Escuela Tecnica Superior de Ingenieria Aeronautica y del Espacio

ACDS - Problems Lecture VI

Author:

Pablo Zapatero Montaña

Professor:

Salvatore Mangano

September 2021

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1 Problem 1

Assume 3 reference frames A,B and I. Let the two reference frames A and B be defined relative to the intertial reference frame I by the orthonormal unit base vectors:

$$\begin{aligned}
\vec{a}_1 &= \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix} \qquad \vec{a}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \vec{a}_3 = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix} \\
\vec{b}_1 &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{b}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{b}_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}
\end{aligned} \tag{1}$$

where the a$_i$ and b$_i$ (i = 1,2,3) vectors components are written in the inertial frame I. Note that the unit base vectors of the inertial frame are

$$\vec{i}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{i}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{i}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{2}$$

a) Check that the unit base vectors a$_i$ respectively b$_i$ (i = 1,2,3) build an orthonormal reference frame.
b) Find the directional cosine matrix C$_{ab}$ that describes the orientation of frame A relative to frame B.
c) Find the directional cosine matrix C$_{ai}$ that describes the orientation of frame A relative to frame I.
d) Find the directional cosine matrix C$_{bi}$ that describes the orientation of frame B relative to frame I.
e) Check if C$_{ab}$=C$_{ai}$(C$_{bi}$)^T.
f) Check if C$_{ab}$(C$_{ab}$)^T=1, where 1 is 3x3 matrix.
g) For given arbitrary matrix A and matrix B check it the do not commute (AB$\neq$BA).
h) Is the following matrix

$$C = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{3}$$

a rotation matrix?

a) In order to assure that three apparent different vectors constitute an orthonormal reference frame is through the scalar product:

$$\begin{aligned}
\vec{a}_1 \cdot \vec{a}_2 = 0 \qquad \vec{a}_2 \cdot \vec{a}_3 = 0 \qquad \vec{a}_3 \cdot \vec{a}_1 = 0 \\
\vec{b}_1 \cdot \vec{b}_2 = 0 \qquad \vec{b}_2 \cdot \vec{b}_3 = 0 \qquad \vec{b}_3 \cdot \vec{b}_1 = 0
\end{aligned} \tag{4}$$

Then is proven that the three vectors from A and the three vectors from B form 90º between them so that it involves an orthogonal reference frame. To be orthonormal, the module of each vector has to be one:

$$
\begin{aligned}
\|\vec{a}_1\| = 1 \qquad \|\vec{a}_2\| = 1 \qquad \|\vec{a}_3\| = 1 \\
\|\vec{b}_1\| = 1 \qquad \|\vec{b}_2\| = 1 \qquad \|\vec{b}_3\| = 1
\end{aligned} \tag{5}
$$

b) To find the rotation matrix using the direction cosine matrix, we use the next expression:

$$
C^{a/b} = \begin{bmatrix} \vec{a}_1 \cdot \vec{b}_1 & \vec{a}_1 \cdot \vec{b}_2 & \vec{a}_1 \cdot \vec{b}_3 \\ \vec{a}_2 \cdot \vec{b}_1 & \vec{a}_2 \cdot \vec{b}_2 & \vec{a}_2 \cdot \vec{b}_3 \\ \vec{a}_3 \cdot \vec{b}_1 & \vec{a}_3 \cdot \vec{b}_2 & \vec{a}_3 \cdot \vec{b}_3 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \end{bmatrix} \tag{6}
$$

c) As the way explained in b):

$$
C^{a/i} = \begin{bmatrix} \vec{a}_1 \cdot \vec{i}_1 & \vec{a}_1 \cdot \vec{i}_2 & \vec{a}_1 \cdot \vec{i}_3 \\ \vec{a}_2 \cdot \vec{i}_1 & \vec{a}_2 \cdot \vec{i}_2 & \vec{a}_2 \cdot \vec{i}_3 \\ \vec{a}_3 \cdot \vec{i}_1 & \vec{a}_3 \cdot \vec{i}_2 & \vec{a}_3 \cdot \vec{i}_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \end{bmatrix} \tag{7}
$$

d) As the way explained in b):

$$
C^{b/i} = \begin{bmatrix} \vec{b}_1 \cdot \vec{i}_1 & \vec{b}_1 \cdot \vec{i}_2 & \vec{b}_1 \cdot \vec{i}_3 \\ \vec{b}_2 \cdot \vec{i}_1 & \vec{b}_2 \cdot \vec{i}_2 & \vec{b}_2 \cdot \vec{i}_3 \\ \vec{b}_3 \cdot \vec{i}_1 & \vec{b}_3 \cdot \vec{i}_2 & \vec{b}_3 \cdot \vec{i}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{8}
$$

e) Considering any vector $\vec{r}$ we can write it in different reference frames, using the rotation matrix:

$$
\vec{r}_i = C^{i/b} \vec{r}_b \tag{9}
$$

$$
\vec{r}_a = C^{a/b} \vec{r}_b \tag{10}
$$

Taking these to expression into:

$$
\vec{r}_a = C^{a/i} \vec{r}_i = C^{a/i} C^{i/b} \vec{r}_b \tag{11}
$$

So that, comparing to [10], and referring the properties of a rotation matrix:

$$
C^{a/b} = C^{a/i} C^{i/b} = C^{a/i} (C^{i/b})^T \tag{12}
$$

f) Using the same general vector we can write:

$$
\vec{r}_a = C^{a/b} \vec{r}_b = C^{a/b} C^{b/a} \vec{r}_a \tag{13}
$$

So:

$$C^{a/b}C^{b/a} = C^{a/b}(C^{a/b})^T = \mathbf{1} \tag{14}$$

g) Using the previous calculated matrix as example:

$$C^{a/i} \cdot C^{b/i} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \end{bmatrix} \tag{15}$$

$$C^{b/i} \cdot C^{a/i} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \end{bmatrix} \tag{16}$$

So:

$$C^{a/i} \cdot C^{b/i} \neq C^{b/i} \cdot C^{a/i} \tag{17}$$

h) It is not, because it does not comply with:

$$C^T = C^{-1} \tag{18}$$

As:

$$C^T = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{19}$$

$$C^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{20}$$

2 Problem 2

Show that the cross vector product can be written as follows:

$$\vec{a} \times \vec{b} = \vec{a}^x \vec{b} \tag{21}$$

where $\vec{a}^x$ is given by the following skew-symmetric matrix formed out of the elements of $\vec{a}$:

$$\vec{a}^x = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{22}$$

This skew-symmetric matrix has the property:

$$(\vec{a}^x)^T = -\vec{a}^x \tag{23}$$

Show also

$$\vec{a}^x\vec{b} = -\vec{b}^x\vec{a} \tag{24}$$

$$\vec{a}^x\vec{a} = 0 \tag{25}$$

where 0 is 3x1 matrix.

a) For this case, we calculate separately each side of the equation. In one hand:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i}_1 & \vec{i}_2 & \vec{i}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix} \tag{26}$$

In the other hand:

$$\vec{a}^x\vec{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix} \tag{27}$$

b) In this case we just need to show that $-\vec{b}^x\vec{a}$ returns the same result as before:

$$-\vec{b}^x\vec{a} = -\begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix} \tag{28}$$

c) For the last:

$$\vec{a}^x\vec{a} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{29}$$

3 Problem 3

a) Find the Euler rotation matrix C_{21} in terms of 3-2-3 Euler angles rotation sequence, with angles Θ_1, Θ_2, Θ_3. Specifically, frame 2 is obtained from frame 1 by:

- **A rotation Θ_1 about z-axis (3 axis) of frame 1.**
- **A rotation Θ_2 about the y-axis (2 axis) of intermediate frame.**
- **A rotation Θ_3 about the z-axis (3 axis) of the transformed frame.**

b) Find from the 3-2-3 Euler rotation matrix the appropriate Euler angles.

c) For the 3-2- Euler sequence, derive the following kinematic differential equation:

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \frac{1}{sin\theta_2} \begin{bmatrix} -cos\theta_3 & sin\theta_3 & 0 \\ sin\theta_3 sin\theta_2 & cos\theta_3 sin\theta_2 & 0 \\ cos\theta_3 cos\theta_2 & -sin\theta_3 cos\theta_2 & sin\theta_2 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \tag{30}$$

d) What are the points of singularity for this Euler rotation?

a) Following the dictated rotation:

$$C_{21}(\theta_1, \theta_2, \theta_3) = C_3(\theta_3)C_2(\theta_2)C_3(\theta_1) = ... \tag{31}$$

$$\begin{pmatrix} cos(\theta_1)cos(\theta_2)cos(\theta_3) - sin(\theta_1)sin(\theta_3) & sin(\theta_1)cos(\theta_2)cos(\theta_3) + cos(\theta_1)sin(\theta_3) & -sin(\theta_2)cos(\theta_3) \\ -cos(\theta_1)cos(\theta_2)sin(\theta_3) - sin(\theta_1)cos(\theta_3) & -sin(\theta_1)cos(\theta_2)sin(\theta_3) + cos(\theta_1)cos(\theta_3) & +sin(\theta_2)sin(\theta_3) \\ +cos(\theta_1)sin(\theta_2) & +sin(\theta_1)sin(\theta_2) & cos(\theta_2) \end{pmatrix} \tag{32}$$

b) Taking a quick look at the previous matrix, we can write:

$$\theta_1 = atan(\frac{C_{32}}{C_{31}}) \tag{33}$$

$$\theta_2 = acos(C_{33}) \tag{34}$$

$$\theta_3 = -atan(\frac{C_{23}}{C_{13}}) \tag{35}$$

c) To derive the kinematic differential equation, we need to define first two intermediate reference frames: A' after we make the first rotation (θ_1) and A" after we make the second rotation (θ_2). With this, we are able to write:

$$\omega^{B/A} = \omega^{B/A''} + \omega^{A''/A'} + \omega^{A'/A} = \dot{\theta}_3\vec{b}_3 + \dot{\theta}_2\vec{a''}_2 + \dot{\theta}_1\vec{a'}_1 \tag{36}$$

Which, once rewritten:

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{bmatrix} + \begin{bmatrix} \vec{a''}_1 & \vec{a''}_2 & \vec{a''}_3 \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\theta}_2 \\ 0 \end{bmatrix} + \begin{bmatrix} \vec{a'}_1 & \vec{a'}_2 & \vec{a'}_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \tag{37}$$

Considering the change between the next reference frames:

$$\begin{bmatrix} \vec{a''}_1 & \vec{a''}_2 & \vec{a''}_3 \end{bmatrix} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} C_3(\theta_3) \tag{38}$$

$$\begin{bmatrix} \vec{a'}_1 & \vec{a'}_2 & \vec{a'}_3 \end{bmatrix} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} C_3(\theta_3)C_2(\theta_2) \tag{39}$$

We finally can write:

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} -sin(\theta_2)cos(\theta_3)\dot{\theta}_1 + sin(\theta_3)\dot{\theta}_2 \\ sin(\theta_2)sin(\theta_3)\dot{\theta}_1 + cos(\theta_3)\dot{\theta}_2 \\ cos(\theta_2)\dot{\theta}_1 + \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} -sin(\theta_2)cos(\theta_3) & sin(\theta_3) & 0 \\ sin(\theta_2)sin(\theta_3) & cos(\theta_3) & 0 \\ cos(\theta_2) & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \tag{40}$$

And also have:

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \frac{1}{sin(\theta_2)} \begin{bmatrix} -cos(\theta_3) & sin(\theta_3) & 0 \\ sin(\theta_2)sin(\theta_3) & sin(\theta_2)cos(\theta_3) & 0 \\ cos(\theta_2)cos(\theta_3) & -cos(\theta_2)sin(\theta_3) & sin(\theta_2) \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \tag{41}$$

d) The singularity appears when the angular rate tends to infinity, which in this case is when:

$$\theta_2 = 0 \tag{42}$$

4 Problem 4

The orientation of an object is given in terms of the 3-2-1 Euler angles (-15º,25º,10º).

a) Write the direction cosine Euler rotation matrix C_{21}.

b) Find the principle Euler eigenaxis rotation angle ϕ.

c) Find the corresponding principal Euler rotation eigenaxis $\vec{e}$. Verify that $C_{21} \cdot \vec{e} = \vec{e}$.

d) Find the corresponding Euler parameters = Quaternions.

e) Is the last expression an unit quaternion? Has it magnitude one?

a) The Euler rotation matrix for a 3-2-1 Euler rotation is:

$$C_{21} = ...$$

$$... = \begin{bmatrix} cos(\theta_2)cos(\theta_3) & cos(\theta_2)sin(\theta_3) & -sin(\theta_2) \\ sin(\theta_1)sin(\theta_2)cos(\theta_3) - cos(\theta_1)sin(\theta_3) & sin(\theta_1)sin(\theta_2)sin(\theta_3) + cos(\theta_1)cos(\theta_3) & sin(\theta_1)cos(\theta_3) \\ cos(\theta_1)sin(\theta_2)cos(\theta_3) + sin(\theta_1)sin(\theta_3) & cos(\theta_1)sin(\theta_2)sin(\theta_3) - sin(\theta_1)cos(\theta_3) & cos(\theta_1)cos(\theta_3) \end{bmatrix} = ...$$

$$... = \begin{bmatrix} 0.8754 & -0.2346 & -0.4226 \\ 0.3258 & 0.9323 & 0.1574 \\ 0.3571 & -0.2755 & 0.8925 \end{bmatrix}$$

b) The expression for the Euler eigenaxis rotation angle is:

$$cos\phi = \frac{1}{2}[C_{11} + C_{22} + C_{33} - 1] \tag{43}$$

Which lead us to:

$$\phi = 31.78^{\circ} \tag{44}$$

c) The expression for the Euler rotation eigenaxis is:

$$\vec{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \frac{1}{2sin\phi} \begin{bmatrix} C_{23} - C_{32} \\ C_{31} - C_{13} \\ C_{12} - C_{21} \end{bmatrix} = \begin{bmatrix} 0.4110 \\ 0.7403 \\ -0.5321 \end{bmatrix} \tag{45}$$

Eigenaxis that satisfy the relation given by:

$$C_{21}\vec{e} = \vec{e} \tag{46}$$

d) The quaternions can be calculated as

$$\vec{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} e_1 sin(\frac{\phi}{2}) \\ e_2 sin(\frac{\phi}{2}) \\ e_3 sin(\frac{\phi}{2}) \end{bmatrix} = \begin{bmatrix} 0.1125 \\ 0.2027 \\ -0.1457 \end{bmatrix}$$
$$q_4 = cos(\frac{\phi}{2}) = 0.96188$$

e) The quaternion has a unit module:

$$q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1 \tag{47}$$

5 Problem 5

Let the orientations of two spacecraft A and B relative to an inertial frame I be given through the 3-2-1 Euler angles rotation sequences Θ_A = (60,-45,30)^T and Θ_B = (-15,25,10)^T degrees. What is the relative orientation of spacecraft A relative to B in terms of 3-2-1 Euler angles?

The Euler rotation matrix for a 3-2-1 Euler rotation, between A relative to I and B relative to I are:

$$C^{A/I} = ...$$
$$... = \begin{bmatrix} cos(\theta_2)cos(\theta_3) & cos(\theta_2)sin(\theta_3) & -sin(\theta_2) \\ sin(\theta_1)sin(\theta_2)cos(\theta_3) - cos(\theta_1)sin(\theta_3) & sin(\theta_1)sin(\theta_2)sin(\theta_3) + cos(\theta_1)cos(\theta_3) & sin(\theta_1)cos(\theta_3) \\ cos(\theta_1)sin(\theta_2)cos(\theta_3) + sin(\theta_1)sin(\theta_3) & cos(\theta_1)sin(\theta_2)sin(\theta_3) - sin(\theta_1)cos(\theta_3) & cos(\theta_1)cos(\theta_3) \end{bmatrix} = ...$$
$$... = \begin{bmatrix} 0.3536 & 0.6124 & -0.7071 \\ -0.9268 & 0.1268 & 0.2500 \\ 0.1268 & -0.7803 & 0.6124 \end{bmatrix}$$

$C^{B/I} = ...$

$$... = \begin{bmatrix} cos(\theta_2)cos(\theta_3) & cos(\theta_2)sin(\theta_3) & -sin(\theta_2) \\ sin(\theta_1)sin(\theta_2)cos(\theta_3) - cos(\theta_1)sin(\theta_3) & sin(\theta_1)sin(\theta_2)sin(\theta_3) + cos(\theta_1)cos(\theta_3) & sin(\theta_1)cos(\theta_3) \\ cos(\theta_1)sin(\theta_2)cos(\theta_3) + sin(\theta_1)sin(\theta_3) & cos(\theta_1)sin(\theta_2)sin(\theta_3) - sin(\theta_1)cos(\theta_3) & cos(\theta_1)cos(\theta_3) \end{bmatrix} = ...$$

$$... = \begin{bmatrix} 0.8754 & -0.2346 & -0.4226 \\ 0.3258 & 0.9323 & 0.1574 \\ 0.3571 & -0.2755 & 0.8925 \end{bmatrix}$$

And the relative orientation of spacecraft A relative to B is given by:

$$C^{A/B} = C^{A/I}(C^{B/I})^T = \begin{bmatrix} 0.4647 & 0.5748 & -0.6735 \\ -0.9472 & -0.1444 & -0.1428 \\ 0.0353 & -0.5898 & 0.8068 \end{bmatrix} \tag{48}$$

6 Problem 6

A spacecraft performs a 45 deg single principle Euler eigenaxis rotation about:

$$\vec{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{49}$$

Find the corresponding rotation matrix C and the corresponding 3-2-1 Euler angles that relate the final attitude to the original attitude.

The expression for the rotation matrix is:

$$\mathbf{C}(\vec{e}, \phi) = cos\phi\mathbf{1} + (1 - cos\phi)\vec{e}\vec{e}^T - sin\phi\mathbf{e}^x = \begin{bmatrix} 0.8047 & 0.5059 & -0.3106 \\ -0.3106 & 0.8047 & 0.5059 \\ 0.5059 & -0.3106 & 0.8047 \end{bmatrix} \tag{50}$$

And the corresponding Euler angles:

$$\theta_1 = atan(\frac{C_{23}}{C_{33}}) = 32.16° \tag{51}$$

$$\theta_2 = -asin(C_{13}) = -18.09° \tag{52}$$

$$\theta_3 = atan(\frac{C_{12}}{C_{11}}) = 32.16° \tag{53}$$