DOC 221 Dinámica orbital y control de actitud

Solutions to Problems Lecture ADCS – IXB

Problem 1:

Here $\hat{o}_3$ is aligned with the cylinder symmetry axis (aligned with the length h direction). The principal inertias for a cylinder are given by

$$I_1 = \frac{m}{12}h^2 + \frac{m}{4}R^2$$

$$I_2 = I_1$$

$$I_3 = \frac{m}{2}R^2$$

The stability condition requires that $I_2 \geq I_1 \geq I_3$. Since $I_2 = I_1$, the first part of this stability condition is automatically satisfied. Thus, we need to make sure that $I_1 \geq I_3$

$$I_1 \geq I_3$$

$$\frac{m}{12}h^2 + \frac{m}{4}R^2 \geq \frac{m}{2}R^2$$

$$h^2 + 3R^2 \geq 6R^2$$

$$h^2 \geq 3R^2$$

$$h \geq \sqrt{3}R$$

Problem 2:

In body coordinates, the gravity-gradient torque is given by

$$\mathbf{T}_{gg,b} = \frac{3\mu}{R_o^5}\mathbf{R}_{o,b}^{\times}\mathbf{I}\mathbf{R}_{o,b}$$

Where $\mathbf{I}$ = diag $\{I_x, I_y, I_z\}$ is the spacecraft inertia matrix, R_o is the spacecraft orbital position vector in body coordinates $R_o=|\mathbf{R}_o|$, and μ is Earth´s gravitational constant. We are given the spacecraft orbital position in ECI coordinates, namely

$$\mathbf{R}_{o,G} = \begin{bmatrix} 0 \\ 0 \\ R_o \end{bmatrix}$$

with R_o = 7000 km.

In body coordinates therefore, we have

$$\mathbf{R}_{o,b} = \mathbf{C}_{bG}\mathbf{R}_{o,G} = \begin{bmatrix} c_\theta c_\psi & c_\theta s_\psi & -s_\theta \\ s_\phi s_\theta c_\psi - c_\phi s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & s_\phi c_\theta \\ c_\phi s_\theta c_\psi + s_\phi s_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi & c_\phi c_\theta \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ R_o \end{bmatrix} = R_o\begin{bmatrix} -s_\theta \\ s_\phi c_\theta \\ c_\phi c_\theta \end{bmatrix}.$$

Now, we are given $\phi = \theta = \psi = \pi/4$. Therefore,

$$\sin\phi = \sin\theta = \sin\psi = \frac{1}{\sqrt{2}} \quad , \quad \cos\phi = \cos\theta = \cos\psi = \frac{1}{\sqrt{2}}.$$

This leads to

$$\mathbf{R}_{o,b} = R_o\begin{bmatrix} -1/\sqrt{2} \\ 1/2 \\ 1/2 \end{bmatrix}.$$

The gravity-gradient torque in body coordinates is now

$$\mathbf{T}_{gg,b} = \begin{bmatrix} -0.3486 \\ -0.2465 \\ -0.2465 \end{bmatrix} \times 10^{-4} \text{ Nm}$$

Problem 3:

From lecture ADCS – IXB , for passive stability, we must have either

$$\begin{gathered} \lambda > I_2 \quad \text{and} \quad \lambda > I_1 \\ \text{or} \\ \lambda < I_2 \quad \text{and} \quad \lambda < I_1 \end{gathered}$$

where

$$\lambda = I_3 + \tfrac{h_s}{\Omega}.$$

Since ω_3 = 0.1 rad/s we have from the first conditions

$$h_s > (I_2 - I_3)\omega_3 = -0.2 \text{ Nms} \quad \text{and} \quad h_s > (I_1 - I_3)\omega_3 = 0.2 \text{ Nms}$$

Therefore, for passive stability, we require

$$h_s > 0.2 \text{ Nms}.$$

From second conditions, we obtain

$$h_s < (I_2 - I_3)\omega_3 = -0.2 \text{ Nms} \quad \text{and} \quad h_s < (I_1 - I_3)\omega_3 = 0.2 \text{ Nms}$$

and therefore

$$h_s < -0.2 \text{ Nms}.$$

Problem 4:

a) For torque-free motion T=0, the equations of motion are

$$\mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega} + h_s \mathbf{a}) = \mathbf{T}$$

where **a** is the wheel spin-axis in body coordinates with $\mathbf{a} = [0 \quad 0 \quad 1]^T$ with $\boldsymbol{\omega} = \mathbf{0}$, the equations of motion become

$$\mathbf{I}\dot{\boldsymbol{\omega}} = \mathbf{0}$$

This indicates that it is an equilibrium condition.

b) The equations of motion are in principal axes given by

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 + h_s\omega_2 &= 0 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 - h_s\omega_1 &= 0 \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 \qquad &= 0 \end{aligned}$$

Consider small perturbations from the equilibrium

$$\omega_1(t) = \varepsilon_1(t) \quad \omega_3(t) = \varepsilon_3(t) \quad \omega_2(t) = \varepsilon_2(t) .$$

Substituting these small perturbations into equations of motion we obtain

$$\begin{aligned} I_1\dot{\varepsilon}_1 + (I_3 - I_2)\varepsilon_2\varepsilon_3 + h_s\varepsilon_2 &= 0 \\ I_2\dot{\varepsilon}_2 + (I_1 - I_3)\varepsilon_1\varepsilon_3 - h_s\varepsilon_1 &= 0 \\ I_3\dot{\varepsilon}_3 + (I_2 - I_1)\varepsilon_2\varepsilon_1 \qquad &= 0 \end{aligned}$$

By linearizing the problem (neglecting products of ε) we get

$$\begin{aligned} I_1\dot{\varepsilon}_1 + h_s\varepsilon_2 &= 0 \\ I_2\dot{\varepsilon}_2 - h_s\varepsilon_1 &= 0 \\ I_3\dot{\varepsilon}_3 \qquad &= 0 \end{aligned}$$

The third equation gives $\varepsilon_3(t) = \varepsilon_3(0)$, which is constant. The first two can be rearranged to give

$$\dot{\varepsilon}_1 + \tfrac{h_s}{I_1}\varepsilon_2 = 0$$
$$\dot{\varepsilon}_2 - \tfrac{h_s}{I_2}\varepsilon_1 = 0$$

Combine perturbed linearized Euler equations on 1- and 2-axis to form one single equation (taking time derivatives) we get

$$\ddot{\varepsilon}_1 = \alpha^2\varepsilon_1$$
$$\ddot{\varepsilon}_2 = \alpha^2\varepsilon_2$$

With $\alpha^2 = -\tfrac{h_s^2}{I_1 I_2}$. We have now $\alpha^2 < 0 \rightarrow \alpha$ is purely imaginary $\alpha = \pm i \tfrac{h_s}{\sqrt{I_1 I_2}}$ and therefore this corresponds to a purely oscillatory behavior of

$$\varepsilon_i(t) = Ae^{\alpha t} + Be^{-\alpha t}, \quad (\alpha \neq 0) \qquad i = 1, 2$$

with frequency

$$\Omega_p = \frac{h_s}{\sqrt{I_1 I_2}}$$