AERO-222: Introduction to Aerospace Computation - Spring 2023
Homework #2 - Due Date: Thursday, 9 March, 2023

Show all work and justify your answers!

Instructions

- *This homework contains both handwritten and coding problems and shall be submitted according to the following guidelines.*
- *Hardcopy:*
  - *Due on CANVAS at 11:59 PM on the day of the deadline.*
  - *Shall include screenshots of any hand-written work.*
  - *For coding problems, the hardcopy shall include any relevant derivations and emphasize the final results (i.e. boxed, highlighted, etc.). INCLUDE ALL CODING RESULTS (including plots, final values) IN THE HARDCOPY.*
  - *Shall be submitted as a single file according to the provided template with the following naming scheme:* "LastNameHW#.pdf"
  - *If preferable, you can put all of your work into a single Jupyter notebook (.ipynb) with photos of your hand-written work as well. Markdown allows for images.*
- *Coding Submission:*
  - *Due on CANVAS at 11:59 PM on the day of the deadline.*
  - *Shall be submitted as a single file according to the provided template with the following naming scheme:* "LastNameHW#.py" or "LastNameHW#.ipynb".
  - *The script shall print out all outputs asked for in the problem.*
- *Late submissions will be accepted with a 10 point deduction per day late.*

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1. **Newton's Method (Coding Problem) (30 pts).** Apply the Newton's method to find the solution to the equation, $\tan^{-1} x = x^2 - e^x$, using $x_0 = 0$ as initial guess.

   (a) For the convergence criteria, use the residual tolerance, $|f(x_k)| < \varepsilon_y = 10^{-8}$. Report the final residual of your solution and the number of iterations (starting at $i = 1$).

   (b) For the convergence criteria use, $|x_{k+1} - x_k| > |x_k - x_{k-1}|$. Report the final error, $\varepsilon_x = |x_{k+1} - x_k|$, and number of iterations. The final error should be the last recorded error above machine precision ($\varepsilon_{x,final} \neq 0$)

   (c) Plot the range of convergence for Newton's method.

(d) Compute the order of convergence (α) and the asymptotic error constant (λ) based on your results from part **b**.

Solution:

(a) $N_{\varepsilon_y} = 4$
$\varepsilon_y = f(x_N) \approx 8.32667 \times 10^{-17}$

(b) $N_{\varepsilon_x} = 5$
$\varepsilon_x = |x_N - x_{N-1}| \approx 5.55112 \times 10^{-17}$

(c)

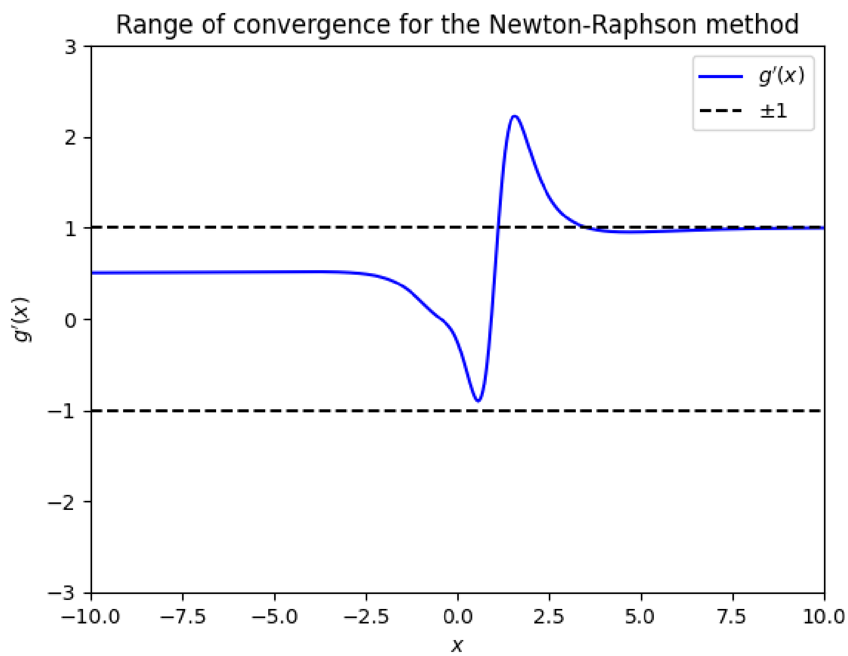

Figure 1: Range of convergence

(d) $\alpha \approx 1.9573$
$\lambda \approx 0.11059$

2. Error Propagation (20 pts). Consider the following two statements,

$$z = 2x - y + \sin(xy^2)$$
$$w = e^z - 2(z^2 - 1)$$

with mean values $\mu_x = 3$ and $\mu_y = 2$, and standard deviations $\sigma_x = 0.02$ and $\sigma_y = 0.01$. Estimate the following parameters using five significant digits:

(a) μ_w

(b) σ_w

Solution:

(a)

$$\begin{aligned}\mu_z &= 2\mu_x - \mu_y + \sin(\mu_x\,\mu_y^2) = 3.4634 \\ \mu_w &= e^{\mu_z} - 2(\mu_z^2 - 1) = 9.9355\end{aligned}$$

(b) Partial derivatives:

$$\begin{aligned}\left.\frac{\partial z}{\partial x}\right|_{\mu_x,\mu_y} &= 2 + \mu_y^2\,\cos(\mu_x\mu_y^2) = 5.3754 \\ \left.\frac{\partial z}{\partial y}\right|_{\mu_x,\mu_y} &= -1 + 2\mu_x\mu_y\,\cos(\mu_x\mu_y^2) = 9.1262 \\ \left.\frac{\partial w}{\partial z}\right|_{\mu_z} &= e^{\mu_z} - 4z = 18.072;\end{aligned}$$

Then, the standard deviation of z becomes,

$$\sigma_z = \sqrt{\left.\frac{\partial z}{\partial x}\right|^2_{\mu_x,\mu_y}\sigma_x^2 + \left.\frac{\partial z}{\partial y}\right|^2_{\mu_x,\mu_y}\sigma_y^2} = 0.14102$$

and the standard deviation of w is,

$$\sigma_w = \sqrt{\left.\frac{\partial w}{\partial z}\right|^2_{\mu_z}\sigma_z^2} = 2.5486$$

3. Gaussian Elimination (20 pts). Use Gaussian Elimination with scaled partial pivoting to solve the following system of equations:

$$\begin{aligned}3x_1 - 4x_3 &= -3 \\ 4x_1 + x_2 + 3x_3 &= 8 \\ x_1 - 3x_2 + 5x_3 &= 2\end{aligned}$$

Solution:

There are a number of different orders of operations, but by rearranging the augmented matrix and substituting back x_3 and x_2 and x_1, the solution is:

$$x_1 \approx 0.65957$$

$$x_2 \approx 1.62766$$

$$x_3 \approx 1.24468$$

4. **An Aerospace Application (Coding Problem) (30 pts).** The following question brings together elements from multiple methods we've covered so far. An Airbus A320 is flying at Mach 0.78. It's Pitot tubes, shown in Figure 2, measure the total, or stagnation, pressure (pressure when the air hits and is stopped by the tube) as well as the static free-stream pressure.

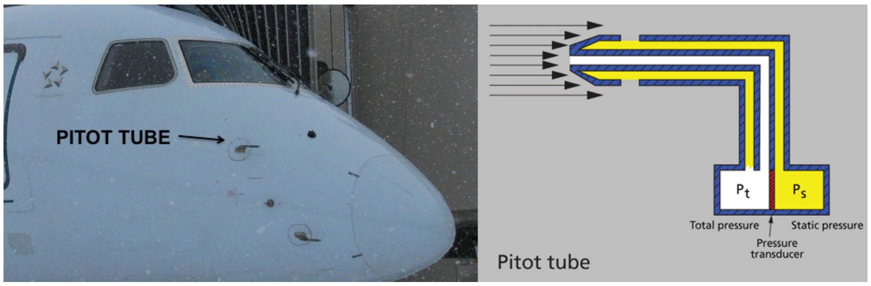

Figure 2: Pitot-Static Tubes: Used to measure the ratio of total pressure to static pressure for determining aircraft airspeed.

The ratio of total to static pressure is determined to be: $p_0/p = 1.364$. The following equation relates this change in pressure to the inlet Mach number, M, and γ, the heat capacity ratio:

$$\frac{p_0}{p} = \left[1 + \frac{\gamma - 1}{2} M^2\right]^{\gamma/(\gamma-1)}$$

Based on the given pressure ratio and Mach number, determine the value of γ that satisfies this equation to *four significant figures* of accuracy. State the method and parameters you used (i.e., Fixed-point method, $g(x)$ =?, initial guess = ?, etc.). Also state how many iterations it took and give your final solution estimate. **Hint:** Don't try to solve analytically for γ. It might be helpful to plot the function $f(\gamma)$ over a range of values to help you pick a good initial first guess. Choose from any of the iterative methods learned in class to solve.

Solution:

First, rearrange the equation so that it equates to zero:

$$f(\gamma) = 0 = \left[1 + \frac{\gamma - 1}{2} M^2\right]^{\gamma/(\gamma-1)} - \frac{p_0}{p}$$

A few different methods are possible to find the root. I found the solution using both implementations of Newton's method from problem 1 to showcase two possible solutions. The derivative was approximated using finite differences with h = 1e-9. While the number of iterations may vary depending on the method, all solutions should yield fairly similar results.

Error Tolerance: $\varepsilon_f = 1 \times 10^{-8}$
Finite Difference: $h = 1e - 9$
Initial Guess: $\gamma_0 = 0$
Given Parameters: $M = 0.7,\ p_0/p = 1.364$

Results from Newton's Method:

(a) Using implementation from Problem 1 part a:
Number of Iterations = 3
$\gamma = 1.024$
$f(\gamma) = 8.397 \times 10^{-12}$

(b) Using implementation from Problem 1 part b:
Number of Iterations = 5
$\gamma = 1.024$
$f(\gamma) = 2.442 \times 10^{-15}$