DOC 221 Dinámica orbital y control de actitud

Solutions to Problems Lecture ADCS - I

Problem 1:

Use:

$$F = mg \Leftrightarrow G\tfrac{mM}{r^2} = mg \Leftrightarrow \tfrac{GM}{r^2} = g \Leftrightarrow \tfrac{\mu}{r^2} = g$$

$$\text{Earth:} \quad \mu = 398600\ \text{km}^3/\text{s}^2 \qquad R_E = 6378\ \text{km}$$

The radial acceleration due to gravitational force is:

- **(a)** g = 9.82 m/s^2
- **(b)** g = 7.75 m/s^2
- **(c)** g = 0.56 m/s^2
- **(d)** g = 0.22 m/s^2

Problem 2:

- **(a)** $E = hc/\lambda$ with $c = \lambda v \rightarrow E = hv$ where v is the frequency.
- **(b)** Use $c \approx 3 \cdot 10^8$ m s^{-1} and $h = 6.63 \cdot 10^{-34}$ m^2 kg s^{-1} and [J] = kg·m^2·s-^{2}
  → Green photon $E = 3.79 \cdot 10^{-19}$ J/photon
  → Radio wave $E = 1.99 \cdot 10^{-25}$ J/photon
  → Gamma ray $E = 1.99 \cdot 10^{-13}$ J/photon

Problem 3:

(a) Stefan-Boltzmann law with $I = \sigma T^4$ describes the power radiation from a black body as a functions of its temperature.
The integration of Planck´s distribution over all wavelengths gives Stefan-Boltzmann law.

$$\text{Planck´s distribution:} \quad \tilde{I}(T,\lambda) = \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1}$$

$$\text{Stefan-Boltmann law:} \quad I(T) = \int_0^\infty \tilde{I}(T,\lambda)\mathrm{d}\lambda = \int_0^\infty \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1}\mathrm{d}\lambda = ... = \sigma T^4$$

(b) To find Wien's law, we notice that for a particular temperature, the Planck spectrum predicts a maximum intensity I_{max} corresponding to λ_{max}, where the slope of the Planck spectrum (I versus λ) is zero. This means take derivative of λ of Planck spectrum.

$$\frac{\mathrm{dI}(T,\lambda)}{\mathrm{d}\lambda} = 0 = -5\frac{2hc^2}{\lambda^6}\frac{1}{\mathrm{e}^{hc/\lambda kT}-1} + \frac{2hc^2}{\lambda^5}\frac{-1}{(\mathrm{e}^{hc/\lambda kT}-1)^2}\mathrm{e}^{hc/\lambda kT}\frac{hc}{\lambda^2 kT}$$
$$= \frac{2hc^2}{\lambda^5}\frac{1}{(\mathrm{e}^{hc/\lambda kT}-1)^2}\left(5-5\mathrm{e}^{hc/\lambda kT} + \frac{hc}{\lambda kT}\mathrm{e}^{hc/\lambda kT}\right)$$

The numerator must equal zero

$$0 = \left(5-5\mathrm{e}^{hc/\lambda kT} + \frac{hc}{\lambda kT}\mathrm{e}^{hc/\lambda kT}\right)$$

$$5 = (5-\frac{hc}{\lambda kT})\mathrm{e}^{hc/\lambda kT} = (5-x)\mathrm{e}^x$$

The solution to this equation gives the value of λ_{max} corresponding to I_{max}.

$$x = hc/\lambda_{\mathrm{max}} kT = 4.96511$$

Using values for h, c and k (k=1.38·10^{-23} m^2 kg s^{-2} K^{-1}) we find

$$\lambda_{\mathrm{max}} = \frac{hc}{4.97 \cdot kT} = \frac{b}{T} \quad \text{with} \quad b = 2.898\times10^{-3}\mathrm{m\ K}$$

Problem 4:

To explain the atmospheric loss we imagine a volume of gas at the top of a planet's atmosphere. The gas will have a range of velocities described by the Maxwell-Boltzmann distribution.

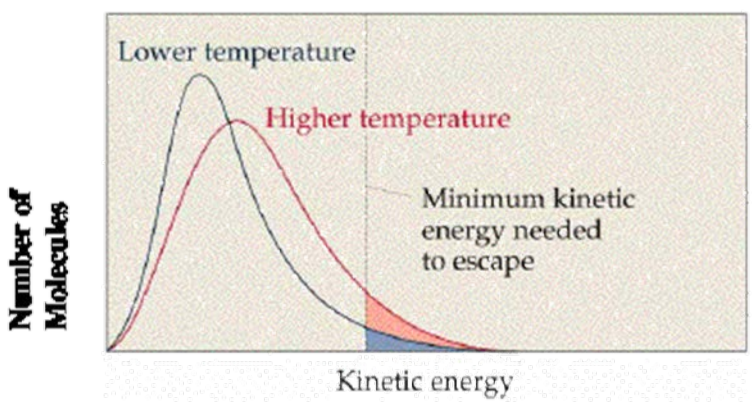
The Maxwell-Boltzmann distribution is given by

$$f(v) = \sqrt{\left(\frac{m}{2\pi kT}\right)^3} 4\pi v^2 e^{-\frac{mv^2}{2kT}}$$

Where *f(v)* gives the probability that a given particle/molecule has the velocity *v*.

The m is the particle mass and *kT* is the product of Boltzmann constant and temperature.

The Maxwell-Boltzmann distribution has following form:



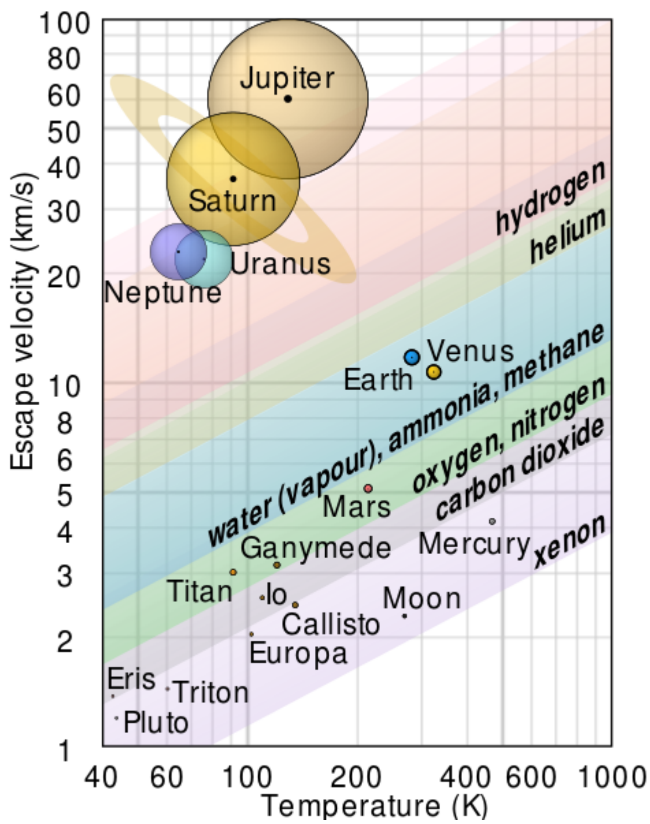
To escape, atmospheric gases must attain escape velocity, which is the minimum needed to overcome gravity.

Therefore, if a particle is moving sufficiently fast (i.e. with a speed greater than the escape velocity) and moving away from the planet, it can escape into space. If the escape velocity is similar to the most probable particle velocity given by thermal velocity, the gas will be depleted fairly quickly. However, typically the escape speed is very far into the high-speed tail of the Maxwell-Boltzmann distribution (see figure) so only a very few particles will be able to escape. But once these particles are lost, the high-speed tail will be replenished and then lost again. Thus, even if the escape speed is well into the high-speed tail, it can slowly in time loose the entire atmosphere.

Moreover, light particles, like hydrogen and helium, typically move faster than heavier ones, like oxygen and nitrogen. The light particles are more likely to reach escape velocity and escape to space.

The constitution of planetary atmospheres has been determined by the interplay between escape velocity and the Maxwell-Boltzmann distribution over the 4600 million years history of the solar system.

From wikipedia: Graphs of escape velocity against surface temperature of some solar system objects showing which gases are retained. The objects are drawn to scale, and their data points are at the black dots in the middle.



	Jupiter	Earth	Mars
Temperature [K]	120	290	210
Escape velocity [km/s]	60	11.4	5.0
Composition	H_2 and He	N_2 and O_2	CO_2

Problem 5:

The equation of hydrostatic equilibrium is given by:

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -g(r)\rho(r) = -G\frac{M(r)}{r^2}\rho_0$$

Express the *M(r)* term simply by the mean density times the volume up to radius *r*:

$$M(r) = \frac{4\pi}{3}\rho_0 r^3$$

Inserting this in above equation:

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{4\pi}{3}G{\rho_0}^2 r$$

Multiplying both sides by d*r* and integrate from radius *r* to *R* with the assumption that *p(R)* = 0 (in space) →

$$\int_{p(r)}^{p(R)} \mathrm{d}p = 0 - p(r) = -\frac{4\pi}{3}G{\rho_0}^2\int_r^R r\mathrm{d}r = -\frac{2\pi}{3}G{\rho_0}^2(R^2 - r^2)$$

$$\Rightarrow p(r) = \frac{2\pi}{3}G{\rho_0}^2(R^2 - r^2)$$