DOC 221 Dinámica orbital y control de actitud

Solutions to Problems Lecture ADCS – IXA

Problem 1:

Euler equations without torques are given by

$$I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = 0$$

$$I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 = 0$$

$$I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = 0$$

The principal moments of inertia (take values of moments from a table) for a cylinder of radius r and height h is given by

$$I_1 = I_2 = \frac{1}{12}m(3r^2 + h^2)$$

$$I_3 = \frac{1}{2}mr^2$$

Since we have h = 4r, we get

$$I_1 = \frac{19}{12}mr^2$$

$$I_2 = \frac{19}{12}mr^2$$

$$I_3 = \frac{1}{2}mr^2$$

Use these values for the Euler equations and we get

$$\frac{19}{12}mr^2\dot{\omega}_1 - \left(\frac{19}{12} - \frac{1}{2}\right)mr^2\omega_2\omega_3 = 0$$

$$\frac{19}{12}mr^2\dot{\omega}_2-\left(\frac{1}{2}-\frac{19}{12}\right)mr^2\omega_1\omega_3=0$$

$$\frac{1}{2}mr^2\dot{\omega}_3=0\Rightarrow\dot{\omega}_3=0$$

The last equation implies that ω_3 is a constant. Using the initial condition on ω_3,

$$\omega_3(0)=\frac{4}{5}\Omega_0\Rightarrow\omega_3(t)=\frac{4}{5}\Omega_0$$

we get

$$\begin{aligned}\frac{19}{12}mr^2\cdot\dot{\omega}_1&=\frac{13}{12}\cdot\omega_2\cdot\frac{4}{5}\Omega_0\\ \frac{19}{12}mr^2\cdot\dot{\omega}_2&=-\frac{13}{12}\cdot\omega_1\cdot\frac{4}{5}\Omega_0\end{aligned}\quad\rightarrow\quad\begin{aligned}\dot{\omega}_1&=\frac{52}{95}\frac{\Omega_0}{mr^2}\omega_2\\ \dot{\omega}_2&=-\frac{52}{95}\frac{\Omega_0}{mr^2}\omega_1\end{aligned}$$

and using $a=\frac{52}{95}\frac{\Omega_0}{mr^2}$ we get

$$\begin{aligned}\dot{\omega}_1&=a\omega_2\\ \dot{\omega}_2&=-a\omega_1\end{aligned}\quad\rightarrow\quad\begin{aligned}\ddot{\omega}_1&=a\dot{\omega}_2=-a^2\omega_1\\ \ddot{\omega}_2&=-a\dot{\omega}_1=-a^2\omega_2\end{aligned}$$

These harmonic oscillation equations have solutions of the following form

$$\begin{aligned}\omega_1(t)&=A_1\cos(at)+A_2\sin(at)\\ \omega_2(t)&=A_3\cos(at)+A_4\sin(at)\end{aligned}$$

From initial conditions t = 0 and $\vec{\boldsymbol{\omega}}(t=0)=\Omega_0\left(\frac{3}{5}\vec{\mathbf{b}}_1+\frac{4}{5}\vec{\mathbf{b}}_3\right)$

$$\begin{aligned}\omega_1(0)&=\frac{3}{5}\Omega_0\\ \omega_2(0)&=0\end{aligned}\quad\rightarrow\quad\begin{aligned}A_1&=\frac{3}{5}\Omega_0\\ A_3&=0\end{aligned}$$

Moreover we have also

$$\begin{aligned}\dot{\omega}_1(0)&=a\omega_2(0)=0\\ \dot{\omega}_2(0)&=-a\omega_1(0)=-\frac{52}{95}\frac{\Omega_0}{mr^2}\cdot\frac{3}{5}\Omega_0\end{aligned}\quad\rightarrow\quad\begin{aligned}A_2&=0\\ A_4&=-\frac{3}{5}\Omega_0\end{aligned}$$

The final results is

$$\omega_1(t) = \frac{3}{5}\Omega_0 \cos\left(\frac{52}{95}\Omega_0 t\right)$$
$$\omega_2(t) = -\frac{3}{5}\Omega_0 \sin\left(\frac{52}{95}\Omega_0 t\right)$$
$$\omega_3(t) = \frac{4}{5}\Omega_0$$

Note that these are the components of ω in a body-fixed frame. The *ω*(t) traces out a cone in the body-fixed frame during one period *T = 2π/a* where the frequency is given by $a = \frac{52}{95}\Omega_0$.

Problem 2:

The principal inertias $I_x = 98\ \mathrm{kg \cdot m^2}$ and $I_y = 102\ \mathrm{kg \cdot m^2}$ are very close. Therefore, we approximate the spacecraft as being axisymmetric with $I_x = I_y = 100\ \mathrm{kg \cdot m^2}$.

(a) The spacecraft will behave like and oblate spinner, with the principal z-axis tracing out a cone in inertial space.

(b) The components of the angular momentum vector are

$$h_x = I_x \omega_x = 10\ \mathrm{Nms}$$
$$h_y = I_y \omega_y = 2\ \mathrm{Nms}$$
$$h_z = I_z \omega_z = 75\ \mathrm{Nms}.$$

The transversal angular momentum is

$$h_t = \sqrt{h_x^2 + h_y^2} = 10.198\ \mathrm{Nms}$$

The total angular momentum is

$$h = \sqrt{h_x^2 + h_y^2 + h_z^2} = 75.690\ \mathrm{Nms}$$

The nutation angle satisfies

$$\sin\gamma = \frac{h_t}{h}$$

which leads to

$$\gamma = \sin^{-1}\left(\frac{h_t}{h}\right) = 0.1351\ \mathrm{rad} = 7.74^{\circ}$$

(c) The precession rate is

$$\Omega_p = \frac{h}{I_t} = 0.7569 \text{ rad} = 43.36°/s$$

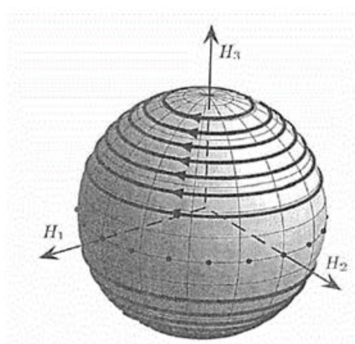
Problem 3:

Axisymmetric case ($I_1 = I_2$):

For the axisymmetric spacecraft assume (without loss of generality) that $I_1 = I_2$. Using Euler equations

$$\begin{aligned}
\dot{\omega}_1 &= -\tfrac{(I_3-I_2)}{I_1}\omega_2\omega_3 \\
\dot{\omega}_2 &= -\tfrac{(I_1-I_3)}{I_2}\omega_1\omega_3 \\
\dot{\omega}_3 &= -\tfrac{(I_2-I_1)}{I_3}\omega_2\omega_1 = 0
\end{aligned}$$

we have $\omega_3(t) = \omega_3(t_O)$ being a constant value. Using $\frac{h_1^2}{2I_1T} + \frac{h_2^2}{2I_2T} + \frac{h_3^2}{2I_3T} = 1$, the kinetic energy ellipsoid has identical semi-axis in the h_1 and h_2 direction, with an unique semi-axis in the h_3 direction. Intersecting this energy constraint surface with the spherical momentum constraint surface leads to circular intersection trajectories about the h_3 as illustrated in the figure below. Because we can assume $I_1 = I_2 > I_3$, the resulting **ω** trajectory orbit the h_3 axis in a clockwise fashion.



a) Axisymmetric body with $I_1 = I_2 > I_3$

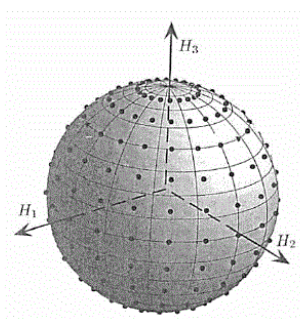
Remark: On the momentum sphere (h_1, h_2) equatorial plane an interesting motion occurs. Note that here $h_3 = \omega_3 = 0$, and for this axisymmetric body we know that $\omega_3(t) = \omega_3(t_O) = 0$. Looking at the Euler´s equation shown above, we find that in this case we have $\dot{\omega}_1 = \dot{\omega}_2 = 0$. This means that if $\omega_3 = 0$, then ω_1 and ω_2 will have constant values. In the polhode plot this constant angular rate condition is illustrated by having dots on the momentum sphere equator instead of trajectories.

Symmetric case ($I_1 = I_2 = I_3$):

In this case the Euler equations are given by

$$\begin{aligned}
\dot{\omega}_1 &= -\tfrac{(I_3-I_2)}{I_1}\omega_2\omega_3 = 0 \\
\dot{\omega}_2 &= -\tfrac{(I_1-I_3)}{I_2}\omega_1\omega_3 = 0 \\
\dot{\omega}_3 &= -\tfrac{(I_2-I_1)}{I_3}\omega_2\omega_1 = 0
\end{aligned}$$

All $\omega_i(t)$ will be constant. Geometrically, this condition indicates that the energy ellipsoid has become a sphere that is identical to the momentum sphere. Thus, every point on the sphere is an intersection of the momentum and energy constraints. The polhode plot is shown in the figure below and illustrates this behavior by plotting a discrete set of solutions as points.



b) Body with $I_1 = I_2 = I_3$

Problem 4:

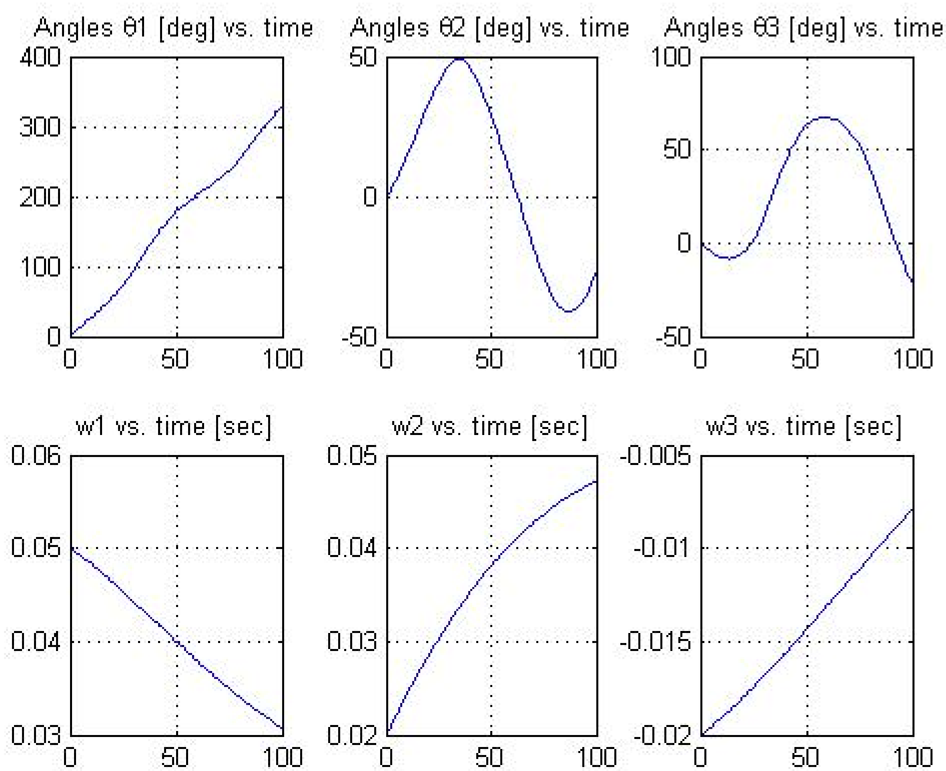
For this problem we need the complete attitude description, therefore we use the solution of Euler´s equation for the ωi and the solution of the attitude kinematics to give a complete description of the satellite.

The angular velocity vector components are integrated using Euler´s equations with torque vector equals zero (**T** = **0**).

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= 0 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 &= 0 \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 &= 0 \end{aligned}$$

The 3-2-1 Euler angles are integrated using

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \frac{1}{\cos\theta_2} \begin{bmatrix} \cos\theta_2 & \sin\theta_1\sin\theta_2 & \cos\theta_1\sin\theta_2 \\ 0 & \cos\theta_1\cos\theta_2 & -\sin\theta_1\cos\theta_2 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}.$$



Matlab script:

```
function ProblemLecture9test
% initial Euler angles and velocities in radians
theta1 = 0;
theta2 = 0;
theta3 = 0;
w1=0.05;
w2=0.02;
w3=-0.02;

[time,dot_x] = ode45(@diff_eqn, [0 100], [theta1;theta2;theta3;w1;w2;w3], odeset('RelTol',1e-6));
%plotting
subplot(231);plot(time,radtodeg(dot_x(:,1)));grid;title('Angles \theta1 [deg] vs. time')
subplot(232);plot(time,radtodeg(dot_x(:,2)));grid;title('Angles \theta2 [deg] vs. time')
subplot(233);plot(time,radtodeg(dot_x(:,3)));grid;title('Angles \theta3 [deg] vs. time')
subplot(234);plot(time,dot_x(:,4));grid;title('w1 vs. time [sec]')
subplot(235);plot(time,dot_x(:,5));grid;title('w2 vs. time [sec]')
subplot(236);plot(time,dot_x(:,6));grid;title('w3 vs. time [sec]')


% dynamic equatation and kinemtaic equation for 3-2-1 sequence
function dot_x = diff_eqn(time,x)
% inertia matrix
I1=210;
I2=200;
I3=118;
theta1 = x(1);
theta2 = x(2);
theta3 = x(3);
w1=x(4);
w2=x(5);
w3=x(6);

dot_w1 = - inv(I1) * (I3-I2)*w2*w3;
dot_w2 = - inv(I2) * (I1-I3)*w1*w3;
dot_w3 = - inv(I3) * (I2-I1)*w1*w2;

dot_angles = 1/cos(theta2)*  [ cos(theta2)    sin(theta1)*sin(theta2)      cos(theta1)*sin(theta2)
                                0             cos(theta1)*cos(theta2)     -sin(theta1)*cos(theta2)
                                0                  sin(theta1)                     cos(theta1) ]* [w1; w2; w3];
dot_x=[dot_angles; dot_w1; dot_w2; dot_w3];
```