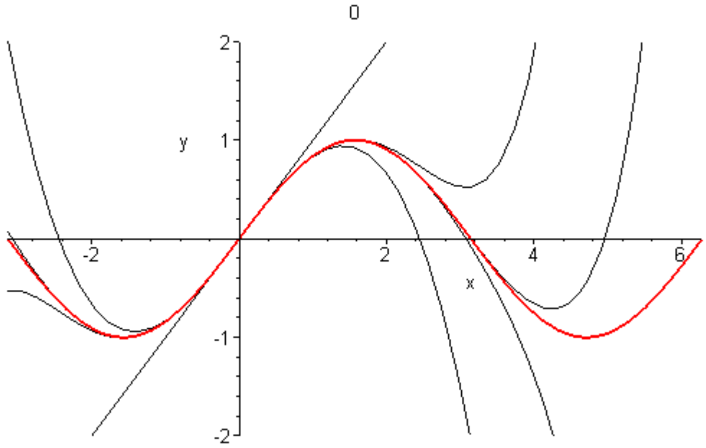
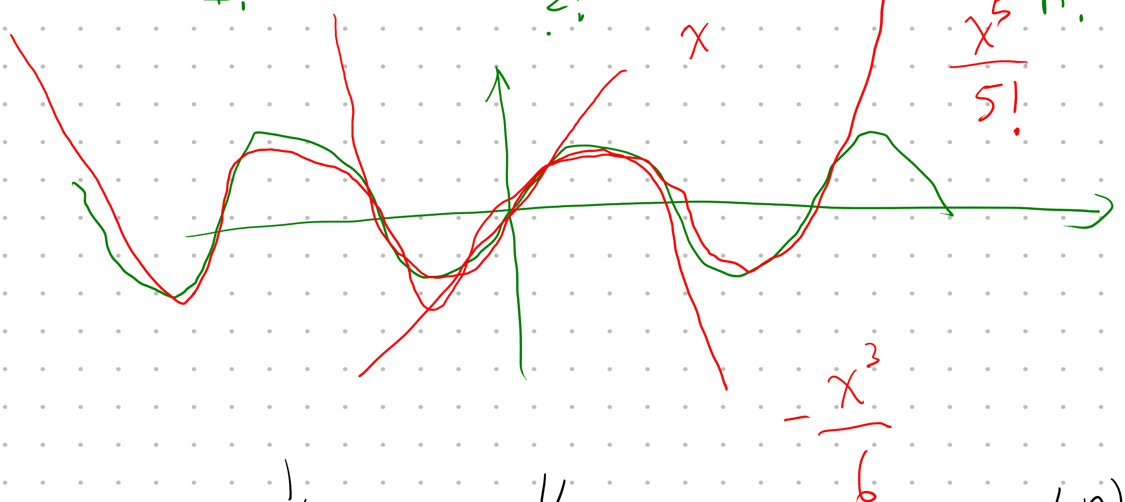
Teorema de Taylor

$f(x) = \text{sen}(x)$



$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}}{n!}(x-a)^n + \dots$$



Dem

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

Proof:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$$

$$f(0) = a_0 \Rightarrow a_0 = f(0)$$

$$g(x) = \ln(x)$$

$$f'(x) = a_1 + 2a_2 x^1 + \dots + n a_n x^{n-1}$$

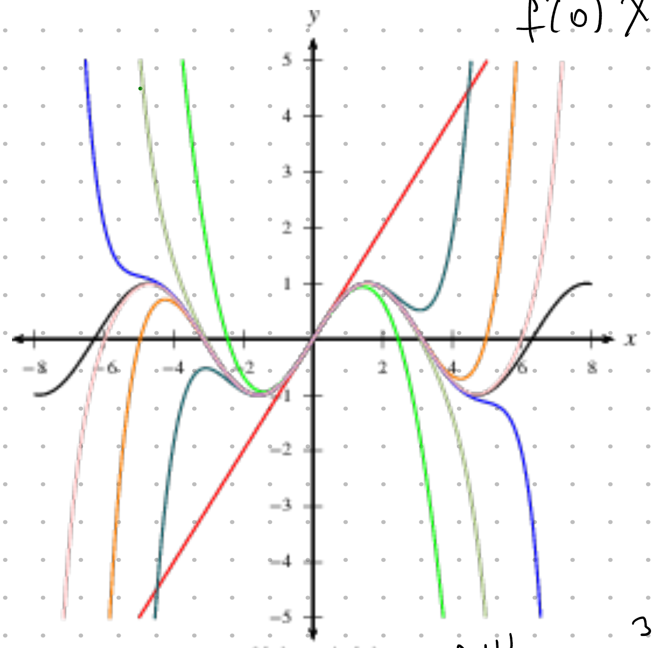
$$g'(x) = \frac{1}{x}$$

$$f'(0) = a_1 \qquad a_1 = f'(0)$$

$$f''(x) = (2\cdot 1)a_2 + (3\cdot 2)a_3 x + \dots + n(n-1)x^{n-2}$$

$$f''(0) = 2!\,a_2 \Rightarrow a_2 = \frac{f''(0)}{2!}$$

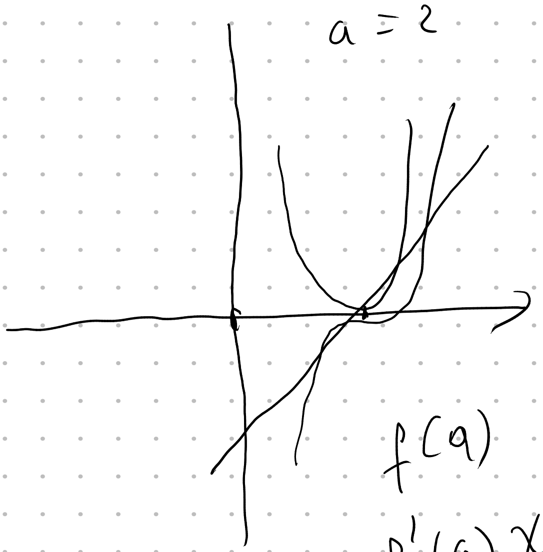
$f'(0)x$



$f(x) = \sin(x)$

$$\frac{f'''(0)x^3}{3!}$$

$a = 2$



$f(a)$

$f'(a)x$

$$\frac{f''(a)x^2}{2!}$$

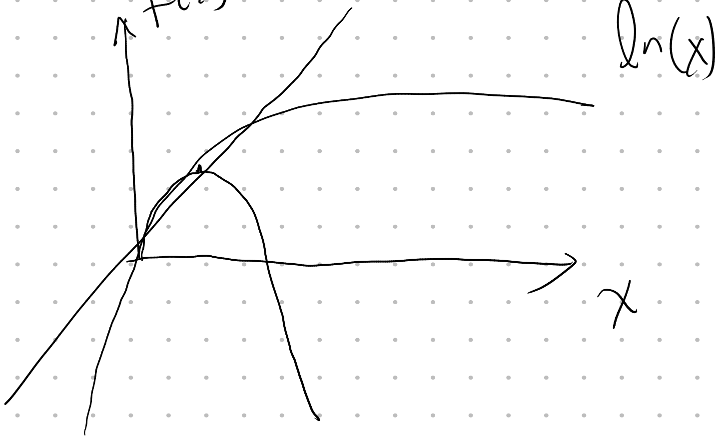
$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

$$f^n(x) = n!\,a_n + (n+1)\cdots(2)\,a_{n+1} x + \cdots$$

$$f^n(0) = n!\,a_n \qquad a_n = \frac{f^{(n)}(0)}{n!}$$

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots +$$

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

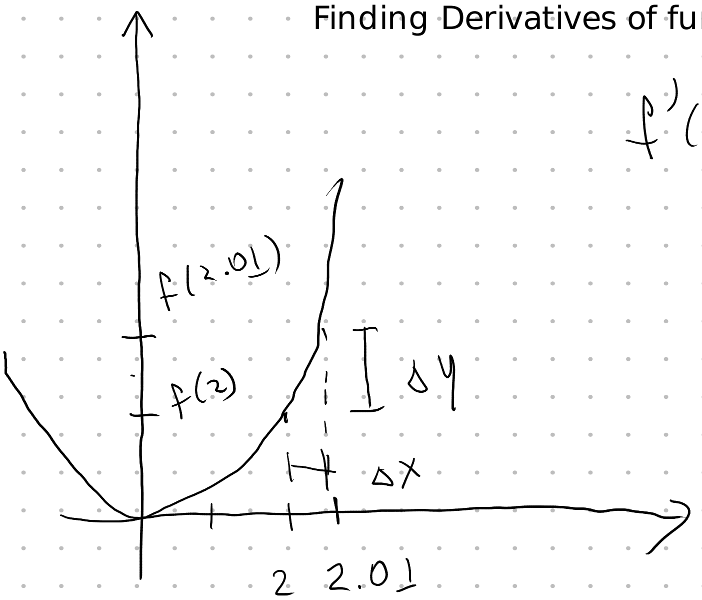


$f(x) = x^2$

$f'(x) = 2x$

$f'(2) = 4$

Finding Derivatives of functions



$$f'(2) = \frac{f(2.01) - f(2)}{0.01}$$

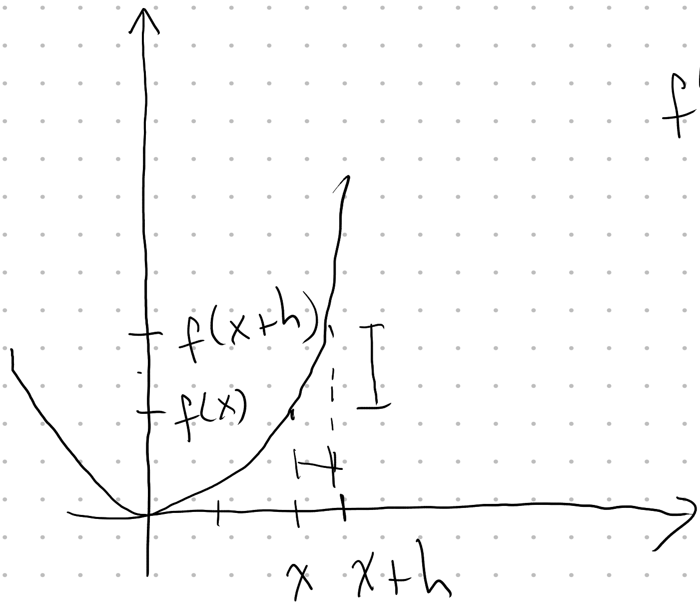
$$= \frac{2.01^2 - 2^2}{0.01} = \frac{4.0401 - 4}{0.01}$$

$$= 4.01$$

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{3!} f'''(x)$$

$$f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{h}{2} f''(x) - \frac{h^2}{3!} f'''(x)$$

$$e \sim h$$



$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

Derivada central

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots$$

$-$

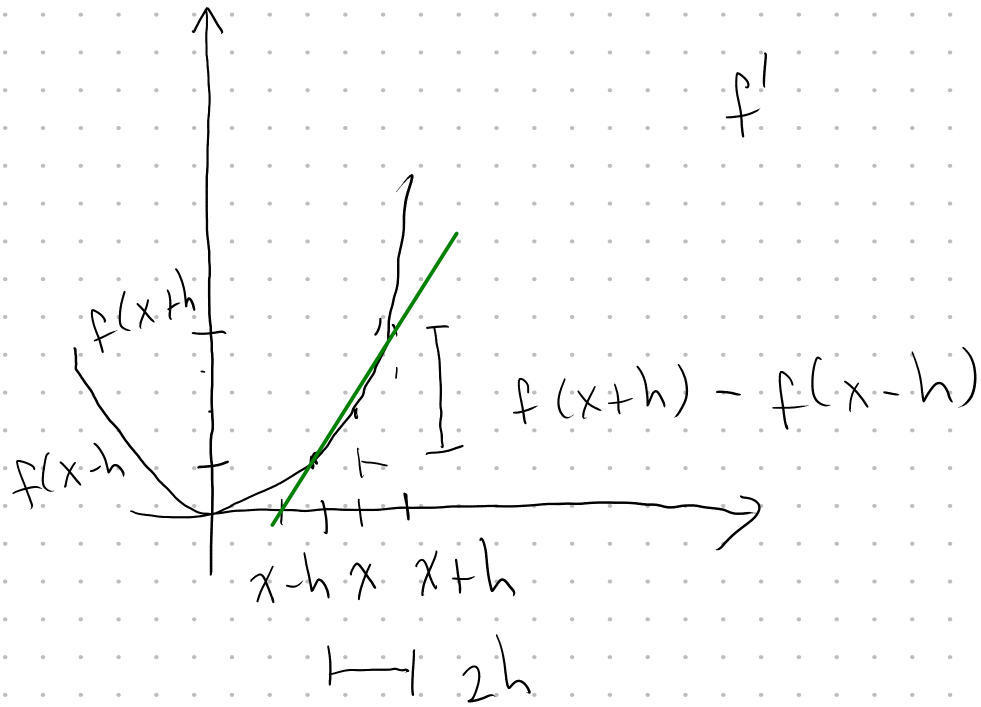
$$f(x-h) = f(x) - h f'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{3!} f'''(x) + \cdots$$

$$f(x+h) - f(x-h) = 2hf'(x) + \frac{2}{6} h^3 f'''(x) + \frac{2}{5!} h^5 f^{(5)}(x)$$

$$2hf'(x) = f(x+h) - f(x-h) - \frac{2}{3!} h^3 f'''(x) - \frac{2}{5!} h^5 f^{(5)}(x)$$

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{1}{3!} h^2 f'''(x) \cdots$$

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + \mathcal{O}(h^2)$$



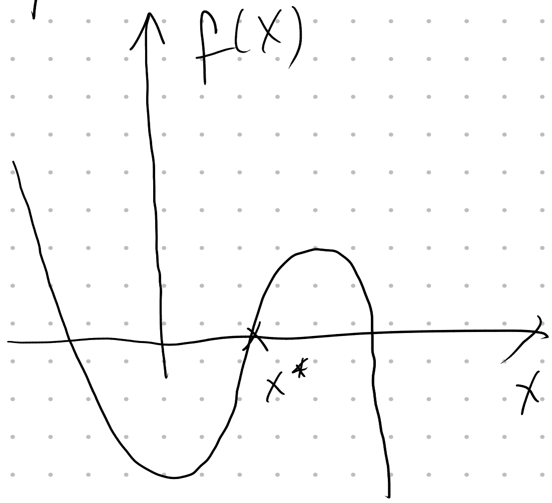
Newton Raphson Method

- Resolver una ecuación
- Encontrar la raiz de una función.

$$f(X) = x^3 + 2X - 3$$

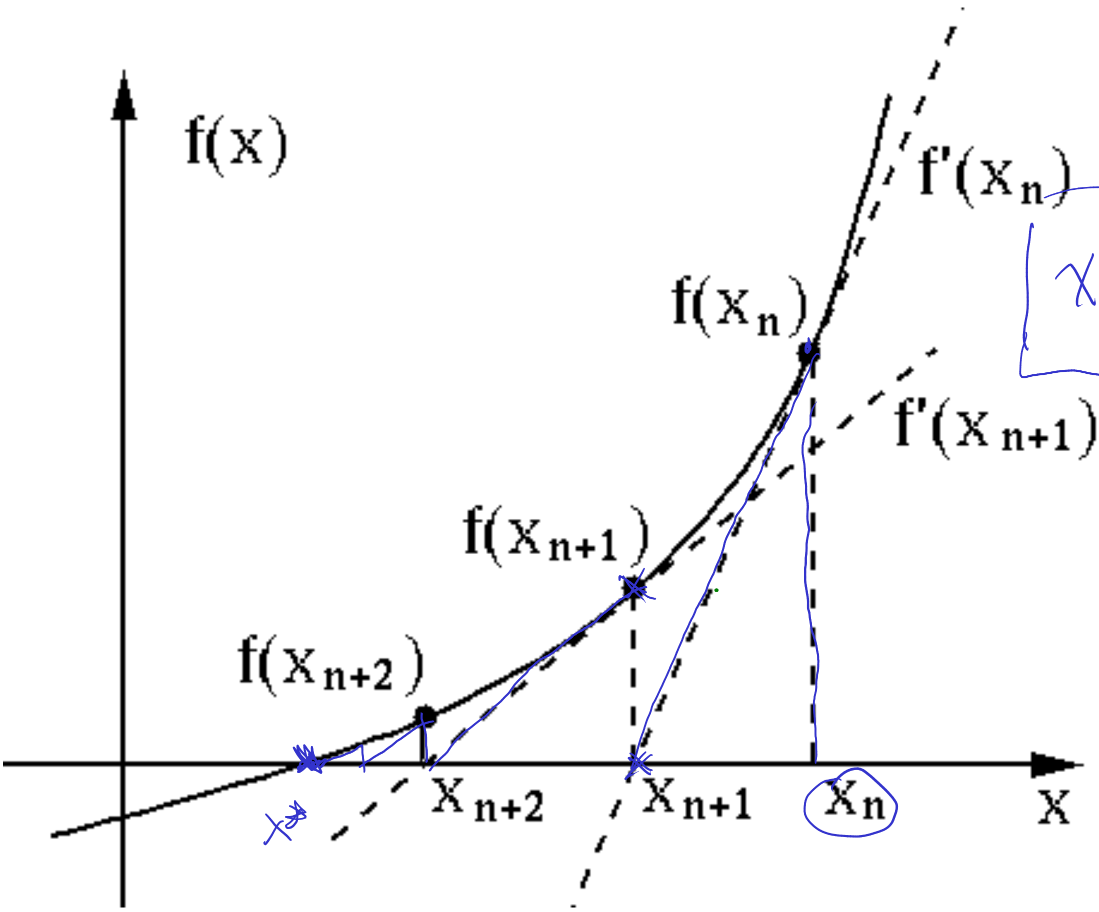
$$f(x^*) = 0$$

$$0 = x^3 + 2X - 3$$



Newton - Raphson

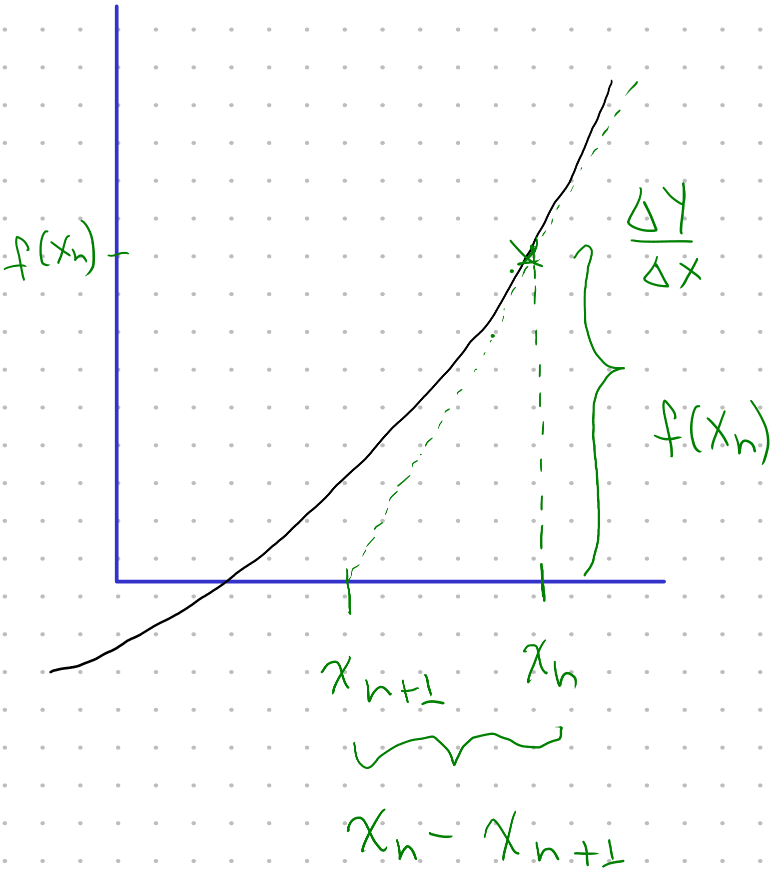
$$x_{n+1} = x_n - \frac{f(X_n)}{f'(X_n)}$$



$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

$f(x^*) = 0$

$f(x^*) = 0$

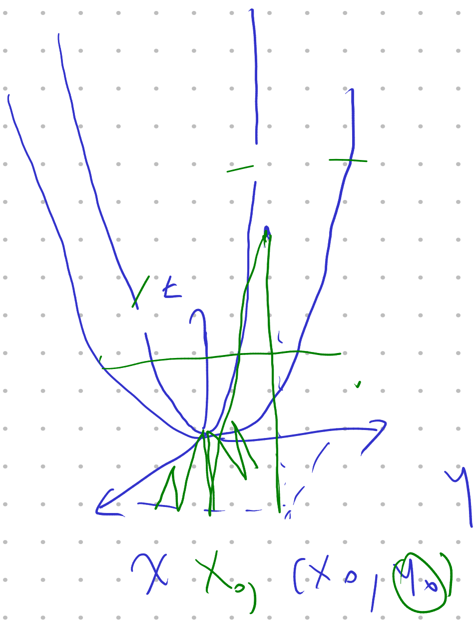


$$\frac{f(x_n)}{x_n - x_{n+1}} = f'(x_n)$$

$$f(x_n) = f'(x_n)\left(x_n - x_{n+1}\right)$$

$$\frac{f(x_n)}{f'(x_n)} = x_n - x_{n+1}$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$



Formal Proof of Newton Raphson Theorem

7.58 Theorem. [NEWTON-RAPHSON].
Suppose that $f : [a,b] \to \mathbf{R}$ is continuous on $[a,b]$ and that $f(c) = 0$ for some $c \in (a,b)$. If f'' exists and is bounded on (a,b) and there is an $\varepsilon_0 > 0$ such that $|f'(x)| \geq \varepsilon_0$ for all $x \in (a,b)$, then there is a closed interval $I \subseteq (a,b)$ containing c such that given $x_0 \in I$, the sequence $\{x_n\}_{n\in\mathbf{N}}$ defined by

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}, \qquad n \in \mathbf{N}, \tag{19}$$

satisfies $x_n \in I$ and $x_n \to c$ as $n \to \infty$.

Proof. Choose $M > 0$ such that $|f''(x)| \leq M$ for $x \in (a,b)$. Choose $r_0 \in (0,1)$ so small that $I = [c - r_0, c + r_0]$ is a subinterval of (a,b) and $r_0 < \varepsilon_0/M$. Suppose that $x_0 \in I$ and define the sequence $\{x_n\}$ by (19). Set $r := r_0 M/\varepsilon_0$ and observe by the choice of r_0 that $r < 1$. Thus it suffices to show that

$$|x_n - c| \leq r^n |x_0 - c| \tag{20}$$

and

$$|x_n - c| \leq r_0 \tag{21}$$

hold for all $n \in \mathbf{N}$.

The proof is by induction on n. Clearly, (20) and (21) hold for $n = 0$. Fix $n \in \mathbf{N}$ and suppose that

$$|x_{n-1} - c| \leq r^{n-1}|x_0 - c| \tag{22}$$

and that

$$|x_{n-1} - c| \leq r_0. \tag{23}$$

Use Taylor's Formula to choose a point ξ between c and x_{n-1} such that

$$-f(x_{n-1}) = f(c) - f(x_{n-1}) = f'(x_{n-1})(c - x_{n-1}) + \frac{1}{2}f''(\xi)(c - x_{n-1})^2.$$

Since (19) implies $-f(x_{n-1}) = f'(x_{n-1})(x_n - x_{n-1})$, it follows that

$$f'(x_{n-1})(x_n - c) = \frac{1}{2}f''(\xi)(c - x_{n-1})^2.$$

Solving this equation for $x_n - c$, we have by the choice of M and ε_0 that

$$|x_n - c| = \left|\frac{f''(\xi)}{2f'(x_{n-1})}\right| |x_{n-1} - c|^2 \leq \frac{M}{2\varepsilon_0}|x_{n-1} - c|^2. \tag{24}$$

Since $M/\varepsilon_0 < 1/r_0$, it follows from (24) and (23) that

$$|x_n - c| \le \frac{M}{\varepsilon_0}|x_{n-1} - c|^2 \le \frac{1}{r_0}|r_0|^2 = r_0.$$

This proves (21). Again, by (24), (22), and the choice of r, we have

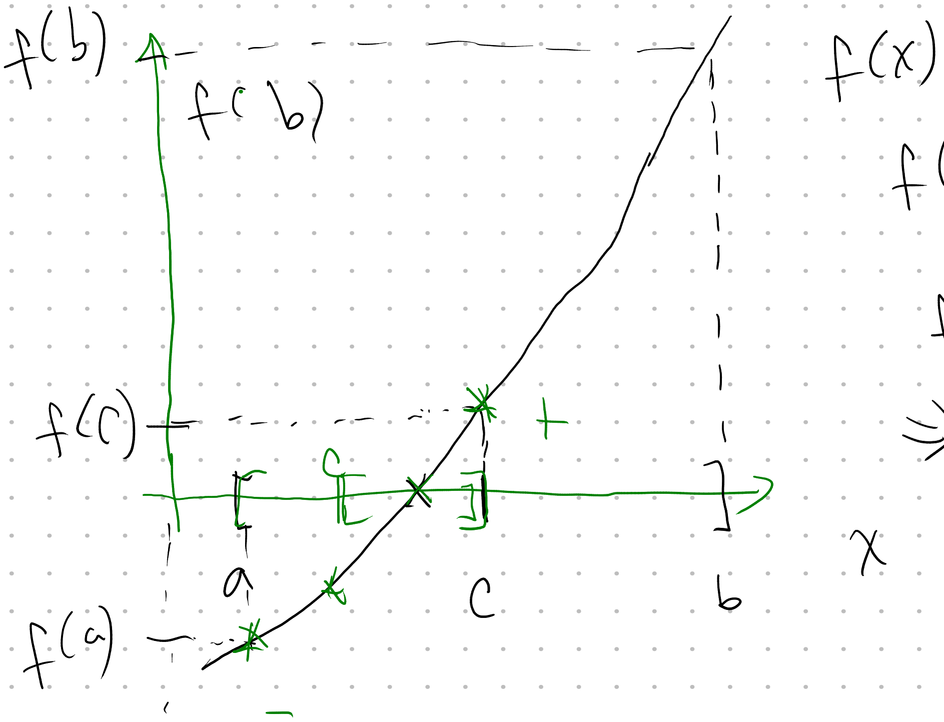
$$|x_n - c| \le \frac{M}{\varepsilon_0}(r^{n-1}|x_0 - c|)^2 = \frac{r}{r_0}(r^{2n-2}|x_0 - c|^2) \le r^{2n-1}|x_0 - c|.$$

Since $r < 1$ and $2n - 1 \ge n$ imply $r^{2n-1} \le r^n$, we conclude that $|x_n - c| \le r^{2n-1}|x_0 - c| \le r^n|x_0 - c|$.

Notice if x_{n-1} and x_n satisfy (19), then x_n is the x-intercept of the tangent line to $y = f(x)$ at the point $(x_{n-1}, f(x_{n-1}))$ (see Exercise 7.5.4). Thus, Newton's method is based on a simple geometric principle (see Figure 7.7). Also notice that, by (24), this method converges very rapidly. Indeed, the number of decimal places of accuracy nearly doubles with each successive approximation.

As a general rule, it is extremely difficult to show that a given nonalgebraic number is irrational. The next result shows how to use infinite series to give an easy proof that certain kinds of numbers are irrational.

Método de Bisección



$$f(x^*) = 0$$

$$f(c)f(b) > 0$$

$$\Rightarrow f(a)f(c) < 0$$

References

https://web.ma.utexas.edu/users/m408s/CurrentWeb/LM11-11-2.php
An introduction to Analysis, William R. Wade