ADCS – IX B
Attitude Dynamics

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Summary of last lecture

Euler´s equation in principal axes frame

Linear and nonlinear differential equations

- Equilibrium, Stability, Characteristic equations
- Stability theorem
- Linearization of nonlinear systems

Symmetric and non-symmetric torque free rotations

- Geometrical and mathematical
- Linearized equation of motion
- Stability

Energy dissipation

- Geometrical and mathematical
- Effect on stability of rotation (major axis rule)

Outline

Spin stabilization

passive attitude control

Dual-spin stabilization

passive attitude control

Gravity-gradient stabilization

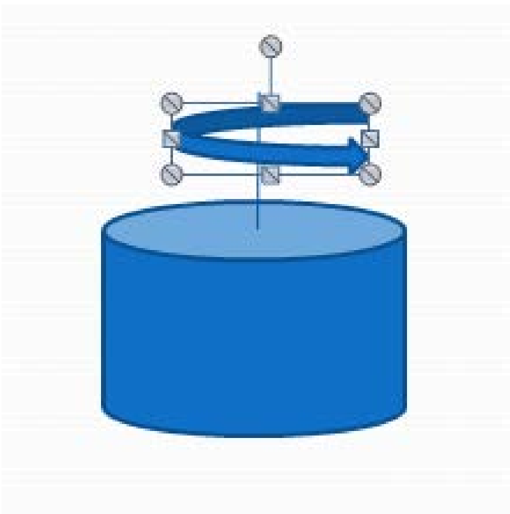
passive attitude control

Three axis stabilization

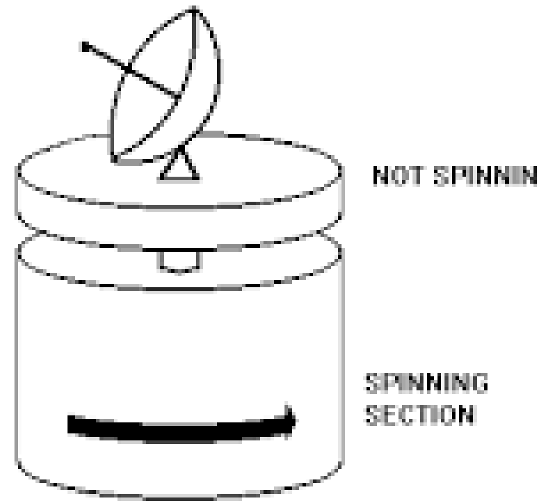
active attitude control

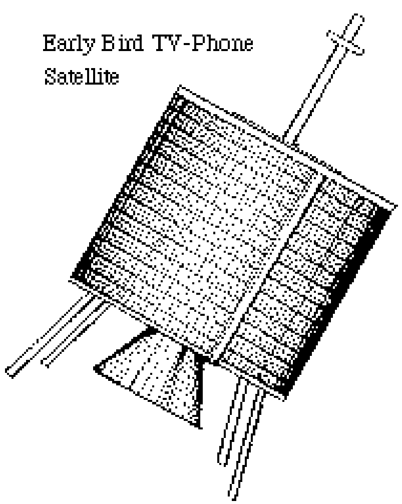
Single spinners and Dual spinners

Single spinner



Dual spinner





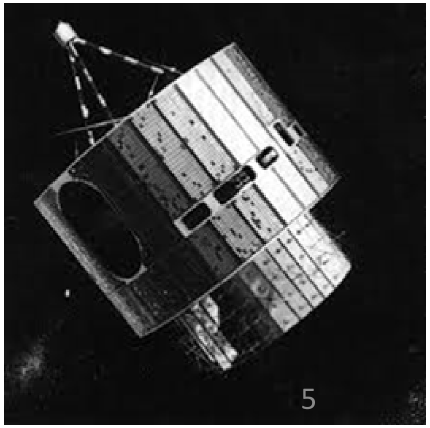
Dual spinners have shape of cylinder
Rotate about its long axis
Two sections:

- Spinning section with solar arrays
- Not-spinning section with communications antennas

Spinning section provides basic stabilization
Not-spinning section can also rotate (e.g. one rotation per orbit to keep antennas pointed at Earth)

Spin stabilization

- Intrinsic gyroscopic stiffness of spinning body used to maintain its orientation in inertial space (conservation of angular momentum)
- Simple and low cost method of attitude stabilization (largely passive)
- Generally not suitable for imaging payloads
- Poor power efficiency since entire spacecraft covered with solar cells
- Satellites intended for GEO are usually spin stabilized (in the past)
- A real (therefore flexible) body can spin stably only about axis of maximum momentum of inertia (major axis rule)
- Spacecraft must be short cylinder rather than long cylinder
- **Passive attitude control**



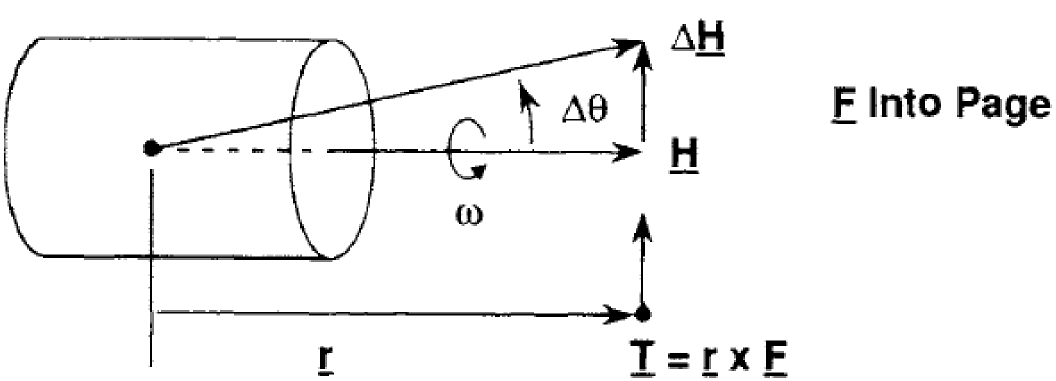
Spin stabilization

What does it mean: **Intrinsic gyroscopic stiffness** of spinning body used to maintain its orientation in inertial space

- If no external torques → angular momentum vector remains fixed in space (constant in magnitude and direction)
- If external torque **T** (components perpendicular and/or parallel to momentum vector **h**)

  - Parallel torque spins spacecraft spin momentum up or down

  - Perpendicular torque causes displacement of **h** in direction of **T**



$$\dot{\mathbf{h}} = \frac{\Delta \mathbf{h}}{\Delta t} = \mathbf{T}$$

If spin → linear growth

$$\Delta\theta = \frac{\Delta \mathbf{h}}{\mathbf{h}} = \frac{\Delta \mathbf{h}}{\mathbf{I}\boldsymbol{\omega}} = \frac{\mathbf{T}\Delta t}{\mathbf{I}\boldsymbol{\omega}}$$

Gyroscopic stability due to angular momentum in denominator
Higher angular momentum →
Smaller perturbation angle ΔΘ

If **no spin** $\mathbf{T} = \mathbf{I}\ddot{\boldsymbol{\theta}} \Rightarrow \theta = \frac{1}{2}\frac{T}{I}t^2$ torque will produce **quadratic growth**

Dual-spin stabilization

Dual-spin stabilization

- Spin-stabilized satellite must be major axis spinners: short and fat
- Two problems:
  - launch vehicles are tall and skinny (thin for aerodynamic reasons)
  - antennas need to point at Earth
- Diameter of spacecraft restricted by cross section of launch vehicle
- Spacecraft lengths limited by stability requirements

→ Short and fat spinner cannot take full advantage of volume in launch vehicle

- Make spacecraft with two parts: one spins relatively fast, other spins slowly or not at all
- Solves both problems: fits in launch vehicle, point despun platform to Earth
- Dual-spin design permits spin stabilization along other than major axis
- Dual-spin configuration consist of rotor and smaller platform joined together along common longitudinal spin axis
- Dual-spin stabilization less common than three axis-stabilization
- Spinning rotor can stabilize any axis
- **Passive attitude control**

Dual-spin system

Consider spacecraft with two rigid bodies:

- Platform
- Wheel

Wheel spin axis fixed within platform with angular velocity $\boldsymbol{\omega}_s$ relative to rigid body

Consider body fixed principal axis frame attached to platform with origin at center of mass of spacecraft (combined platform and wheel)

Assume spin axis of wheel aligned with body 2-axis

I = total moment of inertia matrix of spacecraft

$$\mathbf{I} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

I_s = wheel moment of inertia about spin axis

$\boldsymbol{\omega}$ = angular velocity of platform relative to inertial frame

ω_s = angular velocity of wheel relative to platform

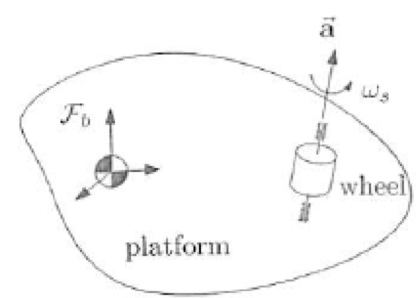


Figure 15.1 Dual-spin spacecraft

Equation of motion

- Total angular momentum of spacecraft (platform and wheel) in body coordinates:

$\mathbf{h} = \mathbf{I}\boldsymbol{\omega} + h_s\mathbf{a} \qquad h_s = I_s\omega_s \qquad \mathbf{a} = [0 \quad 1 \quad 0]^T$ with **a** = wheel spin axis

- Euler equation in inertial frame $\dot{\vec{h}} = \vec{T}$
- Euler equation in rotating body frame $\dot{\vec{h}} + \vec{\omega} \times \vec{h} = \vec{T}$
- Insert total angular momentum $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{I}\boldsymbol{\omega} + h_s\mathbf{a}) + \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega} + h_s\mathbf{a}) = \mathbf{T}$
- Or $\dot{\mathbf{I}}\boldsymbol{\omega} + \mathbf{I}\dot{\boldsymbol{\omega}} + \dot{h}_s\mathbf{a} + h_s\dot{\mathbf{a}} + \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega} + h_s\mathbf{a}) = \mathbf{T}$
- In body coordinates inertia matrix **I** and wheel spin axis **a** are constant

$$\mathbf{I}\dot{\boldsymbol{\omega}} + \dot{h}_s\mathbf{a} + \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega} + h_s\mathbf{a}) = \mathbf{T}$$

- Typically wheel spin angular velocity is kept constant $\dot{h}_s = I_s\dot{\omega}_s = 0$

$$\mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega} + h_s\mathbf{a}) = \mathbf{T}$$

Equation of torque free motion

Dual spin system in body coordinates: $\mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega} + h_s \mathbf{a}) = \mathbf{T}$

With previously defined $\mathbf{I} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$ $\boldsymbol{\omega} = [\omega_1 \quad \omega_2 \quad \omega_3]^T$ $\mathbf{a} = [0 \quad 1 \quad 0]^T$

Torque free **T** = **0** equations of motions are

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 - h_s\omega_3 = 0$$

$$I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 = 0$$

$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 + h_s\omega_1 = 0$$

Equilibrium condition of equation of motion

$$
\begin{aligned}
I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 - h_s\omega_3 &= 0 \\
I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 \qquad\qquad &= 0 \\
I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 + h_s\omega_1 &= 0
\end{aligned}
$$

Search for stability of equilibrium solutions of above equation

Several equilibrium solutions exist (non-linear differential equation)

For comparison with spin stabilization lets look at following solution

$$\omega_1(t) = \omega_3(t) = 0 \quad \omega_2(t) = \Omega$$

Principal 2-axis spin with Ω

Insert in above equation

$$
\begin{aligned}
I_1\dot{\omega}_1 &= 0 \\
I_2\dot{\omega}_2 &= 0 \\
I_3\dot{\omega}_3 &= 0
\end{aligned}
$$

Means principal 2-axis spin is equilibrium condition (angular velocity do not change)

Stability of equilibrium condition

Small perturbation of reference motion with small angular velocities $\varepsilon_i(t)$ i = 1,2,3 with $\varepsilon_i(t) \ll \Omega$

$$\omega_1(t) = \varepsilon_1(t) \quad \omega_3(t) = \varepsilon_3(t) \quad \omega_2(t) = \Omega + \varepsilon_2(t)$$

Equation of motion near equilibrium point

$$I_1\dot{\varepsilon}_1 + (I_3 - I_2)(\varepsilon_2 + \Omega)\varepsilon_3 - h_s\varepsilon_3 = 0$$

$$I_2\dot{\varepsilon}_2 + (I_1 - I_3)\varepsilon_1\varepsilon_3 = 0$$

$$I_3\dot{\varepsilon}_3 + (I_2 - I_1)(\varepsilon_2 + \Omega)\varepsilon_1 + h_s\varepsilon_1 = 0$$

Linearize equation (by neglecting higher order term)

$$I_1\dot{\varepsilon}_1 + [(I_3 - I_2)\Omega - h_s]\varepsilon_3 = 0$$

$$I_2\dot{\varepsilon}_2 = 0$$

$$I_3\dot{\varepsilon}_3 + [(I_2 - I_1)\Omega + h_s]\varepsilon_1 = 0$$

As for spinning satellite 1- and 3-axis are decoupled from 2-axis

$$\varepsilon_2(t) = const$$

Means $\varepsilon_2(t)$ is stable

Linear stability of equilibrium

Do stability analysis for 1- and 3- axis

$$\dot{\varepsilon}_1 + \frac{[(I_3 - I_2)\Omega + h_s]}{I_1}\varepsilon_3 = 0$$

$$\dot{\varepsilon}_3 + \frac{[(I_2 - I_1)\Omega - h_s]}{I_3}\varepsilon_1 = 0$$

For spinning platform $\Omega \neq 0$ and define $\lambda = I_2 + \frac{h_s}{\Omega}$

Equation of motion are

<u>Dual spinner</u>

$$\dot{\varepsilon}_1 + \frac{(I_3 - \lambda)\Omega}{I_1}\varepsilon_3 = 0$$

$$\dot{\varepsilon}_3 + \frac{(\lambda - I_1)\Omega}{I_3}\varepsilon_1 = 0$$

Compare them to spinning spacecraft equation

<u>Single spinner</u>

$$\dot{\varepsilon}_1 + \frac{(I_3 - I_2)\Omega}{I_1}\varepsilon_3 = 0$$

$$\dot{\varepsilon}_3 + \frac{(I_2 - I_1)\Omega}{I_3}\varepsilon_1 = 0$$

Equations of dual spinner are identical to single spinner if I_2 replaced by λ with $\lambda = I_2 + \frac{h_s}{\Omega}$

Presence of wheel with angular momentum h_s augments moment of inertia I_2

Stability is only possible if

$$\boxed{\begin{array}{ccc} \lambda > I_3 & and & \lambda > I_1 \\ & or & \\ \lambda < I_3 & and & \lambda < I_1 \end{array}}$$

Now even intermediate axis spin can be made stable by making wheel stored angular momentum h_s large enough

Example

Assume rigid body with $\mathbf{I} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} = \begin{bmatrix} 300 & 0 & 0 \\ 0 & 350 & 0 \\ 0 & 0 & 400 \end{bmatrix}$ kg m^2

and flywheel with I_s= 10 kg m^2

Is it possible to spin around 2-axis at 2π rad/s ?

2-axis is intermediate axis and so unstable without rotating flywheel

Both conditions $\begin{matrix} \lambda > I_3 & and & \lambda > I_1 \\ & or & \\ \lambda < I_3 & and & \lambda < I_1 \end{matrix}$ with $\lambda = I_2 + \frac{h_s}{\Omega}$ and h_s = 0 are invalid

Need to have $I_2 + \frac{h_s}{\Omega} > I_3 \Leftrightarrow I_2 + \frac{I_s \omega_s}{\Omega} > I_3 \Leftrightarrow \omega_s > \frac{I_3 - I_2}{I_s}\Omega \Rightarrow \omega_s > 10\pi$ rad/s

Flywheel needs to spin faster than ω_s

Note:

If no torque exist→ total angular momentum magnitude is constant

Initial total angular momentum must have same magnitude as final

Disturbance torques

Environmental disturbance torques

- Relax torque-free assumption and consider rotational dynamics of Earth-orbiting spacecraft (introduce disturbance torques)
- Disturbance torques lead to reduction in pointing accuracy of spacecraft
- Torque magnitude depend on spacecraft orbit type and orbit altitude

- Major disturbance torques for spacecraft in vicinity of Earth are:
  - Aerodynamic torque
  - Solar-radiation pressure torque
  - Magnetic torque
  - Gravity-gradient torque

Aerodynamic torque and gravity torque in LEO, solar pressure torque in GEO

Gravity-gradient stabilization

Gravity-gradient

- Extended body placed in non-uniform gravity field may have center of gravity not coincident with center of mass

  → Give rise to torque (gravity-gradient torque)
- Torque will depend on orientation of body:

  - Gradient relates to fact that in analysis gravity field is expanded in Taylor series, keeping terms up to first derivatives (i.e. gradient)

  - Gravity-gradient stabilization involves shaping body so that corresponding gravity-gradient torques maintains body in desired orientation
- Following analysis related to aspects of gravity-gradient stabilization

  - Mainly related to effects of spacecraft shape

  - Circular orbit is simplest
- **Passive attitude control**

$$\frac{\mathrm{d}\vec{h}}{\mathrm{d}t} = \frac{\mathrm{d}(\mathbf{I}\vec{\omega})}{\mathrm{d}t} = \vec{T}_{disturbance}$$

Exploit existing disturbance torques

Gravity-gradient effect on spacecraft attitude

Gravitational attraction:

$f = m/r^2$

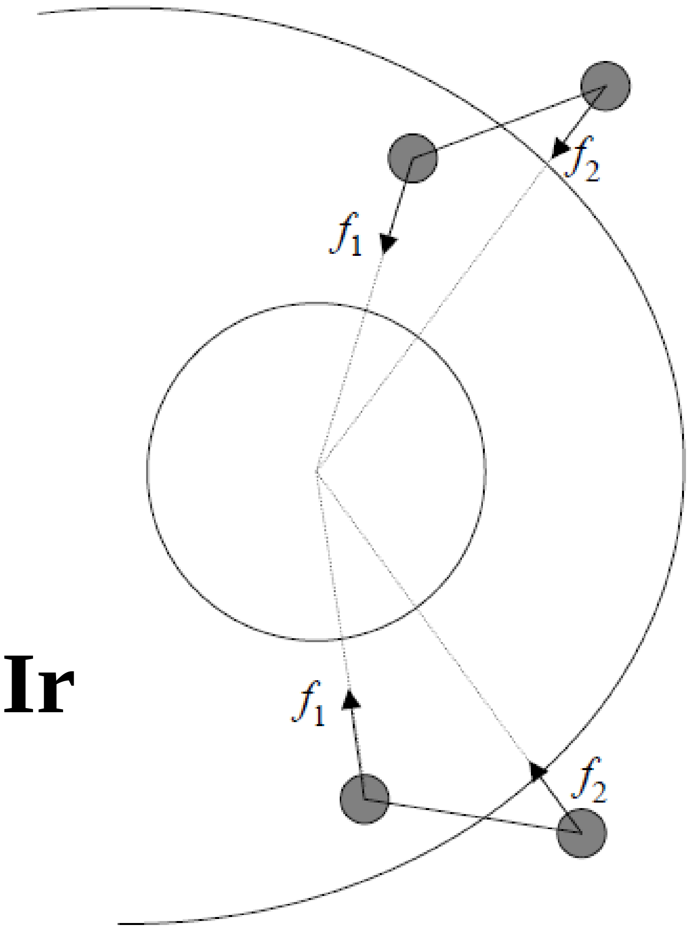
Top: $f_1 > f_2 \rightarrow$ torque is out of page

Bottom: $f_2 < f_1 \rightarrow$ torque is into page

In both cases:

Torque is restoring torque, tending to make spacecraft swing like pendulum

(need of passive or active dumping)



Gravity-gradient torque $$\dot{\mathbf{h}} = \frac{3\mu}{r^5}\mathbf{r} \times \mathbf{I}\mathbf{r}$$

Angular moment about mass center $$\mathbf{T}_g = \dot{\mathbf{h}}$$

Approximate equation of motion for gravity-gradient stabilization problem

For rigid body in central gravitational field equations of motion may approximated by restricted two body problem and momentum about mass center due to gravitational force

Orbital motion

$$\ddot{\mathbf{r}} + \frac{\mu}{r^3}\mathbf{r} = \mathbf{0}$$

r = position vector of mass center of body
with respect to center of gravitational primary
h = angular momentum of body about center of mass

r appears in both equation
→Solution to orbital equation
of motion must be also used
in solving attitude equation

Attitude dynamics

$$\mathbf{T}_g = \dot{\mathbf{h}}$$

$$\dot{\mathbf{h}} = \frac{3\mu}{r^5}\mathbf{r} \times \mathbf{I}\mathbf{r}$$

Attitude dynamics described by Euler equation

$$\mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}\boldsymbol{\omega} = \frac{3\mu}{r^5}\mathbf{r} \times \mathbf{I}\mathbf{r}$$

Spacecraft orbital motion

$$\ddot{\mathbf{r}} + \frac{\mu}{r^3}\mathbf{r} = \mathbf{0}$$

Is simply two-body equation of motion for
point mass orbiting spherical primary
→ Keplerian orbit

To simplify things restrict to circular orbit

Study now attitude motion

Gravity-gradient stabilization

First Euler equation where torque (gravity-gradient) appears

Write Euler equation in principle axes body frame

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = T_1$$
$$I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 = T_2$$
$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 = T_3$$

$$\mathbf{T}_g = \frac{3\mu}{r^5}\mathbf{r}\times\mathbf{I}\mathbf{r}$$

What is end effect of gravity gradient?

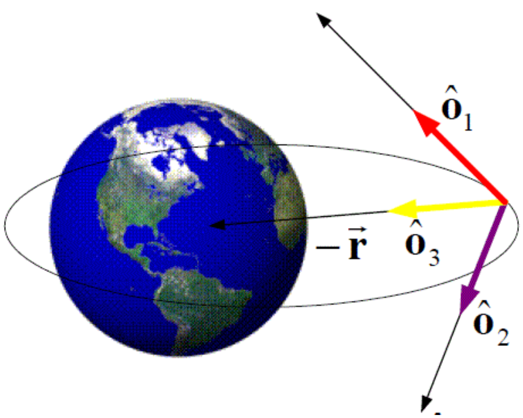
Deal with nonlinear equations

- Search for stable and unstable equilibrium positions
- Perform stability analysis

Reference frames

Approximate rigid satellite in central gravitational field using three reference frames:

1. Inertial frame centered at Earth (not rotating with Earth)
2. Orbiting frame (centered at spacecraft center of mass and rotates with spacecraft in its orbit)



3-axis points always to center of Earth (nadir direction)
2-axis points in negative orbit normal direction (pointing down)
1-axis points in direction of motion (for circular orbits is in velocity vector direction)

Special names for spacecraft dynamics:
1-axis = roll
2-axis = pitch
3-axis = yaw

3. Body frame (body frame slightly angular displaced from orbiting frame)

Circular orbit

- Restrict to circular orbit
- Orbital frame makes one complete turn relative to inertial frame about 2-axis every orbit
- Angular velocity **ω** of orbital frame with respect to inertial frame

$$\boldsymbol{\omega}_{oi} = \begin{bmatrix} 0 & -\omega & 0 \end{bmatrix}^T$$

- For circular orbit $\omega = 2\pi / T$
- With orbital period $T = 2\pi\sqrt{R^3 / \mu}$ given by Kepler´s third law
- Spacecraft position is given by radius *R* of orbit $\mathbf{R}_o = [0 \quad 0 \quad -R]^T$

Body frame to orbiting frame

Spacecraft body frame to orbiting frame given by 3-2-1 rotation matrix

$$\mathbf{C}_{bo}(\theta_1,\theta_2,\theta_3) = \mathbf{C}_1(\theta_1)\mathbf{C}_2(\theta_2)\mathbf{C}_3(\theta_3)$$

$$= \begin{bmatrix} c_2c_3 & c_2s_3 & -s_2 \\ s_1s_2c_3 - c_1s_3 & s_1s_2s_3 + c_1c_3 & s_1c_2 \\ c_1s_2c_3 + s_1s_3 & c_1s_2s_3 - s_1c_3 & c_1c_2 \end{bmatrix}$$

To simplify things consider

body frame is only slightly angular displaced from orbiting frame

For such small angles $\boldsymbol{\theta} = [\theta_1 \quad \theta_2 \quad \theta_3]^T$

Rotation matrix is

$$\mathbf{C}_{bo} = \begin{bmatrix} 1 & \theta_3 & -\theta_2 \\ -\theta_3 & 1 & \theta_1 \\ \theta_2 & -\theta_1 & 1 \end{bmatrix}$$

Body frame nearly aligned with orbital frame

Assume also body frame is principle axis frame

Means for $\boldsymbol{\Theta}$ = $\mathbf{0}$ principle axes of spacecraft are aligned with orbiting frame

Gravity-gradient torque in body fixed frame

Express gravity-gradient torque in body frame $\mathbf{T}_g = \frac{3\mu}{R^5}\mathbf{R}\times\mathbf{IR}$

Spacecraft position vector in orbiting frame is given by

$$\mathbf{R}_o = [0 \quad 0 \quad -R]^T$$

In body frame spacecraft position vector is

$$\mathbf{R}_b = \mathbf{C}_{bo}\mathbf{R}_o = \begin{bmatrix} 1 & \theta_3 & -\theta_2 \\ -\theta_3 & 1 & \theta_1 \\ \theta_2 & -\theta_1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ -R \end{bmatrix} = \begin{bmatrix} \theta_2 R \\ -\theta_1 R \\ -R \end{bmatrix}$$

Gravity gradient torque becomes to

$$\mathbf{T}_g = \tfrac{3\mu}{R^5}\mathbf{R}_b \times \mathbf{IR}_b = \frac{3\mu}{R^5}\begin{bmatrix} 0 & R & -\theta_1 R \\ -R & 0 & \theta_2 R \\ \theta_1 R & -\theta_2 R & 0 \end{bmatrix}\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}\begin{bmatrix} \theta_2 R \\ -\theta_1 R \\ -R \end{bmatrix} = \frac{3\mu}{R^5}\begin{bmatrix} \begin{pmatrix} (I_3 - I_2)R^2\theta_1 \\ (I_3 - I_1)R^2\theta_2 \\ (I_1 - I_2)R^2\theta_2\theta_1 \end{pmatrix} \end{bmatrix}$$

Gravity-gradient torque in body fixed frame

Previous slide

$$\mathbf{T}_g = \frac{3\mu}{R^5}\left[\begin{pmatrix} (I_3 - I_2)R^2\theta_1 \\ (I_3 - I_1)R^2\theta_2 \\ (I_1 - I_2)R^2\theta_2\theta_1 \end{pmatrix}\right]$$

Neglecting second order term $O(\Theta_i\Theta_j)$ and using $\omega^2 = \frac{\mu}{R^3}$

Gravity-gradient torque in body frame

$$\boxed{\mathbf{T}_g = 3\omega^2\left[\begin{pmatrix} (I_3 - I_2)\theta_1 \\ (I_3 - I_1)\theta_2 \\ 0 \end{pmatrix}\right]}$$

Gravity-gradient torque for small angles

For small angles:

Θ_3 does not influence gravity-gradient torque (makes sense because Θ_3 represents rotation around vertical axes and is symmetric)

Angular velocity in body fixed frame

Angular velocity of spacecraft body frame with respect to inertial frame is decomposed into two terms: $\boldsymbol{\omega}_{bi} = \boldsymbol{\omega}_{bo} + \boldsymbol{\omega}_{oi}$

1. Angular velocity of body fixed frame with respect to orbiting frame
2. Angular velocity of orbiting frame with respect to inertia frame

Attitude kinematics relative orbiting frame are given by (seen in lecture VI)

$$\boldsymbol{\omega}_{bo} = \begin{bmatrix} 1 & 0 & -\sin\theta_2 \\ 0 & \cos\theta_1 & \sin\theta_1\cos\theta_2 \\ 0 & \sin\theta_1 & \cos\theta_1\cos\theta_2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

Since assume **small angle** between orbiting frame and body fixed frame →

$$\boldsymbol{\omega}_{bo} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

Angular velocity of orbiting frame relative to inertial frame has already been given in previous slide

$$\boldsymbol{\omega}_{oi} = \begin{bmatrix} 0 & -\omega & 0 \end{bmatrix}^T$$

Inertial angular velocity in body coordinates are

$$\begin{aligned} \boldsymbol{\omega}_{bi} &= \boldsymbol{\omega}_{bo} + \boldsymbol{\omega}_{oi} \\ &= \dot{\boldsymbol{\theta}} + \mathbf{C}_{bo}\boldsymbol{\omega}_{oi} \\ &= \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} + \begin{bmatrix} 1 & \theta_3 & -\theta_2 \\ -\theta_3 & 1 & \theta_1 \\ \theta_2 & -\theta_1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -\omega \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{\theta}_1 - \omega\theta_3 \\ \dot{\theta}_2 - \omega \\ \dot{\theta}_3 + \omega\theta_1 \end{bmatrix} \end{aligned}$$

Assume already stabilized spacecraft where deviations are small

Have angular velocity in terms of Euler angles and their rates

Euler equation

Euler equation and gravitational torque

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{h})_i = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{h})_b + \boldsymbol{\omega}\times\mathbf{h} = \mathbf{T}_g \qquad \mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\times\mathbf{I}\boldsymbol{\omega} = \mathbf{T}_g \qquad \mathbf{T}_g = \frac{3\mu}{R^5}\mathbf{R}\times\mathbf{IR}$$

Angular momentum in body frame

$$\mathbf{h} = \mathbf{I}\boldsymbol{\omega} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 - \omega\theta_3 \\ \dot{\theta}_2 - \omega \\ \dot{\theta}_3 + \omega\theta_1 \end{bmatrix} = \begin{bmatrix} (\dot{\theta}_1 - \omega\theta_3)I_1 \\ (\dot{\theta}_2 - \omega)I_2 \\ (\dot{\theta}_3 + \omega\theta_1)I_3 \end{bmatrix}$$

$$\dot{\mathbf{h}} = \mathbf{I}\dot{\boldsymbol{\omega}} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 - \omega\dot{\theta}_3 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 + \omega\dot{\theta}_1 \end{bmatrix} = \begin{bmatrix} (\ddot{\theta}_1 - \omega\dot{\theta}_3)I_1 \\ \ddot{\theta}_2 I_2 \\ (\ddot{\theta}_3 + \omega\dot{\theta}_1)I_3 \end{bmatrix}$$

Euler equation in body frame and assume principal axis

$$\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 - \omega\dot{\theta}_3 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 + \omega\dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} \dot{\theta}_1 - \omega\theta_3 \\ \dot{\theta}_2 - \omega \\ \dot{\theta}_3 + \omega\theta_1 \end{bmatrix} \times \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 - \omega\theta_3 \\ \dot{\theta}_2 - \omega \\ \dot{\theta}_3 + \omega\theta_1 \end{bmatrix} = 3\omega^2 \left[\begin{pmatrix} (I_3 - I_2)\theta_1 \\ (I_3 - I_1)\theta_2 \\ 0 \end{pmatrix} \right]$$

Substitute into Euler´s equation Dynamic equation for gravity-gradient

$$\mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}\boldsymbol{\omega} = \frac{3\mu}{R^5}\mathbf{R} \times \mathbf{I}\mathbf{R}$$

$$\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 - \omega\dot{\theta}_3 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 + \omega\dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} \dot{\theta}_1 - \omega\theta_3 \\ \dot{\theta}_2 - \omega \\ \dot{\theta}_3 + \omega\theta_1 \end{bmatrix} \times \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}\begin{bmatrix} \dot{\theta}_1 - \omega\theta_3 \\ \dot{\theta}_2 - \omega \\ \dot{\theta}_3 + \omega\theta_1 \end{bmatrix} = 3\omega^2 \left[\begin{pmatrix} (I_3 - I_2)\theta_1 \\ (I_3 - I_1)\theta_2 \\ 0 \end{pmatrix}\right]$$

Use linearization (second order products $O(\Theta_i\Theta_j)$ and their derivations are neglected)

$$I_1\ddot{\theta}_1 - [(I_1 - I_2 + I_3)\omega]\dot{\theta}_3 + [4\omega^2(I_2 - I_3)]\theta_1 = 0$$

$$I_2\ddot{\theta}_2 + [3\omega^2(I_1 - I_3)]\theta_2 = 0$$

$$I_3\ddot{\theta}_3 - [(I_1 - I_2 + I_3)\omega]\dot{\theta}_1 + [\omega^2(I_2 - I_1)]\theta_3 = 0$$

<u>Important:</u>

Equations describing attitude motion of spacecraft relative to orbiting frame are approximations with assumptions: small angles and small angular rates

Stability analysis

Stability analysis of attitude motion
(three coupled linear differential equations)

$$\boxed{\begin{aligned} &I_1\ddot{\theta}_1 - [(I_1 - I_2 + I_3)\omega]\dot{\theta}_3 + [4\omega^2(I_2 - I_3)]\theta_1 = 0 \\ &I_2\ddot{\theta}_2 + [3\omega^2(I_1 - I_3)]\theta_2 = 0 \\ &I_3\ddot{\theta}_3 - [(I_1 - I_2 + I_3)\omega]\dot{\theta}_1 + [\omega^2(I_2 - I_1)]\theta_3 = 0 \end{aligned}}$$

Pitch motion is decoupled from other two motions

Possible:
Check for stability by taking Ansatz (or Laplace transform) and looking for roots

Stability of pitch equation

$$I_2\ddot{\theta}_2 + [3\omega^2(I_1 - I_3)]\theta_2 = 0$$

$$\ddot{\theta}_2 - \lambda^2\theta_2 = 0 \qquad \lambda^2 = 3\omega^2 \frac{I_3 - I_1}{I_2}$$

$\theta(t) = A\mathrm{e}^{\lambda t} + B\mathrm{e}^{-\lambda t}$ $\quad \lambda^2 > 0 \quad (I_3 > I_1)$ Solution grows exponentially with time → Unstable

$\theta(t) = A + Bt$ $\quad \lambda^2 = 0 \quad (I_3 = I_1)$ Solution grows linear with time → Unstable

$\theta(t) = A\mathrm{e}^{\mathrm{i}\lambda t} + B\mathrm{e}^{-\mathrm{i}\lambda t}$ $\quad \lambda^2 < 0 \quad (I_3 < I_1)$ Solution is oscillatory → Motion stable

Pitch motion is stable only if $\boxed{I_1 > I_3}$

Means Earth-pointing axis cannot be major axis

Search solutions for roll/yaw equation

$$I_1\ddot{\theta}_1 - [(I_1 - I_2 + I_3)\omega]\dot{\theta}_3 + [4\omega^2(I_2 - I_3)]\theta_1 = 0$$

$$I_3\ddot{\theta}_3 - [(I_1 - I_2 + I_3)\omega]\dot{\theta}_1 + \quad [\omega^2(I_2 - I_1)]\theta_3 = 0$$

To check stability look for solution of form:

$$\theta_1(t) = C_1 \mathrm{e}^{\lambda t}$$

$$\theta_3(t) = C_3 \mathrm{e}^{\lambda t}$$

Insert assumed solution in above equation

$$I_1 C_1 \lambda^2 - [(I_1 - I_2 + I_3)\omega] C_3 \lambda + [4\omega^2(I_2 - I_3)] C_1 = 0$$

$$I_3 C_3 \lambda^2 - [(I_1 - I_2 + I_3)\omega] C_1 \lambda + \quad [\omega^2(I_2 - I_1)] C_3 = 0$$

Characteristic equation Matrix form of roll/yaw equation

Matrix has non-trivial solution if it is singular

If determinant of matrix is zero

$$\begin{bmatrix} I_1\lambda^2 + [4\omega^2(I_2 - I_3)] & -[(I_1 - I_2 + I_3)\omega]\lambda \\ [(I_1 - I_2 + I_3)\omega]\lambda & I_3\lambda^2 + [\omega^2(I_2 - I_1)] \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Characteristic equation (determinant zero) is polynomial in λ

$$\lambda^4 + (1 + 3k_1 + k_1k_3)\lambda^2\omega^2 + 4k_1k_3\omega^4 = 0$$

$$k_1 = \frac{I_2 - I_3}{I_1} \qquad k_3 = \frac{I_2 - I_1}{I_3}$$

Search condition for stability

With following definitions $s = \frac{\lambda^2}{\omega^2} \quad p = 1 + 3k_1 + k_1 k_3 \quad q = 4k_1 k_3$

Characteristic equation becomes quadratic in *s*

$$s^2 + ps + q = 0$$

Solution of *s* are given by $s = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$

To each solution *s* values of $\lambda = \pm\omega\sqrt{s}$

Now should check four possible solution for *s* (*s* < 0, *s* = 0, *s* > 0 and *s* with imaginary component) → if solutions have some positive real part → unstable

Motion is stable (oscillatory behavior of Θ_1 and Θ_3) only if *s* real and *s* < 0

$\Rightarrow$ Eigenvalues λ must be pure imaginary

Necessary condition to be *s* real and negative is $p^2 - 4q > 0 \quad p > 0 \quad q > 0$

Stability requirements

$$p^2 - 4q > 0 \quad p > 0 \quad q > 0 \qquad p = 1 + 3k_1 + k_1k_3 \quad q = 4k_1k_3$$

Pitch/yaw stability requires following three conditions

$$(1 + 3k_1 + k_1k_3)^2 - 16k_1k_3 > 0$$
$$1 + 3k_1 + k_1k_3 > 0$$
$$k_1k_3 > 0$$

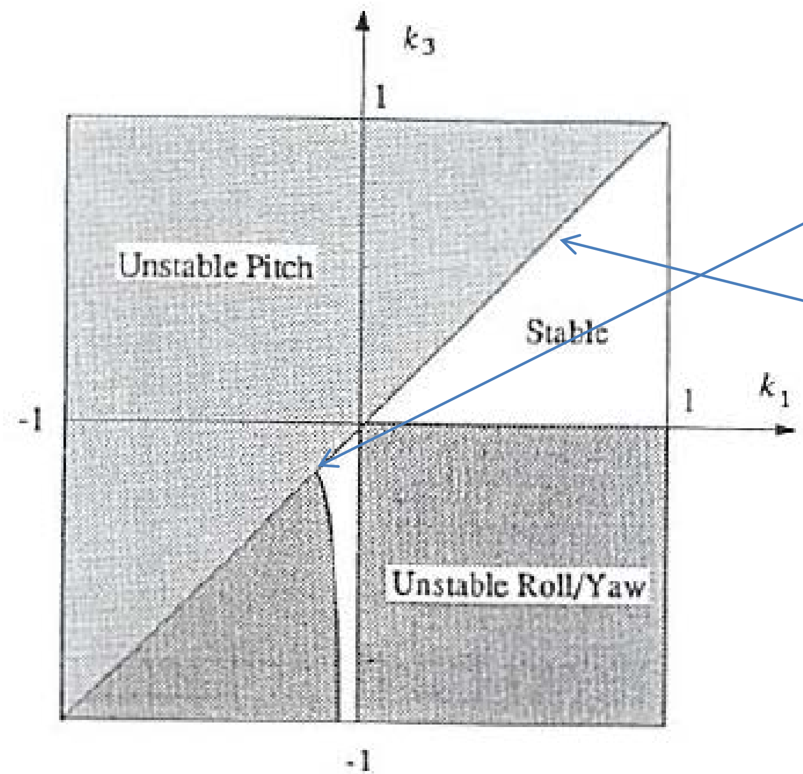
In order to hold these conditions need $k_1 > 0$ and $k_3 > 0$
Using definitions of k_1 and k_3 $\rightarrow$ $I_2 > I_3$ and $I_2 > I_1$
Roll-yaw motion is stable if $I_2 > I_3$ and $I_2 > I_1$

Combine with pitch stability $I_1 > I_3$
Final result is $\boxed{I_2 > I_1 > I_3}$
Spacecraft with principal axes aligned with orbiting frame is stable with respect to gravity-gradient torque if $I_2 > I_1 > I_3$ is satisfied
Means: pitch inertia > roll inertia > yaw inertia

Stability diagram for gravity-gradient in circular orbits

Stability depends only on inertia ratios k_1 and k_3 $\quad k_1 = \frac{I_2 - I_3}{I_1} \quad k_3 = \frac{I_2 - I_1}{I_3}$

Stability can be summarized in following plot:



Can be shown: $|k_1| < 1 \quad |k_3| < 1$

$$(1 + 3k_1 + k_1k_3)^2 - 16k_1k_3 > 0$$

$$1 + 3k_1 + k_1k_3 > 0$$

$$k_1k_3 > 0$$

$$k_1 > k_3$$

Gravity-gradient stabilization in summary

What is end effect of gravity gradient?

Spacecraft tends to align its axis of minimum momentum of inertia vertically

Build gravity-gradient stabilized spacecraft with $\boxed{I_2 > I_1 > I_3}$

Stability condition valid also for noncircular orbits, however situation slightly more complicated

Gravity-gradient stabilization is limited because of weakness of gravity-gradient torque

Damping needed

$$I_1\ddot{\theta}_1 - [(I_1 - I_2 + I_3)\omega]\dot{\theta}_3 + [4\omega^2(I_2 - I_3)]\theta_1 = 0$$

$$I_2\ddot{\theta}_2 + [3\omega^2(I_1 - I_3)]\theta_2 = 0$$

$$I_3\ddot{\theta}_3 - [(I_1 - I_2 + I_3)\omega]\dot{\theta}_1 + [\omega^2(I_2 - I_1)]\theta_3 = 0$$

Undamped motion

Damping needed to remove pendulum like oscillations due to disturbances

Magnetic hysteresis rods or dampers

Discussion of gravity-gradient stabilization with small disturbances

$\Rightarrow$Found equilibrium orientation

$\Rightarrow$However will oscillate about equilibrium point if no energy dissipation

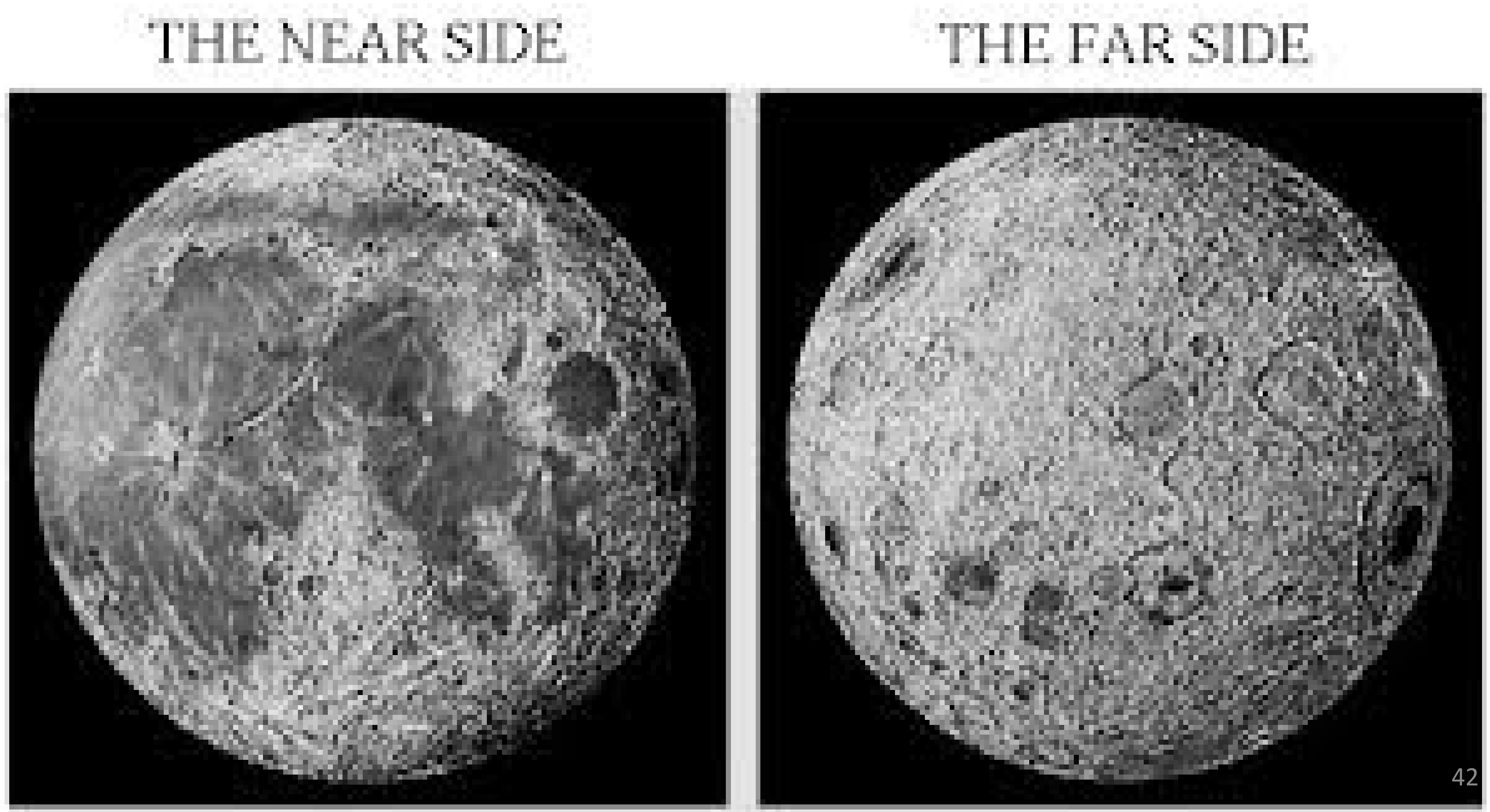
Liquid damper

- Damping accomplished by using various devices that dissipate energy by providing friction
- E.g. can use viscous fluid in partially filled tube to damp oscillation
- Spacecraft oscillation will cause fluid motion along tube

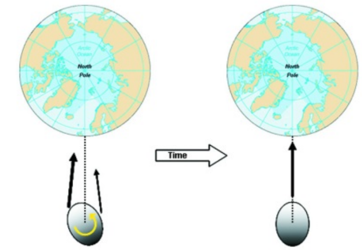
  → Dissipate part of energy through viscous effects or friction

Moon

Same reasoning as gravity-gradient stability answers also why moon keeps same face turned towards Earth all times (Lagrange)



Same side of Moon



In past force of gravity from Earth produced elongated shaped moon

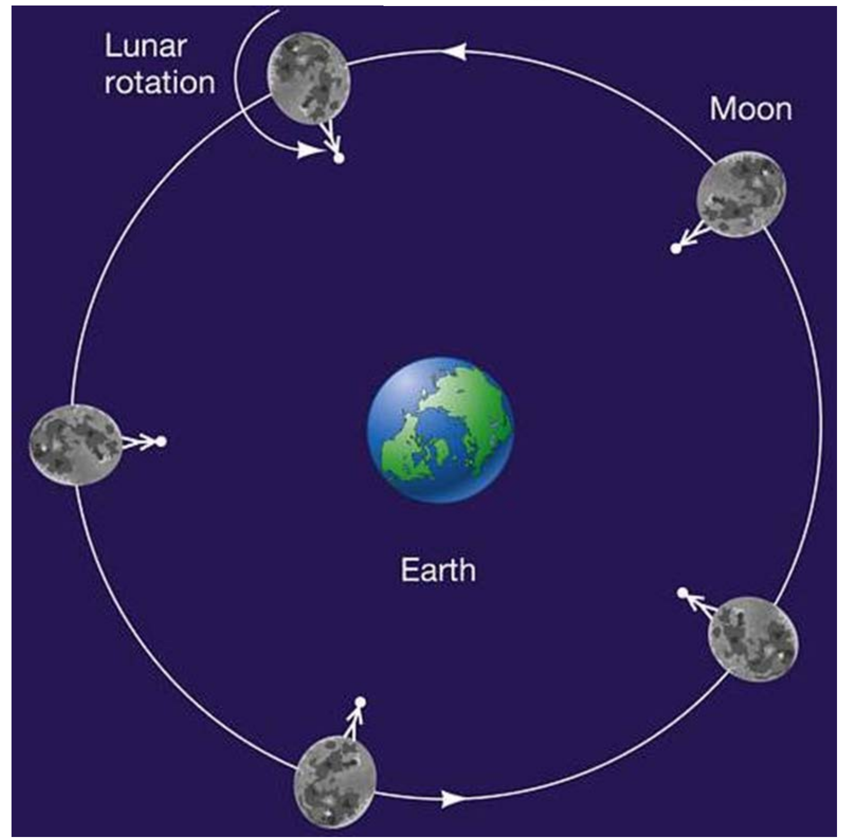
Moon is today slightly elongated along axis which points towards Earth

Moon aligned its axis of minimum momentum of inertia vertically

Local vertical in Earth-Moon system points towards Earth

See always same side of Moon:

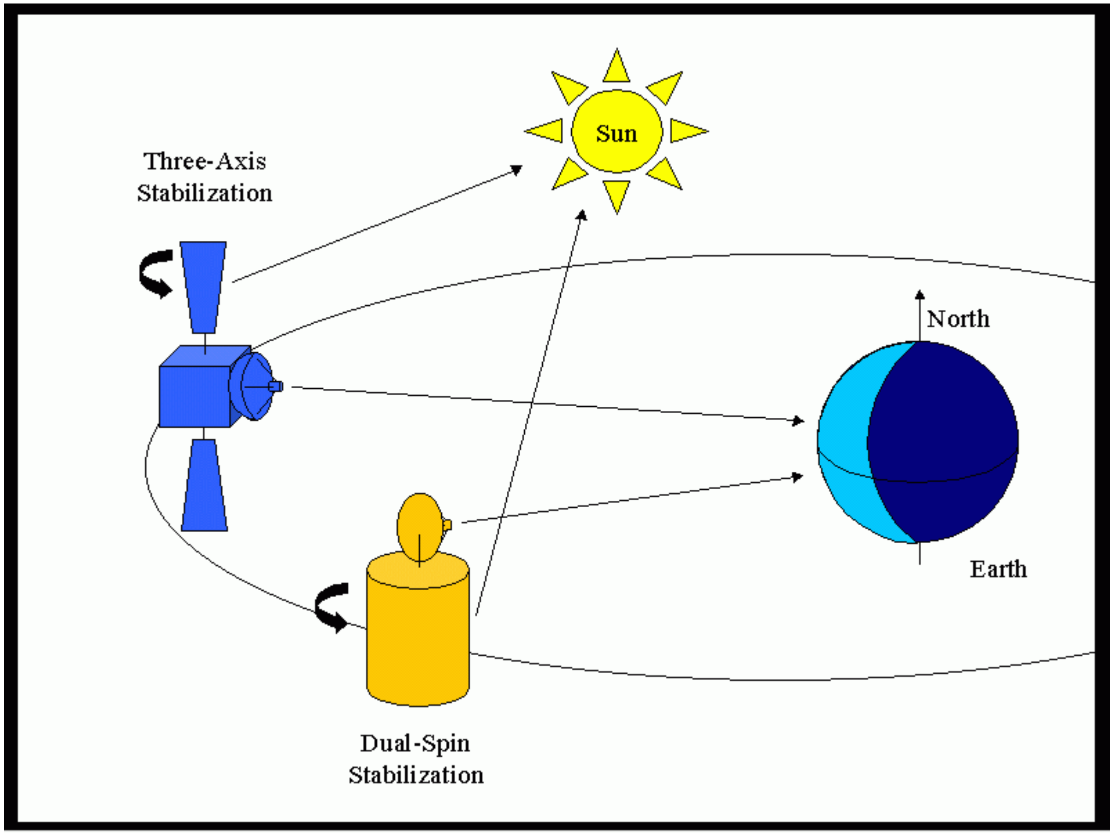
Spin of Moon is equal to its revolution around Earth (27.3 days)



Moon is slightly elongated in shape, with its long axis always pointing toward Earth (Figure not to scale)

Three axis stabilization

Three axis stabilization



Three axis stabilization:
Spacecraft attitude is maintained through use of momentum wheels or control moment gyros
→ **active attitude control**

Body of spacecraft rotates once per orbit to keep antennas pointed at Earth

Solar arrays mounted on paddles also rotate once per orbit to keep them pointed toward Sun

Design spacecraft with passive stability (if possible) and then augment this with active control
Since natural stability → Control system does not need work as hard to maintain required attitude
→ Attitude remains also stable if control system fails

Three axis stabilization

- Instead of keeping only spin axis pointing in specific direction, keep all 3 axes pointed in specific directions
- Can be done with thrusters, reaction wheels, momentum wheels, control momentum gyros, etc.
- 3-axes stabilized spacecraft characterized by body that has roughly box-shape and by deployable solar-array panels
- Many examples exist, e.g. GEO communications satellite, etc.
- Bodies of GEO communications satellite are usually kept inertial stable except for slow rotation induced about one axis to keep payload antennas or sensors continuously pointed toward Earth as satellite orbits
- Solar-array panels are counter rotated relative to spacecraft body to keep them inertial fixed on Sun
- Need of **active attitude control** for three axis stabilization

Three axis stabilization

Active spacecraft attitude control system consists of:

- Attitude sensors
- Attitude actuators
- Program on processor

Attitude sensors take measurements which are used to compute current spacecraft attitude and/or angular velocity

Attitude actuators then supply torques to correct difference between measured and desired attitude

Program on processor has implemented mathematical relationships between measured attitude and corrective torques (so called **control law**)

More on three axis stabilization in lecture ADCS - X

Summary

Spin stabilization (passive attitude control)

Intrinsic gyroscopic stiffness

if spin → perturbation angle with linear growth in time

If no spin → perturbation angle with quadratic growth in time

Dual-spin stabilization (passive attitude control)

Even intermediate axis spins can be made stable by making wheel stored angular momentum h_s large enough

Gravity-gradient stabilization (passive attitude control)

pitch inertia > roll inertia > yaw inertia $\leftrightarrow \quad I_2 > I_1 > I_3$

Three axis stabilization (active attitude control)