ε, σ & ω

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For the EM wave with attenuation (Ohm's law for $\mathbf{J}$) and the plane wave approximation we get

$$k_*^2 = \mu\varepsilon\omega^2 + i\mu\sigma\omega \qquad\qquad k_* = a + ib,$$

for some $a, b \in \mathbb{R}$. a controls velocity and b controls attenuation. We now suppose $\varepsilon, \sigma \in \mathbb{C}$ (why??) with

$$\varepsilon = \varepsilon' - i\varepsilon'' \qquad\qquad \sigma = \sigma' + i\sigma'',$$

so we now have

$$k_*^2 = \mu\omega \underbrace{(\varepsilon'\omega - \sigma'')}_{\varepsilon_e\omega} + i\mu\omega \underbrace{(\varepsilon''\omega + \sigma')}_{\sigma_e}$$

where ε_e, σ_e are *effective* electrical properties, which are the ones we are actually sensitive to in nature (right?). We have

$$\varepsilon_e = \varepsilon' - \frac{\sigma''}{\omega} \qquad\qquad \sigma_e = \sigma' + \varepsilon''\omega,$$

from which we now take limits on ω,

low ω	high ω
$\varepsilon_e \approx -\frac{\sigma''}{\omega}$	$\varepsilon_e \approx \varepsilon'$
$\sigma_e \approx \sigma_{dc}$	$\sigma_e \approx \sigma_{dc} + \varepsilon''\omega$

Horrible formulas for a and b,

$$a = \omega\sqrt{\frac{\mu\varepsilon}{2}}\left(\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right)^{1/2}$$

$$b = \omega\sqrt{\frac{\mu\varepsilon}{2}}\left(\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right)^{1/2},$$

$$|k_*| = \omega\sqrt{\varepsilon\omega\sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2}}$$

We now assume the low-loss approximation $\sigma_e << \omega\varepsilon_e$ and use the Taylor expansion for $\sqrt{x+1}$ to get

$$\begin{aligned} b &= \frac{\sigma_e}{2}\sqrt{\frac{\mu}{\varepsilon_e}} \\ &= \frac{\sigma_{dc} + \varepsilon''\omega}{2}\sqrt{\frac{\mu}{\varepsilon'}} \qquad \leftarrow \text{for high } \omega. \end{aligned}$$