AERO 422 Homework #2

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Due: September 22, 2021 at 12:40p.m.

Fall 2021

(**25 Points**)

1. Consider the function $f(t) = te^{2t}\sin 3t$

   (a) (**2 points**) Find the Laplace transform using the table. Mention which entries from the table are being used.
   Refer to the following properties from the Laplace table

$$\mathcal{L}\left\{t^n f(t)\right\} = (-1)^n \frac{d^n F(s)}{ds^n} \quad (7), \qquad \mathcal{L}\left\{e^{at}\sin kt\right\} = \frac{k}{(s-a)^2+k^2} \quad (22)$$

   Let $f(t) = e^{2t}\sin 3t$. Apply (7) to get

$$\begin{aligned}\mathcal{L}\left\{te^{2t}\sin 3t\right\} &= \mathcal{L}\left\{tf(t)\right\} \\ &= (-1)\frac{dF(s)}{ds} \\ &= (-1)\frac{d\left[\mathcal{L}\left\{e^{2t}\sin 3t\right\}\right]}{ds}\end{aligned}$$

   Apply (22) and simplify using the quotiet rule.

$$\begin{aligned}\mathcal{L}\left\{te^{2t}\sin 3t\right\} &= (-1)\frac{d}{ds}\left\{\frac{3}{(s-2)^2+3^2}\right\} \\ &= (-1)\frac{-3\cdot 2(s-2)}{((s-2)^2+9)^2} \\ &= \frac{6s-12}{((s-2)^2+9)^2}\end{aligned}$$

   (b) (**1 point**) Can we use the F.V.T. to determine $f(\infty)$? Why or why not?
   For the Final Value Theroem to apply, $f(t)$ must approach a finite value in the limit as t approaches ∞. Upon inspection, we see that $f(t) = te^{2t}\sin 3t$ diverges as $t \to \infty$, and thus does not approach a finite value.

   Another way to arrive at this conclusion is to look at the poles of the function. If a pole exists in the right-half plane, then the function will diverge. If the poles are purely imaginary, then the function will oscillate and will not approach a finite value in the limit as $t \to \infty$. Therefore, if there are no poles in the right-half plane, then at least one pole must lie in the left-half plane in order for the function to converge to a steady-state finite value.

   The poles of the transfer function

$$F(s) = \frac{6s-12}{((s-2)^2+9)^2}$$

   are $s = 2 + j3,\ 2 - j3,\ 2 + j3,\ 2 - j3$, which are all in the right-half plane, implying that this function is unstable.

   Therefore, the Final Value Theorem cannot be applied to determine $f(\infty)$.

2. Find the inverse Laplace transform using the table and partial fraction expansion. Show your work.

(a) (**2 points**)

$$F(s) = \frac{s+10}{s^2+2s+10}$$

Refer to the following properties from the Laplace table

$$\mathcal{L}\left\{e^{at}\sin kt\right\} = \frac{k}{(s-a)^2+k^2} \quad (22), \qquad \mathcal{L}\left\{e^{at}\cos kt\right\} = \frac{s-a}{(s-a)^2+k^2} \quad (23)$$

By completing the square, $F(s)$ can be re-written as follows:

$$\begin{aligned} F(s) &= \frac{s+10}{s^2+2s+10} \\ &= \frac{s+10}{(s+1)^2+9} \\ &= \frac{s+1}{(s+1)^2+3^2} + \frac{9}{(s+1)^2+3^2} \\ &= \frac{s+1}{(s+1)^2+3^2} + \frac{3^2}{(s+1)^2+3^2} \end{aligned}$$

Take the inverse Laplace transform of $F(s)$ by using (22) and (23).

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left\{F(s)\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+3^2}\right\} + 3\cdot\mathcal{L}^{-1}\left\{\frac{3}{(s+1)^2+3^2}\right\} \\ &= e^{-t}\cos 3t + 3e^{-t}\sin 3t \\ &= e^{-t}\left(\cos 3t + 3\sin 3t\right) \end{aligned}$$

(b) (**3 points**)

$$F(s) = \frac{s^2+1}{s(s-1)^3}$$

Refer to the following properties from the Laplace table

$$\mathcal{L}\left\{\frac{1}{s}\right\} = 1(t) \quad (1), \qquad \mathcal{L}\left\{\frac{1}{s-a}\right\} = e^{at} \quad (15), \qquad \mathcal{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} \quad (21)$$

$$F(s) = \frac{s^2+1}{s(s-1)^3} \quad \rightarrow \quad \text{Poles at } s_1 = 0 \text{ and } s_2 = 1 \text{ (with multiplicity 3)}$$

The partial fraction expansion takes the form

$$F(s) = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2} + \frac{D}{(s-1^3)}$$

Reduce this form of $F(s)$ into a single rational function to get

$$F(s) = \frac{(A+B+C)s^3 + (-3A-2B-C)s^2 + (3A+B+D)s - A}{s(s-1)^3}$$

Comparing this form of $F(s)$ to the original form yields a set of 4 equations with 4 unknowns as

$$\begin{aligned} A+B+C &= 0 \\ -3A-2B-C &= 1 \\ 3A+B+D &= 0 \\ -A &= 1 \end{aligned}$$

The solution to these equations is $A = -1$, $B = 1$, $C = 0$, and $D = 2$. Then the partial fraction expansion of $F(s)$ is

$$F(s) = -\frac{1}{s} + \frac{1}{s-1} + \frac{2}{(s-1)^3}$$

Apply (1), (15), (21) to find $f(t)$.

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\{F(s)\} \\ &= -\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{(s-1)^3}\right\} \\ &= -1(t) + e^t + t^2 e^t, \quad t \geq 0 \end{aligned}$$

3. A given system is found to have a transfer function that is

$$\frac{Y(s)}{R(s)} = \frac{10(s+2)}{s^2 + 8s + 15}$$

(a) (**3 points**) Using partial fractions, determine $y(t)$ when $r(t)$ is a unit step input. Show your work.
Solution in other document (Problem 3b)

(b) (**1 point**) Can we use F.V.T. to find $y(\infty)$? If the answer is yes, apply F.V.T. If not, explain why.
Solution in other document (Problem 3a)

4. (a) (**3 points**) Using the convolution integral, find the step response of the system whose impulse response is given below

$$h(t) = \begin{cases} te^{-t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

There are only two cases to consider for this problem.

Case (a): for the case $t \leq 0$, the situation is illustrated in Figure 1 as part (c). There is no overlap between the two functions ($u(t-\tau)$ and $h(\tau)$) so the output is zero.

$$y_1(t) = 0, \quad t \leq 0$$

Case (b): for the case $t \geq 0$, the situation is displayed in Figure 1 as part (d). In this figure, the unit step function is denoted as $u(t)$. The output of the system is given by

$$\begin{aligned} y_2(t) &= \int_0^t h(\tau) \cdot u(t-\tau)\, d\tau \\ &= \int_0^t (\tau e^{-\tau})(1)\, d\tau \quad \rightarrow \quad \text{integration by parts} \\ &= 1 - (t+1)e^-t, \quad t \geq 0 \end{aligned}$$

(b) (**2 points**) Now use the Laplace transform table and partial fraction expansion to find $y(t)$.
Refer to the following properties from the Laplace table

$$\mathcal{L}\left\{\frac{1}{s}\right\} = 1(t) \quad (1), \qquad \mathcal{L}\left\{\frac{1}{s-a}\right\} = e^{at} \quad (15), \qquad \mathcal{L}\left\{te^{at}\right\} = \frac{1}{(s-a)^2} \quad (20)$$

Convolution in time-domain is equivalent to multiplication in frequency-domain.

$$\begin{aligned} Y(s) &= H(s) \cdot U(s) \\ &= \mathcal{L}\left\{te^{-t}\right\} \cdot \mathcal{L}\{1\} \\ &= \frac{1}{(s+1)^2} \cdot \frac{1}{s} \end{aligned}$$

The partial fraction expansion of this transfer function takes the form

$$Y(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

Reduce this form of $F(s)$ into a single rational function.

$$\begin{aligned} Y(s) &= \frac{A(s+1)^2 + Bs(s+1) + Cs}{s(s+1)^2} \\ &= \frac{(A+B)s^2 + (2A+B+C)s + A}{s(s+1)^2} \end{aligned}$$

Comparing this form of $F(s)$ to the original form yields a set of 3 equations with 3 unknowns as

$$\begin{aligned} A + B &= 0 \\ 2A + B + C &= 0 \\ A &= 1 \end{aligned}$$

The solution to these equations is $A = 1$, $B = -1$, and $C = -1$. Then the partial fraction expansion of $F(s)$ is

$$F(s) = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}$$

Apply (1), (15), (20) to find $f(t)$.

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\{F(s)\} \\ &= -\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} \\ &= 1 - e^{-t} - te^{-t}, \quad t \geq 0 \\ &= 1 - (t+1)e^{-}t, \quad t \geq 0 \end{aligned}$$

Note this solution is identical to the solution in part a), proving that convolution in time-domain is equivalent to multiplication in frequency-domain.

(c) (**2 points**) Apply I.V.T. and F.V.T. (if applicable) to find $y(0)$ and $y(\infty)$.
By inspection, both Initial Value Theorem and Final Value Theorem can be applied because $y(t)$ results in finite values for $t \to 0$ and $t \to \infty$.

Initial Value Theorem:

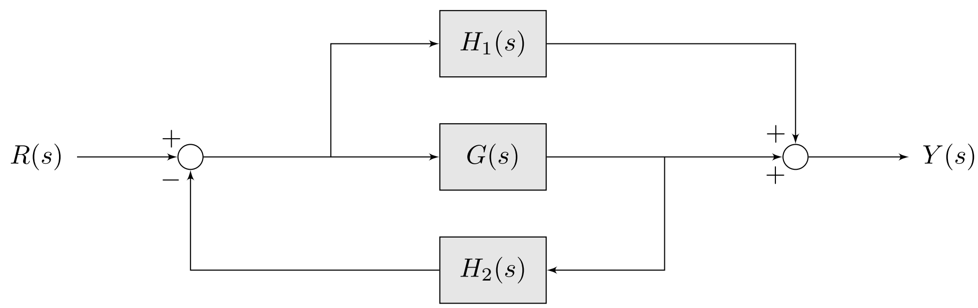
$$\begin{aligned} \lim_{t\to 0} f(t) &= \lim_{s\to\infty} s \cdot Y(s) \\ &= \lim_{s\to\infty} s \cdot \frac{1}{(s+1)^2} \cdot \frac{1}{s} \\ &= \lim_{s\to\infty} \frac{1}{(s+1)^2} \\ &= 0 \end{aligned}$$

Final Value Theorem:

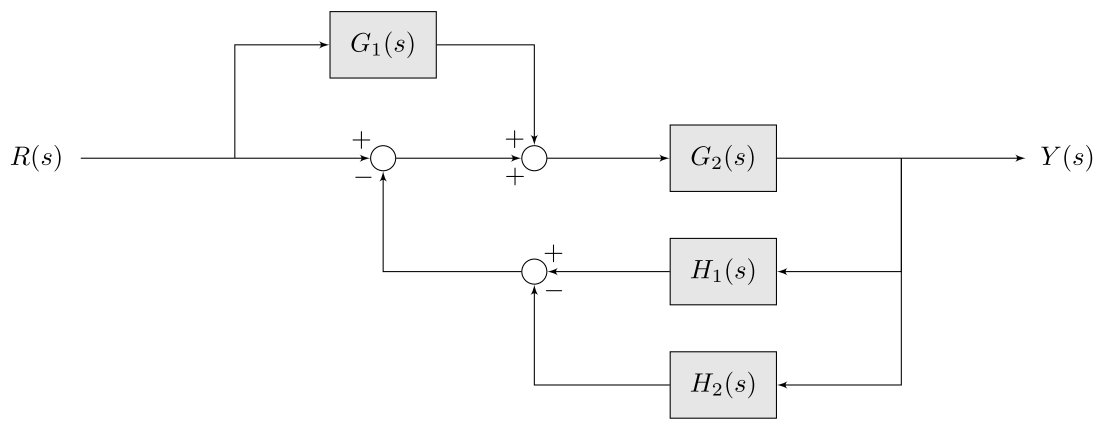
$$\begin{aligned}
\lim_{t\to\infty} f(t) &= \lim_{s\to 0} s \cdot Y(s) \\
&= \lim_{s\to 0} s \cdot \frac{1}{(s+1)^2} \cdot \frac{1}{s} \\
&= \lim_{s\to 0} \frac{1}{(s+1)^2} \\
&= 1
\end{aligned}$$

5. For each of the following block diagrams, reduce the block diagram to find $T(s)$, where $T(s)$ is defined by $Y(s) = T(s)R(s)$.

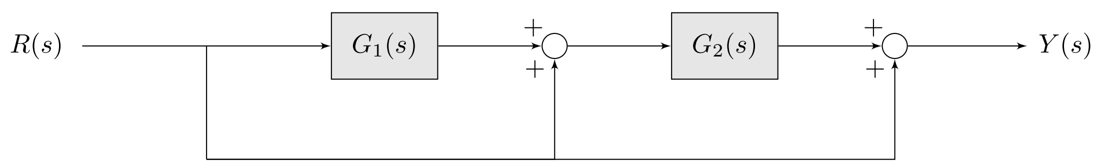
   (a) (**1.5 points**)
   Solution in other document (Problem 4a)



   (b) (**3 points**)
   Solution in other document (Problem 4b)



   (c) (**1.5 points**)
   Solution in other document (Problem 4c)



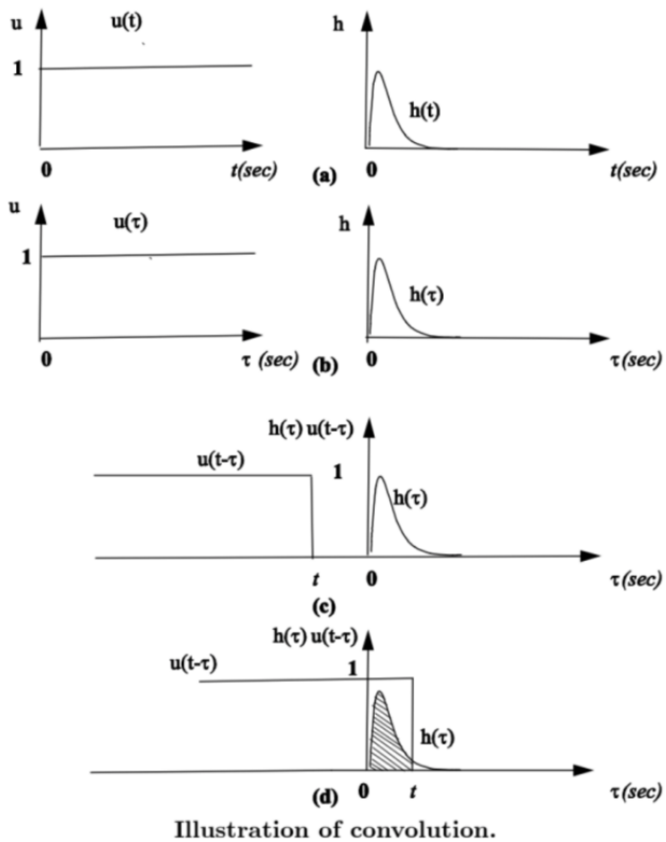


Figure 1: Convolution integral (reference for problem 4)