 1 (Euler axis)

$\vec{e} = [e_1, e_2, e_3]$ $\qquad\qquad$ $\vec{e}^T\vec{e} = e_1^2 + e_2^2 + e_3^2 \equiv 1$

Angle of rotation *Φ* is called Euler angle

Four parameters vector: ***e*** and angle *Φ*

In both frames $\mathbf{C}\vec{e} = \vec{e}$ (***e*** fixed axis)

Euler eigenaxis rotation given by

$$\boxed{\mathbf{C}(\vec{e},\phi) = \cos\phi\mathbf{1} + (1-\cos\phi)\vec{e}\vec{e}^T - \sin\phi\mathbf{e}^\times}$$

Please write
3 x 3 matrix out

$$\mathbf{1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \vec{e}\vec{e}^T = \begin{bmatrix} e_1e_1 & e_1e_2 & e_1e_3 \\ e_2e_1 & e_2e_2 & e_2e_3 \\ e_3e_1 & e_2e_3 & e_3e_3 \end{bmatrix} \quad \mathbf{e}^\times = \begin{bmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{bmatrix}$$

Extracting Euler eigenaxis rotation parameters from known direction cosine matrix

Compute Euler eigenaxis rotation parameters from given direction cosine matrix

Known direction cosine matrix **C** determine ***e*** and Φ

$$\cos\phi = \tfrac{1}{2}[C_{11} + C_{22} + C_{33} - 1]$$

$$\vec{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \tfrac{1}{2\sin\phi}\begin{bmatrix} C_{23} - C_{32} \\ C_{31} - C_{13} \\ C_{12} - C_{21} \end{bmatrix} \quad if \quad \phi \neq 0, \pm\pi, \pm 2\pi, \ldots$$

Euler parameters = Quaternion

Euler parameters or Quaternion

Based on Euler´s eigenaxis rotation theorem construct another set of four parameters → Called Euler parameters or quaternion

Euler parameters or quaternion defined using Euler eigenaxis rotation

Know: $\vec{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$ *and* ϕ $\quad \vec{e}^T\vec{e} = e_1^2 + e_2^2 + e_3^2 \equiv 1$

Define: $$\boxed{\vec{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} e_1 \sin(\phi/2) \\ e_2 \sin(\phi/2) \\ e_3 \sin(\phi/2) \end{bmatrix} \quad , \quad q_4 = \cos\frac{\phi}{2} \qquad \vec{q} = \vec{e}\sin\frac{\phi}{2}}$$

Four parameters are not independent and satisfy

$$\boxed{q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1}$$

Example: Euler parameters or Quaternion

Definition: $\vec{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} e_1 \sin(\phi/2) \\ e_2 \sin(\phi/2) \\ e_3 \sin(\phi/2) \end{bmatrix} \quad , \quad q_4 = \cos\frac{\phi}{2} \qquad \vec{q} = \vec{e}\sin\frac{\phi}{2}$

Unit quaternion represent rotation

Example: Rotation about z-axis with 90º $\qquad \mathbf{C}_3(90^o) \quad \Leftrightarrow \quad \mathbf{q} = \begin{bmatrix} 0 \\ 0 \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$

q and –**q** represent same rotation

From directional cosine matrix to quaternion

Directional cosine matrix can be expressed in terms of quaternion:

$$\mathbf{C}(\vec{e},\phi) = \cos\phi\mathbf{1} + (1-\cos\phi)\vec{e}\vec{e}^T - \sin\phi\mathbf{e}^{\times}$$

$$\boxed{\mathbf{C}(\vec{q},q_4) = (q_4^2 - \vec{q}^T\vec{q})\mathbf{1} + 2\vec{q}\vec{q}^T - 2q_4\mathbf{q}^{\times}}$$

$$= \begin{bmatrix} 1-2(q_2^2+q_3^2) & 2(q_1q_2+q_3q_4) & 2(q_1q_3-q_2q_4) \\ 2(q_2q_1-q_3q_4) & 1-2(q_3^2+q_1^2) & 2(q_2q_3+q_1q_4) \\ 2(q_3q_1+q_2q_4) & 2(q_3q_2-q_1q_4) & 1-2(q_1^2+q_2^2) \end{bmatrix}$$

Inverse relationship is found by looking at matrix

Quaternion expressed by directional cosine matrix

$$q_4 = \pm\tfrac{1}{2}(1+C_{11}+C_{22}+C_{33})^{\frac{1}{2}}$$

$$\vec{q} = \frac{1}{4q_4}\begin{bmatrix} C_{23}-C_{32} \\ C_{31}-C_{13} \\ C_{12}-C_{21} \end{bmatrix}$$

Example: From quaternion to rotation matrix

Write previous quaternion example as rotation matrix $\mathbf{q} = \begin{bmatrix} 0 \\ 0 \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$

Directional cosine matrix can be expressed in terms of quaternion

$$\mathbf{C}(\vec{q}, q_4) = (q_4^2 - \vec{q}^T \vec{q})\mathbf{1} + 2\vec{q}\vec{q}^T - 2q_4\mathbf{q}^\times$$

$$= \begin{bmatrix} 1-2(q_2^2+q_3^2) & 2(q_1q_2+q_3q_4) & 2(q_1q_3-q_2q_4) \\ 2(q_2q_1-q_3q_4) & 1-2(q_3^2+q_1^2) & 2(q_2q_3+q_1q_4) \\ 2(q_3q_1+q_2q_4) & 2(q_3q_2-q_1q_4) & 1-2(q_1^2+q_2^2) \end{bmatrix}$$

Insert:

$$\mathbf{C}(\vec{q}, q_4) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{C}_3(90^o)$$

Quaternion (Euler parameters)

Quaternion is 4 x 1 matrix

vector part $q_{1:3}$

scalar part q_4

$$\mathbf{q} = \begin{bmatrix} \vec{q} \\ q_4 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

Quaternion defined by

rotation axis and rotation angle

Quaternion represents coordinate

transformation from system A to system B

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} e_1 \sin\frac{\phi}{2} \\ e_2 \sin\frac{\phi}{2} \\ e_3 \sin\frac{\phi}{2} \\ \cos\frac{\phi}{2} \end{bmatrix}$$

Quaternion mathematics

Magnitude

$$|\mathbf{q}| = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2}$$

All quaternions for attitude are unit quaternions

$$|\mathbf{q}| = 1$$

Conjugate

$$\mathbf{q}^* = \begin{bmatrix} -q_1 \\ -q_2 \\ -q_3 \\ q_4 \end{bmatrix}$$

Conjugate quaternion has inverted vector part

Quaternion mathematics

Inverse

$$\mathbf{q}^{-1} = \frac{\mathbf{q}^{*}}{|q|}$$

Inverse of quaternion → quaternion is conjugated and normalized

Multiplication

$$\mathbf{q}_1 \otimes \mathbf{q}_2 = \mathbf{q} = \begin{bmatrix} \vec{q} \\ \eta \end{bmatrix} = \begin{bmatrix} \eta_1 \vec{q}_2 + \eta_2 \vec{q}_1 + \vec{q}_1 \times \vec{q}_2 \\ \eta_1 \eta_2 - \vec{q}_1^{\,T} \cdot \vec{q}_2 \end{bmatrix}$$

Product of two quaternions is mix of scalar product and cross product

Euler parameters = quaternion

- Two names: Euler parameters or quaternion
- Quaternion definition introduces a redundant fourth element, which eliminates Euler angle rotation singularity
- Very popular redundant set of attitude coordinates (used in space applications)
- Benefits:
  - Non-singular attitude description
  - Linear differential kinematic equation (see later)
  - Works well for small and large rotations
  - Computationally convenient
- Problems:
  - No intuitively interpretable meaning to human mind
  - Constraint equation must be checked after an operation $|q|=1$
  - Four parameters instead of three as in case of Euler angles

Many more attitude parameterizations

Many attitude parameterization exist:

- Directional Cosine Matrix (DCM)
- Euler axis and angle rotation
- Euler angles (easiest to physically understand)
- Quaternions
- Gibbs vector
- Modified Rodrigues parameters
- Generalized Rodrigues parameters

They are equivalent to each other

Can convert from one to another

Each has its advantages and disadvantages

Parameterization chosen is usually problem dependent

Summary of attitude parameterization

Several equivalent ways to describe one reference frame with respect to another reference frame

- Rotation matrix = Directional Cosines Matrix = DCM =
  vectors of one frame expressed in other =
  dot products of vectors of one frame with those of other frame
  (Orientation defines unique direction cosines matrix, but 6 constrains must be met and is non-intuitive)

- Euler angles: 3 x 2 x 2 = 12 different sets
  If angles are given then orientation is unique, but can have singularity
  Is intuitive and good for analytical work

- Euler axis and angle: Unit vector and angle

- Euler parameters = Quaternion: Unity of four components
  (Computationally robust and ideal for digital control implementation, but not intuitive)

Angular velocity

How does attitude vary with time?

In most cases attitude changes with time

Force, velocity, and acceleration are vector quantities described with respect to a frame of reference

Newton's laws are only valid in inertial systems

Newton´s law **F** = m**a** holds only for forces and acceleration with respect to an inertial reference frame

Rotating reference frames introduce centripetal and Coriolis acceleration terms to Newton´s Law

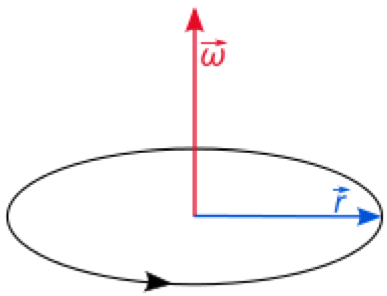
Comments regarding time derivatives

- Observers in all reference frames see same time rate of change of a scalar function of time (e.g. if T is temperature, then dT/dt has one meaning).
- Observers in all reference frame see same vector. It may have different coordinates, because these are reference frame dependent, but its still the same vector:

$$r_{b1}\vec{b}_1 + r_{b2}\vec{b}_2 + r_{b3}\vec{b}_3 = \vec{r} = r_{i1}\vec{i}_1 + r_{i2}\vec{i}_2 + r_{i3}\vec{i}_3$$

- Observers in different reference frames **do not** see same time rate of change of a vector function of time.
- Observers in different reference frames will make different observations about rate of change of a vector function of time.
- Consequently: **Rate of change of a vector** must be qualified by stating explicitly reference frame in which observation and measurement is being made.

Angular velocity vector



Angular velocity **ω** describes speed of rotation and orientation of instantaneous axis about which rotation occurs.

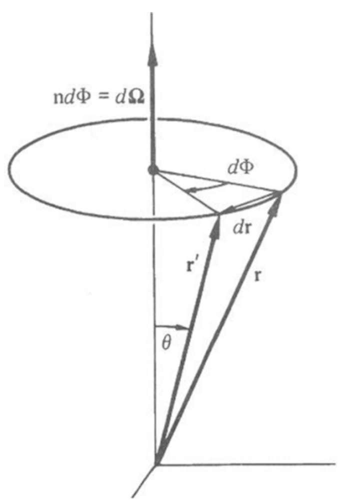
$$\vec{\omega}_{31} = \vec{\omega}_{32} + \vec{\omega}_{21}$$

Addition of angular velocities

Angular velocities of frame 3 with respect to frame 1 is equal to angular velocity of frame 3 with respect to frame 2 plus angular velocity of frame 2 with respect to frame 1. True for vector but caution with their components.

Angular velocity vector of rotating reference frame

Rotating frame of reference is special case of non-inertial reference frame



It can be shown that time derivative of unit vectors $\mathbf{b_i}$ measured in a rotating reference frame are given by:

$$\dot{\vec{b}}_1 = \vec{\omega} \times \vec{b}_1$$

$$\dot{\vec{b}}_2 = \vec{\omega} \times \vec{b}_2$$

$$\dot{\vec{b}}_3 = \vec{\omega} \times \vec{b}_3$$

Relation between time derivatives of unit vectors

Unit vectors **b** fixed in body reference frame rotate with an angular velocity **ω** with respect to inertial frame.

Rate of change of **b** is caused only by **ω** and is normal to both **b** and **ω**.

Transport theorem

Vector **r** expressed in rotating body basis vectors

$$\vec{r} = r_1\vec{b}_1 + r_2\vec{b}_2 + r_3\vec{b}_3$$

Time derivative as seen in inertial frame but expressed in body frame

$$\left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right)_i \equiv \dot{\vec{r}} = \dot{r}_1\vec{b}_1 + \dot{r}_2\vec{b}_2 + \dot{r}_3\vec{b}_3 + r_1\dot{\vec{b}}_1 + r_2\dot{\vec{b}}_2 + r_3\dot{\vec{b}}_3$$

Define time derivative of **r** seen in body frame (because vectors $\mathbf{b}_i$ are fixed (time invariant))

$$\left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right)_b \equiv \dot{r}_1\vec{b}_1 + \dot{r}_2\vec{b}_2 + \dot{r}_3\vec{b}_3$$

Use body-fixed vectors $\mathbf{b}_i$ to find their time derivative in inertial frame $\dot{\vec{b}}_i = \vec{\omega}\times\vec{b}_i \quad i = 1, 2, 3$

velocity of particle seen in frame i

velocity of particle seen in frame b

$$\boxed{\left(\frac{\mathrm{d}}{\mathrm{d}t}\vec{r}\right)_i = \left(\frac{\mathrm{d}}{\mathrm{d}t}\vec{r}\right)_b + \vec{\omega}\times\vec{r}}$$

Transport Theorem

Relation between velocities in two frames
Rotating body reference frame has two components, one from explicit time dependence due to motion of particle itself, and another from frame's own rotation.
If **ω** = 0 then time-derivative of vector seen in both systems is same.

Time derivatives in rotating frame

Transport theorem is much more general

Transport theorem says that derivatives of quantities in an inertial frame i are normal derivatives, whereas in rotating frame b to normal derivatives an additional term **ω**x is used

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\quad\right)_i = \frac{\mathrm{d}}{\mathrm{d}t}\left(\quad\right)_b + \vec{\omega}\times\left(\quad\right)}$$

In open slot you can insert any vector

Angular acceleration

$$\vec{v} = \left(\dot{\vec{r}}\right)_i = \left(\dot{\vec{r}}\right)_b + \vec{\omega} \times \vec{r}$$

$$\ddot{\vec{r}} = \left(\dot{\vec{v}}\right)_i = \left(\dot{\vec{v}}\right)_b + \underbrace{\vec{\omega} \times \vec{v}}$$

$$= \left(\ddot{\vec{r}}\right)_b + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \dot{\vec{r}} + \vec{\omega} \times (\dot{\vec{r}} + \vec{\omega} \times \vec{r})$$

$$= \left(\ddot{\vec{r}}\right)_b + \dot{\vec{\omega}} \times \vec{r} + 2\vec{\omega} \times \dot{\vec{r}} + \vec{\omega} \times \vec{\omega} \times \vec{r}$$

Acceleration of particle
in moving frame b

Coriolis acceleration
due to motion of particle

Angular acceleration

Centripetal acceleration
due to angle between ω and r

If **ω** = constant
→ Rotating system: acceleration acquires coriolis and centripetal components

Newton´s law in rotating frame

In an inertial frame Newton´s law is:

$$m\ddot{\vec{r}} = \vec{F}$$

In a rotating frame Newton´s law becomes:

$$m\left(\ddot{\vec{r}}\right)_b = \vec{F} - m\dot{\vec{\omega}}\times\vec{r} - 2m\vec{\omega}\times\dot{\vec{r}} - m\vec{\omega}\times\vec{\omega}\times\vec{r}$$

Kinematic differential equation

Kinematic differential equation

What do we mean?

- kinematic = position description
- differential equation = time rate equation

Time-dependent relationship between two reference frame is given by kinematic differential equation

Analogy for translation: $\vec{v} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}$ $\vec{r} = \vec{r}_0$

Given velocity of point and initial conditions for position

Compute position as function of time by integrating differential equation

Need to develop equivalent differential equations for attitude when angular velocity is known

Kinematic differential equation

- In most applications attitude changes with time
- Inertial sensors measure angular rate of spacecraft
- Need to integrate angular rate over finite time interval to find change in attitude
- Given initial conditions for attitude and time evolution of angular velocity → compute **C** or any other attitude representation as a function of time
- For each of previous presented parameterization can derive a differential equation which integrated gives attitude change

Kinematic differential equation

Compute position as a function of time by integrating differential equation

$$\dot{\vec{r}} = \vec{v}$$

Equivalent differential equations for attitude when angular velocity is known

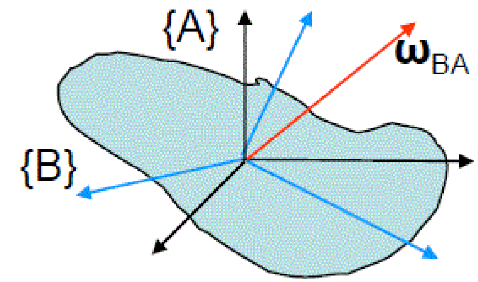
$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \frac{1}{\cos\theta_2} \begin{bmatrix} \cos\theta_2 & \sin\theta_1 \sin\theta_2 & \cos\theta_1 \sin\theta_2 \\ 0 & \cos\theta_1 \cos\theta_2 & -\sin\theta_1 \cos\theta_2 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

Euler 3-2-1

$$\dot{\mathbf{C}} + \boldsymbol{\omega}^{\times} \mathbf{C} = \mathbf{0} \qquad \dot{\mathrm{q}} = \tfrac{1}{2} \Omega \mathrm{q}$$

Kinematic differential equation for direction cosine matrix

How does direction cosine matrix **C** evolve over time? $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{C} = \dot{\mathbf{C}}$



Consider two reference frames A and B,
which are rotating relative to each other

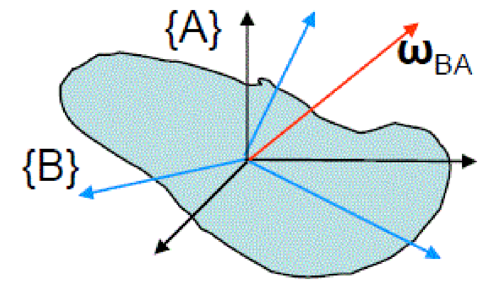
Time dependent angular velocity vector of reference frame B with respect to reference frame A is denoted by $\vec{\omega} \equiv \vec{\omega}^{B/A}$

Rotation rate expressed through angular velocity vector in terms of basis vectors B

$$\vec{\omega} = \omega_1 \vec{b}_1 + \omega_2 \vec{b}_2 + \omega_3 \vec{b}_3$$

Angular velocity vector determines how body will rotate and thus how directional cosine matrix describing orientation will evolve

Kinematic differential equation for direction cosine matrix



With directional cosine matrix $\mathbf{C} \equiv \mathbf{C}^{B/A}$ write:

$$\begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} = \mathbf{C} \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix} \Rightarrow \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix} = \mathbf{C}^{-1} \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} = \mathbf{C}^{T} \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix}$$

Because of rotating reference frames,
elements C_{ij} of directional cosine matrix are function of time
Take time derivatives

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \dot{\mathbf{C}}^{T} \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} + \mathbf{C}^{T} \begin{bmatrix} \dot{\vec{b}}_1 \\ \dot{\vec{b}}_2 \\ \dot{\vec{b}}_3 \end{bmatrix} = \dot{\mathbf{C}}^{T} \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} + \mathbf{C}^{T} \begin{bmatrix} \vec{\omega} \times \vec{b}_1 \\ \vec{\omega} \times \vec{b}_2 \\ \vec{\omega} \times \vec{b}_3 \end{bmatrix} = \dot{\mathbf{C}}^{T} \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} - \mathbf{C}^{T} \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix}$$

Kinematic differential equation for direction cosine matrix

Previous slide $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \dot{\mathbf{C}}^T \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} - \mathbf{C}^T \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix}$

Define skew-symmetric matrix $\boldsymbol{\omega}^\times \equiv \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$ and $\dot{\mathbf{C}} \equiv \begin{bmatrix} \dot{C}_{11} & \dot{C}_{12} & \dot{C}_{13} \\ \dot{C}_{21} & \dot{C}_{22} & \dot{C}_{23} \\ \dot{C}_{31} & \dot{C}_{32} & \dot{C}_{33} \end{bmatrix}$

$$\Rightarrow \quad [\dot{\mathbf{C}}^T - \mathbf{C}^T \boldsymbol{\omega}^\times] \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \dot{\mathbf{C}}^T - \mathbf{C}^T \boldsymbol{\omega}^\times = 0$$

Transpose and use $(\boldsymbol{\omega}^\times)^T = -\boldsymbol{\omega}^\times$

$$\Rightarrow \quad \boxed{\dot{\mathbf{C}} + \boldsymbol{\omega}^\times \mathbf{C} = 0}$$

Kinematic differential equation for direction cosine matrix

Direction cosine matrix differential equation

$$\dot{\mathbf{C}} + \boldsymbol{\omega}^{\times}\,\mathbf{C} = 0$$

If ω_1 ω_2 ω_3 are known as function of time, then orientation of B relative to A as function of time can be determined

Advantages: Linear
No singularities

Disadvantages: 9 differential coupled equations (redundancies)

$$\boldsymbol{\omega}^{\times} \equiv \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

$$\dot{C}_{11} = \omega_3 C_{21} - \omega_2 C_{31} \qquad \dot{C}_{21} = \omega_1 C_{31} - \omega_3 C_{11} \qquad \dot{C}_{31} = \omega_2 C_{11} - \omega_1 C_{21}$$

$$\dot{C}_{12} = \omega_3 C_{22} - \omega_2 C_{32} \qquad \dot{C}_{22} = \omega_1 C_{32} - \omega_3 C_{12} \qquad \dot{C}_{32} = \omega_2 C_{12} - \omega_1 C_{22}$$

$$\dot{C}_{13} = \omega_3 C_{23} - \omega_2 C_{33} \qquad \dot{C}_{23} = \omega_1 C_{23} - \omega_3 C_{13} \qquad \dot{C}_{33} = \omega_2 C_{13} - \omega_1 C_{23}$$

3-2-1 Euler angle kinematic differential equation

Like kinematic differential equation for direction cosine matrix C, orientation of reference frame B relative to reference frame A can also be described by introducing time dependence of Euler angles

Find differential equation of Euler angles $\dot{\theta}_1 \quad \dot{\theta}_2 \quad \dot{\theta}_3$
(i.e. yaw, pitch and roll angles)

Angular rotation is not measured as yaw, pitch and roll rates, but rather through body angular vectors $\vec{\omega} = \omega_1 \vec{b}_1 + \omega_2 \vec{b}_2 + \omega_3 \vec{b}_3$

Find relation between Euler angle rates and body angular velocity components $\dot{\theta}_i \rightleftarrows \omega_i$

Euler angle rates and angular velocity

Consider rotation sequence of $\mathbf{C}_1(\theta_1) \leftarrow \mathbf{C}_2(\theta_2) \leftarrow \mathbf{C}_3(\theta_3)$ to B from A

Represented by

$$\mathbf{C}_3(\theta_3): \quad A' \leftarrow A$$
$$\mathbf{C}_2(\theta_2): \quad A'' \leftarrow A'$$
$$\mathbf{C}_1(\theta_1): \quad B \leftarrow A''$$

Successive rotations are also represented as

$$\vec{\omega}^{A'/A}: \quad A' \leftarrow A$$
$$\vec{\omega}^{A''/A'}: \quad A'' \leftarrow A'$$
$$\vec{\omega}^{B/A''}: \quad B \leftarrow A''$$

Angular velocity vectors $\vec{\omega}^{A'/A}, \vec{\omega}^{A''/A'}$ and $\vec{\omega}^{B/A''}$ are expressed as

Note: Vector quantity same components in either frame

$$\vec{\omega}^{A'/A} = \dot{\theta}_3 \vec{a}_3 = \dot{\theta}_3 \vec{a}'_3$$
$$\vec{\omega}^{A''/A'} = \dot{\theta}_2 \vec{a}'_2 = \dot{\theta}_2 \vec{a}''_2$$
$$\vec{\omega}^{B/A''} = \dot{\theta}_1 \vec{a}''_1 = \dot{\theta}_1 \vec{b}_1$$

To find relation between Euler angle rates and angular velocity do one frame at a time, just like when developed rotation matrices in terms of Euler angles

$$\boxed{\vec{\omega}^{B/A} = \vec{\omega}^{B/A''} + \vec{\omega}^{A''/A'} + \vec{\omega}^{A'/A}} = \dot{\theta}_1 \vec{b}_1 + \dot{\theta}_2 \vec{a}''_2 + \dot{\theta}_3 \vec{a}'_3$$

Euler angle rates and angular velocity

Three angular velocities are expressed in different frames

$$\vec{\omega}^{B/A} = \dot{\theta}_1 \vec{b}_1 + \dot{\theta}_2 \vec{a}''_2 + \dot{\theta}_3 \vec{a}'_3 \qquad \vec{\omega}^{B/A} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \vec{a}''_1 & \vec{a}''_2 & \vec{a}''_3 \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\theta}_2 \\ 0 \end{bmatrix} + \begin{bmatrix} \vec{a}'_1 & \vec{a}'_2 & \vec{a}'_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{bmatrix}$$

To add angular velocities need to write them into same frame

$$\begin{bmatrix} \vec{a}''_1 & \vec{a}''_2 & \vec{a}''_3 \end{bmatrix} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \mathbf{C}_1(\theta_1)$$

$$\begin{bmatrix} \vec{a}'_1 & \vec{a}'_2 & \vec{a}'_3 \end{bmatrix} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \mathbf{C}_1(\theta_1) \mathbf{C}_2(\theta_2)$$

Angular velocity vector is $\vec{\omega} = \omega_1 \vec{b}_1 + \omega_2 \vec{b}_2 + \omega_3 \vec{b}_3 = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$

Therefore:

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} \dot{\theta}_1 \\ 0 \\ 0 \end{bmatrix} + \mathbf{C}_1(\theta_1) \begin{bmatrix} 0 \\ \dot{\theta}_2 \\ 0 \end{bmatrix} + \mathbf{C}_1(\theta_1) \mathbf{C}_2(\theta_2) \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{bmatrix}$$

Do matrix multiplication

3-2-1 Euler angle kinematic differential equation

Do matrix multiplication:
$$\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix}=\begin{bmatrix}\dot\theta_1\\ 0\\ 0\end{bmatrix}+\mathbf{C}_1(\theta_1)\begin{bmatrix}0\\ \dot\theta_2\\ 0\end{bmatrix}+\mathbf{C}_1(\theta_1)\mathbf{C}_2(\theta_2)\begin{bmatrix}0\\ 0\\ \dot\theta_3\end{bmatrix}$$
$$=\begin{bmatrix}1 & 0 & -\sin\theta_2\\ 0 & \cos\theta_1 & \sin\theta_1\cos\theta_2\\ 0 & -\sin\theta_1 & \cos\theta_1\cos\theta_2\end{bmatrix}\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\\ \dot\theta_3\end{bmatrix}$$

Not orthogonal matrix

Singularity if $\Theta_2=\pi/2$

Known

Inverting:
$$\boxed{\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\\ \dot\theta_3\end{bmatrix}=\frac{1}{\cos\theta_2}\begin{bmatrix}\cos\theta_2 & \sin\theta_1\sin\theta_2 & \cos\theta_1\sin\theta_2\\ 0 & \cos\theta_1\cos\theta_2 & -\sin\theta_1\cos\theta_2\\ 0 & \sin\theta_1 & \cos\theta_1\end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix}}$$

Kinematic differential equation for 3-2-1 sequence

Advantages: Only 3 differential equations

Disadvantages: Non-linear

Singularity

3-1-3 Euler angle kinematic differential equation

Consider sequence: $\mathbf{C}_3(\mu) \leftarrow \mathbf{C}_1(\gamma) \leftarrow \mathbf{C}_3(\psi)$

Angular velocity vector: $\vec{\omega} \equiv \vec{\omega}^{B/A} = \omega_1\vec{b}_1 + \omega_2\vec{b}_2 + \omega_3\vec{b}_3 = \dot{\mu}\vec{b}_3 + \dot{\gamma}\vec{a}_1'' + \dot{\psi}\vec{a}_3'$

Do calculation in same frame:

$$\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix} = \begin{bmatrix}0\\0\\ \dot{\mu}\end{bmatrix} + \mathbf{C}_3(\mu)\begin{bmatrix}\dot{\gamma}\\0\\0\end{bmatrix} + \mathbf{C}_3(\mu)\mathbf{C}_1(\gamma)\begin{bmatrix}0\\0\\ \dot{\psi}\end{bmatrix}$$

$$= \begin{bmatrix}\sin\gamma\sin\mu & \cos\mu & 0\\ \sin\gamma\cos\mu & -\sin\mu & 0\\ \cos\gamma & 0 & 1\end{bmatrix}\begin{bmatrix}\dot{\psi}\\ \dot{\gamma}\\ \dot{\mu}\end{bmatrix}$$

Singularity if γ=0

Invert matrix:

$$\boxed{\begin{bmatrix}\dot{\psi}\\ \dot{\gamma}\\ \dot{\mu}\end{bmatrix} = \frac{1}{\sin\gamma}\begin{bmatrix}\sin\mu & \cos\mu & 0\\ \cos\mu\sin\gamma & -\sin\mu\sin\gamma & 0\\ -\sin\mu\cos\gamma & -\cos\mu\cos\gamma & \sin\gamma\end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix}}$$

Kinematic differential equation for 3-1-3 sequence

12 Kinematic differential equation for Euler angles

- For each of 12 Euler rotation sequences exist different kinematic differential equation
- Kinematic differential equation for Euler angle can be derived geometrically, but have to be careful, because some of rates are not orthogonal
- Try 3-2-3 Euler angle sequence
- Write $\omega_i = C_{ij}\, d/dt\, \Theta_j$ (angular velocity = matrix times angle rates)
- Angular velocity as function of Euler angle and Euler rates
- Invert matrix $(C_{ij})^{-1} \omega_i = d/dt\, \Theta_j$ → System of differential equations which can be integrated to yield Euler angle if ω_i is known
- Always some kinematic singularities in differential equation for Euler angles

Quaternion kinematic differential equation

- Without proof $\boxed{\dot{\mathrm{q}} = \tfrac{1}{2}\Omega \mathrm{q}}$ $\Omega = \begin{bmatrix} 0 & \omega_3 & -\omega_2 & \omega_1 \\ -\omega_3 & 0 & \omega_1 & \omega_2 \\ \omega_2 & -\omega_1 & 0 & \omega_3 \\ -\omega_1 & -\omega_2 & -\omega_3 & 0 \end{bmatrix}$
- Written out

$$\boxed{\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & \omega_3 & -\omega_2 & \omega_1 \\ -\omega_3 & 0 & \omega_1 & \omega_2 \\ \omega_2 & -\omega_1 & 0 & \omega_3 \\ -\omega_1 & -\omega_2 & -\omega_3 & 0 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}}$$

- Memo: $\dot{\mathrm{q}} = \tfrac{1}{2}\begin{bmatrix} -\omega^{\times} & \vec{\omega} \\ -\vec{\omega}^T & 0 \end{bmatrix}\mathrm{q}$

Kinematic differential equation for quaternion

- No singularity
- No trigonometric functions
- For quaternion differential equation common to introduce skew matrix of rates (note this is for q_1 q_2 q_3 q_4)

Quaternion kinematic differential equation

Quaternion kinematic differential equation

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & \omega_3 & -\omega_2 & \omega_1 \\ -\omega_3 & 0 & \omega_1 & \omega_2 \\ \omega_2 & -\omega_1 & 0 & \omega_3 \\ -\omega_1 & -\omega_2 & -\omega_3 & 0 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

With

$$\vec{q}_{1:3} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \qquad \vec{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \qquad \vec{\omega} \times \vec{q} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

Can be rewritten as

$$\dot{\vec{q}}_{1:3} = \frac{1}{2}\left(q_4\vec{\omega} - \vec{\omega} \times \vec{q}_{1:3}\right)$$

$$\dot{q}_4 = -\frac{1}{2}\vec{\omega}^T \vec{q}_{1:3}$$

Euler eigenaxis kinematic differential equation

$$\boxed{\begin{aligned} \dot{\phi} &= \vec{e}^{\,T}\vec{\omega} \\ \dot{\vec{e}} &= \tfrac{1}{2}\left[\vec{e}^{\times} - \cot\tfrac{\phi}{2}\,\vec{e}^{\times}\vec{e}^{\times}\right]\vec{\omega} \end{aligned}}$$

From Peter C. Hughes

<u>Problems/Comments:</u>

- Singularity when ϕ = 0 or 2π (both correspond to **C** = **1** and means two frames are identical) → easy to deal with this singularity
- Trigonometric function has to be computed as ϕ varies

Rotation representations

Representation	Par.	Characteristics	Applications
Direction Cosine Matrix	9	• Non- singular • Intuitive • Six redundant parameters • Difficult to maintain orthonormal	Analytical studies Transformation of vectors
Euler angles	3	• Minimal set • Clear physical representation • Trigonometric functions • Singular • Trigonometric functions in kinematic equations	Analytical studies
Quaternions	4	• Easy to maintain orthonormal • No trigonometric functions • Not singular • No clear physical interpretation • One redundant parameter • Linear kinematic equations	Used in simulation Preferred for attitude representation for attitude control

Summary

Reference frames and vectors

Rotation matrix

- Direction Cosine Matrix
- Euler angles
- Euler eigenaxis rotation
- Euler parameters / Quaternion

Angular velocity

- Transport theorem $\frac{\mathrm{d}}{\mathrm{d}t}\left(\ \ \right)_i = \frac{\mathrm{d}}{\mathrm{d}t}\left(\ \ \right)_b + \vec{\omega}\times\left(\ \ \right)$
- Velocity due to rotation → Coriolis and centripetal effect

Kinematic differential equation