DOC 221 Dinámica orbital y control de actitud

Solutions to Problems Lecture ADCS - X

Problem 1:

This is the problem considered in detail from page 31 onwards in the lecture ADCS - X. We saw there that the closed-loop poles are given by

$$s = -\zeta\omega_n \pm \mathrm{j}\omega_d \,,$$

where

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

and

$$\omega_n^2 = \frac{K_p}{I} \quad , \quad 2\zeta\omega_n = \frac{K_d}{I} \,.$$

The maximum overshoot is then given by (page 50 in the lecture)

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \times 100\%$$

The settling time is given by (page 52 in the lecture)

$$t_s = \frac{4.4}{\zeta\omega_n}$$

It is immediately clear that the settling time requirement specifies the real part of the closed-loop poles, that is,

$$\zeta\omega_n = \frac{4.4}{t_s} = 0.0733$$

The maximum overshoot depends on the ratio of real to imaginary part of the closed-loop poles. Since we know the real part already, the maximum overshoot requirement then specifies the imaginary part. That is,

$$M_p = e^{-\pi\zeta\omega_n/\omega_d} \times 100\% = 20\% \,,$$

which rearranges to give

$$\omega_d = \frac{-\pi\zeta\omega_n}{\ln 0.2} = 0.1431 \,.$$

We have now found that to satisfy the transient specifications, the closed-loop poles must be

$$s = -0.0733 \pm \mathrm{j}0.1431.$$

Now, we can compute the gains K_p and K_d. As shown on page 59 in the lecture, $\omega_n = |s|$. Therefore,

$$K_p = I\omega_n^2 = I|s|^2 = 0.0259 \text{ Nm/rad}.$$

Likewise,

$$K_d = 2\zeta\omega_n I = 0.1467 \text{ Nms/rad}.$$

Therefore, the control law becomes

$$u = 0.0259e - 0.1467\dot{y}$$

Problem 2:

(a)

With the given feedback control structure, the closed-loop transfer function form Θ_d to Θ is given by

$$\frac{\hat{\theta}(s)}{\hat{\theta}_d(s)} = \frac{K_p / I}{s^2 + (K_d / I)s + K_p / I} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2},$$

where

$$\omega_n^2 = \frac{K_p}{I}, \quad 2\zeta\omega_n = \frac{K_d}{I}.$$

Therefore, with the given information, we now compute

$$\omega_n = \sqrt{\frac{K_p}{I}} = 0.1 \text{ rad/s}.$$

$$\zeta = \frac{K_d}{2\omega_n I} = 0.25.$$

Since 0 < ζ < 1, it follows that the closed-loop system is underdamped, and the expressions from lecture can be used.

The settling time is now found from (page 52 in the lecture)

$$t_s = \frac{4.4}{\zeta\omega_n} = 176 \text{ s}.$$

(b)

The percent overshoot is given by (page 50 in the lecture)

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \times 100\% = 44.4\%.$$

(c)

The rise time is given by (page 48 in the lecture)

$$t_r = \frac{\pi - \beta}{\omega_d},$$

where

$$\beta = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = 1.318 \text{ rad},$$

and

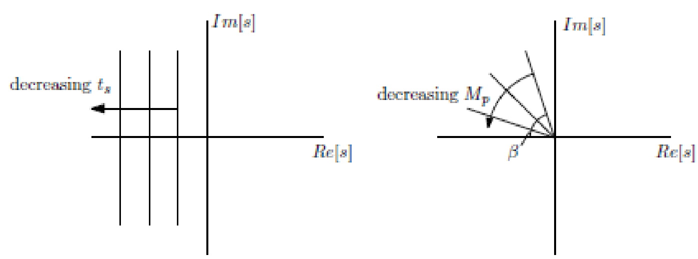
$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 0.0968 \text{ rad/s}.$$

Therefore, the rise-time is

$$t_r = 18.84 \text{ s}.$$

(d)

The settling time depends only on the real of the closed-loop pole. The percent overshoot depends only on the damping ratio, which in turn is specified by the ratio of real to imaginary parts of the closed-loop pole. These are illustrated graphically in the figure below. Therefore, a decrease in both settling time and overshoot require a simultaneous shift left in the closed-loop pole location as well as a reduction in angle β that the pole makes with the negative real-axis.



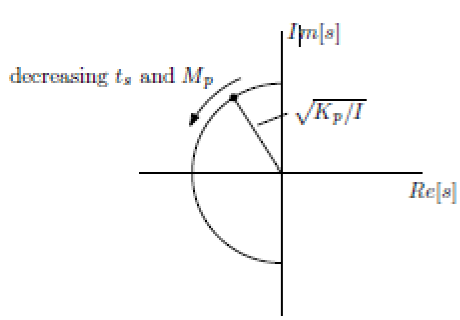
Now, the proportional gain satisfies

$$\frac{K_p}{I} = \omega_n^2 = |s|^2 .$$

Therefore, a line of constant proportional gain K_p in terms of closed-loop pole location is a semi-circle of radius

$$\omega_n = \sqrt{K_p / I} .$$

As shown in the figure below, it is indeed possible to decrease both t_s and M_p without changing K_p, by rotating the closed-loop pole counterclockwise.



Problem 3:

(a)

The actuator has dynamics

$$\dot{u} = -\tfrac{1}{T_a}(u - u_c)$$

Taking Laplace transforms, we have

$$sU(s) = -\frac{1}{T_a}\left[U(s) - U_c(s)\right].$$

which leads to the actuator transfer function

$$\frac{U(s)}{U_c(s)} = \frac{1}{T_a s + 1}.$$

(b)

The attitude dynamics are

$$I\ddot{\theta} = u + T_d.$$

Therefore, with y = Θ, taking Laplace transforms gives

$$s^2 IY(s) = U(s) + \hat{T}_d(s),$$

which leads to

$$Y(s) = \frac{1}{Is^2}\left[U(s) + \hat{T}_d(s)\right],$$

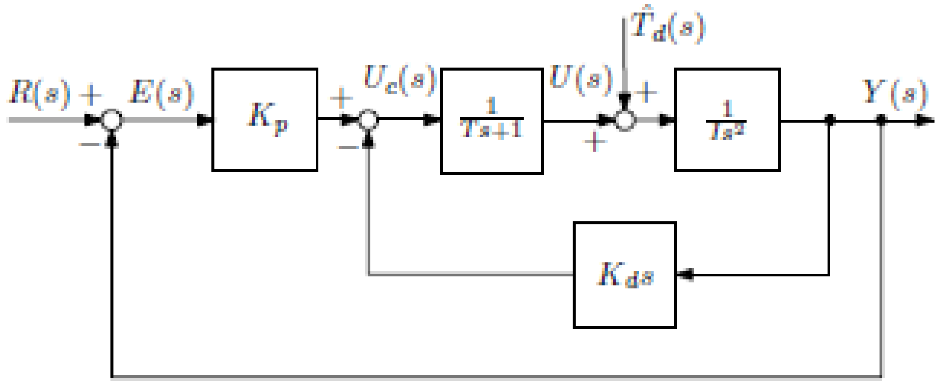
which is a transfer function representation of the plant. The control law is given by

$$u_c(t) = K_p e(t) - K_d \dot{y}(t),$$

where $e = r - y$. Taking Laplace transforms,

$$U_c(s) = K_p E(s) - sK_d Y(s),$$

where $E(s) = R(s) - Y(s)$. The control law has two inputs; the error $E(s)$ and the plant output $Y(s)$. Together with the actuator transfer function obtained in part **(a)**, the block diagram for the closed-loop system can now be drawn as shown in the figure below.



Problem 4:

(a)

When $\omega_x = \varepsilon_x$, $\omega_y = \varepsilon_y$, the equation for the spin-rate becomes

$$I_z\dot{\omega}_z = (I_x - I_y)\varepsilon_x\varepsilon_y + T_{zc} + T_{zd}$$

For small ε_x and ε_y, we neglect their product, to obtain

$$\dot{\omega}_z = \tfrac{T_{zc}}{I_z} + \tfrac{T_{zd}}{I_z}$$

(b)

Setting $y = \omega_z$, and $u = T_{cz}$, taking Laplace transforms of the equation obtained in part **(a)** gives

$$sY(s) = \frac{U(s) + \hat{T}_{dz}(s)}{I_z},$$

which rearranges to give

$$Y(s) = \frac{1}{I_z s}\left[U(s) + \hat{T}_{dz}(s)\right].$$

Therefore, we identify the plant transfer function as

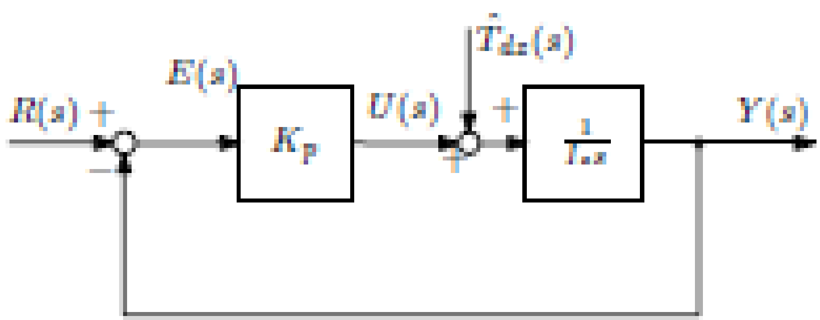
$$G_p(s) = \frac{1}{I_z s}.$$

(c)

With $e = r - y$, the proportional control law in the Laplace domain becomes

$$U(s) = K_p E(s) = K_p\left[R(s) - Y(s)\right].$$

The corresponding block diagram for the closed-loop system is presented in the figure below.



(d)

Assuming that $\hat{T}_{dz}(s) = 0$, we obtain the closed-loop transfer function relationship from R(s) to E(s). First, we compute

$$Y(s) = \frac{G_p(s)G_c(s)}{1 + G_p(s)G_c(s)} R(s).$$

Noting that the control transfer function is simply $G_c(s) = K_p$, we have

$$Y(s) = \frac{K_p}{I_z s + K_p} R(s).$$

Next, noting that $E(s) = R(s) - Y(s)$, we have

$$E(s) = \left(1 + \frac{K_p}{I_z s + K_p}\right) R(s) = \frac{I_z s}{I_z s + K_p} R(s).$$

Next, assuming that R(s) = 0, we obtain the closed-loop transfer function relationship from $\hat{T}_{\mathrm{dz}}(\mathrm{s})$ to E(s). First, we compute

$$Y(s) = \frac{G_p(s)}{1 + G_p(s)G_c(s)} \hat{T}_d(s) = \frac{1}{I_z s + K_p} \hat{T}_{dz}(s).$$

Next, we note that with R(s) = 0, the error becomes E(s) = −Y(s). Therefore,

$$E(s) = -\frac{1}{I_z s + K_p} \hat{T}_{dz}(s).$$

It is clear from the obtained transfer function relationships that the closed-loop system as a single pole, given by

$$s = -\frac{K_p}{I_z}.$$

Therefore, for closed-loop asymptotic stability, the proportional gain must satisfy

$$K_p > 0.$$

(e)

With a step disturbance $T_{\mathrm{dz}}(\mathrm{t}) = \overline{T}_{\mathrm{dz}}$, we have

$$\hat{T}_{dz}(s) = \frac{\overline{T}_{dz}}{s}.$$

From part **(d)**, we then obtain

$$E(s) = \frac{\overline{T}_{dz}}{s(I_z s + K_p)} = -\frac{\overline{T}_{dz}}{I_z} \frac{1}{s(s + K_p / I_z)}.$$

We now make use of the following partial fraction expansion

$$\frac{1}{s(s+K_p/I_z)} = \frac{I_z}{K_p}\left(\frac{1}{s} - \frac{1}{s+K_p/I_z}\right),$$

to obtain

$$E(s) = -\frac{\hat{T}_{dz}}{K_p}\left(\frac{1}{s} - \frac{1}{s+K_p/I_z}\right).$$

Taking inverse Laplace transforms gives the step response

$$e(t) = -\frac{\hat{T}_{dz}}{K_p}\left(1 - e^{-(K_p/I_z)t}\right).$$

The steady-state error is then

$$e_{ss} = \lim_{t\to\infty} e(t) = -\frac{\overline{T}_{dz}}{K_p}.$$

Therefore, to keep the magnitude of the steady-state error below e_{max}, we must have

$$\frac{\left|\overline{T}_{dz}\right|}{K_p} \le e_{\max},$$

which leads to

$$K_p \ge \frac{\left|\overline{T}_{dz}\right|}{e_{\max}}.$$

With e_{max} = 1 deg/s = π /180 rad/s, and $\overline{\mathrm{T}}_{\mathrm{d}} = 10^{-5}\ \mathrm{Nm}$, the proportional gain must satisfy

$$K_p \ge 5.73\times 10^{-4}\ \mathrm{Nms/rad}.$$

(f)

With a step reference signal r(t) = $\overline{v}$, we have

$$R(s) = \frac{\overline{v}}{s}.$$

From part **(d),** we obtain

$$E(s) = \frac{I_z s}{I_z s + K_p} R(s) = \frac{I_z s}{I_z s + K_p} \frac{\overline{v}}{s} = \frac{\overline{v}}{s + K_p / I_z}.$$

Taking inverse Laplace transforms, we obtain the step response

$$e(t) = \overline{v} e^{-(K_p/I_z)t}.$$

To ensure that

$$\frac{|e(t)|}{\overline{v}} \le 0.02,$$

within 10 seconds, we need to choose K_p such that

$$e^{-(K_p/I_z)10} \le 0.02,$$

which leads to

$$K_p \ge \frac{-I_z \ln 0.02}{10} = 3.1296 \text{ Nms/rad}.$$

Problem 5 (optional):

Do the exercise self-control proposed in page 102 of lecture ADCS - X.