AERO-222: Introduction to Aerospace Computation - Spring 2023
Homework #3 - Due Date: Thursday, March 30, 2023

Show all work and justify your answers!

Instructions

- *This homework contains both handwritten and coding problems and shall be submitted according to the following guidelines.*
- *Hardcopy:*
  - *Due on CANVAS at 11:59 PM on the day of the deadline.*
  - *Shall include screenshots of any hand-written work.*
  - *For coding problems, the hardcopy shall include any relevant derivations and emphasize the final results (i.e. boxed, highlighted, etc.). INCLUDE ALL CODING RESULTS (including plots, final values) IN THE HARDCOPY.*
  - *Shall be submitted as a single file according to the provided template with the following naming scheme:* "LastnameHW#.pdf"
  - *If preferable, you can put all of your work into a single Jupyter notebook (.ipynb) with photos of your hand-written work as well. Markdown allows for images.*
- *Coding Submission:*
  - *Due on CANVAS at 11:59 PM on the day of the deadline.*
  - *Shall be submitted as a single file according to the provided template with the following naming scheme:* "LastnameHW#.py" or "LastNameHW#.ipynb".
  - *The script shall print out all outputs asked for in the problem.*
- *Late submissions will be accepted with a 10 point deduction per day late.*

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1. **Linear Least-Squares (Coding Problem) (20 pts).** Create $n = 100$ data points, $[x_k, y_k]$, uniformly distributed in $x \in [-\frac{1}{2}, \frac{3}{2}]$ for the function

$$y(x) = \cos(3x)$$

Corrupt the y_k values by adding Gaussian noise, $\bar{y}_k = y_k + r_k$, where $r_k \sim \mathcal{N}(0, \sigma)$ and using the standard deviation $\sigma = 0.12$. Plot $y(x)$ and $\bar{y}_k$. Perform the least-squares of $\bar{y}_k$ using the fitting functions,

$$\hat{y}_1(x) = c_1 + c_2 x + c_3(2x^2 - 1) + c_4(4x^3 - 3x)$$
$$\hat{y}_2(x) = c_1 \cos^2 x + c_2(1 - 2\sin x) + c_3(\cos 3x \ \sin x) + c_4 \frac{3 - x}{3 + x}$$

by computing the c_k coefficients for each function. In a separate figure, also plot the estimated functions $\hat{y}_1(x)$ and $\hat{y}_2(x)$ on top of the simulated noisy data points, $\bar{y}_k$. Compute the L_1, L_2, and L_∞ norms of the two residual vectors.

2. **Linear Least-Squares (By Hand) (15 pts).** Show all the steps and necessary equations to estimate the coefficients φ, ξ, λ, and σ with linear least-squares for n data points, $[x_k, y_k]$, and using the fitting function

$$\hat{y}(x) = \cos(4x - \varphi) - x + e^{x^2} - \xi \sin(2x + \lambda) + \sigma e^{2x}$$

3. **Weighted linear least-squares (Coding Problem) (25 pts).** Consider the physical system of a damped harmonic oscillator. The equation of motion for this system given certain boundary conditions is

$$x(t) = e^{-t} \cos t$$

Create $n = 100$ data points, $[t_k, x_k]$, uniformly distributed in $t \in [0, 5]$ for this problem. To simulate noisy measurements, corrupt the x_k values by adding scaled Gaussian noise, $\bar{x}_k = x_k + r_k\sqrt{t}$, where $r_k \sim \mathcal{N}(0, \sigma)$ and using the standard deviation $\sigma = 0.05$. The weight function is given by the logistic curve

$$w(t) = \frac{1}{1 + 0.1^{(2.5-t)}}$$

Plot $x(t)$ and $\bar{x}_k$. In a separate figure, plot $w(t)$. Compare these figures and explain why the weight function, $w(t)$, makes sense for this problem. Perform the least-squares of $\bar{x}_k$ using the fitting function

$$\hat{x}(t) = c_1 + c_2 x + c_3(2x^2 - 1) + c_4(4x^3 - 3x)$$

In a new figure, plot $\hat{x}(t)$ on top of the corrupted measurements, $\bar{x}_k$.

Hint: Construct a diagonal weight matrix, W, where the elements of the matrix are given by $w(t_k)$.

4. **Nonlinear least-squares (Coding Problem) (25 pts).** Create $n = 100$ data points, $[x_k, y_k]$, uniformly distributed in $x \in [-1, 1]$, using $a = -2$, $b = 1$, $c = -1$, and $d = 5$ with the following equation,

$$y(x) = 12 - b\,x^2 - e^{-d\,x^2} \sin(c\,x) + a(b\,d - c\,x)$$

Corrupt the y_k data obtained with Gaussian noise, $\bar{y}_k = y_k + r_k$, where $r_k \sim \mathcal{N}(0, \sigma)$ using the standard deviation $\sigma = 0.1$. Estimate the parameters, a, b, c, and d by iterative nonlinear least-squares using $a_0 = -2.2$, $b_0 = 0.3$, $c_0 = -1$, and $d_0 = 5.2$, as a staring point. Plot the L_2 norm of the residual vector as a function of the iterations. Stop the iteration when the L_2 norm of the residual vector is less than 10^{-6}.

5. **Nonlinear least-squares (By Hand) (15 pts).** Consider solving the following system of nonlinear equations using $x_0 = 0.1$, $y_0 = 0.1$, and $z_0 = -0.1$, as starting point,

$$\begin{aligned} 0 &= 3x - \cos(yz) - 0.5 \\ 0 &= x^2 - 81(y + 0.1)^2 + \sin z + 1.06 \\ 0 &= e^{-xy} + 20z + (10\pi - 3)/3 \end{aligned}$$

Write down the first iteration.