AERO-222: Introduction to Aerospace Computation - Spring 2023
Homework #5 - Due Date: Tuesday, May 2, 2023

Show all work and justify your answers!

Instructions

- *This homework contains both handwritten and coding problems and shall be submitted according to the following guidelines.*
- *Hardcopy:*
  - *Due on CANVAS at 11:59 PM on the day of the deadline.*
  - *Shall include screenshots of any hand-written work.*
  - *For coding problems, the hardcopy shall include any relevant derivations and emphasize the final results (i.e. boxed, highlighted, etc.). INCLUDE ALL CODING RESULTS (including plots, final values) IN THE HARDCOPY.*
  - *Shall be submitted as a single file according to the provided template with the following naming scheme:* "LastnameHW#.pdf"
  - *If preferable, you can put all of your work into a single Jupyter notebook (.ipynb) with photos of your hand-written work as well. Markdown allows for images.*
- *Coding Submission:*
  - *Due on CANVAS at 11:59 PM on the day of the deadline.*
  - *Shall be submitted as a single file according to the provided template with the following naming scheme:* "LastnameHW#.py" or "LastNameHW#.ipynb".
  - *The script shall print out all outputs asked for in the problem.*
- *Late submissions will be accepted with a 10 point deduction per day late.*

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1. Integration: Trapezoid and Midpoint (Coding Problem) (25 pts)

Consider the integral $I = \displaystyle\int_0^{\frac{\pi}{2}} e^{-3x}\cos(2x)\mathrm{d}x$, whose exact value is, $I = \dfrac{3}{13}\left(1 + e^{-\frac{3\pi}{2}}\right)$.

(a) For a range of partitions, $n = [10, 50]$, compute an approximation of the integral using the Midpoint and Trapezoid method. Plot the number of partitions versus the value of the integral.

(b) Plot the absolute error with respect to the true value on a semi-log scale.

Solution:

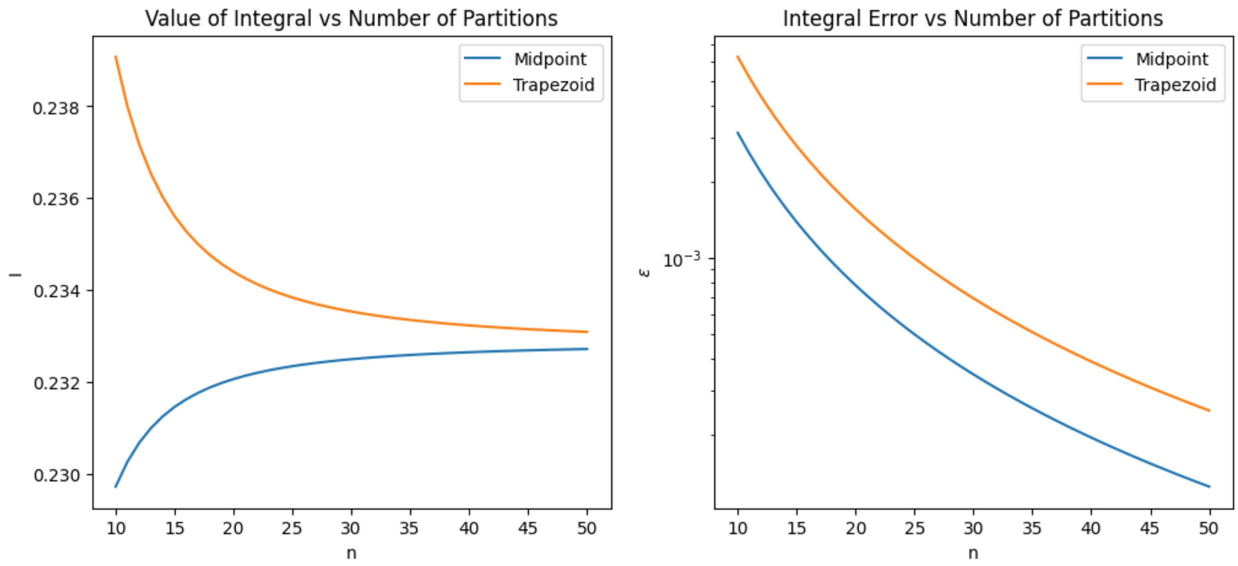

Figure 1: Integral approximation and absolute error using Midpoint and Trapezoid method

2. Integration: Gauss-Legendre (25 pts)

(a) **(By-hand).** Use a 2-point Gauss-Legendre quadrature to approximate the distance covered by a rocket from $t = 5$ sec to $t = 30$ sec, as given by the integral

$$I = \int_5^{30} \left[2,000 \ln\left(\frac{110,000}{110,000 - 1,600\,t}\right) - 9.8\,t\right] \mathrm{d}t.$$

Report the approximate distance.

(b) **(Coding Problem)** Compute the integral with Gauss-Legendre using 4, 6, and 8 points. Report the values of the integral in a table.

Solution:

(a) First, change the limits of integration from $t \in [5, 30]$ to $x \in [-1, 1]$ using

$$t = \frac{(x+1)(b-a)}{2} + a = \frac{25(x+1)}{2} + 5\ , \quad \mathrm{d}t = \frac{b-a}{2}\mathrm{d}x = \frac{25}{2}\mathrm{d}x$$

Substitute into the following equation.

$$\int_a^b f(t)\mathrm{d}t = \int_5^{30} \left[2,000 \ln\left(\frac{110,000}{110,000 - 1,600\,t}\right) - 9.8\,t\right] \mathrm{d}t$$

This gives us

$$\frac{25}{2}\int_{-1}^{1} f\left(\frac{25(x+1)}{2} + 5\right) \mathrm{d}x$$

Apply the 2-point Gauss-Legendre Quadrature to this integral.

$$\frac{25}{2}\int_{-1}^{1} f\left(\frac{25(x+1)}{2}+5\right)\mathrm{d}x \approx \frac{25}{2}\sum_{k=1}^{2} w_k\, f\left(\frac{25(x_k+1)}{2}+5\right)$$

The weighting factors are

$$w_1 = 1 \qquad w_2 = 1$$

and the function argument values are

$$x_1 = -\frac{1}{\sqrt{3}} \qquad x_2 = \frac{1}{\sqrt{3}}$$

Expand the summation defined above and solve.

$$\begin{aligned}\frac{25}{2}\sum_{k=1}^{2} w_k\, f\left(\frac{25(x_k+1)}{2}+5\right) &= 1\cdot f\left(\frac{25(-\frac{1}{\sqrt{3}}+1)}{2}+5\right) + 1\cdot f\left(\frac{25(\frac{1}{\sqrt{3}}+1)}{2}+5\right)\\ &= 223.258057 + 648.843835\\ &= \boxed{10,901.27364565}\end{aligned}$$

(**b**) These values should all be very close, if not identical up to a few decimals, depending on the rounding of the coefficients used.

4 Point	6 Point	8 Point
10,905.39893072	10,905.39943679	10,905.39943687

3. ODE: Improved Euler (Coding Problem) (30 pts)

Solve the following initial value problems:

1. $\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 - 1,$ subject to: $y(0) = 1 \;\rightarrow\; y_{\text{true}}(x) = \frac{1}{3}x^3 - x + 1$
2. $\frac{\mathrm{d}y}{\mathrm{d}x} = -3xy,$ subject to: $y(1) = 2 \;\rightarrow\; y_{\text{true}}(x) = 2e^{-\frac{3}{2}(x^2-1)}$

over the interval $x \in [0, 5]$ (for the first equation) and $x \in [1, 3]$ (for the second equation) using 30 and 300 intervals. You will use the given true solution to determine the absolute error of the methods. For each equation, provide a plot with all 3 absolute errors on a semi-log scale and a plot with all 3 method solutions and the actual solution.

(a) Using Euler's method.

(b) Using Improved Euler's method using average derivative.

(c) Using Improved Euler's method using derivative at the midpoint.

Solution:

Figure 2 contains a subplot of the Euler method, improved Euler methods, and their errors with 30 intervals. The left column corresponds to the first ODE and the right column with the second ODE. Figure 3 contains the same information but for 300 intervals. The students do not need to present their plots in the exact same way. **Give full points if they show similar trends for their solutions**. The important takeaway is that the improved Euler methods have better accuracy than the original Euler method.

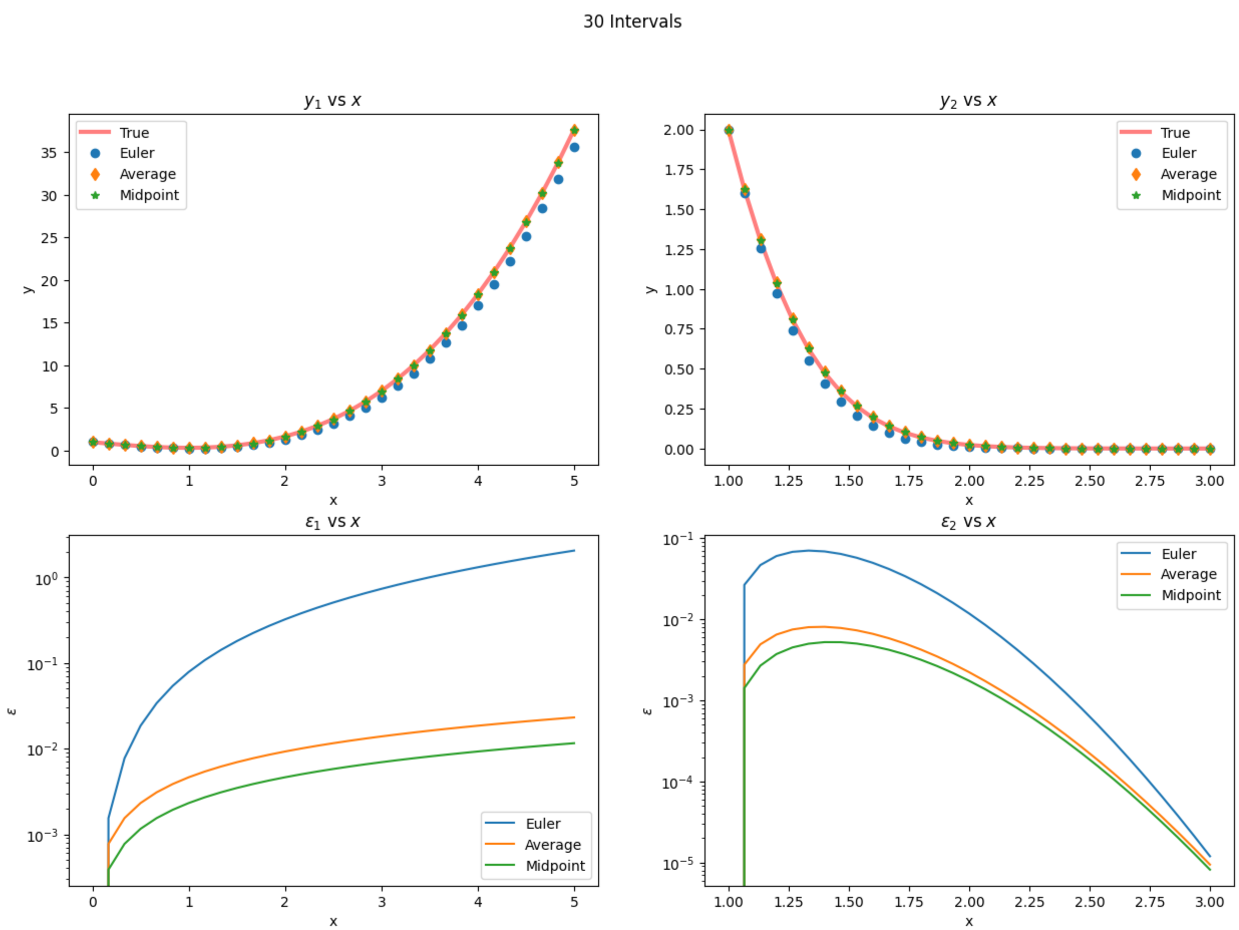

Figure 2: 30 Intervals

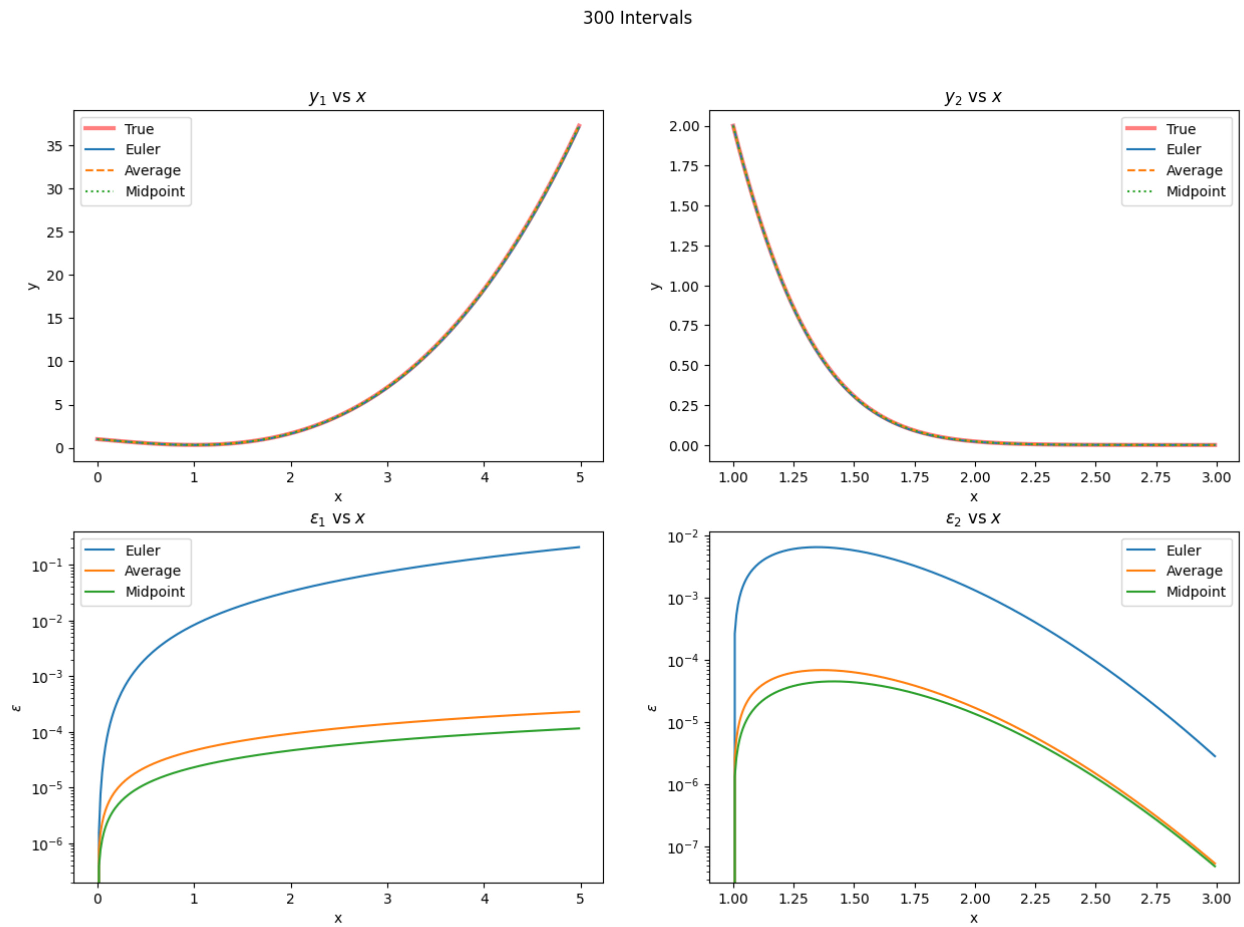

Figure 3: 300 Intervals

4. **ODE: Runge-Kutta (Coding Problem) (20 pts)**

   Solve the two differential equations given in Problem 3 over the same time ranges and intervals. For each equation, provide a plot with both absolute errors on a semi-log scale and a plot with solutions from both methods and the actual solution.

   (a) Using 4th-order Runge-Kutta method.

   (b) Using `scipy.integrate.solve_ivp`.

   Solution:

   The scipy solutions may vary a lot depending on the parameters that were passed into the function, so do not take away points just because the scipy plots don't look similar. **Give full points if the RK4 solutions follow the same trend for the solutions and errors**.

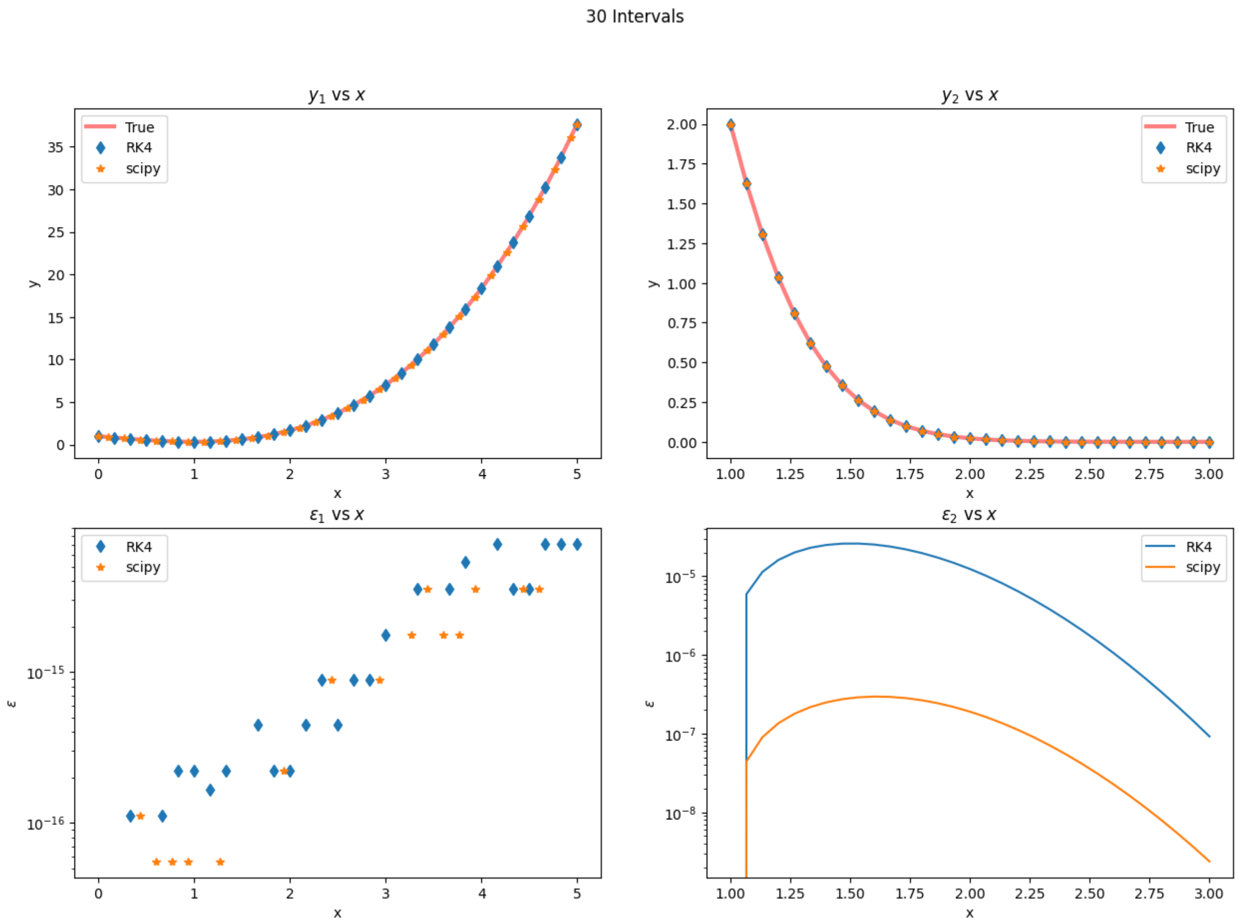

Figure 4: 30 Intervals

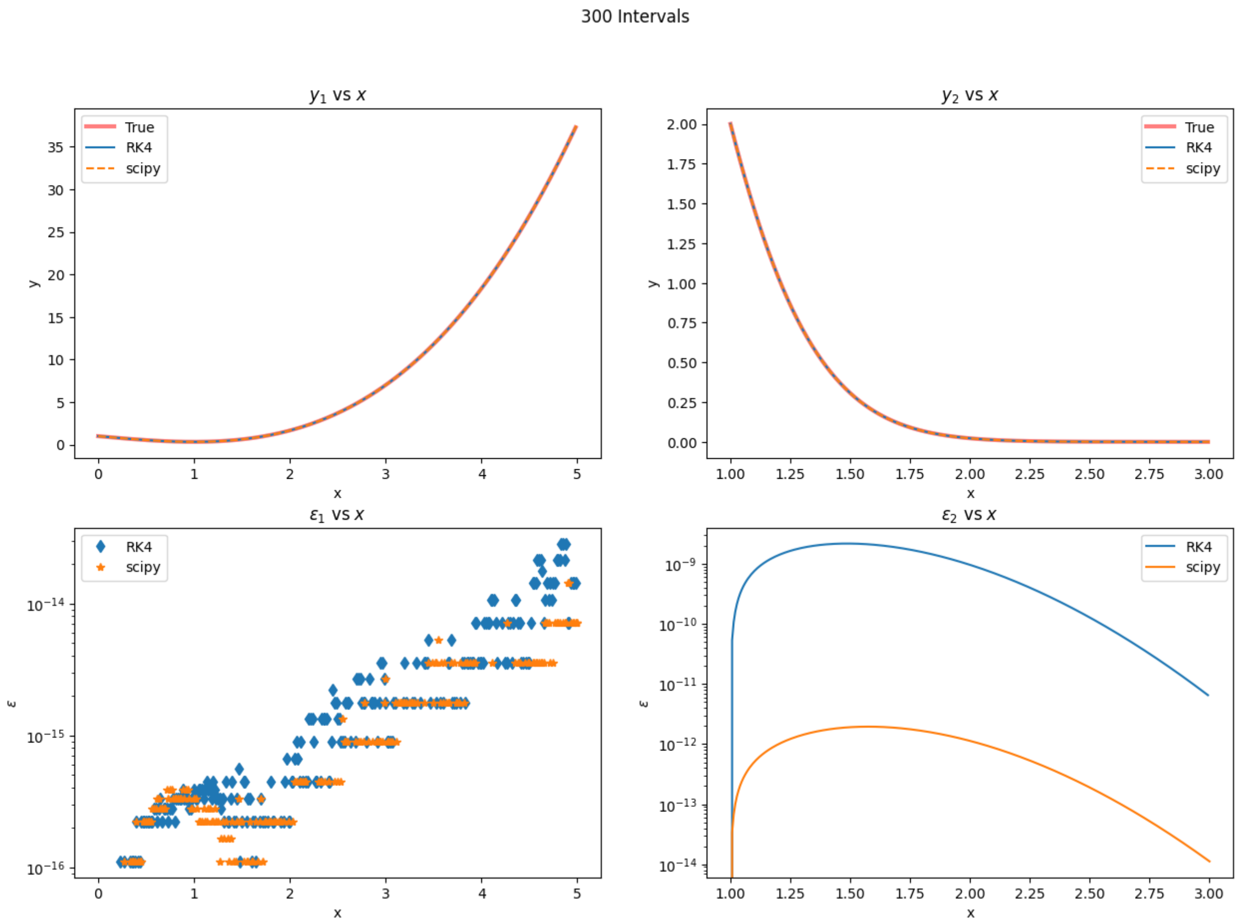

Figure 5: 300 Intervals