DOC 221 Dinámica orbital y control de actitud

Problems Lecture VI

Problem 1:

Assume three reference frames A, B and I. Let the two reference frames A and B be defined relative to the inertial reference frame I by the orthonormal unit base vectors

$$\vec{a}_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix} \quad \vec{a}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \vec{a}_3 = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}$$

$$\vec{b}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \vec{b}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{b}_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix},$$

where the $\mathbf{a}_i$ and $\mathbf{b}_i$ (i = 1, 2, 3) vector components are written in the inertial frame I. Note that the unit base vectors of the inertial frame are

$$\vec{i}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{i}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \vec{i}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

- **(a)** Check that the unit base vectors $\mathbf{a}_i$ respectively $\mathbf{b}_i$ (i = 1, 2, 3) build an orthonormal reference frame.
- **(b)** Find the directional cosine matrix $\mathbf{C}_{ab}$ that describes the orientation of frame A relative to frame B.
- **(c)** Find the directional cosine matrix $\mathbf{C}_{ai}$ that describes the orientation of frame A relative to frame I.
- **(d)** Find the directional cosine matrix $\mathbf{C}_{bi}$ that describes the orientation of frame B relative to frame I.
- **(e)** Check if $\mathbf{C}_{ab} = \mathbf{C}_{ai} (\mathbf{C}_{bi})^T$ holds.
- **(f)** Check if $\mathbf{C}_{ab}(\mathbf{C}_{ab})^T = \mathbf{1}$, where $\mathbf{1}$ is 3x3 unit matrix.
- **(g)** For given arbitrary matrix **A** and matrix **B** check if they do not commute ($\mathbf{AB} \neq \mathbf{BA}$).

(h) Is the following matrix $\mathbf{C} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ a rotation matrix?

Problem 2:

Show that the cross vector product can be written as follows: $\mathbf{a} \times \mathbf{b} = \mathbf{a}^{\times}\mathbf{b}$ where $\mathbf{a}^{\times}$ is given by the following skew-symmetric matrix formed out of the elements of **a:**

$$\mathbf{a}^{\times} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.$$

This skew-symmetric matrix has the property $(\mathbf{a}^{\times})^T = -\mathbf{a}^{\times}$.

Show also $\mathbf{a}^{\times}\mathbf{b} = -\mathbf{b}^{\times}\mathbf{a}$ and $\mathbf{a}^{\times}\mathbf{a} = \mathbf{0}$ where **0** is 3x1 matrix of zeros.

Problem 3:

(a) Find the Euler rotation matrix $\mathbf{C}_{21}$ in terms of 3-2-3 Euler angles rotation sequence, with angles Θ_1, Θ_2 and Θ_3.
Specifically, frame 2 is obtained from frame 1 by:
- A rotation Θ_1 about the z-axis (3-axis) of frame 1,
- a rotation Θ_2 about the y-axis (2-axis) of intermediate frame,
- a rotation Θ_3 about the z-axis (3-axis) of the transformed frame.

(b) Find from the 3-2-3 Euler rotation matrix the appropriate Euler angles.

(c) For the 3-2-3 Euler sequence, derive the following kinematic differential equation $\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \frac{1}{\sin\theta_2} \begin{bmatrix} -\cos\theta_3 & \sin\theta_3 & 0 \\ \sin\theta_3 \sin\theta_2 & \cos\theta_3 \sin\theta_2 & 0 \\ \cos\theta_3 \cos\theta_2 & -\sin\theta_3 \cos\theta_2 & \sin\theta_2 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$

(d) What are the points of singularity for this Euler rotation?

Problem 4:

The orientation of an object is given in terms of the 3-2-1 Euler angles (-15^o, 25^o, 10^o).

- **(a)** Write the direction cosine Euler rotation matrix $\mathbf{C}_{21}$.
- **(b)** Find the principle Euler eigenaxis rotation angle ϕ.
- **(c)** Find the corresponding principal Euler rotation eigenaxis **e**.
  Verify that $\mathbf{C}_{21}\mathbf{e} = \mathbf{e}$.
- **(d)** Find the corresponding Euler parameters = Quaternions.
- **(e)** Is the last expression an unit quaternion? Has it magnitude one?

Problem 5:

Let the orientations of two spacecraft A and B relative to an inertial frame I be given through the 3-2-1 Euler angles rotation sequences $\boldsymbol{\Theta}_A = (60,-45,30)^T$ and $\boldsymbol{\Theta}_B = (-15,25,10)^T$ degrees. What is the relative orientation of spacecraft A relative to B in terms of 3-2-1 Euler angles?

Problem 6:

A spacecraft performs a 45-deg single principle Euler eigenaxis rotation about

$$\vec{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Find the corresponding rotation matrix C and the corresponding 3-2-1 Euler angles that relate the final attitude to the original attitude.