AERO-222: Introduction to Aerospace Computation - Spring 2023
Homework #3 - Due Date: Thursday, March 30, 2023

Show all work and justify your answers!

Instructions

- *This homework contains both handwritten and coding problems and shall be submitted according to the following guidelines.*
- *Hardcopy:*
  - *Due on CANVAS at 11:59 PM on the day of the deadline.*
  - *Shall include screenshots of any hand-written work.*
  - *For coding problems, the hardcopy shall include any relevant derivations and emphasize the final results (i.e. boxed, highlighted, etc.). INCLUDE ALL CODING RESULTS (including plots, final values) IN THE HARDCOPY.*
  - *Shall be submitted as a single file according to the provided template with the following naming scheme:* "LastnameHW#.pdf"
  - *If preferable, you can put all of your work into a single Jupyter notebook (.ipynb) with photos of your hand-written work as well. Markdown allows for images.*
- *Coding Submission:*
  - *Due on CANVAS at 11:59 PM on the day of the deadline.*
  - *Shall be submitted as a single file according to the provided template with the following naming scheme:* "LastnameHW#.py" or "LastNameHW#.ipynb".
  - *The script shall print out all outputs asked for in the problem.*
- *Late submissions will be accepted with a 10 point deduction per day late.*

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1. **Linear Least-Squares (Coding Problem) (20 pts).** Create $n = 100$ data points, $[x_k, y_k]$, uniformly distributed in $x \in [-\frac{1}{2}, \frac{3}{2}]$ for the function

$$y(x) = \cos 3x$$

Corrupt the y_k values by adding Gaussian noise, $\bar{y}_k = y_k + r_k$, where $r_k \sim \mathcal{N}(0, \sigma)$ and using the standard deviation $\sigma = 0.12$. Plot $y(x)$ and $\bar{y}_k$. Perform the least-squares of $\bar{y}_k$ using the fitting functions,

$$\hat{y}_1(x) = c_1 + c_2 x + c_3(2x^2 - 1) + c_4(4x^3 - 3x)$$
$$\hat{y}_2(x) = c_1 \cos^2 x + c_2(1 - 2\sin x) + c_3(\cos 3x \ \sin x) + c_4 \frac{3 - x}{3 + x}$$

by computing the c_k coefficients for each function. In a separate figure, also plot the estimated functions $\hat{y}_1(x)$ and $\hat{y}_2(x)$ on top of the simulated noisy data points, $\bar{y}_k$. Compute the L_1, L_2, and L_∞ norms of the two residual vectors.

Solution:

The first figure should show $y(x) = \cos 3x$ and the corrupted measurements $\bar{y}_k$. They can either show the simulated measurements as a scatter plot or as connected points. Both plots below are valid. The data points will not be identical since the noise is randomly generated, but it should have a similar distribution.

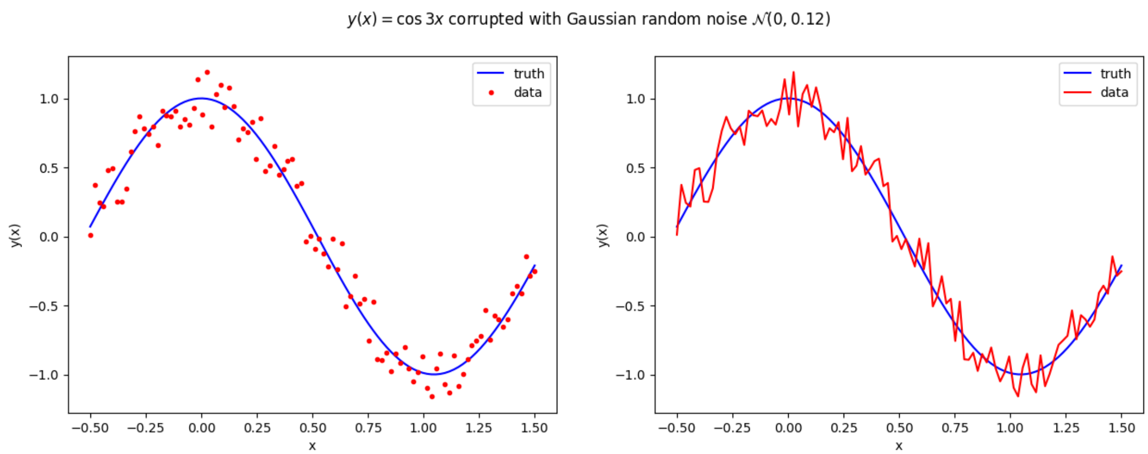

Figure 1: Original Function and Simulated Data Points

The second plot should look something like this. Again, likely not identical because of the randomness in the noise, but the shape should be the same. It is not important if the data points are scattered or connected with lines.

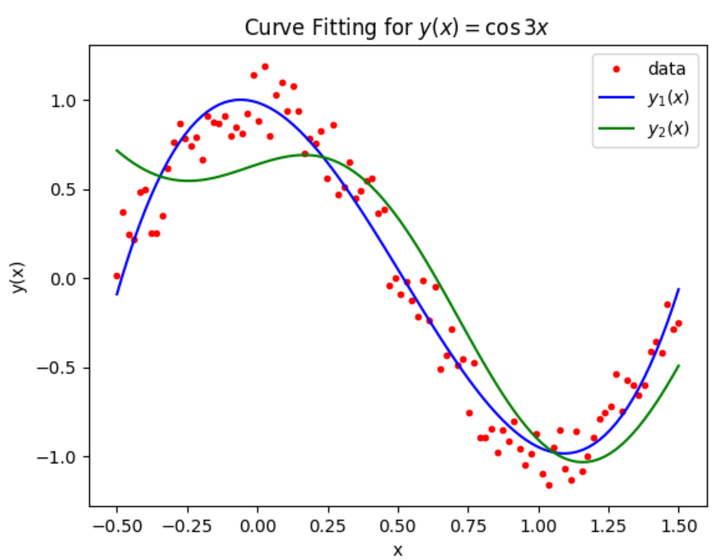

Figure 2: Fitting the Functions onto the Noisy Data

The L_1, L_2, and L_∞ norms should be in a similar range to these solutions

$$\begin{cases} \|\bar{y} - \hat{y}_1\|_1 \approx 11.95482 \\ \|\bar{y} - \hat{y}_2\|_1 \approx 23.67069 \end{cases}$$

$$\begin{cases} \|\bar{y} - \hat{y}_1\|_2 \approx 1.40549 \\ \|\bar{y} - \hat{y}_2\|_2 \approx 2.73101 \end{cases}$$

$$\begin{cases} \|\bar{y} - \hat{y}_1\|_\infty \approx 0.35703 \\ \|\bar{y} - \hat{y}_2\|_\infty \approx 0.70379 \end{cases}$$

2. Linear Least-Squares (By Hand) (15 pts). Show all the steps and necessary equations to estimate the coefficients φ, ξ, λ, and σ with linear least-squares for n data points, $[x_k, y_k]$, and using the fitting function

$$\hat{y}(x) = \cos(4x - \varphi) - x + e^{x^2} - \xi \sin(2x + \lambda) + \sigma e^{2x}$$

Solution:

Start by grouping the terms without unknown coefficients to one side of the equation.

$$\hat{y}(x) + x - e^{x^2} = \cos(4x - \varphi) - \xi \sin(2x + \lambda) + \sigma e^{2x}$$

Recall the angle sum identities for cos and sin

$$\begin{aligned} \cos(x \pm y) &= \cos x \cos y \mp \sin x \sin y \\ \sin(x \pm y) &= \sin x \cos y \pm \cos x \sin y \end{aligned}$$

Apply these identities to the terms $\cos(4x - \varphi)$ and $\sin(2x + \lambda)$.

$$\begin{cases} \cos(4x - \varphi) & = \cos 4x \cos(-\varphi) + \sin 4x \sin(-\varphi) \\ & = \alpha \cos 4x + \beta \sin 4x \end{cases}$$

$$\begin{cases} -\xi \sin(2x + \lambda) & = -\xi \sin 2x \cos \lambda - \xi \cos 2x \sin \lambda \\ & = \gamma \sin 2x + \delta \cos 2x \end{cases}$$

where

$$\begin{aligned} \alpha &= \cos(-\varphi) = \cos \varphi \\ \beta &= \sin(-\varphi) = -\sin \varphi \\ \gamma &= -\xi \cos \lambda \\ \delta &= -\xi \sin \lambda \end{aligned}$$

Set up the following system of equations

$$\begin{bmatrix} y_1 + x_1 - e^{x_1^2} \\ y_2 + x_2 - e^{x_2^2} \\ \vdots \\ y_n + x_n - e^{x_n^2} \end{bmatrix} \approx \begin{bmatrix} \cos 4x_1 & \sin 4x_1 & \cos 2x_1 & \sin 2x_1 & e^{2x_1} \\ \cos 4x_2 & \sin 4x_2 & \cos 2x_2 & \sin 2x_2 & e^{2x_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cos 4x_n & \sin 4x_n & \cos 2x_n & \sin 2x_n & e^{2x_n} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \\ \sigma \end{bmatrix}$$

For the linear least-squares system of equations in the form $y = Ax$, solve for the vector of unknown coefficients, $x = [\alpha, \beta, \gamma, \delta, \sigma]^T$, with the equation

$$x = (A^T A)^{-1} A^T y$$

σ is solved for directly in this step. The other coefficients require an extra step to solve. Using the equations that defined α and β, we can solve for φ

$$\varphi = -\tan^{-1}\left(\frac{\beta}{\alpha}\right)$$

Using the equations that defined γ and δ, we can solve for ξ and λ

$$\xi = \sqrt{\gamma^2 + \delta^2}$$
$$\lambda = \tan^{-1}\left(\frac{\delta}{\gamma}\right)$$

3. **Weighted linear least-squares (Coding Problem) (25 pts).** Consider the physical system of a damped harmonic oscillator. The equation of motion for this system given certain boundary conditions is

$$x(t) = e^{-t}\cos t$$

Create $n = 100$ data points, $[t_k, x_k]$, uniformly distributed in $t \in [0, 5]$ for this problem. To simulate noisy measurements, corrupt the x_k values by adding scaled Gaussian noise, $\bar{x}_k = x_k + r_k\sqrt{t}$, where $r_k \sim \mathcal{N}(0, \sigma)$ and using the standard deviation $\sigma = 0.05$. The weight function is given by the logistic curve

$$w(t) = \frac{1}{1 + 0.1^{(2.5-t)}}$$

Plot $x(t)$ and $\bar{x}_k$. In a separate figure, plot $w(t)$. Compare these figures and explain why the weight function, $w(t)$, makes sense for this problem. Perform the least-squares of $\bar{x}_k$ using the fitting function

$$\hat{x}(t) = c_1 + c_2 x + c_3(2x^2 - 1) + c_4(4x^3 - 3x)$$

In a new figure, plot $\hat{x}(t)$ on top of the corrupted measurements, $\bar{x}_k$.

Hint: Construct a diagonal weight matrix, W, where the elements of the matrix are given by $w(t_k)$.

Solution:

Below are the plots for $x(t)$ and $\bar{x}_k$ (left) and for weighing function, $w(t)$ (right). Again, consider the fact that the random data will not be exactly the same, and it is okay to do a scatter or connected plot for the corrupted measurements.

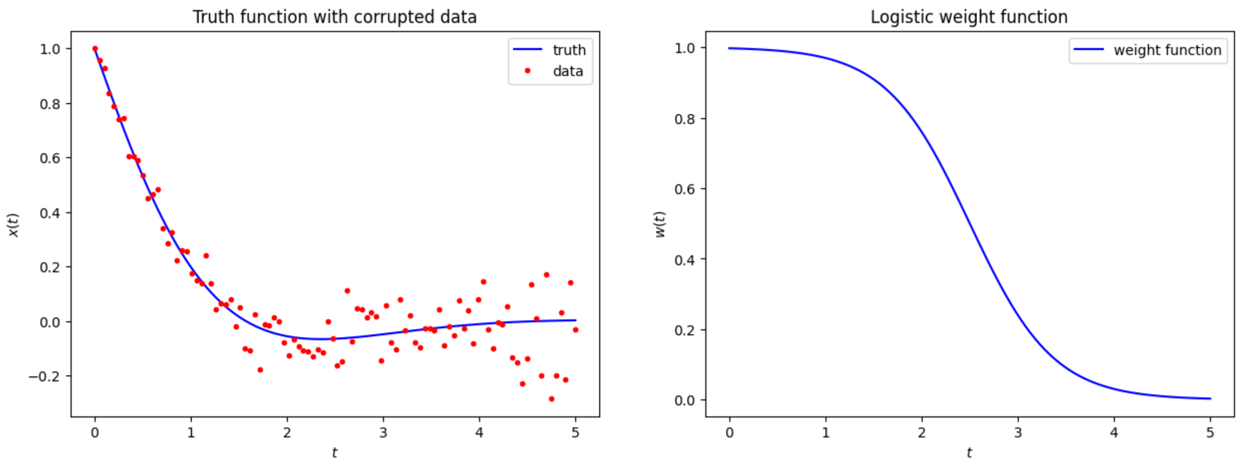

Figure 3: Plotting Original Data and Weight Function

The reason why this weight function makes sense for this problem is because it prioritizes more accurate fitting for lower values of t since the error is increasing with time. Essentially, it trusts the data on the left side of the plot more than the right side.

The final figure should look something like this

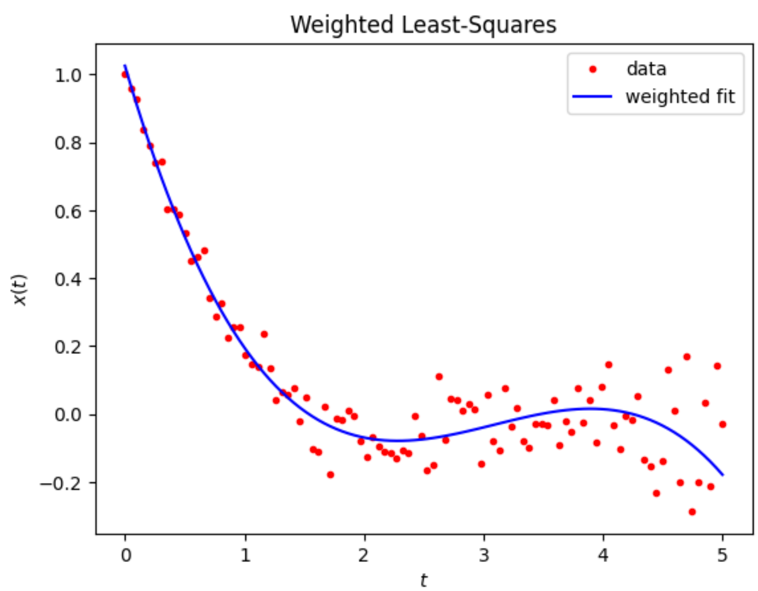

Figure 4: Weighted Least-Squares Fit

4. **Nonlinear least-squares (Coding Problem) (25 pts).** Create $n = 100$ data points, $[x_k, y_k]$, uniformly distributed in $x \in [-1, 1]$, using $a = -2$, $b = 1$, $c = -1$, and $d = 5$ with the following equation,

$$y(x) = 12 - b\,x^2 - e^{-d\,x^2}\sin(c\,x) + a(b\,d - c\,x)$$

Corrupt the y_k data obtained with Gaussian noise, $\bar{y}_k = y_k + r_k$, where $r_k \sim \mathcal{N}(0, \sigma)$ using the standard deviation $\sigma = 0.1$. Estimate the parameters, a, b, c, and d by iterative nonlinear least-squares using $a_0 = -2.2$, $b_0 = 0.3$, $c_0 = -1$, and $d_0 = 5.2$, as a staring point. Plot the L_2 norm of the residual vector as a function of the iterations. Stop the iteration when the L_2 norm of the residual vector is less than 10^{-6}.

Solution:

This first plot is not necessary but can help give students partial credit for this problem. This plot shows the true function along with the corrupted measurements.

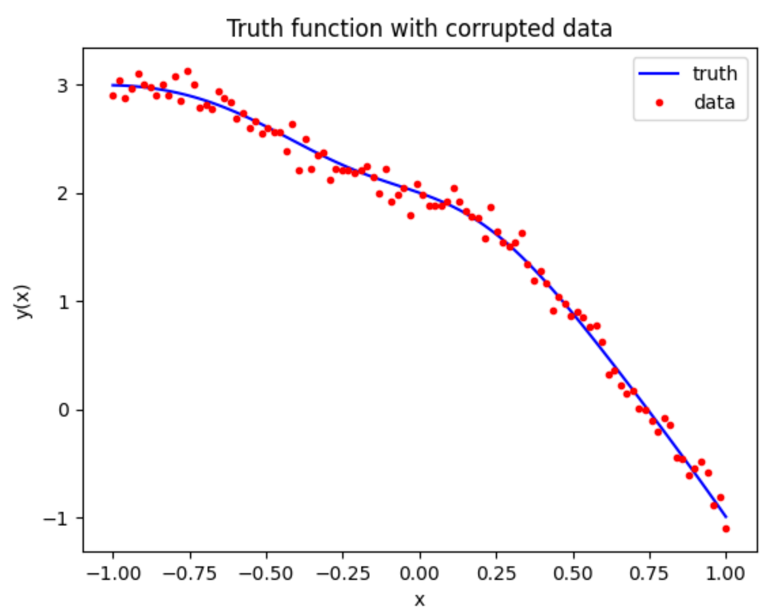

Figure 5: Plotting the True Function and Corrupted Measurements

The plot shown in Figure 6 is what is needed for full points on this problem. Due to the randomness of data generation, this plot can very a lot. The curve may have a different shape and a different number of iterations (I've seen it converge anywhere between 5 and 12 iterations). **The important part of this problem is that their plot converges and meets the residual criteria of 10^{-6}.**

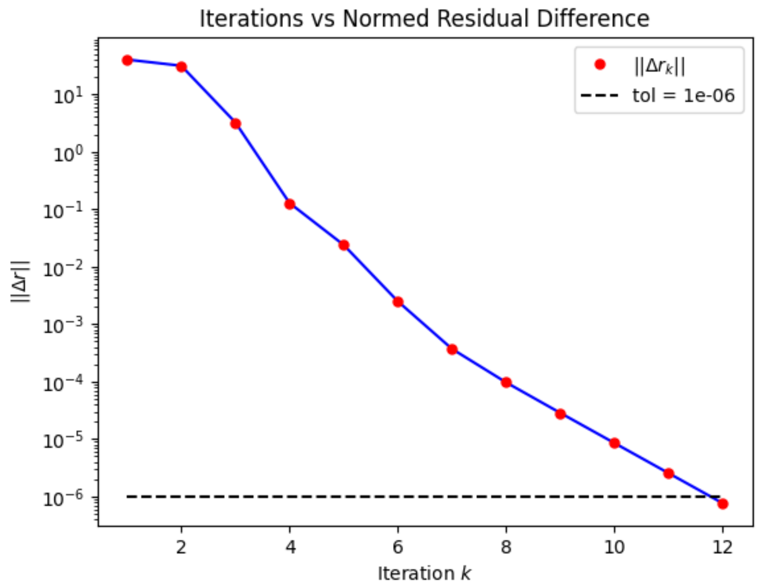

Figure 6: Convergence of Residual Criteria

5. Nonlinear least-squares (By Hand) (15 pts). Consider solving the following system of nonlinear equations using $x_0 = 0.1$, $y_0 = 0.1$, and $z_0 = -0.1$, as starting point,

$$
\begin{aligned}
0 &= 3x - \cos(yz) - 0.5 \\
0 &= x^2 - 81(y + 0.1)^2 + \sin z + 1.06 \\
0 &= e^{-xy} + 20z + (10\pi - 3)/3
\end{aligned}
$$

Write down the first iteration.

Solution:

Define the following vectors

$$
\boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \boldsymbol{f} = \begin{bmatrix} 3x - \cos(yz) - 0.5 \\ x^2 - 81(y + 0.1)^2 + \sin z + 1.06 \\ e^{-xy} + 20z + (10\pi - 3)/3 \end{bmatrix}
$$

Compute the Jacobian matrix

$$
\mathcal{J} = \begin{bmatrix} 3 & z\sin(yz) & y\sin(yz) \\ 2x & -162(y + 0.1) & \cos z \\ -ye^{-xy} & -xe^{-xy} & 20 \end{bmatrix}
$$

The first iteration of the nonlinear least-squares method is given by the equation

$$
\boldsymbol{x}_1 = \boldsymbol{x}_0 - \mathcal{J}_0^{-1}\boldsymbol{f}_0
$$

where $\boldsymbol{x}_0 = [0.1, 0.1, -0.1]^T$. Solving for $\boldsymbol{x}_1$, we get

$$\begin{bmatrix} 0.5002 \\ 0.0195 \\ -0.5215 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.1 \\ -0.1 \end{bmatrix} - \begin{bmatrix} 3 & -0.1\sin(-0.01) & 0.1\sin(-0.01) \\ 0.2 & -32.4 & \cos(-0.1) \\ -0.1e^{-0.01} & -0.1e^{-0.01} & 20 \end{bmatrix}^{-1} \begin{bmatrix} -1.2 \\ -2.0702 \\ 12.462 \end{bmatrix}$$

The important part of this problem is getting the correct Jacobian and setting up the equation. The value of the vector $\boldsymbol{x}_1$ is not so important.