# ADCS - VIII
# Rigid Body Dynamics

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# Summary of last lecture

## Static attitude determination

- Attitude determination is **over-determinated** (if $\geq$2 vectors) or **under-determinated** (if only 1 vector)
- Need of two or more sensor measurements (at given time) which defines distinct vectors known in body and reference frame
- Vectors used in algorithm to estimate attitude (represented by rotation matrix, Euler angles, quaternions, etc.)
- Simplest algorithm **Triad method** (use of only two vectors)
- More accurate methods based on Wahba´s minimization problem (allows arbitrary weights of measurements and allows to use more than two measurements)
- **Analytical solution** to minimization Whaba´s problem **q-method**
- **Approximation** to minimization Whaba´s problem **QUEST-method**

# Outline

**Dynamics of particle**

**Dynamics of system of particles**

**Rigid body dynamics**

- Translational and rotational dynamics
- Angular momentum of rigid body
- Inertia matrix
- Principal axes
- Parallel axis theorem
- Kinetic energy

**Euler´s equations**

# Dynamics of particle

# Particle and point mass

**Particle:**

Particle represents any object whose size, shape and structure is irrelevant in given context

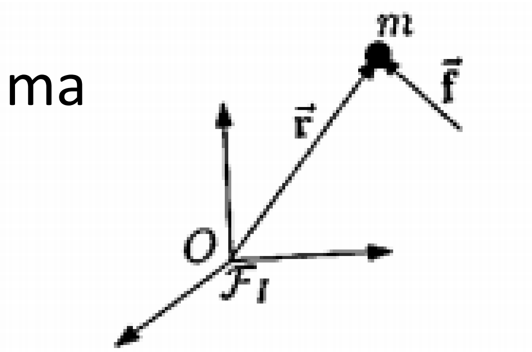
**Point mass:**

Point mass represents body whose dimensions are very small compared to other distances which are relevant to problem

Internal structure of body can be neglected and body is treated as if its mass is concentrated at one point in space

# Newton´s law and momentum

**Newton´s second law** says that for one particle f = ma




In vector form: $\boxed{m\ddot{\vec{\mathbf{r}}} = \vec{\mathbf{f}}}$ or $\dot{\vec{\mathbf{p}}} = \vec{\mathbf{f}}$

where $\boxed{\vec{\mathbf{p}} = m\dot{\vec{\mathbf{r}}}}$ is **momentum** of a particle

In components: $m\frac{\mathrm{d}^2}{\mathrm{d}t^2}x = f_x \quad m\frac{\mathrm{d}^2}{\mathrm{d}t^2}y = f_y \quad m\frac{\mathrm{d}^2}{\mathrm{d}t^2}z = f_z$

Newton´s second law is only valid if force and momentum are defined in an **inertial** coordinate system

What is an inertial coordinate system?

Inertial coordinate system **if it is NOT accelerating or rotating**

But origin of frame can translate at constant linear velocity

# Dynamics of one particle

One particle with mass *m* at position **r** which feels force **f** may be expressed as one-second order differential equation

$$m\ddot{\vec{\mathbf{r}}} = \vec{\mathbf{f}}$$

Or as two first order differential equations

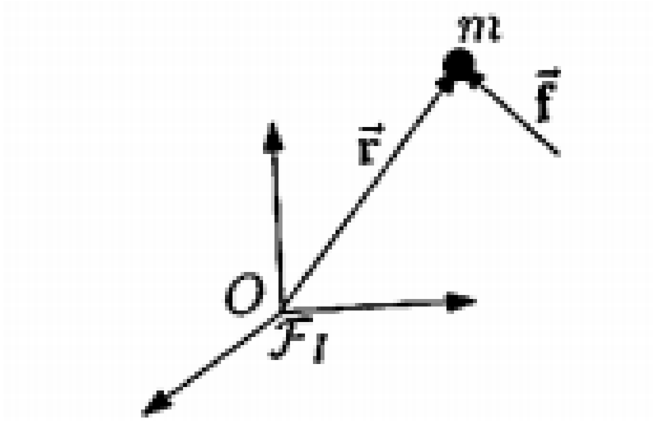
$$\vec{\mathbf{p}} = m\dot{\vec{\mathbf{r}}}$$

$$\dot{\vec{\mathbf{p}}} = \vec{\mathbf{f}}$$

# Angular momentum

**Angular momentum** (about O) is:

$$\boxed{\vec{\mathbf{h}}_O \triangleq \vec{\mathbf{r}}\times\vec{\mathbf{p}}} = m\,\vec{\mathbf{r}}\times\dot{\vec{\mathbf{r}}}$$




Time derivative of angular momentum:

$$\dot{\vec{\mathbf{h}}}_O = m\,\dot{\vec{\mathbf{r}}}\times\dot{\vec{\mathbf{r}}} + m\,\vec{\mathbf{r}}\times\ddot{\vec{\mathbf{r}}} = \vec{\mathbf{r}}\times\vec{\mathbf{f}}$$

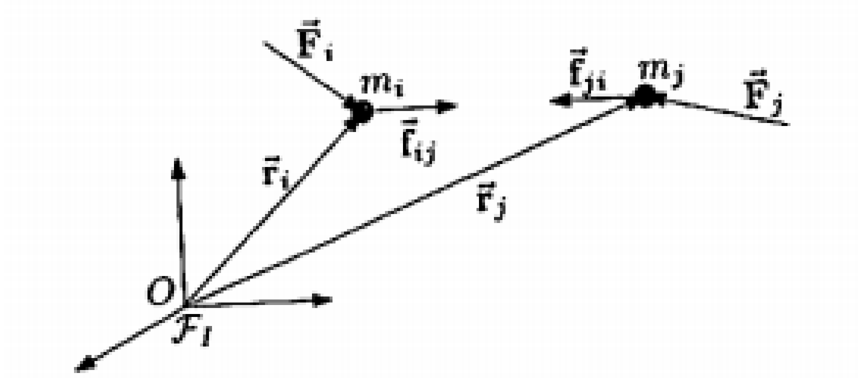
$$\boxed{\dot{\vec{\mathbf{h}}}_O = \vec{\mathbf{t}}_O} \quad , \quad \vec{\mathbf{t}}_O = \vec{\mathbf{r}}\times\vec{\mathbf{f}}$$

Quantity $\vec{\mathbf{t}}_O$ is torque (about O) produced by $\vec{\mathbf{f}}$

Time derivative of angular momentum is equal to external torque

# Dynamics of system of particles

# Application of Newton´s laws on system of particles




System of *N* particles with Masses $m_i$

Force acting on $m_i$ are:

- external forces $\vec{\mathbf{F}}_i$
- internal forces $\vec{\mathbf{f}}_{ij}$ (Newton´s 3rd law $\vec{\mathbf{f}}_{ij} = -\vec{\mathbf{f}}_{ji}$)

Newton´s 2nd law to ith particle $\quad m_i \ddot{\vec{\mathbf{r}}}_i = \vec{\mathbf{F}}_i + \sum_{j=1}^{N} \vec{\mathbf{f}}_{ij}$

Summing over all *N* masses $\quad \sum_{i=1}^{N} m_i \ddot{\vec{\mathbf{r}}}_i = \sum_{i=1}^{N} \vec{\mathbf{F}}_i + \sum_{i=1}^{N}\sum_{j=1}^{N} \vec{\mathbf{f}}_{ij}$

Due to $\quad \sum_{i=1}^{N}\sum_{j=1}^{N} \vec{\mathbf{f}}_{ij} = \vec{0}$ $\qquad \sum_{i=1}^{N} m_i \ddot{\vec{\mathbf{r}}}_i = \sum_{i=1}^{N} \vec{\mathbf{F}}_i$

# Motion for center of mass

Definition of center of mass:

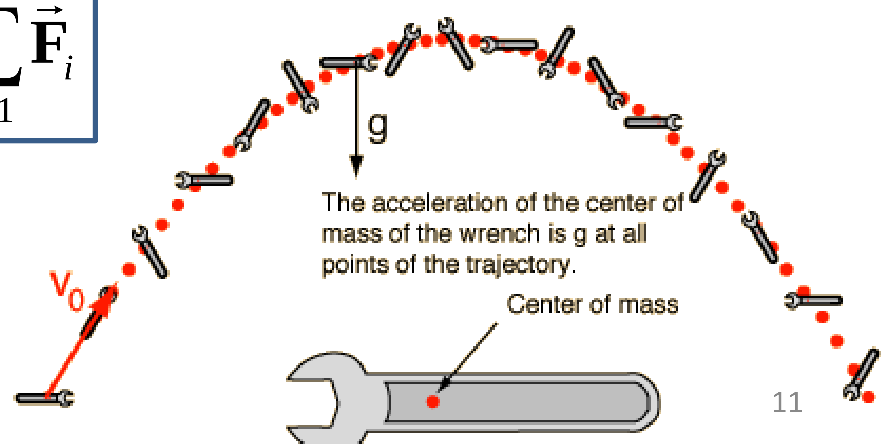
Averaged position of system of particles weighted by their masses

$$\vec{\mathbf{r}}_c = \frac{\sum_{i=1}^{N} m_i \vec{\mathbf{r}}_i}{m} \qquad \sum_{i=1}^{N} m_i = m$$

In last slide $\sum_{i=1}^{N} m_i \ddot{\vec{\mathbf{r}}}_i = \sum_{i=1}^{N} \vec{\mathbf{F}}_i$

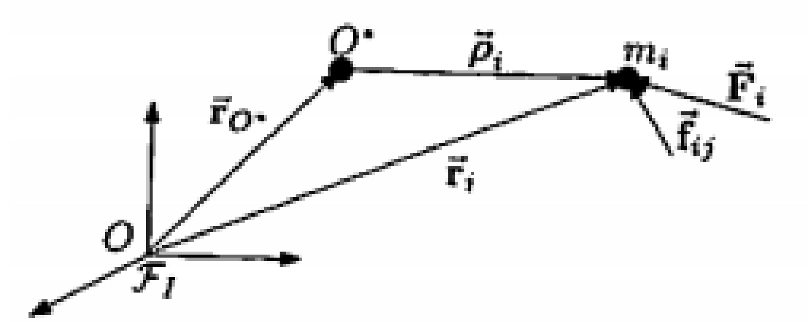
$$\boxed{m\ddot{\vec{\mathbf{r}}}_c = \vec{\mathbf{F}} \quad , \quad \vec{\mathbf{F}} = \sum_{i=1}^{N} \vec{\mathbf{F}}_i}$$

Equation of motion for
center of mass of system of particles
is same as for single particle



# Angular momentum of system of particles

Find angular equation of motion
about an arbitrary position O*




Position of i[th] particle $$\vec{\mathbf{r}}_i = \vec{\mathbf{r}}_{O*} + \vec{\rho}_i \quad , \quad i = 1, \ldots, N$$

Angular momentum of i[th] particle about O* is defined

$$\vec{\mathbf{h}}_{i,O*} \triangleq m_i \vec{\rho}_i \times \dot{\vec{\rho}}_i \quad , \quad i = 1, \ldots, N$$

Angular momentum of system of particles about O*

$$\vec{\mathbf{h}}_{O*} = \sum_{i=1}^{N} \vec{\mathbf{h}}_{i,O*} = \sum_{i=1}^{N} m_i \vec{\rho}_i \times \dot{\vec{\rho}}_i$$

# Angular momentum

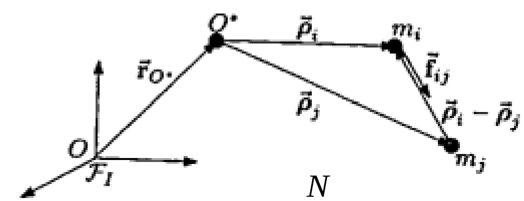
Take time derivative of $\vec{\mathbf{h}}_{O*} = \sum_{i=1}^{N} \vec{\mathbf{h}}_{i,O*} = \sum_{i=1}^{N} m_i \vec{\rho}_i \times \dot{\vec{\rho}}_i$

and use equation of motion of particles $m_i \ddot{\vec{\rho}}_i = \vec{\mathbf{F}}_i + \sum_{j=1}^{N} \vec{\mathbf{f}}_{ij} - m_i \ddot{\vec{\mathbf{r}}}_{O*}$

$$\dot{\vec{\mathbf{h}}}_{O*} = \sum_{i=1}^{N} \vec{\rho}_i \times \vec{F}_i + \sum_{i=1}^{N} \vec{\rho}_i \times \sum_{j=1}^{N} \vec{\mathbf{f}}_{ij} - \sum_{i=1}^{N} m_i \vec{\rho}_i \times \ddot{\vec{\mathbf{r}}}_{O*}$$

But force $\vec{\mathbf{f}}_{ij}$ and vector $(\vec{\rho}_i - \vec{\rho}_j)$ act along same line joining particle i and j

$$\sum_{i=1}^{N} \vec{\rho}_i \times \sum_{j=1}^{N} \vec{\mathbf{f}}_{ij} = \sum_{i=1}^{N} \sum_{j=1}^{N} \vec{\rho}_i \times \vec{\mathbf{f}}_{ij} = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} (\vec{\rho}_i - \vec{\rho}_j) \times \vec{\mathbf{f}}_{ij} = \vec{\mathbf{0}}$$




Angular equation of motion $\dot{\vec{\mathbf{h}}}_{O*} = \vec{\mathbf{T}}_{O*} - \sum_{i=1}^{N} m_i \vec{\rho}_i \times \ddot{\vec{\mathbf{r}}}_{O*}$ , $\vec{\mathbf{T}}_{O*} = \sum_{i=1}^{N} \vec{\rho}_i \times \vec{\mathbf{F}}_i$

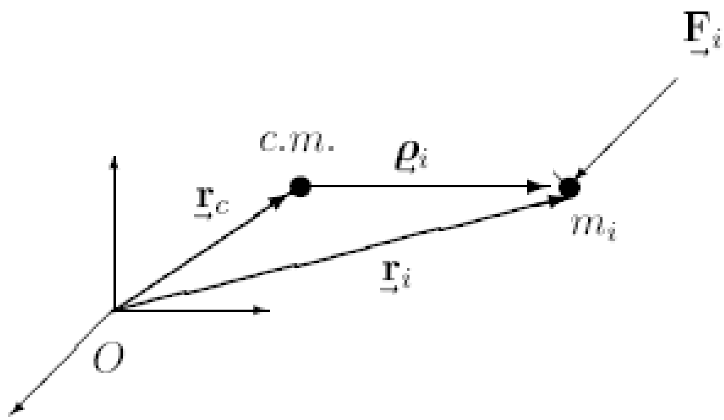
If point O* is center of mass

since by definition $\sum_{i=1}^{N} m_i \vec{\rho}_i = \vec{0}$

$$\boxed{\dot{\vec{\mathbf{h}}}_c = \vec{\mathbf{T}}_c \quad , \quad \vec{\mathbf{T}}_c = \sum_{i=1}^{N} \vec{\rho}_i \times \vec{\mathbf{F}}_i}$$

Torque about center of mass is equal to time derivative of angular momentum with respect to center of mass

# Summary for system of particles




Translational equation: $m\ddot{\vec{\mathbf{r}}}_c = \vec{\mathbf{F}} = \sum_{i=1}^{N} \vec{\mathbf{F}}_i$

Rotational equation: $\dot{\vec{\mathbf{h}}}_c = \vec{\mathbf{T}}_c$

with $\vec{\mathbf{h}}_c = \sum_{i=1}^{N} m_i \vec{\rho}_i \times \dot{\vec{\rho}}_i$ $\quad \vec{\mathbf{T}}_c = \sum_{i=1}^{N} \vec{\rho}_i \times \vec{\mathbf{F}}_i$

# Rigid body dynamics

# Rigid body

Definition:

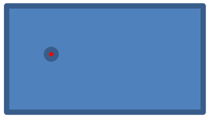
Rigid body is continuum in which distance between any two points on body remains fixed

In other words:

Rigid body is an idealization of a solid body of finite size in which deformation is neglected

Distance between any two given points of rigid body remains constant in time regardless of external forces acting on it

# Number of freedom of rigid body



Rigid body is described by its orientation and location

Location of rigid body is determined by position of any one point of body

Orientation is determined by relative position of all other points
of body relative to first selected point

Position of rigid body can be described by 3 independent coordinates

Orientation of rigid body can be described by 3 independent coordinates

Total number of independent coordinates necessary do
completely describe position and orientation of rigid body is 6

3 translation and 3 rotation = 6

# Rigid body dynamics

Concept of system of particles can be extended to rigid bodies by integration over all particles

Use number of particles become infinitely large and their masses infinitesimal small

$$\vec{\rho}_i \to \vec{\rho}$$

$$m_i \to \mathrm{d}m = \sigma(\vec{\rho})\mathrm{d}V$$

$$\vec{\mathbf{F}}_i \to \vec{\mathbf{f}}(\vec{\rho})\mathrm{d}V$$

$$\sum_{i=1}^{N} \to \int_V$$

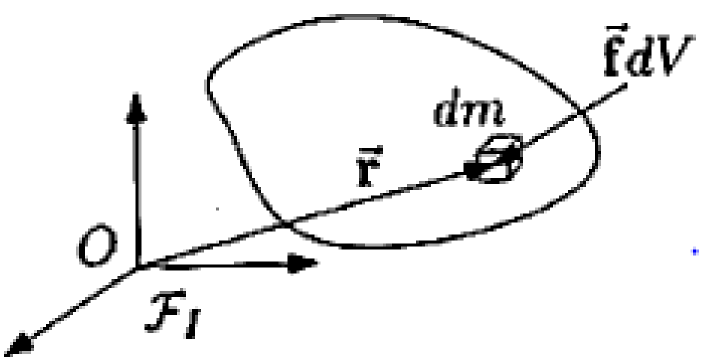
d*m* = infinitesimal mass element

d*V* = infinitesimal volume element

σ(ρ) = mass density at point ρ

f(ρ) = external force per unit volume

Integration is taken over entire volume *V* of body



# Translation dynamics of rigid body

Translation equation **for system of particles** **for rigid body**

$$\sum_{i=1}^{N} m_i \ddot{\vec{\mathbf{r}}}_i = \sum_{i=1}^{N} \vec{\mathbf{F}}_i \quad \rightarrow \quad \int_V \ddot{\vec{\mathbf{r}}}\,\mathrm{d}m = \int_V \vec{\mathbf{f}}\,\mathrm{d}V$$

Center of mass **for system of particles** **for rigid body**

$$m = \sum_{i=1}^{N} m_i \quad \text{and} \quad \vec{\mathbf{r}}_c = \frac{\sum_{i=1}^{N} m_i \vec{\mathbf{r}}_i}{m} \quad \rightarrow \quad m = \int_V \mathrm{d}m \quad \text{and} \quad \vec{\mathbf{r}}_c = \frac{\int_V \vec{\mathbf{r}}\,\mathrm{d}m}{m}$$

Differentiating twice

$$m\ddot{\vec{\mathbf{r}}}_c = \int_V \ddot{\vec{\mathbf{r}}}\,\mathrm{d}m$$

$$\boxed{m\ddot{\vec{\mathbf{r}}}_c = \vec{\mathbf{F}} \quad , \quad \vec{\mathbf{F}} \triangleq \int_V \vec{\mathbf{f}}\,\mathrm{d}V}$$

**Translation dynamics of rigid body**

**F** is total force acting on rigid body

Center of mass of rigid body behaves in translation like one point mass

# Rigid body motion

Motion of rigid body can be described by:

- Define $x_b$-$y_b$-$z_b$ frame (body frame) attached to rigid body

  Same direction as $x_i$-$y_i$-$z_i$ frame (inertial frame) at $t = 0$

  Origin fixed at one point of rigid body (e.g. center of mass)

- Use $T_{Translation}(t)$ to describe motion of origin
- Use $\mathbf{C}(t)$ to describe rotation of $x_b$-$y_b$-$z_b$ frame

  Use rotation matrix with $\theta_1$, $\theta_2$, $\theta_3$

  $\mathbf{C}(0) = \mathbf{1}$ with rotation matrix with $\theta_1(0) = \theta_2(0) = \theta_3(0) = 0$

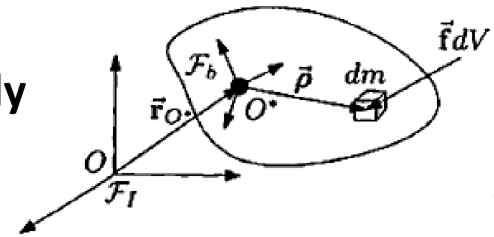
6 independent coordinates ($x$, $y$, $z$, $\theta_1$, $\theta_2$, $\theta_3$)

# Angular momentum of rigid body

Angular momentum **of system of particles** with respect to point O*

$$\vec{\mathbf{h}}_{O*} = \sum_{i=1}^{N} m_i \vec{\rho}_i \times \dot{\vec{\rho}}_i$$

**of rigid body**




$$\vec{\mathbf{h}}_{O*} = \int_V \vec{\rho} \times \dot{\vec{\rho}} \mathrm{d}m$$

Rate of change of vector (as seen in inertial frame)

$$\left\{\dot{\vec{\rho}}\right\}_i = \left\{\dot{\vec{\rho}}\right\}_b + \vec{\omega} \times \vec{\rho}$$

***ω*** = angular velocity of rigid body with respect to inertial frame

For rigid body in rotating frame all distances between points of rigid body are fixed

$$\left\{\dot{\vec{\rho}}\right\}_b = 0 \quad \Rightarrow \quad \left\{\dot{\vec{\rho}}\right\}_i = \vec{\omega} \times \vec{\rho}$$

**Angular momentum of rigid body:**

$$\boxed{\vec{\mathbf{h}}_{O*} = \int_V \vec{\rho} \times (\vec{\omega} \times \vec{\rho}) \mathrm{d}m}$$

# Moment of inertia matrix

Previous slide: $\vec{\mathbf{h}}_{O*} = \int_V \vec{\rho} \times (\vec{\omega} \times \vec{\rho}) \mathrm{d}m$

$$\vec{\rho} = \begin{bmatrix} \rho_x \\ \rho_y \\ \rho_z \end{bmatrix} \quad \vec{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

In body-fixed frame

Term $\vec{\rho} \times (\vec{\omega} \times \vec{\rho})$ can be written as

$$\begin{aligned} \vec{\rho} \times (\vec{\omega} \times \vec{\rho}) = \omega_x(\rho_y^2 + \rho_z^2) - \rho_x\rho_y\omega_y - \rho_x\rho_z\omega_z \\ -\rho_x\rho_y\omega_x + (\rho_x^2 + \rho_z^2)\omega_y - \rho_y\rho_z\omega_z \\ -\rho_x\rho_z\omega_x - \rho_y\rho_z\omega_y + \omega_z(\rho_y^2 + \rho_x^2) \end{aligned}$$

Double cross-product can be written down with help of

$$\vec{a} \times \vec{b} \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

Moment of inertia matrix about O* defined by

$$\mathbf{J} \triangleq \int_V \begin{bmatrix} (\rho_y^2 + \rho_z^2) & -\rho_x\rho_y & -\rho_x\rho_z \\ -\rho_x\rho_y & (\rho_x^2 + \rho_z^2) & -\rho_y\rho_z \\ -\rho_x\rho_z & -\rho_y\rho_z & (\rho_x^2 + \rho_y^2) \end{bmatrix} \sigma(\rho_x, \rho_y, \rho_z) \mathrm{d}V$$

# Alternative derivation of moment of inertia matrix

$$\vec{\mathbf{h}}_{O*} = \int_V \vec{\rho}\times(\vec{\omega}\times\vec{\rho})\mathrm{d}m = -\int_V \vec{\rho}\times(\vec{\rho}\times\vec{\omega})\mathrm{d}m$$

Angular momentum of rigid body about O*

$$\mathbf{h}_{O*} = -\int_V (\boldsymbol{\rho}^\times\boldsymbol{\rho}^\times\boldsymbol{\omega})\mathrm{d}m = \left[-\int_V \boldsymbol{\rho}^\times\boldsymbol{\rho}^\times\mathrm{d}m\right]\boldsymbol{\omega}$$

Angular momentum in body-fixed frame

$$= -\int_V \begin{bmatrix} 0 & -\rho_z & \rho_y \\ \rho_z & 0 & -\rho_x \\ -\rho_y & \rho_x & 0 \end{bmatrix}\begin{bmatrix} 0 & -\rho_z & \rho_y \\ \rho_z & 0 & -\rho_x \\ -\rho_y & \rho_x & 0 \end{bmatrix}\mathrm{d}m\boldsymbol{\omega}$$

Use cross-product-equivalent matrix

$$= \int_V \begin{bmatrix} (\rho_y^2+\rho_z^2) & -\rho_x\rho_y & -\rho_x\rho_z \\ -\rho_x\rho_y & (\rho_x^2+\rho_z^2) & -\rho_y\rho_z \\ -\rho_x\rho_z & -\rho_y\rho_z & (\rho_x^2+\rho_y^2) \end{bmatrix}\mathrm{d}m\boldsymbol{\omega}$$

$$\mathbf{J} \triangleq -\int_V \boldsymbol{\rho}^\times\boldsymbol{\rho}^\times\mathrm{d}m$$

$$= \int_V \begin{bmatrix} (\rho_y^2+\rho_z^2) & -\rho_x\rho_y & -\rho_x\rho_z \\ -\rho_x\rho_y & (\rho_x^2+\rho_z^2) & -\rho_y\rho_z \\ -\rho_x\rho_z & -\rho_y\rho_z & (\rho_x^2+\rho_y^2) \end{bmatrix}\sigma(\rho_x,\rho_y,\rho_z)\mathrm{d}V$$

Define moment of inertia matrix about O*

Therefor: $\boxed{\mathbf{h}_{O*} = \mathbf{J}\,\boldsymbol{\omega}}$

# Angular momentum and inertia matrix

$$\vec{\mathbf{h}}_{O*} = \int_V \vec{\rho} \times (\vec{\omega} \times \vec{\rho}) \mathrm{d}m$$

Angular momentum

$$\mathbf{h}_{O*} = \mathbf{J}\,\boldsymbol{\omega}$$

(3x3) matrix J is called inertia matrix

$$= \begin{bmatrix} J_{xx} & J_{xy} & J_{xz} \\ J_{yx} & J_{yy} & J_{yz} \\ J_{zx} & J_{zy} & J_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

Inertia matrix is real symmetric matrix (only 6 independent elements): $J_{ik} = J_{ki}$

$$\mathbf{J} \triangleq \int_V \begin{bmatrix} (\rho_y^2 + \rho_z^2) & -\rho_x \rho_y & -\rho_x \rho_z \\ -\rho_x \rho_y & (\rho_x^2 + \rho_z^2) & -\rho_y \rho_z \\ -\rho_x \rho_z & -\rho_y \rho_z & (\rho_x^2 + \rho_y^2) \end{bmatrix} \sigma(\rho_x, \rho_y, \rho_z) \mathrm{d}V$$

Moment of inertia matrix depends upon position and direction of axis of rotation

$$\mathbf{h}_c = \mathbf{I}\boldsymbol{\omega}$$

Special case: Point O* chosen as center of mass then label **I** instead of **J**

# Angular momentum with respect to center of mass

Angular momentum of system of particles
with respect to center of mass

Exactly same derivation as for O*
Choose O* as center of mass

$$\vec{\mathbf{h}}_c = \sum_{i=1}^{N} m_i \vec{\rho}_i \times \dot{\vec{\rho}}_i$$

$$\vec{\mathbf{h}}_c = \int_V \vec{\rho} \times \dot{\vec{\rho}} \mathrm{d}m$$

Rate of change of a vector
(as seen in inertial frame)

$$\left\{\dot{\vec{\rho}}\right\}_i = \left\{\dot{\vec{\rho}}\right\}_b + \vec{\omega} \times \vec{\rho}$$

***ω*** = angular velocity of rigid body
with respect to inertial frame

For rigid body in rotating frame all distances between points of rigid body are fixed

$$\left\{\dot{\vec{\rho}}\right\}_b = 0 \quad \Rightarrow \quad \left\{\dot{\vec{\rho}}\right\}_i = \vec{\omega} \times \vec{\rho}$$

Angular momentum becomes:

$$\vec{\mathbf{h}}_c = \int_V \vec{\rho} \times (\vec{\omega} \times \vec{\rho}) \mathrm{d}m$$

# Angular momentum with respect to center of mass

Choose center of mass as origin
about which angular momentum
is calculated (= center of rotation)

$$\mathbf{h}_c = \mathbf{I}\boldsymbol{\omega}$$

**I** = moment of inertia matrix about center of mass

$$\mathbf{I} \triangleq \int_V \begin{bmatrix} (\rho_y^2 + \rho_z^2) & -\rho_x\rho_y & -\rho_x\rho_z \\ -\rho_x\rho_y & (\rho_x^2 + \rho_z^2) & -\rho_y\rho_z \\ -\rho_x\rho_z & -\rho_y\rho_z & (\rho_x^2 + \rho_y^2) \end{bmatrix} \sigma(\rho_x, \rho_y, \rho_z)\mathrm{d}V$$

Same definition as for moment of inertia matrix **J**

Note:

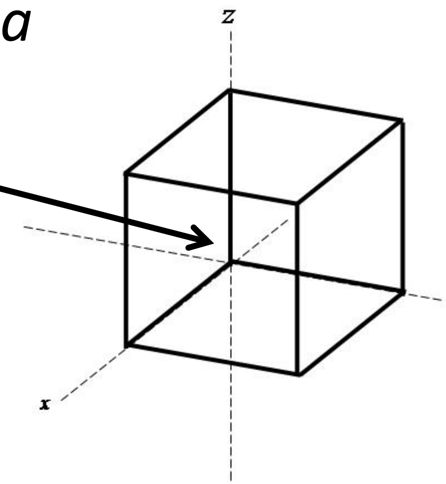
Inertia matrix derives from equal effect of angular rate on all particles

# Example: inertia matrix of homogeneous cube

Consider homogeneous cube of mass *M* and side *a*

**Choose origin at one of cube's corners**

(not center of mass)




$$\mathbf{J} \triangleq \int_V \begin{bmatrix} (\rho_y^2 + \rho_z^2) & -\rho_x\rho_y & -\rho_x\rho_z \\ -\rho_x\rho_y & (\rho_x^2 + \rho_z^2) & -\rho_y\rho_z \\ -\rho_x\rho_z & -\rho_y\rho_z & (\rho_x^2 + \rho_y^2) \end{bmatrix} \sigma(\rho_x, \rho_y, \rho_z)\mathrm{d}V$$

$$J_{xx} = \int_V (\rho_y^2 + \rho_z^2)\sigma(\rho_x, \rho_y, \rho_z)\mathrm{dV} = \sigma\int_0^a\int_0^a\int_0^a(\rho_y^2 + \rho_z^2)\mathrm{d}\rho_x\mathrm{d}\rho_y\mathrm{d}\rho_z$$

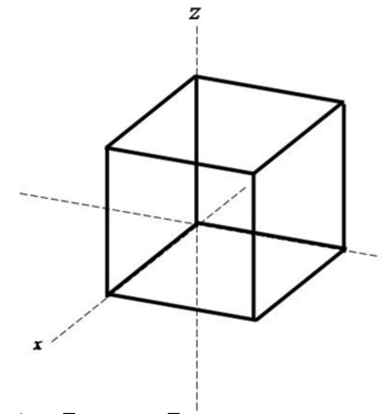
$$= \sigma a\int_0^a\int_0^a(\rho_y^2 + \rho_z^2)\mathrm{d}\rho_y\mathrm{d}\rho_z = \sigma a\int_0^a\int_0^a \rho_y^2\mathrm{d}\rho_y\mathrm{d}\rho_z + \sigma a\int_0^a\int_0^a \rho_z^2\mathrm{d}\rho_y\mathrm{d}\rho_z$$

$$= \frac{2\sigma a^5}{3} = \frac{2Ma^2}{3} = J_{yy} = J_{zz}$$

$$\sigma = \frac{M}{a^3}$$

# Example: inertia matrix of homogenous cube

$$\mathbf{J} \triangleq \int_V \begin{bmatrix} (\rho_y^2+\rho_z^2) & -\rho_x\rho_y & -\rho_x\rho_z \\ -\rho_x\rho_y & (\rho_x^2+\rho_z^2) & -\rho_y\rho_z \\ -\rho_x\rho_z & -\rho_y\rho_z & (\rho_x^2+\rho_y^2) \end{bmatrix} \sigma(\rho_x,\rho_y,\rho_z)\mathrm{d}V$$




$$J_{xy} = \int_V -(\rho_x\rho_y)\sigma(\rho_x,\rho_y,\rho_z)\mathrm{dV} = -\sigma a \int_0^a \int_0^a (\rho_x\rho_y)\mathrm{d}\rho_x \mathrm{d}\rho_y$$

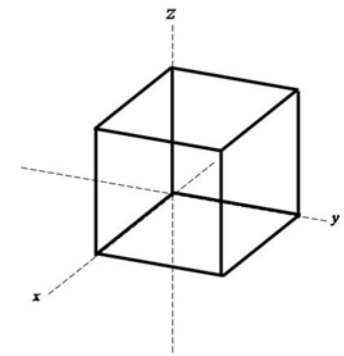
$$= -\frac{\sigma a^5}{4} = \frac{-Ma^2}{4} = J_{xy} = J_{yx} = J_{xz} = J_{zx} = J_{yz} = J_{zy}$$

$$\mathbf{J} = \frac{Ma^2}{12}\begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix}$$

Inertia matrix through cube´s corner

Find angular momentum for rotation about any axis through this corner

# Example: rotation of homogenous cube




Rotation about x axis of cube $\boldsymbol{\omega} = \omega \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$\mathbf{h}_{O*} = \mathbf{J}\,\boldsymbol{\omega} = \frac{Ma^2\omega}{12}\begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{Ma^2\omega}{12}\begin{bmatrix} 8 \\ -3 \\ -3 \end{bmatrix}$$

$\mathbf{h}_{O*}$ is not in same direction as rotation axis ω

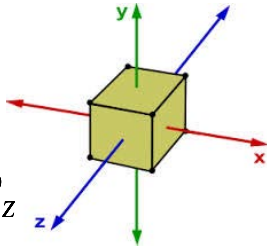
Rotation about diagonal through cube´s corner O* $\boldsymbol{\omega} = \frac{\omega}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\mathbf{h}_{O*} = \mathbf{J}\,\boldsymbol{\omega} = \frac{Ma^2\omega}{12\sqrt{3}}\begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{Ma^2\omega}{12\sqrt{3}}\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \frac{Ma^2}{6}\boldsymbol{\omega}$$

$\mathbf{h}_{O*}$ is in same direction as rotation axis ω

# Example: inertia matrix of homogeneous cube

Calculate matrix **I** for **center of mass**, shift origin to center of cube




$$I_{xx} = \int_V (\rho_y^2 + \rho_z^2)\sigma(\rho_x, \rho_y, \rho_z)\mathrm{d}V = \sigma \int_{-a/2}^{a/2}\int_{-a/2}^{a/2}\int_{-a/2}^{a/2} (\rho_y^2 + \rho_z^2)\mathrm{d}\rho_x \mathrm{d}\rho_y \mathrm{d}\rho_z$$

$$= \sigma a \int_{-a/2}^{a/2}\int_{-a/2}^{a/2} (\rho_y^2 + \rho_z^2)\mathrm{d}\rho_y \mathrm{d}\rho_z = \sigma a \int_{-a/2}^{a/2}\int_{-a/2}^{a/2} \rho_y^2 \mathrm{d}\rho_y \mathrm{d}\rho_z + \sigma a \int_{-a/2}^{a/2}\int_{-a/2}^{a/2} \rho_z^2 \mathrm{d}\rho_y \mathrm{d}\rho_z$$

$$= \frac{2\sigma a^2 2(a/2)^3}{3} = \frac{Ma^2}{6} = I_{yy} = I_{zz}$$

$$I_{xy} = \int_V -(\rho_x \rho_y)\sigma(\rho_x, \rho_y, \rho_z)\mathrm{d}V = -\sigma a \int_{-a/2}^{a/2}\int_{-a/2}^{a/2} (\rho_x \rho_y)\mathrm{d}\rho_x \mathrm{d}\rho_y$$

$$= 0 = I_{xy} = I_{yx} = I_{xz} = I_{zx} = I_{yz} = I_{zy}$$

Inertia matrix is diagonal

$$\mathbf{I} = \frac{Ma^2}{6}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

No matter what direction rotation axis **ω** has angular momentum **h** is always parallel to **ω**

$$\mathbf{h}_c = \mathbf{I}\boldsymbol{\omega} = \frac{Ma^2}{6}\boldsymbol{\omega}$$

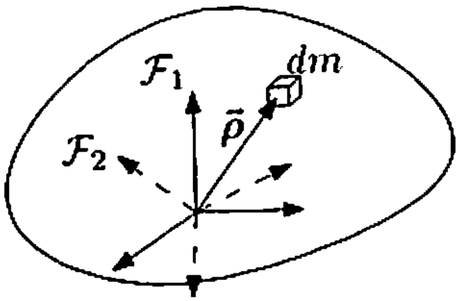
# Facts about inertia matrix

- Inertia matrix can be defined with respect to any body-fixed coordinate system
$$\mathbf{h}_{O*} = \mathbf{J}\,\boldsymbol{\omega}$$
- Matrix form
$$\begin{bmatrix} h_{xO*} \\ h_{yO*} \\ h_{zO*} \end{bmatrix} = \begin{bmatrix} J_{xx} & J_{xy} & J_{xz} \\ J_{yx} & J_{yy} & J_{yz} \\ J_{zx} & J_{zy} & J_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$
- If frame has its origin at center of mass
$$\mathbf{h}_c = \mathbf{I}\boldsymbol{\omega}$$
- Inertia matrix is **real symmetric** matrix $\quad \mathbf{J} = \mathbf{J}^T$
- Inertia matrix is **positive-definite** $\quad \mathbf{x}^T\mathbf{J}\mathbf{x} > 0 \quad (\mathbf{x} \neq \mathbf{0})$

  → **Eigenvalues of J are real and positive**
- Moment of inertia matrix is constant property of rigid body
- Moment of inertia matrix (**J**) has same role in rotational dynamics as mass (m) in translational dynamic

# Rotational transformation theorem

Consider two body-fixed frames 1 and 2
with same origin but different orientation




Components of angular momentum and angular velocity related by

$$\mathbf{h}_{1O*} = \mathbf{J}_1 \boldsymbol{\omega}_1$$

$$\mathbf{h}_{2O*} = \mathbf{J}_2 \boldsymbol{\omega}_2$$

$\mathbf{J}_1$ is inertia matrix in frame 1 and $\mathbf{J}_2$ inertia matrix in frame 2

Use rotation matrix $\mathbf{C}_{21}$

$$\mathbf{h}_{2O*} = \mathbf{C}_{21}\mathbf{h}_{1O*} \qquad \boldsymbol{\omega}_2 = \mathbf{C}_{21}\boldsymbol{\omega}_1$$

With:

$$\mathbf{C}_{21}\mathbf{h}_{1O*} = \mathbf{J}_2 \mathbf{C}_{21}\boldsymbol{\omega}_1 \quad \Leftrightarrow \quad \mathbf{h}_{1O*} = \mathbf{C}_{21}^T \mathbf{J}_2 \mathbf{C}_{21}\boldsymbol{\omega}_1$$

Rotational transformation
theorem for inertia matrix

$$\boxed{\mathbf{J}_1 = \mathbf{C}_{21}^T \ \mathbf{J}_2 \mathbf{C}_{21}} \quad or \quad \mathbf{J}_2 = \mathbf{C}_{21} \mathbf{J}_1 \mathbf{C}_{21}^T$$

# Principal axes of inertia

**Definition: Principal axes frame is body-fixed frame in which moment of inertia is diagonal**

$$\mathbf{J} = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix}$$

Diagonal elements $J_i$ are called **principal moments of inertia**

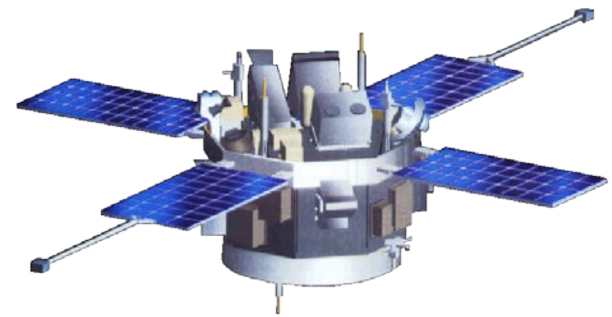
If inertia matrix is real, symmetric and positive definite

→ Inertia matrix can always be diagonalized

Linear algebra theorem: Real symmetric n-dimensional matrix possesses n real eigenvalues with n associated real eigenvectors, which are or can be chosen to be mutually orthogonal.

→ For any shape of rigid body exist always principal axes frame

# Principle axes for any shape of rigid body Moment of inertia matrix about center of mass

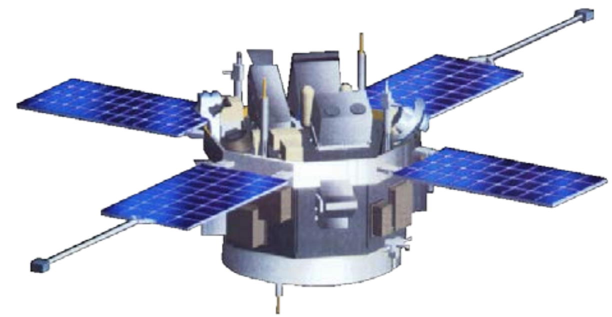


$$\mathbf{I} = \begin{bmatrix} \frac{Ma^2}{6} & 0 & 0 \\ 0 & \frac{Ma^2}{6} & 0 \\ 0 & 0 & \frac{Ma^2}{6} \end{bmatrix}$$

$$\mathbf{I} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

Often possible to find principal axes by just looking at object

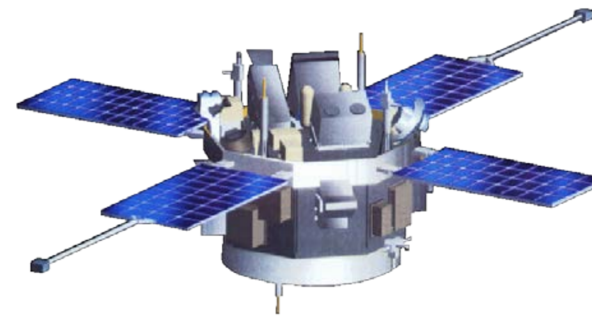
# Principle axes for any shape of rigid body Moment of inertia matrix about O*



$$\mathbf{J} = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix}$$

Principal axis with smallest, intermediate and largest inertia are often called minor, intermediate and major axes

# Arbitrary orthogonal for any shape of rigid body
# Moment of inertia matrix about O*



$$\mathbf{J} = \begin{bmatrix} J_{xx} & J_{xy} & J_{xz} \\ J_{yx} & J_{yy} & J_{yz} \\ J_{zx} & J_{zy} & J_{zz} \end{bmatrix}$$

How to find diagonal inertia matrix?

First calculate inertia matrix then if needed diagonalize matrix

# Find principal axis eigenvalue equations

Find directions of principal axes by finding directions for eigenvectors **ω**

Find diagonal matrix with eigenvalue equation $\mathbf{J}\boldsymbol{\omega} = \lambda\boldsymbol{\omega}$

Number λ is eigenvalue

Vector **ω** is eigenvector

Find three eigenvectors with three eigenvalue $\mathbf{J}\mathbf{e}_i = \lambda_i \mathbf{e}_i \quad , \quad i = 1,2,3$

Use identity matrix **1** $\quad \mathbf{J}\boldsymbol{\omega} = \lambda\mathbf{1}\boldsymbol{\omega} \quad \Leftrightarrow \quad (\mathbf{J} - \lambda\mathbf{1})\boldsymbol{\omega} = \mathbf{0}$

Non-trivial solution if $\quad det(\mathbf{J} - \lambda\mathbf{1}) = 0 \quad$ (cubic equation for λ)

To diagonalize inertia matrix , find solution of characteristic equation

$$det\begin{bmatrix} J_{xx} - \lambda & J_{xy} & J_{xz} \\ J_{yx} & J_{yy} - \lambda & J_{yz} \\ J_{zx} & J_{zy} & J_{zz} - \lambda \end{bmatrix} = 0$$

# Example: inertia tensor of homogenous cube

Inertia tensor of homogenous cube **with origin at one corner** and coordinate axes along edges

$$\mathbf{J} = \tfrac{Ma^2}{12}\begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix} \qquad det\begin{bmatrix} 8k-\lambda & -3k & -3k \\ -3k & 8k-\lambda & -3k \\ -3k & -3k & 8k-\lambda \end{bmatrix} = 0 \quad k = \tfrac{Ma^2}{12}$$

Solution of characteristic equation (Matlab [factor(det(A)) ] or hand)

Eigenvalues

$$\lambda_1 = \lambda_2 = \frac{11Ma^2}{12} \qquad \lambda_3 = \frac{Ma^2}{6}$$

In this case two eigenvalues are repeated

To find directions of principle axes find directions of eigenvectors

Lets consider $\lambda_3 = \frac{Ma^2}{6}$

Solve equation $(\mathbf{J} - \lambda_3 \mathbf{1})\boldsymbol{\omega}_3 = \mathbf{0}$

# Example: inertia tensor of homogenous cube

(Origin at one corner)

Solve equation for each eigenvalue

$$\lambda_3 = 2k$$

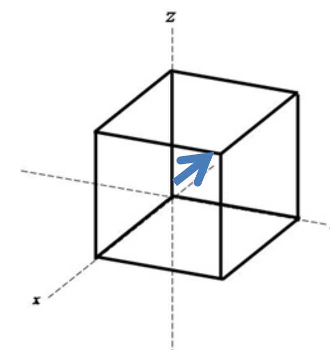
$$(\mathbf{J} - \lambda_3 \mathbf{1})\boldsymbol{\omega}_3 = \mathbf{0} \Rightarrow \begin{bmatrix} 8k-\lambda & -3k & -3k \\ -3k & 8k-\lambda & -3k \\ -3k & -3k & 8k-\lambda \end{bmatrix} \begin{bmatrix} \omega_{13} \\ \omega_{23} \\ \omega_{33} \end{bmatrix} = k \begin{bmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{bmatrix} \begin{bmatrix} \omega_{13} \\ \omega_{23} \\ \omega_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution to this set of equations is $\omega_{13} = \omega_{23} = \omega_{33}$

Unit vector along this directions is

which is diagonal of cube

$$\mathbf{e}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$




Other solution is

$$k \begin{bmatrix} -3 & -3 & -3 \\ -3 & -3 & -3 \\ -3 & -3 & -3 \end{bmatrix} \begin{bmatrix} \omega_{11} \\ \omega_{21} \\ \omega_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \omega_{11} + \omega_{21} + \omega_{31} = 0$$

Any vector perpendicular to $\mathbf{e}_3$ ($\boldsymbol{\omega}\cdot\mathbf{e}_3=0$) fulfills condition

# Obtain principle axes by inspection

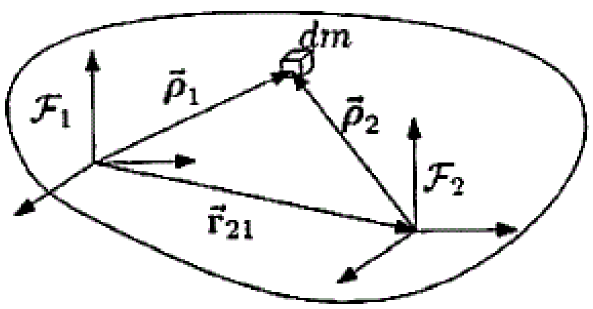
Previous example illustrated the needed to do calculations to determine principle axes and inertias in a general situation

Principle axes for homogenous rigid bodies with regular geometric shapes can be obtained by inspection

Useful rules:

- The axis of a body of revolution is a principle axis, and any transverse axis passing through the center of mass is also a principle axis.
- The plane of symmetry contains two principle axes with a third being normal to this plane.
- In general, the three principal axes passing through the center of mass include the axes of maximum and minimum inertia. These are referred to as the *major* and *minor* axes, respectively.

# Parallel axis theorem




- Requirement: reference frame stationary with respect to body frame
- Origin of reference frame may be at any point within body (E.g. rotation may occur not around center of mass, but around some other point fixed at given moment of time)
- Moment of inertia about frame 1 $\mathbf{J}_1 = -\int_V \boldsymbol{\rho}_1^\times \boldsymbol{\rho}_1^\times \mathrm{d}m$

$$\boldsymbol{\rho}_2 = \boldsymbol{\rho}_1 - \mathbf{r}_{21}$$

- Moment of inertia about frame 2 $\mathbf{J}_2 = -\int_V \boldsymbol{\rho}_2^\times \boldsymbol{\rho}_2^\times \mathrm{d}m$

$$= -\int_V (\boldsymbol{\rho}_1 - \mathbf{r}_{21})^\times (\boldsymbol{\rho}_1 - \mathbf{r}_{21})^\times \mathrm{d}m$$

$$= -\int_V \boldsymbol{\rho}_1^\times \boldsymbol{\rho}_1^\times \mathrm{d}m + \int_V \mathbf{r}_{21}^\times \underbrace{\boldsymbol{\rho}_1^\times \mathrm{d}m} + \int_V \boldsymbol{\rho}_1^\times \mathbf{r}_{21}^\times \mathrm{d}m - \int_V \mathbf{r}_{21}^\times \mathbf{r}_{21}^\times \mathrm{d}m$$

First moment of mass about origin frame 1

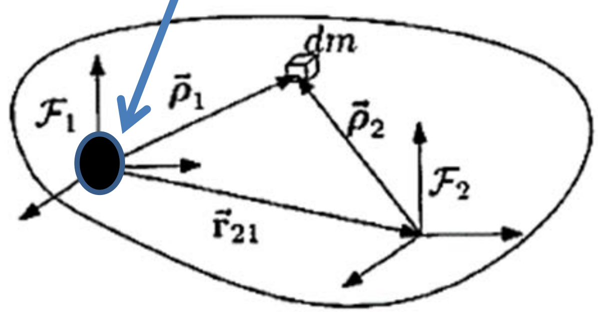
- Parallel axis theorem for inertia matrices:

$$\mathbf{J}_2 = \mathbf{J}_1 + \mathbf{r}_{21}^\times \mathbf{c}_1^\times + \mathbf{c}_1^\times \mathbf{r}_{21}^\times - m\,\mathbf{r}_{21}^\times \mathbf{r}_{21}^\times$$

# Parallel axis theorem for center of mass

$$\mathbf{J}_2 = \mathbf{J}_1 + \mathbf{r}_{21}^{\times}\mathbf{c}_1^{\times} + \mathbf{c}_1^{\times}\mathbf{r}_{21}^{\times} - m\,\mathbf{r}_{21}^{\times}\mathbf{r}_{21}^{\times}$$

If center of mass $\mathbf{c}_1 = \mathbf{0} = \boldsymbol{\rho}_1$




$$\boxed{\mathbf{J}_2 = \mathbf{I}_1 - m\,\mathbf{r}_{21}^{\times}\mathbf{r}_{21}^{\times}}$$

Parallel axis theorem:

Moment of inertia about given axis is equal to

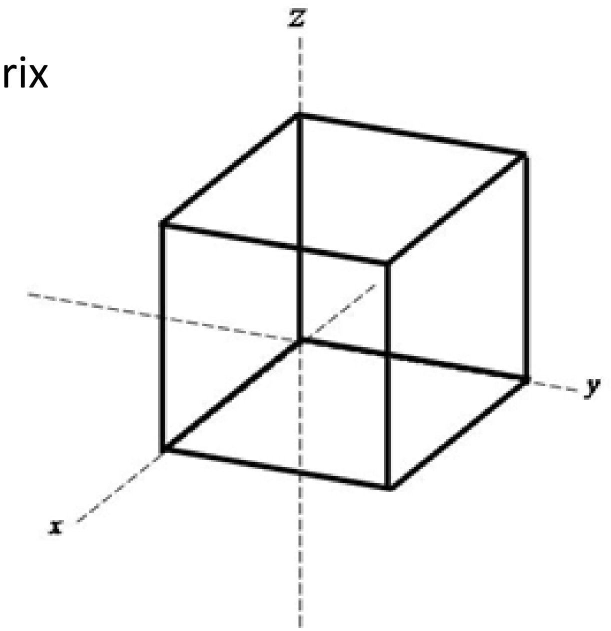
moment of inertia about parallel axis through center of mass

minus moment of inertia of body, as if concentrated at center of mass, with respect to original axis

# Example: Parallel axis theorem for homogenous cube

$$\mathbf{J}_2 = \mathbf{I}_1 - m\,\mathbf{r}_{21}^{\times}\,\mathbf{r}_{21}^{\times}$$

Calculate inertia matrix for cube´s corner




$$\mathbf{I} = \begin{bmatrix} \frac{Ma^2}{6} & 0 & 0 \\ 0 & \frac{Ma^2}{6} & 0 \\ 0 & 0 & \frac{Ma^2}{6} \end{bmatrix}$$

$$\mathbf{r}_{21} = -\tfrac{a}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

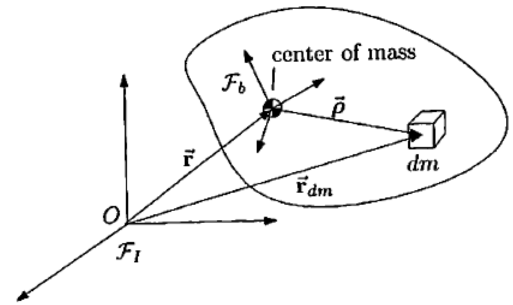
$$\mathbf{r}_{21}^{\times} = -\tfrac{a}{2}\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\mathbf{r}_{21}^{\times}\mathbf{r}_{21}^{\times} = \tfrac{a^2}{4}\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} = \tfrac{a^2}{4}\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$$\mathbf{J}_2 = \begin{bmatrix} \frac{Ma^2}{6} & 0 & 0 \\ 0 & \frac{Ma^2}{6} & 0 \\ 0 & 0 & \frac{Ma^2}{6} \end{bmatrix} - \frac{Ma^2}{12}\begin{bmatrix} -6 & 3 & 3 \\ 3 & -6 & 3 \\ 3 & 3 & -6 \end{bmatrix}$$

Compare result for $\mathbf{J_2}$
with previous calculation
for inertia matrix at cube´s corner

# Kinetic energy of rigid body




Origin of body-fixed frame is center of mass

Vector of mass element $$\vec{\mathbf{r}}_{\mathrm{d}m} = \vec{\mathbf{r}} + \vec{\rho}$$

Inertial velocity of mass element $$\dot{\vec{\mathbf{r}}}_{\mathrm{d}m} = \dot{\vec{\mathbf{r}}} + \dot{\vec{\rho}}$$

Inertial time derivative related to time derivative seen in body frame

$$\left\{\dot{\vec{\rho}}\right\}_i = \left\{\dot{\vec{\rho}}\right\}_b + \vec{\omega} \times \vec{\rho}$$

$$\left\{\dot{\vec{\rho}}\right\}_b = 0 \quad \Rightarrow \quad \left\{\dot{\vec{\rho}}\right\}_i = \vec{\omega} \times \vec{\rho}$$

**v** is velocity of center of mass $$\dot{\vec{\mathbf{r}}}_{\mathrm{d}m} = \vec{\mathbf{v}} + \vec{\omega} \times \dot{\vec{\rho}}$$

Kinetic energy of mass element $$\mathrm{d}T = \frac{1}{2}\dot{\vec{\mathbf{r}}}_{\mathrm{d}m} \cdot \dot{\vec{\mathbf{r}}}_{\mathrm{d}m}\, \mathrm{d}m$$

# Kinetic energy of rigid body

Kinetic energy of mass element

$$\mathrm{d}T = \frac{1}{2}(\vec{\mathbf{v}} + \vec{\omega}\times\vec{\rho})\cdot(\vec{\mathbf{v}} + \vec{\omega}\times\vec{\rho})\mathrm{d}m$$

Integrate:

$$T = \frac{1}{2}\int_B \vec{\mathbf{v}}\cdot\vec{\mathbf{v}}\,\mathrm{d}m + \int_B \vec{\mathbf{v}}\cdot(\vec{\omega}\times\vec{\rho})\mathrm{d}m + \frac{1}{2}\int_B (\vec{\omega}\times\vec{\rho})\cdot(\vec{\omega}\times\vec{\rho})\mathrm{d}m$$

$$\frac{1}{2}m\,\vec{\mathbf{v}}\cdot\vec{\mathbf{v}} = \frac{1}{2}mv^2$$

**v** is constant

$$\int_B \vec{\rho}\,\mathrm{d}m = \vec{\mathbf{0}}$$

Origin is center of mass

$$\frac{1}{2}\int_B (\vec{\omega}\times\vec{\rho})\cdot(\vec{\omega}\times\vec{\rho})\mathrm{d}m = \frac{1}{2}\int_B (\omega^\times\rho)^T(\omega^\times\rho)\mathrm{d}m$$

$$= \frac{1}{2}\int_B (\rho^\times\omega)^T(\rho^\times\omega)\mathrm{d}m$$

$$= \frac{1}{2}\omega^T\left[-\int_B \rho^\times\rho^\times\mathrm{d}m\right]\omega$$

$$\frac{1}{2}\int_B (\vec{\omega}\times\vec{\rho})\cdot(\vec{\omega}\times\vec{\rho})\mathrm{d}m = \frac{1}{2}\boldsymbol{\omega}^T\,\mathbf{I}\,\boldsymbol{\omega}$$

# Kinetic energy of rigid body

Total kinetic energy of translating and rotating rigid body is

$$\boxed{T = T_t + T_r}$$

**Rotational kinetic energy**
in center of mass

$$\boxed{T_r = \frac{1}{2}\boldsymbol{\omega}^T \mathbf{I}\,\boldsymbol{\omega}}$$

**Translational kinetic energy**
of all mass

$$\boxed{T_t = \frac{1}{2} m v^2}$$

In principal axes frame
rotational kinetic energy becomes:

$$T_r = \frac{1}{2}(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2)$$

Used:

$$\mathbf{I} = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \quad , \quad \boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

# Rotation around one principal axis

When rotation occurs around one principal axis, then only one non-zero component $\omega_n$ exists

In this case, angular momentum has only one component

$$h_n = I_n \omega_n$$

In this case, rotational kinetic energy has only one term

$$T_r = \frac{1}{2} I_n \omega_n^{\ 2}$$

# Euler´s equation

# Inertia matrix for rotating body expressed in inertial frame

Note:

Inertial matrix for rotating body expressed in inertial frame is **NOT** constant

Rotational equation of motion of body about center of mass in inertial frame

$$\frac{\mathrm{d}\mathbf{h}_c}{\mathrm{d}t} = \frac{\mathrm{d}(\mathbf{I}\boldsymbol{\omega})}{\mathrm{d}t} = \mathbf{T}_c$$

Rate of change of angular momentum equals total external torque

$$\frac{\mathrm{d}(\mathbf{I}\boldsymbol{\omega})}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{I}}{\mathrm{d}t}\boldsymbol{\omega} + \mathbf{I}\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}$$

Chain rule

$$\frac{\mathrm{d}\mathbf{I}}{\mathrm{d}t} \neq \mathbf{0}$$

In inertial frame d**I**/dt is not constant

How to remove term d**I**/dt in rotational equation of motion?

→ Write rotation equation in body-fixed frame

For rotating body-fixed frame inertial matrix of rigid body is constant

But for rotating frame rotational equation must be modified

# Euler´s equation derivation

$$\dot{\vec{\mathbf{h}}}_c = \vec{\mathbf{T}}_c$$

Rate of change of angular momentum
with respect to center of mass
equals total external torque
with respect to center of mass

$$\left\{\dot{\vec{\mathbf{h}}}_c\right\}_i = \left\{\dot{\vec{\mathbf{h}}}_c\right\}_b + \vec{\omega}\times\vec{\mathbf{h}}_c$$

Relationship between vector derivative
in inertial frame and rotating body-fixed frame

$$\dot{\vec{\mathbf{h}}}_c + \vec{\omega}\times\vec{\mathbf{h}}_c = \vec{\mathbf{T}}_c$$

Rotation equation of motion in body fixed frame

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{I}\boldsymbol{\omega}) + \boldsymbol{\omega}\times\mathbf{I}\boldsymbol{\omega} = \mathbf{T}_c$$

Use $\mathbf{h}_c = \mathbf{I}\boldsymbol{\omega}$

$$\cancel{\frac{\mathrm{d}\mathbf{I}}{\mathrm{d}t}\boldsymbol{\omega}} + \mathbf{I}\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} + \boldsymbol{\omega}\times\mathbf{I}\boldsymbol{\omega} = \mathbf{T}_c$$

System of coordinates fixed with rotating rigid body,
inertia matrix is constant

$$\boxed{\mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\times\mathbf{I}\boldsymbol{\omega} = \mathbf{T}_c}$$

Euler´s equation

# Euler´s equation

$$\mathbf{I}\,\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}\,\boldsymbol{\omega} = \mathbf{T}_c$$

We have seen that for any rigid body exist always principle axes frame

Choose body-fixed principle axes frame:

$$\mathbf{I} = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \quad , \quad \boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \quad , \quad \mathbf{T}_c = \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix}$$

Euler´s equation in principle axes frame:

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = T_1$$

$$I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 = T_2$$

$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 = T_3$$

# Complete attitude description

## Attitude dynamics

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = T_1$$

$$I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 = T_2$$

$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 = T_3$$

## Attitude kinematics

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} 1 & \sin\theta_1 \tan\theta_2 & \cos\theta_1 \tan\theta_2 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 \sec\theta_2 & \cos\theta_1 \sec\theta_2 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

Solution of Euler´s equation for $\omega_i$ and
solution of attitude kinematics
give complete description of attitude

# Summary I

**Dynamics of particles**

**Dynamics of system of particles**

**Rigid body dynamics**

- Angular momentum $\vec{\mathbf{h}}_{O*} = \int_V \vec{\rho} \times (\vec{\omega} \times \vec{\rho}) \mathrm{d}m$
- Defined inertia matrix $\mathbf{J} \triangleq -\int_V \boldsymbol{\rho}^{\times} \boldsymbol{\rho}^{\times} \mathrm{d}m$

$$= \int_V \begin{bmatrix} (\rho_y^2 + \rho_z^2) & -\rho_x\rho_y & -\rho_x\rho_z \\ -\rho_x\rho_y & (\rho_x^2 + \rho_z^2) & -\rho_y\rho_z \\ -\rho_x\rho_z & -\rho_y\rho_z & (\rho_x^2 + \rho_y^2) \end{bmatrix} \sigma(\rho_x, \rho_y, \rho_z) \mathrm{d}V$$

- Calculated angular momentum and rotational kinetic energy

$$\mathbf{h}_{O*} = \mathbf{J}\,\boldsymbol{\omega} \qquad T_r = \frac{1}{2} \boldsymbol{\omega}^T \mathbf{I}\,\boldsymbol{\omega}$$

# Summary II

- Inertia matrix **J** can be made diagonal

$$\mathbf{J}_2 = \mathbf{C}_{21}\mathbf{J}_1\mathbf{C}_{21}^T = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix}$$

**C**$_{21}$ is rotation matrix

$J_i > 0$

- Principal axes:
  - Inertia matrix is real symmetric matrix

    → Such matrices can be made diagonal
  - Principal axes are coordinate axes in which inertia matrix is diagonal
  - To diagonalize inertia matrix, find solutions of characteristic equation $det(\mathbf{J} - \lambda\mathbf{1}) = 0$
  - Eigenvalues of characteristic equations are components of principal moment of inertia

$$\lambda_i = J_1, J_2, J_3$$

- Kinematic properties of rigid body are fully described by its mass, principal axes and inertia moments

# Summary III

## Euler equation

- Equation of motion in body-fixed frame $\mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}\boldsymbol{\omega} = \mathbf{T}_c$
- Principle axes frame
  - After diagonalization of inertia matrix,
    equations of motion for rotation of rigid body looks like

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = T_1$$
$$I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 = T_2$$
$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 = T_3$$

  - After diagonalization of inertia matrix,
    rotational kinetic energy of rigid body looks like

$$T_r = \frac{1}{2}\boldsymbol{\omega}^T\,\mathbf{I}\,\boldsymbol{\omega}$$