AERO-222: Introduction to Aerospace Computation - Fall 2021
Homework #2 - Due Date: Wednesday, October 6, 2021

<u>**Show all work and justify your answers!**</u>

Instructions

- *This homework contains both handwritten and coding problems and shall be submitted according to the following guidelines.*
- *Hardcopy:*
  - *Due on CANVAS at 11:59 PM on the day of the deadline.*
  - *Shall include screenshots of any hand-written work.*
  - *For coding problems, the hardcopy shall include any relevant derivations and emphasize the final results (i.e. boxed, highlighted, etc.).*
  - *Shall be submitted as a single file according to the provided template with the following naming scheme:* "LastnameHW#.pdf"
- *Coding Submission:*
  - *Due on CANVAS at 11:59 PM on the day of the deadline.*
  - *Shall be submitted as a single file according to the provided template with the following naming scheme:* "LastnameHW#.py"
  - *The script shall print out all outputs asked for in the problem.*
- *Late submissions will be accepted with a 10 point deduction per day late.*

---

1. **Newton's Method (15 pts) Code.** Apply the Newton method to find the root of the equation, $\sqrt{x+1} = e^x - 1$, using $x_0 = 0$ as an initial guess.
   1. Report the final estimated error (ε_x), **not the residual**, of your solution using the prescribed residual tolerance: $|f(x_k)| < \varepsilon_y = 10^{-12}$. Report also the number of iterations.
   2. Repeat the same problem, but stop the iterations when $|x_{k+1} - x_k| > |x_k - x_{k-1}|$ is satisfied. Report the number of iterations.
   3. Plot the convergence criteria as a function of iteration number from parts 1) and 2)

2. **Homotopy Continuation (10 pts) Code.** The Newton's method applied to find the root of the equation, $f(x) = 2x - 4 + \sin(2\pi x) = 0$, diverges if $x_0 = 0$ is selected as an initial guess. Solve this problem using homotopy continuation with step $\delta t = 0.025$. Provide subplots of the following as functions of t: (1) the number of iterations needed to converge at each step; (2) the estimated root. Use the residual tolerance $\varepsilon_y = 10^{-12}$ and $N_{\max} = 100$ as the maximum number of iterations.

2. Gaussian Elimination (15 pts) By-hand. Use Gaussian Elimination with scaled partial pivoting to solve the following system of equations:

$$\begin{cases} x_1 + 3x_2 + x_3 = -1 \\ 2x_1 + 2x_2 - 6x_3 = 2 \\ 3x_1 - x_2 + 2x_3 = 3 \end{cases}$$

3. An Aerospace Application (15 pts) Code. The following question brings together elements from multiple methods we've covered so far. An Airbus A320 is flying at Mach 0.7. It's Pitot tubes, shown in Figure **??**, measure the total, or stagnation, pressure (pressure when the air hits and is stopped by the tube) as well as the static free-stream pressure.

The ratio of total to static pressure is determined to be: $p_0/p = 1.289$. The following equation relates this change in pressure to the inlet Mach number, M, and γ, the heat capacity ratio:

$$\frac{p_0}{p} = \left[1 + \frac{\gamma - 1}{2} M^2\right]^{\gamma/(\gamma-1)}$$

Based on the given pressure ratio and Mach number, determine the value of γ that satisfies this equation to *four significant figures* of accuracy. State what method you used, how many iterations it took and give your final solution estimate. **Hint:** Don't try to solve analytically for γ. It might be helpful to plot the function $f(\gamma)$ over a range of values to help you pick a good initial first guess. Choose from any of the iterative methods learned in class to solve.

4. Fixed Point Iteration and Range of Convergence (10 pts) By-hand. A fixed point is a point where $x = g(x)$. We will use fixed point iteration to find these points and then apply concepts from fixed point iteration in order to determine the range of convergence for root solving methods.

(a) Given $x^2 - 4x = 6x + 1$ give two functions, $g_1(x)$ and $g_2(x)$, for which we can perform fixed point iteration to solve for the roots of the equation above.

(b) Compute and draw the range of convergence (if any) on the interval $[-3, 3]$ for both functions, $g_1(x)$ and $g_2(x)$.

5. Newton's Method (20 pts) Code. Given the function $f(x) : -x^4 + 2x^3 = e^{-x} - 1$,

(a) Find the root using Newton's method. Use $x_0 = 1.6$ and perform 5 iterations after the initial guess. Save the x_k from each iteration to a vector.

(b) Determine if Newton's Method converges for the range of $x \in [-3, 3]$ by using the convergence check for fixed point iteration. Provide a plot of $|g'(x)|$ to determine the range of convergence.

(c) Use the results from your code to provide a table of the estimated x_k values from your Newton method iterations. Compute the order of convergence (α) and the asymptotic error constant (λ) as accurately as possible using the x_k values you have saved.

6. Order/Rate of Convergence (10 pts) By-hand. By studying how the x_k values generated by iterative methods change, we can see how quickly root solving methods converge.

(a) What is the order of convergence, α, for Newton's method? What if it's a tangential (double) root?

(b) For a linear method with an asymptotic error constant, λ, of 0.5 what is the ratio between successive errors (in x) after every iteration?

(c) Some really fast root solving method have cubic convergence with an asymptotic error constant $\lambda = 1/2$. Note $\lambda = \dfrac{e_k}{e_{k-1}^{\alpha}}$. If the initial error is $e_0 = 1$ give the error in fractional form for one, two, and three iterations.

7. Aitken Acceleration (15 pts) Code. Using Aitken acceleration with a fixed point, $E = g(E)$, solve Kepler's Equation: $M = E - e \sin E$, for a convergence tolerance defined as $|E_k - E_{k-1}| = 10^{-12}$. Assume $e = 0.65$, $M = \pi/16$, and the initial guess is $E_0 = M$. Compare the results obtained using Aitken acceleration to those obtained without Aitken acceleration. Plot the convergence of E both with and without Aitken acceleration.