MATH 422 HW 3 Diego Torres

1. $\vec{V} = y\hat{\imath} + 2x\hat{\jmath} + z\hat{k}$ C is circle $x^2+y^2=1$, $x=\cos t$, $y=\sin t$, $0\le t\le 2\pi$ S is area bounded by C

$$\iint_S \vec{\nabla}\cdot(\vec{V}\times\hat{n})\,d\sigma = \iint_S \hat{n}\cdot(\vec{\nabla}\times\vec{V})\,d\sigma \ , \ \hat{n}=\hat{k}$$

$$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & 2x & z \end{vmatrix} = \hat{\imath}\,0 - \hat{\jmath}\,0 + \hat{k}(2-1) = \hat{k}$$

$$\Rightarrow \iint_S \hat{k}\cdot\hat{k}\,d\sigma = \iint_S d\sigma = \boxed{\pi}$$

$$\oint_C \vec{V}\cdot d\vec{r} = \oint_C \vec{V}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

$\vec{r}(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath}$

$\dfrac{d\vec{r}}{dt} = -\sin t\,\hat{\imath} + \cos t\,\hat{\jmath}$

$\vec{V}(\vec{r}(t)) = \sin t\,\hat{\imath} + 2\cos t\,\hat{\jmath} + 0\,\hat{k}$

$$= \oint_C \langle \sin t, 2\cos t, 0\rangle\cdot\langle -\sin t, \cos t, 0\rangle\,dt$$

$$= \oint_C (-\sin^2 t + 2\cos^2 t)\,dt = \int_0^{2\pi}(\cos 2t + \cos^2 t)\,dt = \int_0^{2\pi}\left(\cos 2t + \frac{1}{2} + \frac{\cos 2t}{2}\right)dt$$

$$= \int_0^{2\pi}\left(\frac{1}{2} + \frac{3}{2}\cos 2t\right)dt = \left(\frac{1}{2}t + \frac{3}{4}\sin 2t\right)\Big|_0^{2\pi} = \boxed{\pi}$$

Stoke's Thm is True!

2. Determine $\iint_S \hat{n}\cdot\vec{\nabla}\times\vec{V}\,d\sigma$ over unit sphere above xy plane

$x^2+y^2+z^2=1$

when $\vec{V}=y\hat{i}$

$0\le t\le 2\pi$

$$\iint_S \hat{n}\cdot\vec{\nabla}\times\vec{V}\,d\sigma = \oint_C \vec{V}\cdot d\vec{r} = \oint_C \vec{V}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

$\vec{r}(t)=x\hat{i}+y\hat{j}+z\hat{k}$, $x=\cos t$, $y=\sin t$

$\vec{r}(t)=\cos t\,\hat{i}+\sin t\,\hat{j}$, $\vec{r}\,'(t)=-\sin t\,\hat{i}+\cos t\,\hat{j}$

$\vec{V}(\vec{r}(t))=\sin t\,\hat{i}$

$$\oint_C \langle \sin t,0,0\rangle\cdot\langle -\sin t,\cos t,0\rangle\,dt = \int_0^{2\pi} -\sin^2 t\,dt = -\int_0^{2\pi}\left(\frac{1}{2}-\frac{\cos 2t}{2}\right)dt$$

$$=\left(-\frac{1}{2}t+\cancel{\frac{\sin(2t)}{4}}\right)\Bigg|_0^{2\pi} = \boxed{-\pi}$$

3. Solve

(a) $\dfrac{d^2y}{dx^2}+k^2y=\sin(x)$

Char. eq.

$m^2+k^2=0$ $\quad m=\pm\sqrt{-k^2}=\pm ik$

$y_H=C_1\cos(kx)+C_2\sin(kx)$

$\Updownarrow$ They're lin. indep. y_p stays same form

$y_p=A\cos(x)+B\sin(x)$

$y_p'=-A\sin(x)+B\cos(x)$

$y_p''=-A\cos(x)-B\sin(x)$

$-A\cos(x)-B\sin(x)+Ak^2\cos(x)+Bk^2\sin(x)$
$=\sin(x)$

$A=0$

$Bk^2-B=1\Rightarrow B=\dfrac{1}{k^2-1}$

$y_p=\dfrac{1}{k^2-1}\sin(x)$

$$\boxed{\Rightarrow y=C_1\cos(kx)+C_2\sin(kx)+\frac{1}{k^2-1}\sin(x)}$$

(b) $\frac{d^2y}{dx^2} - y = e^x$

Char. eq:

$m^2 - 1 = 0$

$m = \pm 1$

$y_H = C_1 e^x + C_2 e^{-x}$ Family: $\{e^x\}$, need to become $\{xe^x\}$

$y_p = Axe^x$

$y_p' = A(xe^x + e^x)$

$y_p'' = A(xe^x + 2e^x)$

$\cancel{Axe^x} + 2Ae^x - \cancel{Axe^x} = e^x$

$A = \frac{1}{2} \Rightarrow y_p = \frac{1}{2}xe^x$

$$\Rightarrow \boxed{y = C_1 e^x + C_2 e^{-x} + \frac{1}{2}xe^x}$$

(c) $\frac{d^2y}{dx^2} - y = xe^x$

$y_H = C_1 e^x + C_2 e^{-x}$ Family: $\{e^x\} \cdot \{x, 1\} \rightarrow \{xe^x, e^x\} \rightarrow \{x^2e^x, xe^x\}$

$y_p = Ax^2e^x + Bxe^x$

$y_p' = A(2xe^x + x^2e^x) + B(e^x + xe^x)$

$y_p'' = A(2e^x + 2xe^x + 2xe^x + x^2e^x) + B(e^x + e^x + xe^x)$

$= A(2e^x + 4xe^x + x^2e^x) + B(2e^x + xe^x)$

$\Rightarrow 2Ae^x + 4Axe^x + \cancel{Ax^2e^x} + 2Be^x + \cancel{Bxe^x} - \cancel{Ax^2e^x} - \cancel{Bxe^x} = xe^x$

$4A = 1 \rightarrow A = 1/4$

$2A = -2B \rightarrow B = -1/4$

$\Rightarrow y_p = \frac{1}{4}x^2e^x - \frac{1}{4}xe^x$

$$\boxed{\Rightarrow y = C_1 e^x + C_2 e^{-x} + \frac{1}{4}x^2e^x - \frac{1}{4}xe^x}$$

4. $u_1 = e^x$ given $(x-1)y'' - xy' + y = 1$

Homogeneous: $(x-1)e^x - xe^x + e^x = 0 \Rightarrow \cancel{xe^x} - \cancel{e^x} - \cancel{xe^x} + \cancel{e^x} = 0$ ✓

St. form

$h(x) = \frac{1}{x-1}$

$$y'' - \frac{x}{x-1}y' + \frac{1}{x-1}y = \frac{1}{x-1}$$

$$u_2 = u_1 \int^x \frac{e^{-\int^{x'} -\frac{x''}{x''-1}dx''}}{e^{2x'}}\,dx' \rightarrow e^{\int \frac{x}{x-1}dx} \quad \begin{matrix} x-1=u \\ x=u+1 \end{matrix} \quad \int\left(\frac{u}{u} + \frac{1}{u}\right)du = u + \ln|u| = x-1+\ln|x-1|$$

$$u_2 = e^x \int^x \frac{e^{x'-1}(x-1)}{e^{2x'}}\,dx' = e^{x-1}\int^x e^{x'-2x'}(x'-1)\,dx' = e^{x-1}\int e^{-x'}(x'-1)\,dx'$$

	D	I
+	$x-1$	e^{-x}
−	1	$-e^{-x}$
+	0	e^{-x}

$$u_2 = e^{x-1}\left((x-1)(-e^{-x}) - e^{-x}\right) = e^{x-1}\left(-xe^{-x} + \cancel{e^{-x}} - \cancel{e^{-x}}\right)$$

$$u_2 = -xe^{-1} = C_2 x \quad (e^{-1}) \text{ gets absorbed}$$

$y_H = C_1 e^x + C_2 x$, y_p, family: $\{1\}$

$y_p = A$
$y_p' = 0$
$y_p'' = 0$

Plugging into DE:

$A = 1$, $y_p = 1$

$$\boxed{y = C_1 e^x + C_2 x + 1}$$