2d to 2.5d transform

We assume that in nature there is no lateral variation along the y axis ($\partial_y = 0$), and we model the synthetic dc electric potential in a true 2-dimensional setting. In 3d, we want to model

$$-\nabla \cdot \sigma(x,z)\nabla\, u(x,y,z) = s(x,y,z). \tag{1}$$

In the Fourier k_y-domain we have,

$$-\nabla \cdot \sigma\nabla\, \tilde{u}(x,k_y,z) + k_y^2\, \sigma\, \tilde{u}(x,k_y,z) = \frac{1}{2}s(x,y,z). \tag{2}$$

The 3d solution on the xz-plane is thus

$$u(x,y=0,z) = \frac{2}{\pi}\int_0^\infty \tilde{u}\, \mathrm{d}k_y. \tag{3}$$

Discretized, we have

$$u = \frac{2}{\pi}\sum_i \tilde{u}(k_i)\, \omega_i, \tag{4}$$

but what are k_i and ω_i? We follow Pidlisecky and Knight [2008] and proceed to find them by noting that the Green's function solution (for homogeneous σ) of (1) on the half xz-plane is

$$u(x,y=0,z) = \frac{\mathbf{i}}{2\pi\sigma}\underbrace{\left(\frac{1}{\underbrace{||x-s_+||_2}_{r+}} - \frac{1}{\underbrace{||x-s_-||_2}_{r-}}\right)}_{1/R}, \tag{5}$$

bringing back to the Fourier domain,

$$\tilde{u} = \int_0^\infty u\, \cos(y\, k_y)\, \mathrm{d}y = \frac{\mathbf{i}}{2\pi\sigma}(B_o(k_y r_+) - B_o(k_y r_-)), \tag{6}$$

where B_o is the zero order modified Bessel function of the second kind. By plugging in equations 5 and 6 into equation 4 we discretize by

$$1 \approx \sum_j \underbrace{\frac{2R}{\pi}\{B_o(k_j\, r_+) - B_o(k_j\, r_-)\}}_{K_{ij}}\, \omega_j \tag{7}$$

$$K = \frac{2}{\pi}R\,\{B_o(k\, r_+) - B_o(k\, r_-)\} \tag{8}$$

$$v \approx K\omega, \tag{9}$$

where $K = K(k,\, s)$ is a matrix of size $(n_R \times n_k)$, v is a vector of length n_R whose entries should approximate 1, and $k = (k_{yi})$, $\omega = (\omega_i)$ are vectors of length n_k. We minimize

$$\Phi(k) = ||1 - \underbrace{K\,\underbrace{(K^T K)^{-1} K^T}_{\omega}}_{v(k)}||_2^2 = ||1 - v(k)||_2^2,$$

using a regularized Newton method. Note that both k and ω are geometry dependent and not parameter dependent.

Finding k_y and ω for a given s

1. initial guess for $k =$ (some numbers) $\cdot\, \Delta x$ and build $K(k, s)$.
2. $v \leftarrow K(K^\top K)^{-1} K^\top \cdot 1$,
3. compute $J = \nabla_k v$ using n_k finite differences,
4. $\nabla_k \Phi^\top \leftarrow J^\top (1 - v) + \beta k$,
5. $\Delta k = (J^\top J + \beta I)^{-1} \cdot \nabla_k \Phi^\top$,
6. $k \leftarrow k + \alpha\, \Delta k$,
7. build $K(k, s)$
8. check if v is almost 1,
9. repeat $2 - 8$
10. $\omega = (K^\top K)^{-1} K^\top \cdot 1$,
11. correct for flatness $k \leftarrow k \cdot \Delta x$
12. return k and $\frac{2}{\pi}\omega$.

Finding the 2.5d electric potential u for given s and σ

Given a source s and conductivity σ, the forward model is computed as follows,

1. retrieve from memory (or compute) k and ω,

2. choose $k_i \in k$ and build L_i with the right boundary conditions for that k_i,

$$
\begin{aligned}
L^i &\approx -\nabla \cdot \sigma \nabla, \\
L_i &= L^i + k_i^2\, \sigma,
\end{aligned}
$$

3. solve $L_i \tilde{u}_i = \frac{s}{2}$ for $\tilde{u}_i$ and store,
4. repeat 3-4 until all k has been used,
5. $u = \sum_i \omega_i\, \tilde{u}_i \rightarrow d = Mu \rightarrow e = d - d^o$.

2.5d inversion

Each 2d forward model is,

$$
\begin{aligned}
L_i &= L^i + k_i^2\, \sigma, \\
L_i \tilde{u}_i &= \frac{s}{2}, \\
\tilde{d}_i &= M \tilde{u}_i.
\end{aligned}
$$

We can write the 2.5d data and its Jacobian as a linear combination of each 2d problem,

$$
\begin{aligned}
d &= M \underbrace{\sum_i \omega_i \tilde{u}_i}_{u} = \sum_i \omega_i \underbrace{M \tilde{u}_i}_{\tilde{d}_i} = \sum_i \omega_i \tilde{d}_i, \\
\nabla_\sigma d &= \underbrace{\sum_i \omega_i J_i}_{J},
\end{aligned}
$$

where

$$
J_i = M L_i^{-1} S_i^\top, \qquad \text{and} \qquad S_i = -\left((\nabla_\sigma L^i)\tilde{u}_i\right)^\top - k_i^2\, \mathsf{diag}(\tilde{u}_i)^\top.
$$

We can now write the gradient of the 2.5d data as a linear combination of the 2d gradients,

$$
g = \left(\sum_i \omega_i J_i\right)^\top e = \sum_i \omega_i \underbrace{J_i^\top e}_{g_i} = \sum_i \omega_i g_i,
$$

where

$$
g_i = S_i a_i, \qquad \text{and} \qquad L_i^\top a_i = M^\top e.
$$

Finding g for given s and σ

Given a source s, conductivity σ and weights $\{k, \omega\}$ the gradient is computed as follows,

1. compute the 2.5d forward model to get $\{\tilde{u}_i\}$ and e,
2. choose $k_i \in k$,
3. build L_i and $S_i = -\left((\nabla_\sigma L^i)\tilde{u}_i\right)^\top - k_i^2 \,\text{diag}(\tilde{u}_i)^\top$,
4. solve $L_i^\top a_i = M^\top e$ for a_i,
5. compute and store $g_i = S_i a_i$,
6. repeat 2-5 until all k has been used,
7. $g = \frac{2}{\pi} \sum_i \omega_i \, g_i$.

Routines

- `dc_kfourier.m` for given source s outputs $\{k, \omega\}$.
- `dc_fwd2_5d.m` for a given source s and for each $k_i \in k$ performs `dc_fwd_k.m` and solves u by weight-stacking $\{\tilde{u}_{k_i}\}$. Outputs L, u, $\{\tilde{u}_{k_i}\}$, $d = Mu$ and $e = d - d^o$.
    - `dc_fwd_k.m` for a given source s and a given $k_i \in k$ solves the 2d fwd problem $L_{k_i}\tilde{u}_{k_i} = s$. Outputs L and $\tilde{u}_{k_i}$.
- `dc_gradient2_5d.m` for a given source s, its weights $\{k, \omega\}$, its 2d potentials $\{\tilde{u}_{k_i}\}$, its matrix L and its 2.5d error e, outputs g as a weighted stack of $\{g_{k_i}\}$.

Verification with analytical models

anal-homo and anal-bi.

References

Adam Pidlisecky and Rosemary Knight. Fw2_5d: A matlab 2.5-d electrical resistivity modeling code. *Computers & Geosciences*, 34(12):1645–1654, 2008.