AERO-222: Introduction to Aerospace Computation - Spring 2023
Homework #4 - Due Date: Tuesday, April 18, 2023

Show all work and justify your answers!

Instructions

- *This homework contains both handwritten and coding problems and shall be submitted according to the following guidelines.*
- *Hardcopy:*
  - *Due on CANVAS at 11:59 PM on the day of the deadline.*
  - *Shall include screenshots of any hand-written work.*
  - *For coding problems, the hardcopy shall include any relevant derivations and emphasize the final results (i.e. boxed, highlighted, etc.). INCLUDE ALL CODING RESULTS (including plots, final values) IN THE HARDCOPY.*
  - *Shall be submitted as a single file according to the provided template with the following naming scheme:* "LastnameHW#.pdf"
  - *If preferable, you can put all of your work into a single Jupyter notebook (.ipynb) with photos of your hand-written work as well. Markdown allows for images.*
- *Coding Submission:*
  - *Due on CANVAS at 11:59 PM on the day of the deadline.*
  - *Shall be submitted as a single file according to the provided template with the following naming scheme:* "LastnameHW#.py" or "LastNameHW#.ipynb".
  - *The script shall print out all outputs asked for in the problem.*
- *Late submissions will be accepted with a 10 point deduction per day late.*

---

1. **Numerical Differentiation (By Hand) (25 pts).** Show all steps to derive the MOST accurate formula of the second derivative, f_i'', using all of the following points

$$f_{i-1}, \qquad f_i, \qquad f_{i+1}, \qquad f_{i+2}, \qquad f_{i+3},$$

via matrix inversion.

Solution:
Begin with a Taylor expansion of the four off-centered points.

$$
\begin{aligned}
f_{i-1} &= f_i - hf_i' + \frac{h^2}{2!}f_i'' - \frac{h^3}{3!}f_i''' + \frac{h^4}{4!}f_i'''' + \mathcal{O}(h^5) \\
f_{i+1} &= f_i + hf'(x_i) + \frac{h^2}{2!}f_i'' + \frac{h^3}{3!}f_i''' + \frac{h^4}{4!}f_i'''' + \mathcal{O}(h^5) \\
f_{i+2} &= f_i + 2hf_i' + 4\frac{h^2}{2!}f_i'' + 8\frac{h^3}{3!}f_i''' + 16\frac{h^4}{4!}f_i'''' + \mathcal{O}(h^5) \\
f_{i+3} &= f_i + 3hf_i' + 9\frac{h^2}{2!}f_i'' + 27\frac{h^3}{3!}f_i''' + 81\frac{h^4}{4!}f_i'''' + \mathcal{O}(h^5)
\end{aligned}
$$

Set up the following matrix-vector system of equations to solve for the coefficients that will get us the most accurate formula for the second derivative.

$$
\begin{bmatrix} -1 & 1 & 2 & 3 \\ 1 & 1 & 4 & 9 \\ -1 & 1 & 8 & 27 \\ 1 & 9 & 16 & 81 \end{bmatrix}
\begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix} \implies \begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix} = \begin{Bmatrix} 11/24 \\ 1/4 \\ 1/6 \\ -1/24 \end{Bmatrix}
$$

Multiply the Taylor expansions by the four coefficients and add them together.

$$
\frac{11}{24}f_{i-1} + \frac{1}{4}f_{i+1} + \frac{1}{6}f_{i+2} - \frac{1}{24}f_{i+3} = \frac{5}{6}f_i + \frac{h^2}{2!}f_i'' + \mathcal{O}(h^5)
$$

$$
\frac{11f_{i-1} - 20f_i + 6f_{i+1} + 4f_{i+2} - f_{i+3}}{24} = \frac{h^2}{2!}f_i'' + \mathcal{O}(h^5)
$$

$$
\boxed{\frac{11f_{i-1} - 20f_i + 6f_{i+1} + 4f_{i+2} - f_{i+3}}{12h^2} = f_i'' + \mathcal{O}(h^3)}
$$

2. Numerical Differentiation (Coding Problem) (25 pts). The derivative of the function,

$$
f(x) = 3\cos(5x) - 2x^3 + x^2 - 4x + 16
$$

is the function, $f'(x) = -15\sin(5x) - 6x^2 + 2x - 4$. Discretize the function $f(x)$ using $N = 100$ points, $[x_k, f_k]$, that are uniformly distributed in $x \in [0, 4]$. Evaluate the first numerical derivative using the 5-point difference formula. Plot the absolute error between true and numerical derivatives. Pay attention to the two extremes: you cannot always use the 5-point **central** difference formula.

Solution:

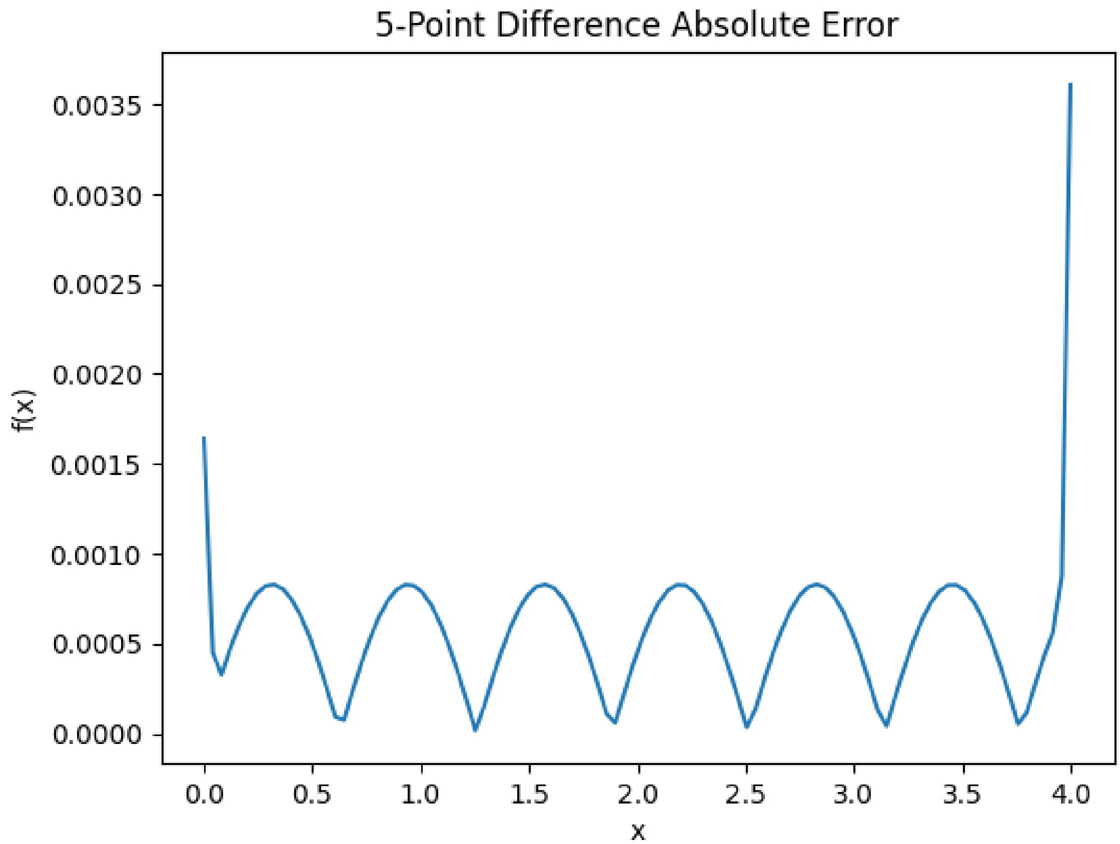

Figure 1: 5-Point Difference Absolute Error

3. Richardson Extrapolation (By Hand) (25 pts). Using the function given in problem #2, estimate the 3-point central second derivative at $x = 3$, using $h_1 = 0.1$ and $h_2 = 0.01$. Use Richardson extrapolation to refine your estimate and provide your final result (Note: $h_1 \neq 2h_2$). Compute the absolute error with respect to the true solution.

Solution:

Richardson extrapolation assumes the following form for a 3-point central difference approximation for a second-order derivative:

$$f''_{RE} = f''_{h_1} + ch_1^2$$

$$f''_{RE} = f''_{h_2} + ch_2^2$$

Solve for c.

$$f''_{h_1} + ch_1^2 = f''_{h_2} + ch_2^2$$

$$f''_{h_1} + c\left(\frac{h_1}{h_2}\right)^2 h_2^2 = f''_{h_2} + ch_2^2$$

$$ch_2^2\left[\left(\frac{h_1}{h_2}\right)^2 - 1\right] = f''_{h_2} - f''_{h_1}$$

$$c = \frac{f''_{h_2} - f''_{h_1}}{h_2^2\left[\left(\frac{h_1}{h_2}\right)^2 - 1\right]}$$

Substitute back into the second equation.

$$f''_{RE} = f''_{h_2} + \frac{f''_{h_2} - f''_{h_1}}{\left(\frac{h_1}{h_2}\right)^2 - 1}$$

The equation for 3-point central finite difference for a second-order derivative is

$$f''(x_i) \approx \frac{f(x_{i-1}) - 2f(x_i) + f(x_{i+1})}{h^2}$$

Apply this equation at $x = 3$ for $h_1 = 0.1$ and $h_2 = 0.01$ for the function $f(x) = 3\cos(5x) - 2x^3 + x^2 - 4x + 16$.

$$f''(3)|_{h_1} \approx \frac{f(2.9) - 2f(3) + f(3.1)}{(0.1)^2} = 21.7994$$

$$f''(3)|_{h_2} \approx \frac{f(2.99) - 2f(3) + f(3.01)}{(0.01)^2} = 22.9647$$

Apply the Richardson extrapolation with these solutions.

$$f''_{RE} = 22.9647 + \frac{22.9647 - 21.7994}{\left(\frac{0.1}{0.01}\right)^2 - 1} = \boxed{22.976495}$$

The true value of $f''(3)$ is 22.976593 and the absolute error of the Richardson extrapolation approximation is

$$\boxed{\varepsilon_{RE} = 9.8473 \times 10^{-5}}$$

4. **An Aerospace Application (Coding Problem) (25 pts).** An airfoil is placed in a wind tunnel and heated to 80°C before lowering the surface temperature to 20.7°C using convective cooling. A thermocouple is placed on the surface of the airfoil and measures the surface temperature at discrete time steps. The following temperatures are recorded,

t (sec)	0	5	10	15	20	25
T (°C)	80	44.5	30	24.1	21.7	20.7

Use numerical differentiation to compute the first derivative $T'(t_k)$ at each time step given in the table above. Specifically, apply the three-point forward, backward or central finite-difference method wherever appropriate in order to estimate the airfoil's temperature gradient, $T'(t_k)$. Provide a table of values.

Solution:

t (sec)	0	5	10	15	20	25
T' (°C/sec)	-9.2	-5.0	-2.02	-0.83	-0.34	-0.06