DOC 221 Dinámica orbital y control de actitud

Solutions to Problems Lecture ADCS - II

Problem 1:

For the satellite in a Keplerian orbit around the Earth and with perigee height above the Earth of h_p = 300 km, an apogee height above the Earth of h_a = 10000 km, R_E = 6378 km and μ = 398600 km^3 s^{-2} we have the following results:

$r_p = R_E + h_p$ = 6678 km

$r_a = R_E + h_a$= 16378 km

(a) The semi-major axis a is $a = (r_p+r_a)/2$ = 11528 km.

(b) The eccentricity e *is* $e = (r_a-r_p)/(r_a+r_p)$ = 0.421.

(c) Use vis-viva equation $\frac{v_p^2}{2} - \frac{\mu}{r_p} = -\frac{\mu}{2a}$

→ The velocity at perigee v_p *is* v_p= 9.21 km/s.

(d) Use vis-viva equation $\frac{v_a^2}{2} - \frac{\mu}{r_a} = -\frac{\mu}{2a}$

→ The velocity at apogee v_a is v_a = 3.75 km/s.

(e) The orbital period T is $T = 2\pi\sqrt{a^3/\mu}$ = 12307 s = 205.1 min.

Problem 2:

(a) The semi-major axis a is given by the vis-viva equation

$$\frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} \rightarrow a = 8450 \text{ km.}$$

(b) The eccentricity e is given by $r_p = a(1\text{-}e) \rightarrow e = 0.151.$

(c) The maximum radius of this orbit r_a is given by

$$r_a = a(1+e) \rightarrow r_a = 9729 \text{ km.}$$

(d) The maximum altitude h_{max} is given by

$$h_{max} = r\text{-}R_E \rightarrow h_{max} = 3351 \text{ km.}$$

Problem 3:

(a) The radial acceleration is given by: $a_r = -\frac{\partial U}{\partial r}$

The North-South acceleration is given by: $a_\phi = -\frac{1}{r}\frac{\partial U}{\partial \phi}$

The East-West acceleration is given by: $a_\lambda = -\frac{1}{r\cos\phi}\frac{\partial U}{\partial \lambda}$

(b) Use $P_n(x) = \frac{1}{(-2)^n n!}\frac{\partial^n}{\partial x^n}(1-x^2)^n$ and set $P_{20}(x)|_{x=\sin\phi} = P_{20}\left(\sin\phi\right)$

Legendre polynomial is $P_2(x) = \frac{1}{(-2)^2 2!}\frac{\partial^2}{\partial x^2}(1-x^2)^2 = \frac{1}{4\cdot 2}\frac{\partial^2}{\partial x^2}(1-2x^2+x^4) = -\frac{1}{2}+\frac{3}{2}x^2$

$$\rightarrow P_2(\sin\phi) = -\frac{1}{2}+\frac{3}{2}\sin\phi^2$$

The gravitational potential for the U_{J2} term is

$$U_{J_2}(r,\phi) = \frac{GM}{r}\left(\frac{R_e}{r}\right)^2 J_2 P_{20}\left(\sin\phi\right) = \frac{GM}{r}\left(\frac{R_e}{r}\right)^2 J_2\left(\frac{3}{2}\sin\phi^2-\frac{1}{2}\right)$$

$$U_{J_2}(r,\phi) = \frac{GM}{r}\left(\frac{R_e}{r}\right)^2 J_2 P_{20}\left(\sin\phi\right) = \frac{GM}{r}\left(\frac{R_e}{r}\right)^2 J_2\left(\frac{3}{2}\sin\phi^2-\frac{1}{2}\right)$$

The radial acceleration due to J_2 term is

$$a_{r,J_2} = -\frac{\partial U_{J_2}}{\partial r} = 3\frac{GM}{r^4}R_e{}^2 J_2\left(\frac{3}{2}\sin^2\phi-\frac{1}{2}\right)$$

The radial acceleration due to J_3 term is

$$a_{r,J_3} = -\frac{\partial U_{J_3}}{\partial r} = 4\frac{GM}{r^5}R_e{}^3 J_3\left(\frac{5}{2}\sin^3\phi-\frac{3}{2}\sin\phi\right)$$

(c) The North-South acceleration due to J_2 term is

$$a_{\phi,J_2} = -\frac{1}{r}\frac{\partial U_{J_2}}{\partial \phi} = -\frac{GM}{r^4}R_e{}^2 J_2(3\sin\phi\cos\phi)$$

The North-South acceleration due to J_3 term is

$$a_{\phi,J_3} = -\frac{1}{r}\frac{\partial U_{J_3}}{\partial \phi} = -\frac{GM}{r^5}R_e{}^3 J_3\left(\frac{15}{2}\sin^2\phi\cos\phi-\frac{3}{2}\cos\phi\right)$$

(d) The East-West acceleration due to J_2 term is

$$a_{\lambda,J_2} = -\frac{1}{r\sin\phi}\frac{\partial U_{J_2}}{\partial \lambda} = 0$$

The East-West acceleration due to J_3 term is

$$a_{\lambda,J_3} = -\frac{1}{r\sin\phi}\frac{\partial U_{J_3}}{\partial \lambda} = 0$$

(e) $a_{r,J_2}(400\text{ km}) = 0.025\cdot 10^{-3}\left(\frac{3}{2}\sin^2\phi-\frac{1}{2}\right)\text{ km/s}^2$

Problem 4:

(a) The orientation of the orbital plane with respect to the direction towards the Sun is constant over time.

(b) The secular rates of changes for Ω and ω are given by

$$\dot{\Omega}_{avg} = -\frac{3J_2R_e^2}{2(1-e^2)^2}\sqrt{\frac{\mu}{a^7}}\cos i$$

$$\dot{\omega}_{avg} = \frac{3\pi J_2R_e^2}{4(1-e^2)^2}\sqrt{\frac{\mu}{a^7}}(5\cos^2 i - 1)$$

For a sun-synchronous orbit $\dot{\Omega}_{avg} = \frac{360^o}{year} = \frac{2\pi}{year} = 1.99\cdot 10^{-7}$ rad/s

For a frozen orbit (constant in the argument of the perigee), $\dot{\omega}_{avg} = 0$

This leads to $5\cos^2 i - 1 = 0 \Leftrightarrow \cos i = \pm\sqrt{\frac{1}{5}}$.

For a sun-synchronous orbit $\dot{\Omega}_{avg} > 0$ and therefore from equation for $\dot{\Omega}_{avg}$, it must be that $\cos i < 0 \Rightarrow \cos i = -\sqrt{\frac{1}{5}}$.

Now, we have for the sun-synchronous orbit

$$\dot{\Omega}_{avg} = -\frac{3J_2R_e^2}{2(1-e^2)^2}\sqrt{\frac{\mu}{a^7}}\cos i = \frac{3J_2R_e^2}{2(1-e^2)^2}\sqrt{\frac{\mu}{a^7}}\sqrt{\frac{1}{5}}$$

$$\rightarrow \frac{3J_2R_e^2}{2\dot{\Omega}_{avg}}\sqrt{\frac{\mu}{5a^7}} = (1-e^2)^2$$

$$\rightarrow e = \left[1 - \left(\frac{3J_2R_e^2}{2\dot{\Omega}_{avg}}\sqrt{\frac{\mu}{5a^7}}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}$$

(c) Use $J_2 = 1.083\cdot 10^{-3}$, $R_e = 6378$ km, $\mu = 398600$ km^3 s^{-2} and

$$\dot{\Omega}_{avg} = 1.99\cdot 10^{-7} \text{ rad/s}.$$

For semi-major axis a = 10000 km,

the values are $\frac{3J_2R_e^2}{2\dot{\Omega}_{avg}} = 3.32\cdot 10^{11}$ km^2 s $\quad \sqrt{\frac{\mu}{5a^7}} = 2.82\cdot 10^{-12}$ km^{-2} s^{-1}

and the eccentricity is e = 0.18.

The radius of perigee is $r_p = a(1-e)$=8200 km.

Note that the radius of perigee is higher than the Earth´s radius R_e=6378 km, so this is a feasible orbit.

For semi-major axis a = 15000 km, the eccentricity is e = 0.72.

The radius of perigee is $r_p = a(1-e)$ = 4200 km.

Note that the radius of perigee is lower than the Earth´s radius R_e = 6378 km, so this is not a feasible orbit.

For semi-major axis a = 20000 km, the eccentricity is e = 0.84.
The radius of perigee is $r_p = a(1-e)$=3200 km.
Note that the radius of perigee is lower than the Earth´s radius R_e = 6378 km, so this is not a feasible orbit.

(d) See following figure

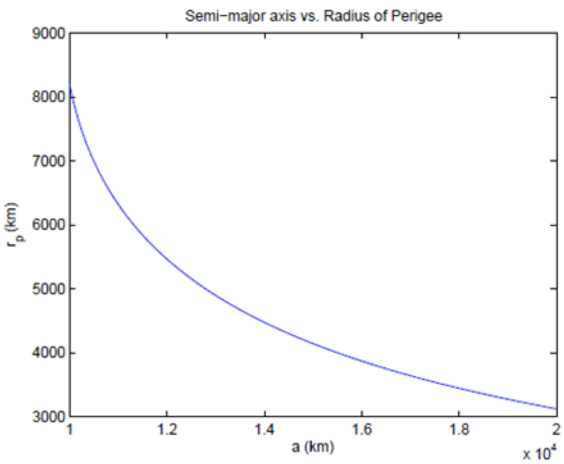

Figure 7.1 Semi-major axis vs. radius of perigee for a sun-synchronous frozen orbit

(e) From part **(b)**, clearly eccentricity decreases as semi-major axis increases. The minimum semi-major axis therefore occurs when $e = 0$, in which case the orbit is circular. From part **(b)**, this occurs when

$$1=\left(\frac{3J_2R_e^2}{2\dot{\Omega}_{avg}}\sqrt{\frac{\mu}{5a^7}}\right)^{\frac{1}{2}}$$

This can be solved for a as $a=\left(\frac{3J_2R_e^2}{2\dot{\Omega}_{avg}}\sqrt{\frac{\mu}{5}}\right)^{\frac{2}{7}}=9811\,\text{km}$.

(f) As seen in part **(d),** the radius of perigee decreases with increasing semi-major axis. Therefore, to determine the maximum semi-major axis, we set $r_p = a(1-e) = R_e$ + 200 km = 6578 km.
Solving iteratively for a, we obtain a = 10784 km.