Modeling 2.5d with 2d PDE solvers

for wave and dc energy transfers

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2.5d nature and 2d numerical code

We assume that in nature there is no lateral variation along the y axis $(\partial_y = 0)$. Because the xz-plane is embedded in xyz-space, the energy transfer is not truly 2d and so we refer to this assumption as "2.5d", i.e. we assume nature transfers energy in 2.5d.

We note that even though under the assumption $\partial_y = 0$ the energy transfer is 2.5d, all the relevant information of the material properties is 2d and lies in the xz-plane.

Numerically it is less computationally expensive to model 2d rather than 3d phenomena, therefore we either have to transform observed 2.5d data (d^o) into 2d data (d^o_{2d}) or conversely, transform the 2d synthetic field u_{2d} to a 2.5d field u. Table 1 gives quick reference to the methods used for both wave and dc energy transfers.

	nature		code	transform
wave	2.5d	$\rightarrow$	2d	$\tilde{d}^o_{2d} = \sqrt{\frac{2\pi v_o \Delta sr}{\lvert\omega\rvert}} \exp\left\{ \frac{\imath}{4} \operatorname{sgn}(\omega) \right\} \tilde{d}^o$
dc	2.5d	$\leftarrow$	2d	$u = \sum_i w_i u^i_{2d}$

Table 1: Quick 2d-2.5d transform reference table.

2.5d wave solver

For the wave energy transfer, we observe data in 2.5d, we solve the governing PDE in 2d, and we transform the observed data into 2d. We follow Bleistein

[1986] and Ernst et al. [2007].

Let $u = u(x, y, z, t, v_o;\, s)$ be the 2.5d wavefield on the xyz-space at time t due to a source $s = (s_x, 0, s_z, t)$ and a constant velocity v_o. Let u_{2d} be the 2d wave solved numerically, then the 2.5d→2d transform is given by,

$$\tilde{u}_{2d} = \sqrt{\frac{2\pi v_o\, \Delta sr_{xz}}{|\omega|}} \exp\left\{\frac{\imath}{4}\,\mathrm{sgn}(\omega)\right\}\, \tilde{u}, \tag{1}$$

where ω is angular frequency and Δsr_{xz} is distance between source and a point $(x, 0, z)$.

This 2d→2.5d asymptotic conversion assumes isotropic media, is valid in the high frequency eikonal regime, and does not take intrinsic attenuation into account.

In practice we only perform this transform on the observed data, i.e. only on receiver locations r and we approximate v_o with the average of the heterogeneous velocity model v,

$$\tilde{d}^o_{2d} = \sqrt{\frac{2\pi v_o\, \Delta sr}{|\omega|}} \exp\left\{\frac{\imath}{4}\,\mathrm{sgn}(\omega)\right\}\, \tilde{d}^o. \tag{2}$$

2d inversion

After transforming the 2.5d observed data into 2d, we proceed with the inversion by minimizing the error $e = d^o_{2d} - d$ where d is the synthetic data. This transform is performed only once before the inversion and never again.

The imaged parameters after the inversion will be in 2d, which in the case of material properties 2d is also 2.5d.

Routines

- `w_bleistein.m`

$$\tilde{b} = \sqrt{\frac{1}{|f|}} \exp\left\{\frac{\imath}{4}\,\mathrm{sgn}(f)\right\} \cdot \sqrt{\Delta sr}.$$

  $\tilde{b}$ is a matrix of size $(n_f \times n_r)$ and f is the frequency discretization.

- `w_2_5d_2d.m`

  Takes d_w^o and $\tilde{b}$ and performs the Bleistein filter in the frequency domain but returns d_w^o in the time domain.

2.5d dc solver

For the dc energy transfer, we observe data in 2.5d, we solve many (more than one at least) governing PDEs in 2d, and we transform these 2d solutions into 2.5d. We follow Pidlisecky and Knight [2008].

In 2.5d, we want to model

$$-\nabla \cdot \sigma(x,z)\nabla\, u(x,y,z) = s(x,y,z). \tag{3}$$

In the Fourier k_y-domain we have,

$$-2\nabla \cdot \sigma \nabla\, \tilde{u}(x,k_y,z) + 2k_y^2\, \sigma\, \tilde{u}(x,k_y,z) = s(x,y,z). \tag{4}$$

The 3d solution on the xz-plane is thus

$$u(x,y=0,z) = \frac{2}{\pi}\int_0^\infty \tilde{u}\, \mathrm{d}k_y. \tag{5}$$

Discretized, we have

$$u = \frac{2}{\pi}\sum_i \tilde{u}(k_i)\, \omega_i, \tag{6}$$

but what are k_i and ω_i? We follow Pidlisecky and Knight [2008] and proceed to find them by noting that the Green's function solution (for homogeneous σ) of (3) on the half xz-plane is

$$u(x,y=0,z) = \frac{\mathbf{i}}{2\pi\sigma}\underbrace{\left(\frac{1}{\underbrace{||x-s_+||_2}_{r+}} - \frac{1}{\underbrace{||x-s_-||_2}_{r-}}\right)}_{1/R}, \tag{7}$$

bringing back to the Fourier domain,

$$\tilde{u} = \int_0^\infty u\, \cos(y\, k_y)\, \mathrm{d}y = \frac{\mathbf{i}}{2\pi\sigma}(B_o(k_y r_+) - B_o(k_y r_-)), \tag{8}$$

where B_o is the zero order modified Bessel function of the second kind. By plugging in equations 7 and 8 into equation 6 we discretize by

$$1 \approx \sum_j \underbrace{\frac{2R}{\pi}\{B_o(k_j r_+) - B_o(k_j r_-)\}}_{K_{ij}} \omega_j \tag{9}$$

$$K = \frac{2}{\pi} R \{B_o(k\, r_+) - B_o(k\, r_-)\} \tag{10}$$

$$v = K\omega, \tag{11}$$

where $K = K(k, s)$ is a matrix of size $(n_R \times n_k)$, v is a vector of length n_R whose entries should approximate 1, and $k = (k_i)$, $\omega = (\omega_i)$ are vectors of length n_k. We minimize

$$\Phi(k) = ||1 - \underbrace{K \underbrace{(K^T K)^{-1} K^T}_{\omega}}_{v(k)} ||_2^2 = ||1 - v(k)||_2^2,$$

using a regularized Newton method. Note that both k and ω are geometry dependent and not parameter dependent.

Finding k_y and ω for a given s

1. initial guess for $k =$ (some numbers) $\cdot \Delta x$ and build $K(k, s)$.
2. $v \leftarrow K(K^\top K)^{-1} K^\top \cdot 1$,
3. compute $J = \nabla_k v$ using n_k finite differences,
4. $\nabla_k \Phi^\top \leftarrow J^\top (1 - v) + \beta k$,
5. $\Delta k = (J^\top J + \beta I)^{-1} \cdot \nabla_k \Phi^\top$,
6. $k \leftarrow k + \alpha\, \Delta k$,
7. build $K(k, s)$
8. check if v is almost 1,
9. repeat $2 - 8$
10. $\omega = (K^\top K)^{-1} K^\top \cdot 1$,
11. correct for flatness $k \leftarrow k \cdot \Delta x$
12. return k and $\frac{2}{\pi}\omega$.

Finding the 2.5d electric potential u for given s and σ

Given a source s and conductivity σ, the forward model is computed as follows,

1. retrieve from memory (or compute) k and ω,
2. choose $k_i \in k$ and build L_i with the right boundary conditions for that k_i,

$$\begin{aligned} L^i &\approx -\nabla \cdot \sigma \nabla, \\ L_i &= L^i + k_i^2 \, \sigma, \end{aligned}$$

3. solve $L_i \tilde{u}_i = \frac{s}{2}$ for $\tilde{u}_i$ and store,
4. repeat 3-4 until all k has been used,
5. $u = \sum_i \omega_i \, \tilde{u}_i \to d = Mu \to e = d - d^o$.

2.5d inversion

Each 2d forward model is,

$$\begin{aligned} L_i &= L^i + k_i^2 \, \sigma, \\ L_i \tilde{u}_i &= \frac{s}{2}, \\ \tilde{d}_i &= M \tilde{u}_i. \end{aligned}$$

We can write the 2.5d data and its Jacobian as a linear combination of each 2d problem,

$$\begin{aligned} d &= M \underbrace{\sum_i \omega_i \tilde{u}_i}_{u} = \sum_i \omega_i \underbrace{M \tilde{u}_i}_{\tilde{d}_i} = \sum_i \omega_i \tilde{d}_i, \\ \nabla_\sigma d &= \underbrace{\sum_i \omega_i J_i}_{J}, \end{aligned}$$

where

$$J_i = M L_i^{-1} S_i^\top, \qquad \text{and} \qquad S_i = -\left((\nabla_\sigma L^i) \tilde{u}_i \right)^\top - k_i^2 \, \mathsf{diag}(\tilde{u}_i)^\top.$$

We can now write the gradient of the 2.5d data as a linear combination of the 2d gradients,

$$g = \left(\sum_i \omega_i J_i \right)^\top e = \sum_i \omega_i \underbrace{J_i^\top e}_{g_i} = \sum_i \omega_i g_i,$$

where

$$g_i = S_i a_i, \qquad \text{and} \qquad L_i^\top a_i = M^\top e.$$

Finding g for given s and σ

Given a source s, conductivity σ and weights $\{k, \omega\}$ the gradient is computed as follows,

1. compute the 2.5d forward model to get $\{\tilde{u}_i\}$ and e,
2. choose $k_i \in k$,
3. build L_i and $S_i = -\left((\nabla_\sigma L^i)\tilde{u}_i\right)^\top - k_i^2\, \text{diag}(\tilde{u}_i)^\top$,
4. solve $L_i^\top a_i = M^\top e$ for a_i,
5. compute and store $g_i = S_i a_i$,
6. repeat 2-5 until all k has been used,
7. $g = \sum_i \omega_i\, g_i$.

Routines

- `dc_kfourier.m` for given source s outputs $\{k, \omega\}$.
- `dc_fwd2_5d.m` for a given source s and for each $k_i \in k$ performs `dc_fwd_k.m` and solves u by weight-stacking $\{\tilde{u}_{k_i}\}$. Outputs L, u, $\{\tilde{u}_{k_i}\}$, $d = Mu$ and $e = d - d^o$.
  - `dc_fwd_k.m` for a given source s and a given $k_i \in k$ solves the 2d fwd problem $L_{k_i}\tilde{u}_{k_i} = s$. Outputs L and $\tilde{u}_{k_i}$.
- `dc_gradient2_5d.m` for a given source s, its weights $\{k, \omega\}$, its 2d potentials $\{\tilde{u}_{k_i}\}$, its matrix L and its 2.5d error e, outputs g as a weighted stack of $\{g_{k_i}\}$.

Verification with analytical models

anal-homo and anal-bi.

References

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