Two-dimensional waves

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2D wave equation

Let $\nabla = (\partial_x, \partial_z)$ be a row vector.

$$\mathbf{C}\dot{\mathbf{v}} = \nabla u - \mathbf{A}\mathbf{v} + \mathbf{g} \tag{1}$$
$$c\dot{u} = \nabla \cdot \mathbf{v} - au + f \tag{2}$$

Velocity is determined by $\mathbf{C}$ and c. Intrinsic attenuation is determined by $\mathbf{A}$ and a.

In expanded-matrix form we have,

$$\begin{pmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} \dot{v}_x \\ \dot{v}_z \\ \dot{u} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \partial_x \\ 0 & 0 & \partial_z \\ \partial_x & \partial_z & 0 \end{pmatrix} \begin{pmatrix} v_x \\ v_z \\ u \end{pmatrix} - \tag{3}$$
$$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a \end{pmatrix} \begin{pmatrix} v_x \\ v_z \\ u \end{pmatrix} + \begin{pmatrix} g_x \\ g_z \\ f \end{pmatrix}. \tag{4}$$

Note that this equation can handle anisotropy and attenuation acting on **v**.

2D EM wave equation

$$\boldsymbol{\mu}\dot{\mathbf{H}} = -\nabla \times \mathbf{E} - \boldsymbol{\sigma}^m \mathbf{H} + \mathbf{M}_s \tag{5}$$

$$\boldsymbol{\varepsilon}\dot{\mathbf{E}} = \nabla \times \mathbf{H} - \boldsymbol{\sigma}\mathbf{E} - \mathbf{J}_s \tag{6}$$

Assuming quasi-isotropic materials and $\partial_y = \boldsymbol{\sigma}^m = \mathbf{M}_s = 0$, in expanded-matrix form we have the TE mode,

$$\begin{pmatrix} \mu_z & 0 & 0 \\ 0 & \mu_x & 0 \\ 0 & 0 & \varepsilon \end{pmatrix} \begin{pmatrix} -\dot{H}_z \\ \dot{H}_x \\ \dot{E}_y \end{pmatrix} = \begin{pmatrix} 0 & 0 & \partial_x \\ 0 & 0 & \partial_z \\ \partial_x & \partial_z & 0 \end{pmatrix} \begin{pmatrix} -H_z \\ H_x \\ E_y \end{pmatrix} - \tag{7}$$

$$\sigma \begin{pmatrix} 0 \\ 0 \\ E_y \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ J_y \end{pmatrix}. \tag{8}$$

The 2D EM wave code

The actual code models the 2D EM wave assuming $\partial_z = 0$ and quasi-isotropic materials with attenuation for both E_z and $\mathbf{H}$,

$$\begin{pmatrix} \mu_y & 0 & 0 \\ 0 & \mu_x & 0 \\ 0 & 0 & \varepsilon_z \end{pmatrix} \begin{pmatrix} \dot{H}_y \\ -\dot{H}_x \\ \dot{E}_z \end{pmatrix} = \begin{pmatrix} 0 & 0 & \partial_x \\ 0 & 0 & \partial_y \\ \partial_x & \partial_y & 0 \end{pmatrix} \begin{pmatrix} H_y \\ -H_x \\ E_z \end{pmatrix} - \tag{9}$$

$$\begin{pmatrix} \sigma_y^m & 0 & 0 \\ 0 & \sigma_x^m & 0 \\ 0 & 0 & \sigma_z \end{pmatrix} \begin{pmatrix} H_y \\ -H_x \\ E_z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -J_z \end{pmatrix}. \tag{10}$$

2D elastic SV wave equation

Let ∇_{el} be the matrix,

$$\nabla_{el} = \begin{pmatrix} \partial_x & 0 & \partial_z \\ 0 & \partial_z & \partial_x \end{pmatrix} \qquad \nabla_{el}^\top = \begin{pmatrix} \partial_x & 0 \\ 0 & \partial_z \\ \partial_z & \partial_x \end{pmatrix}, \tag{11}$$

and the particle velocity and stress be $\mathbf{v}^{el}$ and $\mathbf{T}$ respectively,

$$\mathbf{v}^{el} = \begin{pmatrix} v_x^{el} \\ v_z^{el} \end{pmatrix} \qquad \mathbf{T} = \begin{pmatrix} \sigma_{xx} \\ \sigma_{xz} \\ \sigma_{zz} \end{pmatrix}. \tag{12}$$

The 2D elastic SV polarized wave equation is given by,

$$\mathbf{s}\dot{\mathbf{T}} = \nabla_{el}^\top \cdot \mathbf{v} - \boldsymbol{\tau}\mathbf{T} \tag{13}$$

$$\rho\dot{\mathbf{v}}^{el} = \nabla_{el} \cdot \mathbf{T} + \mathbf{f} \tag{14}$$

where $\mathbf{s}$ and $\boldsymbol{\tau}$ denote the inverse of stiffness and viscosity respectively [Carcione and Cavallini, 1995]. We have (how does $\boldsymbol{\tau}^{-1}$ look like??),

$$\mathbf{s}^{-1} = \begin{pmatrix} \lambda + 2\mu & \lambda & 0 \\ 0 & 0 & \mu \\ \lambda & \lambda + 2\mu & 0 \end{pmatrix} \tag{15}$$

$$v_p = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \qquad v_s = \sqrt{\frac{\mu}{\rho}}. \tag{16}$$

$$\lambda = (v_p^2 - 2v_s^2)\rho \qquad \mu = \rho v_s^2 \tag{17}$$

The *first* and *second* Lamé parameters are λ and μ respectively.

Following Groos et al. [2017],

$$200 \leq v_p\,(m/s) \leq 500 \tag{18}$$

$$80 \leq v_s\,(m/s) \leq 400 \tag{19}$$

$$1700 \leq \rho\,(Kg/m^3) \leq 2000 \tag{20}$$

$$-1.4 \times 10^8 \leq \lambda\,(Kg/m/s^2) \leq 4.6 \times 10^7 \tag{21}$$

$$10^7 \leq \mu\,(Kg/m/s^2) \leq 3.2 \times 10^8 \tag{22}$$

At $20°C$ and 1 atm (101.325 kPa), air-speed is $v_p = 343.21m/s$ with a density of $\rho = 1.2041Kg/m^3$.

References

JoséM Carcione and Fabio Cavallini. On the acoustic-electromagnetic analogy. *Wave motion*, 21(2):149–162, 1995.

Lisa Groos, Martin Schäfer, Thomas Forbriger, and Thomas Bohlen. Application of a complete workflow for 2d elastic full-waveform inversion to recorded shallow-seismic rayleigh wavesworkflow for fwi of rayleigh waves. *Geophysics*, 82(2):R109–R117, 2017.