ADCS – X
Attitude Control

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Summary of last lecture

Spin stabilization (passive attitude control)

Intrinsic gyroscopic stiffness

if spin → perturbation angle with linear growth in time

If no spin → perturbation angle with quadratic growth in time

Dual-spin stabilization (passive attitude control)

Even intermediate axis spins can be made stable by making wheel stored angular momentum h_s large enough

Gravity-gradient stabilization (passive attitude control)

pitch inertia > roll inertia > yaw inertia $\leftrightarrow$ $I_2 > I_1 > I_3$

Three axis stabilization (active attitude control)

Outline

Overview of attitude control and terminology

Mathematical formulation of attitude control

- Formulate attitude control problem
- Transfer function
- Block diagrams
- Control laws
- Time-domain specification

Steady-state specifications

Laplace transform

Attitude determination and control

Previous lectures showed how current attitude of spacecraft (angular position with respect to reference frame) is determined and also seen passive attitude control

This lecture describes process by which know attitude is corrected to desired attitude → active attitude control

Two attitude control methods:

Passive attitude control

$$\frac{\mathrm{d}\vec{h}}{\mathrm{d}t} = \frac{\mathrm{d}(\mathbf{I}\vec{\omega})}{\mathrm{d}t} = \vec{T}_{disturbance}$$

Exploit existing disturbance torques

Active attitude control

$$\frac{\mathrm{d}\vec{h}}{\mathrm{d}t} = \frac{\mathrm{d}(\mathbf{I}\vec{\omega})}{\mathrm{d}t} = \vec{T}_{control}$$

Apply with actuators control torque

Hardware to control system

To control system two types of hardware component are needed:

- **Sensors** (Measure, sense state of system)
  Measurement used for attitude determination
  Translate sensor response into parameter values
  → describe attitude parameterization
- **Actuators** (Used to control and adjust state of system)
  Control used for dynamical response by applying torques
  → describe dynamics (satellite, actuators, disturbance torques)

Hardware and software to control system

Active spacecraft attitude control system consists of:

- Attitude sensors
- Attitude actuators
- Program on processor

Attitude sensors take measurements which are used to compute current spacecraft attitude and/or angular velocity

Attitude actuators then supply torques to correct difference between measured and desired attitude

Program on processor has implemented mathematical relationships between measured attitude and corrective torques (so called **control law**)

(Today´s lecture)

Typical spacecraft control system

- Satellite attitude is measured and compared with a desired value

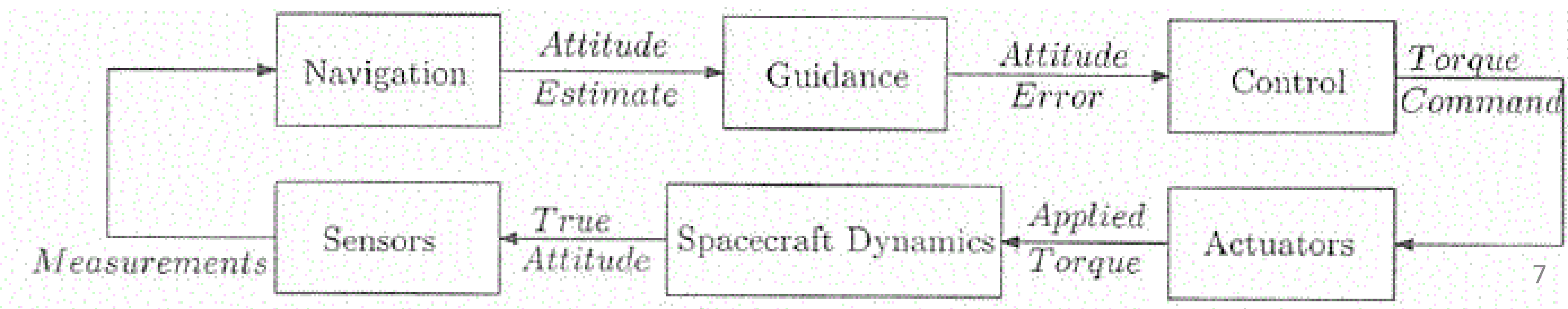
  → Attitude error
- Attitude error is used to determine corrective torque to be applied by onboard actuators

  → New attitude
- Cycle continuous indefinitely because:

  - disturbance torque occur

  - attitude measurement is imperfect

  - attitude correction is imperfect



Attitude control system

Attitude **control system** is based on three major parts:

- Navigation – where am I ?
- Guidance – where do I want to be ?
- Control – how do I get there ?

In other words:

- Attitude sensor (provide measurement of satellite position)
- Feedback control system (corrects measured attitude to desired attitude)
- Actuator (provide desired control torque)

But what do we mean with control system?

What is control system?

Control system example: Shower

Find shower temperature: Not too hot, not too cold

Adjust water temperature to within acceptable tolerance

(Do not burn yourself and do not need to much time)

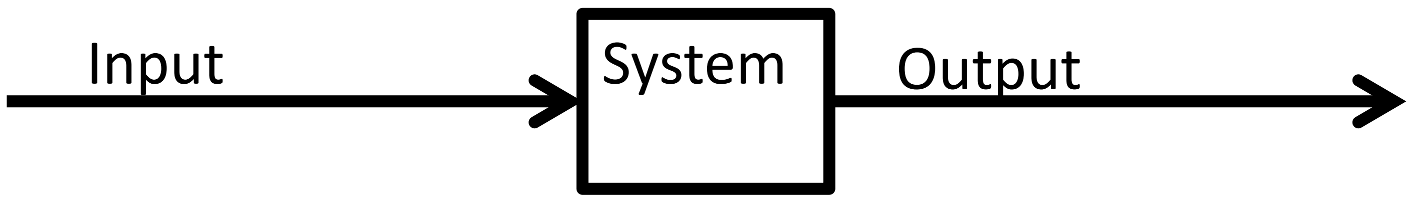
How to do it? (Human-loop control)

- Too cold → rotate water tap clockwise
- Too hot → rotate water tap counterclockwise
- Much too cold → rotate faster
- Much too hot → rotate faster

If too fast → overshoot

Abstract system using block diagrams

System

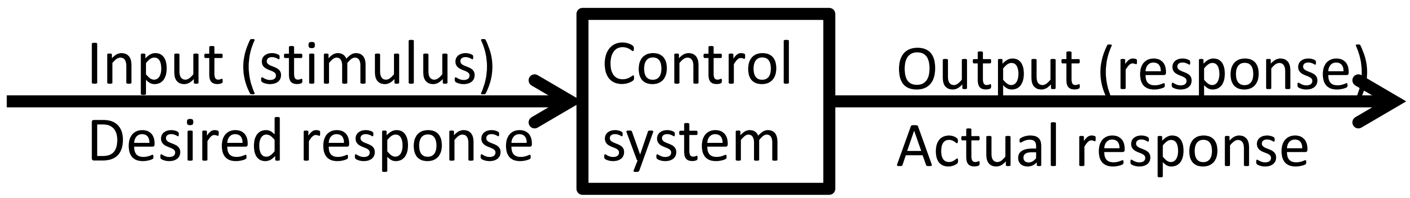


System is anything with inputs and outputs

Any system to be controlled has parameters the control system can change to make the system achieve desired behavior

The system being controlled is called **plant**

Control system



Control system is system which modifies inputs to plant to produce desired outputs

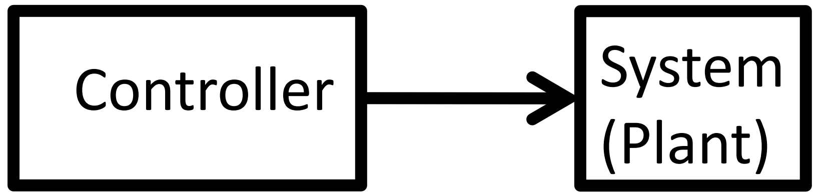
For satellite control:

- Inputs: Actuators or disturbances
- Outputs: Sensor measurements

- Control system is the brain of any system
- Purpose of control system is to make system perform desired task
- For satellite: purpose of control system is to achieve or maintain desired attitude

Controller

Actuator is the mechanism by which **controller** affects input of the plant



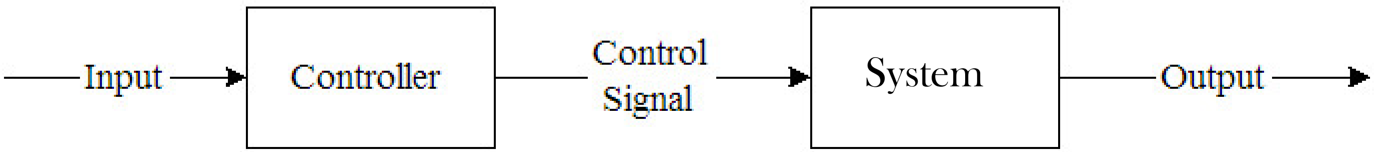
Two types of control exist:

- **Open-loop control**
  Controller actuation independent of output
  Use of actuators
- **Closed-loop control**
  Input to controller modified based on measurement from output
  Use of sensor and actuators

Sensor is the mechanism by which controller detects output of the plant

Control law is a principle on which controller is designed to achieve desired overall system performance

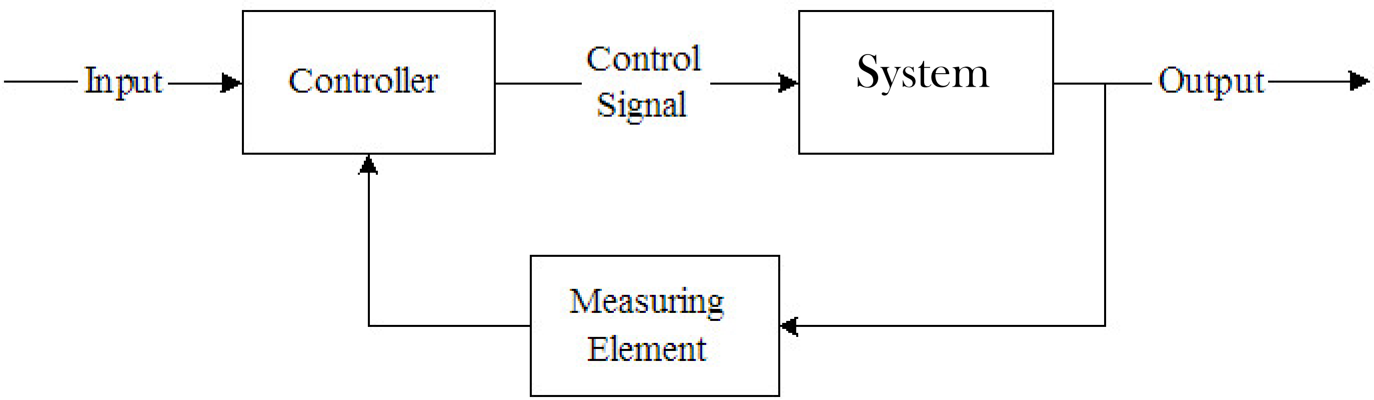
Open-loop versus closed-loop



Open Loop System

No feedback of information to control system

Like driving car with closed eyes



Closed Loop System

If any error or disturbance
$\Rightarrow$ Loosing control

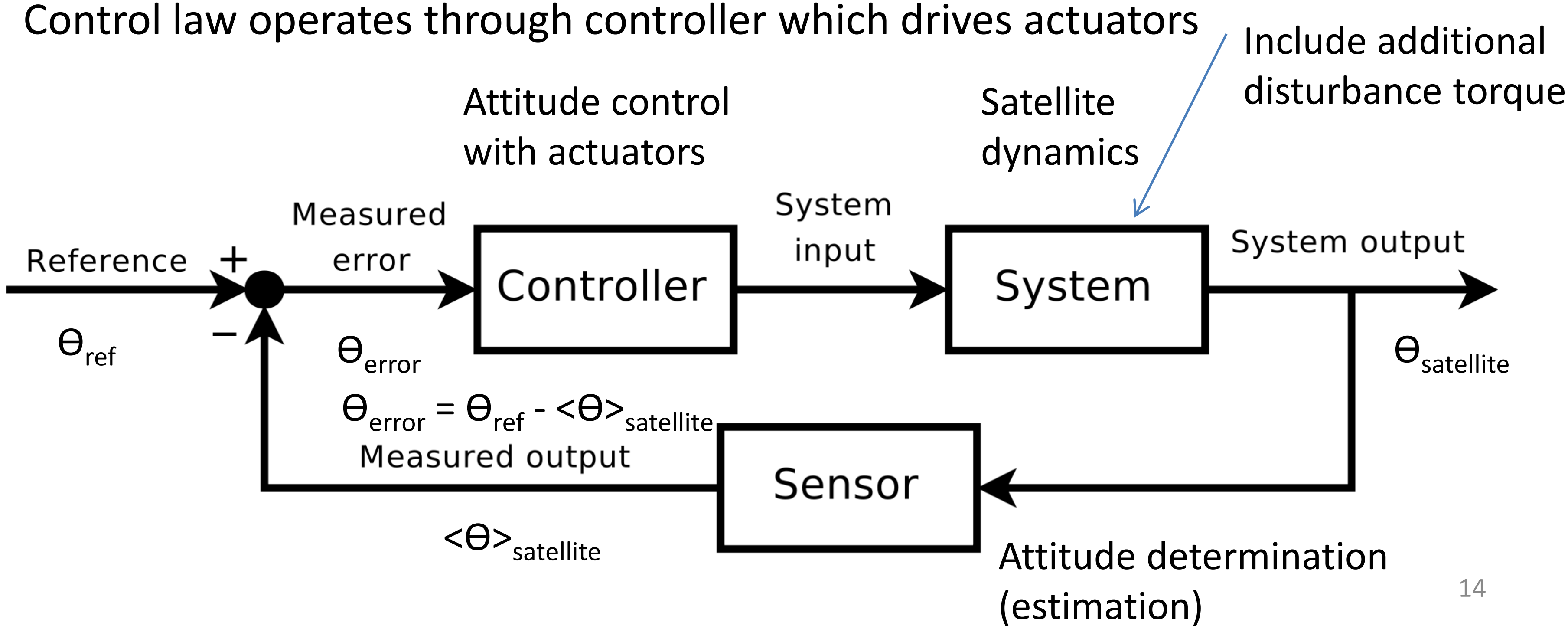
Satellite needs control system with feedback to correct for any errors

Closed-loop control system also called **feedback control** system

Typical feedback control system

Feedback control provides data on difference between desired and actual attitude

- Typical feedback control for satellite attitude with on-board control
- Require desired (reference) attitude (e.g. point antenna to Earth)
- Sensor estimate actual attitude of satellite to generate attitude error
- Attitude error, input to control law
- Control law operates to drive error signal to zero within some limit
- Control law operates through controller which drives actuators



Mathematical formulation of attitude control

Formulate attitude control problem

- Formulate attitude control problem in case of body fixed frame relative to initially fixed reference frame
- Assume: 3-2-1 Euler sequence and body principle axis frame

Kinematic equation

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\theta_2 \\ 0 & \cos\theta_1 & \sin\theta_1\cos\theta_2 \\ 0 & -\sin\theta_1 & \cos\theta_1\cos\theta_2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

nonlinear equations

Dynamic equation

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = T_{1c} + T_{1d}$$
$$I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 = T_{2c} + T_{2d}$$
$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 = T_{3c} + T_{3d}$$

External torque split into two components:

- T_c = control torque (can be applied by actuators)
- T_d = disturbance torque (present but unwanted)

Linearization of attitude control problem

- Restrict to control design of **linear system**
- Assume small angles and small angle rates

Kinematic equation

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

Dynamic equation

$$I_1\ddot{\theta}_1 = T_{1c} + T_{1d}$$
$$I_2\ddot{\theta}_2 = T_{2c} + T_{2d}$$
$$I_3\ddot{\theta}_3 = T_{3c} + T_{3d}$$

Neglect second order terms

Dynamic equations are **decoupled** and have same form

$\Rightarrow$ Consider only one axis at a time

$\Rightarrow$ Can be written as $$I\ddot{\theta} = T_c + T_d$$

Use of control system terminology

Use of control system terminology:

1. System being controlled is called plant
2. Plant is described by spacecraft attitude dynamics $I\ddot{\theta} = T_c + T_d$
3. Interested quantity to control is attitude angle *Θ*
4. Angle *Θ* called **plant output** and generally labeled as *y*(*t*)
   → $$y(t) = \theta(t)$$
5. Torqueing devices or actuators is mechanism by which controller affects attitude and is called **plant input** (generally labeled as *u*(*t*))
   → $$u(t) = T_c(t)$$

Transfer function representation of system

- Control system design interested in response of output to **zero initial conditions**
- **Laplace transform *L*** (LT) of $I\ddot{\theta} = T_c + T_d$ with zero initial conditions→

$$Is^2\hat{\theta}(s) = \hat{T}_c(s) + \hat{T}_d(s)$$

- Use of output $Y(s) = \hat{\theta}(s)$ and input $U(s) = \hat{T}_c(s)$ give relationship between output and input $Y(s) = \frac{1}{Is^2}\left(U(s) + \hat{T}_d(s)\right)$
- **Transfer function** *G*(*s*) is defined from LT of plant output to LT of input

$$\boxed{G(s) \equiv \frac{L(output)}{L(input)}} = \frac{Y(s)}{U(s)}\bigg|_{\hat{T}_d(s)=0} = \frac{1}{Is^2}$$

where *G*(*s*) describes how output behaves in response of given input

- In particular $Y(s) = \frac{1}{Is^2}\left(U(s) + \hat{T}_d(s)\right)$ becomes $Y(s) = G(s)\left(U(s) + \hat{T}_d(s)\right)$

Form of transfer function

For linear time-invariant system with one input and one output the **transfer function** will in general have following form

$$G(s) = \frac{a_0 s^m + a_1 s^{m-1} + \ldots + a_m}{s^n + b_1 s^{n-1} + \ldots b_n}$$

- Ratio of two polynomials in *s* (complex) with $m \leq n$ and a_i, b_i real coefficients
- *G*(*s*) can be factored into $G(s) = K \frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{i=1}^{n}(s - p_i)}$

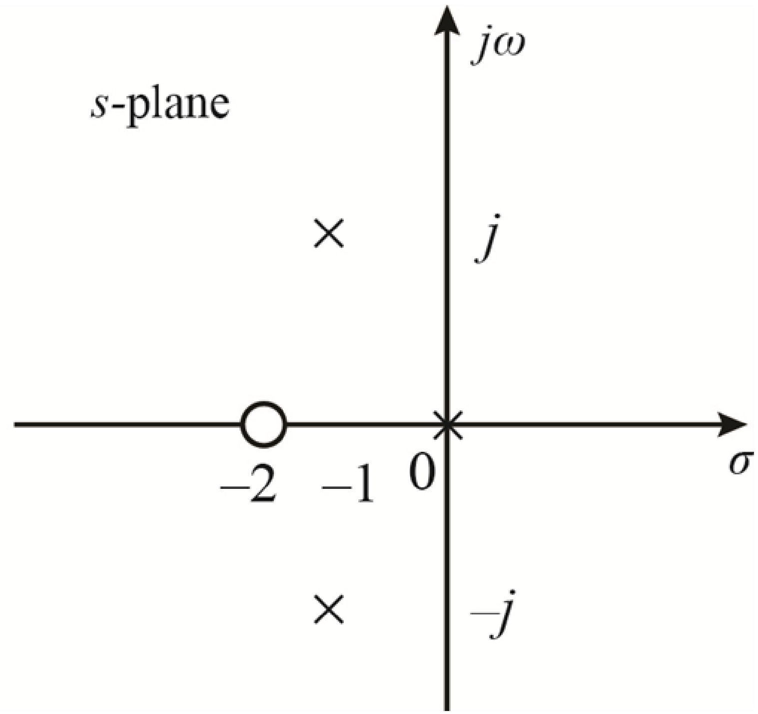
  where z_i are zeros of polynomial, called **zeros** of *G*(*s*) and

  where p_i are zeros of denominator polynomial, called **poles** of *G(s)*
- Transfer function is an algebraic equation in *s* which is derived from differential equation representing system dynamics
- Remark: **Poles and zeros completely characterize transfer function** (and therefore system) except for an overall constant *K*

Pole-zero plots

Transfer function for given system:

$$G(s)=\frac{10(s+2)}{(s)(s^2+2s+2)}$$
$$=\frac{10(s+2)}{s(s+1+j)(s+1-j)}$$



Poles of function can be determined by equating denominator to zero

$$s(s+1+j)(s+1-j)=0$$
$$s=0,-1+j,-1-j$$

Poles of function

Zeros of function can be determined by equating numerator to zero

$$(s+2)=0$$
$$s=-2$$

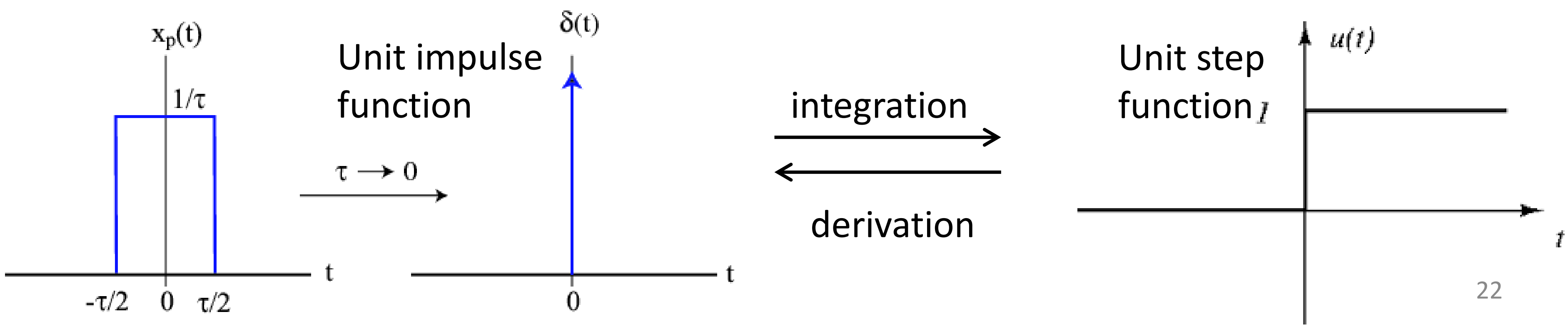
Zero of function

Pole position denoted by **x** and zeros drawn as **o**

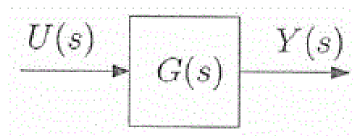
Pole-zero plot is used extensively in control theory and system dynamics to provide qualitative indication of dynamics behavior of system

Common input signals used in control

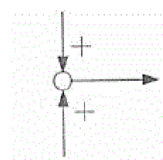
- In analyzing and designing control systems, one must have a basis of comparison of performance of various control systems. This basis may be setup by specifying particular **test input signals** and by comparing responses of various systems to theses input signals.
- Two classes of inputs signals used to characterize the performance of feedback control systems:
  - **unit impulse** function (Dirac delta function)
    (used to characterize response of system to brief intense input)
  - **unit step** function
    (used to characterize system transient response to sudden change)

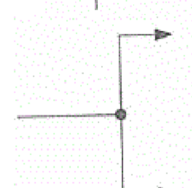


Block diagrams

- A **block diagram** of a system is a pictorial representation of the functions performed by each component and of the flow of signals
- Block diagram helps to show the signal flow of the system (in contrast to abstract mathematical representation)
- **Transfer function *G*(*s*) relates inputs *U*(*s*) to outputs *Y*(*s*)**
  - Mathematical representation $Y(s) = G(s)U(s)$
  - Block diagram representation 

Signal represented by arrow $\longrightarrow$

Summing point represented by 

Branch point represented by 

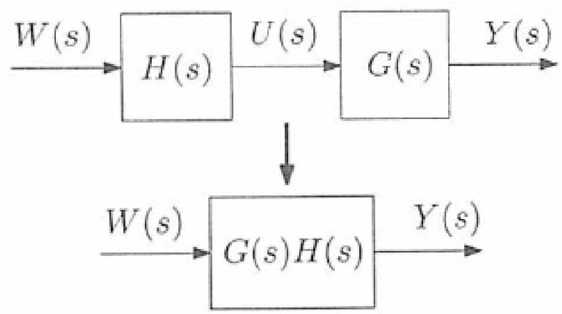
Block diagrams

- **Series connection**:

  Suppose *U(s)* is output of another system with transfer function *H*(*s*) with input *W*(*s*)

  → *U(s) = H(s)W(s)*

  → Have *Y(s) = G(s)H(s)W(s)* and transfer function for combined system is $\boxed{G(s)H(s)}$



Means:

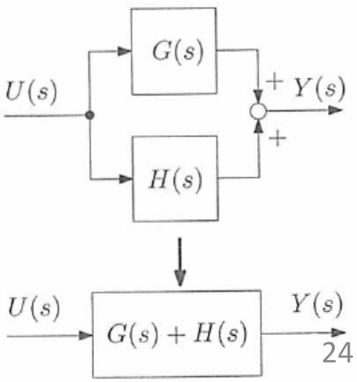
Block diagrams represent flow of information with **transfer function blocks** operating on inputs using *multiplication* to yield outputs

- **Parallel connection**:

  Parallel connection between *G(s)* and *H(s)* is

  Y(s) = G(s)U(s) + H(s)U(s) = (G(s) + H(s))U(s)

  and transfer function is $\boxed{G(s) + H(s)}$



Block diagrams

- **Feedback interconnection or closed loop system**:

  Mathematics is as follows:

  Let be *A(s) = H(s)Y(s)* be output of H(s)

  Let be *B(s) = U(s) - A(s)*

  → *Y*(*s*) = *G(s)B(s)*

  = *G*(*s*)[*U(s)* – *A*(s)]

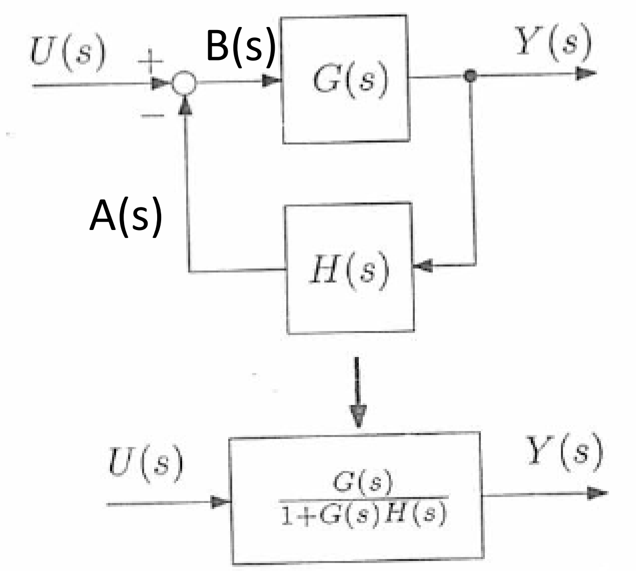
  = *G(s)*[*U*(s) – *H(s)Y(s)*]

  = *G(s)U(s)* – *G(s)H(s)Y(s)*

  → Y(s) = $\frac{G(s)}{1+G(s)H(s)} U(s)$

  Therefore **transfer function for feedback interconnection** is given by

$$\boxed{\frac{G(s)}{1+G(s)H(s)}}$$



Feedback control with disturbance

- Plant (spacecraft attitude dynamics) is represented by

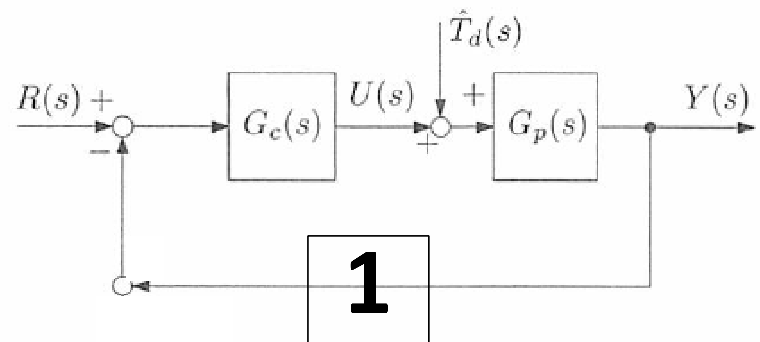
$$Y(s) = G_p(s)\left(U(s) + \hat{T}_d(s)\right)$$

with $G_p(s) = \frac{1}{Is^2}$ and subscript *p* means transfer function from plant

- Output $Y(s) = \Theta(s)$ is attitude

- Input $U(s) = T_c(s)$ is control torque and $T_d(s)$ is disturbance torque

- Goal is to make output $y(t)$ (attitude $\Theta(t)$) follow reference signal $r(t)$ (desired attitude $\Theta_d(t)$) with Laplace transform $R(s) = \Theta_d(s)$
- Control input must correct output error (error in attitude $e(t) = \Theta_d(t) - \Theta(t) = r(t) - y(t)$); means goal is to drive error $e(t)$ to zero
- Let represent control law by transfer function $G_c(s)$ such that $U(s) = G_c(s)E(s)$ with $E(s) = L(e(t))$
- Assume here $H(s) = 1$ (***unity feedback***)

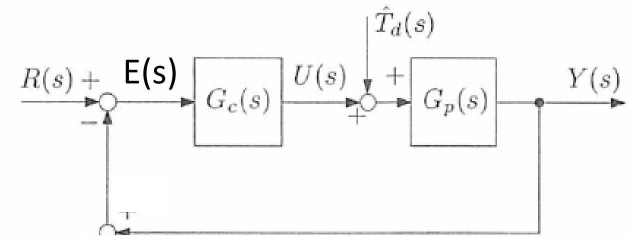


From now on drop the **1**

Feedback control with disturbance

- System has two inputs:
  - Reference signal R(s)
  - Disturbance T_d(s)

$$Y(s) = G_p(s)\left(U(s) + \hat{T}_d(s)\right)$$
$$U(s) = G_c(s)E(s)$$



- **System is linear** → output of simultaneous application of reference signal and disturbance is obtained by adding individual response
- Let Y_r(s) be response to reference signal R(s) with T_d(s) = 0

$$Y_r(s) = G_p(s)G_c(s)E(s) \quad \text{with} \quad \mathrm{E(s)=R(s)\text{-}Y}_r\mathrm{(s)} \;\Leftrightarrow\; Y_r(s) = G_p(s)G_c(s)(\mathrm{R(s)\text{-}Y}_r\mathrm{(s)})$$

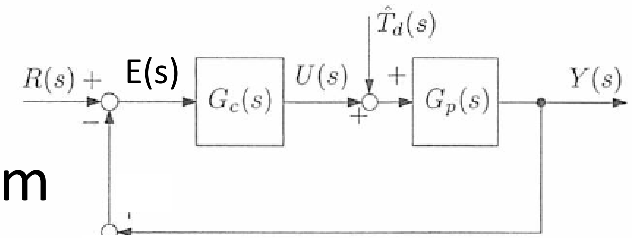
$$\rightarrow \quad Y_r(s) = \frac{G_p(s)G_c(s)}{1+G_p(s)G_c(s)}R(s)$$

- Let Y_d(s) be response to reference signal T_d(s) with R(s) = 0

$$Y_d(s) = G_p(s)\left(U(s) + \hat{T}_d(s)\right) = G_p(s)G_c(s)E(s) + G_p(s)\hat{T}_d(s) \quad with \quad E(s) = -Y_d(s)$$

$$\rightarrow \quad Y_d(s) = \frac{G_p(s)}{1+G_p(s)G_c(s)}\hat{T}_d(s)$$

Feedback control with disturbance



- Since system is linear

  → combined response to R(s) and T_d(s) is sum

$$Y(s) = Y_r(s) + Y_d(s) = \frac{G_p(s)G_c(s)}{1+G_p(s)G_c(s)} R(s) + \frac{G_p(s)}{1+G_p(s)G_c(s)} \hat{T}_d(s)$$

- Goal: Make Y(s) ≈ R(s) (means E(s) small despite disturbance)

  In words: Output Y(s) should match reference signal R(s)

  G_p(s) is plant and cannot be changed

  G_c(s) is controller and can be changed

  → Make control large $|G_c(s)| \to \infty$

$$\frac{G_p(s)G_c(s)}{1+G_p(s)G_c(s)} \to 1 \quad and \quad \frac{G_p(s)}{1+G_p(s)G_c(s)} \to 0$$

  → Get perfect tracking Y_r(s) = R(s)

  → Response to disturbance becomes suppressed (Y_d(s) = 0)

- Disadvantage of feedback, noise also affected (not shown here)

Typical control laws

Proportional control $\boxed{u(t) = K_p e(t)}$

Proportional gain

Control input is proportional to error signal with $K_p > 0$

Associated control transfer function $G_c(s) = K_p$

$$Y(s) = \frac{1}{Is^2} G_c(s) E(s)$$

Equation of motion for spacecraft $I\ddot{y}(t) = u(t) = K_p e(t) = K_p (r - y(t))$

Spring-mass system → If reference attitude $r(t)$ constant →

→ Closed loop behavior is undamped oscillatory attitude motion

Proportional derivative control $\boxed{u(t) = K_p e(t) + K_d \dot{e}(t)}$

Add damping to control input with $K_d > 0$ (Derivative gain)

Associated control transfer function $G_c(s) = K_p + sK_d$

Equation of motion of spacecraft $I\ddot{y}(t) = K_p e(t) + K_d \dot{e}(t) = K_p (r - y(t)) - K_d \dot{y}(t)$

Spring-mass-damper system→ If reference attitude $r(t)$ constant →

→ Oscillatory behavior reduced by damping term

General requirement of control system

Useful control system must satisfy following requirements:

- Most important requirement: **Control system must be stable**
- In addition: Control system must have relative stability, means response must show damping
- In addition: Speed of response must be reasonably fast
- In addition: Control system must be capable of reducing errors to zero or some small tolerable value

Asymptotic stability of closed-loop system

Most important requirement to control laws:

Control law must provide asymptotical stability to closed-loop system

Assume excite closed-loop system $Y(s) = \frac{G_p(s)G_c(s)}{1+G_p(s)G_c(s)}R(s)$ with unit impulse $r(t) = \delta(t)$

Asymptotical stability means that impulse response $y(t) = \mathcal{L}^{-1}\left(\frac{G_p(s)G_c(s)}{1+G_p(s)G_c(s)}\right)$ must asymptotically go to zero

Means that **poles** of closed-loop transfer function $T(s) = \frac{G_p(s)G_c(s)}{1+G_p(s)G_c(s)}$ must have **negative real parts**

Performance of closed-loop system

When design controller, take into account:

1. Asymptotically stable closed-loop system
2. Then quantify performance of close-loop control system

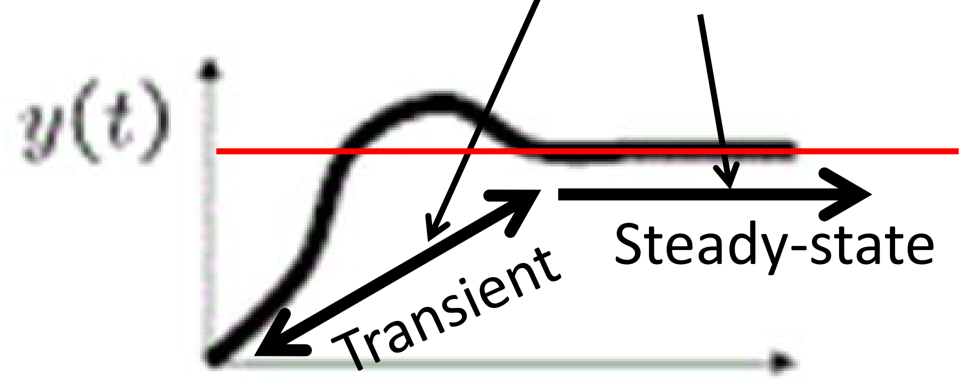
Two components are important to specify desired time-domain behavior:

- **Transient** behavior (behavior that dies out, i.e. t_r , t_p , t_r and M_p)
- **Steady-state** behavior (behavior that persist)

Transient behavior of control system often studied by considering response of closed-loop system to **unit step** command

Transient and steady-state behavior

Time response divided as $y(t) = y_t(t) + y_{ss}(t)$

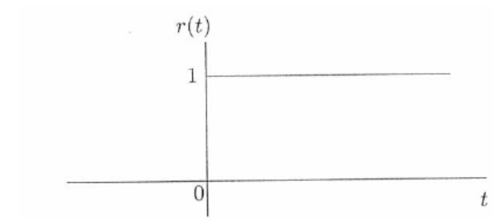


Let us first look to transient behavior

Transient behavior of control system often studied by considering response of closed-loop system to **unit step** command

Time-domain specification

- **Transient** specification often given in terms of response of closed-loop system to **unit step** command
- Reference attitude given by $r(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$



- **Modified proportional derivative** control law given by $u(t) = K_p e(t) + K_d \dot{y}(t)$
- Remark: Include derivative term in plant $G_p(s)$ rather than in control $G_c(s)$
- $T_d(s) = 0 \rightarrow$ **augmented plant transfer function**

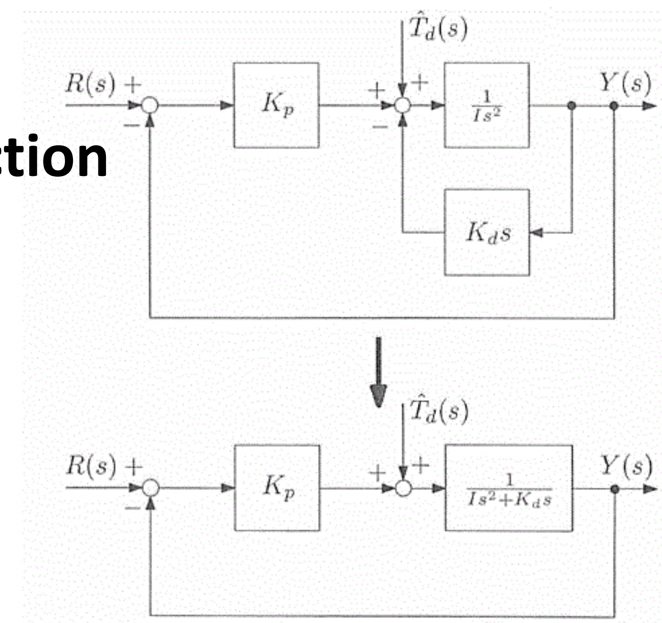
$$G_p(s) = \frac{1}{Is^2 + K_d s}$$

- Effective control transfer function

$$G_c(s) = K_p$$

- Response to reference command $R(s)$

$$Y(s) = T(s)R(s)$$



Time-domain specification

- Closed-loop response to reference command $R(s)$ with $T_d(s) = 0$

$$Y(s) = T(s)R(s)$$

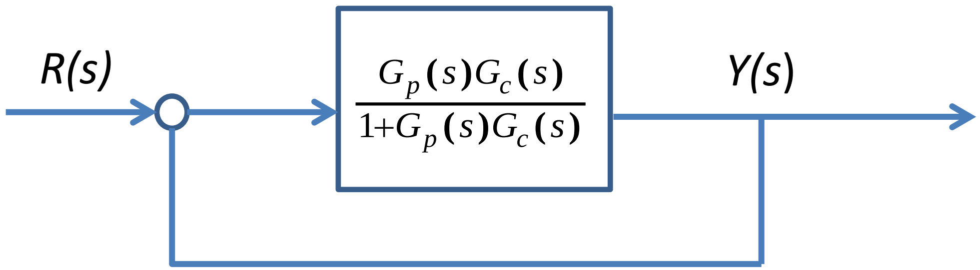
with closed-loop transfer function $T(s)$

$$T(s) = \frac{G_p(s)G_c(s)}{1 + G_p(s)G_c(s)} = \frac{K_p/(Is^2 + K_d s)}{1 + K_p/(Is^2 + K_d s)}$$
$$= \frac{(K_p/I)}{s^2 + (K_d/I)s + (K_p/I)}.$$

- Define $\boxed{\omega_n^2 = \frac{K_p}{I}, \quad 2\zeta\omega_n = \frac{K_d}{I}}$
- Transfer function becomes $T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$
- Quantity ω_n → undamped natural frequency (is frequency of oscillation of closed-loop system without damping)
- Quantity ζ→ damping ratio (is measure of resistance to change system output)

Time-domain specification

- General second-order system is characterized by following transfer function



$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

- **With proportional-derivative control the closed-loop spacecraft attitude system is second-order**
- Closed-loop poles can be obtained using quadratic equation solution

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

- Exist four possible cases

Time-domain specification

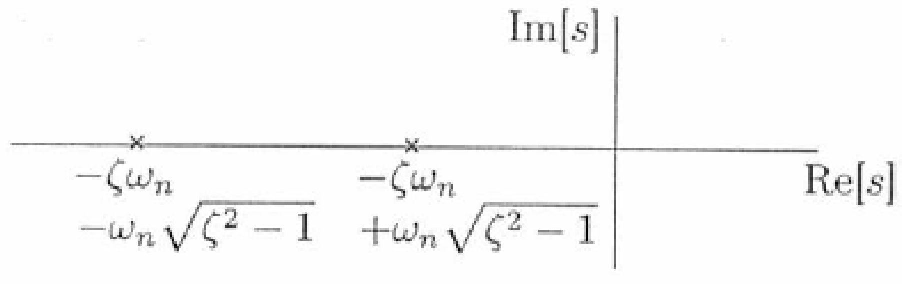
Two poles of system are $s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

According to value ζ second-order system can be into one of four categories

Case 1: $\zeta > 1$

System called **overdamped**

System has **two real distinct negative poles** → $s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$



Time-domain specification

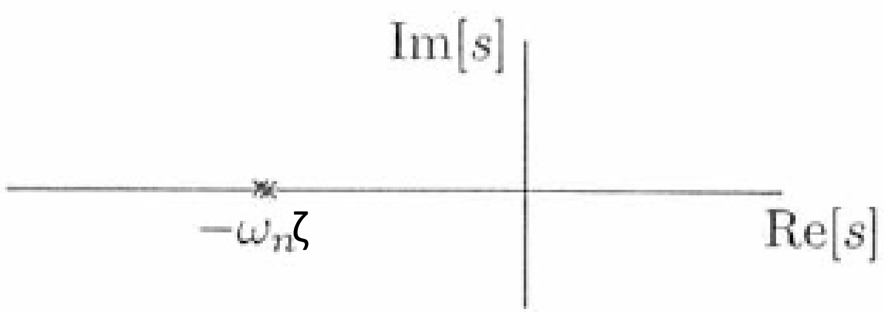
Two poles of system are $s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

According to value ζ second-order system can be into one of four categories

Case 2: ζ = 1

System called **critically damped**

System has **two real but equal poles** → $s = -\zeta\omega_n, -\zeta\omega_n$



Time-domain specification

Two poles of system are $s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

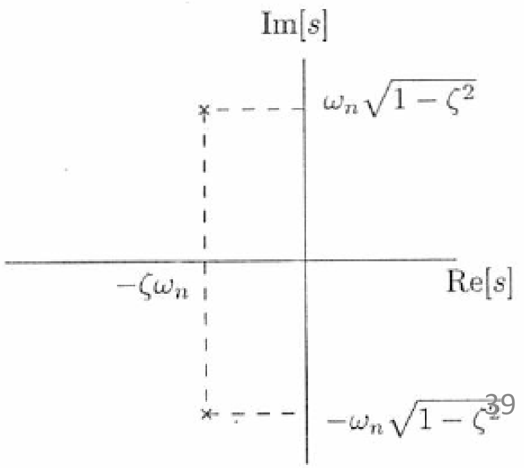
According to value ζ second-order system can be into one of four categories

Case 3: $0 < \zeta < 1$

System called **underdamped**

System has **two complex conjugate poles** → $s = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$,

With $\boxed{\omega_d = \omega_n\sqrt{1-\zeta^2}}$ **damped natural frequency**



Poles lead to **decaying oscillatory** behavior at frequency ω_d

Time-domain specification

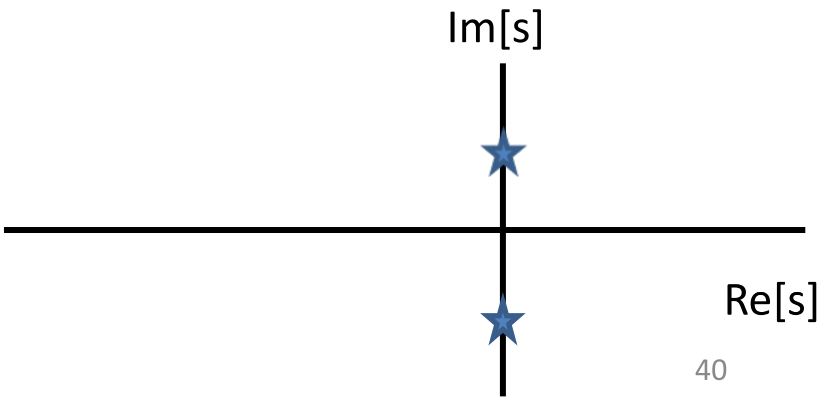
Two poles of system are $s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

According to value ζ second-order system can be into one of four categories

Case 4: $\zeta = 0$

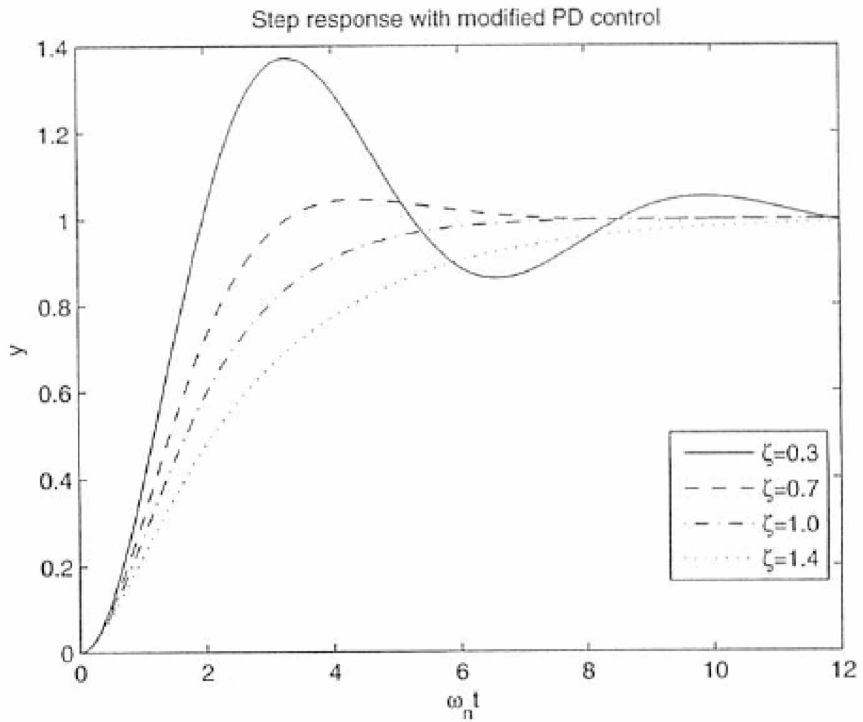
System called **undamped**

System has **two pure imaginary poles** → $s = \pm j\omega_n$



Time-domain specification

Figure shows **unit step** response for different values of ζ

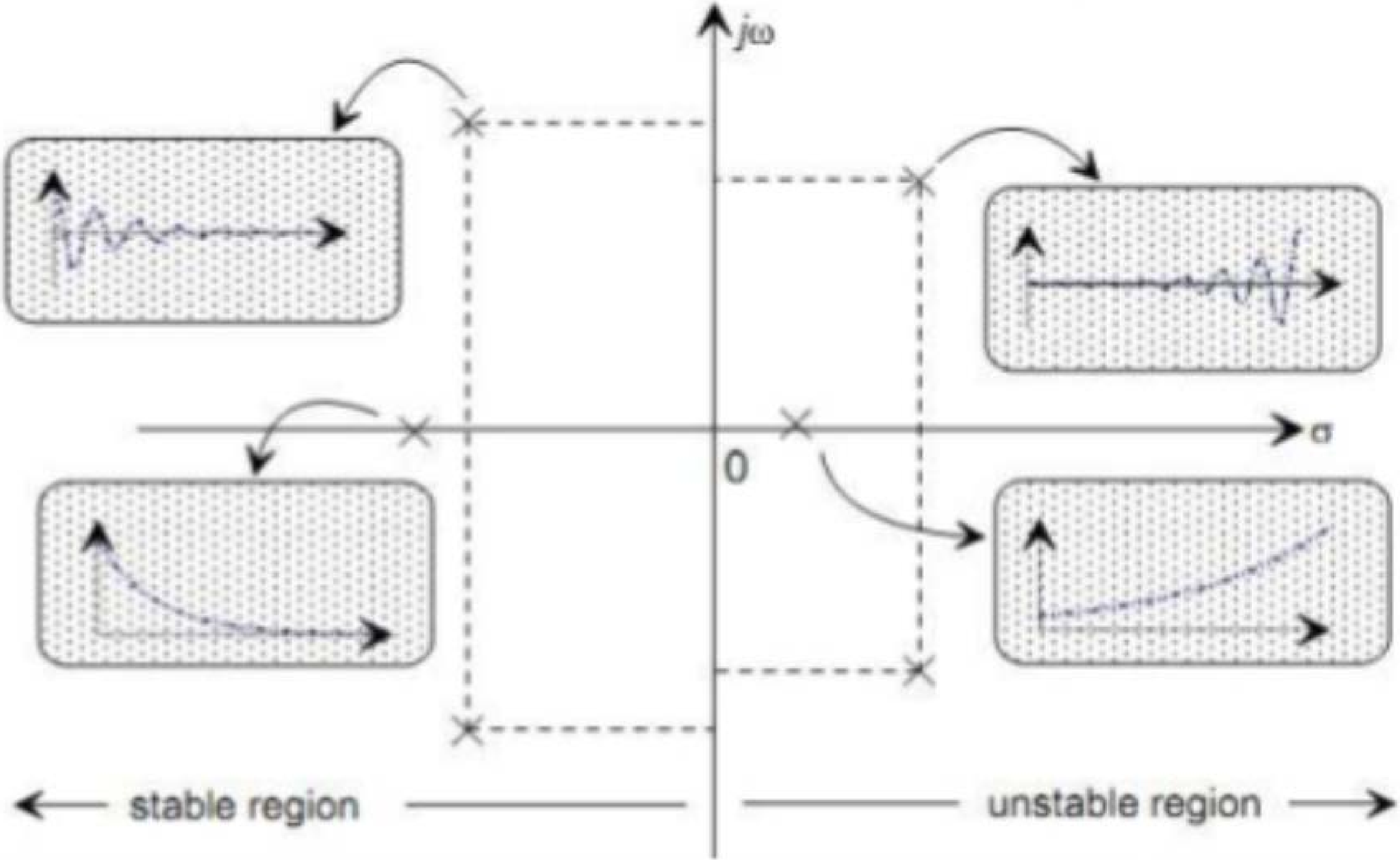


Possible responses of 2nd order system due to unit step input

- Case with $0 < \zeta < 1$ has oscillatory component with some overshoot of steady state value
- Cases with $\zeta \geq 1$ no oscillatory component and no overshoot
- For spacecraft attitude control some overshoot allowed (faster response than other cases) → interested in underdamped case

Effect of poles in pole-zero plot

Effect of pole locations in pole-zero plot on function in time domain



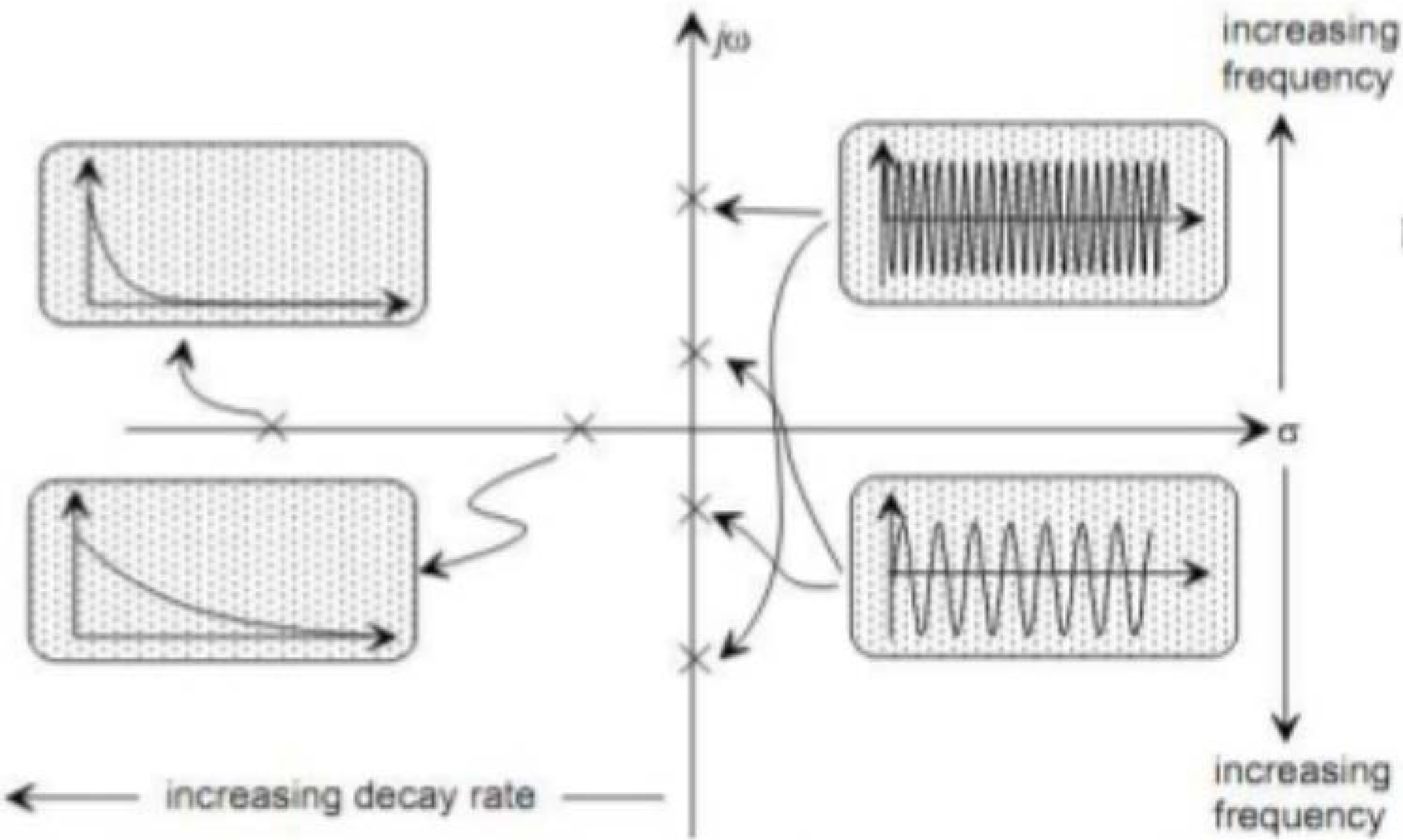
Effect of poles in pole-zero plot

Effect of pole locations in pole-zero plot on function in time domain

Rate of decay/growth determined by **real part of pole**

Oscillatory frequency determined by **imaginary part of pole pair**

Note: For pure complex conjugate pole pairs only oscillation (no decay)



Step response of underdamped system

- Study of step response in underdamped case
- Reference command $R(s) = 1/s$ → Step response given by

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\frac{1}{s}.$$

- Expand $Y(s)$ in **partial fraction expansion**

$$Y(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{s} - \frac{s + \zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} - \frac{\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Complete square in denominator

$$\begin{aligned} s^2 + 2\zeta\omega_n s + \omega_n^2 &= \left(s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2\right) + \omega_n^2 - \zeta^2\omega_n^2, \\ &= (s + \zeta\omega_n)^2 + \omega_n^2\left(1 - \zeta^2\right), \\ &= (s + \zeta\omega_n)^2 + \omega_d^2. \end{aligned}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

- Step response ready for Laplace transformation

$$Y(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{\omega_d}\frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}.$$

Step response of underdamped system

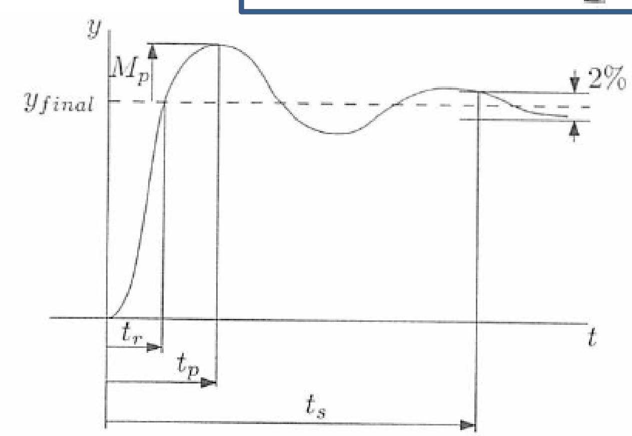
- Study of step response in underdamped case
- Step response ready for Laplace transformation
- Inverse Laplace transform can be obtained easier if *Y*(*s*) is

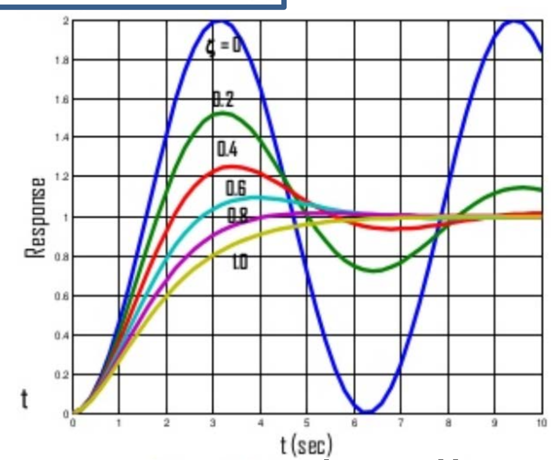
$$Y(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{\omega_d}\frac{\omega_d}{(s+\zeta\omega_n)^2 + \omega_d^2}.$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

- Step response in time domain

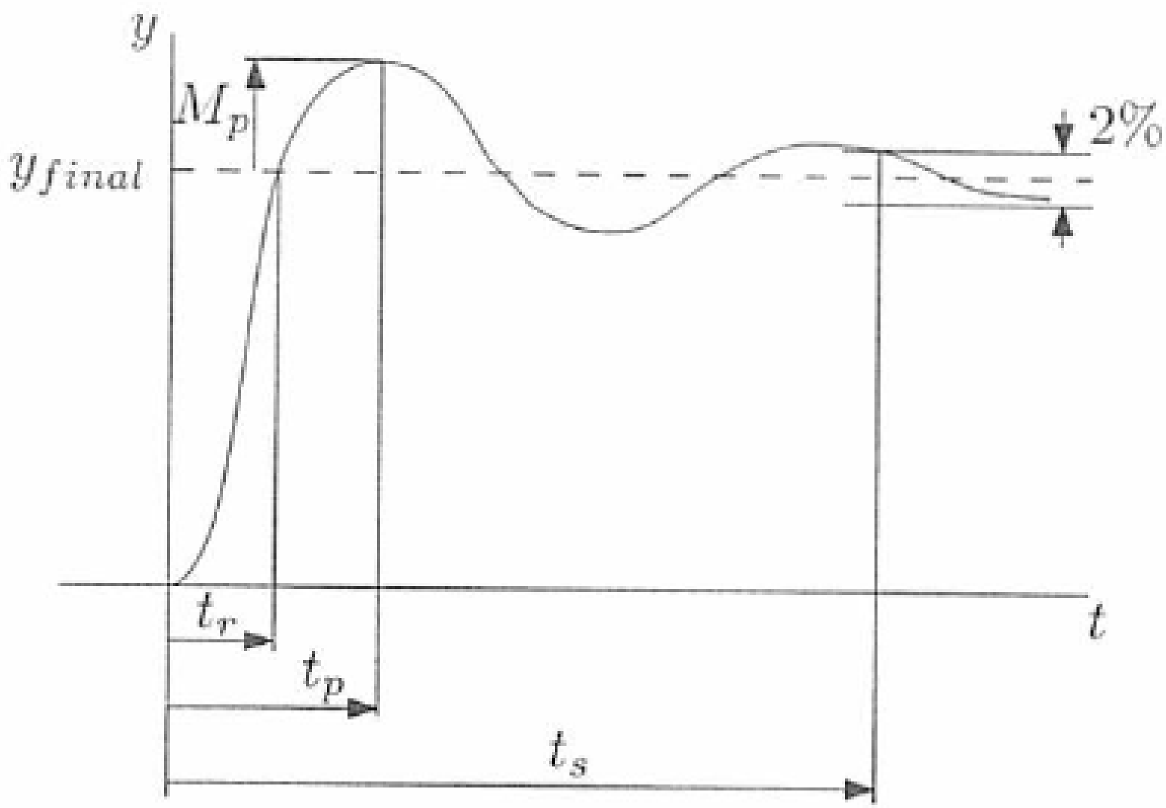
$$\boxed{y(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right]}$$





- When ζ = 0 → $\omega_d = \omega_n$ → $y(t) = 1 - \cos\omega_n t$ (oscillation)

Time-domain specifications of underdamped system based on unit step response



- **Rise time** t_r – time taken for step response to first reach final value
- **Peak time** t_p – time taken to first achieve peak response
- **Maximum overshoot** M_p - maximum percentage overshoot from final value
- **Settling time** t_s – time taken for output to get to within 2% of final value and stay

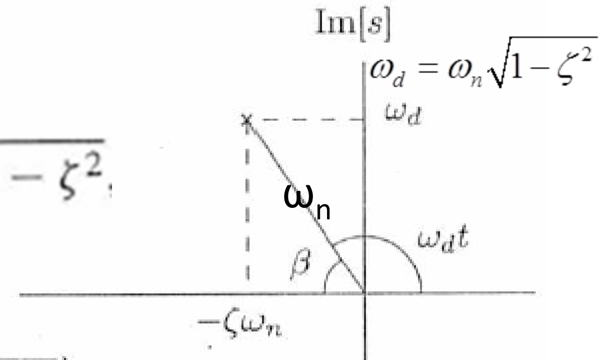
Time-domain specification (Rise time)

$$y(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right]$$

Rise time t_r – time taken for step response to first reach final value

Put t = t_r in above equation and set $y(t_r) = 1$ → $\cos\omega_d t_r + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t_r = 0.$

Rewrite equation as $\tan\omega_d t_r = \frac{\omega_d}{-\zeta\omega_n} = \frac{\omega_n\sqrt{1-\zeta^2}}{-\omega_n\zeta}.$



Pole location obtained from figure $\omega_d t_r = \pi - \beta.$

Note: $-\omega_n\zeta$ is real part of pole $s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}.$

ω_d us imaginary part of pole

Expression for rise time $\boxed{t_r = \frac{\pi - \beta}{\omega_d}}$ with $\beta = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$

Relationship between pole and rise time

Translate **rise time** into allowable pole region

$\bar{t}_r$=maximum allowable rise-time

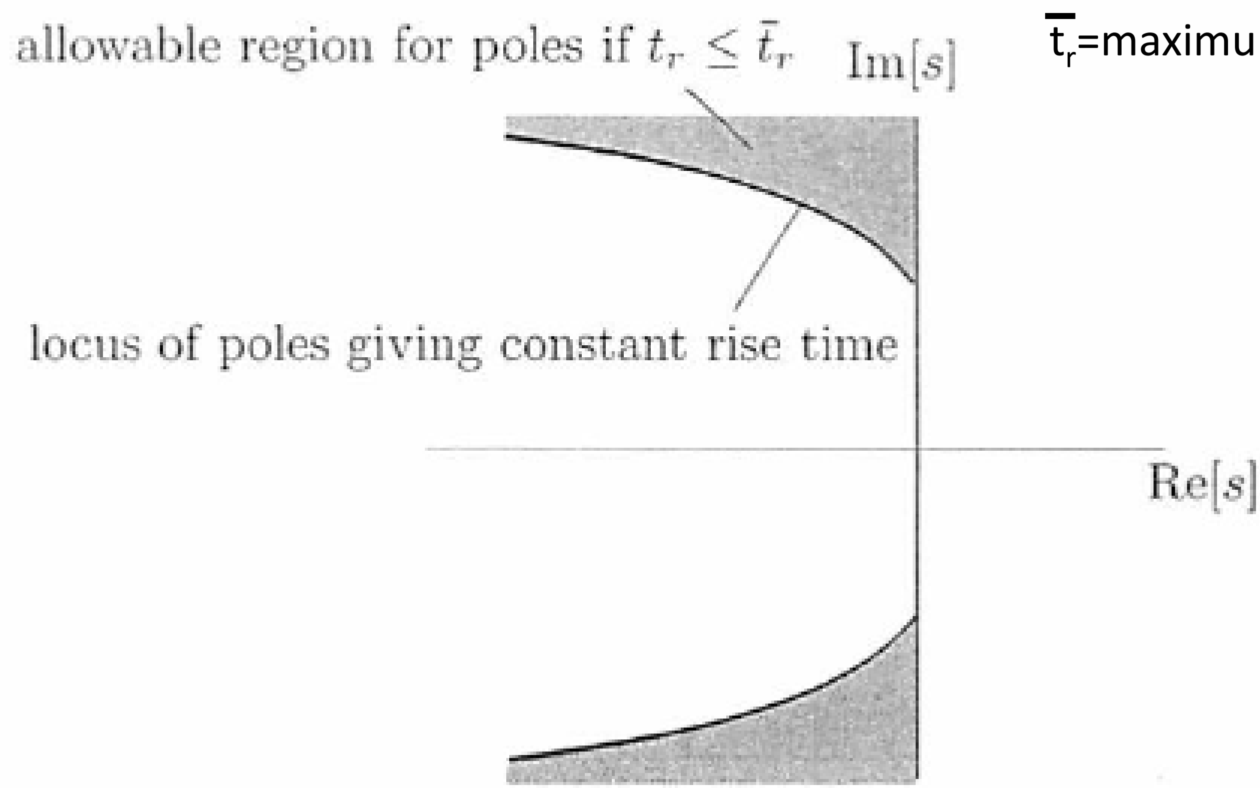


Figure 17.22 Relationship between pole locations and rise-time

Farther away pole from origin → smaller rise time

Time-domain specification (Peak time)

$$y(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right]$$

Peak time t_p – time taken to first achieve peak response

In order to find peak time differentiate above equation and set zero
Peak time occurs at first time when d*y*/d*t* = 0

$$\begin{aligned}\frac{dy}{dt} &= \zeta\omega_n e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right] - e^{-\zeta\omega_n t}\left[-\omega_d\sin\omega_d t + \frac{\zeta\omega_n}{\omega_d}\omega_d\cos\omega_d t\right]\\ &= e^{-\zeta\omega_n t}\sin\omega_d t\left[\frac{\zeta^2\omega_n^2}{\omega_d} + \omega_d\right].\end{aligned}$$

Find peak time t = t_p when $\sin\omega_d t_p = 0.$ → $\omega_d t = 0, \pi, 2\pi, 3\pi, \ldots.$

For underdamped system first peak gives peak time $\boxed{t_p = \dfrac{\pi}{\omega_d}}$

Note: Peak time depends only from imaginary part of pole
($s = -\zeta\omega_n \pm j\omega_d$)

Relationship between pole and peak time

Translate **peak time** into allowable pole region

$\bar{t}_p$ = maximum allowable peak-time

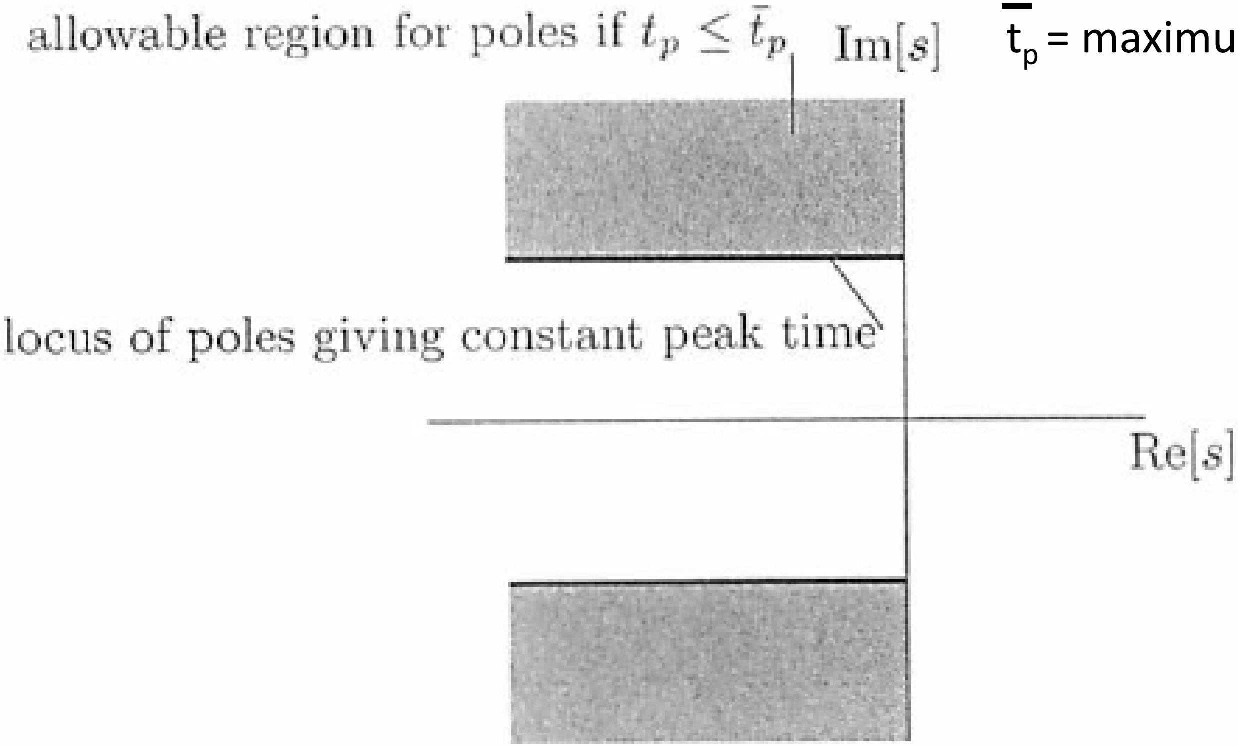


Figure 17.23 Relationship between pole locations and peak-time

Larger imaginary part of pole → smaller peak time $t_p = \dfrac{\pi}{\omega_d}$.

Time-domain specification (Maximum overshoot)

$$y(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right]$$

Maximum overshoot M_p- maximum percentage overshoot from final value

Maximum overshot occurs at peak time t = t_p with maximum response

$$y_p = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t_p + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t_p\right]$$

Know from peak time that $\omega_d t_p = \pi$ → $\cos\omega_d t_p = -1$ and $\sin\omega_d t_p = 0$

Maximum percentage overshoot of final values

$$\boxed{M_p} = \frac{y_p - 1}{1} \times 100\% \boxed{= e^{-\pi\zeta/\sqrt{1-\zeta^2}} \times 100\%}$$

Maximum overshoot depends only on damping ration ζ

Relationship between pole and overshoot

Translate **percent overshoot** into allowable pole region

$\bar{M}_p$ = maximum allowable percent overshoot

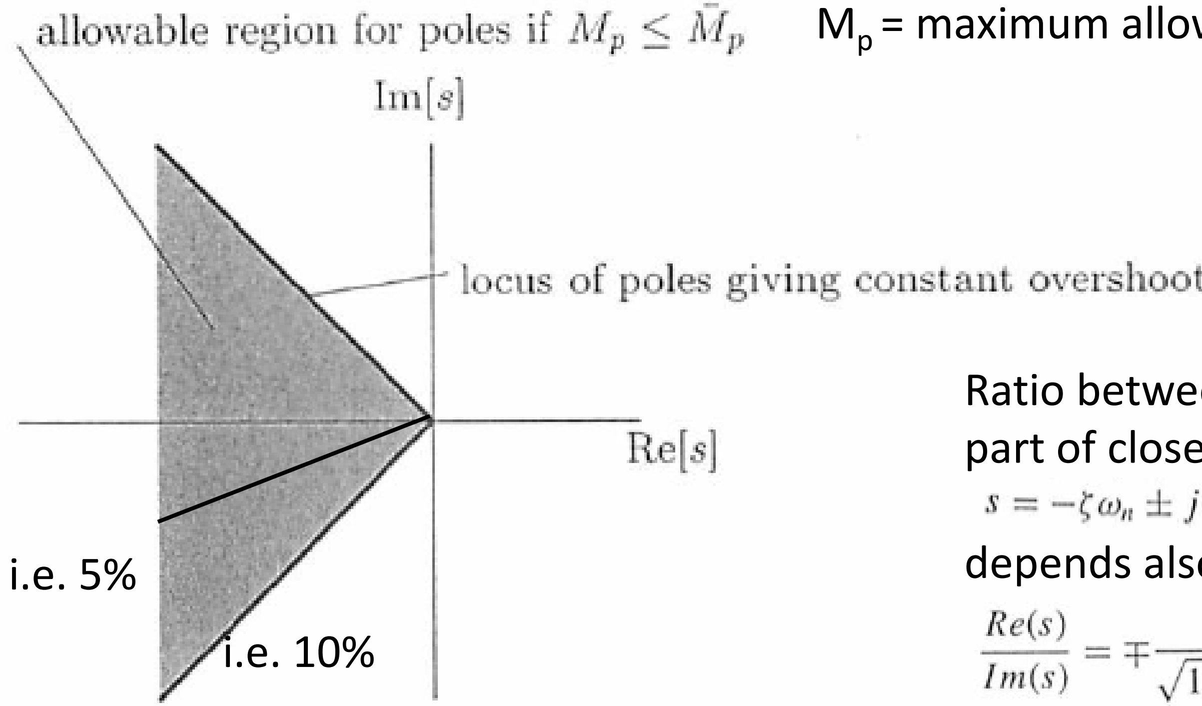


Figure 17.24 Relationship between pole locations and overshoot

Ratio between real and imaginary part of closed loop poles

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

depends also only on damping ratio

$$\frac{Re(s)}{Im(s)} = \mp\frac{\zeta}{\sqrt{1-\zeta^2}}.$$

Smaller angle $\rightarrow$ smaller percent overshoot

Time-domain specification (Settling time)

$$y(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right]$$

Settling time t_s – time taken for output to get to within 2% of final value and stay there

From equation above, difference between current and final value of *y*(t)

$$y(t) - 1 = -e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right]$$

After some manipulations $y(t) - 1 = -\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \phi)$ and with $|\sin(\omega_d t + \phi)| \le 1$

$\rightarrow$ $|y(t) - 1| \le \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}$

Percentage deviation from final values $\frac{|y(t) - 1|}{1} \times 100\% \le 2\%$, $\rightarrow$ $\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \le 0.02.$

Settling time $\frac{e^{-\zeta\omega_n t_s}}{\sqrt{1-\zeta^2}} = 0.02$, $\rightarrow$ $t_s = \frac{\ln\left(0.02\sqrt{1-\zeta^2}\right)}{-\zeta\omega_n}$

For $0.1 < \zeta < 0.9$ settling time given by $\boxed{t_s \approx \frac{4.4}{\zeta\omega_n}}$

Note: Settling time depends primarily on real part of pole

Relationship between pole and settling time

Translate **settling time** into allowable pole region

$\bar{t}_s$=maximum allowable settling-time

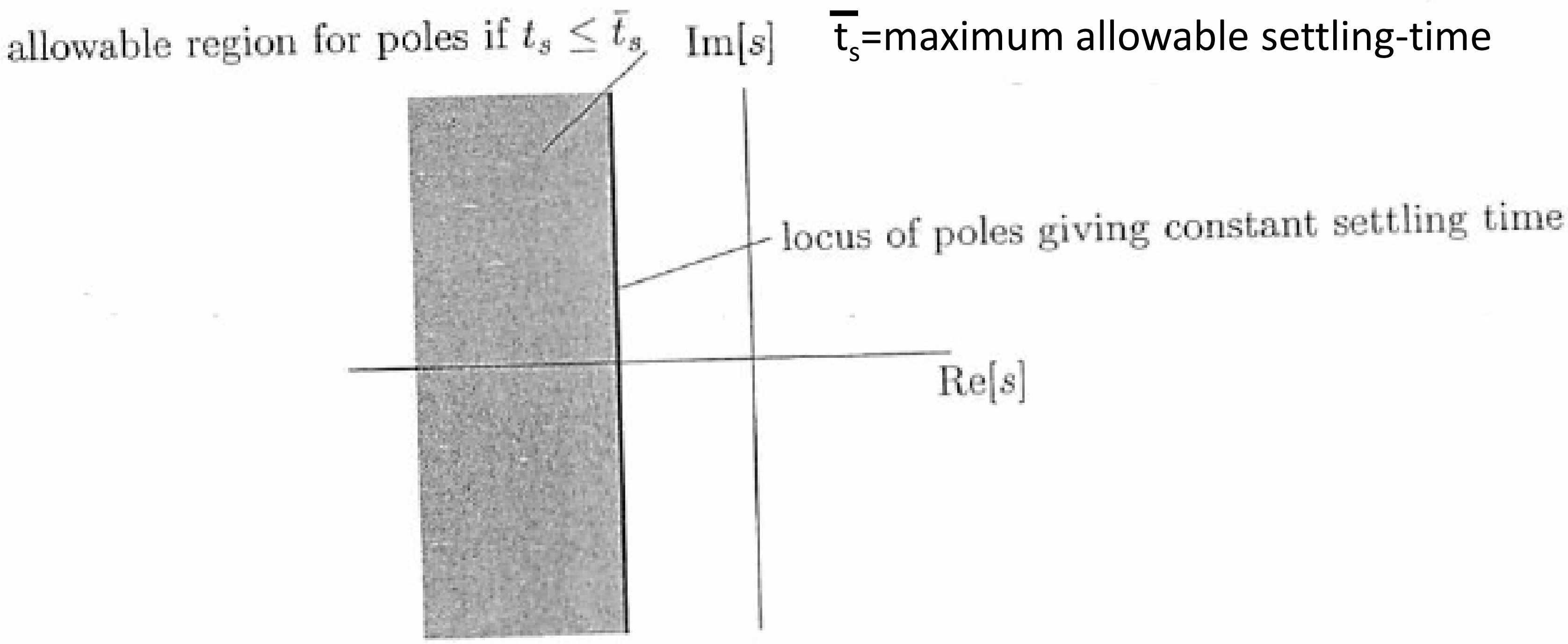


Figure 17.25 Relationship between pole locations and settling time

Smaller real part of pole → smaller settling time

Second order system with step input

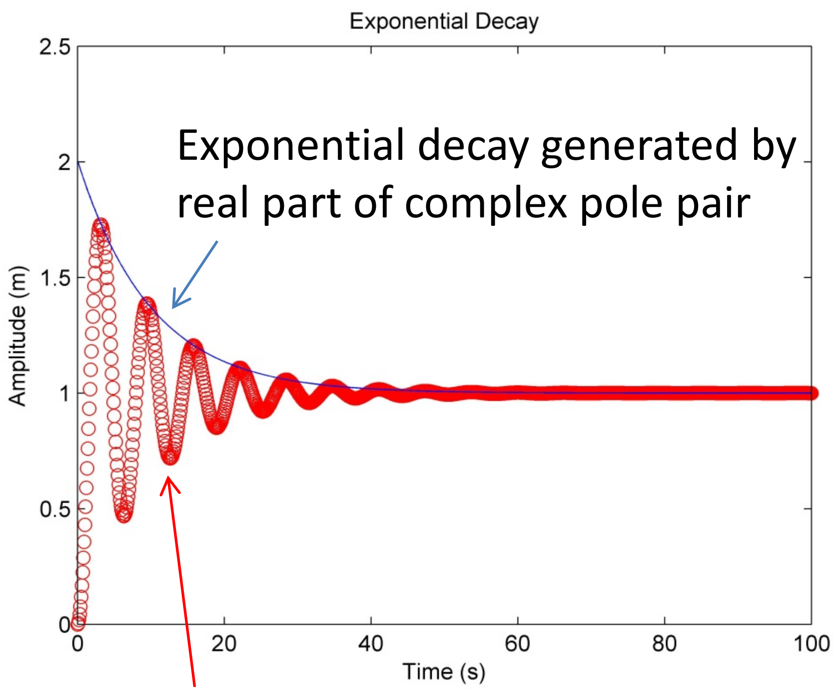
Solution of second order system with step input

$$y(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right]$$

Underdamped case 0< ζ < 1

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2},$$

Real part **Imaginary part**



Exponential decay generated by real part of complex pole pair

Oscillation generated by imaginary part of complex pole pair

Summary of time domain specifications

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

Rise time

$$t_r = \frac{\pi - \beta}{\omega_d}$$

$$\beta = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

Peak time

$$t_p = \frac{\pi}{\omega_d}.$$

Maximum overshoot

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \times 100\%$$

Settling time

$$t_s \approx \frac{4.4}{\zeta\omega_n}.$$

Example

Example 17.1 (PD controller design for spacecraft attitude control) *Suppose that our spacecraft has moment of inertia $I = 1\ kg{\cdot}m^2$, and that the following time-domain design specifications are given.*

1. *Rise-time constraint, $t_r \leq 30$ seconds.*
2. *Maximum overshoot constraint, $M_p \leq 30\%$.*
3. *Settling-time constraint, $t_s \leq 100$ seconds.*

Find simple controller design using desired pole region

Following steps facilitate choice of allowable parameter (proportional-derivative control gains)

1. Translate transient performance into desired pole region
2. Plot graphically poles of constraint system as function of parameter into allowable region
3. Place poles of constraint system inside allowable region

This graphical method is called **root locus analysis**

Example

Rise time constraint → $\frac{\pi - \beta}{\omega_d} \leq 30,$

→ $\beta \geq \pi - 30\omega_d.$

Allowable region for closed-loop poles to satisfy rise time constraint

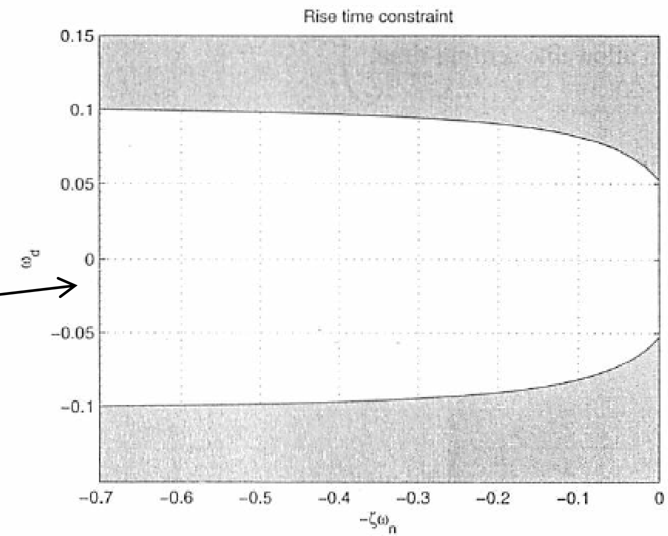


Figure 17.26 Allowable region for rise time constraint

Maximum overshoot constraint

→ $e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \leq 0.3,$ → $\frac{\zeta}{\sqrt{1-\zeta^2}} \geq -\frac{\ln 0.3}{\pi}.$

⇒ $\frac{\sqrt{1-\zeta^2}}{\zeta} \leq -\frac{\pi}{\ln 0.3}$ and with $\beta = \tan^{-1}(\frac{\sqrt{1-\zeta^2}}{\zeta})$

⇒ $\beta \leq \tan^{-1}\left(-\frac{\pi}{\ln 0.3}\right) = 1.205\,(69^\circ)$

Allowable region for closed-loop poles to satisfy overshoot requirement

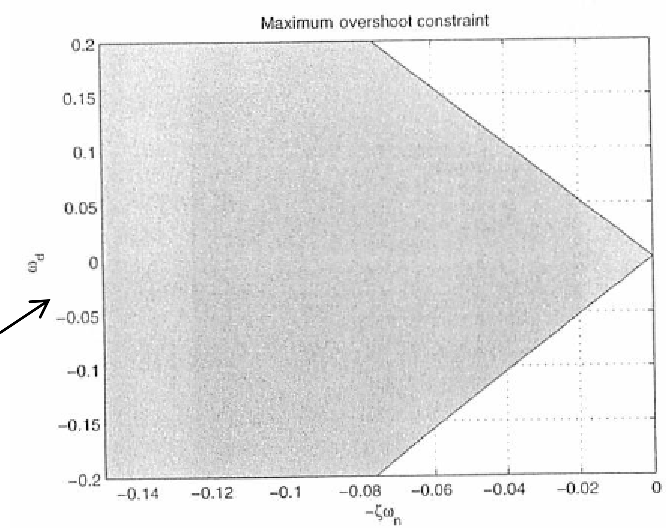


Figure 17.27 Allowable region for overshoot constraint

Example

Settling time constraint → $\frac{4.4}{\zeta\omega_n} \le 100,$

→ $\zeta\omega_n \ge \frac{4.4}{100} = 0.044.$

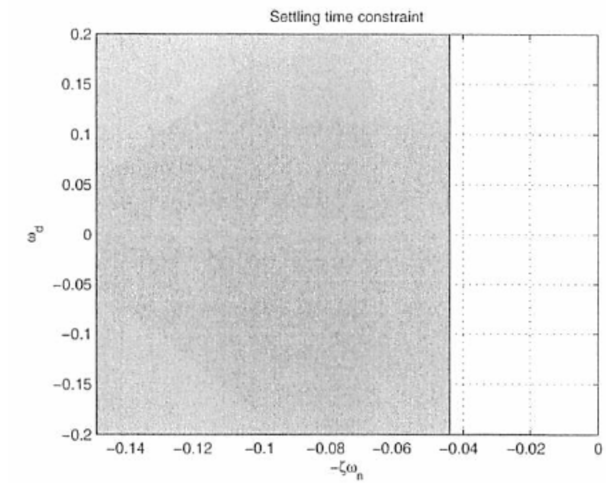


Figure 17.28 Allowable region for settling-time constraint

Allowable region for closed-loop poles to satisfy settling time constraint

Combine regions found by three constraint

$\Rightarrow$ Allowable closed-loop pole region for combined constraint

$\Rightarrow$ Intersection of three regions

$\Rightarrow$ Choose any poles inside shaded region

$\Rightarrow$ I.e. $s = -0.05 \pm j0.1$ satisfy all constraint

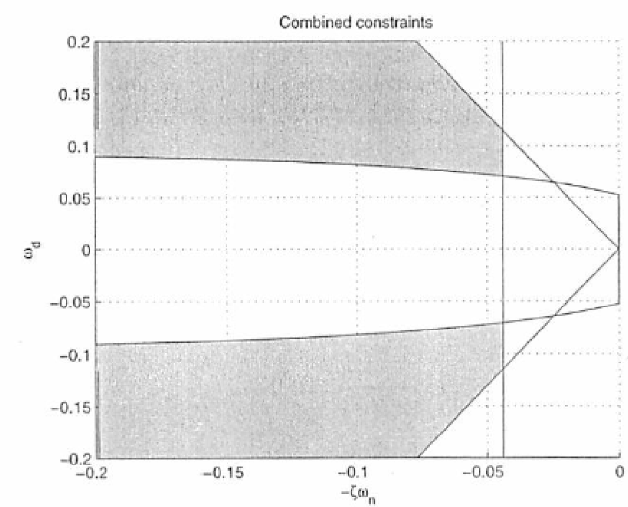


Figure 17.29 Allowable region for combined constraints

Example

General expression for closed loop $s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$, $\rightarrow$ $\omega_n = |s|$.

For case $s = -0.05 \pm j0.1 \rightarrow$ $\omega_n^2 = |s|^2 = 0.05^2 + 0.1^2 = 0.0125$.

From definition $\omega_n^2 = \frac{K_p}{I}, \quad 2\zeta\omega_n = \frac{K_d}{I}$

calculate proportional-derivative (PD) control gains

$$K_p = \omega_n^2 I = 0.0125, \quad K_d = 2\zeta\omega_n I = 2 \times 0.05 = 0.1.$$

Control law that satisfy time-domain specification is given by

$$u(t) = 0.0125e(t) - 0.1\dot{y}(t).$$

Unit step response of spacecraft attitude with designed PD controller

Note: Met all time-domain specification

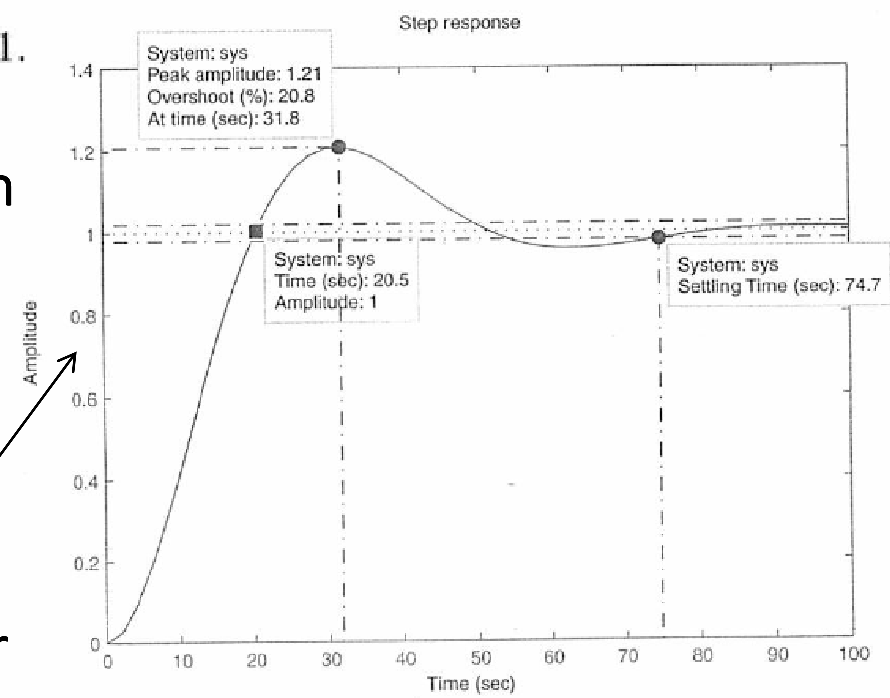


Figure 17.30 Step response with our PD design

Effects of addition of poles and zeros

Two factors can modify transient behavior:

1. Presence of zero in closed-loop transfer function
2. Presence of additional poles in closed-loop transfer function

General effect of addition of zeros is to improve stability

General effect of addition of poles is to lower stability

Effects of addition zeros

modified PD control

$$u(t) = K_p e(t) + K_d \dot{y}(t)$$

transfer function

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

standard PD control

$$u(t) = K_p e(t) + K_d \dot{e}(t)$$

transfer function

$$T(s) = \frac{2\zeta\omega_n s + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2},$$

Poles exactly same

$$s = \zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

No zeros

Zero at $s = -\omega_n/(2\zeta)$

Undamped case, response given by

$$y(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t - \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right]$$ (with $\omega_d = \omega_n\sqrt{1-\zeta^2}$)

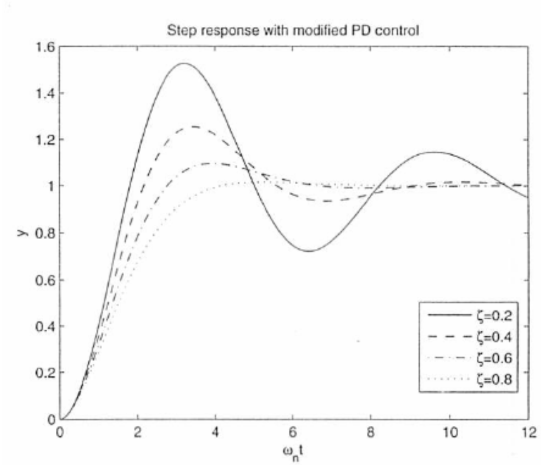
Frequency of oscillation ω_d and rate of decay $\zeta\omega_n$ are same, because determined by poles

However, coefficients of terms are different (due to zeros)

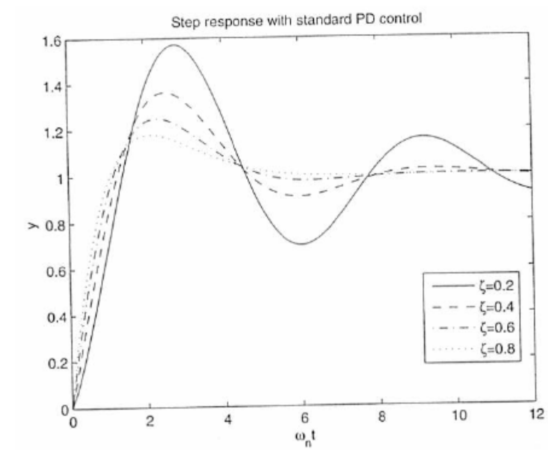
Effects of addition zeros

Zero change transient characteristic

Modified PD control



Standard PD control



Settling time and rate of oscillation similar

Rise and peak times with standard PD control faster than with modified PD control

In general, zeros far away of poles do not have significant effect on transient behavior. Zeros close to poles do have an effect on transient behavior. Zero very close to pole tends to reduce contribution of that pole since pole-zero cancelation almost occurs in transfer function.

Short: General effect of addition of zeros is to improve stability

Effects of addition of poles

General form of closed-loop transfer function

$$T(s) = K \frac{\prod_{m}^{i=1}(s - z_i)}{\prod_{n}^{i=1}(s - p_i)}.$$

with $\mathrm{m} \leq n$ and z_i are zeros and p_i are poles

Control law has asymptotically stability if Re(p_i) < 0

Assume distinct poles: form of partial fraction expansion

$$Y(s) = \frac{a}{s} + \sum_{i=1}^{n} \frac{b_i}{s - p_i}$$

Step response in time-domain

$$y(t) = a + \sum_{i=1}^{n} b_i e^{p_i t}.$$

Contribution of each pole to responds depends on associated coefficient b_i and how quickly it decays (how negative Re (p_i) is)

Short: General effect of addition of poles is to lower stability

Effects of addition of poles

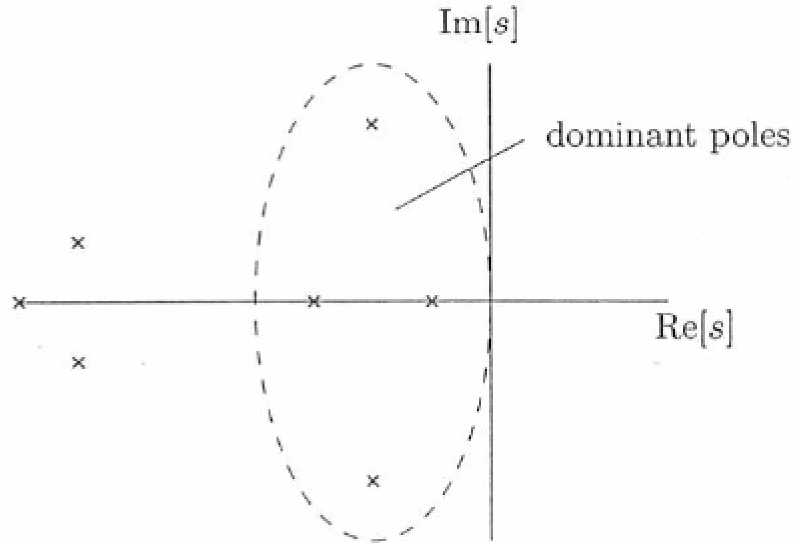


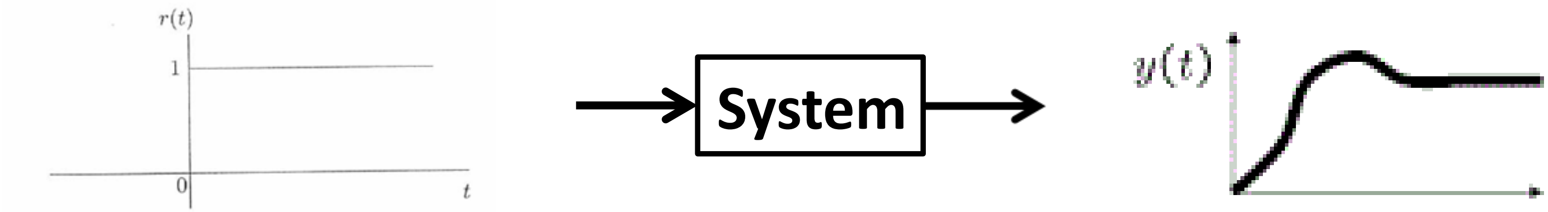
Figure 17.33 Dominant poles

Poles near imaginary axes dominate response (since their contribution decays slowest)

Short: General effect of addition of poles is to lower stability

Steady-state specification

Steady-state specification



Apply reference input r(t) and study time response y(t)

Two components are important to specify desired time-domain behavior:
Transient behavior (behavior that dies out, i.e. t_r , t_p , t_r and M_p)
Steady-state behavior (behavior that persist)

Time response divided as $y(t) = y_t(t) + y_{ss}(t)$

Transient behavior

$\lim_{t \to \infty} y_t(t) = 0$

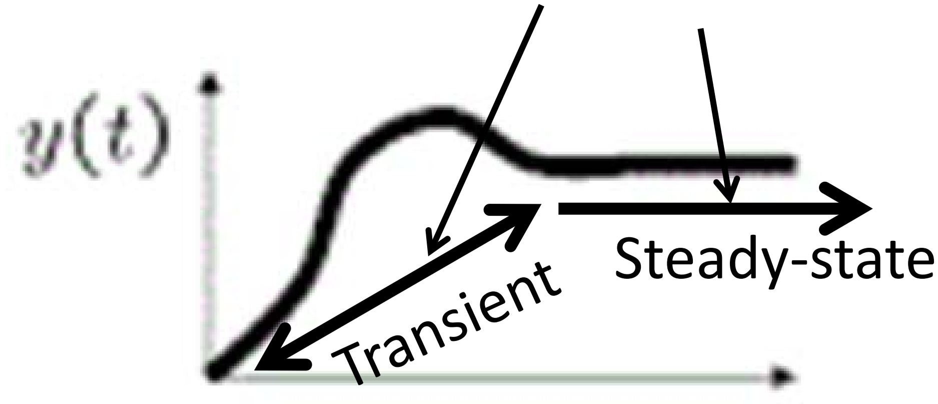
Steady-state behavior

(after $y_t(t)$ dies out)

Lets look at steady-state part

Transient and steady-state behavior

Time response divided as $y(t) = y_t(t) + y_{ss}(t)$

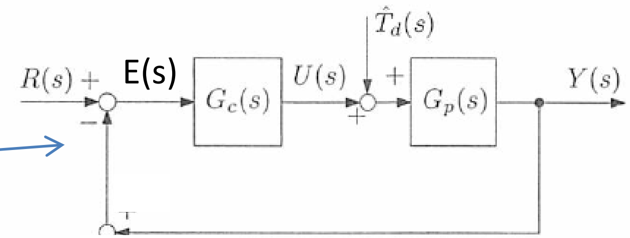


Long-term (steady state) objective is for output y(t) to match reference signal r(t)
If transient behavior dies out → want to have error $e(t) = r(t) – y(t) \approx 0$
→ means actual attitude match desired attitude

Problems:
- Non-vanishing disturbance $T_d(t)$ may cause non-zero steady-state error
- Closed-loop system may not be capable of track reference signal r(t)

Feedback control (closed-loop system)

Previously seen that closed-loop system corresponds to following figure



This closed-loop system is given by

linear sum of response *R*(s) and disturbance T_d(s)

$$Y(s) = Y_r(s) + Y_d(s) = \frac{G_p(s)G_c(s)}{1+G_p(s)G_c(s)} R(s) + \frac{G_p(s)}{1+G_p(s)G_c(s)} \hat{T}_d(s)$$

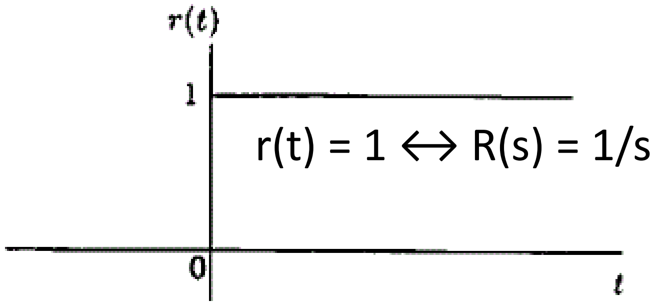
Since error *E*(s) is $E(s) = R(s) - Y(s)$

$$\rightarrow \quad E(s) = \frac{1}{1+G_p(s)G_c(s)} R(s) - \frac{G_p(s)}{1+G_p(s)G_c(s)} \hat{T}_d(s)$$

Typical reference signals

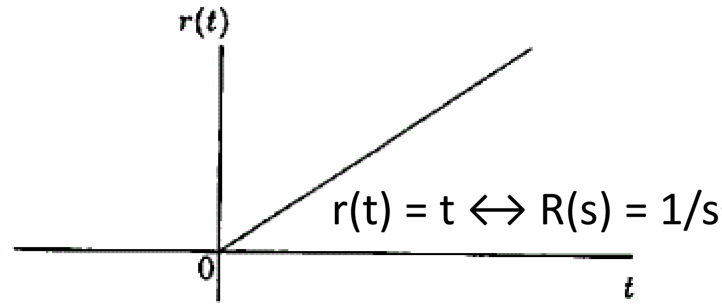
Most common type of reference signal is step input, but in practice test signal may not be as simple

Examine closed-loop response to test signals, which are similar to expected reference signal



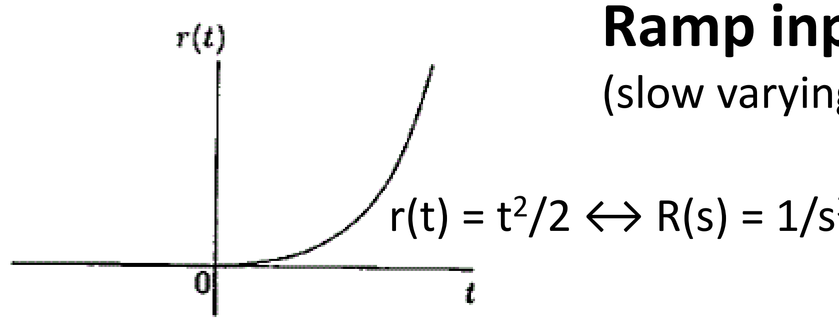
Step input

(most popular)



Ramp input

(slow varying signal)



Acceleration input

(not commonly used)

System type

If closed-loop system behaves well for test signals, it will most probable behave well for real signals

Main questions are:

What type of signals the closed-loop system with equation

$$E(s) = \frac{1}{1+G_p(s)G_c(s)} R(s) - \frac{G_p(s)}{1+G_p(s)G_c(s)} \hat{T}_d(s)$$

- can track perfectly
- can track but with some steady-error
- cannot track at all

→ **System type**

System type

What kind of signal the closed loop system can track?

Consider transfer function from reference signal R(s) to error E(s)

(ignore disturbance $T_d(s)$) $E(s) = \frac{1}{1+G_p(s)G_c(s)} R(s) - \frac{G_p(s)}{1+G_p(s)G_c(s)} \hat{T}_d(s)$

$$E(s) = \frac{1}{1+G_o(s)} R(s) \quad with \quad G_o(s) = G_p(s) G_c(s)$$

Open-loop transfer function

(product of all transfer functions around loop)

Generally open-loop transfer function can be written as

$$G_o(s) = \bar{K} \frac{(s - z_1) \ldots (s - z_m)}{s^N (s - p_1) \ldots (s - p_n)}$$

where z_i = open-loop zeros, $p_i \neq 0$ non − zero open loop poles

There are also *N* open-loop poles at *s* = 0

Type of system is now classified as value of *N* (e.g. *N* = 0 - type 0 system

N = 1 - type 1 system

N = 2 - type 2 system)

Rewrite open-loop transfer function

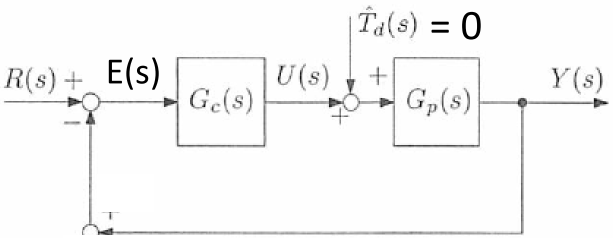
$$G_o(s) = \bar{K} \frac{(s - z_1) \ldots (s - z_m)}{s^N (s - p_1) \ldots (s - p_n)}$$

Can be written also as

$$G_o(s) = K \frac{(T_{a1} s + 1) \ldots (T_{am} s + 1)}{s^N (T_{b1} s + 1) \ldots (T_{bn} s + 1)}.$$

With $K = \bar{K} \frac{(-z_1) \ldots (-z_m)}{(-p_1) \ldots (-p_n)}$, and $T_{a1} = \frac{-1}{z_1}, \ldots, T_{b1} = \frac{-1}{p_1}, \ldots$

Steady state error (closed-loop system)



Assume closed-loop system asymptotical stable

(1 + $G_o(s)$ = 0 only has roots with negative real parts)

- Suppose output y(t) to track r(t)
- Error e(t) = r(t) – y(t)
- **Steady-state error**

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{s}{1 + G_o(s)} R(s).$$

Final value theorem

(Suppose closed-loop system stable)

Examine now steady-state error for different types of inputs

Error constant definitions

- Step-error: static position error constant

$$K_{sp} = \lim_{s \to 0} G_o(s).$$

- Ramp-error: static velocity error constant

$$K_{sv} = \lim_{s \to 0} s\,G_o(s).$$

- Acceleration-error: static acceleration error constant

$$K_{sa} \equiv \lim_{s \to 0} s^2 G_o(s)$$

- Constants K_{sp}, K_{sv}, K_{sa} : ability to reduce steady-state error

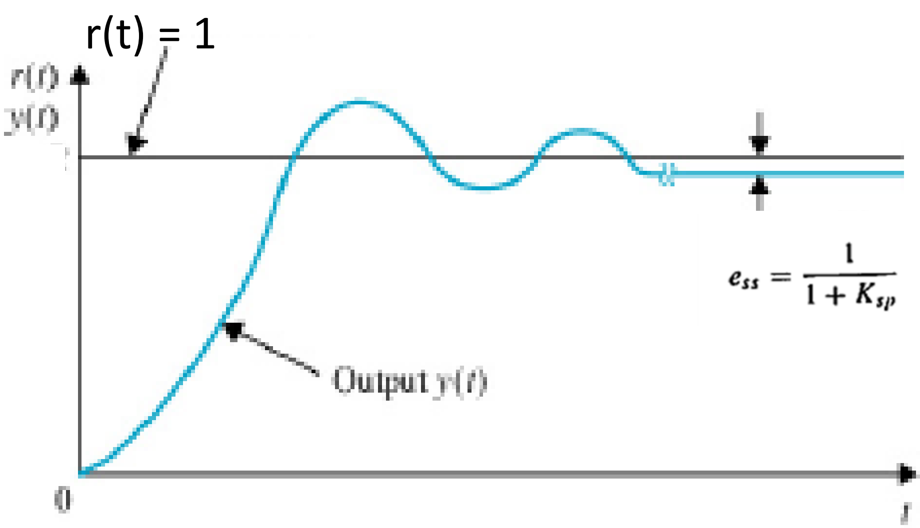
Steady-state error for step input r(t)

$r(t) = 1 \leftrightarrow R(s) = 1/s$

$$e_{ss} = \frac{1}{1 + K_{sp}}$$

$$e_{ss} = \lim_{s \to 0} \frac{s}{1 + G_o(s)} \frac{1}{s} = \frac{1}{1 + \lim_{s \to 0} G_o(s)}$$

$$K_{sp} = \lim_{s \to 0} G_o(s).$$



For type 0 system ($N = 0$)

$$K_{sp} = \lim_{s \to 0} K \frac{(T_{a1}s + 1) \ldots (T_{am}s + 1)}{(T_{b1}s + 1) \ldots (T_{bn}s + 1)} = K.$$

For type 1 or higher system ($N = 1$)

$$K_{sp} = \lim_{s \to 0} K \frac{(T_{a1}s + 1) \ldots (T_{am}s + 1)}{s^N (T_{b1}s + 1) \ldots (T_{bn}s + 1)} = \infty$$

Type 0 systems → steady-state error to step input is which is non-zero but finite

$$e_{ss} = \frac{1}{1 + K_{sp}}$$

Type 1 or higher systems can track step inputs perfectly and steady state error is

$$e_{ss} = \frac{1}{1 + \infty} = 0.$$

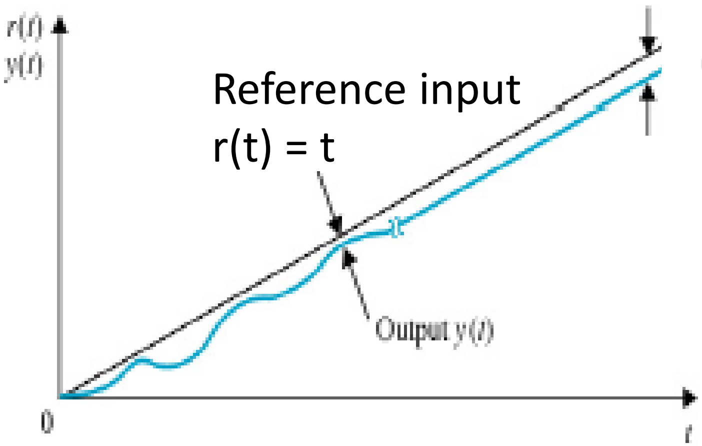
Steady-state error for ramp input r(t)

$r(t) = t \leftrightarrow R(s) = 1/s^2$

$$e_{ss} = \frac{1}{K_{sv}}.$$

$$e_{ss} = \lim_{s\to 0} \frac{s}{1+G_o(s)}\frac{1}{s^2} = \lim_{s\to 0}\frac{1}{s+sG_o(s)} = \frac{1}{\lim_{s\to 0} sG_o(s)}.$$

$$K_{sv} = \lim_{s\to 0} sG_o(s).$$



$$e_{ss} = \frac{1}{K_{sv}}.$$

For type 0 system (N = 0)

$$K_{sv} = \lim_{s\to 0} K\frac{s(T_{a1}s+1)\ldots(T_{am}s+1)}{(T_{b1}s+1)\ldots(T_{bn}s+1)} = 0.$$

For type 1 system (N = 1)

$$K_{sv} = \lim_{s\to 0} K\frac{(T_{a1}s+1)\ldots(T_{am}s+1)}{(T_{b1}s+1)\ldots(T_{bn}s+1)} = K$$

For type 2 or higher system ($N \geq 2$)

$$K_{sp} = \lim_{s\to 0} K\frac{(T_{a1}s+1)\ldots(T_{am}s+1)}{s^{N-1}(T_{b1}s+1)\ldots(T_{bn}s+1)} = \infty.$$

Type 0 systems → steady-state error to ramp input is $e_{ss} = \frac{1}{0} = \infty,$
which shows that type 0 system cannot track ramp input

Type 1 systems can track step inputs perfectly $e_{ss} = \frac{1}{K}$
which is non-zero but finite

Type 2 or higher systems can track ramp inputs perfectly and steady state error is $e_{ss} = \frac{1}{\infty} = 0.$

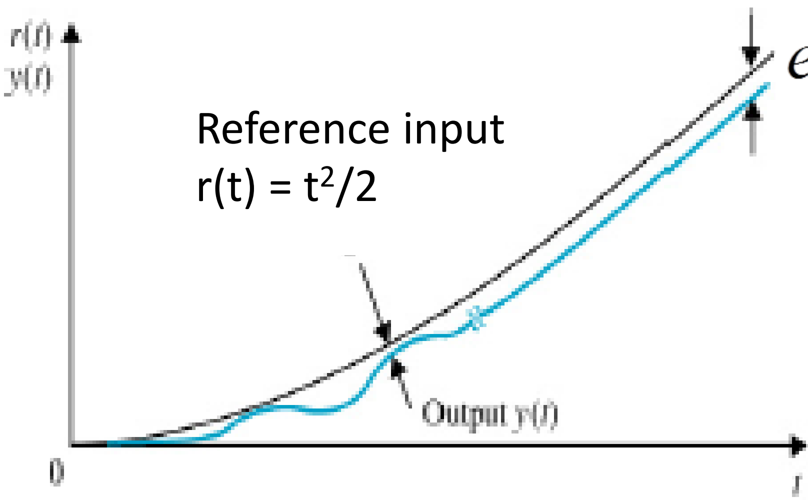
Steady-state error for acceleration input r(t)

$r(t) = t^2/2 \leftrightarrow R(s) = 1/s^3$

$$e_{ss} = \frac{1}{K_{sa}}$$

$$e_{ss} \equiv \lim_{s\to 0} \frac{s}{1+G_o(s)} R(s) = \lim_{s\to 0} \frac{s}{1+G_o(s)} \frac{1}{s^3} = \lim_{s\to 0} \frac{1}{s^2 G_o(s)}$$

$$K_{sa} \equiv \lim_{s\to 0} s^2 G_o(s)$$



$$e_{ss} = \frac{1}{K_{sa}}$$

Zero steady-state error

If error constant is infinite → achieve zero steady-state error
→ accurate tracking

- For step input r(t)

$$K_{sp} = \lim_{s \to 0} G_o(s) = \infty \Leftrightarrow G_o(s) \text{ is of at least type } 1$$

- For ramp input r(t)

$$K_{sv} = \lim_{s \to 0} sG_o(s) = \infty \Leftrightarrow G_o(s) \text{ is of at least type } 2$$

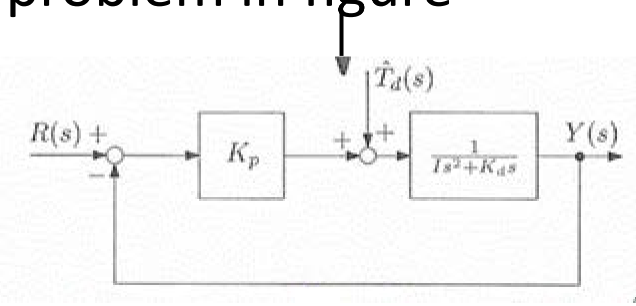
- For acceleration input r(t)

$$K_{sa} = \lim_{s \to 0} s^2 G_o(s) = \infty \Leftrightarrow G_o(s) \text{ is of at least type } 3$$

To decrease the steady-state error to a reference input, we can either increase the open-loop gain, K, or increase the type of the system by adding integrators (1/s).

Example

Examine proportional-derivative attitude control problem in figure



Plant transfer function is $G_p(s) = \dfrac{1}{Is^2 + K_d s} = \dfrac{1}{K_d s\left(\frac{I}{K_d}s + 1\right)}.$

Control transfer function is $G_c(s) = K_p.$

Therefore open-loop transfer functions is $G_o(s) = G_p(s)G_c(s) = \dfrac{K_p}{K_d s\left(\frac{I}{K_d}s + 1\right)}$

→ Type 1 system

→ Can track step attitude commands with zero steady-state error

Static velocity error constant is $K_{sv} = \lim_{s\to 0} sG_o(s) = \dfrac{K_p}{K_d}.$ and therefore can track ramp inputs with steady-state error $e_{ss} = \dfrac{1}{K_{sv}} = \dfrac{K_d}{K_p}.$

Laplace transform

Laplace transform

Laplace transform is a tool used to **convert**
an operation of a real **time domain** variable (t) **into**
an operation of a **complex domain** variable (s)

By operating on transformed complex domain rather
than original real time domain problems can be simplified:

- **Linear differential equations**
- Convolutions
- Etc.

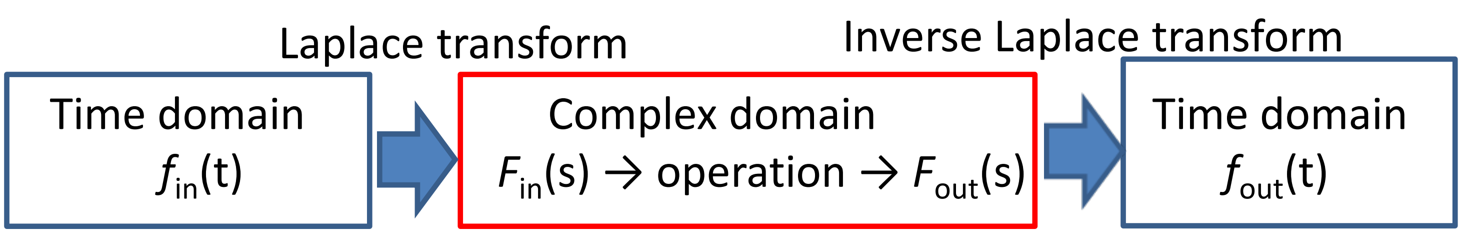
Idea of Laplace transform

Laplace transform converts
differential equations into algebraic equations

Operations on signals involving linear differential equations may be difficult to perform strictly in time domain

Operations may be simplified by:

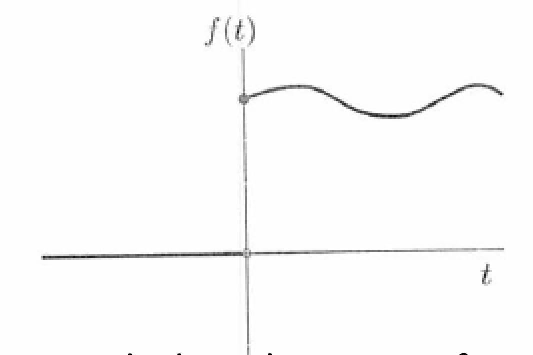
- Converting signal to complex domain
- Performing simpler equivalent operations
- Transforming back to time domain



Laplace transform definition

Definition of Laplace transform

$$F(s) = \mathcal{L}(f(t)) = \int_{0^-}^{\infty} e^{-st} f(t)\,dt$$



One-sided Laplace transform

t = real variable (time)

f(t) = continuous (possibly complex) function

F(s) = Laplace transform of *f*(t)

s = complex variable

Convergence

Finding Laplace transform requires integration of function from zero to infinity

$$F(s) = \mathcal{L}(f(t)) = \int_{0^-}^{\infty} e^{-st} f(t) dt$$

- **For *F*(s) to exist, integral must converge**
- Convergence means that area under integral is finite

Formal:
Laplace transform is guaranteed to exist if *f*(t) is piecewise continuous, and of exponential order, that is $|f(t)| \leq Ae^{tB}$ for all $t \geq 0$, for some $A \geq 0$, and real constant B. The Laplace integral converges if Re(s) > B.

Inverse Laplace transform

Inverse Laplace transform is used to compute $f(t)$ from $F(s)$

Exist a mathematical definition for inverse Laplace transform
(for which computation is complicated)

In practice, the inverse Laplace transform of a rational function using **partial fraction expansions** is calculated using a method of **look-up tables**

Common Laplace transforms

Short **look-up table** to find **inverse Laplace transforms**

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^n}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

Properties of Laplace transform

- Linear combination
- Multiplication by an exponential
- Scaling of time
- Laplace transform of a derivative (Differentiation in time domain)
- Laplace transform of an integral (Integration in time domain)
- Convolution
- Final value theorem

Linear combination

Laplace transform is a **linear** operation

Superposition principle can be applied

Proof:

$$\begin{aligned}\mathcal{L}(\alpha f_1(t) + \beta f_2(t)) &= \int_0^\infty e^{-st}(\alpha f_1(t) + \beta f_2(t))\,dt,\\ &= \alpha \int_0^\infty e^{-st} f_1(t)dt + \beta \int_0^\infty e^{-st} f_2(t)dt,\\ &= \alpha \mathcal{L}(f_1(t)) + \beta \mathcal{L}(f_2(t)).\end{aligned}$$

Multiplication by an exponential

Time domain signal f(t) **multiplied by an exponential function of *at***
→ Laplace transform of f(t) is **shifted** in Laplace s-domain **by *a***

<u>Proof:</u>

$$
\begin{aligned}
\mathcal{L}\left(e^{-at} f(t)\right) &= \int_0^{\infty} f(t) e^{-at} e^{-st} dt, \\
&= \int_0^{\infty} f(t) e^{-(s+a)t} dt, \\
&= F(s+a),
\end{aligned}
$$

where $F(s) = \mathcal{L}(f(t))$.

Scaling of time

In time domain: **times are scaled by *a***

→ In Laplace s-domain: **complex values are scaled by 1/*a***

<u>Proof:</u>

$$\mathcal{L}(f(at)) = \int_0^\infty f(at)e^{-st}dt = \frac{1}{a}\int_0^\infty f(\tau)e^{-(s/a)\tau}d\tau = \frac{1}{a}F\left(\frac{s}{a}\right)$$

where $\tau = at$, and $dt/d\tau = 1/a$,

Laplace transform of a derivative

- Laplace transform of the n^{th} derivative of a continuous function *f*(t) is given by

$$\boxed{\mathcal{L}\left(\frac{d^n}{dt^n}f(t)\right) = s^n F(s) - s^{n-1} f(0) - s^{n-2} \left.\frac{df}{dt}\right|_{t=0} - \dots - \left.\frac{d^{n-1}f}{dt^{n-1}}\right|_{t=0}}$$

- 1^{st} derivative example: $\boxed{\mathcal{L}\left(\frac{df}{dt}\right) = sF(s) - f(0).}$

- 2^{nd} derivative example: $\boxed{L\left(\frac{d^2 f(t)}{dt^2}\right) = s^2 F(s) - sf(0) - \left.\frac{df(t)}{dt}\right|_{t=0}}$

Laplace transform of a derivative

Laplace transform of the 1st derivative of a continuous function *f*(t) is given by

$$\mathcal{L}\left(\frac{df}{dt}\right) = sF(s) - f(0).$$

<u>Proof:</u>

$$\begin{aligned}\mathcal{L}\left(\frac{df}{dt}\right) &= \int_0^\infty \frac{df}{dt} e^{-st}\, dt, \\ &= f(t)e^{-st}\Big|_0^\infty - \int_0^\infty f(t)(-s)e^{-st}\, dt, \\ &= sF(s) - f(0).\end{aligned}$$

Integration by parts

Laplace transform of an integral

Laplace transform of the **integral** of a time domain function *f*(t) is the function **Laplace transform** *F(s)* **divided by *s***

$$\mathcal{L}\left(\int_0^t f(\tau)d\tau\right) = \frac{F(s)}{s}.$$

<u>Proof:</u>

$$\mathcal{L}\left(\int_0^t f(\tau)d\tau\right) = \int_0^\infty \int_0^t f(\tau)d\tau e^{-st}dt.$$

$$\mathcal{L}\left(\int_0^t f(\tau)d\tau\right) = \left[-\frac{1}{s}e^{-st}\int_0^t f(\tau)d\tau\right]\Bigg|_0^\infty - \int_0^\infty f(t)\left(\frac{-1}{s}\right)e^{-st}dt = \frac{F(s)}{s}$$

0

Integration by parts

Convolution

Convolution in time domain becomes **multiplication** in Laplace s-domain

Definition of convolution integral

$$f_1(t) * f_2(t) = \int_0^t f_1(\tau) f_2(t-\tau)\,d\tau = \int_0^t f_1(t-\tau) f_2(\tau)\,d\tau.$$

Without proof

$$\mathcal{L}(f_1(t) * f_2(t)) = F_1(s) F_2(s).$$

Final value theorem

Steady-state value of the signal *f*(t) can be determined using the Laplace transform (limiting behavior of a system governed by ordinary differential equation)

$$\boxed{\lim_{s \to 0} sF(s) = \lim_{t \to \infty} f(t)}$$

Important:
This relationship is only guaranteed if integral for d*f*/d*t* is convergent for s = 0

<u>Proof:</u>

Consider the limit

$$\lim_{s \to 0} \mathcal{L}\left(\frac{df}{dt}\right) = \lim_{s \to 0} \int_0^\infty \frac{df}{dt} e^{-st}\, dt.$$

Change integration and limit

$$\begin{aligned}
\lim_{s \to 0} \mathcal{L}\left(\frac{df}{dt}\right) &= \int_0^\infty \lim_{s \to 0} \frac{df}{dt} e^{-st}\, dt, \\
&= \int_0^\infty \frac{df}{dt}\, dt, \\
&= \lim_{t \to \infty} f(t) - f(0).
\end{aligned}$$

Know also that

$$\mathcal{L}\left(\frac{df}{dt}\right) = sF(s) - f(0).$$

Example: Final value theorem

Example A.2 *Consider the function* $f(t) = e^{-2t} + 3$. *Then,* $df/dt = -2e^{-2t}$, *which clearly satisfies* $|df/dt| \leq 2e^{Bt}$ *for any* $B \geq -2$. *Therefore, the final value theorem may be applied. The Laplace transform of* $f(t)$ *is readily found to be*

$$F(s) = \frac{1}{s+2} + \frac{3}{s}.$$

Applying the final value theorem, we have

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s) = \lim_{s \to 0} \frac{s}{s+2} + 3 = 3,$$

which is exactly the same as would have been found by taking the limit $\lim_{t\to\infty} e^{-2t} + 3$ *directly.*

Example of using Laplace transforms to solve a linear differential equation

Consider the linear time-invariant differential equation

$$\ddot{x} + 3\dot{x} + 2x = 0, \quad x(0) = a, \quad \dot{x}(0) = b.$$

Note:
Laplace transform method applicable only if one has initial conditions

Then,

$$\mathcal{L}(x(t)) = X(s),$$
$$\mathcal{L}(\dot{x}(t)) = sX(s) - x(0) = sX(s) - a,$$
$$\mathcal{L}(\ddot{x}(t)) = s^2X(s) - sx(0) - \dot{x}(0), = s^2X(s) - sa - b.$$

Therefore, taking the Laplace transform of the differential equation gives

$$\mathcal{L}(\ddot{x} + 2\dot{x} + 2x) = \mathcal{L}(0),$$

which leads to

$$s^2X(s) - sa - b + 3(sX(s) - a) + 2X(s) = 0,$$

Example of using Laplace transforms to solve a linear differential equation

which can be rewritten as

$$\left(s^2 + 3s + 2\right) X(s) = as + b + 3a.$$

Solving for $X(s)$, we have

$$X(s) = \frac{as + b + 3a}{s^2 + 3s + 2}.$$

Let us now find the poles of $X(s)$, by solving

$$s^2 + 3s + 2 = 0.$$

From the quadratic equation, we have

$$s = \frac{-3 \pm \sqrt{3^2 - 4 \times 2}}{2} = -2, -1.$$

That is, we can write

$$X(s) = \frac{as + b + 3a}{(s + 2)(s + 1)}.$$

Example of using Laplace transforms to solve a linear differential equation

Let us now expand this in a partial fraction expansion, i.e.

$$X(s) = \frac{c_1}{s+2} + \frac{c_2}{s+1}.$$

$$c_i = (s - p_i)X(s)|_{s=p_i}$$

Since the poles are distinct, we can use to compute c_1 and c_2. We have

$$\begin{aligned} c_1 &= (s+2)X(s)|_{s=-2}, \\ &= \left.\frac{(s+2)(as+b+3a)}{(s+2)(s+1)}\right|_{s=-2}, \\ &= \frac{-2a+b+3a}{-1} = -(a+b), \end{aligned}$$

and likewise for c_2 we obtain

$$\begin{aligned} c_2 &= (s+1)X(s)|_{s=-2}, \\ &= \left.\frac{(s+1)(as+b+3a)}{(s+2)(s+1)}\right|_{s=-1}, \\ &= \frac{-a+b+3a}{1} = 2a+b. \end{aligned}$$

Example of using Laplace transforms to solve a linear differential equation

So, we find the partial fraction expansion of $X(s)$ to be

$$X(s) = \frac{-(a+b)}{s+2} + \frac{2a+b}{s+1}.$$

From Section A.5, we find that

$$\mathcal{L}\left(e^{-at}\right) = \frac{1}{s+a}.$$

Making use of this, we can find $x(t)$ by taking the inverse Laplace transform of $X(s)$. We have,

$$\begin{aligned} x(t) = \mathcal{L}^{-1}(X(s)) &= \mathcal{L}^{-1}\left(\frac{-(a+b)}{s+2}\right) + \mathcal{L}^{-1}\left(\frac{2a+b}{s+1}\right), \\ &= -(a+b)e^{-2t} + (2a+b)e^{-t}, \end{aligned}$$

and the differential equation is solved.

This demonstrates the usefulness of Laplace transforms. They turn differential equations in the time domain into algebraic equations in the Laplace domain which are easier to solve. Finally, taking inverse Laplace transforms yields the solution in the time domain.

Self control

End of final lecture

Please have again a look to following videos:

“Spacecraft stabilization and control 1968”

https://www.youtube.com/watch?v=NJL1ey0zpZg

https://www.youtube.com/watch?v=NROrBUp96o0

Write short summary

Compare old and new summaries

Hope you can recognize a learning curve

Summary

Overview of attitude control and terminology

Mathematical formulation of attitude control

- Formulate attitude control problem
- Transfer function
- Block diagrams
- Control laws
- Time-domain specification

Steady-state error

- For unity feedback (stable) systems, system type (number of integrators in the loop) determine if steady-state error is zero
- Key tool is final value theorem

Laplace transform

- Properties of Laplace transform to simplify time domain operations such as differentiation