ADCS – IX A
Attitude Dynamics

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Summary of last lecture

Dynamics of particle

Dynamics of system of particles

Rigid body dynamics

- Translational and rotational dynamics
- Angular momentum of rigid body
- Inertia matrix
- Principal axes
- Parallel axis theorem
- Kinetic energy

Euler´s equations

$$\mathbf{I}\,\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}\,\boldsymbol{\omega} = \mathbf{T}_c$$

Outline

Euler´s equation

- Equation of motion in body-fixed frame
- Principle axes frame

Differential equations

- Equilibrium, Stability, Characteristic equations

Torque free motion (axial symmetric)

Torque free motion (non-symmetric)

- Geometrical
- Mathematical

Spin stabilization

Spin stabilization with energy dissipation

Euler´s equations

Torque equal to rate of change of angular momentum in inertial frame

$$\mathbf{T} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{h}$$

Angular momentum of rigid body in body frame with constant **I**

$$\mathbf{h} = \mathbf{I}\boldsymbol{\omega}$$

If body-fixed frame rotating with rotation vector **ω**, then for any vector **v**, d/d*t* **v** in inertial frame given by

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v})_i = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v})_b + \boldsymbol{\omega} \times \mathbf{v}$$

Time derivative of vector with respect to inertial frame and with respect to body frame

Apply to angular momentum law

$$\mathbf{T} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{I}\boldsymbol{\omega}) + \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega})$$

In rotating frame inertia matrix is constant →

$$\mathbf{T} = \mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}\boldsymbol{\omega}$$

Euler equations describe attitude dynamics of a rigid body in body frame

Euler equations in principal axes

Euler rotational equation of motion of a rigid body is given by

$$\mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}\boldsymbol{\omega} = \mathbf{T}$$

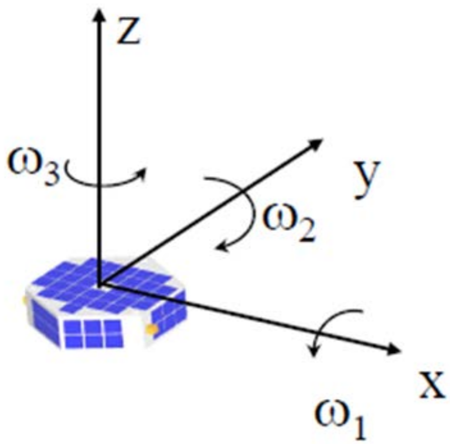
Inertia matrix simplest in principal axes frame

$$\mathbf{I} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

Euler equation

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = T_1$$
$$I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 = T_2$$
$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 = T_3$$

Spacecraft body axes



Euler equations are
three coupled nonlinear first order differential equation

Euler´s equations (special cases)

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = T_1$$
$$I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 = T_2$$
$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 = T_3$$

Special cases:

$\omega_1 = \omega_2 = 0 \quad \rightarrow \quad I_i\dot{\omega}_i = T_i \quad i = 1, 2, 3$

$I_1 = I_2 = I_3 \quad \rightarrow \quad I_i\dot{\omega}_i = T_i \quad i = 1, 2, 3 \quad \rightarrow$

Before looking at more general cases of Euler´s coupled nonlinear first order differential equations

Short review about differential equations

Differential equations

Information from:
http://tutorial.math.lamar.edu/Classes/DE/DE.aspx (Paul´s online math note)
http://www.math.umn.edu/~olver/am_/odz.pdf (Peter Olver´s home page)

Differential equations

Euler equations are three coupled nonlinear first order differential equation. What does it mean?

Motion of dynamical systems are normally described by **differential equations** e.g.

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(x(t))$$

This is

- first-order differential equation
- *x* variable of interest (e.g. position, angle, velocity, etc.)
- *f* possibly nonlinear function

Note: Euler equations have this form if $\mathbf{x} = \boldsymbol{\omega} = [\omega_1\ \omega_2\ \omega_3]$

Note: Differential equation can be higher order or have multiple variable of interest

Linear differential equation

Linear differential equation $$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lambda x(t)$$

with λ constant (previous set $f(x) = \lambda x$)

Solution $$x(t) = \mathrm{e}^{\lambda t}$$

Main questions: Existence, Uniqueness, Equilibrium, Stability

Focus on these two subjects

Stability:

Check solution behavior if $t \rightarrow$ infinity

- Stable if $\lambda < 0$
- Unstable if $\lambda > 0$

Initial value

Differential equation $\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lambda x(t)$ does not determine unique solution

Differential equation specifies slope dx/dt of solution at each point

Infinite family of possible functions solve differential equation

To find particular solution:

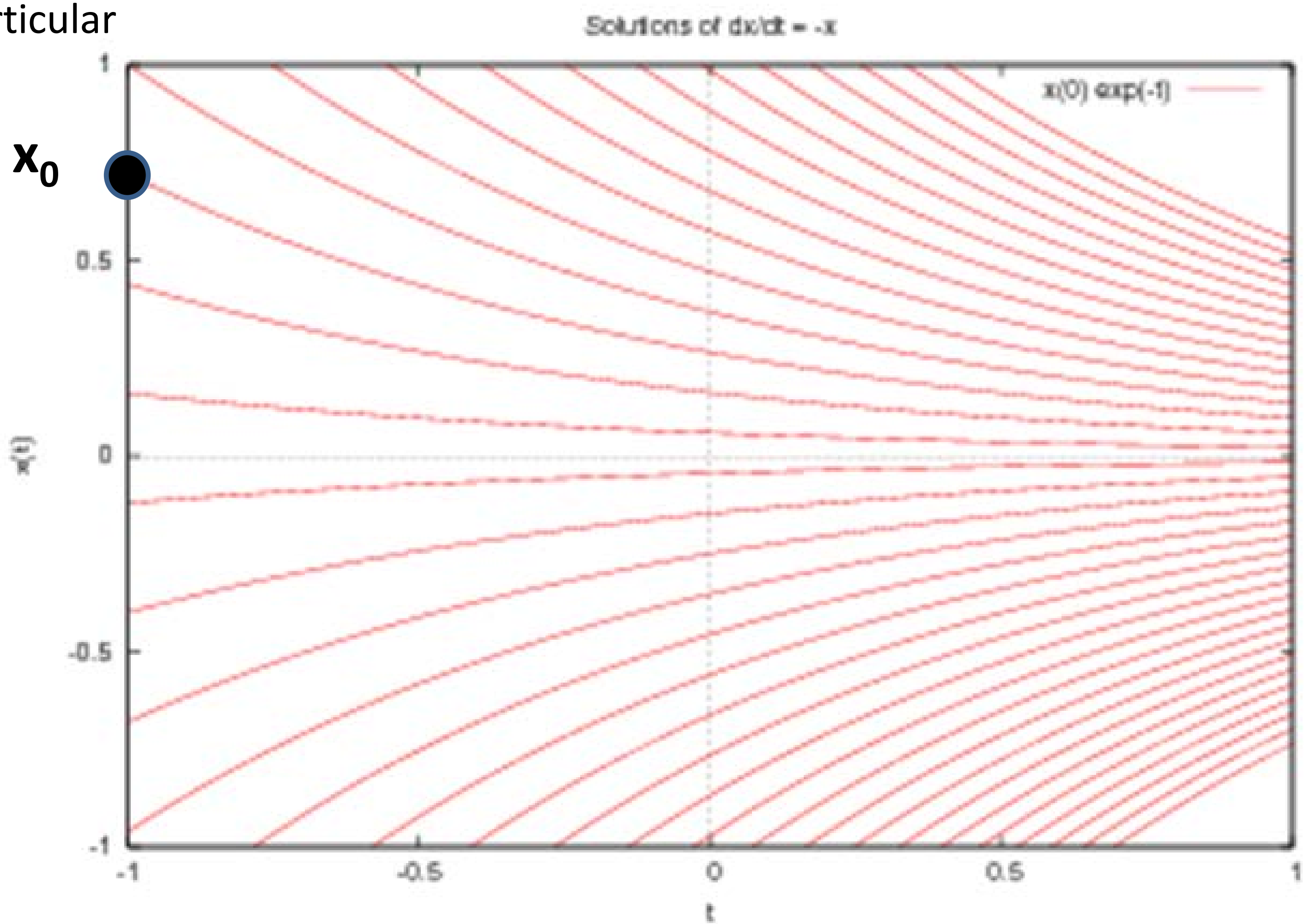
- Need initial values x_0 at specific initial time t_0 → $x(t_0) = x_0$
- Differential equation describes evolution of system in time from state x_0 at time t_0 to any other state
- Example $\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lambda x(t)$ and initial condition $x(t_0) = x_0$
  Solution $x(t) = c\mathrm{e}^{\lambda t}$ with $t_0 = 0$ and $c = x_0$ → Solution $x(t) = x_0\mathrm{e}^{\lambda t}$

Example

Family of solutions for $\dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = -x(t)$

Select particular solution

x_0



Higher order differential equation

Higher order differential equation are of form: $\ddot{x} = a\dot{x} + bx$

Commonly obtained from Newton´s third law $F = m\ddot{x}$

Different techniques exist to solve higher order differential equations
Possible method: Reduce any higher order differential equation to equivalent system of first order differential equations

Method:

1. Define new variable for every higher order term except for highest

$$y_1 = x \quad y_2 = \dot{x}$$

2. Add new first order differential equation for each variable $\dot{y}_1 = \dot{x} = y_2$

$$\dot{y}_2 = a\dot{x} + bx$$

For example above →
$$\dot{y}_1 = y_2$$
$$\dot{y}_2 = ay_2 + by_1$$

System of linear differential equations

Variety of physical systems are modeled by system of differential equations (coupled variables)

$$\dot{y}_1 = ay_1 + by_2$$

$$\dot{y}_2 = cy_1 + dy_2$$

Motion of y_1 affects motion of y_2 and vice-versa

Write system of first order equations using matrix-formulation

$$\dot{\vec{y}} = \mathbf{A}\vec{y}$$

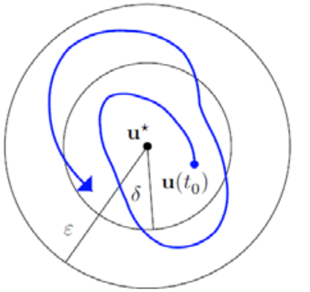
$$with \quad vector \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad square \quad matrix \quad \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Equation describes motion of vector

Stability of solutions

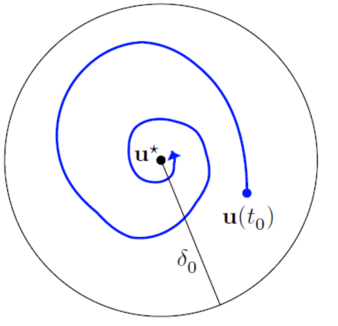
Solution of differential equation is:

Stable
if solutions resulting from small changes (perturbations) of initial value remain close to original solution



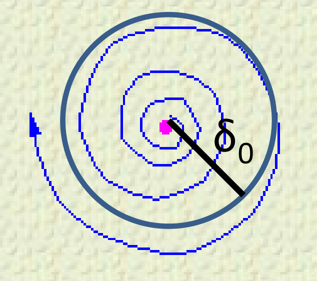
Stability

Asymptotical stable
if solutions resulting from small changes (perturbations) of initial value converge back to original solution

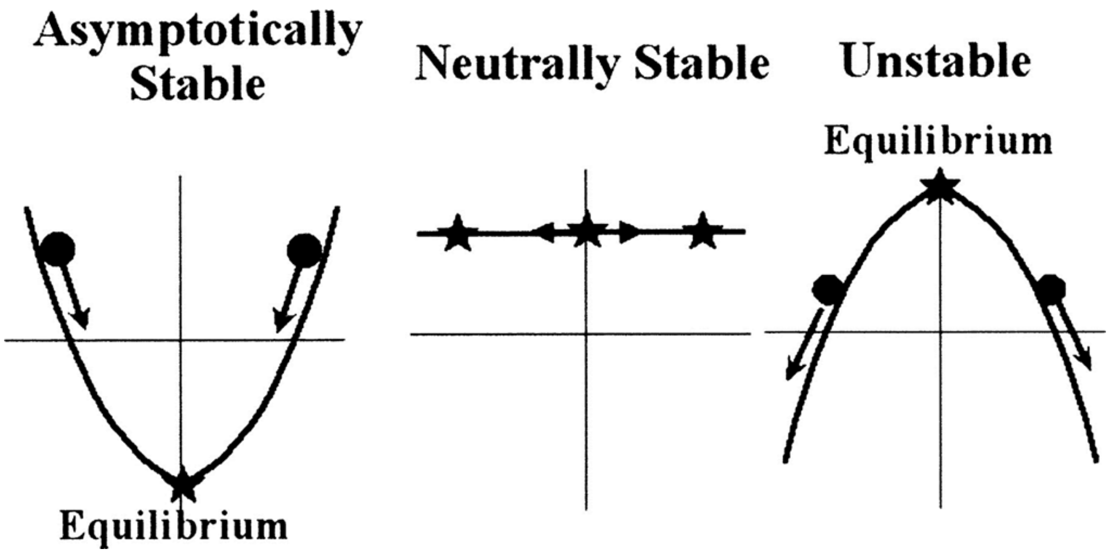


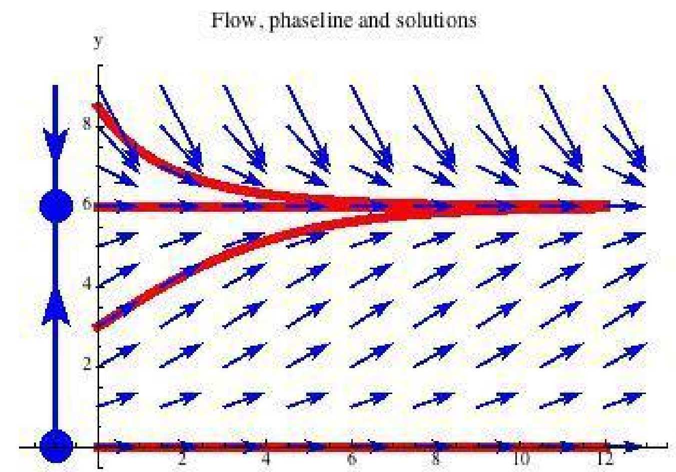
Asymptotic Stability

Unstable
if solutions resulting from small changes (perturbations) diverge away from original solution



Stability





Schematic representations of:

- asymptotic stability around equilibrium point (star)
- neutral stability with continuum of equilibrium points
- instability around equilibrium

Axes represent any two state variables

What is equilibrium point?

Equilibrium

- Equilibrium solutions correspond to physical system which does not move
- Real physical system exist only if equilibrium is stable
- Unstable equilibrium do not exist in practice, because any tiny perturbation will move system fare away from initial equilibrium

Speak formally, with $\dot{x} = f(x)$

Equilibrium solution constant means $x(t) = x^*$ for all t

→ Derivatives must vanish $dx/dt = 0$

→ Every equilibrium solution has $f(x^*) = 0$

x^* is **Equilibrium point** of $\dot{x} = f(x)$ if $f(x^*) = 0$ (Means $dx/dt=f(x^*)=0$)

Note: Nonlinear systems may have many equilibrium points

Linear systems with one variable only have one equilibrium point

Example: stability of solution

Consider scalar differential equation $$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lambda x(t)$$

Solution with some initial values $$x(t) = \mathrm{e}^{\lambda t}$$

For real $\boldsymbol{\lambda}$:

- $\lambda > 0$: all solutions grow exponentially → unstable
- $\lambda < 0$: all solutions decay exponentially → asymptotically stable
- $\lambda = 0$: all solutions stay constant → stable

For complex $\boldsymbol{\lambda}$:

→ **Oscillation** $\mathrm{e}^{\lambda t} = \mathrm{e}^{(\alpha + \mathrm{i}\beta)t} = \mathrm{e}^{\alpha t}\mathrm{e}^{\mathrm{i}\beta t} = \mathrm{e}^{\alpha t}(\cos\beta + \mathrm{i}\sin\beta)$

- $\mathrm{Re}(\lambda) > 0$ → unstable
- $\mathrm{Re}(\lambda) < 0$ → asymptotically stable
- $\mathrm{Re}(\lambda) > 0$ → stable

Characteristic equation for higher-order system

Both higher-order and linear systems of differential equations have characteristic equation

Characteristic equation found by using

- **Ansatz** for solution $x(t) = x_0 e^{\lambda t} \quad with \quad x_0 = \text{constant}$
- Laplace transform ($x(t) \rightarrow x(\lambda)$, $d/dt(x(t)) \rightarrow \lambda x(\lambda)$, $d^2/dt^2 x(t) \rightarrow \lambda^2 x(\lambda)$,…)

Characteristic equation of homogeneous scalar equation e.g.

$$a\ddot{x} + b\dot{x} + cx = 0 \rightarrow a\lambda^2 + b\lambda + c = 0 \begin{cases} x(t) = x_0 e^{\lambda_1 t} \\ x(t) = x_0 e^{\lambda_2 t} \end{cases}$$

Roots of characteristic equation determine motion of differential equation (Roots will appear in complex conjugate pairs $\lambda = \alpha + \beta i$)

Equilibrium and

- Roots have all negative real part → stable
- At least one root has positive real part → unstable
- At least one nonzero imaginary part → oscillation

Characteristic equation for linear system of differential equations

System of differential equations: $$\dot{\vec{y}} = \mathbf{A}\vec{y}$$

For system of differential equation use Ansatz (or Laplace transform) to get: $$\vec{y}(t) = \vec{y}_0 \mathrm{e}^{\lambda t}$$

$$\lambda \vec{y}_0 = \mathbf{A}\vec{y}_0 \qquad (\lambda \mathbf{1} - \mathbf{A})\vec{y}_0 = \vec{\mathbf{0}}$$

Characteristic equation of matrix is given by following determinant

$$\det(\lambda \mathbf{1} - \mathbf{A}) = 0 \qquad \text{e.g.} \qquad \det\begin{bmatrix} \lambda - a & -b \\ -c & \lambda - d \end{bmatrix} = 0$$

Determinant is polynomial in λ

Equation $\lambda \vec{y}_0 = \mathbf{A}\vec{y}_0$ is also an eigenvalue problem

Therefore: Eigenvalues of matrix **A** are roots of characteristic equation

Stability of solutions for matrix system of differential equations

- Suppose **A** has eigenvalues λ_i and corresponding eigenvectors $\mathbf{v}_i$
- Express $\boldsymbol{y}_0$ as linear combination $\vec{y}_0 = \sum_{i=0}^{n} \alpha_i \vec{v}_i \Rightarrow \vec{y}(t) = \sum_{i=0}^{n} \alpha_i \vec{v}_i e^{\lambda t}$
  and is solution to differential equation with initial condition $\boldsymbol{y}(0)=\boldsymbol{y}_0$
- Eigenvalues of matrix **A** with positive real parts yield exponential growing solutions
- Eigenvalues with negative real parts yield exponential decaying solutions
- Eigenvalue with zero real components (only imaginary) yield oscillatory solutions

Solutions are:

stable if $\mathrm{Re}(\lambda_i) \leq 0$ for every eigenvalue

asymptotically stable if $\mathrm{Re}(\lambda_i) < 0$ for every eigenvalue

unstable if $\mathrm{Re}(\lambda_i) > 0$ for every eigenvalue

Stability theorem for linear systems

Found both higher-order and linear system of differential equations have characteristic equation

$$\frac{\mathrm{d}^n x}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1}x}{\mathrm{d}t^{n-1}} + \ldots + a_2\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + a_1\frac{\mathrm{d}x}{\mathrm{d}t} + a_0 x = 0 \quad \text{and} \quad \dot{\vec{x}} = \mathbf{A}\vec{x}$$

<u>Theorem 1</u>: Equilibrium solution ***x****(*t*)=0 of above equations are **stable** if roots of characteristic equation are distinct and **Re(**$\boldsymbol{\lambda_i}$**)** $\leq \mathbf{0}$.

<u>Theorem 2</u>: Equilibrium solution ***x****(*t*)=0 of above equations are **asymptotically stable** if roots of characteristic equation satisfy **Re(**$\boldsymbol{\lambda_i}$**)<0**.

(Eigenvalues need not to be distinct)

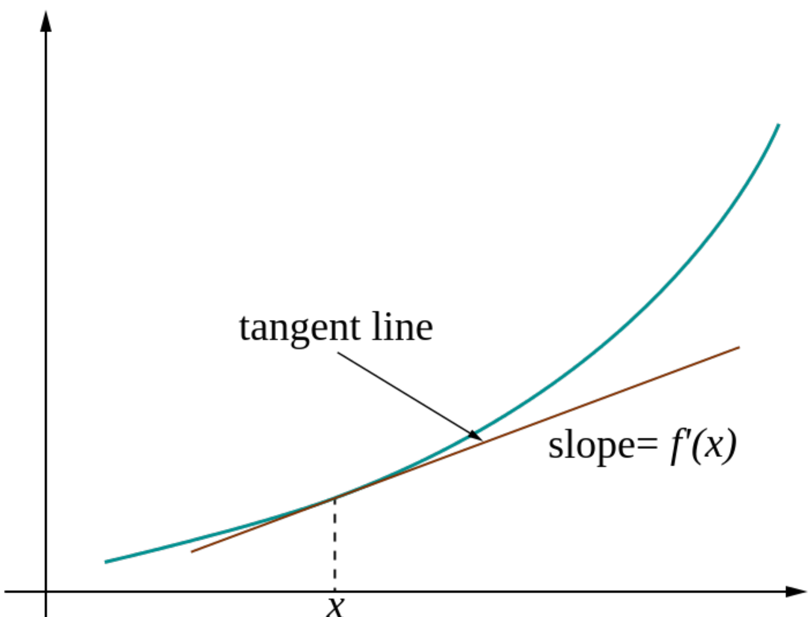
<u>Theorem 3</u>: Equilibrium solution ***x****(*t*)=0 of above equations are **unstable** if **any** roots of characteristic equation satisfy **Re(**$\boldsymbol{\lambda_i}$**)** $> \mathbf{0}$.

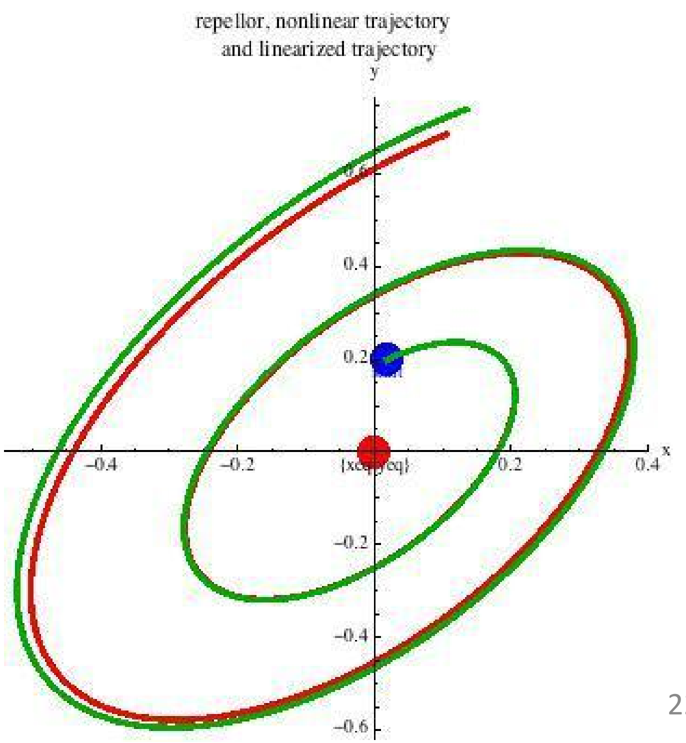
(Eigenvalues need not to be distinct)

Note: If roots of characteristic equation are same then general solution is e.g. $x(t) = x_0 e^{\lambda t} + x_1 t e^{\lambda t}$

Linearization of nonlinear system

- Remind: solutions of linear systems may be found explicitly
- But real problems may only be modeled by **nonlinear** systems
- Behavior of nonlinear system around an equilibrium point is mystery
- Idea: approximate nonlinear system by linear one (equilibrium point)
- Behavior of solutions of linear system will be same as nonlinear one (not always true)





Linearization of nonlinear system

For general nonlinear system of differential equations like Euler equations determining stability of solutions is more complicated

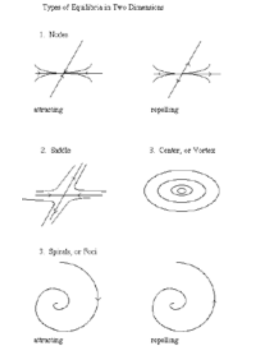
Differential equation can be linearized locally about (equilibrium) solution **x***(*t*) by first-order Taylor series yielding linear differential equations with partial derivatives given by Jacobian matrix

- Assume $\frac{\mathrm{d}\vec{x}(t)}{\mathrm{d}t} = f(\vec{x})$ and expand solution as $\vec{x}(t) = \vec{x}^* + \vec{\varepsilon}(t)$
- Use first-order Taylor $\dot{\vec{x}} = \dot{\vec{\varepsilon}} = f(\vec{x}^* + \vec{\varepsilon}(t)) = f(\vec{x}^*) + \frac{\mathrm{d}f}{\mathrm{d}\vec{\varepsilon}}\Big|_{x=x^*} \vec{\varepsilon} + \ldots.$
- Linearize system $\dot{\vec{\varepsilon}} = \mathbf{A}\vec{\varepsilon}$ $\quad \mathbf{A} = \frac{\mathrm{d}f}{\mathrm{d}\vec{\varepsilon}}\Big|_{x=x^*}$
- Check for stability of linearized system $\vec{\varepsilon}(t) = \vec{\varepsilon}_0 \mathrm{e}^{\lambda t} \Rightarrow \lambda\vec{\varepsilon}_0 = \mathbf{A}\vec{\varepsilon}_0$
- Solution is $\vec{\varepsilon}(t) = \sum_{i=0}^{n} \alpha_i \vec{v}_i \mathrm{e}^{\lambda t}$

Eigenvalues of **A** determine stability locally, but may not be valid globally

Asymptotic stability or instability of linearized equation imply same properties for nonlinear system

Summary of linearization technique



Summary of linearization technique:

1. Find equilibrium point of nonlinear system
2. Linearize: Find partial derivatives and write down Jacobian matrix
3. Find eigenvalues of Jacobian matrix
4. Imply from eigenvalues behavior of solutions around equilibrium
   - If eigenvalues are negative or complex with negative real part
     → Equilibrium point is sink (solutions converge to equilibrium point)
     → If eigenvalue are complex (solutions spiral around equilibrium)
   - If eigenvalue are positive or complex with positive real part
     → Equilibrium point is source (solutions move away from equilibrium)
     → If eigenvalue are complex (solutions spiral away from equilibrium)
   - If eigenvalues are real numbers with opposite sign
     → Equilibrium point is saddle (some solutions move away from equilibrium others approach equilibrium point)

Torque free motion
(axial symmetric)

Euler equation without external torque

Free dynamics of rigid body defined by Euler equations:

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = 0$$

$$I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 = 0$$

$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 = 0$$

Conserved quantities:

Rotational kinetic energy:

$$T_r = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$$

Magnitude of angular momentum:

$$h^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$$

Please prove that these quantities are conserved for free rigid body

Euler equation without external torque

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Conserved quantities:

Rotational kinetic energy:

$$T_r = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$$

Magnitude of angular momentum:

$$h^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$$

Please prove that these quantities are conserved for free rigid body

(Hint: Time derivation of e.g. kinetic energy and use Euler equation)

Axial symmetric body

Assume perfectly symmetric spacecraft with transverse momentum of inertia

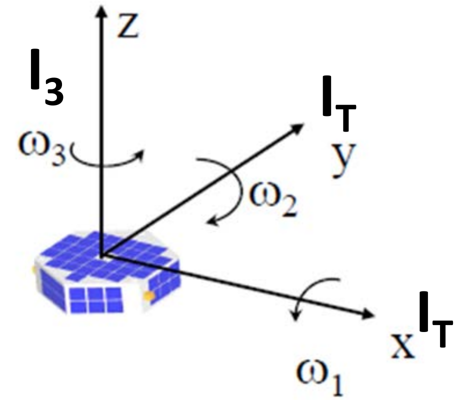
$$I_T = I_1 = I_2$$

Space environment is torque free: assume no external torques acting on spacecraft **T** = **0** → Euler equation given by

$$0 = I_T \dot{\omega}_1 + (I_3 - I_T)\omega_2\omega_3$$

$$0 = I_T \dot{\omega}_2 + (I_T - I_3)\omega_1\omega_3$$

$$0 = I_3 \dot{\omega}_3$$



Even in absence of external torque axis of rotation **ω** evolves

Initial conditions are

$$\omega_1(0) = \omega_{01}$$

$$\omega_2(0) = \omega_{02}$$

$$\omega_3(0) = \omega_{03}$$

Third equation shows that 3-axis spin rate is constant

What about other two spin rates?

Torque-free motion (axial symmetric)

Integrate last equation $\omega_3(t) = \omega_{03}$

3-axis spin rate constant → means angular velocity component about axis of symmetry constant

Other two equations with $\Omega = \frac{I_T - I_3}{I_T}\omega_{03}$ (**Ω = relative spin rate**) are

$$\dot{\omega}_1 - \Omega\omega_2 = 0$$
$$\dot{\omega}_2 + \Omega\omega_1 = 0$$

Linear differential equation
→ Linear systems have nice solutions

Differentiating first equation with respect to time and substituting in second for $\dot{\omega}_2$ gives

$$\ddot{\omega}_1 + \Omega^2\omega_1 = 0$$

Harmonic oscillation equation

Torque-free motion (axial symmetric)

Similarly for other equation (combine Euler equation for 1- and 2-body axes to form one single equation)

$$\ddot{\omega}_2 + \Omega^2 \omega_2 = 0$$

Solutions of these equations are given by

$$\boxed{\begin{aligned} &\omega_1(t) = \omega_{01} \cos \Omega t + \omega_{02} \sin \Omega t \\ &\omega_2(t) = \omega_{02} \cos \Omega t - \omega_{01} \sin \Omega t \\ &\omega_3(t) = \omega_{03} \end{aligned}}$$

Have oscillatory solution for 1- and 2-axis angular velocity

Check validity of: $\dot{\omega}_1 - \Omega \omega_2 = 0$ $\qquad \dot{\omega}_2 + \Omega \omega_1 = 0$

Torque-free motion (axial symmetric)

Define: $\omega_T = (\omega_1{}^2(t) + \omega_2{}^2(t))^{\frac{1}{2}}$

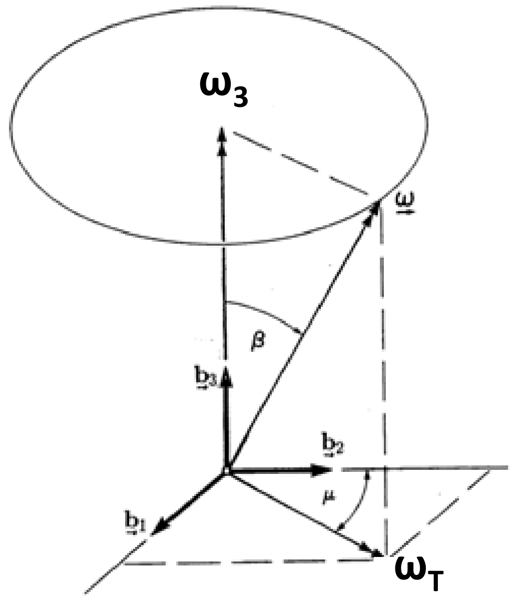
$$\omega_T = (\omega_{01}\cos\Omega t + \omega_{02}\sin\Omega t)^2 + (\omega_{02}\cos\Omega t - \omega_{01}\sin\Omega t)^2$$

$$= (\omega_{01}{}^2 + \omega_{02}{}^2)^{\frac{1}{2}}$$ Transversal angular velocity is constant

Shown in body-axis frame:

- **ω_3 constant**
  rotation about axis of symmetry
- **ω_T constant**
  rotation perpendicular to axis of symmetry

→ Circular motion in body-fixed frame

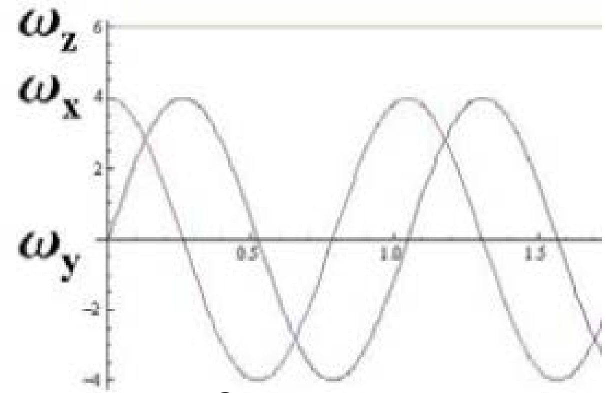


Circular motion in body-fixed frame (axial symmetric)

Solution of harmonic oscillation equation $\ddot{\omega}_1 + \Omega^2 \omega_1 = 0 \qquad \ddot{\omega}_2 + \Omega^2 \omega_2 = 0$

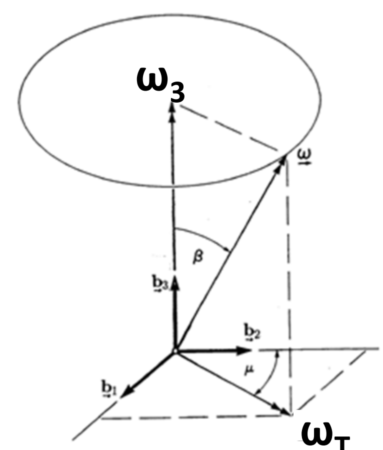
can also be written as

$$\boxed{\begin{aligned} \omega_1(t) &= \omega_T \sin[\Omega(t-t_0)] \\ \omega_2(t) &= \omega_T \cos[\Omega(t-t_0)] \\ \omega_3(t) &= \omega_{03} \end{aligned}}$$



This is complete solution of Euler´s equations for torque free axial-symmetric rigid body (in body frame)

Vector $\boldsymbol{\omega}_T$ rotates uniformly about body symmetry-axis with **relative spin rate Ω**



This is description of angular vector with respect to body frame

What is description of motion in inertial frame?

Motion in inertial frame (axial symmetric)

Interested in body motion in inertial frame

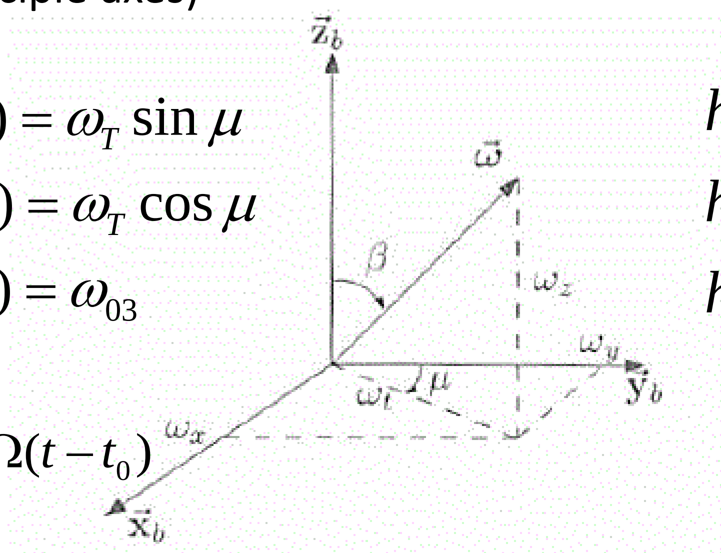
Calculate angular momentum vector in body-fixed coordinates: $\mathbf{h} = \mathbf{I}\boldsymbol{\omega}$

Angular velocity in body coordinate (principle axes)

$$\omega_1(t) = \omega_T \sin\mu$$
$$\omega_2(t) = \omega_T \cos\mu$$
$$\omega_3(t) = \omega_{03}$$

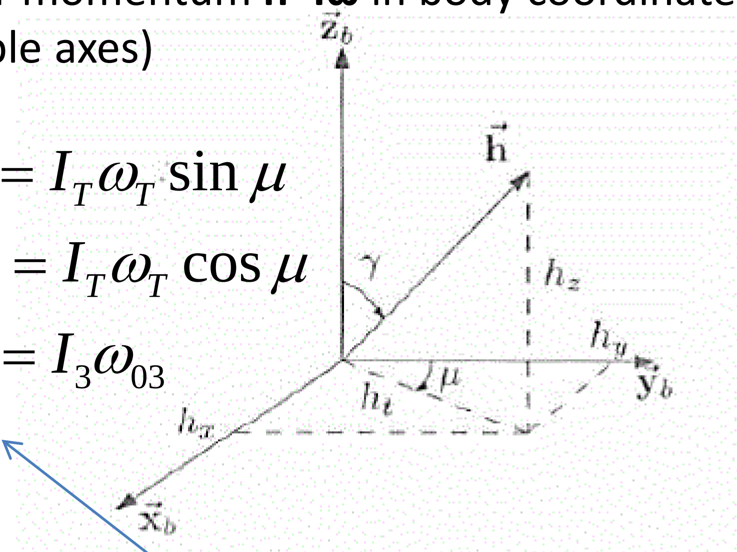
with

$$\mu = \Omega(t - t_0)$$



Angular momentum **h=Iω** in body coordinate (principle axes)

$$h_1 = I_T\omega_1(t) = I_T\omega_T \sin\mu$$
$$h_2 = I_T\omega_2(t) = I_T\omega_T \cos\mu$$
$$h_3 = I_3\omega_3(t) = I_3\omega_{03}$$



For torque free motion, angular momentum vector is constant with respect to inertial frame $\mathbf{T} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{h} = \mathbf{0}$

But see how angular momentum vector rotates in body frame

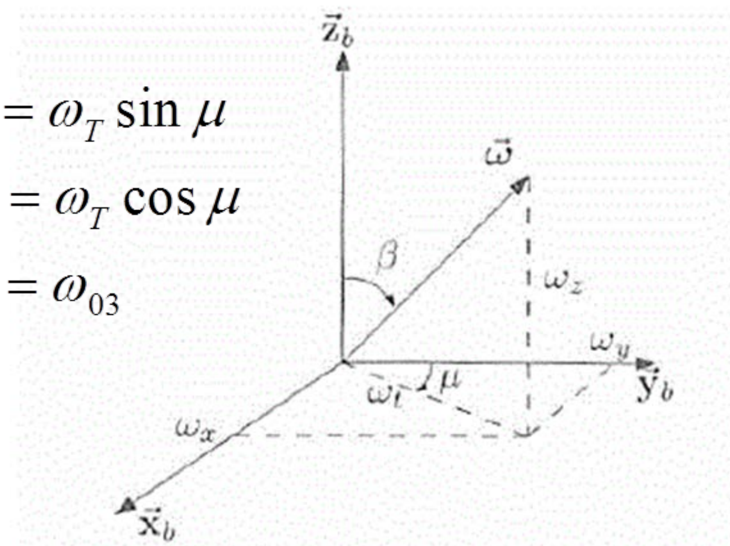
Motion in inertial frame (axial)

Find orientation of vector **ω**, **h** and body symmetry axis

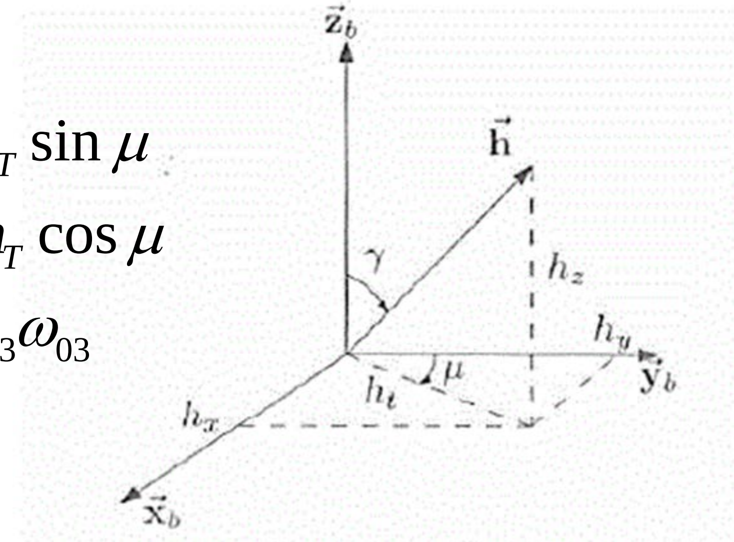
Transverse angular momentum and total angular momentum→ constant

$$h_T = (h_1^2 + h_2^2)^{\frac{1}{2}} = I_T \omega_T \qquad h = (h_1^2 + h_2^2 + h_3^2)^{\frac{1}{2}} = (h_T^2 + h_3^2)^{\frac{1}{2}}$$

$$\omega_1(t) = \omega_T \sin \mu$$
$$\omega_2(t) = \omega_T \cos \mu$$
$$\omega_3(t) = \omega_{03}$$



$$h_1 = h_T \sin \mu$$
$$h_2 = h_T \cos \mu$$
$$h_3 = I_3 \omega_{03}$$



From figure → **h**, **ω** and body symmetry 3-axes lie always in same plane

Just like angular velocity vector, also angular momentum vector rotates about symmetry 3-axes at rate Ω, but do not have same direction

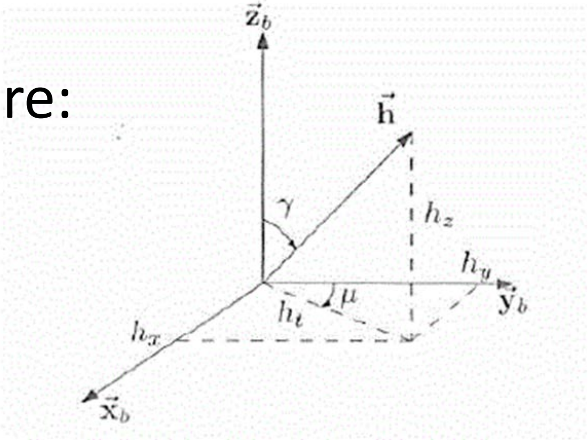
Motion in inertial frame: Nutation angle ϒ

Define two important angles

Have two constants of motion

- ϒ=Angle between symmetry axis and angular momentum vector **h**
- β=Angle between symmetry axis and angular velocity vector **ω**

From figure:



$$h_3 = h\cos\gamma \qquad h_T = h\sin\gamma$$

Angle ϒ = nutation angle is constant

$$\tan\gamma = \frac{h_T}{h_3} = \frac{I_T\omega_T}{I_3\omega_3}$$

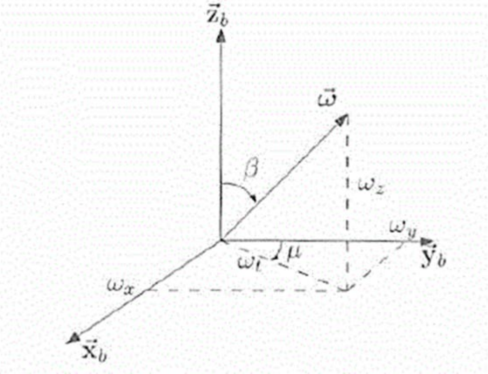
Since transverse angular momentum $\mathbf{h}_T$ and total angular momentum **h** are constant
→ ϒ constant

Motion in inertial frame: Angle β

Second angle:

β=Angle between symmetry axis and angular velocity vector **ω**

From figure:



$$\tan\beta = \frac{\omega_T}{\omega_3}$$

Since $\boldsymbol{\omega}_T$ and $\boldsymbol{\omega}_3$ are constant → β constant

Relationship between angles:

$$\boxed{\frac{I_T}{I_3}\tan\beta = \tan\gamma}$$

Consider two cases:

- Case 1: $I_3 < I_T \rightarrow \Upsilon > \beta$
- Case 2: $I_3 > I_T \rightarrow \Upsilon < \beta$

Prolate and Oblate

Direction of relative spin rate $\Omega = \frac{I_T - I_3}{I_T}\omega_{03}$ depends on shape of body

Definition: (Asparagus can)

An axial symmetric (about 3-axis) rigid body is **Prolate** if $I_3 < I_T$.

Definition: (Tuna can)

An axial symmetric (about 3-axis) rigid body is **Oblate** if $I_3 > I_T$.

Case 1 with $\Omega > 0$ is called **prograd precession** (if object is prolate)

Case 2 with $\Omega < 0$ is called **retrograd precession** (if object is oblate)

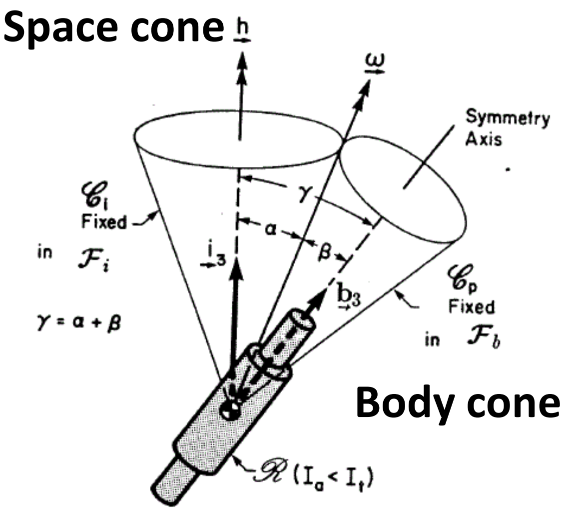
Note: These rotations are in body-fixed frame

Geometrical interpretation

$$\frac{I_T}{I_3}\tan\beta = \tan\gamma$$

Space cone and body cone with fixed angles

Motion of body about fixed point is same as motion of body cone rolling without slipping on space cone



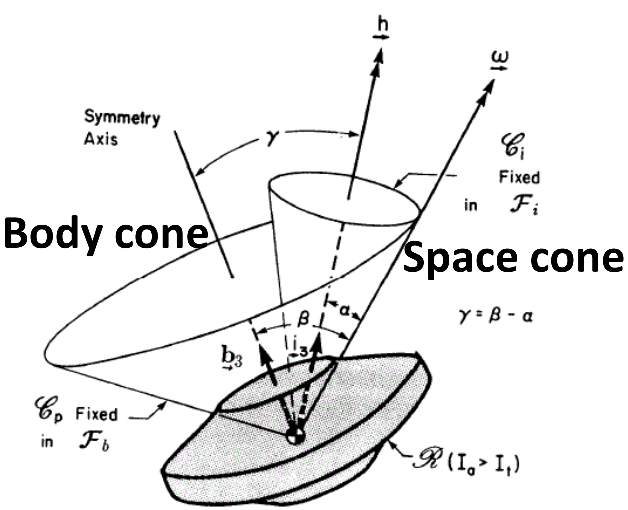
Prolate spinner ($I_3 < I_T$):

Case 1: Elongated body with ϒ>β

Space cone and body cone have external tangent

Tangent along angular velocity vector **ω**

Prograde precession (relative spin rate Ω>0)



Oblate spinner ($I_3 > I_T$):

Case 2: Flattened body with ϒ<β

Space cone is inside body cone

Tangent along angular velocity vector **ω**

Retrograde precession (relative spin rate Ω<0)

Prograde versus retrograde precession

Movies:

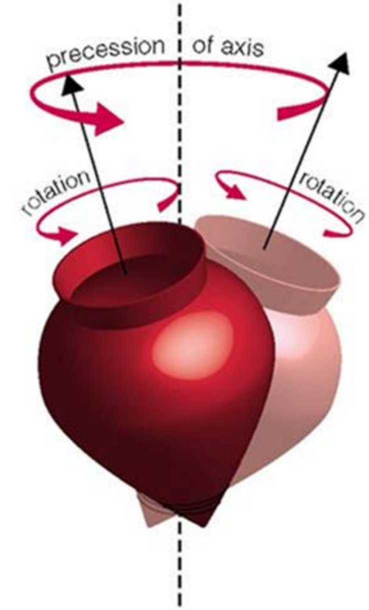
Prolate (prograde) precession:

https://www.youtube.com/watch?v=r_EgzvIMWQw

Oblate (retrograde) precession:

https://www.youtube.com/watch?v=EwcP36CCe5Q

What is precession rate?



Precession rate corresponds to rate at which symmetry body 3-axis rotates about inertial-fixed direction **h** (angular momentum vector).

Orientation of body-fixed frame into inertial frame is given by sequence of Euler rotations

Since angular momentum vector **h** inertially fixed ($\mathbf{T} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{h} = \mathbf{0}$)
$\Rightarrow$ choose inertial frame with 3-axis aligned with **h**

Map spacecraft body-fixed frame onto inertial frame through 3-1-3 Euler rotation sequence (where angles μ and ϒ are two of 3-1-3 Euler angles)

$$\mathbf{C}_{bi}(\mu,\gamma,\psi) = \mathbf{C}_3(\mu)\mathbf{C}_1(\gamma)\mathbf{C}_3(\psi)$$

Euler rotation

3-1-3 rotation matrix describes body frame relative to inertial frame

$$\mathbf{C}_{bi}(\mu,\gamma,\psi) = \mathbf{C}_3(\mu)\mathbf{C}_1(\gamma)\mathbf{C}_3(\psi)$$

Same as in Lecture VI

ψ = C_3 rotation about body-fixed 3-axis called **precession angle**

ϒ = C_1 rotation to make angular momentum vector **h**

same as body-fixed 3-axis (Seen that ϒ is fixed angle, $\dot{\gamma} = 0$)

μ = C_3 rotation about angular momentum vector **h**

Euler angles related to angular velocity vector as

$$\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ \dot{\mu}\end{bmatrix} + \mathbf{C}_3(\mu)\begin{bmatrix}\dot{\gamma}\\ 0\\ 0\end{bmatrix} + \mathbf{C}_3(\mu)\mathbf{C}_1(\gamma)\begin{bmatrix}0\\ 0\\ \dot{\psi}\end{bmatrix}$$

With:

$$\mathbf{C}_3(\mu) = \begin{bmatrix}\cos\mu & \sin\mu & 0\\ -\sin\mu & \cos\mu & 0\\ 0 & 0 & 1\end{bmatrix} \qquad \mathbf{C}_1(\gamma) = \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\gamma & \sin\gamma\\ 0 & -\sin\gamma & \cos\gamma\end{bmatrix}$$

Precession rate (axial symmetric)

Relation of Euler angles and angular velocity
$$\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix}=\begin{bmatrix}\dot{\psi}\sin\gamma\sin\mu\\ \dot{\psi}\sin\gamma\cos\mu\\ \dot{\mu}+\dot{\psi}\cos\gamma\end{bmatrix}$$

Compare with previously derived result for angular velocity

$$\omega_1(t)=\omega_T\sin\mu$$
$$\omega_2(t)=\omega_T\cos\mu$$
$$\omega_3(t)=\omega_{03}$$

$$\begin{bmatrix}\dot{\psi}\sin\gamma\sin\mu\\ \dot{\psi}\sin\gamma\cos\mu\\ \dot{\mu}+\dot{\psi}\cos\gamma\end{bmatrix}=\begin{bmatrix}\omega_T\sin\mu\\ \omega_T\cos\mu\\ \omega_{03}\end{bmatrix}$$

From first component:

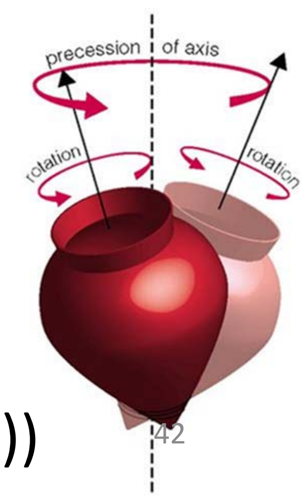
$$\dot{\psi}\sin\gamma=\omega_T \Leftrightarrow \dot{\psi}=\frac{\omega_T}{\sin\gamma}\Leftrightarrow \dot{\psi}=\frac{h}{I_T}\quad with\quad \sin\gamma=\frac{h_T}{h}$$

Define **precession rate**:

$$\boxed{\Omega_p \triangleq \dot{\psi}=\frac{h}{I_T}}$$

Precession rate Ω_p corresponds to rate at which symmetry body 3-axis rotates about inertial-fixed direction **h**

Note: Total angular velocity vector **ω** also precesses around inertial fixed direction **h** (since **h**, **ω** and 3-axis in plan))



Precession rate and relative spin rate

Compare third component of
$$\begin{bmatrix} \dot{\psi}\sin\gamma\sin\mu \\ \dot{\psi}\sin\gamma\cos\mu \\ \dot{\mu}+\dot{\psi}\cos\gamma \end{bmatrix} = \begin{bmatrix} \omega_T\sin\mu \\ \omega_T\cos\mu \\ \omega_{03} \end{bmatrix}$$

$$\dot{\mu}+\dot{\psi}\cos\gamma=\omega_{03} \Leftrightarrow \Omega+\Omega_p\cos\gamma=\omega_{03} \quad \textit{with} \quad \Omega=\dot{\mu}$$

$$\Omega_p=\dot{\psi}=\frac{\omega_{03}-\Omega}{\cos\gamma}=\frac{I_3}{(I_T-I_3)\cos\gamma}\Omega \quad \textit{with} \quad \Omega=\frac{I_T-I_3}{I_T}\omega_{03}$$

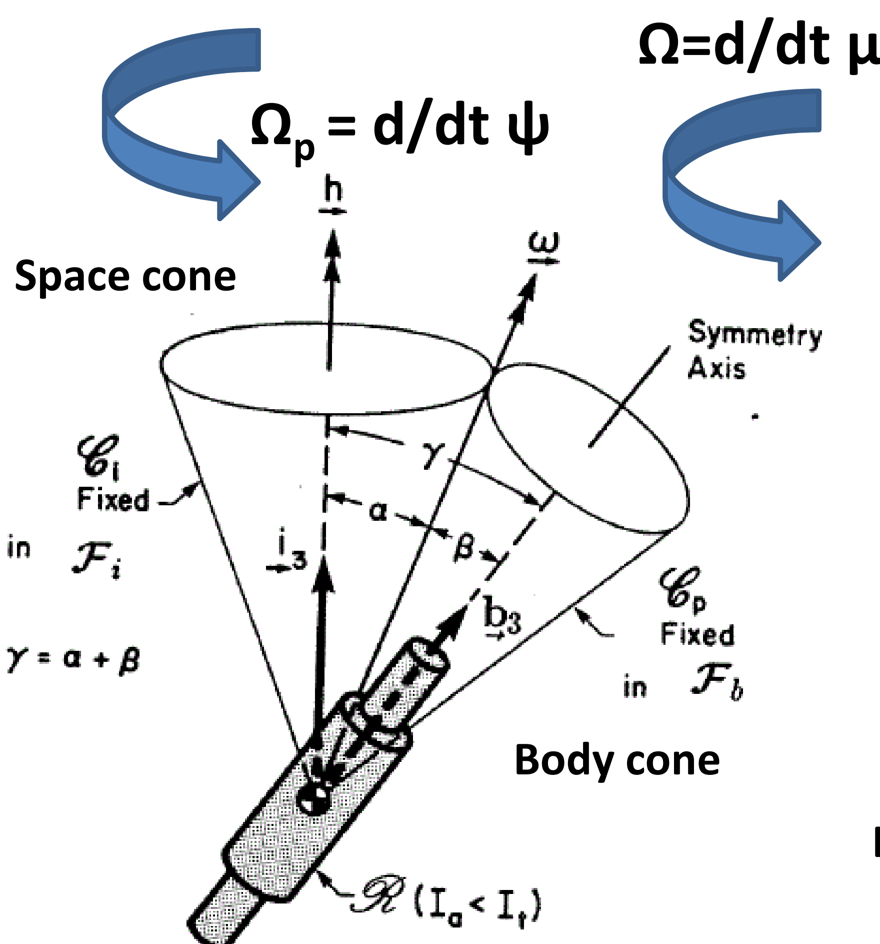
This relates precession rate Ω_p with relative spin rate Ω

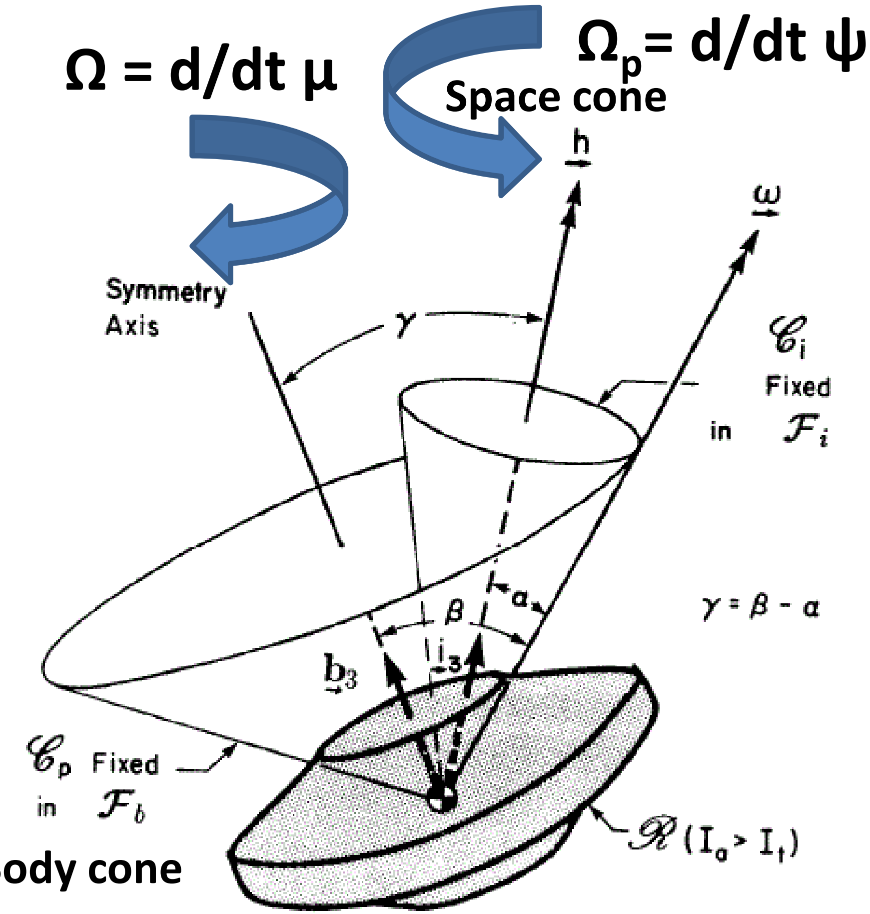
$$\boxed{\Omega_p=\frac{I_3}{(I_T-I_3)\cos\gamma}\Omega}$$

Therefore two cases:

- Case 1: $I_3 < I_T$ → Prograde precession (Ω_p and Ω have same sign)
- Case 2: $I_3 > I_T$ → Retrograde precession (Ω_p and Ω opposite sign)

Euler rates: prolate and oblate case



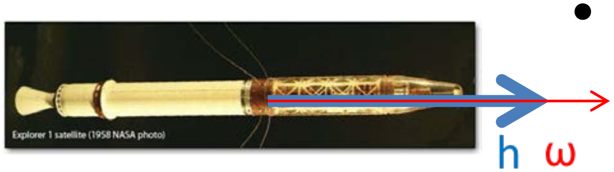


Prolate case ($I_3 < I_T$)
Prograde precession
Spin and precession have same sign

Oblate case ($I_3 > I_T$)
Retrograde precession
Spin and precession opposite sign

Torque-free motion without precession

If no forces and axial-symmetric spacecraft only two possible motions without precession:



- **Spacecraft spins about its axis of symmetry**

  $\omega_T = h_T = 0$

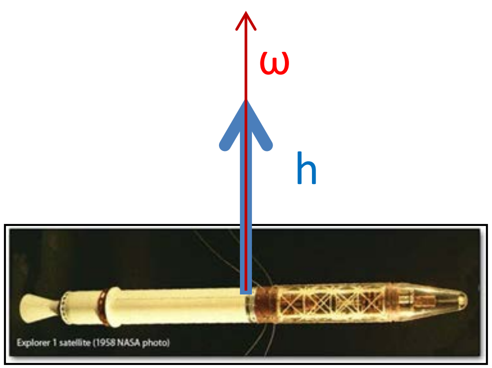
  $\vec{h}$ *and* $\vec{\omega}$ are aligned

  Look at:

  $$h_1 = I_T \omega_1(t) = I_T \omega_T \sin[\tfrac{I_T - I_3}{I_T}(t - t_0)]$$

  $$h_2 = I_T \omega_2(t) = I_T \omega_T \cos[\tfrac{I_T - I_3}{I_T}(t - t_0)]$$

  $$h_3 = I_3 \omega_3(t) = I_3 \omega_{03}$$



- **Spacecraft spins about its transverse axis**

  $\omega_3 = h_3 = 0$

  $\vec{h}$ *and* $\vec{\omega}$ are aligned

Torque-free motion and their constants

Transversal projection of **ω** (ω_T) is constant $\omega_T = (\omega_1^{\,2}(t) + \omega_2^{\,2}(t))^{\frac{1}{2}}$

Magnitude of angular vector is constant $\omega = (\omega_1^{\,2}(t) + \omega_2^{\,2}(t) + \omega_3^{\,2}(t))^{\frac{1}{2}}$

Angles β and ϒ are fixed

Magnitude of angular moments (h^2) is constant (because scalar)

(h^2 is constant in any frame!) $h^2 = h_1^{\,2} + h_2^{\,2} + h_3^{\,2} = I_T^{\,2}\omega_T^{\,2} + I_3^{\,2}\omega_3^{\,2}$

Angular momentum vector is constant in inertial frame

but it rotates in body frame (not constant)

$$h_1 = I_T\omega_1(t)$$
$$h_2 = I_T\omega_2(t)$$
$$h_3 = I_3\omega_3(t)$$

From $\dot{\vec{h}} = \vec{T}$ and torque-free motion

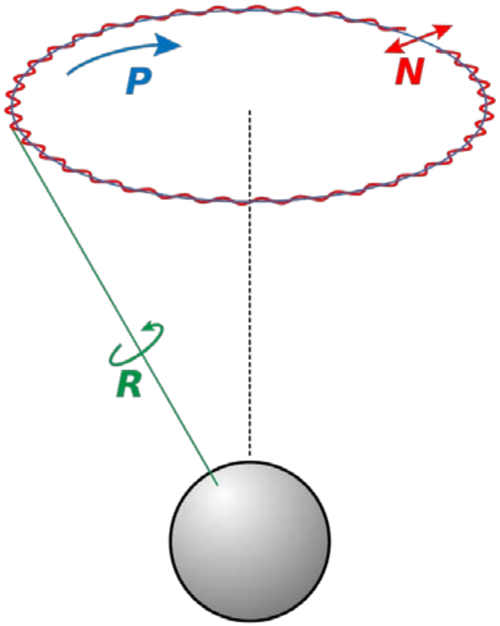
direction of angular momentum vector (**h**) is constant in inertial frame

Remark

Confusing:

Some books use nutation instead of precession (perhaps because of nutation angle)

In physics for torque free motion there exist no change of nutation



Rotation (green)
Precession (blue)
Nutation in obliquity (red)

For nutation of planet need gravitational attraction of other bodies

Torque free motion (non-symmetric)

Torque-free motion (non symmetric)

Torque free motion of body without symmetry (different principal moments of inertia)

Solve all three Euler´s equations:

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = 0$$
$$I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 = 0$$
$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 = 0$$

Conservation of rotational kinetic energy

$$T_r = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$$

Conservation of magnitude of angular momentum

$$h^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$$

Complete analytical solution exist with **Jacobian elliptical functions** (unfamiliar periodic functions)

Circular functions of time for axissymmetric body become elliptical function for non-symmetric body

Torque-free motion (non symmetric) Geometrical view: Poinsot construction

Torque free motion of non-symmetric body

- Instead to deal with unfamiliar elliptical functions use geometrical trick (L. Poinsot)
- Geometrical solution permits to obtain qualitative understanding of general torque free rigid body motion
- Poinsot construction:

  - Poinsot construction used to visualize how endpoint of angular vector ω moves

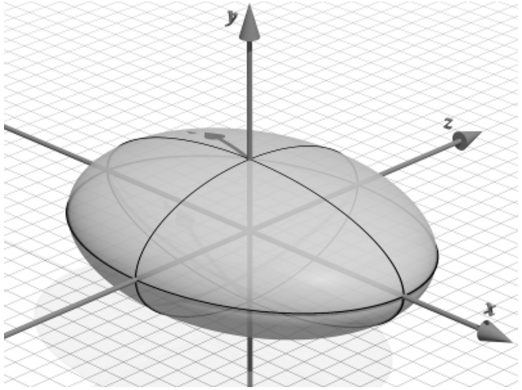
  - Use **kinetic energy conservation** and **magnitude of angular momentum** to constraint motion of angular velocity ω

Kinetic energy ellipsoid

In principal axis system:
kinetic energy T = 1/2**ωIω** and if torque free motion T is constant

$$T = \tfrac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$$

$$\boxed{\frac{\omega_1^2}{2T/I_1} + \frac{\omega_2^2}{2T/I_2} + \frac{\omega_3^2}{2T/I_3} = 1}$$



Angular vector ω must lie on surface of **kinetic energy ellipsoid**

Prove that energy is constant:

$$\dot{T} = I_1\omega_1\dot{\omega}_1 + I_2\omega_2\dot{\omega}_2 + I_3\omega_3\dot{\omega}_3$$

Euler equation

$$= -\omega_1[(I_3 - I_2)\omega_2\omega_3] - \omega_2[(I_1 - I_3)\omega_1\omega_3] - \omega_3[(I_2 - I_1)\omega_1\omega_2]$$

$$= 0$$

Angular momentum ellipsoid

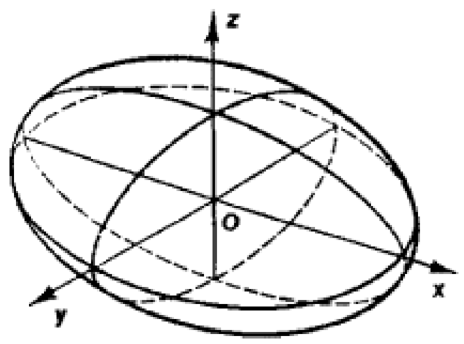
For torque-free motion angular momentum vector **h** = **Iω** is fixed in inertial space → magnitude **h**^{2} = (**Iω**)^2 must be constant (in any frame)

Angular vector **ω** must lie on surface of **angular momentum ellipsoid**

In principal axes system:

$$h^2 = h_1^2 + h_2^2 + h_3^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$$

$$\boxed{\frac{\omega_1^2}{h^2/I_1^2} + \frac{\omega_2^2}{h^2/I_2^2} + \frac{\omega_3^2}{h^2/I_3^2} = 1}$$



Energy and angular momentum ellipsoid are not same, since their axis length are not same

Axis length of angular momentum ellipsoid are : h/I_1, h/I_2, h/I_3

Axis length of kinetic energy ellipsoid are: $\sqrt{2T/I_1}, \sqrt{2T/I_2}, \sqrt{2T/I_3}$

Polhode

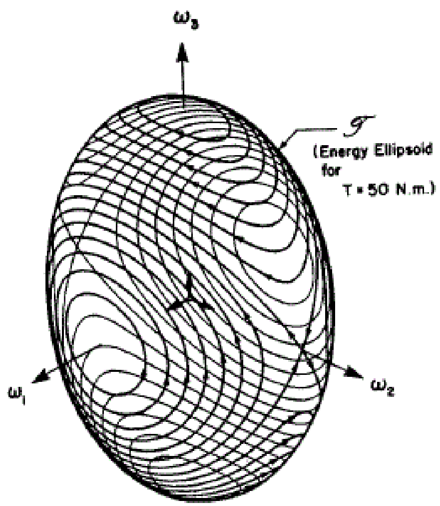
Polhode:

Angular velocity components [ω_1,ω_2,ω_3] have to lie on surface of energy and angular momentum ellipsoids

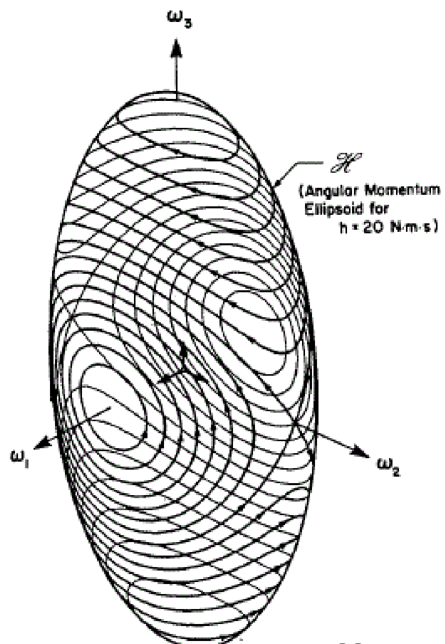
Locus must lie on curve of intersection between two ellipsoids

Curve traced by angular velocity vector **ω** is called **polhode**

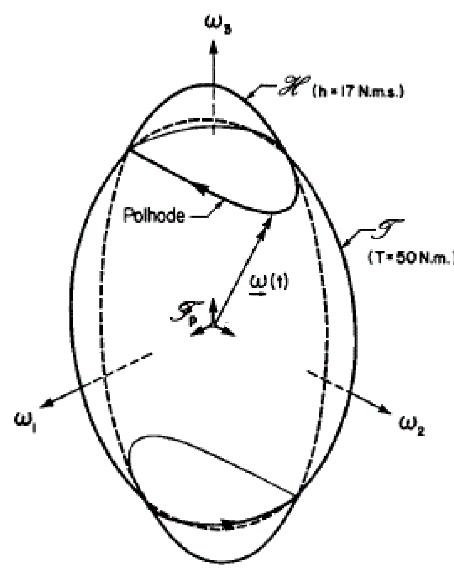
Polhode shows possible path of angular vector **ω** seen from body frame (no information about speed of movements of **ω** vector)



Energy ellipsoid



Momentum ellipsoid



Intersecting polhodes

Intersection of kinetic and angular momentum ellipsoid

Visualization of intersection of two ellipsoids by using **angular momentum vector components h_i** instead of body angular velocity vector component ω_i

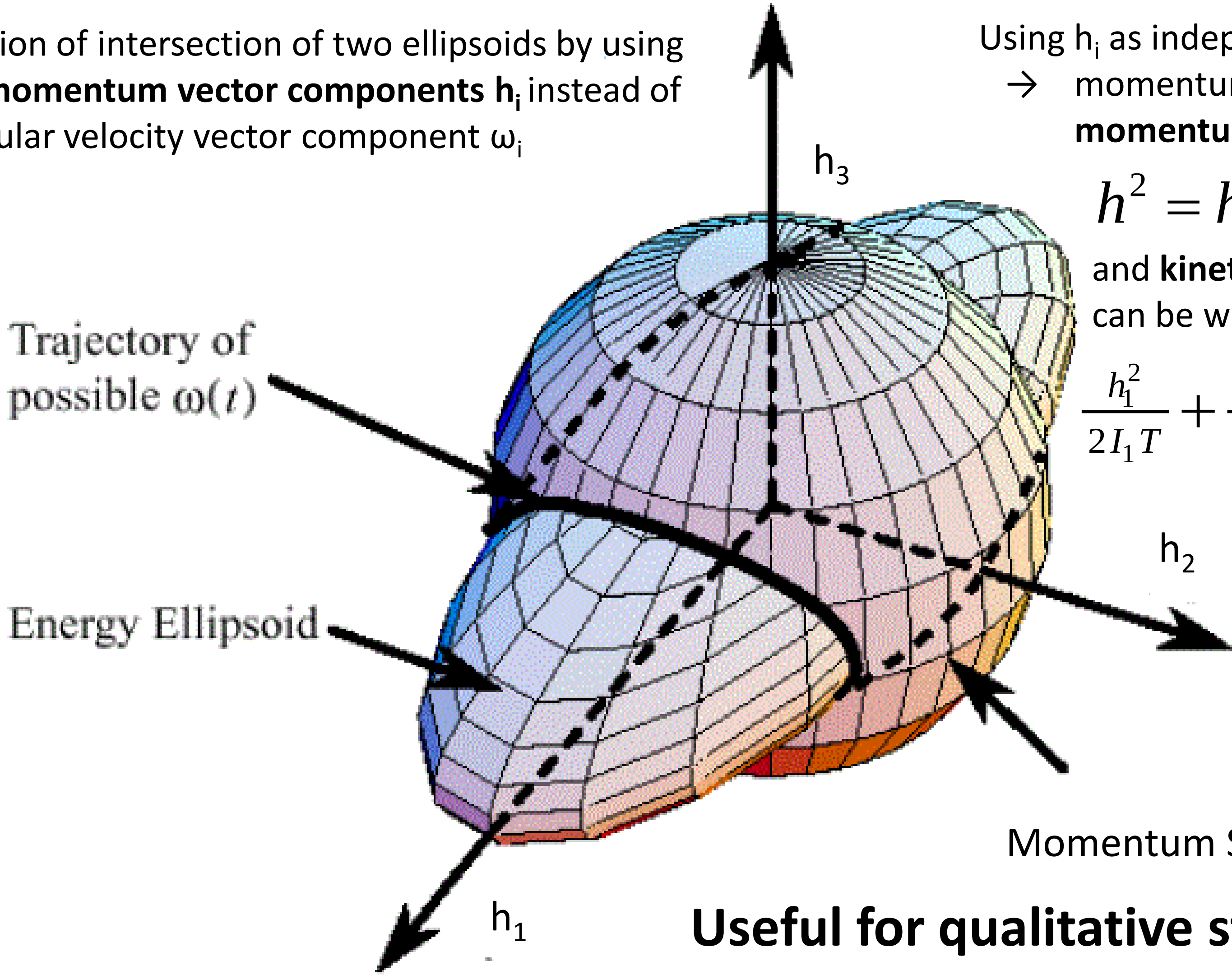
Using h_i as independent coordinates

→ momentum ellipsoid becomes **momentum sphere**

$$h^2 = h_1^2 + h_2^2 + h_3^2$$

and **kinetic energy ellipsoid** can be written as

$$\frac{h_1^2}{2I_1T} + \frac{h_2^2}{2I_2T} + \frac{h_3^2}{2I_3T} = 1$$

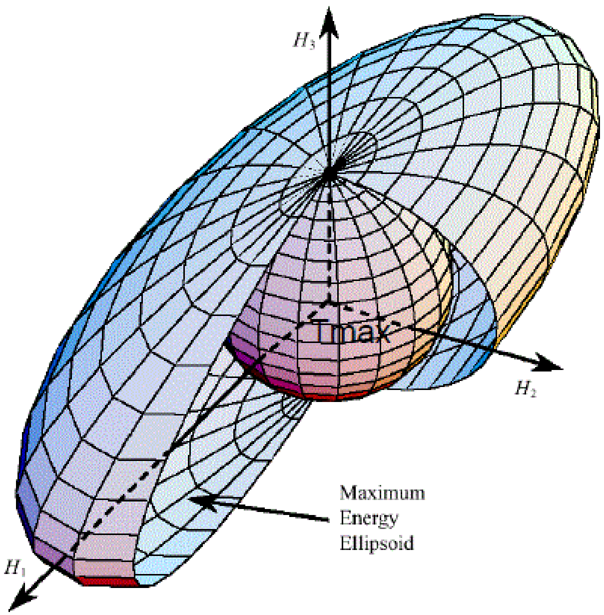


Useful for qualitative studies

Intersection of kinetic and angular momentum ellipsoid (special case)

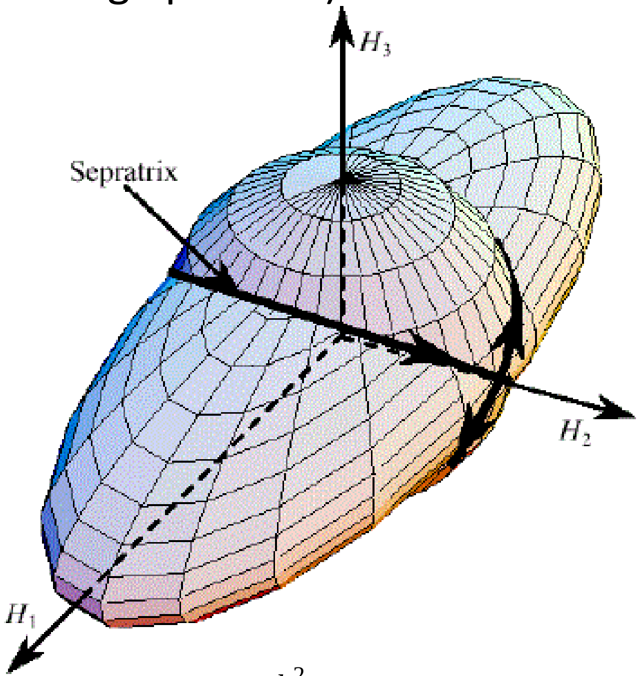
Fix magnitude of angular momentum (h^2)
→ vary kinetic energy (only certain range possible)

Assume: $I_1 \geq I_2 \geq I_3$



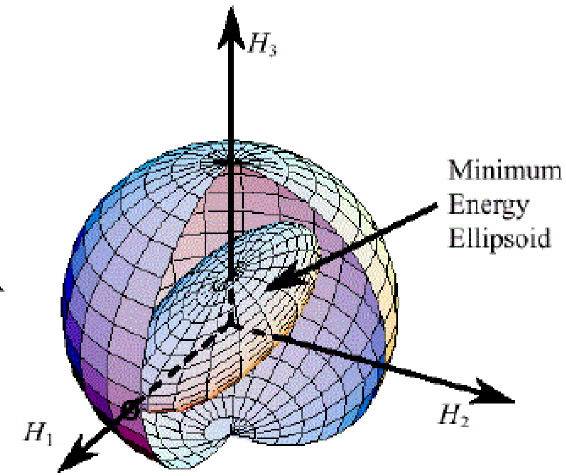
$$T_{\max} = \frac{h^2}{2I_3}$$

Maximum energy
Polhodes small circle
around 3-axis



$$T_{\mathrm{int}} = \frac{h^2}{2I_2}$$

Intermediate energy
Polhodes large circle
over entire ellipsoid



$$T_{\min} = \frac{h^2}{2I_1}$$

Minimum energy
Polhode small circle
around 1-axis

Stability of torque free motion: Geometrical view

(small perturbations to non symmetric body)

Special case with steady rotation

Steady rotation (constant angular velocity ω)

$(d\omega_i / dt) = 0 \qquad (i=1,2,3)$

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = 0$$
$$I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 = 0$$
$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 = 0$$

From Euler´s equation it follows:

$$(I_3 - I_2)\omega_2\omega_3 = (I_1 - I_3)\omega_1\omega_3 = (I_2 - I_1)\omega_2\omega_1 = 0$$

All components of **ω** can be constant only if at least two of $\omega_i = 0$
Means: Vector **ω** can be constant only if it is along one principle axis!

But not all rotations with **ω** along principle axis are stable!

Stable rotation means that small perturbation causes rotation axis of body to move only slightly away from principle axis

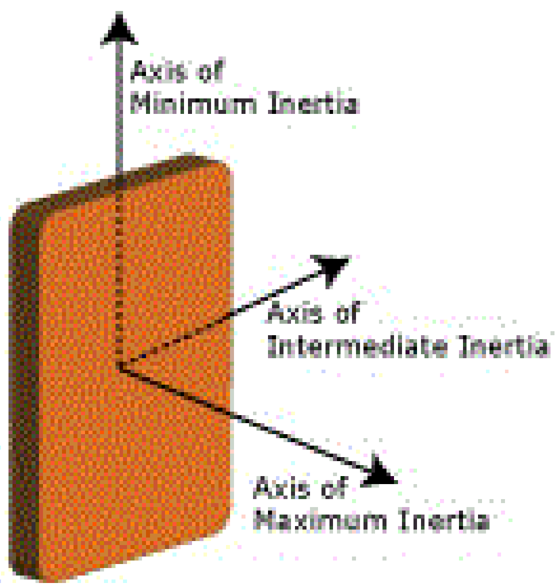
Example of stable rotation (minor axis)

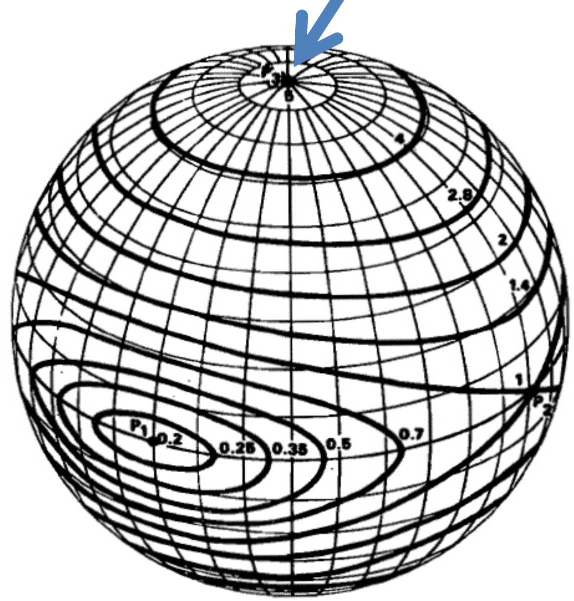
Stable and steady rotation about principle 3-axis
(minimal principal inertia I_3 , 3-axis)

$$I_1 \geq I_2 \geq I_3$$

Figure shows intersection of angular momentum and kinetic energy ellipsoid
Begin with pure spin about smallest axis of inertia (point at top or bottom, two possible spins)
If decrease slightly energy (closed curves around 3-axis)

Motion is stable because ω vector is never far from its initial position (minor axis)





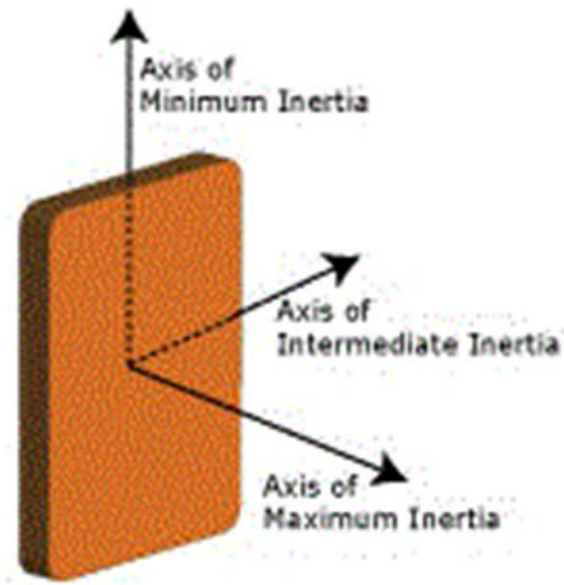
Example of stable rotation (major axis)

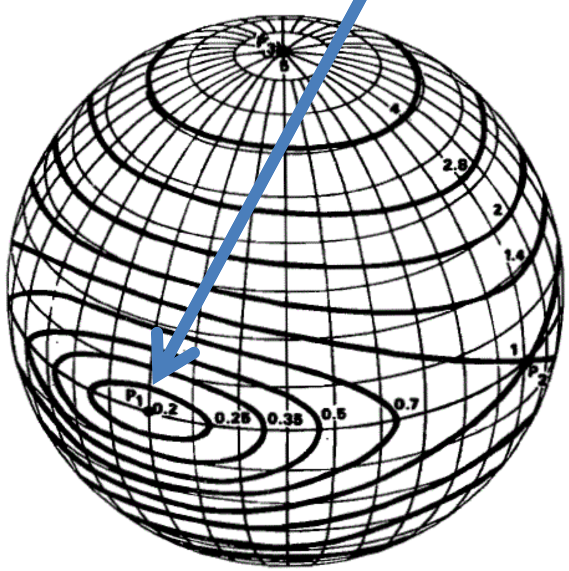
Stable and steady rotation about principle 1-axis
(largest principle inertia I_1 , 1-axis)

$$I_1 \geq I_2 \geq I_3$$

Figure shows intersection of angular momentum and kinetic energy ellipsoid
Begin with pure spin about largest axis of inertia (point at front or behind, two possible spins)
If change slightly energy (closed curves around 1-axis)

Motion is stable because ω vector is never far from its initial position (major axis)





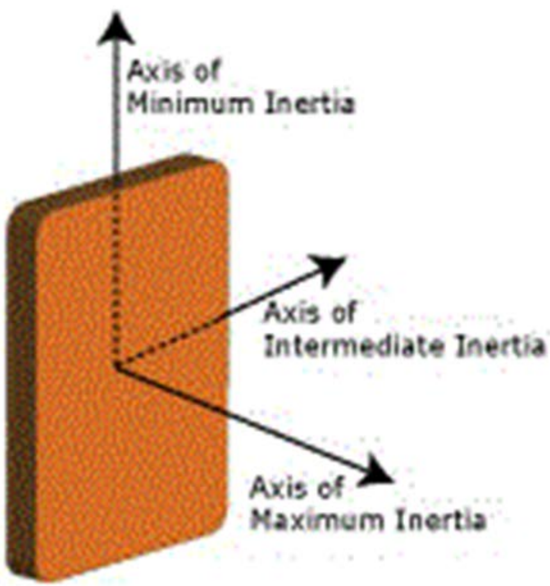
Example of unstable rotation (intermediate axis)

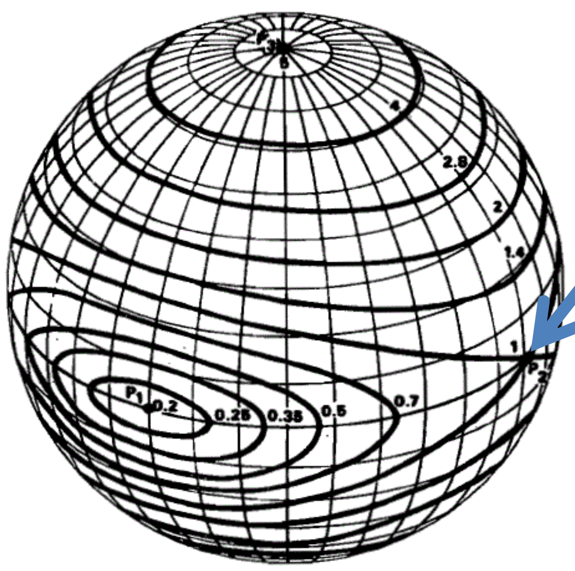
Start stable and steady rotation about principle 2-axis
(intermediate principle inertia I_2 , 2-axis)

$$I_1 \geq I_2 \geq I_3$$

Figure shows intersection of angular momentum and kinetic energy ellipsoid
Begin with pure spin about intermediate axis of inertia (point at right or behind, two spins)
If smallest deviations (two closed curves which circle around and cross each other at 2-axis)
ω has long path on surface and deviates significantly → tumbling motion

Motion is unstable because ω vector deviates significantly from its initial position



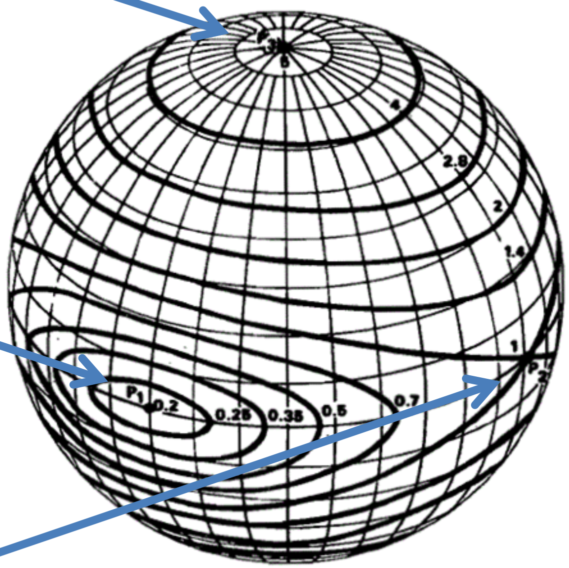


Use polhodes to see stability

Near
minimal principle inertia axis
we have little loops
stable

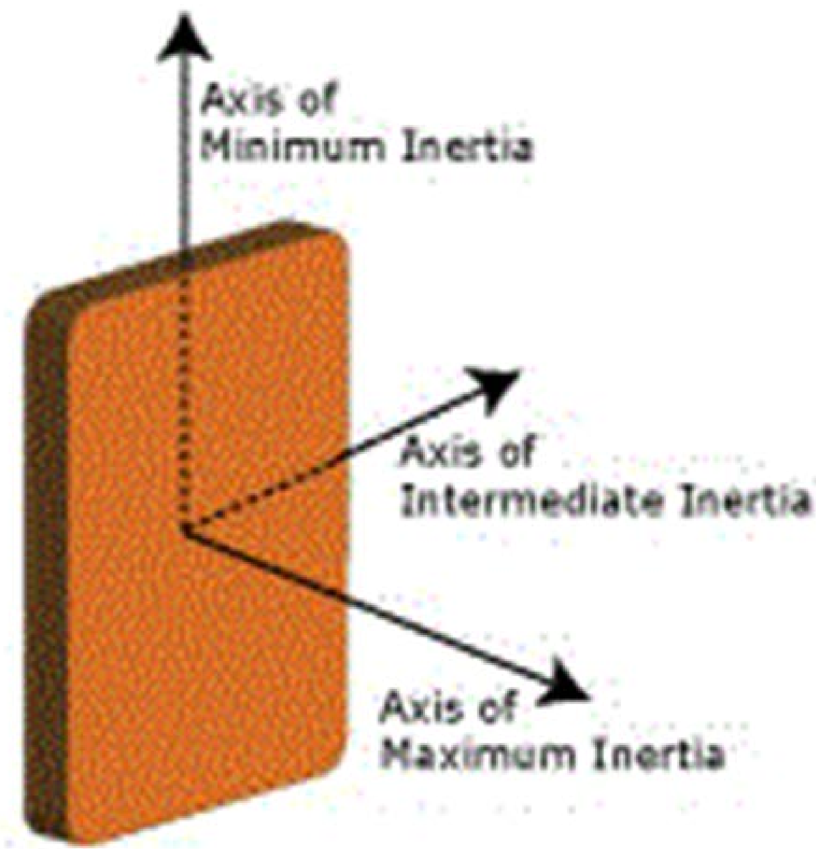
Near
maximum principle inertia axis
we have little loops
stable

Near
intermediate principle inertia axis
we get send away
unstable



Small deviations will eventually get big!

Stability of torque free motion



Theorem:
Torque free motion of rigid body is stable if spin of body is about axis of maximum or minimum principle moment of inertia.
If spin is about axis of intermediate principle moment of inertia, then motion is unstable.

$I_1 > I_2 > I_3$

Major (maximum) axis of inertia is stable

Minor (minimum) axis of inertia is stable

Intermediate axis of inertia is unstable

Stability of book (non-symmetric)

Stability of torque free motion

Motion of freely rotating rigid body:

https://www.youtube.com/watch?v=iTRbeQpXJfE

Stability of torque-free motion: Mathematically

(small perturbations non-symmetric body)

Stability of rotation about principle axes

Up to now geometrical view of stability problem

Let´s have more mathematical view

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = 0$$
$$I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 = 0$$
$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 = 0$$

From Euler´s equation without external torque and with spins about principle axis of inertia:

$$\omega_1(t) = const \quad if \quad \omega_2(t) = \omega_3(t) = 0$$
$$\omega_2(t) = const \quad if \quad \omega_1(t) = \omega_3(t) = 0$$
$$\omega_3(t) = const \quad if \quad \omega_1(t) = \omega_2(t) = 0$$

Permanent rotations seem to be possible about each of principle axes

Are they stable? How do solution behave as t → infinity?

Under what conditions does spacecraft spin remain stable?

If perturbed, does motion remain bounded (stable)? Grow without bound (unstable)?

Does motion always tend to particular equilibrium (asymptotically stable)?

Stability of Euler´s equations

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = 0$$
$$I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 = 0$$
$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 = 0$$

Symmetric body: $I_1 = I_2 = I_3$

- d/dt $\omega_j = 0$ → ω_j is constant (j = 1,2,3)

Axial symmetric body: $I_T = I_1 = I_2$

- d/dt $\omega_3 = 0$ → ω_3 is constant
- Euler equations are linear
- Linear equations have nice solutions

$$\ddot{\omega}_1 + \Omega^2\omega_1 = 0$$
$$\ddot{\omega}_2 + \Omega^2\omega_2 = 0$$

$$\omega_1(t) = \omega_T \sin[\Omega(t - t_0)]$$
$$\omega_2(t) = \omega_T \cos[\Omega(t - t_0)]$$

Asymmetric body: inertias mutually distinct

- Euler equations are nonlinear
- Need to linearize nonlinear equations
- Linearization around an equilibrium

Equilibrium

- Need to find an equilibrium for nonlinear Euler´s equations
- **Linearization of nonlinear system** allows to consider small deviation around equilibrium and so study stability of system
- **Equilibrium: Pure spins about a principle axis** (rotation about 2-axis)

$$\begin{bmatrix} \omega_1(t) \\ \omega_2(t) \\ \omega_3(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \Omega \\ 0 \end{bmatrix} \quad \text{with} \quad \Omega = \text{constant}$$

- This is equilibrium of Euler´s equations because

$$\dot{\omega}_1 = -\frac{(I_3 - I_2)}{I_1}\omega_2\omega_3 = 0$$

$$\dot{\omega}_2 = -\frac{(I_1 - I_3)}{I_2}\omega_1\omega_3 = 0$$

$$\dot{\omega}_3 = -\frac{(I_2 - I_1)}{I_3}\omega_2\omega_1 = 0$$

Perturbation from equilibrium

Consider rotation about 2-axes with angular velocity Ω

$$\omega_2(t) = \Omega, \ \omega_1(t) = \omega_3(t) = 0$$

Now perturb state of pure spin such that

with $\varepsilon_i \ll \Omega$ small disturbances to equilibrium

$$\omega_1(t) = \varepsilon_1(t)$$
$$\omega_2(t) = \Omega + \varepsilon_2(t)$$
$$\omega_3(t) = \varepsilon_3(t)$$

Euler´s equation is now:

$$I_1\dot{\varepsilon}_1 + (I_3 - I_2)(\varepsilon_2 + \Omega)\varepsilon_3 = 0$$
$$I_2\dot{\varepsilon}_2 + (I_1 - I_3)\varepsilon_1\varepsilon_3 = 0$$
$$I_3\dot{\varepsilon}_3 + (I_2 - I_1)(\varepsilon_2 + \Omega)\varepsilon_1 = 0$$

Linearize Euler equation

Neglect second order term because perturbation ε_j are small

Linearization:

$$I_1\dot{\varepsilon}_1 + (I_3 - I_2)\Omega\varepsilon_3 = 0$$

$$I_2\dot{\varepsilon}_2 = 0$$

$$I_3\dot{\varepsilon}_3 + (I_2 - I_1)\Omega\varepsilon_1 = 0$$

Equations 1 and 3 are decoupled from equation 2

$$\varepsilon_2(t) = const$$

ε_2(t) = constant → represents constant perturbation to angular velocity component Ω about spin 2-axis

$$\omega_2(t) = \Omega + \varepsilon_2(t)$$

2^{nd} order differential equation

Combine perturbed linearized Euler equations on 1- and 3-axis to form one single equation (taking time derivatives)

$$I_1\ddot{\varepsilon}_1 = (I_2 - I_3)\Omega\dot{\varepsilon}_3$$
$$I_3\ddot{\varepsilon}_3 = (I_1 - I_2)\Omega\dot{\varepsilon}_1$$

From previous slide

$$I_1\dot{\varepsilon}_1 + (I_3 - I_2)\Omega\varepsilon_3 = 0$$
$$I_3\dot{\varepsilon}_3 + (I_2 - I_1)\Omega\varepsilon_1 = 0$$

Obtain single 2^{nd} order differential equation with constant coefficient

$$\ddot{\varepsilon}_1 = \alpha^2\varepsilon_1$$

$$\ddot{\varepsilon}_3 = \alpha^2\varepsilon_3$$

$$\alpha^2 = \frac{(I_2 - I_3)(I_1 - I_2)}{I_1 I_3}\Omega^2$$

Condition for spin axis stability

Solution for 2nd order differential equation ε_i(t) are:

$$\ddot{\varepsilon}_1 = \alpha^2 \varepsilon_1 \qquad \varepsilon_i(t) = Ae^{\alpha t} + Be^{-\alpha t}, \quad (\alpha \neq 0) \qquad i = 1,3$$

$$\ddot{\varepsilon}_3 = \alpha^2 \varepsilon_3 \qquad \varepsilon_i(t) = A + Bt, \quad (\alpha = 0) \qquad i = 1,3$$

Stability analysis with three cases:

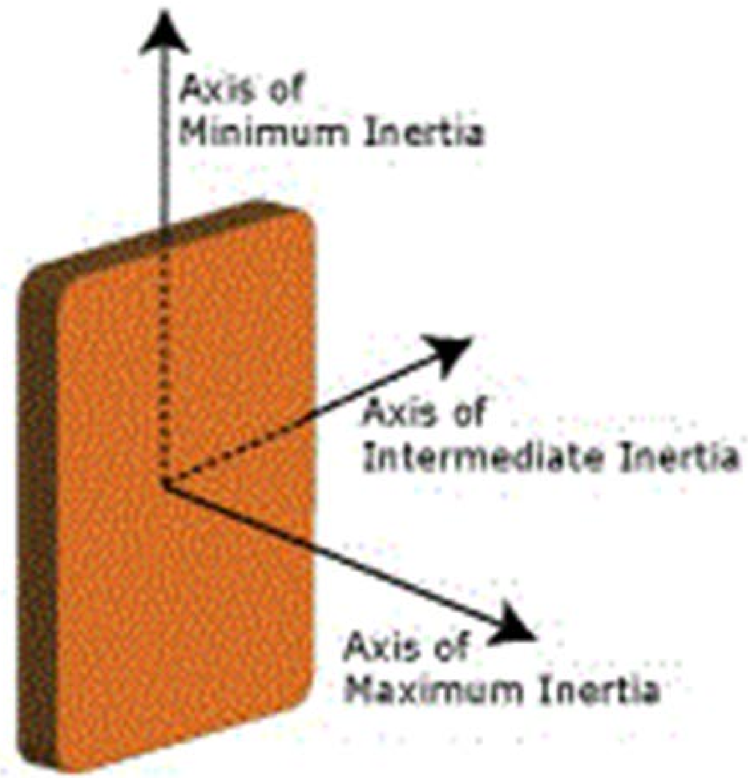
1. $\alpha^2 > 0 \rightarrow \alpha$ real → Exist exponent with positive real part
   Solution grows without bound → Motion is unstable
2. $\alpha^2 < 0 \rightarrow \alpha$ imaginary → Solution is periodic → Motion is stable
3. $\alpha^2 = 0 \rightarrow$ Solution grows linearly with time → Motion unstable

$$\alpha^2 = \frac{(I_2 - I_3)(I_1 - I_2)}{I_1 I_3} \Omega^2 < 0$$

For stability is therefore required $(I_2 - I_3)(I_2 - I_1) > 0$

$$\boxed{\begin{array}{ccc} I_2 > I_3 & \mathit{and} & I_2 > I_1 \\ & \mathit{or} & \\ I_2 < I_3 & \mathit{and} & I_2 < I_1 \end{array}}$$

Showed stability of torque free motion of asymmetric body



Obtain stable oscillatory solution if coefficient $\alpha^2<0$ (otherwise unstable solution)

Theorem:

Torque free motion of rigid body is stable if spin of body is about axis of maximum or minimum principal moment of inertia.

If nominal spin is about axis of intermediate principal moment of inertia, then motion is unstable.

$I_1 > I_2 > I_3$

Major (maximum) axis of inertia is stable ($\alpha^2<0$)

Minor (minimum) axis of inertia is stable ($\alpha^2<0$)

Intermediate axis of inertia is unstable ($\alpha^2>0$)

Instability of intermediate axis

Instability of intermediate axes:
https://www.youtube.com/watch?v=fPI-rSwAQNg

Movie of angle: 4ejes

Major axis rule

- Energy dissipation change this results

  → Minor axis becomes unstable

  Simple spins about major axis of inertia are asymptotically stable
- Previous theorem restricted to case for **torque free** motion of **rigid body**
- Internal torques can produce structural deformation of body that rotates in a complex motion and create loss of energy in form of heat (energy dissipation comes from kinetic energy that is decreasing quantity)

Energy sink hypothesis:

A quasi-rigid body in a complex rotation motion will dissipate energy until a state of minimum kinetic energy is reached. For torque free motion, the angular momentum is conserved.

(Simple motion around a principle axis does not produce mechanical energy dissipation)

Major axis rule:

Spin about major axis is asymptotically stable

Spin about any other axis is unstable

Energy dissipation: Energy sink hypotheses

- Flexible elements of spacecraft can deform due to internal torques associated to complex motions
- Structural deformations result in energy dissipation through heat
- Total energy is conserved, therefore heat must be taken from kinetic energy

Quasi-rigid body dissipates energy until reaches state of minimum kinetic energy
Angular momentum (***h***) is conserved for torque-free motion

$$T_{maj} = \tfrac{1}{2} I_{maj} \omega_{maj}^2 = \frac{h^2}{2 I_{maj}} \qquad h = I_{maj}\omega_{maj} \quad \text{for major axis spin} \qquad \boxed{2T_{maj} = \frac{h^2}{I_{maj}}}$$

$$T_{\min} = \tfrac{1}{2} I_{\min} \omega_{\min}^2 = \frac{h^2}{2 I_{\min}} \qquad h = I_{\min}\omega_{\min} \quad \text{for minor axis spin} \qquad \boxed{2T_{\min} = \frac{h^2}{I_{\min}}}$$

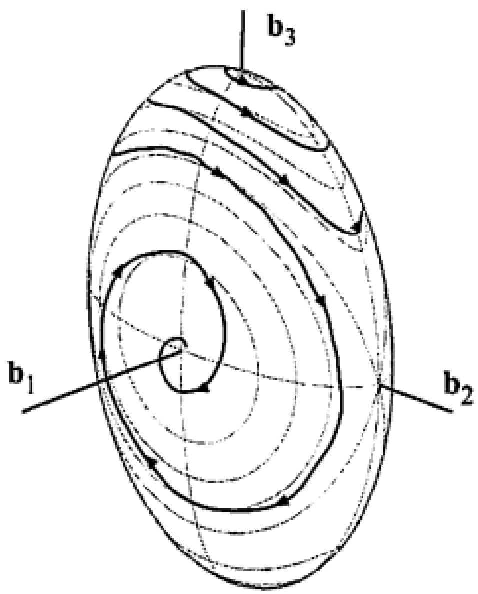
$I_{\min} < I_{maj}$ and for given angular momentum *h* $\rightarrow$ $T_{maj} < T_{\min}$

Kinetic energy is minimized for major axis spin

Real (therefore flexible) spacecraft can spin stably only about axis of maximum moment of inertia

Energy dissipation: Polhode drift for quasi-rigid body

$$I_1 \geq I_2 \geq I_3$$



Destabilization caused by energy dissipation:

Polhode of general rigid body modified by energy dissipation

Assume vector $\boldsymbol{\omega}$ starts in pure spine about minor inertia axis $\mathbf{b}_3$

Energy dissipation:
Vector $\boldsymbol{\omega}$ moves away from minor inertia axis $\mathbf{b}_3$

Amplitude of precession increases

When vector $\boldsymbol{\omega}$ is near intermediate axis
larger precession with increased rate of energy dissipation

Vector $\boldsymbol{\omega}$ converges to state of pure spin
about major inertia axis $\mathbf{b}_1$

Only spin about major axis of inertia is stable

Energy dissipation for axisymmetric body: Mathematical

Know/assume:

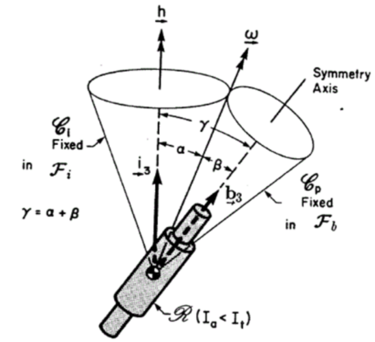
1. Angular momentum (***h***) conserved for torque-free motion
2. Kinetic energy not conserved
3. But use energy sink hypotheses

Take energy sink hypothesis for special case of torque free motion of axisymmetric body

Questions:

1. How to find angular vector **ω** when minimize energy (energy sink hypotheses $\dot{T} < 0$) with angular momentum **h** = constant ?
2. What is end effect of internal kinetic energy loss on stability for axisymmetric body?

Energy dissipation for axisymmetric body: Mathematical

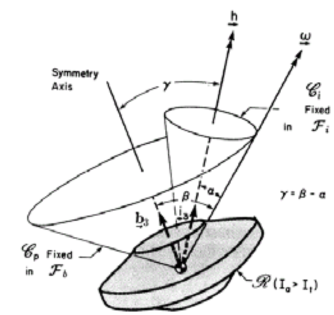


Consider kinetic energy with axisymmetric body

$$T = \tfrac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) = \tfrac{1}{2}(I_T\omega_T^2 + I_3\omega_3^2)$$

Angular momentum $h_t = I_T\omega_t = h\sin\gamma \Leftrightarrow \omega_t = \frac{h}{I_T}\sin\gamma$

$$h_3 = I_3\omega_3 = h\cos\gamma \Leftrightarrow \omega_3 = \frac{h}{I_3}\cos\gamma$$



Describe angular velocity components in terms of angular momentum and nutation angle

Substitute into expression for energy

$$T = \tfrac{1}{2}\left(I_T\left(\tfrac{h}{I_T}\sin\gamma\right)^2 + I_3\left(\tfrac{h}{I_3}\cos\gamma\right)^2\right) = \tfrac{1}{2}h^2\left(\tfrac{1}{I_T}\sin^2\gamma + \tfrac{1}{I_3}\cos^2\gamma\right)$$

Take time derivative of kinetic energy

$$\dot{T} = \frac{\mathrm{dT}}{\mathrm{d}\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \tfrac{1}{2}h^2\left(\tfrac{1}{I_T}2\sin\gamma\cos\gamma - \tfrac{1}{I_3}2\cos\gamma\sin\gamma\right)\dot{\gamma} = \tfrac{1}{2}h^2\sin 2\gamma\left(\frac{I_3 - I_T}{I_3 I_T}\right)\dot{\gamma}$$

Energy dissipation for axisymmetric body

$$\boxed{\dot{T} = \tfrac{1}{2} h^2 \sin 2\gamma \left(\tfrac{I_3 - I_T}{I_3 I_T}\right) \dot{\gamma}}$$

Using energy sink hypotheses $\dot{T} < 0$

Implies sign d/dt ϒ must correspond to sign of $(I_T - I_3)$

There are two cases:

- **Case 1:** If $(I_T - I_3) < 0 \rightarrow$ d/dt ϒ < 0

  Nutation angle will decrease and $I_T < I_3$ corresponds to major axis spin

  $$\text{For} \quad \text{minimum} \quad T \quad \Rightarrow \quad \dot{T} = 0 \Rightarrow \sin 2\gamma = 0 \quad \Rightarrow \quad \gamma = 0 \quad or \quad \gamma = \pi / 2$$

  From $T = \frac{1}{2} h^2 \left(\frac{1}{I_T} \sin^2 \gamma + \frac{1}{I_3} \cos^2 \gamma\right)$

  $$T \,|_{\gamma=0} = \frac{h^2}{2I_3} \quad \text{or} \quad T \,|_{\gamma=\pi/2} = \frac{h^2}{2I_T} \quad \Rightarrow \quad T \,|_{\min} = \frac{h^2}{2I_3} \quad \text{when} \quad \gamma = 0$$

- **Case 2:** If $(I_T - I_3) > 0 \rightarrow$ d/dt ϒ > 0

  Nutation angle will increase and $I_T > I_3$ corresponds to minor axis spin

  $$\text{Nutation angle increase until} \quad \dot{T} = 0 \quad at \quad \gamma = \pi / 2$$

  $$T \,|_{\min} = \frac{h^2}{2I_T} < \frac{h^2}{2I_3}$$

Energy dissipation for axisymmetric body

$$\dot{T} = \frac{1}{2} h^2 \sin 2\gamma \left(\frac{I_3 - I_T}{I_3 I_T}\right)\dot{\gamma}$$

Using energy sink hypotheses $\dot{T} < 0$

Implies sign d/dt ϒ of must correspond to sign of (I_T - I_3)

There are two cases:

- **Case 1:** If (I_T - I_3) < 0 → d/dt ϒ < 0

  Nutation angle will decrease and $I_T < I_3$ corresponds to major axis spin
  → 3-axis spin is **stable**

- **Case 2:** If (I_T - I_3) > 0 → d/dt ϒ > 0
  Nutation angle will increase and $I_T > I_3$ corresponds to minor axis spin
  → 3-axis spin is **unstable**

Energy dissipation causes nutation angle die away for major axis spins
Destabilization for minor axis spin because of energy dissipation

Energy dissipation

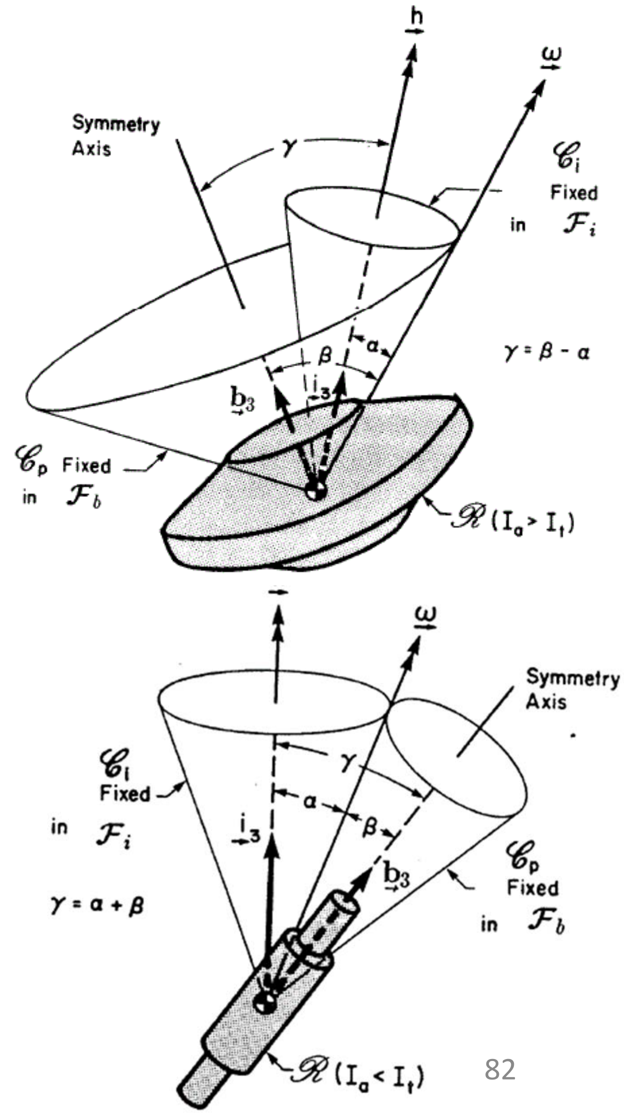
Energy dissipation causes nutation angle die away for major axis spins

Since $\frac{I_T}{I_3}\tan\beta = \tan\gamma$

$$\gamma = 0 \Rightarrow \beta = 0$$

Therefore:
Angular momentum vector
Angular velocity vector
Symmetry axis
are aligned

Minor axis spins are unstable
Nutation angle grows
until major axis spin with ϒ = π/2 is obtained



Explorer

Explorer 1 (first USA satellite in 1958) designed as minor axis spinner

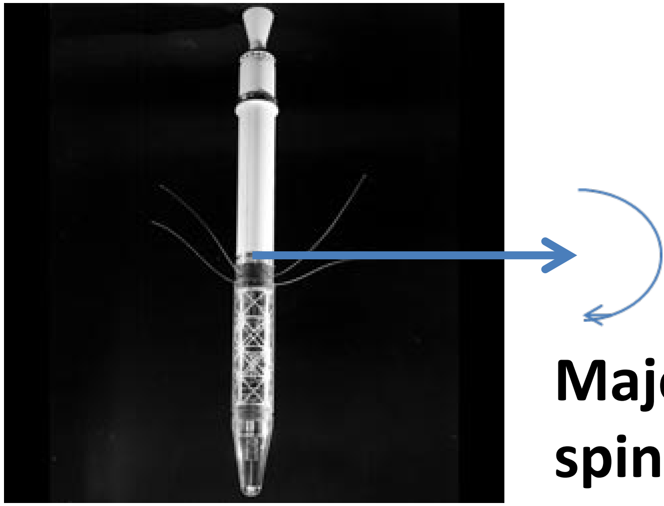
Through energy dissipation in short time spacecraft converted in major axis spinner

Energy dissipation mostly caused by flexible wire antennas on spacecraft

All real spacecraft have always some non-rigid properties (elastic structural deflection/ sloshing)

Energy dissipation effect:

Minor axis spinner



Major axis spinner

Summary: general torque free motion

Angular velocity vector must lie at same time on

1) angular momentum ellipsoid
2) kinetic energy ellipsoid

Intersection: polhode (seen from body-fixed reference frame)

Analytical closed-form solution in torque-free motion of an asymmetric rigid body is expressed in terms of Jacobi elliptic functions

Stability of torque-free motion about principal axes:

1) major axis: always stable
2) intermediate axis: unstable
3) minor axis: stable only if no energy dissipation

Summary

Euler´s equation in principle axes frame

Linear and nonlinear differential equations

- Equilibrium, Stability, Characteristic equations
- Stability theorem
- Linearization of nonlinear systems

Symmetric and non-symmetric torque free rotations

- Geometrical and mathematical
- Linearized equation of motion
- Stability

Energy dissipation

- Geometrical and mathematical
- Effect on stability of rotation (major axis rule)