Forward modeling of electromagnetic waves and steady currents: First steps for a GPR and ER joint inversion

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Abstract

Electromagnetic methods are an important tool for imaging material properties in the subsurface. With the longterm goal of performing a joint inversion using electromagnetic data from different types of energy transport, we developed our own forward models for the *electrical resistivity* (ER) and *ground penetrating radar* (GPR) experiments.

Regarding the transport of electromagnetic energy, we assume the subsurface is made by linear isotropic materials where Ohm's Law holds, there is no free charge and no significant variation along the y axis (horizontal to the ground),

$$\nabla \times \mathbf{E} = -\mu \dot{\mathbf{H}} \qquad \nabla \cdot \mathbf{E} = 0$$
$$\nabla \times \mathbf{H} = \varepsilon \dot{\mathbf{E}} + \sigma \mathbf{E} - \mathbf{J}_s \qquad \nabla \cdot \mathbf{H} = 0.$$

We distinguish two variants of electromagnetic energy transport: steady case (for the ER experiment) and wave propagation (for the GPR experiment). We model both experiments in two dimensions and numerically solve for their energy response using a *finite volume method* (FVM) for ER and a *finite difference time domain* (FDTD) for GPR.

Steady current (ER) continuous

We denote Ω the halfplane perpendicular to the ground which will be our domain.

By writting $\mathbf{E} = -\nabla\varphi$ we have

$$-\nabla \cdot \sigma \nabla \varphi = \nabla \cdot \mathbf{J}_s.$$

Our forward model equations are

$$\begin{aligned} \nabla \cdot \sigma \nabla \varphi &= \boldsymbol{i}\,(\delta^2_{\mathbf{x}-\mathbf{s}_-} - \delta^2_{\mathbf{x}-\mathbf{s}_+}) && \text{in } \Omega \\ \sigma \nabla \varphi \cdot \hat{\mathbf{n}} &= 0 && \text{on } \Gamma_N \\ \sigma \nabla \varphi \cdot \hat{\mathbf{n}} + \alpha \varphi &= 0 && \text{on } \Gamma_R \end{aligned}$$

where Γ_N is the ground-air interface and Γ_R is the boundary everywhere else.

Wave (GPR) continuous

We do a change of variables

$$\mathbf{v} = (H_z,\ -H_x)^t \qquad u = E_y$$
$$\mathbf{s} = (0,\ 0,\ J_y)^t \qquad \mathbf{w} = (\mathbf{v},\, u)^t$$

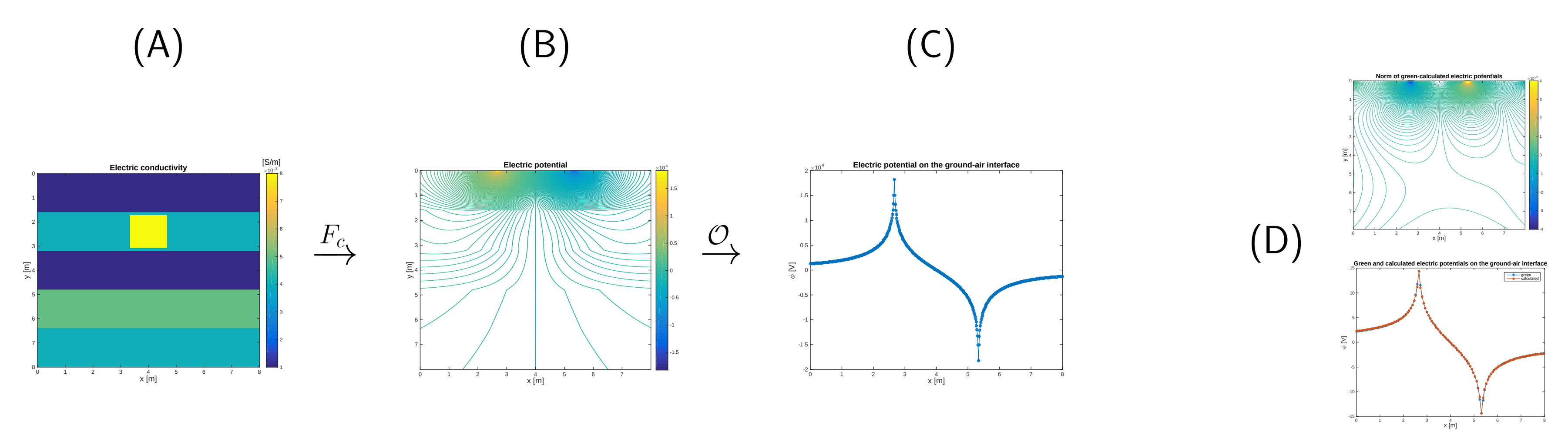
to write our forward model in Ω as an advection transport,

$$\dot{\mathbf{w}} = \begin{bmatrix} \frac{1}{\mu} I_2 & 0 \\ 0 & \frac{1}{\varepsilon} \end{bmatrix} \begin{bmatrix} 0 & \nabla^t \\ \nabla & 0 \end{bmatrix} \mathbf{w} + \begin{bmatrix} 0 & 0 \\ 0 & -\frac{\sigma}{\varepsilon} \end{bmatrix} \mathbf{w} + \mathbf{s}.$$

On $\partial\Omega$ we impose a *perfectly matched layer* (PML),

$$\dot{\mathbf{v}} = \frac{1}{\mu} \nabla u - \Sigma \mathbf{v}$$
$$\begin{bmatrix} \dot{u}_x \\ \dot{u}_z \end{bmatrix} = \frac{1}{\varepsilon} \begin{bmatrix} \partial_x & 0 \\ 0 & -\partial_z \end{bmatrix} \mathbf{v} - \Sigma \begin{bmatrix} u_x \\ u_z \end{bmatrix}.$$

Steady current (ER) experiment

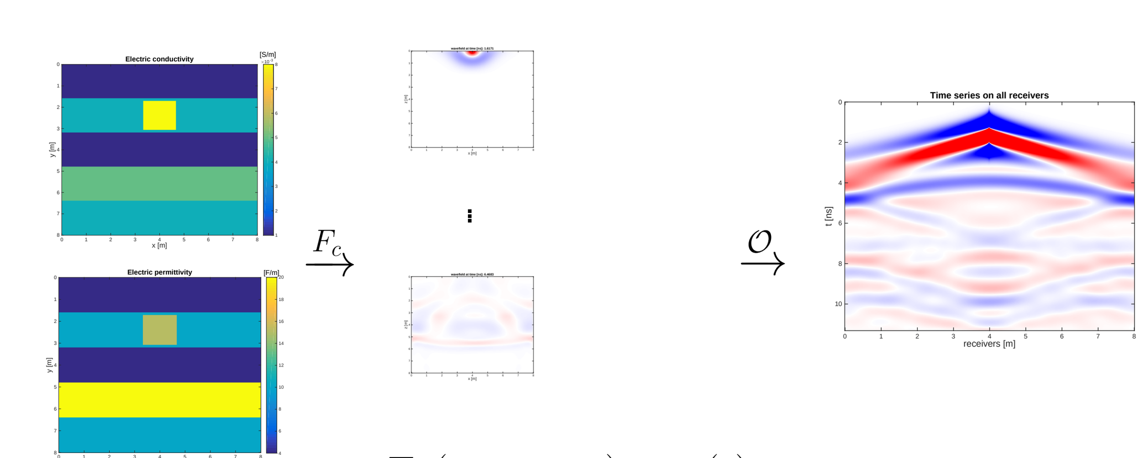


Figures (A)-(B)-(C) show a flow diagram of how the ER experiment is conducted. Figure (A) shows a synthetic distribution of electric conductivity σ that varies in space. After an electric current $\boldsymbol{i}$ has been injected at source and sink locations 2.7m and 5.7m respectively, we solve for the electric potential φ shown in Figure (B). We denote the process of solving for φ given σ and $\boldsymbol{i}$ by F_c. Figure (C) shows the electric potential that is $\mathcal{O}$bserved on the air-ground interface φ^o,

$$F_c(\sigma, \boldsymbol{i}) = \varphi \qquad \mathcal{O}(\varphi) = \varphi^o.$$

Figures in (D) show a comparison between our solved electric potential and Green's solution for the electric potential in the case of homogeneous conductivity $\sigma = 1$. The error is in the order of 10^{-3}.

Wave (GPR) experiment

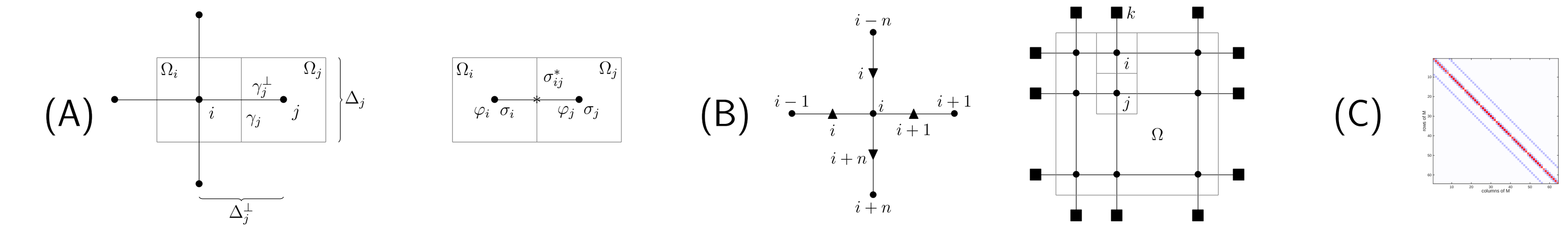


Given a synthetic distribution of electric permittivity ε and conductivity σ that vary in space, we calculate the electromagnetic wavefield u generated by our source at every point and time in our domain. We denote this process by F_w.

$$F_w(\sigma, \varepsilon, s_y, t) = u(t) \qquad \mathcal{O}(u) = u^o$$

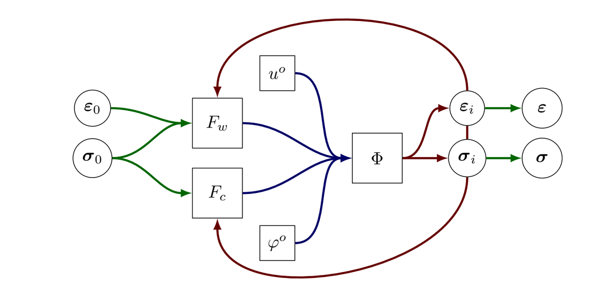
Our antenna receivers located on the air-ground interface observe u and we express their record as a time series which we denote by $\mathcal{O}(u) = u^o$.

FVM discretization



We number the nodes in our discretized domain with integers in the order of reading a book. Figure (A) shows a region Ω_i with its neighbor Ω_j where values for φ are taken at its center nodes and values for σ are taken on the interface of the regions. Figure (B) *left* shows the numbering scheme for interior nodes and edges with ●, ▲ and ▼ denoting nodes, horizontal edges and vertical edges respectively. Figure (B) *right* shows our discretized region Ω with interior nodes ●, ghost nodes (for implementing boundary conditions) ■, and typical regions Ω_i, Ω_j. Figure (C) shows an example of the matrix $\mathbf{M}$ on an 8×8 grid exposing its sparsity and diagonal structure.

Joint inversion (next steps)



We propose to image the material properties ε and σ by joining the steady current (ER) and wavefield (GPR) data since they are both sensitive to σ in complementary ways: ER is most sensitive to low frequency conductivity while GPR is most sensitive to high frequency conductivity. This allows us to build an objective function Φ that takes into account both data in order to mutually regularize the inversion for ER and GPR, while honoring the physics and avoiding biased regularization. The diagram shown is the work flow of how this inversion will take place, where σ_0 and ε_0 are initial guesses for the material properties, and the red and blue arrows indicate the iterative procedures that will take place.

Conclusions and future work

In order to perform a joint inversion of ER and GPR data, we have implemented FVM and FDTD methods to numerically solve for the electric potential and electric wavefield respectively. We perform the discretization on a single grid (both F_c and F_w are solved in the same grid) that is allowed to vary the size of its cells. Both methods are accurate and relatively fast to run on a common laptop for moderate sized problems. In future work we will implement a 2 dimensional joint inversion using a *full wavefield inversion* (FWI) approach to handle the GPR data, using a regularizing term explicitly dependent on both forward models.

Steady current (ER) discretization

We use a finite volume method (FVM) by exploiting the identity

$$\nabla \cdot \mathbf{F}|_{\mathbf{x}} = \lim_{A \to \{\mathbf{x}\}} \oint \frac{\mathbf{F} \cdot \hat{\mathbf{n}}}{A}\, \mathrm{d}\Gamma,$$

where $\mathbf{F}$ is a vector field and A is the area of a region Ω enclosing $\mathbf{x}$. For a region $\Omega_i \subset \Omega$ with $\partial\Omega_i = \Gamma_i$ we have,

$$\int_{\Omega_i} \nabla \cdot \sigma \nabla \varphi \ \mathrm{d}\Omega_i = \oint_{\Gamma_i} \sigma \nabla \varphi \cdot \hat{\mathbf{n}}_{\Gamma_i} \ \mathrm{d}\Gamma_i$$
$$\approx \sum_j \sigma^*_{ij} \frac{\Delta_j}{\Delta_j^{\perp}} (\varphi_j - \varphi_i)$$

where $\Gamma_i = \sum_j \gamma_j$, and j runs over all neighbors of Ω_i. We rearrange the grid values of φ into a vector, and write

$$\mathbf{M}(\sigma)\varphi = q \qquad \Rightarrow \varphi = \mathbf{M}^{-1} q$$

where q is a vector with nonzero entries only where the source and sink are located.

Wave (GPR) discretization

We use a finite difference time domain (FDTD) approach on a staggered space-time grid and discretize as

$$\begin{aligned} \dot{v}_z^t &\to \frac{v_z^{t+1/2} - v_z^{t-1/2}}{2\Delta t} \\ \partial_x u_{i+1/2} &\to \frac{u_{i+1} - u_i}{2\Delta x} \\ \dot{u}^{t+1/2} &\to \frac{u^{t+1} - u^t}{2\Delta t} \\ \partial_x v_{x,i-1/2} &\to \frac{v_{x,i} - v_{x,i-1}}{2\Delta x} \end{aligned}$$

Since a Ricker wavelet is the first derivative of a gaussian, we use the following expression for s_y

$$s_{y,i}^{t+1/2} \to -\sqrt{\frac{2e}{\omega_o}}(t - t_o) e^{-\frac{(t-t_o)^2}{\omega_o^2}}.$$

References

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